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Hyperbolic Geometry With and Without Models

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Chad Kelterborn

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HYPERBOLIC GEOMETRY WITH AND WITHOUT MODELS

CHAD KELTERBORN

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Abstract

We explore the development of hyperbolic geometry in the 18th and early 19th following the works of Legendre, Lambert, Saccheri, Bolyai, Lobachevsky, and Gauss. In their attempts to prove Euclid's parallel postulate, they developed hyperbolic geometry without a model. It was not until later in the 19th century, when Felix Klein provided a method (which was influenced by projective geometry) for viewing the hyperbolic plane as a disk in the Euclidean plane, appropriately named the "Klein disk model". Later other models for viewing the hyperbolic plane as a subset of the Euclidean plane were created, namely the Poincaré disk model, Poincaré spherical model, and Poincaré upper half-plane model. In proving various theorems of hyperbolic geometry, the thesis focuses on the Klein disk model because this model allows us to view hyperbolic lines as Euclidean chords. We then establish the isomorphisms between the various models of hyperbolic geometry. And in the end, we consider a fifth model, the Minkowsky space-time model from the Special Theory of Relativity (STR), and its connection/isomorphism to the Klein disk and the Poincaré disk models of hyperbolic geometry.

Acknowledgements

I would like to thank the Math department for the continued support and encouragement that they have provided me throughout my studies at Eastern Illinois University. I would also like to thank my adviser for all of the enlightening discussions we shared on hyperbolic geometry.

To my family.

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Introduction

1.1 A Little Bit of History

The development of non-Euclidean geometry evolved from the attempts of several prominent mathematicians of the 17th and 18th centuries to prove Euclid's 5th postulate. In the 4th century BC, a Greek mathematician, named Euclid, set out to formulate certain statements which could be accepted as indisputable truths, axioms, or postulates. These axioms laid the foundation for the development of what today is known as Euclidean geometry.

Euclid's five axioms are as follows:

(E_1) Any line contains at least 2 distinct points and any 2 points determine the line uniquely.

(E_2) Any segment can be extended as far as you wish.

(E_3) For every point O and for every line segment CD, there exists a circle $c = c(O, OA)$ centered at point O of radius OA, where $OA \simeq CD$.

(E_4) Any two right angles are congruent to each other.

(E_5) For every line l and for every point P not lying on l, there exists a unique line m passing through P such that the lines m and l are parallel.

The first four Euclidean axioms are so natural that they are indeed indisputable. As for the 5th postulate, Euclid assumed that it had to be true, yet he was unable to prove it as a theorem from the first four postulates. He attempted to prove the fifth postulate, since unlike the first four postulates, the fifth postulate is not self-evident. All aspects of Euclidean geometry follow from these five postulates. Proving Euclid's parallel postulate became the focus of many mathematicians work for centuries to come.

In Euclidean geometry there are two methods for constructing a rectangle.

Method 1:

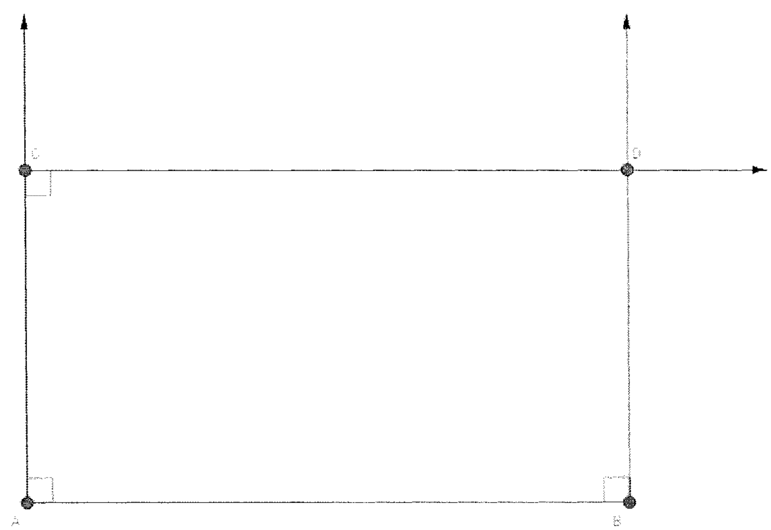

Figure 1.1: Method 1 for constructing a rectangle.

Start with the segment AB. Then erect the ray perpendicular to AB at A and take a point C lying on that ray. Now erect the ray perpendicular to the segment AB at point B and the ray perpendicular to the segment AC at point C. These two rays intersect at a point which we will call D. As a result, we obtain the quadrilateral $ACDB$. So, we know the measure of the three angles $\angle A = \angle B = \angle C = 90°$, as well as the lengths of two adjacent sides AB and AC. To prove that the resulting quadrilateral is a rectangle, it remains to show that $CD = AB$, $BD = AC$, and $\angle D = 90°$.

Begin by drawing the diagonal BC joining points B and C. Then we have two triangles, $\triangle ABC$ and $\triangle BCD$. We denote $\angle ACB = \alpha$, $\angle ABC = \beta$, $\angle BCD = \gamma$, and $\angle DBC = \delta$. Since we are in Euclidean geometry, we know that the angle sum of each triangle is 180°: $\sum(\triangle ABC) = \sum(\triangle BCD) = 180°$. Then from the angle sum of triangle $\triangle ABC$ we find that

$$\sum(\triangle ABC) = 90° + \alpha + \beta = 180° \implies \alpha + \beta = 90°.$$

Applying this result to the angle sum of triangle $\triangle BCD$, we find that

$$\begin{aligned}\sum(\triangle BCD) &= \gamma + \delta + \angle D = 180° \\ &= (90° - \alpha) + (90° - \beta) + \angle D = 180° \\ &\implies \angle D = \alpha + \beta = 90°.\end{aligned}$$

Therefore, $\triangle ABC \cong \triangle BCD$ by the angle-side-angle axiom: $\alpha = \delta$, $BC = BC$, and $\gamma = \beta$. Hence, $CD = AB$ and $AC = BD$, as well as $\angle A = \angle B = \angle C = \angle D = 90°$. We conclude that the quadrilateral $ACDB$ is indeed a rectangle.

Method 2:

Start with the segment AB. Erect the ray perpendicular to AB at point A, and take a point C lying in that ray. So, we have segment $AC \perp AB$. Now erect the ray perpendicular to AB at the point B, and along the ray layoff the segment BD from point B congruent to segment AC. Draw the line passing through the points C and D, we know such a line exists and is unique by the postulate (E_1). Thus, we have constructed the quadrilateral $ACDB$. In our construction, we know the measure of two angles: $\angle A = \angle B = 90°$, as well as the lengths of three adjacent sides: $AC = BD$, and the length of AB. To show that the quadrilateral $ACDB$ is a rectangle, it remains to show that the segments CD and AB are congruent, and that the angles $\angle C$ and $\angle D$ are right angles: $\angle C = \angle D = 90°$.

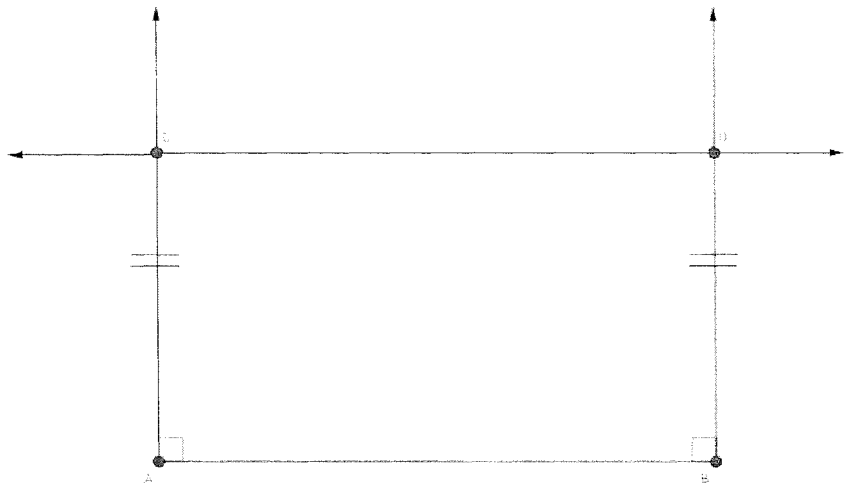

Figure 1.2: Method 2 for constructing a rectangle.

Begin by drawing the diagonal AD. Then our quadrilateral is decomposed into two triangles: $\triangle ACD$ and $\triangle ABD$. We denote the angles as follows: $\angle BAD = \alpha$, $\angle ADB = \beta$, $\angle ADC = \gamma$, and $\angle DAC = \delta$. Since we are in Euclidean geometry, we know that the angle sum of the triangle $\triangle ABD$ is 180°:

$$\sum(\triangle ABD) = 90° + \alpha + \beta = 180° \implies \alpha + \beta = 90°$$

Additionally, we know that the $\angle A = 90° = \alpha + \delta$. Subtracting these two relations we find that $\beta = \delta$. Therefore, we know that the triangles $\triangle ABD \cong \triangle ACD$ by the side-angle-side axiom ($AD = AD$, $\beta = \delta$, and $AC = BD$). Therefore, by congruent triangles we conclude that $\angle C = \angle B = 90°$, $\gamma = \alpha$, and $CD = AB$. Since $90° = \alpha + \beta = \gamma + \delta = \angle D$, it follows that $\angle A = \angle B = \angle C = \angle D = 90°$, and $AB = CD$, $AC = BD$. Therefore, we conclude that the quadrilateral $ACDB$ is a rectangle.

Two millennia after Euclid proposed his five postulates of geometry, an Italian mathematician, *Giovanni Girolemo Saccheri* (1667-1733) became one of the first mathematicians to make great progress in working with Euclid's 5th postulate. Shortly before his death, Saccherri published his work on non-Euclidean geometry encapsulating the progress that he was able to make in working with Euclid's parallel postulate. Setting out to validate Euclid's claim regarding the validity of the parallel postulate via a *reductio ad absurdum argument*, Saccheri first noticed that the parallel postulate was equivalent to stating that the angle sum of a triangle was equal to 180°. Continuing with his idea of proof by the absurd, Saccheri considered the negation of the statement *"The angle sum of a triangle is equal to 180 °"*. In mathematical logic, the negation of the statement $x = y$ is $x \neq y$. As a result, there are two cases to consider: $x < y$ or $x > y$. With this in mind, Saccheri arrived at the two cases: "The angle sum of a triangle is greater than 180°" or "The angle sum of a triangle are less than 180°". Saccheri quickly dispensed of the first statement (angle sum of a triangle is greater than 180°), proving that under this assumption lines would be finite, which he accepted as a contradiction. Today we understand that spherical geometry is consistent under this assumption. So, Saccheri then set off to find a contradiction in assuming that the angle sum of a triangle is less than 180°. To this end, Saccheri attempted to construct a rectangle following the procedure outlined in **Method 2**. Although he was able to prove that the summit angles of a rectangle are congruent, he could not arrive at a contradiction for having acute summit angles.

Several decades later it was the Swiss mathematician *Johann Heinrich Lambert* (1728-1777) who set out to prove, like Saccheri, Euclid's parallel postulate by looking at quadrilaterals. Following the method of constructing a rectangle via **Method 1**, Lambert studied quadrilaterals having at least 3 right angles. If he could show that the measure of the fourth angle of the quadrilateral was necessarily 90°, then the parallel postulate would be proven. Despite his work, Lambert was unable to find a proof. He was, however, able to show that the measure of the fourth angle of such a quadrilateral was necessarily less than or equal to 90°. These quadrilaterals are called *Lambert quadrilaterals*. In fact, they are closely related to Saccheri quadrilaterals. Reflecting a Lambert quadrilateral across its side with two right angles will create an equal Lambert quadrilateral, and the union of these two Lambert quadrilaterals form a Saccheri quadrilateral. Moreover, Lambert was able to prove that if one accepted the negation of Euclid's 5th postulate, the angle sum of a triangle is less than 180°, then it followed that similar triangles were in fact congruent, implying that there was an idea of a universal length in this new, non-Euclidean geometry. Additionally, he showed that the defect of a triangle is proportional to its area.

Shortly after Lambert completed his progress, the stalwart researcher *Adrien-Marie Legendre*, a French mathematician (1752-1833), made numerous attempts to prove Euclid's parallel postulate, which he published in his textbook "Geometry". Each time after he published one of his proofs of (E_5), he found an error in the proof. This caused Legendre to look for a new proof. In the end, after his 14th attempt, Legendre was unsuccessful in his many attempts to prove Euclid's parallel postulate. After all of his work, Legendre could only claim that the angle sum of a triangle is less than or equal to 180° in neutral geometry: $\sum(\triangle) \leq 180°$ in $\mathbb{N}^2$.

These three mathematicians, unbeknown to them, laid the groundwork for the development of non-Euclidean geometry. In the early 19th century, a Hungarian mathematician named *János Bolyai* (1802-1860) developed the theory of non-Euclidean geometry by using familiar constructions from Euclidean geometry and exploring similar constructions under the assumption that the angle sum of a triangle was strictly less than 180°. He published in 1831 his discovery as an appendix to his father's book the *Tentament*. The book itself was his father's attempt to prove Euclid's parallel postulate. János was able to develop non-Euclidean geometry and show that it was possible to have consistent geometries independent of the parallel postulate.

At the same time that János was developing his theory, *Carl Gauss* (1777-1855), a German mathematician often called the greatest mathematician of his time (he had the title "King of Mathematicians"), also spent a great deal of time thinking about the consequences of negating Euclid's 5th postulate. Although he never formally published his ideas, Gauss claimed to have independently arrived at and developed the same notions as János. Gauss said as much in a letter to János' father.

A Russian mathematician named *Nikolai Lobachevsky* (1792-1856), developed independently of János Bolyai a non-Euclidean geometry. Completing his work in 1823, it largely remained unpublished until 1909. This provided János the opportunity to publish his own work several years later. Unlike Bolyai, Lobachesky only focused on one geometry, which is today called *hyperbolic geometry* or *Lobachevskian geometry*. His formulation stemmed from the negation of Euclid's fifth postulate: "There exists more than one line through any point P not on line l that is parallel to line l". Additionally, he formulated the idea of the angle of parallelism, and he showed that in hyperbolic geometry, often denoted $\mathbb{H}^2$, the angle sum of a triangle is strictly less than 180°.

1.2 The Postulate (E_5), its Negation (H_5), and the Formulation of Theorem 1.1

Euclid's first four postulates formulate what is called neutral geometry. Both Euclidean and hyperbolic geometry are contained in neutral geometry. As we will come to find out, it is in accepting either the statement (E_5) or its negation that will lead to the different geometries, Euclidean and hyperbolic, respectively. Since these geometries are contained in $\mathbb{N}^2$ if we can prove a theorem in neutral geometry, then the theorem will be true in both Euclidean and hyperbolic geometry. These proofs are independent of models and are very strong formulations. We will keep this fact in mind as we develop hyperbolic geometry.

Euclid's 5th postulate:

For every line l and for every point P not lying on l, there exists a unique line m passing through P such that the lines m and l are parallel.

The negation of this statement is:

There exists a line l and there exists a point P not lying on l, such that there are (at least) two distinct lines, m and n, passing through P parallel to l.

There are strong and weak forms of (E_5) in Euclidean geometry as well as of (H_5) in hyperbolic geometry. The negation of the strong form of (E_5) is the weak form of (H_5) (the parallel postulate in hyperbolic geometry). Similarly, the negation of the weak form of (E_5) is the strong form of (H_5). It is quite evident that the strong form of each statement implies the weak form in the same geometry. On the other hand, for many statements the weak form does not imply the strong form; and, it is not obvious that we can recover the strong form of the parallel postulate in each geometry from its weak form. Here are the two forms of (E_5)

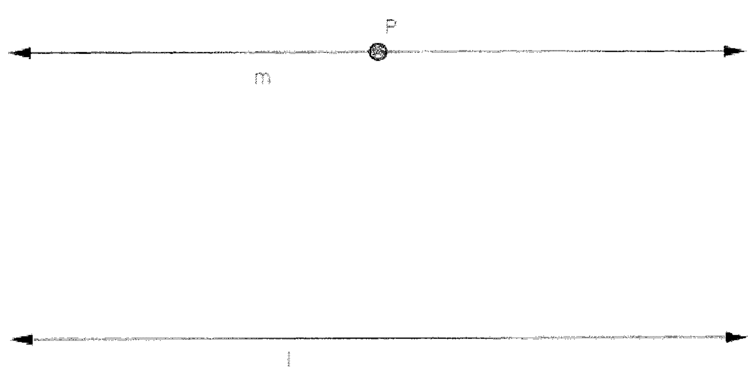

Figure 1.3: Euclid's parallel postulate in $\mathbb{E}^2$.

Postulate 1.1 (Strong (E_5)).

For every line l and for every point $P \notin l$, there exists a unique line $m \parallel l$.

Postulate 1.2 (Weak (E_5)).

There exists a line l_0 and there exists a point $P_0 \notin l_0$ such that there exists a unique line $m_0 \parallel l_0$ through P_0.

Likewise, there are two forms of the postulate (H_5).

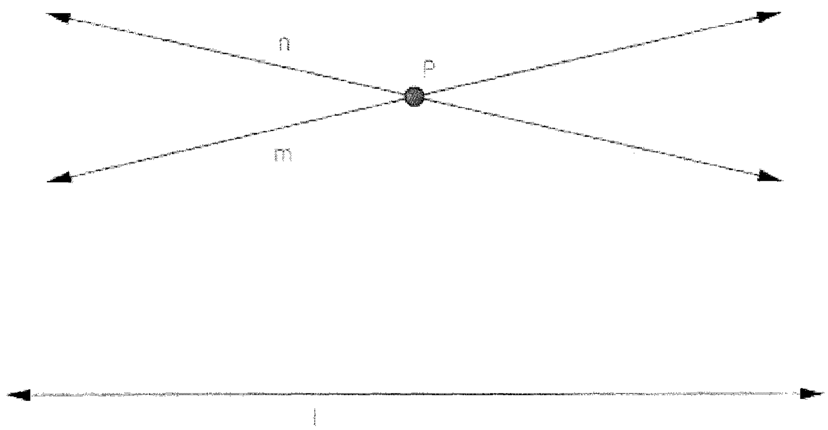

Figure 1.4: The negation of Euclid's parallel postulate in $\mathbb{H}^2$.

Postulate 1.3 (Strong (H_5)).

For every line l and for every point $P \notin l$, there exists at least two distinct lines m and n such that $m \parallel l$ and $n \parallel l$.

Postulate 1.4 (Weak (H_5)).

There exists a line l_1 and there exists a point $P_1 \notin l_1$ such that there exists two distinct lines m_1 and n_1 such that $m_1 \parallel l_1$ and $n_1 \parallel l_1$.

$$\text{Strong form } (E_5) \implies \text{Weak form } (H_5)$$

$$\Downarrow \Uparrow$$

$$\text{Weak form } (E_5) \implies \text{Strong form } (H_5)$$

Theorem 1.1 (Weak $(E_5) \implies$ Strong (E_5)). *If there exists a line l_0 and there exists a point $P_0 \notin l_0$ such that there exists a unique line $m_0 \parallel l_0$ containing P_0, then for every line l and for every point $P \notin l$, there exists a unique line $m \parallel l$ containing P.*

The proof that the Weak form of (E_5) implies the Strong form of (E_5) consists of several steps. We will first show that the Weak form of (E_5) implies that there exists a triangle whose angle sum is 180°. Then we will show that if one such triangle exists, then every triangle has angle sum 180°. This implies that we can construct a special rectangle and then a rectangle. Then introducing the notion of the defect of a polygon, we will show that for any triangle one can construct a rectangle that contains this triangle; hence, the defect of the triangle is 0, and thus, the angle sum of the triangle is 180°. The proof of Theorem 1.1 is long; it takes the remaining sections (1.3, 1.4, 1.5, 1.6) of this chapter. Special notations will be used in our proof.

1.2.1 Notations

Throughout this paper will we appeal to using certain representations for the sake of brevity. We will use $\sum(\triangle ABC)$ to denote the sum of the angles of triangle $\triangle ABC$, and we will refer to it as "the angle sum of $\triangle ABC$". By $area(\triangle ABC)$ and $\delta(\triangle ABC)$ we denote the area and the defect of the triangle $\triangle ABC$, respectively. From time to time, for example Lemma 1.2 and Theorem 1.4, we will need to discuss the ordering of points on the line $l = \overleftrightarrow{AB}$. We denote a point $H \in l$ lying between points A and B by $A * H * B$. Often times when discussing neutral geometry we will appeal to using the shorthand $\mathbb{N}^2$ to represent 2-dimensional neutral geometry. In a similar fashion we will denote 2-dimensional Euclidean geometry by $\mathbb{E}^2$, and 2-dimensional hyperbolic geometry by $\mathbb{H}^2$.

1.3 Proof of Theorem 1.1: The Beginning

Lemma 1.2. *The Weak form of (E_5) implies that there exists a triangle such that the sum of its angles is* 180°.

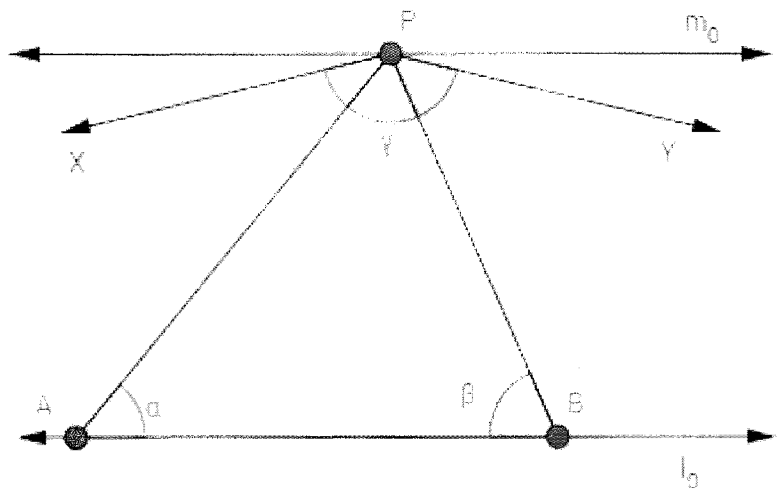

Figure 1.5: Triangle $\triangle APB$ has angle sum 180° as proved in Lemma 1.2

Proof. Suppose that Weak form of (E_5) is true. Consider a line l_0 and any two points on that line, $A, B \in l_0$. Now take a point not lying on the line l_0, $P \notin l_0$. Begin by connecting points A, B with P to form the triangle $\triangle APB = \triangle_0$, as depicted in Figure 1.5. We denote the three angles of the triangle as follows $\angle A$ by α, $\angle B$ by β, and $\angle P$ by γ. By our assumption, we know such a triangle exists. We will now show that the sum of the angles $\alpha + \beta + \gamma = 180°$.

By our assumption, we also know that there exists a unique line m_0 through the point P which is parallel to the line l_0. We will construct two rays emanating from the point P, so that one ray will be laid off an angle α from the segment AP and the second will be by an angle β from the segment BP. We will then show that these two rays form the line m_0.

Draw the ray $\overrightarrow{PX}$ such that $\angle XPA = \angle PAB = \alpha$. Similarly, draw the ray $\overrightarrow{PY}$ such that $\angle YPB = \angle PBA = \beta$. By the Exterior Angle Theorem we know that both rays are parallel to the line l_0. That is we have that $\overrightarrow{PX} \parallel l_0$ and $\overrightarrow{PY} \parallel l_0$. Since there exists a unique line m_0 parallel to the line l_0 through the point P, then it follows that $X * P * Y$ and $\overleftrightarrow{XPY} = m_0$. This implies that $\angle XPY = 180°$. But $\angle XPY = \angle XPA + \angle APB + \angle BPY = 180°$, which implies that $\alpha+\beta+\gamma=180°$. Hence the angle sum of our triangle $\sum(\triangle_0)=180°=\alpha+\beta+\gamma=\angle A+\angle B+\angle P$. $\square$

In the preceding Lemma 1.2 we made use of the Exterior Angle Theorem. We will now formulate the Exterior Angle Theorem in neutral geometry.

Definition 1.3. For a given triangle $\triangle ABC$ we say that the **exterior angle** for the angle $\angle A$ is $ext(\angle A)=\varphi:=180°-\angle A=180°-\alpha$, where $\angle A=\alpha$.

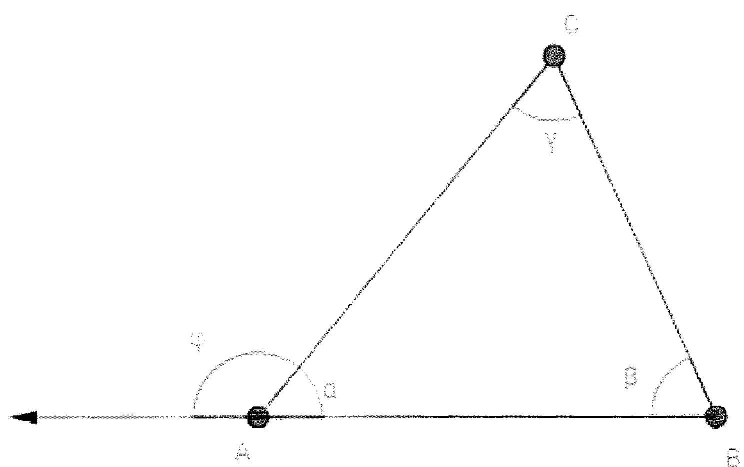

Figure 1.6: The exterior angle φ of the triangle $\triangle ABC$.

Theorem 1.4. *For the triangle $\triangle ABC$ with angles $\angle A=\alpha$, $\angle B=\beta$, $\angle C=\gamma$, and exterior angle $ext(\angle A)=\varphi$, then the exterior angle is greater than an interior remote angle; that is, the following inequalities hold: $\varphi>\gamma$ and $\varphi>\beta$.*

Proof. There are two main instruments for this proof:

1. The axiom (E_1): the uniqueness of a geodesic, for any point A and for any point B there exists a unique line $\overleftrightarrow{AB}$.

2. Triangle inequality: for every triangle, $\triangle ABC$, the sum of the length of any two sides is greater than the length of the third, $a+b>c$ for sides a,b,c.

We will construct a proof of Theorem 1.4 by contradiction, assuming that the exterior angle is smaller than or equal to the remote interior angles, β or γ. Since this proof is the same for either angle β or γ then without loss of generality we may consider the angle β. Suppose that the exterior angle $\varphi \leq \beta$. Then there are two cases that we must consider, the case when $\varphi = \beta$, and the case when $\varphi < \beta$. In both cases, it is our goal to arrive at contradicting statements.

Case 1: The exterior angle equals a remote interior angle, $\varphi = \beta$

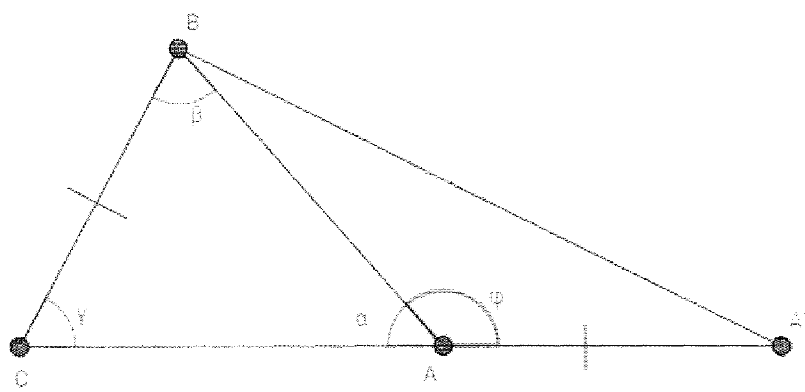

Figure 1.7: Case 1: the exterior angle φ equals the remote interior angle β leads to a contradiction.

Consider the triangle $\triangle ABC$. Begin by laying off the segment AA' on the ray $\overrightarrow{CA}$ so that the points are situated as $C * A * A'$ and the segments $AA' = BC = a$. Then connect the points B and A', forming a new triangle $\triangle ABA'$. We see that $\triangle ABC = \triangle ABA'$ by the side-angle-side axiom. By construction we have that the segment $BC = a = AA'$, by assumption the angle $\beta = \varphi$, and the shared side $BA = c = AB$. Since the triangles are similar, it follows that the segment $BA' = b = AC$. Now since B is not contained in the line $\overleftrightarrow{CA}$, $B \notin \overleftrightarrow{CA}$, then clearly B is not contained in the ray $\overrightarrow{CA}$, and hence, B is not contained in the ray $\overrightarrow{CA'}$, $B \notin \overrightarrow{CA'}$. So, the triangle inequality for the triangle $\triangle CBA'$ holds and we have that $CB + BA' > CA'$. But by similar triangles we have that

$CB + BA' = a + b$, and we have that $CA' = b + a$ by construction. So, $CB + BA' = CA'$. But this contradicts the axiom (E_1), the uniqueness of a line through two points. Therefore, our assumption is false, and $\varphi \neq \beta$.

Case 2: The exterior angle is less than a remote interior angle, $\phi < \beta$

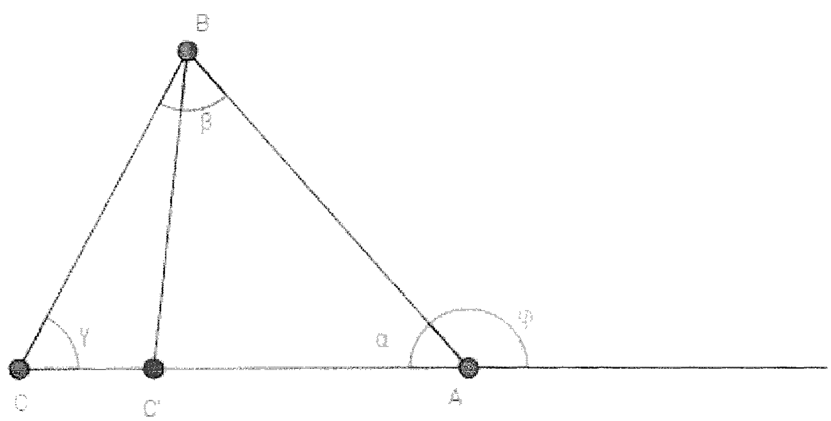

Figure 1.8: Case 2: the exterior angle φ is less than the remote interior angle β leads to a contradiction.

Draw the ray $\overrightarrow{BC'}$ such that $\angle ABC' = ext(\angle A) = \varphi$. This is possible since $\varphi < \beta$. Moreover, the ray $\overrightarrow{BC'}$ is inside $\angle ABC$. This implies that $\overrightarrow{BC'} \bigcap CA \neq \phi$, and in fact their intersection is a point, $\overrightarrow{BC'} \bigcap CA = C'$. We also know that the points on CA are situated so that C' lies between C and A, $C * C' * A$. For the triangle $\triangle ABC'$, we find that the exterior angle $ext(\angle A) = \varphi = \angle ABC'$. But this contradicts **Case 1**. Thus, our assumption that $\varphi < \beta$ is false, and thus, $\varphi \not< \beta$.

In conclusion, neither **Case 1** nor **Case 2** can take place. Therefore, we conclude that the exterior angle of a triangle is strictly greater than a remote interior angle, $\varphi > \beta$; and, hence, $\varphi > \alpha$. $\square$

We have just shown the proof of the Exterior Angle Theorem in neutral geometry which will help us prove our next theorem. In Euclidean geometry, we know that every triangle has angle sum of 180°. Additionally, it was shown by Legendre that the angle sum of a triangle is $\leq 180°$ in neutral geometry. Later, Lobachevsky stated that in hyperbolic geometry the angle sum of a triangle is strictly less than 180°. So, we have seen cases in which the angle sum of a triangle could be less than 180°, or equal to 180°. The third case is the angle sum of a triangle is greater than 180°. This case corresponds to spherical geometry, which we will not discuss in this text. One question to think about is the possibility of having two triangles in the same geometry satisfying different angle sum restrictions? For example, is it possible to have a triangle whose angle sum is strictly less than 180° and a triangle whose angle sum is equal to 180° exist in the same geometry? The following theorem provides us with insight to this question.

Theorem 1.5. *If there exists a triangle $\triangle_0$ with angle sum $\sum(\triangle_0) = 180°$, then for every triangle $\triangle$, the angle sum $\sum(\triangle) = 180°$.*

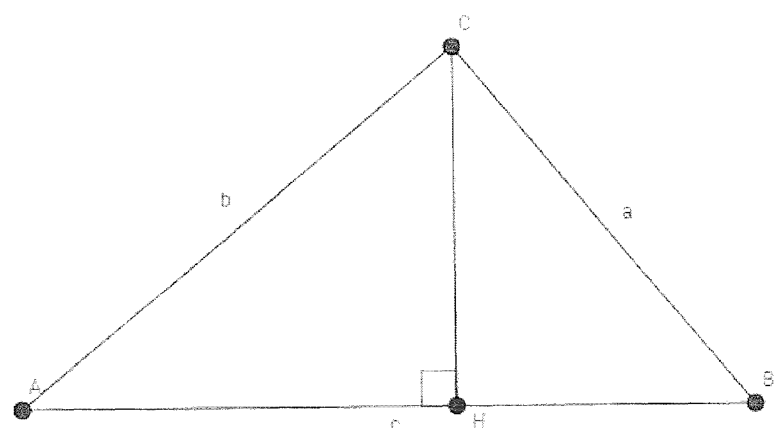

Figure 1.9: Construction of a right triangle with angle sum 180°.

Proof. The proof of this theorem will require several steps, requiring the formulation of several lemmas and theorems below. Our first step is to show that the assumption "there exists a triangle $\triangle_0$ such that the angle sum of triangle $\triangle_0$ is 180°" implies that "there exists a right triangle whose angles sum to 180°".

Consider the triangle $\triangle_0 = \triangle ABC$. Let AB be the side of greatest length. Then we have that $c = AB \geq b = AC$, $c = AB \geq a = BC$. Then from vertex C drop the perpendicular segment $CH \perp AB$. We claim that the point $H \in AB$.

Claim 1: $H \in AB$ and $A * H * B$

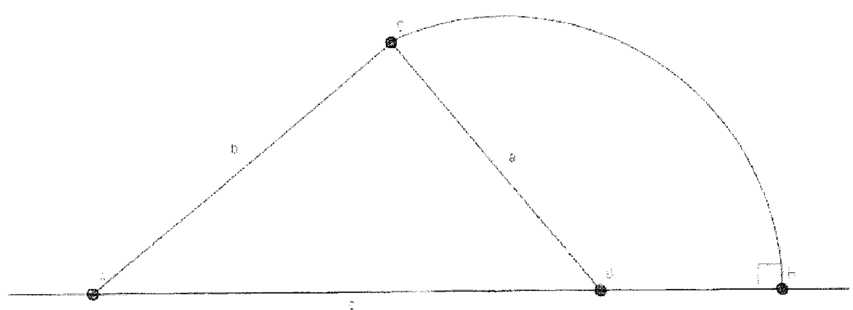

Figure 1.10: The foot point of the altitude CH is situated outside the triangle ABC.

We will show this by contradiction. Suppose without loss of generality that $A * B * H$, as depicted in Figure 1.10. Then the triangle $\triangle BCH$ is a right triangle with $\angle CHB = 90°$, since the segment $CH \perp AB$. Then considering triangle $\triangle BCH$, the angle $\angle CBA = ext(\angle CBH)$. So, by Theorem 1.4 the angle $\angle CBA > \angle CHB = 90°$. But from our assumption we know that triangle $\triangle ABC$ has angle sum $\angle A + \angle B + \angle C = 180°$. Moreover, the side $c \geq b$ which implies that $\angle C \geq \angle B = 90°$. It follows that the angle sum of triangle $\triangle ABC$ is now $\sum(\triangle ABC) = \angle A + \angle B + \angle C > \angle A + 90° + 90° > 180°$. But this contradicts our initial assumption that $\sum(\triangle ABC) = 180°$. Therefore, our assumption that $A * B * H$ is false. Thus, $H \in AB$ and $A * H * B$.

Claim 2: $\sum(\triangle ACH) = 180°$ and $\sum(\triangle BCH) = 180°$

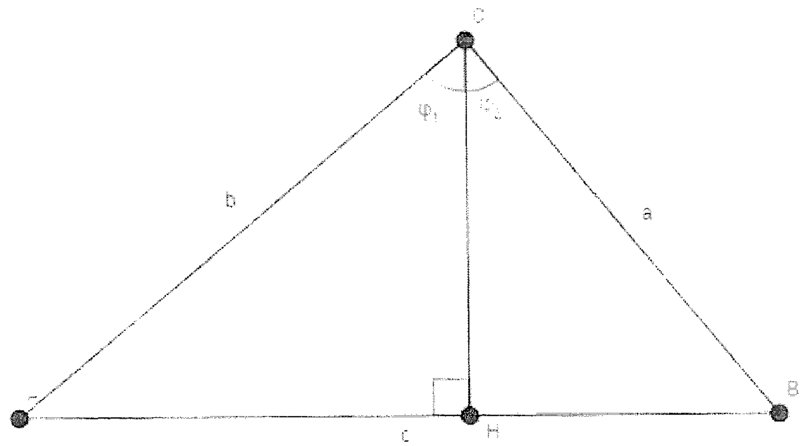

Figure 1.11: The foot point of the altitude CH of triangle $\triangle ABC$ lies between the points A and B.

Let $\angle ACH = \varphi_1$ and $\angle BCH = \varphi_2$, so that $\angle C = \varphi_1 + \varphi_2$. Since $\sum(\triangle ABC) = 180°$, it follows that $\angle A + \varphi_1 + \varphi_2 + \angle B = 180°$. This implies that

$$\begin{aligned}\sum(\triangle ACH) + \sum(\triangle BCH) &= (\angle A + \varphi_1 + 90°) + (\angle B + \varphi_2 + 90°)\\ &= (\angle A + \varphi_1 + \varphi_2 + \angle B) + 90° + 90°\\ &= 180° + 90° + 90°\\ &= 360°\end{aligned}$$

So, we have $\sum(\triangle ACH) + \sum(\triangle BCH) = 360°$. Then by Legendre-Saccheri's Theorem (see Theorem 1.6 below), we have the angle sum of the triangles $\sum(\triangle ACH) \le 180°$ and $\sum(\triangle BCH) \le 180°$. This implies that $\sum(\triangle ACH) + \sum(\triangle BCH) \le 360°$, where the equality $\sum(\triangle ACH) + \sum(\triangle BCH) = 360°$ holds if and only if $\sum(\triangle ACH) = \sum(\triangle BCH) = 180°$. Indeed, if say $\sum(\triangle ACH) < 180°$, then $\sum(\triangle BCH) > 180°$ which contradicts the inequality $\sum(\triangle BCH) \le 180°$. Therefore, the angle sum $\sum(\triangle ACH) = \sum(\triangle BCH) = 180°$. $\square$

We have just shown that given a triangle whose angle sum is 180°, there exists a right triangle whose angle sum is 180°. One key element in this proof was the Legendre-Saccheri Theorem in $\mathbb{N}^2$, which we will now prove.

Theorem 1.6 (Legendre-Saccheri Theorem). *In neutral geometry, $\mathbb{N}^2$, for every triangle $\triangle$, the sum of its angles does not exceed* 180°*:* $\sum(\triangle) \leq 180°$.

Proof. We prove by contradiction. Suppose that there exists a triangle $\triangle_0 \in \mathbb{N}^2$ such that $\sum(\triangle_0) > 180°$. Then we can say that the triangle has the angle sum of 180° plus an additional amount ε; that is $\sum(\triangle_0) = 180° + \varepsilon$, where $\varepsilon > 0$. Consider the triangle $\triangle ABC = \triangle_0$ and the line $l = \overleftrightarrow{AC}$. We denote the side lengths of triangle $\triangle ABC$ by $BC = a$, $AC = b$, and $AB = c$ and its angles by $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$. We want to construct a chain of triangles identical to $\triangle ABC$ along the line l.

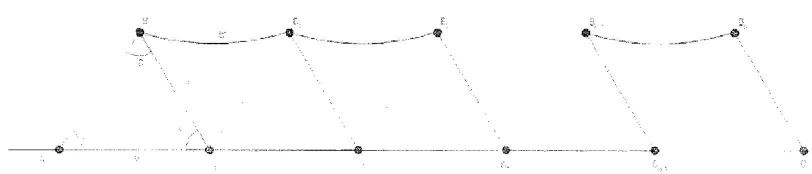

Figure 1.12: A chain of $n-1$ congruent triangles for the proof of Theorem 1.6.

Using compass and straightedge we construct the next triangle in our chain $\triangle A_1B_1C_1$. Along the line l from the point $C = A_1$ layoff a segment of length equal to the length of the segment AC, terminating at a point C_1 and resulting in the segment A_1C_1. To find the point B_1, draw the circles $c(A_1, AB)$ and $c(C_1, BC)$. Then, these two circles intersect at a point, B_1, above the line l. Joining the points A_1B_1 and B_1C_1, we construct the triangle $\triangle A_1B_1C_1$. In fact, by the side-side-side axiom, the triangle $\triangle A_1B_1C_1 = \triangle ABC$ since $AB = A_1B_1$, $AC = A_1C_1$, and $BC = B_1C_1$ by construction. Therefore, $\angle B_1A_1C_1 = \angle A = \alpha$, $\angle A_1B_1C_1 = \angle B = \beta$, and $\angle A_1C_1B_1 = \angle C = \gamma$. In a similar fashion, we construct a chain of $n-1$ triangles where $\triangle A_nB_nC_n$ is the $(n-1)$st triangle.

Note that $C_i = A_{i+1}$. As a result, we have a chain of congruent triangles

$$\triangle ABC = \triangle A_1B_1C_1 = \triangle A_2B_2C_2 = ... = \triangle A_nB_nC_n$$

Draw the segments joining the points B_i to B_{i+1} for $1 \leq i < n$, and forming the triangles $\triangle BCB_1$, $\triangle B_1C_1B_2$, ..., $\triangle B_{n-1}C_{n-1}B_n$. Then by the side-angle-side axiom

$$\triangle BCB_1 = \triangle B_1C_1B_2 = ... = \triangle B_{n-1}C_{n-1}B_n$$

since $BC = B_1C_1 = ... = B_{n-1}C_{n-1}$, $CB_1 = C_1B_2 = ... = C_{n-1}B_n$, and $\angle BCB_1 = \angle B_1C_1B_2 = ... = \angle B_{n-1}C_{n-1}B_n = \beta'$. Then $\gamma + \beta' + \alpha = 180°$, $\gamma + \beta + \alpha = 180° + \varepsilon$. From these two equations it follows that $\beta > \beta'$. So, the angle measure $\beta' = 180° - \alpha - \gamma$. Comparing the triangles $\triangle ABC$ and $\triangle BCB_1$ we find a relation between the side lengths b and b'.

Observe that the angles $\angle ABC$ and $\angle BCB_1$ have legs of equal length, $BC = BC$ and $AB = CB_1$. Then since $\beta > \beta'$ it follows that $AC > BB_1$, that is $b > b_1$. So, we may write $b' = b - \delta$ for some $\delta > 0$. Applying the triangle inequality on the chain of triangles, we find that

$$\text{length of broken line}(ABB_1B_2...B_nC_n) > \text{length of segment}(AC_n)$$

So, we compute

$$\begin{aligned} c + nb' + a > (n+1)b &\Longrightarrow c + n(b-\delta) + a > nb + b \\ &\Longrightarrow c + nb - n\delta + a > nb + b \\ &0 < c + a - b > n\delta \quad \forall n \end{aligned}$$

But for some n, $n\delta > c + a - b$, a contradiction. Therefore, our supposition that there exists a triangle with angle measure greater than 180° is false. Thus, $\forall\triangle$, the angle sum $\sum(\triangle) \leq 180°$. □

We have thus shown that the angle sum of a triangle in $\mathbb{N}^2$ is $\leq 180°$.

1.4 The Defect of a Triangle and of a Polygon

In neutral geometry $\mathbb{N}^2$ following from Theorem 1.6 triangles can have angle sum of at most 180°. There is, however, the opportunity for triangles to have angle sum less than 180°. For such a triangle having angle sum less than 180°, it is helpful to know by how much the angle sum of the triangle differs from the expected angle sum of 180°. It is this difference that we now look to define.

Definition 1.7. The **defect** of a triangle $\triangle$, denoted $\delta(\triangle)$, is defined as

$$\delta(\triangle) = 180° - \sum(\triangle)$$

It follows from Theorem 1.6 that the defect is non-negative, $\delta(\triangle) \geq 0$. An immediate consequence of the defect of a triangle pertains to its additive nature.

Theorem 1.8 (Additivity of the defect). *If a triangle $\triangle$ is made up of finitely many smaller triangles, $\triangle = \bigcup_{i=1}^{N} \triangle_i$, then its defect equals the sum of the defects of the smaller triangles: $\delta(\triangle) = \sum_{i=1}^{N} \delta(\triangle_i)$.*

Proof. We will prove this theorem via induction. Consider the triangle $\triangle = \triangle ABC$ which is made up of two smaller triangles $\triangle_1 = \triangle ABH$ and $\triangle_2 = \triangle CBH$.

Then the defect of triangle $\triangle$ is

$$\delta(\triangle) = 180° - (\alpha + \beta_1 + \beta_2 + \gamma)$$

adding in and subtracting out the supplementary angles φ and ψ we find that

$$\begin{aligned}
\delta(\triangle) &= 180° - (\alpha + \beta_1 + \varphi) - (\beta_2 + \gamma + \psi) + (\varphi + \psi) \\
&= [180° - (\alpha + \beta_1 + \varphi)] + [180° - (\beta_2 + \gamma + \psi)] \\
&= \delta(\triangle_1) + \delta(\triangle_2)
\end{aligned}$$

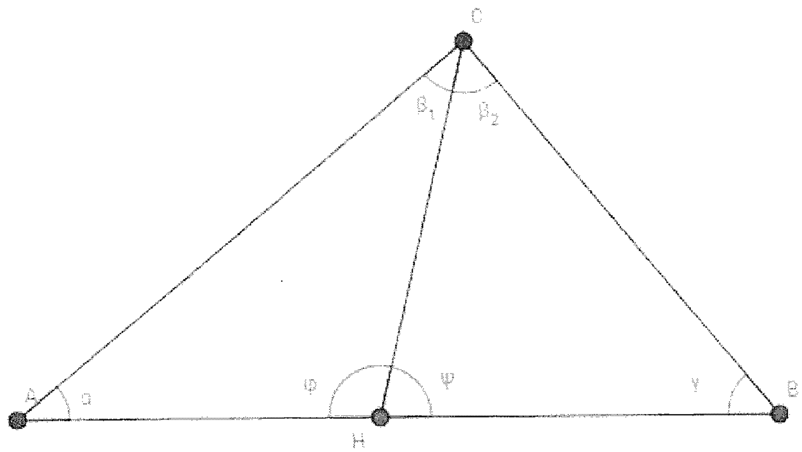

Figure 1.13: The base case in proving the additivity of the defect of the triangle $\triangle ABC$, Theorem 1.8.

Now we need to prove the inductive step. Suppose that the triangle $\triangle$ is comprised of $n+1$ smaller triangles, $\triangle = \bigcup_{i=1}^{n+1} \triangle_i$. Here the triangle $\triangle_i$ has the angles α_{2i-1}, α_{2i}, and φ_i.

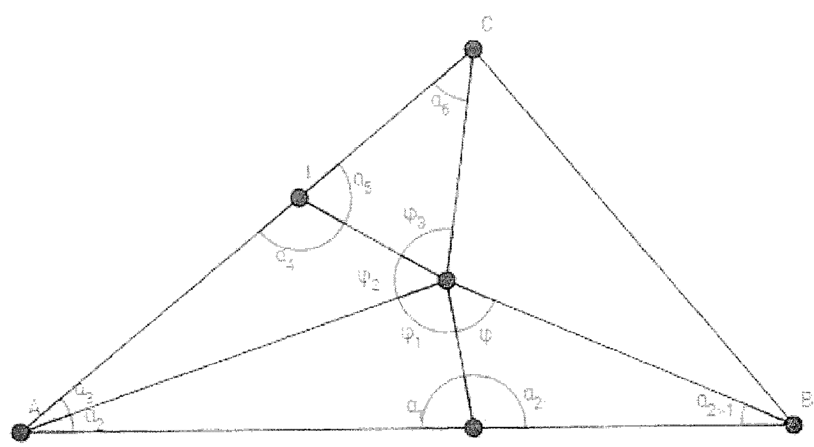

Figure 1.14: The inductive step proving the additivity of the defect of a triangle, Theorem 1.8.

Then the defect of this triangle is

$$\delta(\triangle) = 180^\circ - (\alpha_1 + \alpha_2 + ... + \alpha_n + ... + \alpha_{2n+1} + \alpha_{2n+2})$$

We know that the angle sum $\varphi_1 + \varphi_2 + ... + \varphi_n = 360^\circ$. So, adding in and subtracting out the φ_i we find that

$$\begin{aligned}
\delta(\triangle) &= 180^\circ - (\alpha_1 + ... + \alpha_{2n+2}) - (\varphi_1 + ... + \varphi_n) + (\varphi_1 + ...\varphi_n) \\
&= [180^\circ - (\alpha_1 + \alpha_2 + ... + \alpha_{2n-3} + \alpha_{2n-2} + \varphi_1 + ... + \varphi_{n-1})] + \\
&+ [180^\circ - (\alpha_{2n-1} + \alpha_{2n} + \varphi_n)] + [180^\circ - (\alpha_{2n+1} + \alpha_{2n+2} + \varphi_{n+1})] \\
&= \sum_{i=1}^{n-1} \delta(\triangle_i) + \delta(\triangle_n) + \delta(\triangle_{n+1}) \\
&= \sum_{i=1}^{n+1} \delta(\triangle_i)
\end{aligned}$$

□

Now we wish to generalize the notion of the defect of a triangle to that of a polygon.

Definition 1.9. Given an n-sided polygon $P^{(n)}$, its defect is the non-negative quantity

$$\delta(P^{(n)}) = 180(n-2) - \sum(P^{(n)})$$

where $\sum(P^{(n)})$ denotes the sum of the angles of the polygon $P^{(n)}$ (shortly: "the angle sum of $P^{(n)}$").

Similar to the case of triangles, we can dissect a polygon into finitely many smaller disjoint polygons and consider the sum of the defects of these smaller polygons. We expect that this sum is equal to the defect of the whole polygon $P^{(n)}$. The following theorem justifies our expectation.

Theorem 1.10. *If P is a polygon and $P = \bigcup_{i=1}^{N} P_i$, then $\delta(P) = \sum_{i=1}^{N} \delta(P_i)$.*

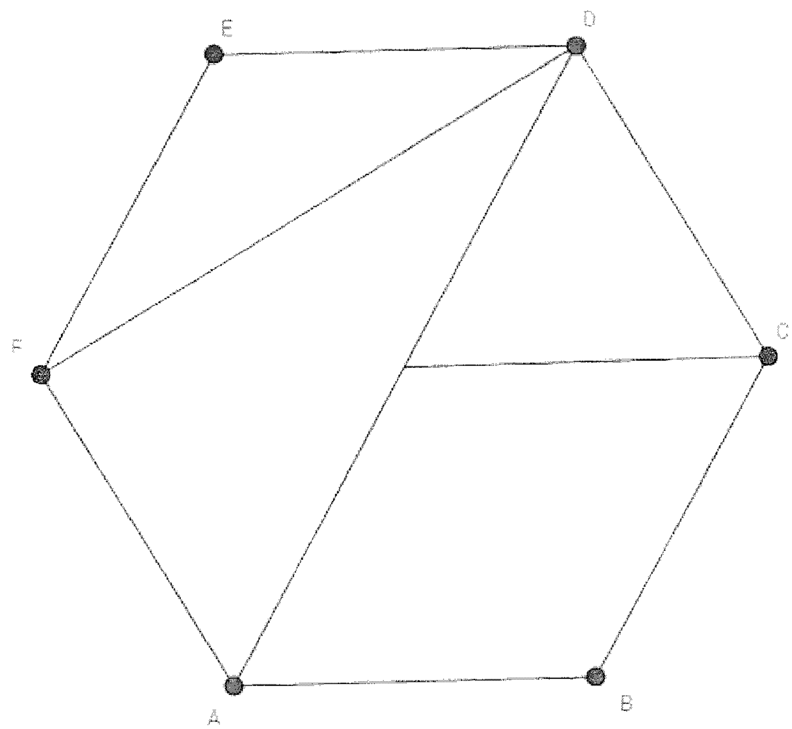

Figure 1.15: The additivity of the defect of a polygon, Theorem 1.10.

We will not prove the additivity of the defect of a polygon. Note only that the defect $\delta(P)$ does not depend on the dissection of the polygon P into pieces (polygons).

The next ingredient in proving Theorem 1.5 is showing a relation between the existence of right triangles whose angle sum is 180°, and the existence of rectangles in our geometry.

1.5 The Existence of Rectangles

Lemma 1.11. *If there exists a right triangle, $\triangle_1$, and the angle sum $\sum(\triangle_1) = 180°$, then there exists a rectangle $R_1 = ABCD$ with 4 right angles: $\angle A = \angle B = \angle C = \angle D = 90°$.*

Proof. Start with the right triangle $\triangle AHC$, where $\angle H = 90°$. Then we draw segments AD and CD such that $AD = CH$ and $CD = AH$, note that this can be accomplished using a

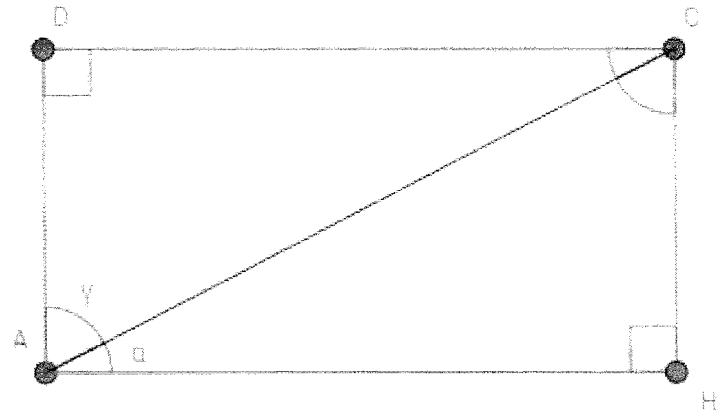

Figure 1.16: Rectangle of the proof of Lemma 1.11.

compass. What results is triangle $\triangle ADC$. Then by side-side-side axiom, we know that $\triangle AHC = \triangle ADC$. Since the triangles are congruent, it follows that $\angle DCA = \angle CAH = \alpha$, $\angle DAC = \angle ACH = \gamma$, and $\angle D = \angle H = 90°$. Summing the angles of triangle $\triangle AHC$ we find that $\alpha + \gamma + 90° = \sum(\triangle AHC) = 180°$, by assumption. This immediately implies that $\alpha + \gamma = 90°$. So, $\angle A = \alpha + \gamma = 90°$ and $\angle C = \alpha + \gamma = 90°$. Therefore, $\angle A = \angle B = \angle C = \angle D = 90°$ which implies that $ADCH$ is a rectangle. □

Now that we are able to construct a rectangle given that there exists a right triangle whose angle sum is 180°, it would be beneficial to be able to construct a rectangle of any size. For if such a construction is possible, then for any triangle in our geometry we could always find a rectangle which contains it. Then the defect of the triangle would be at most equal to the defect of the rectangle.

Lemma 1.12. *There exists a rectangle R_2 of arbitrary size: the side lengths of R_2 can be as big as one wishes.*

Proof. Begin with a rectangle $R_1 = ADCH$, following from Lemma 1.11. To show that we can construct a rectangle of arbitrary size, we will show that we can tile the plane with rectangles of equal size. Extend the segments AH, DC, CH, and AD, so that we now

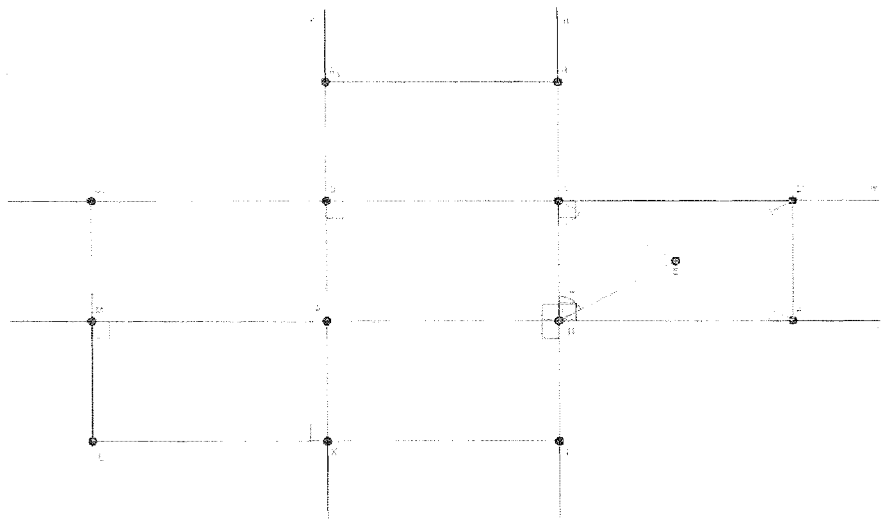

Figure 1.17: Tiling the plane with rectangles, Lemma 1.12.

have two pairs of parallel lines containing the segments of the rectangle. Denote the line containing segment DC by m, and the line containing the segment AH by l. Since $AH = DC$, layoff segment DC from points C and H, along lines m and l, respectively, obtaining points D' and A'. We need to show that $\angle A' = \angle D' = 90°$.

First we draw the diagonals HD' and CA', which intersect at the point E. Then $\angle ECH = \angle EHC = \varphi$ since $\triangle A'HC = \triangle D'CH$ by side-angle-side axiom: $A'H = D'C$, $\angle C = \angle H$, and $CH = HC$. So the triangle $\triangle ECH$ is an isosceles triangle which implies that $EC = EH$. Moreover, $\angle ECD' = \angle EHA' = 90 - \varphi$. So, it follows by the side-angle-side axiom that $\triangle EHA' = \triangle ECD'$, since $EH = EC$, $\angle EHA' = \angle ECD' = 90 - \varphi$, $CD' = HA'$. Then the angle $HA'E = \angle CD'E = \beta$ and $A'E = D'E$. This implies that $\angle A'D'E = \angle D'A'E = \lambda$. But $\angle A' = \beta + \lambda = \angle D'$. Thus, angle $\angle A' = \angle D'$. Let $\alpha = \beta + \lambda$. Then by Legendre-Saccheri Theorem (Theorem 1.6) in neutral geometry $\mathbb{N}^2$ we have that $2\alpha + 2 \cdot 90° \leq 360°$. This implies that the angle $\alpha \leq 90°$. We know that in Euclidean geometry $\mathbb{E}^2$ the angle $\alpha = 90°$ and in hyperbolic geometry $\mathbb{H}^2$ the angle $\alpha < 90°$. Since we are working in $\mathbb{E}^2$, it follows that $\angle D' = \angle A' = \angle H = \angle C = 90°$. Therefore, $HCD'A'$ is a rectangle.

This argument can be extended so that we can construct a rectangle in the horizontal strip. Additionally, we can apply a similar method to show that the quadrilateral $DEFC$ is a rectangle. Here the segments $DE = CF$ are obtained by laying off the segment AD from the points D and C along the lines k and n, respectively. Then we can extend the argument to obtain any rectangle in the vertical strip as well. The last case that remains to be verified is that of constructing a rectangle that is diagonal to the rectangle $ADCH$.

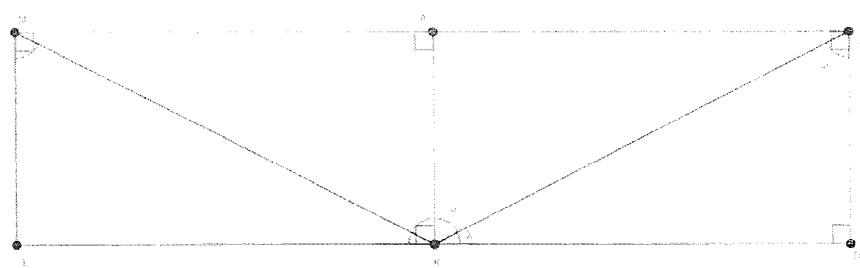

Figure 1.18: The quadrilateral $LMAK$ diagonal to the rectangle $ADCH$ is a rectangle.

Consider the quadrilateral $LMAK$ and its neighboring rectangle $KAHN$. Immediately, we have that $\angle M = \angle A = \angle K = 90°$. Additionally, from our previous arguments we know that the segments $MA = AH = KN$ and $AK = HN$. We begin by constructing the diagonals MK and KH. If we can show that the segment AK is the perpendicular bisector of the segments MH and LN, then we will be done. By the side-angle-side axiom, we have that the triangles $\triangle MAK = \triangle HAK$: $\angle A = \angle A$, $MA = HA$, $AK = AK$. It follows that $\angle KHA = \angle KMA = 90 - \varphi$, and that $\angle MKA = \angle HKA = 90 - \lambda = \phi$. Then we have by complimentary angles that the angle $\angle KML = \angle KHN = \varphi$, and $\angle MKL = \angle HKN = \lambda$. To see that the segment AK is the perpendicular bisector to the segment LN, consider the set of points $S = \{P \mid PM = PH\}$. We begin by noting that $LN^{\perp} \subset S$.

Next consider a point $X \notin LN^{\perp}$. Without loss of generality we may take $X \notin LN^{\perp}$ as shown, that is $X \in KAHN$. Then we see that $MX \bigcap LN^{\perp} = P$. So we have that $MX = MP + PX$. Connecting the points H and P, then by the triangle inequality for $\triangle HPX$ we

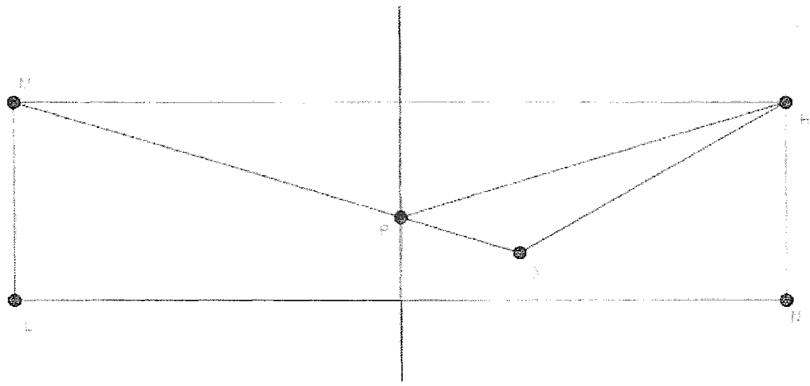

Figure 1.19: The set S coincides with the perpendicular bisector $LN^{\perp}$.

have that

$$MX = MP + PX = HP + PX > HX$$

Thus, $MX \neq HX$. Since we're working in $\mathbb{E}^2$, then $\sum(\triangle LMK) = 180°$ which implies that $\angle L = 180° - \varphi - \lambda$ and $\sum(\triangle NHK) = 180°$ which implies $\angle N = 180° - \varphi - \lambda$. So, $\angle L = \angle N = 90°$. Thus, by angle-side-angle axiom, $\triangle LMK = \triangle NHK$. By congruent triangles we have that $LK = KN$ and $ML = HN$. Therefore, the quadrilateral $LMAK$ is a rectangle. By extension of the same type, we can now construct a rectangle of arbitrary size. □

1.6 Proof of Theorem 1.1

We now return to the proof of Theorem 1.5. At the onset of this proof, we made the assumption that there exists a triangle $\triangle_0$ such that $\sum(\triangle_0) = 180°$. Following from the preceding Theorems and Lemmas, we can construct an arbitrarily large rectangle. Now we want to prove that for every triangle, $\triangle$, the angle sum of this triangle is 180°: $\sum(\triangle) = 180°$. To prove this, construct a rectangle $R \supset \triangle$ (we know that from 1.12 such a rectangle exists). Then comparing the defect of the rectangle R and the defect of the triangle $\triangle$ yields $\delta(R) \geq \delta(\triangle)$ by Theorem 1.10. Also, the defect of the rectangle $\delta(R) = 180(4-2) - \sum(R) = 180(2) - 90(4) = 0$. But we know from Legendre-Saccheri's Theorem that $\delta(\triangle) \geq 0$. Thus $0 = \delta(R) \geq \delta(\triangle) \geq 0$ which implies $0 \geq \delta\triangle \geq 0$, and thus,

$\delta(\triangle) = 0$. Therefore, every triangle in our geometry has the angle sum 180°: $\sum(\triangle) = 180°$.

1.6.1 The End of the Proof of Theorem 1.1

Proof. From the preceding Lemmas and Theorems, we see that the Weak form of (E_5) ($\exists l_0\ \exists P_0 \notin l_0,\ \exists! m_0 \parallel l_0,\ P_0 \in m_0$) implies that for every triangle $\triangle$ its angle sum is $\sum(\triangle) = 180°$ by the uniqueness of the line $m \parallel l$. We need to show that changing the quantifiers $\exists l_0$ and $\exists P_0 \notin l_0$ for $\forall l$ and $\forall P \notin l$ gives the strong form of (E_5): for any line l and any point $P \notin l$, there exists a unique line $m \parallel l$ such that m passes through P. We first show the existence of such a line, and then we will prove its uniqueness.

Existence of $m \parallel l$

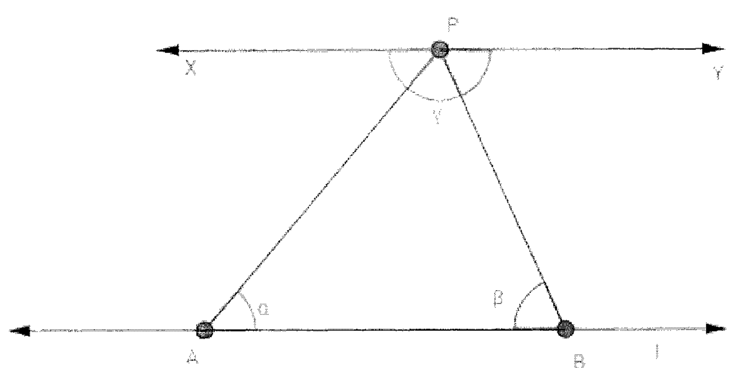

Figure 1.20: The existence of the line m through the point P parallel to the line l.

Now for a given line-point pair (l, P) construct an arbitrary triangle $\triangle APB$ with $A, B \in l$. Then from the preceding step we know that any such triangle has angle sum $\triangle APB = 180°$. For convenience we denote $\angle A = \alpha$, $\angle P = \gamma$, and $\angle B = \beta$. Draw the ray $\overrightarrow{PX}$ such that angle $\angle XPA = \alpha$, and the ray $\overrightarrow{PY}$ such that the angle $\angle YPB = \beta$. Then by the Exterior Angle Theorem (Theorem 1.4) we know that the rays $\overrightarrow{PX} \parallel l$ and $\overrightarrow{YP} \parallel l$. Since the angle sum of the triangle $\sum(\triangle APB) = 180°$, then we see that $\alpha + \beta + \gamma = 180°$. But this means that $\angle XPY = \alpha + \beta + \gamma = 180°$. So, $X * P * Y$ which implies that $\overleftrightarrow{XPY} = m$

is a line parallel to the line l. Thus the existence of the line $m \parallel l$ is proved.

Uniqueness of $m \parallel l$

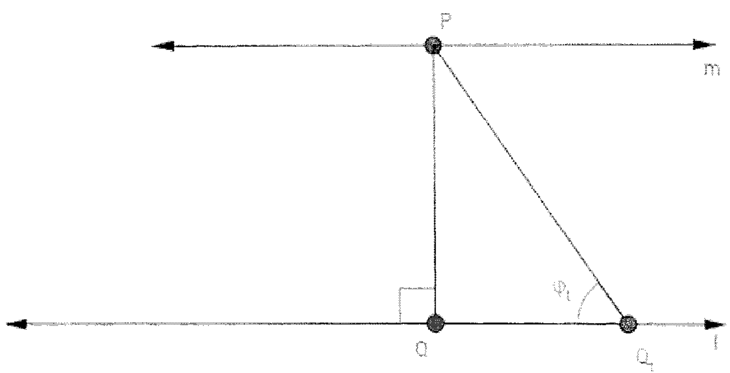

Figure 1.21: The uniqueness of the line m through the point P parallel to the line l.

To prove the uniqueness of the line $m \parallel l$, $P \in m$, we drop the perpendicular line $\overleftrightarrow{PQ} \perp l$ where $Q \in l$. Then force the point Q to move along the line l. Here, as depicted in Figure 1.21, we send point Q to the right. We can think of point $Q(t)$ as the point Q moving along the line l for $t \in [0, +\infty)$, where $Q(0) = Q$. Then denote the angle $\angle PQ(t)Q = \varphi(t)$.

Theorem 1.13 (Legendre's Angle Theorem)**.** *When the foot point $Q(t)$, with $t \in [0, +\infty)$, moves along the line l from $Q = Q(0)$ to infinity, the angle $\angle PQ(t)Q = \varphi(t)$ tends to zero: $\varphi(t) \longrightarrow 0$ as $t \longrightarrow \infty$.*

Proof of Legendre's Angle Theorem. Instead of considering the whole continuous trace of the point $Q(t)$ on l, we consider only a very special, discrete sequence of points Q_1, Q_2, Q_3, ..., $\in l$ for which we will prove that $\angle PQ_nQ \longrightarrow 0$ as $n \longrightarrow \infty$. This sequence is sufficient for us to prove the theorem, for if $Q_n \longrightarrow \infty$, then for any point $Q(t)$, with $Q < Q_n < Q(t)$, we have (by the Exterior Angle Theorem) $\angle PQ(t)Q < \angle PQ_nQ \longrightarrow 0$ as $n \longrightarrow \infty$.

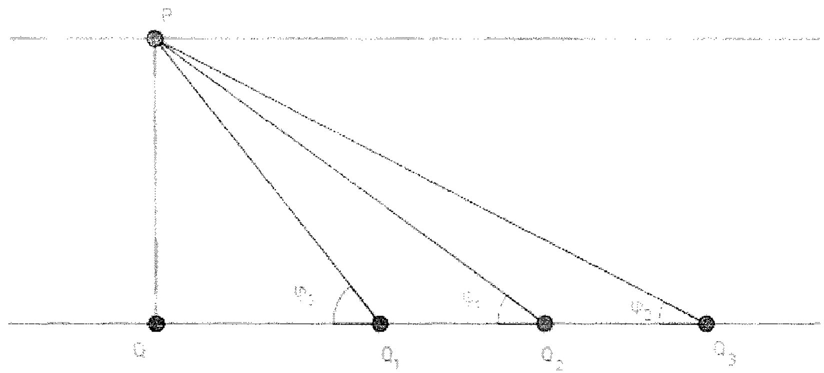

Figure 1.22: The first three points of the sequence $\{Q_n\}$ corresponding to Legendre's Trick.

To prove Legendre's Angle Theorem, we will employ Legendre's Trick which we now describe. Since every triangle $\triangle$ has angle sum $\sum(\triangle) = 180°$, then we have the following:

$$\begin{aligned}
&(a_1) \text{ For } \triangle PQQ_1 : 2\varphi_0 = 180° - 90° \implies \varphi_0 = 45°;\\
&(a_2) \text{ For } \triangle PQ_1Q_2 : 2\varphi_1 = \varphi_0 \implies \varphi_1 = \frac{1}{2} \cdot 45°;\\
&(a_3) \text{ For } \triangle PQ_2Q_3 : 2\varphi_2 = \varphi_1 \implies \varphi_2 = \frac{1}{2} \cdot \varphi_1 = \frac{1}{2^2} \cdot 45°;\\
&\qquad \vdots\\
&(a_n) \text{ For } \triangle PQ_{n-1}Q_n : 2\varphi_n = \varphi_{n-1} \implies \varphi_n = \frac{1}{2}\varphi_{n-1} = \frac{1}{2^n} \cdot 45°.
\end{aligned}$$

We see that $\varphi_n \longrightarrow 0$ as $n \longrightarrow \infty$, and the sequence of angles $\{\varphi_n\}$ approaches 0 as a geometric sequence with the common ratio $\frac{1}{2}$. So, $\psi_n = \varphi_0 + \varphi_1 + \varphi_2 + ... + \varphi_n$ and $\psi_n = 45°(1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n})$. It follows that

$$\begin{aligned}
\psi &= \lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k\\
&= 45° \cdot \sum_{k=0}^{\infty} \frac{1}{2^k}\\
&= 45° \cdot 2 = 90°.
\end{aligned}$$

□

So, $\varphi(t) \longrightarrow 0$ means that the limiting position of the ray $\overrightarrow{PQ(t)}$ (that intersects line l at $Q(t)$) is exactly the ray r^+. Hence, there exists only one right ray, namely r^+, which is parallel to l passing through P. This means that the right limiting ray r^+ is unique. The same reasoning shows that the left limiting ray, r^-, is also unique. Then the union of the two limiting rays, $r^- \bigcup r^+$, is determined uniquely. But since the two rays make the right angle with PQ, and $90° + 90° = 180°$, we conclude that $m = r^- \bigcup r^+$ is the unique line parallel to l. Thus, Theorem 1.1 is proved.

$\square$

Chapter 2

Theorems on $\mathbb{H}^2$

By taking constructions from Euclidean geometry, we set out to formulate the results that we obtain by performing similar constructions in hyperbolic geometry. In $\mathbb{H}^2$, the negation of Euclid's 5th postulate tells us that given a line l and a point P not on l, then there are many lines passing through point P which are parallel to line l. Since we no longer have a unique parallel line as in the Euclidean case, we need to understand now how our parallel lines behave. As we will come to find there are two types of parallel lines in hyperbolic geometry, asymptotically parallel and divergently parallel.

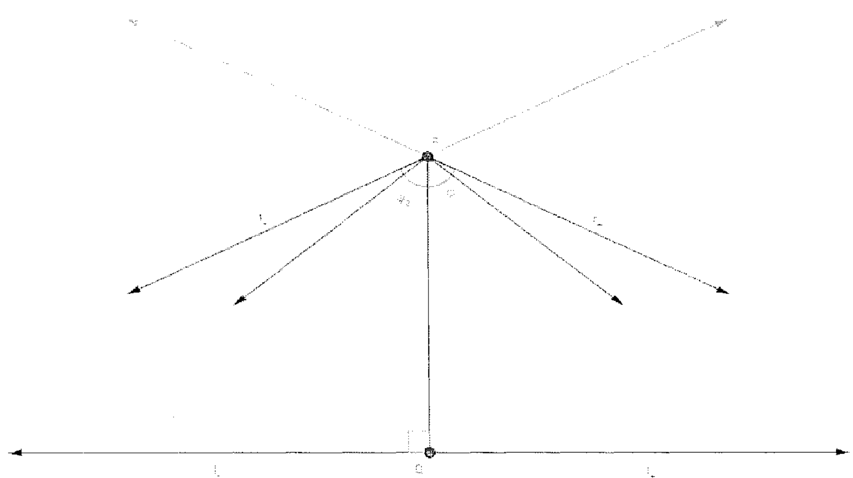

Figure 2.1: Asymptotically parallel lines in hyperbolic geometry.

Given a line l, a point $Q \in l$, and a point $P \notin l$ such that PQ is perpendicular to line l. Consider the part of line l to the right of the line $\overleftrightarrow{PQ}$, call it l^+. Draw rays out of point P and to the right. We call r^+ the first ray that is parallel to line l. Repeat the same process for the left side of $\overleftrightarrow{PQ}$, and denote the left part of l by l^-. The two rays, r^+ and r^-, form a an angle $\angle r^- P r^+$ with the vertex P. Extend the rays r^- and r^+ to the two lines $\overleftrightarrow{r^-}$ and $\overleftrightarrow{r^+}$. They are called, respectively, the *left* and *right asymptotic parallel* lines to l. The ray $\overrightarrow{PQ}$ splits the angle $\angle r^- P r^+$ into two acute angles: $\varphi_1 = \angle QPr^+$ and $\varphi_2 = \angle QPr^-$. It turns out that PQ is, actually, the angle bisector of the angle $\angle r^- P r^+$.

Lemma 2.1. *The two acute angles from the right and left asymptotic parallel lines φ_1 and φ_2, respectively, are equal: $\varphi_1 = \varphi_2$.*

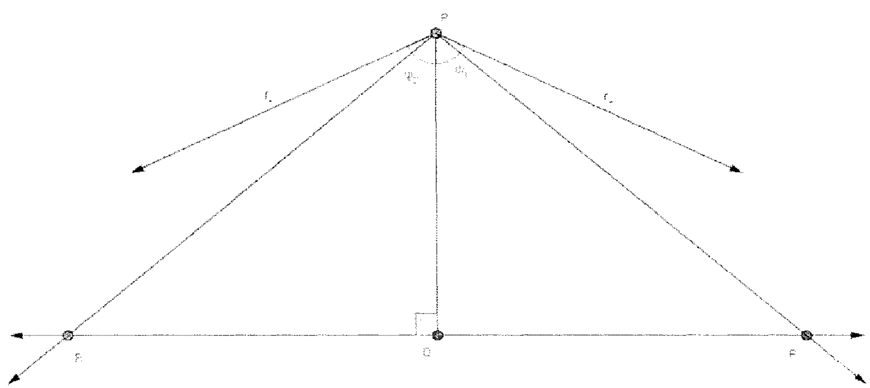

Figure 2.2: The angles φ_1 and φ_2 are equal.

Proof. Suppose that $\varphi_1 \neq \varphi_2$. Without loss of generality, let $\varphi_2 > \varphi_1$. Then draw the ray $\overrightarrow{Pq}$ inside the angle $\angle QPr^-$ such that $\angle QPq = \varphi_1$. Since r^- is the asymptotic parallel ray to l, we conclude that the ray $\overrightarrow{Pq}$ must intersect at some point $R \in l^-$. Now reflect the triangle $\triangle PQR$ in the line $\overleftrightarrow{PQ}$ and we get $\triangle PQR'$, where $R' \in l^+$, congruent to $\triangle PQR$. Hence, $\angle QPR' = \varphi_1$ which means that the point R' lies on the ray r^+. Thus we get that ray r^+ meets line l at point R'; that is, $\overleftrightarrow{r^+}$ is not parallel to l, a contradiction. Hence, the angles $\varphi_1 = \varphi_2$. □

This Lemma allows us to give the following fundamental definition,

Definition 2.2. The angle $\angle QPr^+ = \varphi$ is said to be the **angle of parallelism** for the pair (P, l).

As a consequence, every line m through point P that does not intersect the angle $\angle r^- Pr^+$ and its vertical angle is called *divergently parallel* to line l.

By the homogeneity of the hyperbolic plane, the angle of parallelism $\varphi = \varphi(P, l)$ does not depend on the position of the pair (*point* P, *line* l) as a rigid body in the plane. It depends only on the distance d between P and l; that is, $\varphi = function(\mathrm{dist}(P, l))$. Denoting $\mathrm{dist}(P, l) = d$, we obtain, due to Lobachevsky's notation, the function Π:

$$\varphi = \Pi(d) \tag{2.1}$$

The function Π is called the Bolyai-Lobachevsky function. Thus, $\Pi(d)$ is the angle of parallelism for the pair (P, l); and therefore, $\angle r^- Pr^+ = 2\Pi(d)$.

The following question arises: is the distance d a function of the angle of parallelism? That is, does there exist a function Π^{-1} such that $d = \Pi^{-1}(\varphi)$? In other words, is the Lobachevsky function Π a one-to-one function? The following theorem answers this question affirmatively.

Theorem 2.3 (Bolyai-Lobachevsky)**.** *$\forall$ distance d $\exists!$ angle $\varphi = \Pi(d)$, the angle of parallelism. Hence: $\forall \varphi \in (0, \frac{\pi}{2})$ $\exists!$ d such that $\varphi = \Pi(d)$; that is $d = \Pi^{-1}(\varphi)$.*

Moreover, the function Π has the following additional properties:

1. $\Pi(d)$ is a strictly decreasing function; that is,

$$d_1 < d_2 \Longleftrightarrow \varphi_1 = \Pi(d_1) > \varphi_2 = \Pi(d_2)$$

2. $d \to 0 \Longleftrightarrow \varphi \to \frac{\pi}{2}$; $\quad d \to \infty \Longleftrightarrow \varphi \to 0$

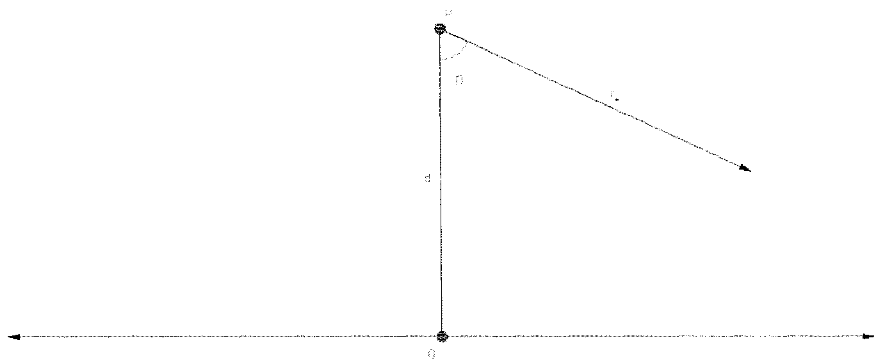

Figure 2.3: The Bolyai-Lobachevsky function for the angle of parallelism.

3. The function Π satisfies the following two equivalent relationships:

$$\cos(\varphi) = \tanh(d) \text{ and}$$
$$\tan(\frac{\varphi}{2}) = e^{-d}$$

We will not prove this theorem now, instead we will only show the equivalency of these two equations.

Proof. We want to show the equivalency of the equations

$$\tanh(d) = \cos(\varphi) \tag{2.2}$$

$$e^{-d} = \tan\left(\frac{\varphi}{2}\right) \tag{2.3}$$

So, we compute

$$\begin{aligned}\tanh(d) = \frac{\sinh(d)}{\cosh(d)} &= \frac{\frac{e^d - e^{-d}}{2}}{\frac{e^d + d^{-d}}{2}} \\ &= \frac{e^d - e^{-d}}{e^d + e^{-d}} \\ &= \frac{e^{2d} - 1}{e^{2d} + 1} < 1\end{aligned}$$

Let $t = \tanh(d) = \dfrac{e^{2d}-1}{e^{2d}+1}$. Then we have that

$$\begin{aligned} e^{2d-1} &= t(e^{2d}+1) \\ &\implies e^{2d} - 1 = te^{2d} + t \\ &\implies e^{2d}(1-t) = 1+t \\ &\implies e^{2d} = \frac{1+t}{1-t} \end{aligned}$$

So, $e^{-d} = \sqrt{\dfrac{1-t}{1+t}}$, and equation (2.3) can be rewritten as follows:

$$\sqrt{\frac{1-t}{1+t}} = \frac{\sin(\frac{\varphi}{2})}{\cos(\frac{\varphi}{2})}$$

Now we want to show that plugging $t = \cos(\varphi)$ from equation (2.2) into the left hand side (LHS) of equation (2.3) yields the right hand side (RHS) of equation (2.3). From equation (2.2) we have that

$$\begin{aligned} \text{LHS}(2.3) = \sqrt{\frac{1-t}{1+t}} &= \sqrt{\frac{1-\cos(\varphi)}{1+\cos(\varphi)}} \\ &= \sqrt{\frac{2\sin^2(\frac{\varphi}{2})}{2\cos^2(\frac{\varphi}{2})}} \\ &= \frac{\sin(\frac{\varphi}{2})}{\cos(\frac{\varphi}{2})} = \tan\left(\frac{\varphi}{2}\right) = \text{RHS}(2.3) \end{aligned}$$

The proof of the equivalency of equations (2.2) $\iff$ (2.3) is complete. $\square$

Chapter 3

Geometric Structure of Lines and Special Curves of $\mathbb{H}^2$

In Chapter 1 we introduced the idea of two types of special quadrilaterals, the Saccheri quadrilateral and the Lambert quadrilateral. The Saccheri quadrilateral had two right angles, and one pair of opposite congruent sides. On the other hand, the Lambert quadrilateral had three right angles. Moreover, these quadrilaterals will be used to establish the geometric structure of $\mathbb{H}^2$. Our first step is to make precise their definitions.

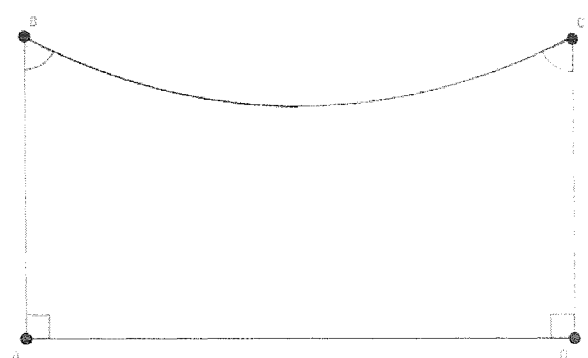

Figure 3.1: The quadrilateral $ABCD$ is a Saccheri quadrilateral.

Earlier, we saw that there are two methods for constructing a rectangle in $\mathbb{E}^2$. We learned that Saccheri used the construction outlined in **Method 2** in his attempt to prove

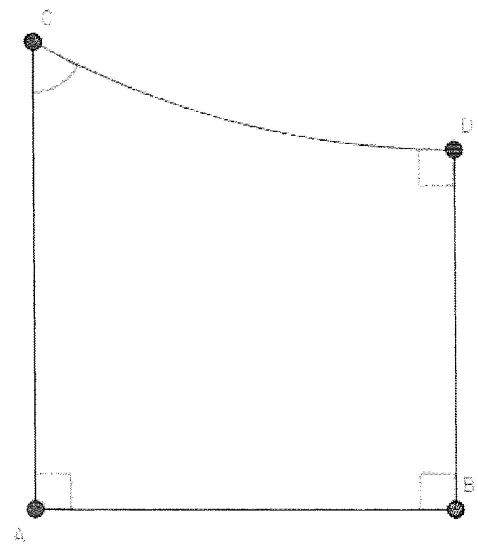

Figure 3.2: The quadrilateral $ABCD$ is a Lambert quadrilateral.

(E_5), ultimately finding the Saccheri quadrilateral. Start with a base segment, called AD, and two perpendicular segments of equal length called AB and CD. After joining the vertices B and C by segment BC, we have constructed the quadrilateral $ABCD$ having the following two properties:

$$\begin{cases} 1)\ AB &= CD \\ 2)\ \angle A &= \angle D = 90^\circ \end{cases}$$

In $\mathbb{E}^2$, we saw that $\angle B = \angle C = 90°$ and $BC = AD$. What remains to be understood is if in $\mathbb{H}^2$ there is any relation between the two remaining angles, $\angle B$ and $\angle C$, as well as what is the relation between the segment BC and the segment AD. As a matter of taste, we refer to angles $\angle A$ and $\angle D$ as the base angles of the quadrilateral, the segment AD as the base, and the segment BC as the summit. Right away we can see that the base and summit are not of equal length, for if it were the case, then we would have an Euclidean rectangle. So, we need to see if the summit is of length greater than or less than the length of the base. As for the angles, $\angle B$ and $\angle C$, intuition might lead you to believe that these angles are equal; and in fact, the following theorem will show that this is precisely the case.

Theorem 3.1. *The two summit angles, $\angle B$ and $\angle C$, of a Saccheri quadrilateral, $ABCD$, are congruent: $\angle B = \angle C$.*

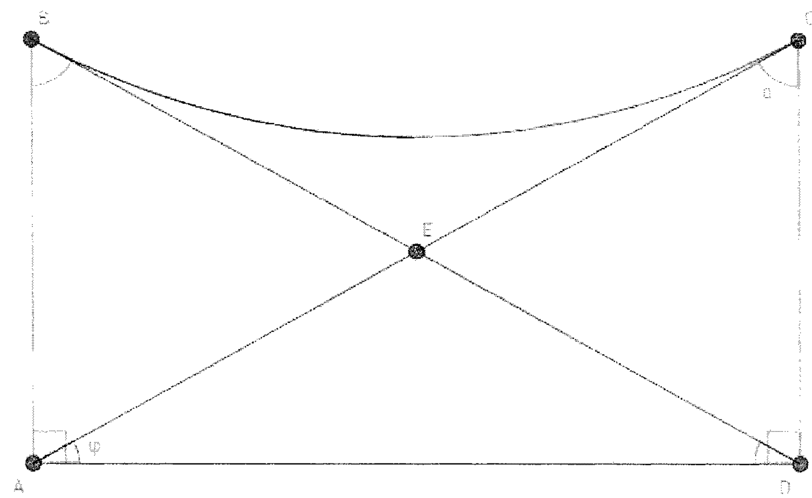

Figure 3.3: The summit angles, $\angle B$ and $\angle C$, of a Saccheri quadrilateral are equal.

Proof. Consider the Saccheri quadrilateral $ABCD$ with $\angle A = \angle D = 90°$, and $AB = CD$. Construct the diagonals AC and BD. These two diagonals intersect at a point, called E. Observe that if we can show that the triangles $\triangle ABC$ and $\triangle DCB$ are congruent, then we are done. In the resulting figure, we know that $\angle A = \angle CAB + \angle CAD$ and $\angle D = \angle BDA + \angle BDC$. Then the triangles $\triangle ACD$ and $\triangle ABD$ are congruent by the side-angle-side axiom: $CD = AB$, $\angle A = \angle D$, and $AD = AD$. As a result, we know that $\angle CAD = \angle BDA$, and that $AC = BD$. Now since $\angle CAD = \angle BDA = \varphi$, it follows that $\angle CAB = 90° - \varphi = \angle BDC$.

Now consider the triangles $\triangle ABC$ and $\triangle DCB$. These two triangles are also congruent by the side-angle-side axiom, since $AC = BD$, $\angle CAB = \angle BDC$, and $AB = CD$. Therefore, we can conclude that the angles $\angle B = \angle C$. □

We have shown that the two summit angles of a Saccheri quadrilateral are equal. In $\mathbb{N}^2$, it follows from Legendre's Theorem (Theorem 1.6) applied to the two triangles $\triangle ABD$ and $\triangle DCB$ that $2\alpha + 2\cdot 90° \leq 360°$. Solving this equation for α, we find that $\alpha \leq 90°$. This reduces to two cases. If $\alpha = 90°$, then the quadrilateral $ABCD$ is a rectangle, and we are in $\mathbb{E}^2$. On the other hand, if $\alpha < 90°$, then we are in $\mathbb{H}^2$, and we have a Saccheri quadrilateral. Formalizing our result, a Saccheri quadrilateral $ABCD$ is a quadrilateral in $\mathbb{H}^2$ satisfying the properties:

$$\begin{cases} 1) AB = CD \\ 2) \angle A = \angle D = 90° \\ 3) \angle B = \angle C < 90° \end{cases}$$

The other method for constructing a rectangle, **Method 1**, was used by Lambert in his attempt to prove Euclid's parallel postulate, ultimately creating the Lambert quadrilateral. To build such a quadrilateral, first take a segment AD which will function as the base of our quadrilateral. Construct the unique perpendicular lines to segment AD through each of the points A and D. Choose some point B on the perpendicular line passing through point A, and from there erect the perpendicular line to AB passing through B. This new perpendicular line will intersect the perpendicular line through the point D, and we denote this point of intersection C. The resulting figure $ABCD$ is a quadrilateral having the following properties:

$$\begin{cases} 1)\ \angle A = \angle B = \angle D = 90° \\ 2)\ \angle C < 90° \end{cases}$$

In Euclidean geometry $\mathbb{E}^2$, we saw that following this construction the angle measure $\angle C = 90°$ and pairs of opposite sides had equal length. Since the angle measure $\angle C$ is now less than $90°$ in this new setting, there is reason to believe that the pairs of opposite sides are no longer congruent. Indeed, the following theorem explains the relations between the side lengths of opposite sides.

Theorem 3.2. *Given a Lambert quadrilateral $ABCD$. The following inequalities hold:*

$$\begin{cases} 1)\ BC > AD \\ 2)\ CD > BA \end{cases}$$

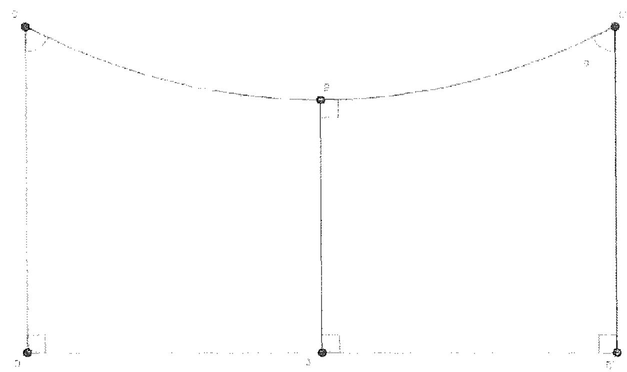

Figure 3.4: The Saccheri quadrilateral $D'C'CD$ obtained via the reflection of the Lambert quadrilateral $ABCD$.

Proof. We will prove the first inequality for a Lambert quadrilateral, $BC > AD$; proving the second inequality $CD > BA$ follows similar steps. Reflect the Lambert quadrilateral $ABCD$ across the line segment AB to obtain the Saccheri quadrilateral $D'C'CD$ where $\angle D' = \angle D = 90°$ and $\angle C' = \angle C = \alpha$. We denote the summit BC by $s = BC$ and the base AD by $b = AD$. Moreover, the segments $C'B = BC$ and $D'A = AD$. We will show that $(BC > AD) \iff (C'C > D'D)$, or $s > b$. We construct a proof by contradiction. Suppose that $s \not> b$. Then either $s = b$ or $s < b$. We consider both cases.

Case 1 $(s = b)$

Suppose that $s = b$. Then construct the diagonal $C'A$. By the angle-side-angle axiom, the triangles $\triangle AD'C' = \triangle ABC'$ since $AD' = b = s = BC'$, $\angle D' = 90° = \angle ABC'$, and $\angle AC'D' = \angle BAC'$. As a result, the angle $\angle D'AC' = BC'A$. But we know that

$$90° = \angle A = \angle BAC' + \angle D'AC' = \angle BC'A + \angle AC'D' = \angle C'$$

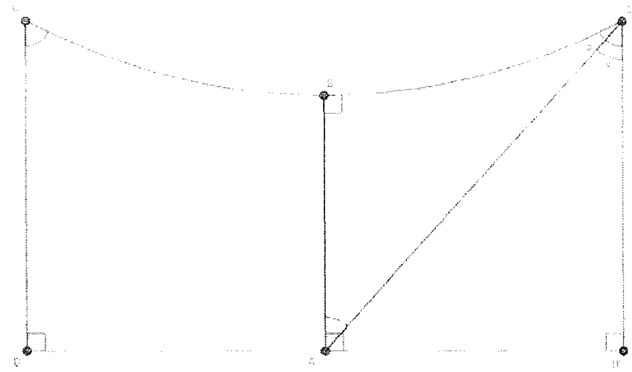

Figure 3.5: The summit length s equals the base length b leads to a contradiction.

Since the quadrilateral $D'C'CD$ is a Sachheri quadrilateral, then by definition $\angle C = \angle C' = 90°$. Therefore, the quadrilateral $ABCD$ is a rectangle which implies that we are in Euclidean geometry $\mathbb{E}^2$. But this is a contradiction with our initial assumption that we are in $\mathbb{H}^2$. Thus, $\angle C' = \alpha \neq 90°$; hence, $s \neq b$.

Case 2 $(s < b)$

Figure 3.6: The summit length s is less than the base length b leads to a contradiction.

Suppose that $s < b$. Then we construct a Saccheri quadrilateral $X_1X_2Y_1Y_2$ and reflect it across X_2Y_2, yielding the equal Saccheri quadrilateral $X_2X_3Y_2Y_3$. Again we reflect the quadrilateral $X_2X_3Y_2Y_3$ across X_3Y_3, and we continue reflecting the quadrilaterals in this fashion until we have a chain of $n-1$ equivalent quadrilaterals (the last one being $X_{n-1}X_nY_{n-1}Y_n$). Then by the triangle inequality we know that the shortest distance between two points in the plane is the straight line distance. So, the length of the broken line is longer than the length of the bottom segment,

$$length(X_1 Y_1 Y_2 ... Y_{n-1} Y_n X_n) > length(X_1 X_2 ... X_n)$$

and each of the angles $\angle Y_i = \alpha + \alpha < 90° + 90° = 180°$. Then we have that

$$\begin{aligned} &2h + ns > nb \\ &\implies h > n\frac{b-s}{2} = n\varepsilon \\ &\implies n < \frac{h}{\varepsilon} \end{aligned}$$

$\forall n \in \mathbb{N}$. Here we denote $\varepsilon = \dfrac{b-s}{2}$ which is non-negative since $b > s$ and ε is a fixed number. Also, h is a fixed number. So, the quantity $\dfrac{h}{\varepsilon}$ is a fixed number. Take $n_0 > \dfrac{h}{\varepsilon}$ as n. We know such an $n \in \mathbb{N}$ exists. Then we will have that $h < n_0\dfrac{b-s}{2}$. But $h > n\dfrac{b-s}{2}$, a contradiction. Thus, $s \not< b$. Combining the results from Case 1 and Case 2 we conclude that since the summit length does not equal the base length, $s \neq b$, and the summit length is not less than the base length, $s \not< b$, then summit length is greater than the base length, $s > b$, in a Saccheri quadrilateral. Therefore, we conclude that $BC > AD$ in the Lambert quadrilateral $ABCD$ since $BC = \frac{1}{2}s > \frac{1}{2}b = AD$.

□

Having established the relations between side lengths of opposite sides of a Lambert quadrilateral, we now formalize the requirements of a Lambert quadrilateral. We say that a quadrilateral $ABCD$ is a Lambert quadrilateral if it satisfies the following properties:

$$\begin{cases} 1) \angle A = \angle B = \angle D = 90° \\ 2) \angle C < 90° \\ 3) BC > AD \\ 4) CD > BA \end{cases}$$

We now explore the connections between the two types of quadrilaterals in hyperbolic geometry, Lambert quadrilateral and Saccheri quadrilateral.

Theorem 3.3. *The only common perpendicular segment for the base and summit of a Saccheri quadrilateral is* MN*, where* M *is the midpoint of* BC*, and* N *is the midpoint of* AD*.*

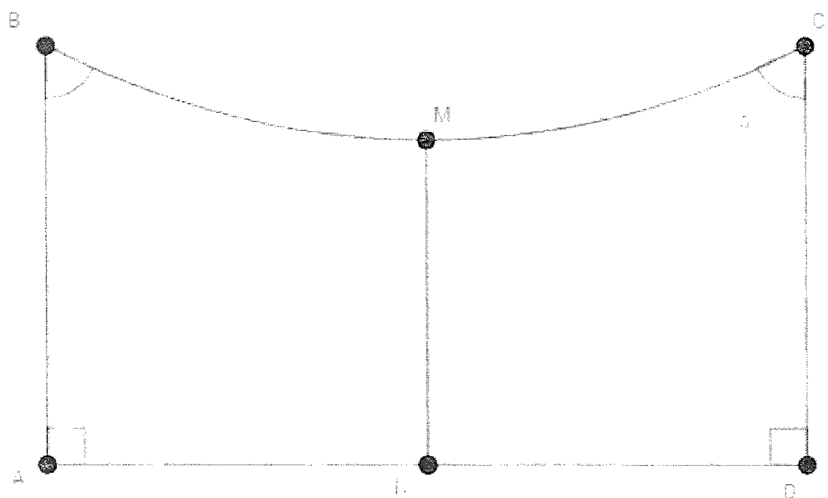

Figure 3.7: The segment MN is the perpendicular bisector of the Saccheri quadrilateral $ABCD$.

Proof. Let $ABCD$ be a Saccheri quadrilateral. Let M and N be the midpoints of segments BC and AD, respectively. Then $BM = CM$ and $AN = DN$. Draw the segment MN which joins the two midpoints. Our first goal is to show that MN is perpendicular to both BC and AD. We will do this by proving that the resulting angles $\angle BMN = \angle CMN = \angle ANM = \angle DNM = 90°$. Then we will show that MN is the unique common perpendicular.

The next step in showing that the segment MN is perpendicular to both BC and AD is to construct the two diagonal segments BN and CN. Since $ABCD$ is a Saccheri quadrilateral, we know that $\angle B = \angle C = \alpha$, $\angle A = \angle D = 90°$, and $BA = CD$. The triangles

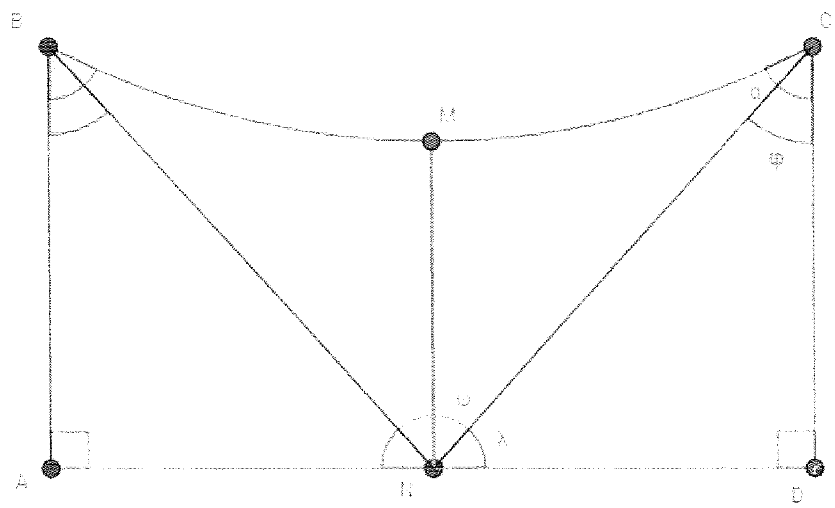

Figure 3.8: The segment MN is perpendicular to the sides BC and AD.

$\triangle BAN$ and $\triangle CDN$ are congruent by the side-angle-side axiom, since $BA = CD$, $\angle A = \angle D = 90°$, and $AN = DN$. It follows that the angles $\angle ABN = \angle DCN = \varphi$, $\angle DNC = \angle ANB = \lambda$, and $BN = CN$. Observe that the angles $\angle NBM = \angle NCM = \alpha - \varphi$. Then we see that the triangles $\triangle NMB = \triangle NCM$ by the side-angle-side axiom, since $BN = CN$, $\angle NBM = \angle NCM = \alpha - \varphi$, and $BM = CM$. As a result, the angles $\angle BMN = \angle CMN$ and $\angle BNM = \angle CNM = \psi$. Combining the last statement with the result above we find that

$$\angle ANM = \angle ANB + \angle BNM = \lambda + \psi = \angle DNC + \angle CNM = \angle DNM$$

The last step is to show that the segment MN lies on the line perpendicular to BC, called $BC^{\perp}$.

Construct the line perpendicular to BC through the point M. We need to show that the intersection $BC^{\perp} \bigcap AD = N$. Let the set of points equidistant from the points B and C be the set $S = \{P \mid PB = PC\}$. We will show that S is the perpendicular bisector to BC. Let P be a point such that $P \in BC^{\perp}$. Then after drawing the segments BP and PC we have two congruent triangles $\triangle BPM \cong \triangle CPM$ by the side-angle-side axiom, since $BM = CM$, $\angle BMP = \angle CMP$, and $MP = MP$ is a shared side. Therefore, we know that

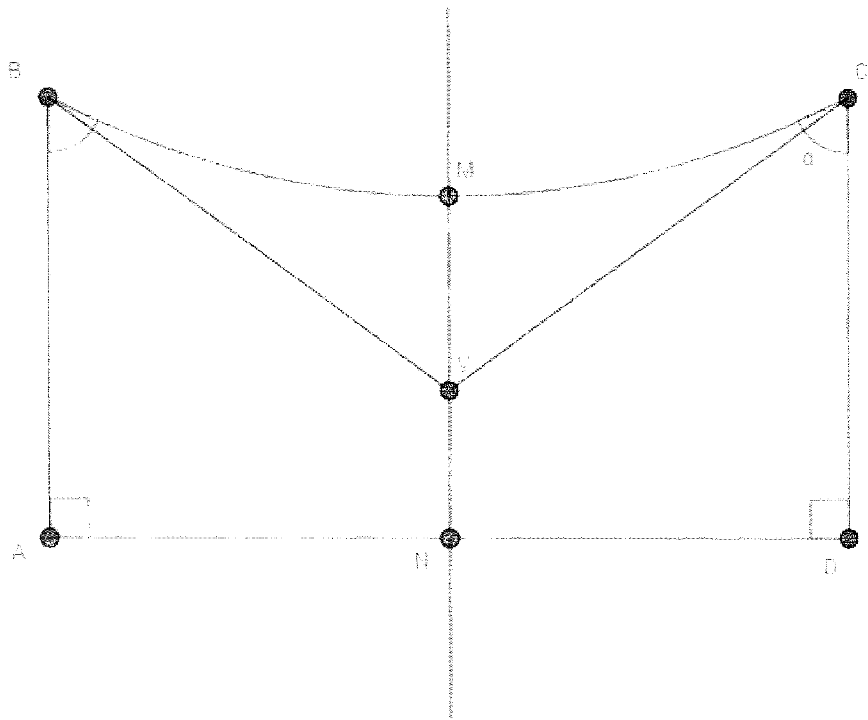

Figure 3.9: The set of points S equidistant to the points B and C coincides with the perpendicular bisector to BC.

the sides $BP = PC$. Hence, the set $S \supset BC^{\perp}$. To show that $S \subset BC^{\perp}$, we will prove that contrapositive statement that if a point $X \notin BC^{\perp}$, then $BX \neq CX$.

Let X be a point not on $BC^{\perp}$. Without loss of generality we may assume that the point X lies as shown in Figure 3.10. Then draw the segments BX and CX. The segment BX intersects the line $BC^{\perp}$ at some point P. So, we can say that $BX = BP + PX$. Since P is a point on $BC^{\perp}$, then from the preceding argument we know that $BP = PC$. Substituting, into the relation for BP we find that $BX = PC + PX$. Applying the triangle inequality on triangle $\triangle PCX$ we find that $PC + PX > CX$. So, $BX = PC + PX > CX$; and hence, $X \notin S$. Therefore, $S \subset BC^{\perp}$. Having shown containment in both directions, we can conclude that $S = BC^{\perp}$. More importantly, we see that since $CN = BN$, we conclude that $\angle ANM = \angle DNM = 90°$. $\square$

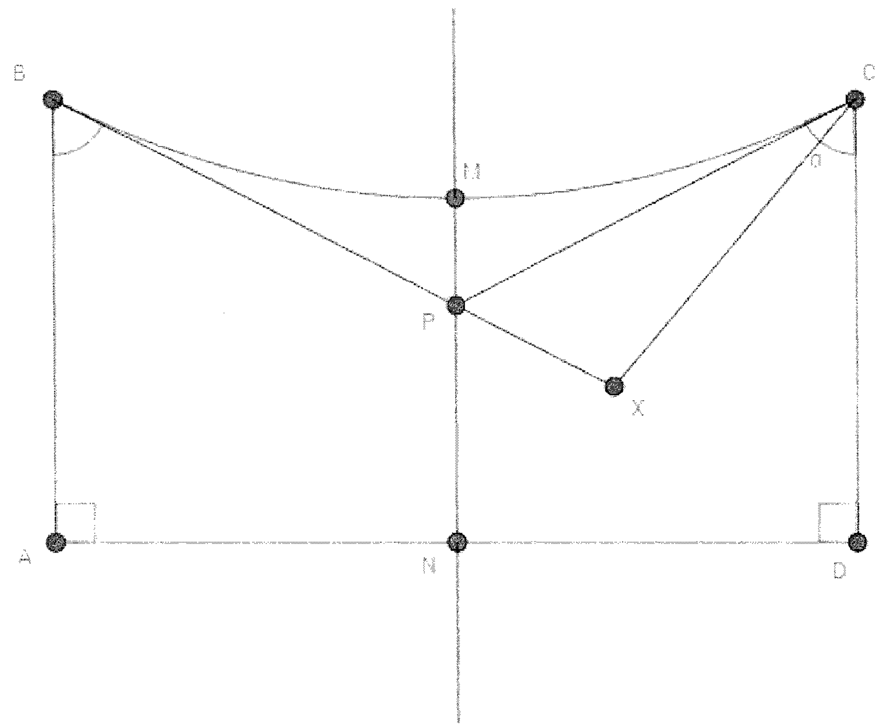

Figure 3.10: The segment MN is the perpendicular bisector of segment BC.

After drawing the common perpendicular, MN, for the base and summit of a Saccheri quadrilateral, $ABCD$, we create two quadrilaterals, $ABMN$ and $DCMN$. These two quadrilaterals are congruent and are in fact Lambert quadrilaterals. This idea leads us to our next discovery: starting with a Lambert quadrilateral, $ABMN$, and reflecting it in the line $\overleftrightarrow{MN}$ yields another Lambert quadrilateral, $DCMN$. And together, these two quadrilaterals form a Saccheri quadrilateral. So, we have established a connection between Lambert quadrilaterals and Saccheri quadrilaterals in $\mathbb{H}^2$.

Corollary 3.4. *Given Saccheri quadrilateral $ABCD$ and the common perpendicular segment MN to the base and the summit, $MN < AB = CD$.*

3.1 Perpendicular in Saccheri Quadrilateral

In a Saccheri quadrilateral $ABCD$, the two summit angles, $\angle B = \angle C = \alpha$, have angle measure less than $90°$. Something to consider is at what point on the line $\overleftrightarrow{AB}$ does the line perpendicular to $\overleftrightarrow{AB}$ passing through C intersect the line $\overleftrightarrow{AB}$. Does this intersection point lie above or below the point B? Suppose that $B' = AB^{\perp} \bigcap AB$ lies above B. Then by the exterior angle theorem (Theorem 1.4) on $\triangle BB'C$ the angle $\alpha > 90° = \angle B'$. But this is a contradiction since $\alpha < 90°$. Thus, we conclude that the point B' lies below B.

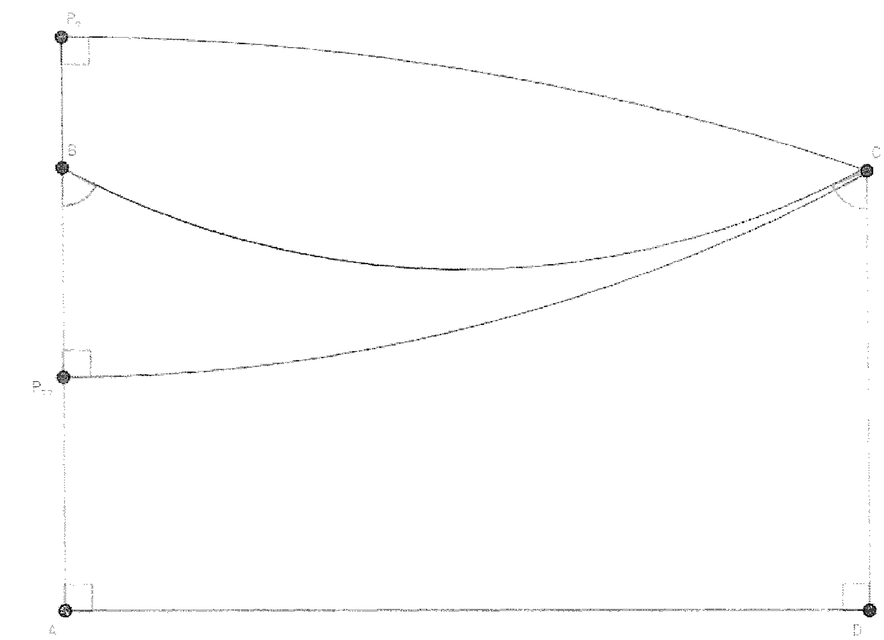

Figure 3.11: The location of the line perpendicular to the segment AB passing through the point C of a Saccheri quadrilateral.

Extending this construction to a Lambert quadrilateral, $ABCD$ with $\angle C = \alpha < 90°$, we deduce that the intersection point between the line $\overleftrightarrow{CD}$ and the line perpendicular to $\overleftrightarrow{CD}$ passing through B lies below the point C, such that $B * C * D$. Call this point of intersection C', then we have two Lambert quadrilaterals, namely $ABCD$ and $ABC'D$ with $\angle ABC' = \alpha' < 90°$. What is the relation between the angle measure α and α'. We explore this relation in the following construction.

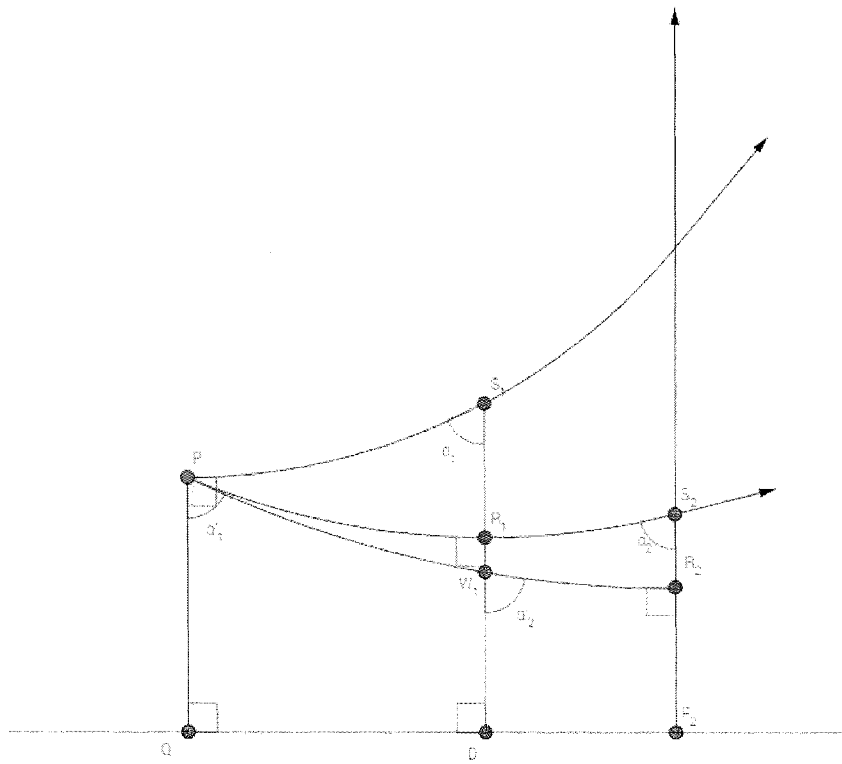

Figure 3.12: Successive perpendiculars in a Lambert quaderilateral.

We are interested in taking successively perpendiculars in a Lambert quadrilateral. Consider the Lambert quadrilateral QQ_1S_1P as depicted in Figure 3.12. Let $a_1 = \overleftrightarrow{PS_1}$. Construct the line perpendicular to S_1Q_1 through the point P_1. We know from above that this line, call it a_2, lies below S_1. What is the relation between the angles $\alpha_1 = \angle Q_1S_1P$ and $\alpha_1' = \angle QPR_1$?

From the defect of a quadrilateral we have that $area(QPS_1Q_1) > area(QPR_1Q_1)$. It follows that

$$2\pi - (\frac{3\pi}{2} + \alpha_1) > 2\pi - (\frac{3\pi}{2} + \alpha_1')$$
$$\Longrightarrow \alpha_1' > \alpha_1$$

Erect the perpendicular at Q_2, then we have the Lambert quadrilateral $Q_1R_1S_2Q_2$. Construct the perpendicular line to S_2Q_2 through P. Then we can show that $\alpha_2' > \alpha_2$, as depicted in Figure 3.12.

Question: Does $h > h' \Longrightarrow \alpha_1' < \alpha_1''$

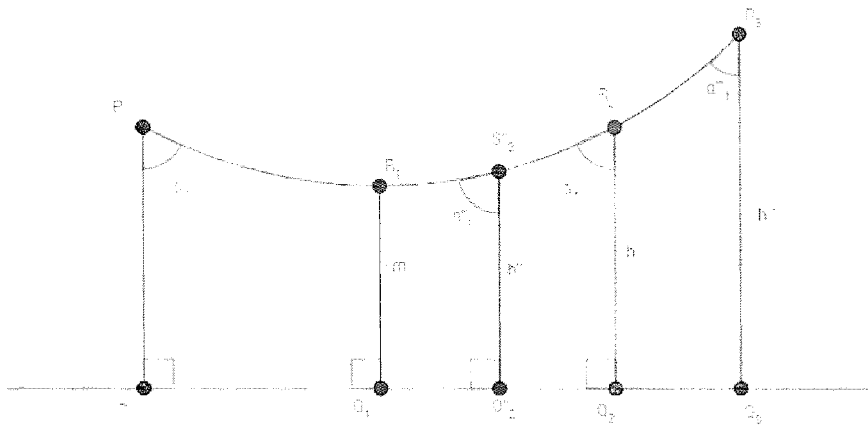

Figure 3.13: The angle $\angle\alpha_1''$ is greater than the angle $\angle\alpha_1'$.

Prior to answering this question, note that the defect of a Lambert quadrilateral can be simplified

$$\delta(QPR_1Q_1) = 2\pi - (\frac{3\pi}{2} + \alpha_1') = (2\pi - \frac{3\pi}{2}) - \alpha_1' = \frac{\pi}{2} - \alpha_1'$$

Consider the Lambert quadrilaterals QPR_1Q_1 and $Q_1R_1S_2''Q_2''$. Then we have that $\delta(QPR_1Q_1) > \delta(Q_1R_1S_2''Q_2'')$. This follows from reflecting the quadrilateral QPR_1Q_1 across the segment R_1Q_1 resulting in the mirrored Lambert quadrilateral $Q_1R_1S_2Q_2$. So, it follows that

$$\frac{\pi}{2} - \alpha_1' > \frac{\pi}{2} - \alpha_1'' \Longrightarrow \alpha_1'' > \alpha_1'$$

and $S_2Q_2 = PQ = h > h'' > m$ where m is the common perpendicular between the lines a_2 and b. For a subsequent height h''', we first need to reflect the quadrilateral QPR_1Q_1

across R_1Q_1 to yield the mirrored Lambert quadrilateral $Q_1R_1S_2Q_2$. Then we determine which quadrilateral has the larger defect (area) which in turn yields the following cases:

$$\begin{cases} h > h''' & \Longrightarrow \alpha_1' < \alpha_1''' \\ h''' > h & \Longrightarrow \alpha_1''' < \alpha_1' \end{cases}$$

This observation yields the following theorem.

Theorem 3.5. *If the lines a and b are divergently parallel, then the orthogonal projection of the hyperbolic line a onto the hyperbolic line b, denoted $proj_b a$, is such that $proj_b a \in b$ and the following properties of the angle $\alpha(t)$, depicted in Figure 3.13, hold:*

1. *$\alpha(t)$ increases monotonically to $\frac{\pi}{2}$ on the left of the common perpendicular MN.*
2. *$\alpha(t) = \frac{\pi}{2}$ on the common perpendicular MN.*
3. *$\alpha(t)$ decreases monotonically to 0 on the right of the common perpendicular MN.*

Theorem 3.6. *$proj_l(m) =$ an open interval $\subset l$.*

Proof. The proof of this theorem will be provided in Chapter 4 with the help of the Klein model. □

Theorem 3.7. *If the line m is asymptotically parallel to line l, then $proj_l(m)$ is a ray.*

Theorem 3.8. *If the line m intersects line l, then the projection $proj_l(m)$ is an open interval.*

Chapter 4

Klein Model of $\mathbb{H}^2$

Up to this point we have developed the theory of hyperbolic geometry by extending various constructions used in Euclidean geometry (such as constructing a rectangle). Additionally, we have attempted to understand what hyperbolic geometry looks like by considering the special curves of $\mathbb{H}^2$. Taking it a step further, it would be nice to be able to visualize the hyperbolic plane in terms of something with which we are familiar, namely the Euclidean plane. One way to accomplish this goal is through the use of a model. Here a model is a subset of the Euclidean plane $\mathbb{E}^2$, or the typical plane $\mathbb{R}^2$. Formally speaking, there is a function, f, which maps the hyperbolic plane to the Euclidean plane: $f : \mathbb{H}^2 \longrightarrow \mathbb{E}^2$, taking the entire hyperbolic plane and mapping it to a subset of $\mathbb{E}^2$. There are, however, some caveats to this visualization process. We are not able to recover the entire structure of the hyperbolic plane within our model. As a result, there are different models which preserve different aspects of the hyperbolic plane. The information that interests us determines which model we use.

Since we will often talk about these models, we adopt the following shorthand notation: the Klein disk model - $\mathbb{K}^2$; the Poincaré spherical model - $\mathbb{S}^2$; the Poincaré disk model - $\mathbb{P}^2$; the Poincaré upper half-plane model - $\mathbb{U}^2$; and finally, the Minkowski hyperboloid model - $\mathbb{M}^2$. When discussing a hyperbolic line in a particular model, we will use the shorthand k-line for a hyperbolic line in the Klein disk model, p-line for a hyperbolic line in the Poincaré disk model, u-line for a hyperbolic line in the Ponicaré upper half-plane model, and m-line for a hyperbolic line in the Minkowski hyperboloid model.

The first model that we will consider is called the Klein disk model, and it is a model which preserves Euclidean lines, but distorts the hyperbolic angles. We denote the disk of the Klein model by ω, and its boundary by $\partial\omega$. The advantage of working in the Klein model is that we can imagine hyperbolic lines as typical Euclidean lines; however, when two of these lines in our model intersect, for the most part, the angles that they form are not the actual hyperbolic angles that we would see in $\mathbb{H}^2$. The case when the Euclidean angle in the Klein disk model agrees with the hyperbolic angle in the hyperbolic plane occurs when the vertex of the angle is located at the center of the disk ω.

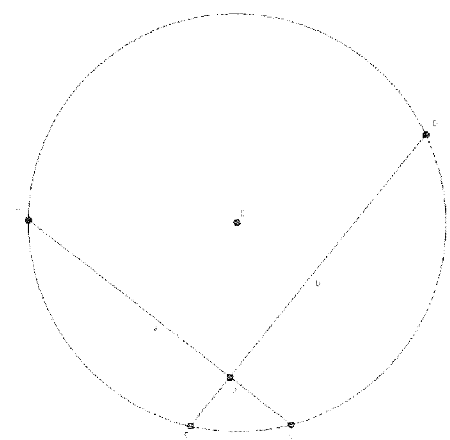

Figure 4.1: The regular lines AB and CD in the Klein disk model.

For a pair of lines in the Klein disk model, we want to know where their point of intersection can occur. Consider two distinct lines, m and l, in the Klein model. These two lines intersect at a point, Q. So, there are three cases to consider: Q lies inside ω, Q lies on the boundary $\partial\omega$ of the disk ω, or Q lies outside ω. According to these three cases, we say that the lines m and l are regular (Figure 4.1), asymptotically parallel (Figure 4.2), and divergently parallel (Figure 4.3), respectively.

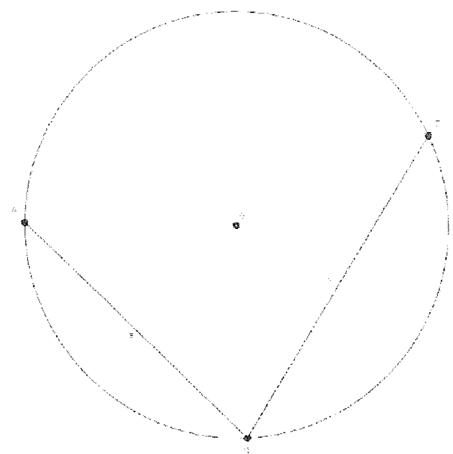

Figure 4.2: The asymptotically parallel lines AB and CD in the Klein disk model.

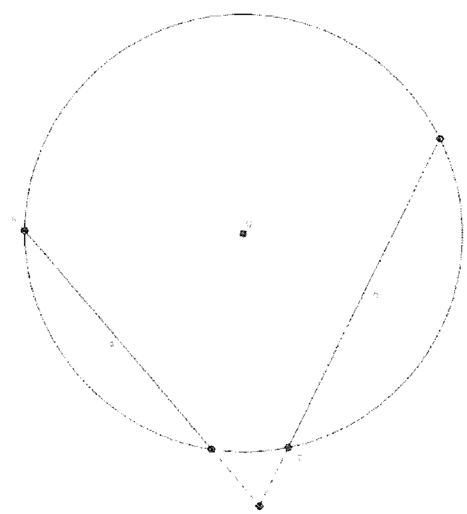

Figure 4.3: The divergently parallel lines AB and CD in the Klein disk model.

Understanding now how two lines can intersect in the Klein model, it is now time to consider if given a k-line, how to construct a line that is perpendicular to it. The following definition explains a nice property of the set of perpendicular lines to a given k-line.

Definition 4.1. Given a k-line $\Sigma\Omega$ in the Klein model, the **pole** of the k-line $\Sigma\Omega$, denoted $P(\Sigma\Omega)$, is the point through which extensions of all lines perpendicular to $\Sigma\Omega$ pass through. It is the point of intersection of the lines tangent to the disk at points Σ and Ω.

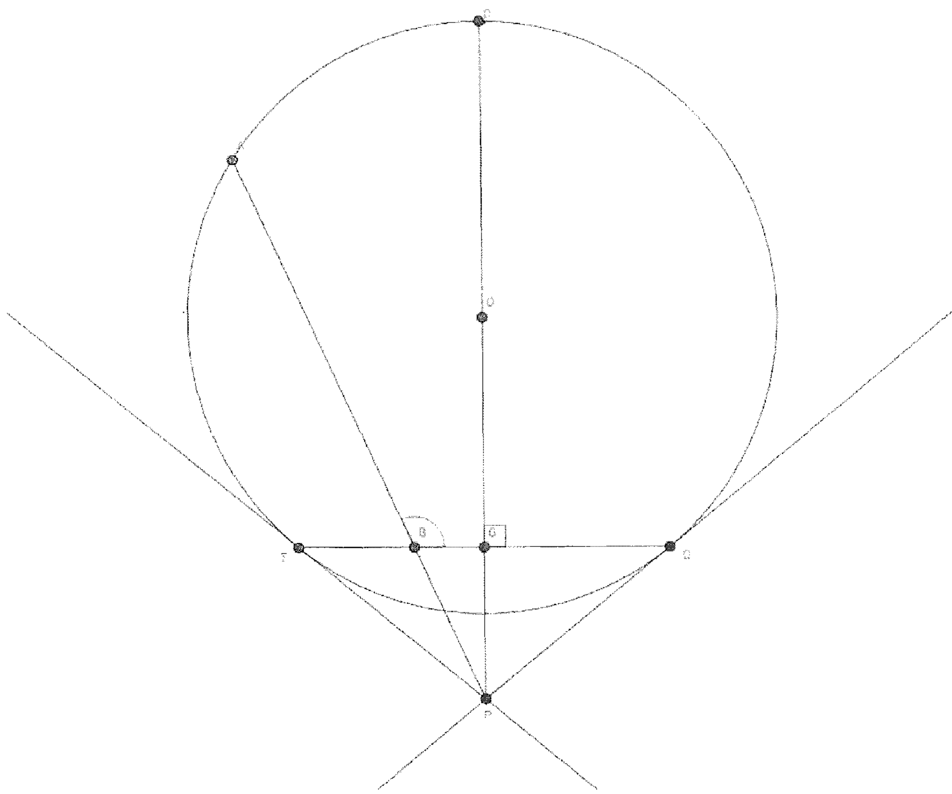

Figure 4.4: Point P is the pole of the k-line $\Sigma\Omega$, here $\angle B = \angle G = 90°$.

Note that if the k-line is a diameter, then the two tangent lines are parallel and do not intersect. In this case, the lines that are perpendicular to the diameter in the Klein model coincide with the Euclidean lines perpendicular to the diameter. The pole is an instrumental tool in working in the Klein model, and we will rely heavily upon it during subsequent constructions and proofs.

4.1 Projection

Orthogonal projection of h-lines in $\mathbb{H}^2$ understood through the Klein model $\mathbb{K}^2$. We will snow that for two divergently parallel lines a and b, the orthogonal projection of a onto b, $proj_b(a)$, is an open interval in b. Showing this fact will prove Theorem 3.6.

Let's consider the diameter $\Sigma\Omega$ and a k-line $\Sigma'\Omega'$ as depicted in Figure 4.5 . We denote $a = \Sigma'\Omega'$ and $b = \Sigma\Omega$. To project a point X in the k-line a onto b, we construct

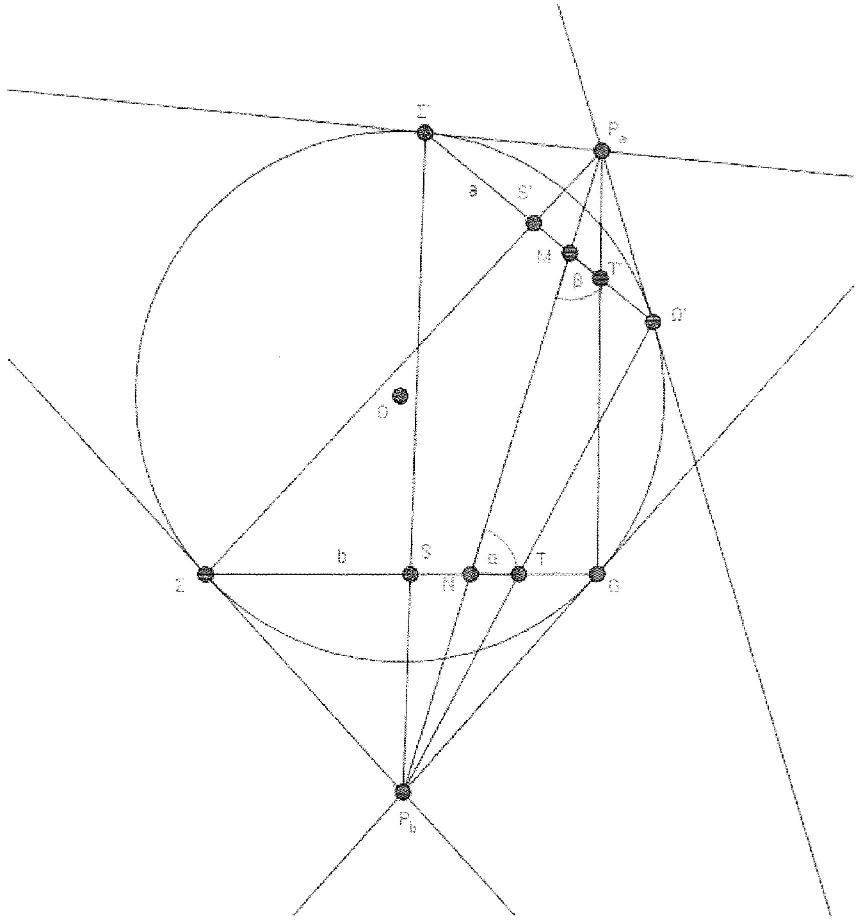

Figure 4.5: The orthogonal projection of a k-line.

the unique line passing through X perpendicular to b. Constructing the pole $P = P(\Sigma\Omega)$ of the k-line $\Sigma\Omega$ allows us to do just that. Note that the k-line b is a diameter, so its pole is located at infinity. So, the k-lines perpendicular to b coincide with the Euclidean lines perpendicular to b. Then the projection of X onto b is the point $Y \in b$ such that the Euclidean perpendicular to b passing through x intersects b at Y.

Performing an orthogonal projection of the k-line $\Sigma'\Omega'$ onto the diameter $\Sigma\Omega$, we construct the pole of the k-line $\Sigma'\Omega'$. Observe that only a portion of the k-line $\Sigma'\Omega'$ projects onto the diameter $\Sigma\Omega$. This can be seen since the points S' projects onto Σ and T' projects onto Ω. The observation to be made here is that we can project the k-line $\Sigma'\Omega'$ onto the entire diameter. From this example, we now want to consider the two possible orthogonal projections between two arbitrary k-lines.

Let a and b be two arbitrary, divergently parallel k-lines, with $a = \Sigma'\Omega'$ and $b = \Sigma\Omega$. Construct the pole for each of the k-lines a and b, yielding the points $P(a)$ and $P(b)$, respectively. Then the common perpendicular, the line passing through the points $P(a)$ and $P(b)$, projects orthogonally the point M onto N. That is we have the relation between the hyperbolic angles α and β: $\angle\alpha = \angle\beta = 90°$. As in the previous case, we want to see what portion of the k-line a onto the k-line b. Drawing the Euclidean lines passing through the pole $P(a)$ and each of the endpoints, Σ and Ω, of the line b, we see that $P(a)\Sigma \bigcap a = S'$ and $P(a)\Omega \bigcap a = T'$. Since the point S' projects orthogonally onto the point at infinity σ and the point T' projects orthogonally onto the point at infinity Ω, it follows that we can project the segment $S'T' \subset a$ onto the entire k-line b. On the other hand, erecting the k-lines perpendicular to b passing through the endpoints, Σ' and Ω', of the k-line a, we find that $P(b)\Sigma' \bigcap b = S$ and $P(b)\Omega' \bigcap b = T$. So, we can only project the k-line b up to some barriers, here the points S and T. This follows from S and T are mapped to the points at infinity of the k-line a; every point $Y \in a$ where $\Sigma' * Y * \Omega'$ must be the image under orthogonal projection of a point $X \in b$ where $S * X * T$.

Above, we saw that in $\mathbb{K}^2$ you can always drop the perpendicular passing through a point P in the k-line a to the k-line b; that is, the projection of the k-line a spans the entire k-line b. On the other hand, you can only erect the perpendicular to the k-line b up to some barrier; that is, we can only project the points on the hyperbolic segment $ST \subset b$ onto the k-line a. Every point on the k-line b that lies outside the segment ST is mapped under orthogonal projection outside of the disk ω.

4.2 Reflection

4.2.1 Reflection in a k-line

Suppose that we are in the Klein model and we want to reflect a point about a given k-line. Recall the case of the Euclidean plane. Consider a point A about an arbitrary line l,

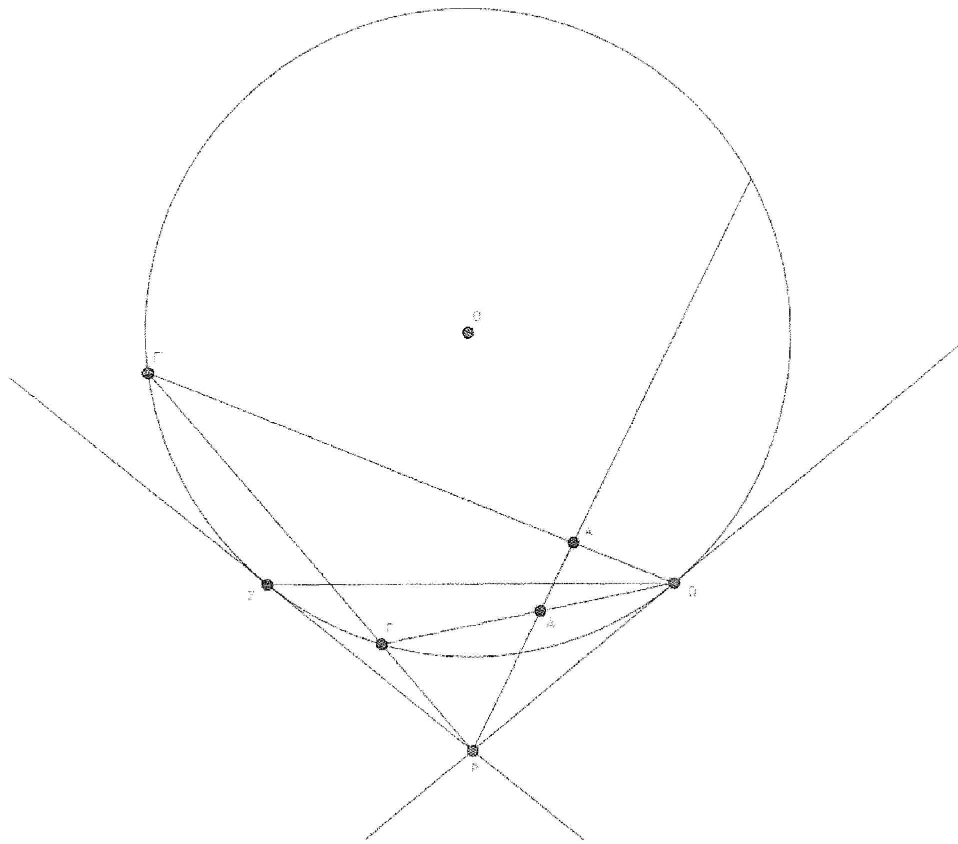

Figure 4.6: The reflection of a point in the Klein model.

where $A \notin l$. To reflect the point A about the line l, drop the perpendicular line from A to l. This perpendicular line, $l^{\perp}$, intersects l at a point $Q \in l$. Lay off from point Q a segment of length AQ along the line $l^{\perp}$. Then the end of this segment is called $A' = \sigma_l(A)$. Note that $\sigma_l^2 = \sigma_l \cdot \sigma_l = id$, performing a reflection through the same line twice is equal to the identity, i.e. not performing a reflection at all.

Now in the Klein model, we are going to employ a similar method. Namely, given a k-line, $l = \Sigma\Omega$, and a point $A \notin \Sigma\Omega$, find the line which is perpendicular to $\Sigma\Omega$ that passes through A. Then reflect this point A through the line $\Sigma\Omega$ to get the point A'. First, we construct the pole of the k-line $\Sigma\Omega$, $P(\Sigma\Omega)$. Since the extensions of the k-lines orthogonal to $\Sigma\Omega$ all pass through the pole $P(\Sigma\Omega)$, we can draw the line k which passes through $P(\Sigma\Omega)$ and A. Now, we know that the point $A' = \sigma_l(A)$, the mirror image of A about the k-line l, lies somewhere on this line k. Next we draw the asymptotic parallel line through A and Ω (note that we could just as easily have chosen Σ and A), call it m. Then m intersects the boundary $\partial\omega$ of the Klein model at the point Γ. Since Γ is a point at infinity, then we know that its reflection, $\sigma_l(\Gamma) = \Gamma$', is a point Γ' which is also at infinity (i.e. Γ' lies on $\partial\omega$). Now we connect Γ' and Ω, and this k-line intersects the k-line k at the point $\sigma_l(A) = A'$. Here we note that the reflection $\sigma_l(\Omega) = \Omega$. The point A' is the reflection of the point A through the k-line $\Sigma\Omega$.

Above, when we reflected the point A about l to find A', we projected the points Γ to Γ' and Ω to Ω. In actuality we projected the entire line $m = \Gamma\Omega$ through the line l, and found the line $\Gamma'\Omega$. With this in mind, we can generalize the preceding construction of reflecting a point through a line so that we can reflect any k-line containing the point A.

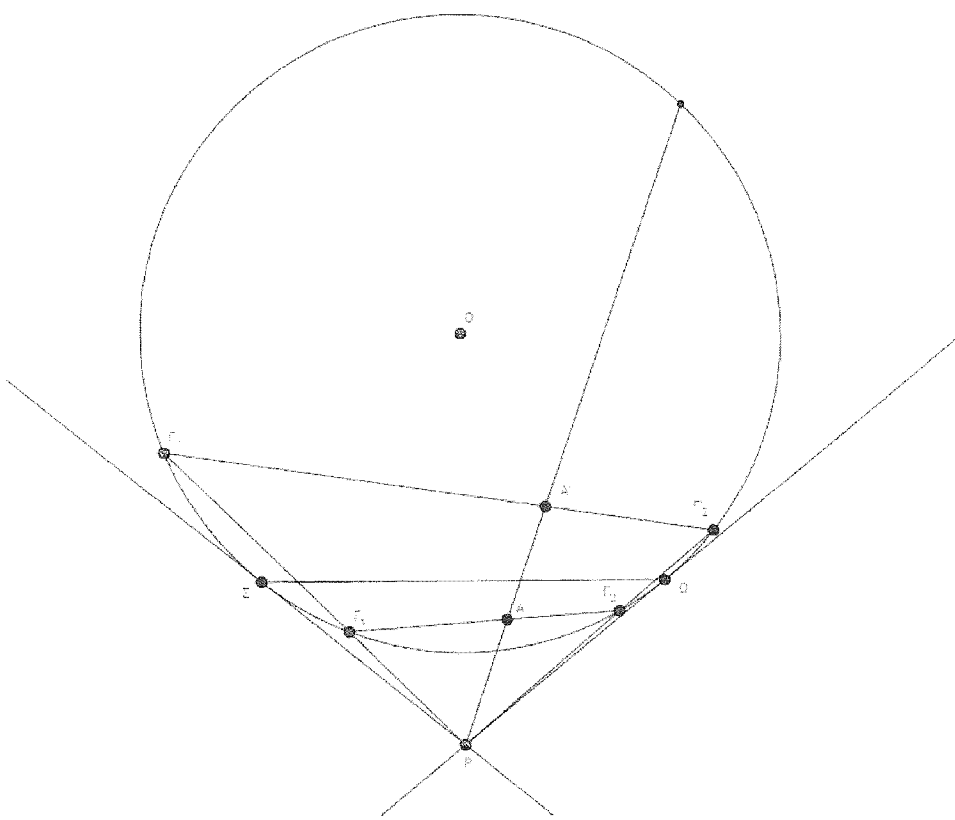

Figure 4.7: The reflection of a k-line through another k-line.

Consider the k-line $m = \Gamma_1\Gamma_2$ that passes through A (this is one of many possible k-lines). We know that the reflection through the line l of the point A is contained in the reflection through l of the line m, $A' = \sigma_l(A) \in \sigma_l(m)$. So, we have that $\Gamma_1' = \sigma_l(\Gamma_1)$ and $\Gamma_2' = \sigma_l(\Gamma_2)$. Since Γ_1 and Γ_2 are both located at infinity, then their reflections Γ_1' and Γ_2' are both also at infinity. Joining the points Γ_1' and Γ_2', we find $\sigma_l(m)$. Now $A'A \in \overrightarrow{PA}$ since there is a unique perpendicular passing through A and A'; and, the pole $P = P(\Sigma\Omega)$ contains all of the extensions of the k-lines orthogonal to l. It follows that $A' = \sigma_l(m) \bigcap \overrightarrow{PA}$.

4.2.2 Reflection of an Angle

Suppose that we are given an angle α whose vertex is not the origin of our disk. Then our angle is distorted from the angle that we would measure in $\mathbb{H}^2$. If we could reflect the angle so that its vertex was at the origin, then we would find its actual hyperbolic angle. We now describe such a method.

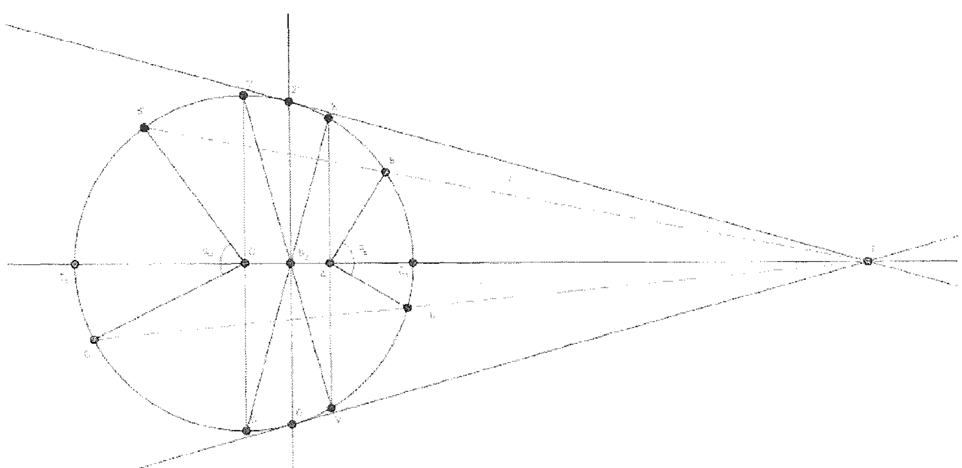

Figure 4.8: The reflection of a distorted h-angle to its actual h-angle.

Consider the diameter $\Sigma\Omega$ and a point $A \in \Sigma\Omega$ which is the vertex of angle $\alpha = \angle BAC$. We want to reflect angle α so that its vertex is at the origin.

1. We want to construct the k-line perpendicular to $\Sigma\Omega$ at the point O (the origin). So, we construct the Euclidean perpendicular line to $\Sigma\Omega$ at O. Since the pole $P(\Sigma\Omega)$ is at ∞, then the perpendicular k-line coincides with the Euclidean perpendicular line. We denote the resulting line VW.

2. Then we construct the Euclidean line perpendicular to $\Sigma\Omega$ through the point A, and call the resulting line XY.

3. To find the mirror in which the point A reflects to the origin O, we connect the points VY and WX.

4. These two lines, VY and WX, intersect the line $\Sigma\Omega$ at a point called M. We now construct the Euclidean line perpendicular to $\Sigma\Omega$ at the point M, called $\Sigma'\Omega'$.

5. The line $\Sigma'\Omega'$ is the mirror through which we will reflect the angle α. We now construct the pole of the line $\Sigma'\Omega'$. One way to do this is to draw the lines tangent to the disk at points Σ' and Ω', and find their point of intersection $P(\Sigma'\Omega')$. Another way follows from realizing that when reflecting through a mirror a point at infinity must be sent to another point at infinity. So, Y goes to W, and X goes to V. Extending the lines YW and VX, they intersect at a point, which is the pole of the k-line $\Sigma'\Omega'$, $P(\Sigma'\Omega')$.

6. To reflect the angle α through the mirror $\Sigma'\Omega'$, we project the points A, B, and C through the pole $P(\Sigma'\Omega')$. So, we draw the rays $\overrightarrow{P(\Sigma'\Omega')A}$, $\overrightarrow{P(\Sigma'\Omega')B}$, and $\overrightarrow{P(\Sigma'\Omega')C}$. Then A maps to the origin O, B maps to B', and C maps to C'.

7. Connect the points forming the lines OB' and OC'. These lines form the legs of the angle $\angle B'OC' = \beta$. The angle β is the undistorted hyperbolic angle of angle α.

4.3 Distance in the Klein Model

Recall in Euclidean geometry, the distance between two points $X, Y \in \mathbb{E}^2$ is the absolute value of the difference: $dist_{\mathbb{E}^2} = |X - Y|$. We will see that a similar distance between two points exists in $\mathbb{H}^2$. From Lobachevsky's formula

$$\tan\left(\frac{\varphi}{2}\right) = e^{-d} = \frac{1}{e^d}$$

Note that $\tan\left(\frac{\varphi}{2}\right) \neq \dfrac{QR}{QP}$ in the right hyperbolic triangle $\triangle PQR$. Instead, $\tan\left(\frac{\varphi}{2}\right) = e^{-d}$ corresponds to the triangle in the Euclidean plane $\mathbb{E}^2$ with a vertex at the center of the circle and corresponding angle $\frac{\varphi}{2}$.

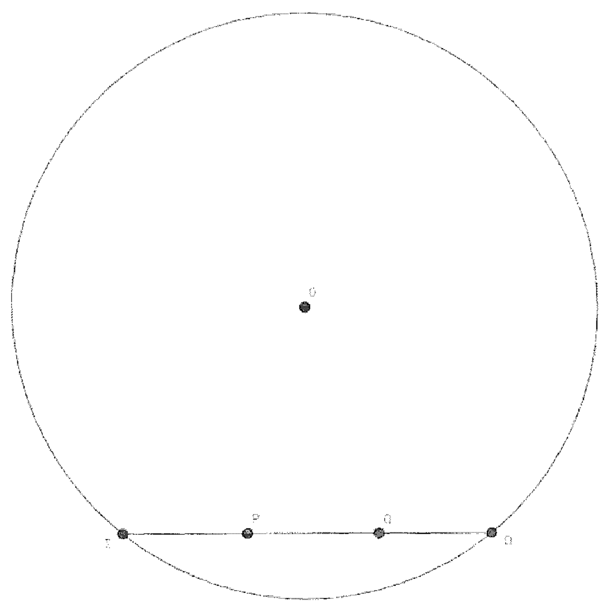

Figure 4.9: The distance between two points in the Klein disk model is given by the cross-ratio.

Suppose we have a k-line $\Sigma\Omega$ in $\mathbb{K}^2$, and two points $P, Q \in \Sigma\Omega$. How can we determine the hyperbolic distance between points P and Q? We first need to consider each point P and Q as a coordinate. Regarding point P, we can describe "coordinate" (P) as a ratio of ΣP and ΩP as follows:

$$0 < \text{"coordinate"}\ (P) = \frac{\Sigma P}{\Omega P} < \infty$$

Here ΣP and ΩP denote the Euclidean length of the segments in the Klein disk model. Note that the coordinate changes from 0 to ∞, but the coordinate must have the range $(-\infty, \infty)$. This can be achieved by involving the natural logarithm, ln, in our consideration:

$$-\infty < \text{“coordinate"}\ (P) = \ln\left(\frac{\Sigma P}{\Omega P}\right) < \infty$$

So, we have $x_P = \ln\left(\frac{\Sigma P}{\Omega P}\right)$, and $x_Q = \ln\left(\frac{\Sigma Q}{\Omega Q}\right)$. Now, we can define the hyperbolic distance between points P and Q in the standard way as the absolute value of the difference of the coordinates P and Q:

$$\begin{aligned} dist_{\mathbb{H}^2} &= \frac{1}{2}|x_P - x_Q| \\ &= \frac{1}{2}\left|\ln\left(\frac{\Sigma P}{\Omega P}\right) - \ln\left(\frac{\Sigma Q}{\Omega Q}\right)\right| \\ &= \frac{1}{2}\left|\ln\left(\frac{\Sigma P}{\Omega P} : \frac{\Sigma Q}{\Omega Q}\right)\right| := \frac{1}{2}|\ln(\Sigma\Omega : PQ)| \end{aligned} \tag{4.1}$$

Here $(\Sigma\Omega, PQ) = (PQ, \Sigma\Omega) :=$ cross-ratio for the pair $(P, Q) \subset (\Sigma, \Omega)$, defined by

$$(PQ, \Sigma\Omega) = \frac{\Sigma P}{\Omega P} : \frac{\Sigma Q}{\Omega Q} = \frac{\Sigma P}{\Omega P} \cdot \frac{\Omega Q}{\Sigma Q} = \frac{\Sigma P}{\Sigma Q} : \frac{\Omega P}{\Omega Q}$$

This establishes a method for computing distance in $\mathbb{K}^2$.

4.4 The Butterfly Theorem

Consider the k-line $\Sigma\Omega$ in $\mathbb{K}^2$, and two points $A, B \in \Sigma\Omega$. Let $\partial\omega$ denote the boundary of the Klein model. Our goal is to lay off a segment AB along $\Sigma\Omega$ from point B. In other words, starting at point B construct a segment of length equal to AB along the k-line $\Sigma\Omega$.

We begin by constructing the pole $P(\Sigma\Omega) = P$. Then we draw the ray $\overrightarrow{PA}$ which intersects $\partial\omega$ at points Ω_1 and Ω_2. Then we draw the unique k-lines containing the points Ω_1 and B, and Ω_2 and B. The k-line containing Ω_2 and B intersects $\partial\omega$ at the point Γ_1. Similarly, the line containing Ω_1 and B intersects $\partial\omega$ at the point Γ_2. Note that $\Omega_2\Gamma_1$ is the mirror image of $\Omega_1\Gamma_2$ through $\Sigma\Omega$. By construction we have $P * \Omega_1 * A * \Omega_2$, which implies $\Omega_2 * B * \Gamma_1$ and $\Omega_1 * B * \Gamma_2$. It follows that $P * \Gamma_1 * \Gamma_2$. We will now show that $\angle\Omega_1 BA = \angle\Omega_2 BA$.

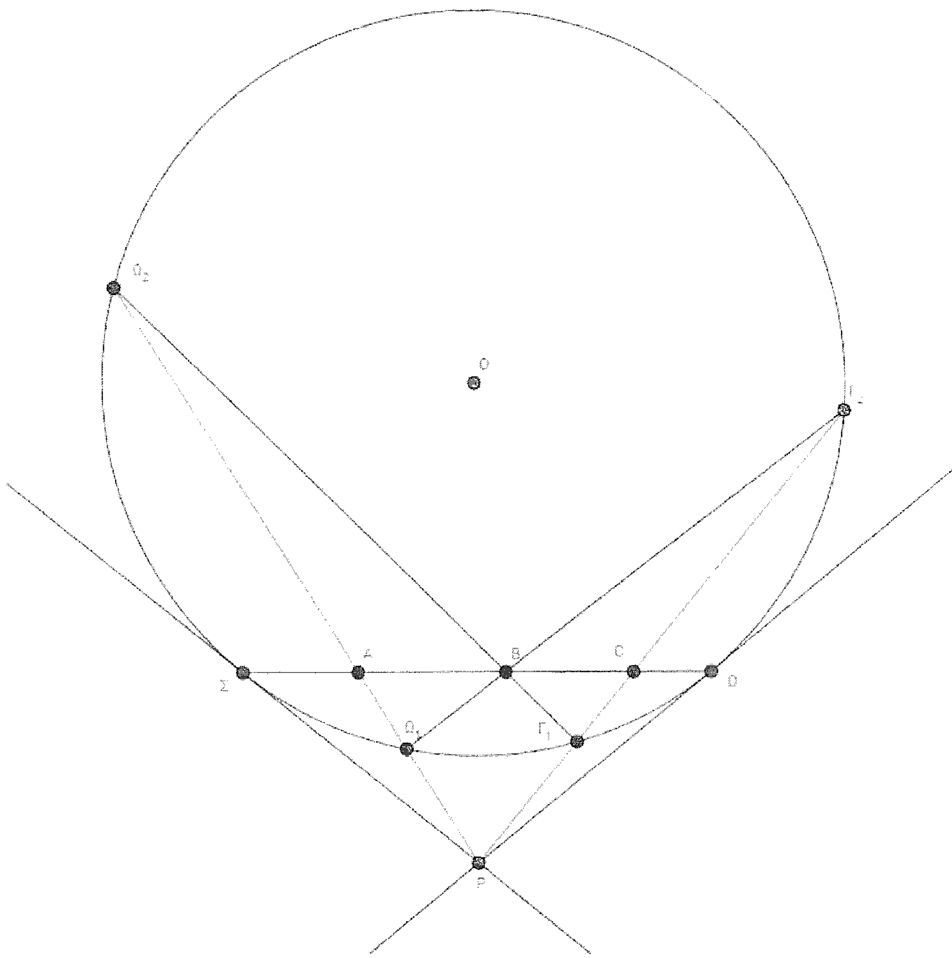

Figure 4.10: The Butterfly Theorem in the Klein disk model.

In Figure 4.10, the rays $\overrightarrow{B\Omega_1}$ and $\overrightarrow{B\Omega_2}$ are limiting rays of the k-line $\overleftrightarrow{\Omega_1\Omega_2}$. From Chapter 2, recall the discussion about the angle of parallelism, specifically Lemma 2.1. It follows that the angles $\angle\Omega_1 BA = \angle\Omega_2 BA = \varphi$. Additionally, by vertical angles we deduce the following equality:

$$\angle\Gamma_2 BC = \angle\Omega_1 BA = \varphi = \angle\Omega_2 BA = \angle\Gamma_1 BC$$

where $C = \overrightarrow{P\Gamma_1} \bigcap \Sigma\Omega$. Then the triangle $\Delta\Omega_1 BA = \Delta\Gamma_1 BC$, and thus, $AB = BC$.

In the preceding paragraph, we claimed that the triangles $\triangle\Omega_1 BA$ and $\triangle\Gamma_1 BC$ were equal. This result follows from two ideas. First, the theorem regarding similar triangles in $\mathbb{H}^2$.

Theorem 4.2. *In $\mathbb{H}^2$, there does not exist similar triangles, with $k \neq 1$. That is, if two triangles are similar in $\mathbb{H}^2$, then they are equal (if $\triangle_1 \sim \triangle_2$, then $\triangle_1 = \triangle_2$).*

In addition, to the standard theorems that we have in Euclidean geometry for proving two triangles are equal, Theorem 4.2 allows us prove equivalent triangles in $\mathbb{H}^2$ via the angle-angle-angle axiom for similar triangles. Returning our attention to the previous proof, we will show that the triangles $\triangle\Omega_1 BA$ and $\triangle\Gamma_1 BC$ are equal via the angle-angle-angle axiom. From previous arguments we know that $\angle\Omega_1 BA = \angle\Gamma_1 BC = \varphi$, and $\angle\Omega_1 AB = \angle\Gamma_1 CB = 90°$. It remains to show that $\angle A\Omega_1 B = \angle C\Gamma_1 B$. This is quickly remedied by realizing that $B\Gamma_1$ and $C\Gamma_1$ are asymptotically parallel lines; hence, the angle $B\Gamma_1 C = 0°$. A similar argument can be made for the lines $A\Omega_1$ and $B\Omega_1$. Therefore, $\angle A\Omega_1 B = \angle C\Gamma_1 B = 0°$. We conclude that the triangles $\triangle\Omega_1 BA$ and $\triangle\Gamma_1 BC$ are similar in $\mathbb{H}^2$, and thus by Theorem 4.2, they are equal.

The Butterfly Theorem as described above is a special case of a more general theorem.

Theorem 4.3. *Generalized Butterfly Theorem.*

Construction: Consider the points consider the points Σ, Σ', Ω, $\Omega' \in \partial\omega$, the k-lines joining these points $\Sigma\Sigma'$, $\Omega\Sigma'$, $\Omega\Omega'$, and $\Sigma\Omega'$. Let M denote the point of intersection of the k-lines $\Omega\Sigma'$ and $\Sigma\Omega'$; that is, $M = \Sigma'\Omega \bigcap \Omega'\Sigma$. Now draw the k-line XY such that it intersects the four existing k-lines at 4 distinct points. Let $A = XY \bigcap \Sigma\Sigma'$, $B = XY \bigcap \Omega\Sigma'$, $C = XY \bigcap \Sigma\Omega'$, and $D = XY \bigcap \Omega\Omega'$. We will now show $AB = CD$.

Proof. We will compute the hyperbolic length of segments AB and CD, and show that these two lengths are equal. Draw the k-lines $X\Sigma'$, $Y\Sigma'$, $X\Omega'$, and $Y\Omega'$. Then angle $\angle X\Sigma' B = \angle X\Omega' C = \alpha$ since they subtend the same arc, $\widehat{X\Sigma} = 2\alpha$. Similarly, $\angle A\Sigma' B =$

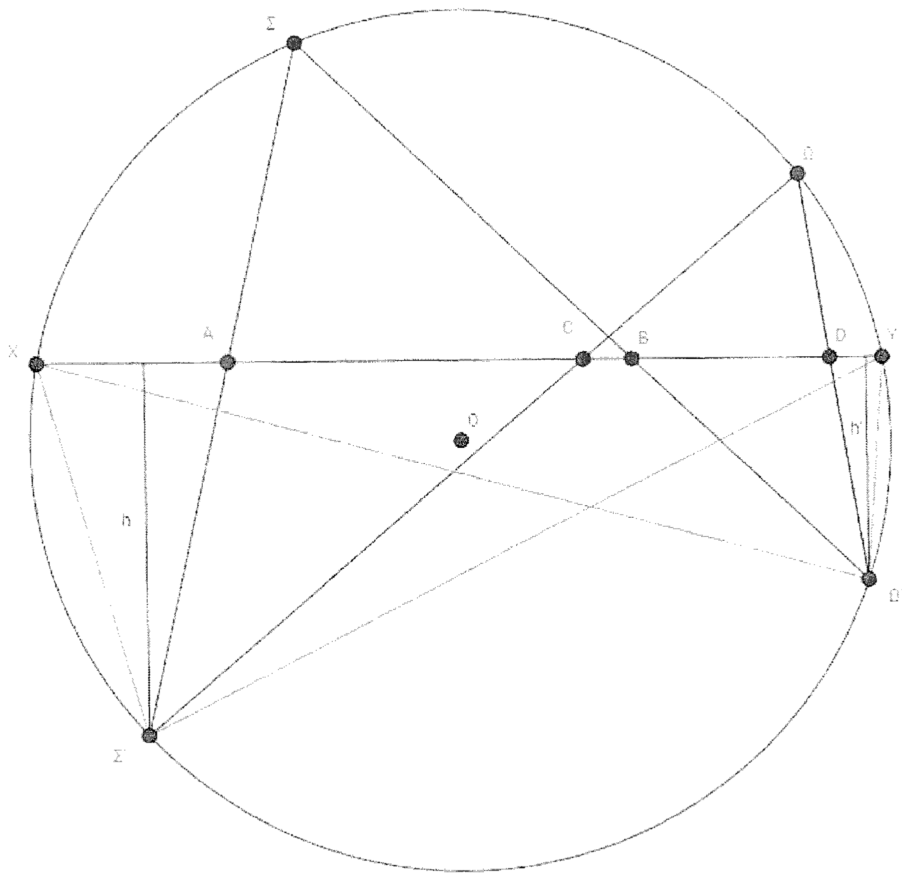

Figure 4.11: The Generalized Butterfly Theorem.

$\angle C\Omega' D = \beta$, $\angle B\Sigma' Y = \angle D\Omega' Y = \gamma$, and $\widehat{\Sigma\Omega} = 2\beta$, and $\widehat{\Omega Y} = 2\gamma$. Now we compute the hyperbolic lengths of AB and CD.

$$\begin{aligned}
||AB||_{\mathbb{H}^2} &= \frac{1}{2}\left|\ln\left(\frac{XA}{XB} : \frac{YA}{YB}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{\frac{1}{2}XA\cdot h}{\frac{1}{2}XB\cdot h} : \frac{\frac{1}{2}YA\cdot h}{\frac{1}{2}YB\cdot h}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{area(\triangle X\Sigma' A)}{area(\triangle X\Sigma' B)} : \frac{area(\triangle Y\Sigma' A)}{area(\triangle Y\Sigma' B)}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{\frac{1}{2}\Sigma' X\cdot\Sigma' A\sin\alpha}{\frac{1}{2}\Sigma' X\cdot\Sigma' B\sin(\alpha+\beta)} \cdot \frac{\frac{1}{2}\Sigma' Y\cdot\Sigma' B\sin\gamma}{\frac{1}{2}\Sigma' Y\cdot\Sigma' A\sin(\beta+\gamma)}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{\sin\alpha}{\sin(\alpha+\beta)} \cdot \frac{\sin\gamma}{\sin(\beta+\gamma)}\right)\right|
\end{aligned}$$

Similarly, we compute the hyperbolic length of the segment CD.

$$
\begin{aligned}
||CD||_{\mathbb{H}^2} &= \frac{1}{2}\left|\ln\left(\frac{XC}{XD} : \frac{YC}{YD}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{\frac{1}{2}XC\cdot h'}{\frac{1}{2}XD\cdot h'} : \frac{\frac{1}{2}YC\cdot h'}{\frac{1}{2}YD\cdot h'}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{area(\triangle X\Omega' C)}{area(\triangle X\Omega' D)} : \frac{area(\triangle Y\Omega' C)}{area(\triangle Y\Omega' D)}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{\frac{1}{2}\Omega' X\cdot\Omega' C\sin\alpha}{\frac{1}{2}\Omega' X\cdot\Omega' D\sin(\alpha+\beta)} \cdot \frac{\frac{1}{2}\Omega' Y\cdot\Omega' D\sin\gamma}{\frac{1}{2}\Omega' Y\cdot\Omega' C\sin(\beta+\gamma)}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{\sin\alpha}{\sin(\alpha+\beta)} \cdot \frac{\sin\gamma}{\sin(\beta+\gamma)}\right)\right|
\end{aligned}
$$

Therefore, we conclude $||AB||_{\mathbb{H}^2} = ||CD||_{\mathbb{H}^2}$. □

The hyperbolic lengths are a function of the angles α, β, and γ. This means (keeping the points Σ, Ω, and Ω' fixed) that we have the freedom to move the point Σ' along the boundary $\partial\omega$ of the disk until it coincides with the point Ω'. As Σ' moves towards the point Ω', the point A moves along the k-line XY to the point C. Similarly, the point B moves along XY towards point D. At the moment when Σ' meets Ω', then the points $A = C$ and $B = D$. Thus, showing that the segments AB and CD have equal hyperbolic length.

4.4.1 Shifting a segment on a k-line

We have formulated all of the tools that we will need to successfully shift a segment along a k-line in the Klein model $\mathbb{K}^2$.

Construction: In the $\mathbb{K}^2$ model, start with a k-line XY and points $A, B \in XY$. We are interested in shifting the segment AB some distance we will call AA' along XY. That is we want to shift the points A and B to A' and B', respectively, so that $A'B' = AB$. To complete this shift, we will lay off a segment of length AA' from point B along the k-line XY.

Proof. Take another k-line $\Sigma\Omega$. The one depicted in Figure 4.12, for example. Then draw the k-lines $X\Omega$ and $Y\Sigma$, extending them until they intersect at a point, call it S. Draw the unique k-line contained in $\overleftrightarrow{SA}$. Then the line $\overleftrightarrow{SA}$ intersects the absolute $\partial\omega$ at two

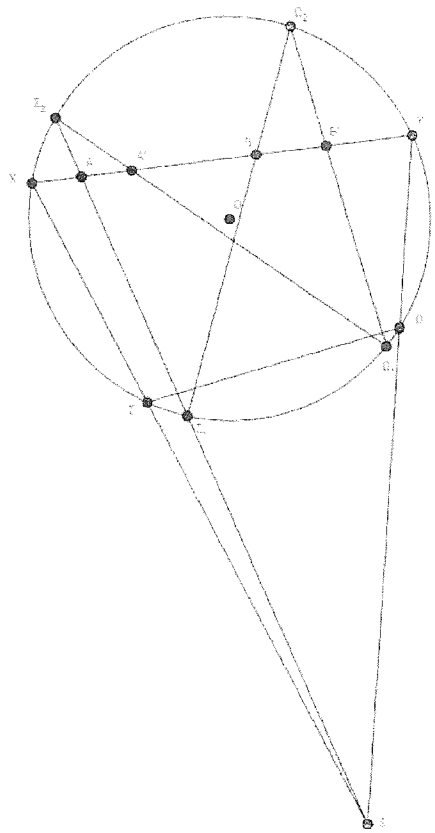

Figure 4.12: Shifting the hyperbolic segment AB along the k-line XY by a distance AA'.

points, denoted Σ_1 and Σ_2. Draw the k-lines $\Sigma_1 B$ and $\Sigma_2 A'$. Observe that $\Sigma_1 B$ intersects $\partial\omega$ at Ω_2, and $\Sigma_2 A'$ intersects $\partial\omega$ at Ω_1. As the last step of this construction, draw the line $\Omega_1\Omega_2$. This line intersects XY at a point: $\Omega_1\Omega_2 \bigcap XY = B'$.

Claim: $AA' = BB'$

Indeed, this result follows from the Butterfly Theorem. Therefore, we conclude that $AB = A'B'$, $A'B$ is a shared segment of the segments AB and $A'B'$ ($A'B = AB \bigcap A'B'$). □

In summary, we have developed the construction of reflecting a point in a k-line. Additionally, we discussed a method of reflecting an angle so that its vertex is at the center of the disk ω. This reflection allows us to measure the undistorted hyperbolic angle. As in Euclidean geometry, we found that the hyperbolic distance between two points in the Klein model was of the form of the absolute value of the difference of the two points.

Chapter 5

Some Hyperbolic Theorems Established with the Klein Model

In Euclidean geometry we are familiar with the often used Pythagorean Theorem, Law of Sines, and Law of Cosines. We seek to formulate the equivalent theorems in hyperbolic geometry through the use of the Klein model.

Lemma 5.1. *A segment of length x in the hyperbolic plane has Euclidean length* $\tanh(x) \leq 1$ *in the Klein disk, whenever it is laid off from the center of the disk* ω.

Lemma 5.2. *Given a right Euclidean triangle $\triangle ABC$ in the Klein disk $\mathbb{K}^2$ with a vertex at the origin, $A \equiv \mathcal{O}$, $\|BC\|_{\mathbb{H}^2} = a$, $\|AC\|_{\mathbb{H}^2} = b$, the Euclidean side length of the leg BC is given by* $\|BC\|_{\mathbb{E}^2} = \tanh(a) \cdot sech(b)$.

Proof. Given a right triangle $\triangle ABC$ in the hyperbolic plane $\mathbb{H}^2$ with known hyperbolic side lengths $AB = c$, $BC = a$, and $AC = b$. We want to find the corresponding Euclidean side lengths of the Euclidean triangle (which we denote for the sake of simplicity by the same letters A, B, and C) in the Klein model $\mathbb{K}^2$. Placing the vertex A at the center $\mathcal{O}$ of the disk ω, then immediately we have two of the side lengths $\|AC\|_{\mathbb{E}^2} = \tanh(b)$ and $\|AB\|_{\mathbb{E}^2} = \tanh(c)$. It remains to compute the Euclidean length of the side BC in the Klein model.

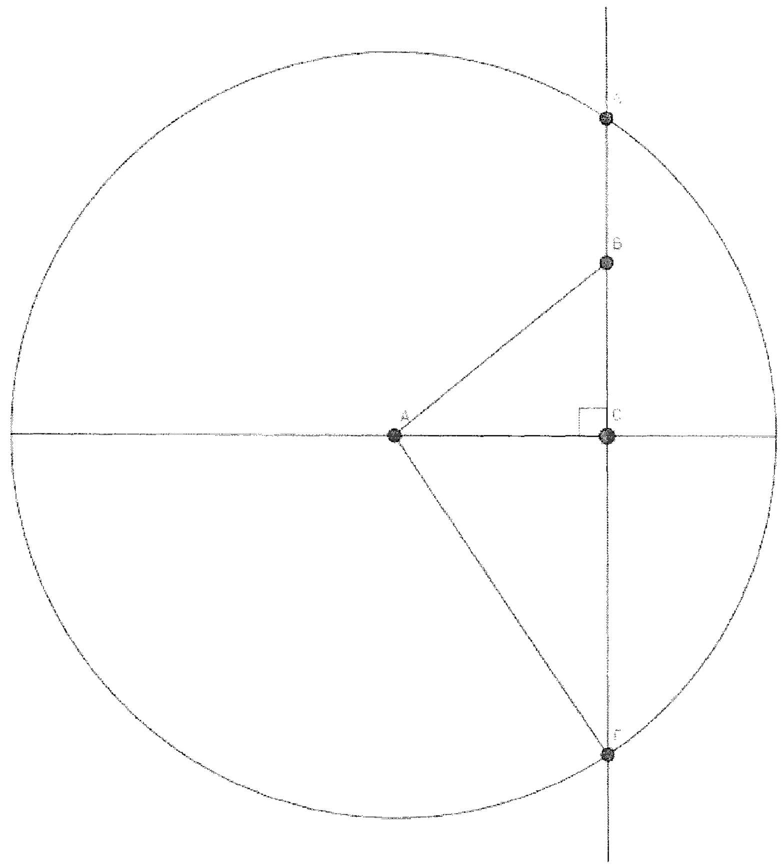

Figure 5.1: The triangle $\triangle ABC$ is a right triangle in the Klein disk model.

Extend the segment BC to the chord $\Lambda\Gamma$ and draw $\mathcal{O}\Gamma$, creating the triangle $\triangle\mathcal{O}C\Gamma$ with $\|\mathcal{O}\Gamma\|_{\mathbb{E}^2} = 1$. Then using the Pythagorean Theorem for $\triangle ABC$ in the Klein model, we compute the Euclidean side length $C\Gamma$:

$$
\begin{aligned}
\|C\Gamma\|_{\mathbb{E}^2} &= \sqrt{1 - AC^2} \\
&= \sqrt{1 - \tanh^2(b)} \\
&= \sqrt{\frac{\cosh^2(b) - \sech^2(b)}{\cosh^2(b)}} \quad = \frac{1}{\cosh(b)} = \sech(b)
\end{aligned}
$$

By symmetry it follows that $C\Lambda = C\Gamma = \dfrac{1}{\cosh(b)}$. Now we compute the Euclidean length $x = \|BC\|_{\mathbb{E}^2}$ of the segment BC via its hyperbolic length $a = \|BC\|_{\mathbb{H}^2}$:

$$
\begin{aligned}
a = \|BC\|_{\mathbb{H}^2} = & \ \frac{1}{2}\left|\ln\left(\frac{\Lambda B}{\Lambda C} : \frac{\Gamma B}{\Gamma C}\right)\right| \\
= & \ \frac{1}{2}\left|\ln\left(\frac{\Lambda C - x}{\Lambda C}\cdot\frac{\Gamma C}{\Gamma C + x}\right)\right| \quad (\Gamma C \simeq \Lambda C \text{ by symmetry}) \\
= & \ \frac{1}{2}\ln\frac{\Gamma C + x}{\Gamma C - x} \\
= & \ \frac{1}{2}\ln\frac{\frac{1}{\cosh(b)} + x}{\frac{1}{\cosh(b)} - x} \quad .
\end{aligned}
$$

Solving for x in terms of a and b, we find that

$$
\begin{aligned}
\frac{\frac{1}{\cosh(b)} + x}{\frac{1}{\cosh(b)} - x} &= e^{2a} \\
\Longrightarrow \quad \frac{1}{\cosh(b)} + x &= e^{2a}\frac{1}{\cosh(b)} - e^{2a}x \\
\Longrightarrow \quad x(e^{2a}+1) &= (e^{2a}-1)\frac{1}{\cosh(b)} \\
\Longrightarrow \quad x &= \frac{e^{2a}-1}{e^{2a}+1}\frac{1}{\cosh(b)} = \tanh(a)\frac{1}{\cosh(b)} \quad .
\end{aligned}
$$

Therefore, the Euclidean side length BC in the Klein model $\mathbb{K}^2$ is expressed as

$$
x = \|BC\|_{\mathbb{E}^2} = \frac{\tanh(a)}{\cosh(b)} = \tanh(a)\cdot\operatorname{sech}(b)
$$

□

We have thus proven Lemma 5.2.

5.1 The Hyperbolic Pythagorean Theorem

Applying the Euclidean Pythagorean Theorem to the right Euclidean triangle $\triangle ABC$ in the Klein model $\mathbb{K}^2$, which is the image of the hyperbolic triangle $\triangle ABC$ in $\mathbb{H}^2$, we obtain

$$
\tanh^2(b) + x^2 = \tanh^2(c) \quad .
$$

In $\mathbb{K}^2$, the Euclidean triangle $\triangle ABC$ has side lengths $\|AC\|_{\mathbb{E}^2} = \tanh b$, $\|BC\|_{\mathbb{E}^2} = x$, and $\|AB\|_{\mathbb{E}^2} = \tanh c$.

Substituting for x from Lemma 5.2, we find the first form of the hyperbolic Pythagorean Theorem for the hyperbolic triangle $\triangle ABC$:

$$\tanh^2(b) + \frac{\tanh^2(a)}{\cosh^2(b)} = \tanh^2(c) \tag{5.1}$$

This expression is very close to the Euclidean Pythagorean Theorem, but it has a little bit more complicated form with respect to the standard formula $a^2 + b^2 = c^2$. Now we drastically simplify the expression (5.1) to get the simple expression (5.2) in the form of the theorem. The Pythagorean Theorem for a right hyperbolic triangle does not have the same form as the Euclidean Pythagorean Theorem.

Theorem 5.3 (Pythagorean Theorem). *Let $\triangle ABC$ be a right hyperbolic triangle with the legs $BC = a$, $AC = b$, and the hypotenuse $AB = c$, then*

$$\cosh(c) = \cosh(a) \cdot \cosh(b) \tag{5.2}$$

Proof. Subtracting 1 from both sides of Equation 5.1 leads us finally to the desired formula 5.2:

$$\begin{aligned}
\Longrightarrow (\tanh^2(b) - 1) + \frac{\tanh^2(a)}{\cosh^2(b)} &= \tanh^2(c) - 1 \\
\Longrightarrow \frac{-1}{\cosh^2(b)} + \frac{\tanh^2(a)}{\cosh^2(b)} &= \frac{-1}{\cosh^2(c)} \\
\Longrightarrow \frac{1 - \tanh^2(a)}{\cosh^2(b)} &= \frac{1}{\cosh^2(c)} \\
\Longrightarrow \frac{\frac{1}{\cosh^2(a)}}{\cosh^2(b)} &= \frac{1}{\cosh^2(c)} \\
\Longrightarrow \frac{1}{\cosh^2(a)\cosh^2(b)} &= \frac{1}{\cosh^2(c)} \\
\Longrightarrow \cosh(c) &= \cosh(a) \cdot \cosh(b)
\end{aligned}$$

□

5.1.1 Trigonometric Relationships for a Right Hyperbolic Triangle

For an arbitrary triangle $\triangle ABC$ we formulate the equivalent hyperbolic trigonometric rules. Drop the perpendicular AH from point A to segment BC. Then H is the foot point of the altitude AH; denote $h = \|AH\|_{\mathbb{H}^2}$. Additionally, we denote $|k| = \|HC\|_{\mathbb{H}^2}$. The hyperbolic directed length of the segment HC is positive, zero, or negative depending on the measure of angle $\angle C$ defined by:

$$\begin{cases} HC = k > 0 \Longleftrightarrow \angle C < 90^\circ \\ HC = k = 0 \Longleftrightarrow \angle C = 90^\circ \\ HC = k < 0 \Longleftrightarrow \angle C > 90^\circ \end{cases}$$

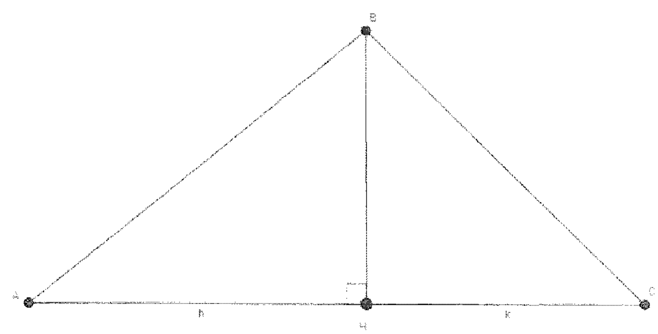

Figure 5.2: Case 1: The hyperbolic directed length $HC = k$ is greater than zero.

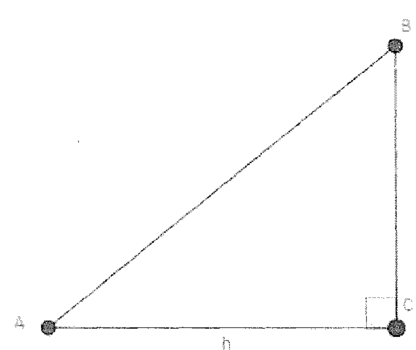

Figure 5.3: Case 2: The hyperbolic directed length $HC = k$ is equal to zero.

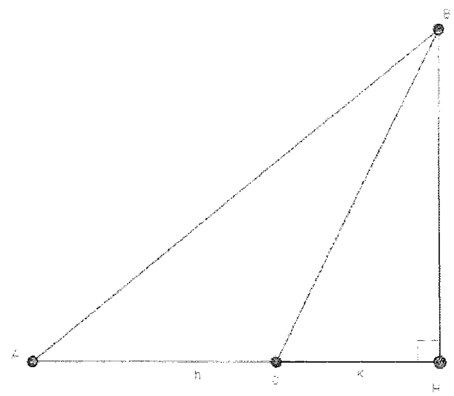

Figure 5.4: Case 3: The hyperbolic directed length $HC = k$ is less than zero.

We formulate the trigonometric relations that express the angles A and B of a right triangle $\triangle ABC$ with $\angle C = 90°$ in terms of the Euclidean side lengths. Then we switch the roles and find the trigonometric relations that express the side lengths via the angles A and B.

1. $\sin A = \dfrac{\|BC\|_{\mathbb{E}^2}}{\|AB\|_{\mathbb{E}^2}} = \dfrac{\tanh(a)\operatorname{sech}(b)}{\tanh(c)} = \dfrac{\sinh(a)\cosh(c)}{\cosh(a)\cosh(b)\sinh(c)} = \dfrac{\sinh(a)}{\sinh(c)}$

   In $\mathbb{E}^2$, $\sin A = \frac{a}{c}$; however, in $\mathbb{H}^2$ we have to add sinh to both the numerator and the denominator. Similarly, we see that $\sin B = \dfrac{\sinh(b)}{\sinh(c)}$.

2. $\cos A = \dfrac{\|AC\|_{\mathbb{E}^2}}{\|AB\|_{\mathbb{E}^2}} = \dfrac{\tanh(b)}{\tanh(a)}$

   In $\mathbb{E}^2$, $\cos A = \frac{b}{c}$; in $\mathbb{H}^2$ we have to add tanh to both the numerator and denominator. Likewise, $\cos B = \dfrac{\tanh(a)}{\tanh(c)}$.

3. $\tan A = \dfrac{\|BC\|_{\mathbb{E}^2}}{\|AC\|_{\mathbb{E}^2}} = \dfrac{\tanh(a)\operatorname{sech}(b)}{\tanh(b)} = \dfrac{\tanh(a)}{\dfrac{\sinh(b)}{\cosh(b)}\cosh(b)} = \dfrac{\tanh(a)}{\sinh(b)}$

   In $\mathbb{E}^2$, $\tan A = \frac{a}{b}$; and, in $\mathbb{H}^2$ we have to add tanh to the numerator and sinh to the denominator. Likewise, $\tan B = \dfrac{\tanh(b)}{\sinh(a)}$.

Now we look to formulate the relations between the side lengths of the right hyperbolic triangle $\triangle ABC$ and the angles A and B. We will use the second form of the hyperbolic Pythagorean Theorem, $\cosh(c) = \cosh(a)\cosh(b)$.

4. $\|BC\|_{\mathbb{E}^2} = \cosh(a) = \dfrac{\cosh(c)}{\cosh(b)} = \dfrac{\sinh(b)\cosh(c)}{\cosh(b)\sinh(c)} \cdot \dfrac{\sinh(c)}{\sinh(b)} = \dfrac{\dfrac{\tanh(b)}{\tanh(c)}}{\dfrac{\sinh(b)}{\sinh(c)}} = \dfrac{\cos A}{\sin B}$

5. $\|AC\|_{\mathbb{E}^2} = \cosh(b) = \dfrac{\cosh(c)}{\cosh(a)} = \dfrac{\dfrac{\tanh(a)}{\tanh(c)}}{\dfrac{\sinh(a)}{\sinh(c)}} = \dfrac{\cos B}{\sin A}$

6. $\|AB\|_{\mathbb{E}^2} = \cosh(a)\cosh(b) = \dfrac{\cos A}{\sin B} \cdot \dfrac{\cos B}{\sin A} = \dfrac{1}{\tan A \tan B} = \cot A \cot B$

The trigonometric relationships defined in relations 4., 5., and 6. do not have equivalent relations for a right Euclidean triangle. The values h and k will play crucial roles in the following proofs of the Law of Sines and the 1st Law of Cosines, respectively.

5.2 Law of Sines

Theorem 5.4. *Let a, b, c be the side lengths of the hyperbolic triangle $\triangle ABC$, and A, B, and C be its interior angles. Then*

$$\frac{\sinh(a)}{\sin A} = \frac{\sinh(b)}{\sin B} = \frac{\sinh(c)}{\sin C} \tag{5.3}$$

Proof.

$$\sin B = \frac{\sinh(h)}{\sinh(c)}$$

$$\sin C = \frac{\sinh(h)}{\sinh(b)}$$

It follows that

$$\sinh(c)\sin(B) = \sinh(b)\sin(C) = \sinh(h)$$

Then we arrive at

$$\frac{\sinh(c)}{\sin(C)} = \frac{\sinh(b)}{\sin(B)} = \frac{\sinh(a)}{\sin(A)}$$

or, multiplying throughout by the quantity 2π we have another formulation of the Law of Sines:

$$\frac{O(a)}{\sin(A)} = \frac{O(b)}{\sin(B)} = \frac{O(c)}{\sin(C)}$$

where $O(x) = 2\pi \sinh(x)$ is the circumference of a circle of radius x. □

5.3 The Two Hyperbolic Laws of Cosines

5.3.1 The First Hyperbolic Law of Cosines

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(C) \tag{5.4}$$

Proof. Suppose we have an arbitrary hyperbolic triangle $\triangle ABC$ as depicted in the Figure 5.2. After dropping the perpendicular AH from A to segment BC, we consider the right hyperbolic triangle ABH. Then applying the Pythagorean Theorem to triangle $\triangle ABH$ we obtain $\cosh(c) = \cosh(h) \cdot \cosh(a - k)$. Similarly, for triangle $\triangle ACD$ we get $\cosh(b) = \cosh(h) \cdot \cosh(k)$. Then solving for $\cosh(c)$ we compute

$$\begin{aligned}
\cosh(c) &= \frac{\cosh(b)}{\cosh(k)} \cdot \cosh(a - k) \\
&= \frac{\cosh(b) \cdot (\cosh(a)\cosh(k) - \sinh(a)\sinh(k)}{\cosh(k)} \\
&= \cosh(a)\cosh(b) - \sinh(a)\cosh(b)\tanh(k) \\
&= \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\tanh(b)cos(C) \\
&= \cosh(a)\cosh(b) - \sinh(a)\sinh(b)cos(C)
\end{aligned}$$

□

5.3.2 The Second Hyperbolic Law of Cosines

The first law of cosines provided a method for determining the side lengths (a, b, and c) of the triangle ABC from its angles (A, B, and C). Unlike in Euclidean geometry, there is a second formulation of the law of cosines which provides a method for computing an angle from the side lengths. We will not derive the second hyperbolic law of cosines.

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh(c) \tag{5.5}$$

Chapter 6

The Conformal Poincare Models

The Klein disk model allowed us to visualize the hyperbolic plane as a disk in the Euclidean plane where hyperbolic lines were Euclidean lines. There was a cost, however, the angles that we measured in the model were not equal to the actual hyperbolic angles (except when the vertex of the angle was located at the center of the disk). Suppose that we want to have a model which preserves angles; what would it look like? Such a model is called *conformal*.

Our goal is to construct a conformal model of $\mathbb{H}^2$. Start with the Klein disk model, and attach a hemisphere below the disk. Lines in the Klein model (k-lines) are Euclidean chords. Consider one such k-line lying in our disk. Then intersect the hemisphere with the vertical plane which intersects the disk along the given k-line. The resulting intersection of the plane with the hemisphere is a semicircle which is orthogonal to the disk. Repeating this process for any line in our disk, we see that the intersection with the hemisphere always results in a semicircle on that hemisphere. Now our goal is to create a model which preserves angles, so we need to see how the angle formed by the intersection of two k-lines in our disk changes when we determine the corresponding angle on the hemisphere.

Suppose we have two k-lines, l and m, in our disk which intersect forming angle α. To map them on the hemisphere, we take the two planes (the vertical plane containing

l, and the vertical plane containing m) and intersect them with the hemisphere. Two intersecting semicircles are formed, denoted l' and m'. The intersection of l' and m' forms an angle β: this is the angle between the tangent lines to l' and m' in the tangent plane to the hemisphere at the intersection point = $l' \cap m'$. Note that this angle is equal to the angle between the two vertical planes which is equal to the angle between the two k-lines in $\mathbb{K}^2$. So, we have an equal angle on the hemisphere.

Now, add the hemisphere above the disk so that we have an entire sphere. Consider the plane, Π, tangent to the sphere at the south pole S. From the north pole N, project stereographically the southern hemisphere onto the plane Π. If our sphere has radius r, then southern hemisphere projects to a disk, ω, on the plane whose radius is $R = 2r$ (which follows from considering similar triangles). The equator of the sphere projects to the boundary $\partial\omega$ of the disk ω in that plane. This boundary, $\partial\omega$, represents the absolute of our model. The lines (semicircles) that we had constructed earlier on the southern hemisphere project to circular arcs on the plane which are perpendicular to $\partial\omega$, or they project to diameters of ω, which correspond to arcs through the south pole of the sphere. Since stereographic projection is a mapping that preserves angles, the angle β between the arcs on the hemisphere is projected to an angle β between the projected arcs in the disk. This new model is called the *conformal Poincaré disk model*, denoted by $\mathbb{D}^2$.

A couple of differences to note between the Klein disk model and the Poincaré disk model are examined in Table 6.1.

Table 6.1: The differences between the Klein disk model and the Poincaré disk model.

	Klein Model ($\mathbb{K}^2$)	Poincaré Disk Model ($\mathbb{D}^2$)
Angles	Distorts hyperbolic angles	Preserves hyperbolic angles
Lines	Euclidean chords	circular arcs and Euclidean diameters
Distance	$dist_{\mathbb{K}^2} = \frac{1}{2}\lvert\ln(\Sigma\Omega, PQ)\rvert$	$dist_{\mathbb{D}^2} = \lvert\ln(\Sigma\Omega, PQ)\rvert$

One subtle difference highlighted in the table is that of the distance between two points P and Q on the respective hyperbolic line $\Sigma\Omega$. Observe that unlike in the Klein model, there is no $\frac{1}{2}$ factor in computing the distance between two points in the

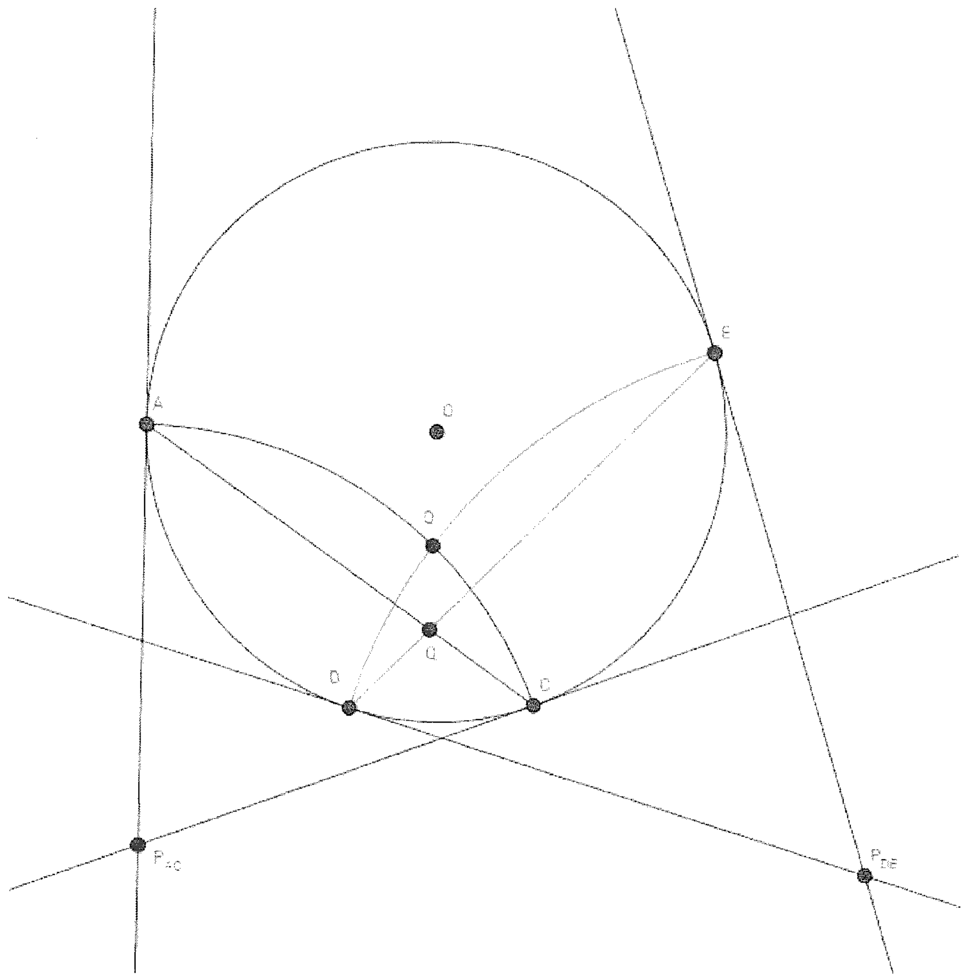

Figure 6.1: Poincaré lines intersecting at regular points.

Poincaré disk model. The rationale behind this comes from our earlier construction of the Poincaré disk model. Recall that when we projected the southern hemisphere onto the disk ω located in the tangent plane Π, the equator of the sphere projected to the boundary of the disk, $\partial\omega$, a circle of twice the radius of the equatorial circle, and also the sphere.

6.1 Isomorphisms between the Three Models

The three models, $\mathbb{K}^2$, $\mathbb{S}^2$, and $\mathbb{D}^2$, provide different perspectives of the hyperbolic plane in the Euclidean plane. One question that arises is since we have these different ways of

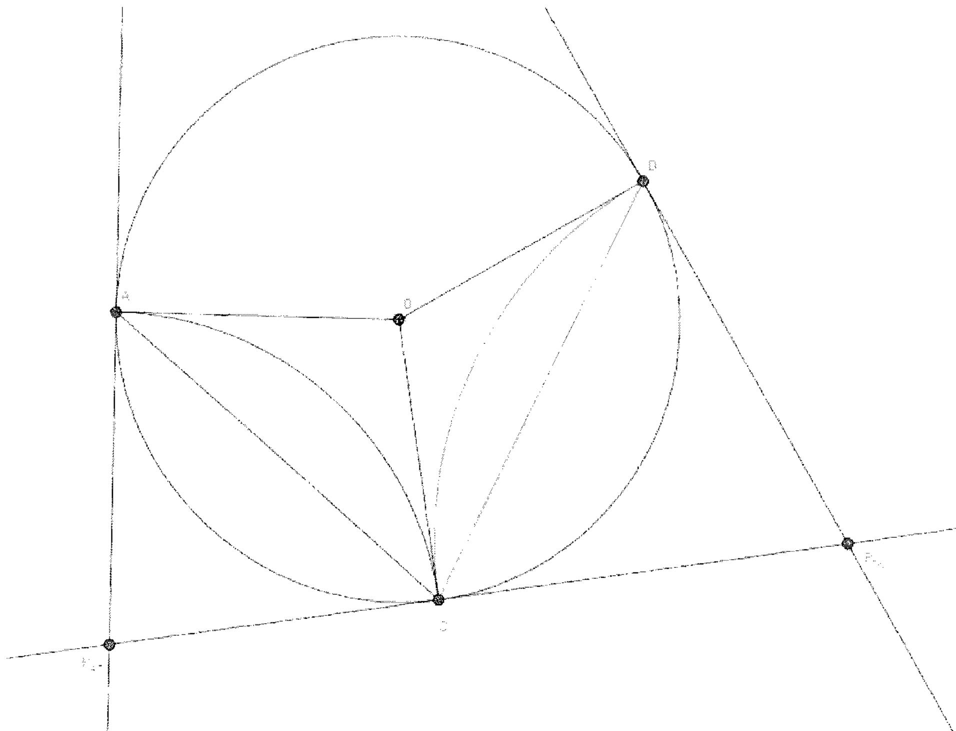

Figure 6.2: Asymptotically parallel lines in the Poincaré disk model.

viewing the hyperbolic plane, is it possible to move freely between these different models? The quick answer is yes. Now in order to move between two hyperbolic models, we will map one into the other, preserving the underlying structure of the hyperbolic plane. That is such a map should send hyperbolic points to hyperbolic points, preserve hyperbolic angles, and send h-lines to corresponding h-lines. This last criterion is necessary since there are three types of lines in hyperbolic geometry, namely regular, asymptotically parallel, and divergently parallel lines. A map between two models satisfying these criteria has a special name.

Definition 6.1. We say that there exists an **isomorphism** between two models (or two models are **isomorphic**) if there is a map between the two models satisfying the following criteria:

$$\begin{cases} \text{1) one-to-one correspondence between objects} \\ \text{2) relation between objects is preserved} \end{cases}$$

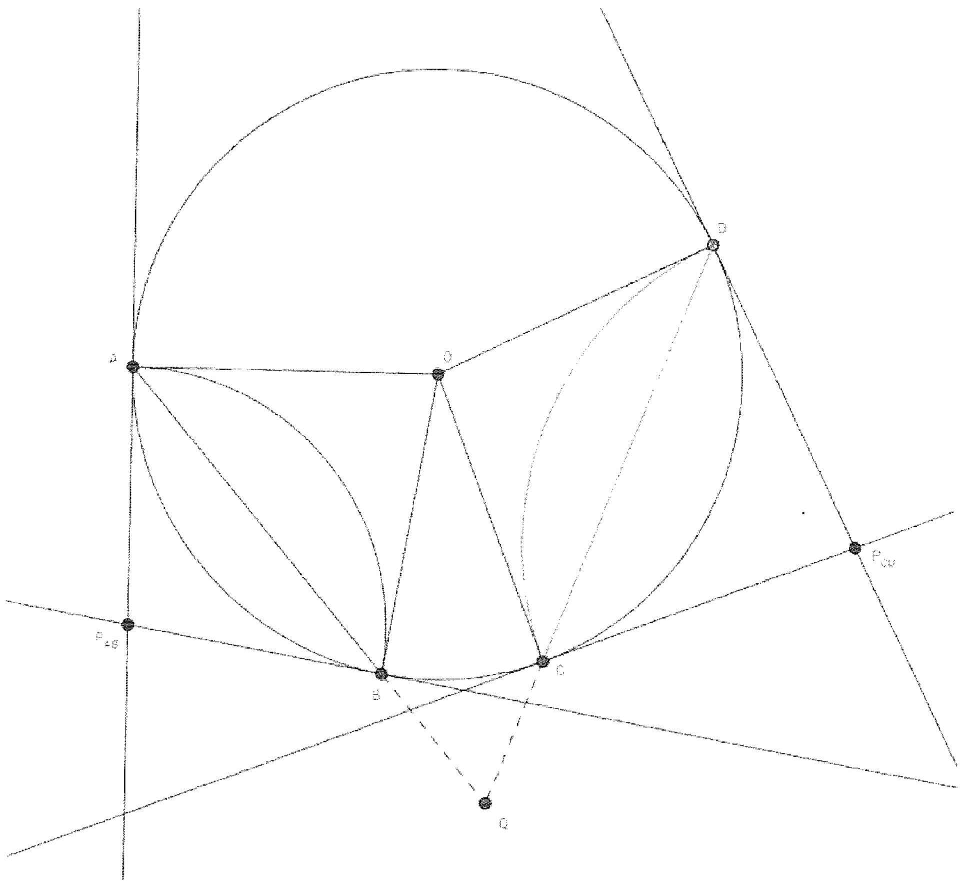

Figure 6.3: Divergently parallel lines in the Poincaré disk model.

The first condition of Definition 6.1 requires for example that points are sent to points and hyperbolic lines are sent to hyperbolic lines. As for the second condition, for example, it requires two congruent angles in one model to have under an ismorphism images which are also congruent in the second model. Note that this last condition does not require for an angle and its image under an isomorphism to be congruent.

We have already been exposed to one isomorphism, the one between the Klein disk model $\mathbb{K}^2$ and the Poincaré disk model $\mathbb{D}^2$ discussed in the construction of $\mathbb{D}^2$.

$$\mathbb{K}^2 \xrightarrow{\text{vertical planes}} \mathbb{S}^2 \xrightarrow{\text{stereographic projection}} \mathbb{D}^2$$

Another isomorphism between the Klein disk model and the Poincaré disk model is the radial isomorphism. Since $\mathbb{K}^2$ and $\mathbb{D}^2$ both take place in disks, it would make sense that we could go directly from one model to the other without having to use the intermediate step of the sphere. The following construction provides us with some insight in how to move between the Klein model and the Poincaré disk model via a radial isomorphism.

6.1.1 The Radial Isomorphism between $\mathbb{K}^2$ and $\mathbb{D}^2$

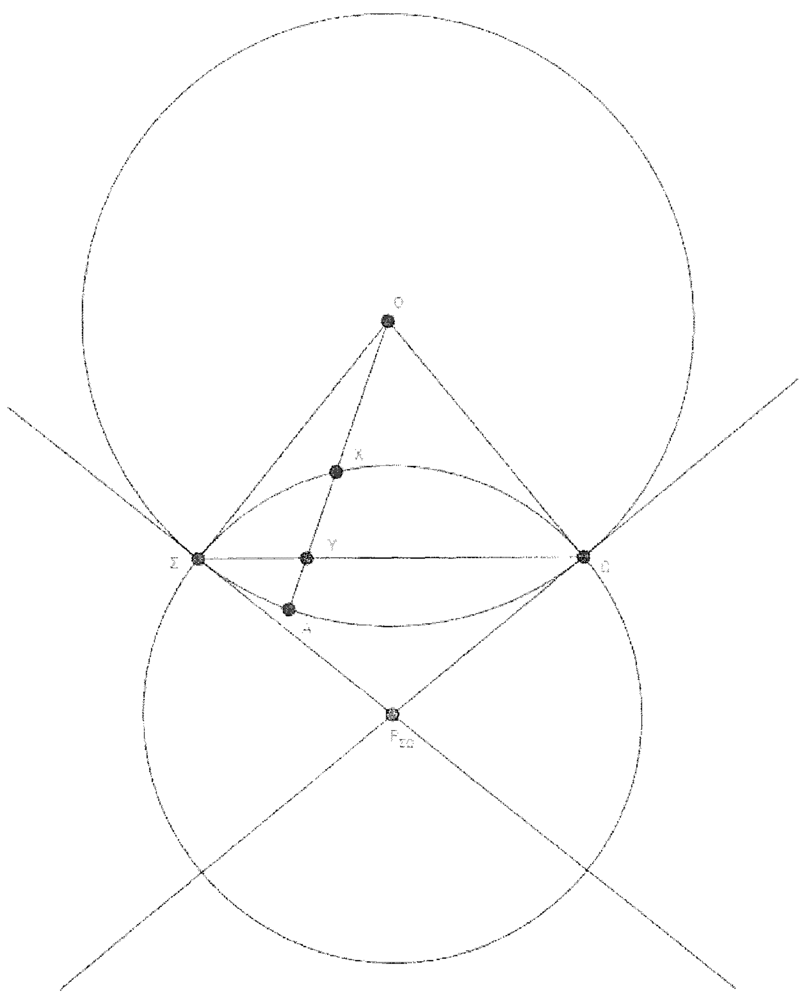

Figure 6.4: The radial isomorphism between the Klein disk model and the Poincaé disk model.

Take a line $\Sigma\Omega$ in the Klein model. We denote the center of the disk by $\mathcal{O}$ and the absolute by $\partial\omega$. Construct the pole of the k-line $\Sigma\Omega$, $P = P(\Sigma\Omega)$. Then construct the lines $\Sigma\mathcal{O}$ and $\Omega\mathcal{O}$. We want to find the Poincaré line, p-line, which corresponds to the k-line $\Sigma\Omega$. Draw the circle, $c(P, P\Sigma)$, centered at the pole P of radius $P\Sigma$. The p-line is the arc $\widehat{\Sigma\Omega}$ which is the portion of the circle $c(P, P\Sigma)$ located inside of the disk ω. For any point $X \in p$-line, the corresponding point Y in the k-line is the point of intersection of the radius through X and the chord $\Sigma\Omega$: $Y = OX \bigcap \Sigma\Omega$. This process is bijective, every point in the p-line is mapped to a unique point in the k-line, and vice-verse. So, we have established a bijection between the points of these two types of hyperbolic lines. A fact which we will not prove is that the radial isomorphism preserves hyperbolic angle.

Chapter 7

The Poincare Upper Half-Plane Model

Up until now, we have developed several models in Euclidean space to aid us in viewing and understanding the hyperbolic plane $\mathbb{H}^2$. Two of the models, the Klein disk model and the Poincaré disk model in the Euclidean plane, allowed us to view the hyperbolic plane $\mathbb{H}^2$ as disks in $\mathbb{E}^2$ provided that certain properties were met. The main distinction between these two models was that of conformality. The Poincaré disk model afforded us a way to see hyperbolic angles in the Euclidean plane by taking lines in the disk to be diameters and circular arcs orthogonal to the absolute. In both cases, the models viewed $\mathbb{H}^2$ as a bounded subset of $\mathbb{E}^2$. That is the planar structure of the hyperbolic plane was more difficult to see in the disk models. Suppose now that we want to use the Euclidean planar structure for the hyperbolic plane. It would be especially beneficial if we could eventually obtain the structure of the standard complex plane. What would this new model look like? In fact, this new model will be a conformal model, which views the hyperbolic plane as the upper half-plane in $\mathbb{E}^2$.

7.1 Construction of the Poincaré Upper Half-Plane Model

7.1.1 The Projection of the Poincare Disc

Start with a sphere, with the north and south poles labeled N and S respectively, along with the plane Π tangent to the sphere at S. On the sphere, we draw the prime meridian (0th meridian) and its antimeridian (180th meridian) forming a great circle passing through the north and south poles. For simplicity, from here on when we discuss the prime meridian, we are in fact referring to the great circle containing the prime meridian. Additionally, unless otherwise stated, when we talk about a meridian in general, we are considering the great circle containing the meridian in question. Consider circular arcs whose center resides on the prime meridian and are located in the right hemisphere. For visualization purposes, we are considering the sphere discussed in the construction of the Poincaré disk model which has been rotated 90°. Then from the north pole N, we project stereographically onto the plane Π. As a result, the prime meridian projects to a line l, and the north pole projects itself to some point at infinity. So, the circular arcs on the sphere will project to semicircles in the plane centered on the line l or to lines perpendicular to l. The latter case occurs when the circular arc on the sphere contains the north pole. The semicircles appear because the stereographic projection preserves angles; hence, the circular arcs on the sphere must be projected to circular arcs orthogonal to line l in the plane, i.e. they must be semicircles with centers on the line l.

Let's understand to what we have projected the right hemisphere. Suppose for a moment that we have a sphere, with north and south poles N and S, and a plane Π tangent to the sphere at S. If we were to project stereographically from N the entire sphere onto Π, then we would project onto the entire Euclidean plane with an additional point at infinity, corresponding to the stereographic projection of the north pole itself. In the plane, we can view the south pole as the origin. Additionally, two orthogonal meridians will project to orthogonal lines in the plane. Choosing them nicely, we can view these two lines in the plane as the x and y axes. Returning to our construction above, if we choose the prime meridian to project stereographically onto the x-axis in the plane, then every point in the right hemisphere projects to points above the x-axis. Thus, we get all of the projections located in the Euclidean upper half-plane.

7.1.2 Lines in the Upper Half-Plane $\mathbb{U}^2$

In this new setting, by "plane" we mean the upper half-plane of $\mathbb{E}^2$; that is the set of points $\{(x, y) : y > 0\}$. Note that the line l is not a part of this model and is called the absolute, as the projection of the absolute of the Poincaré model. A hyperbolic "point" is a Euclidean point which lies above the x-axis (the absolute). A hyperbolic "line" is a ray with its vertex on the absolute and perpendicular to the absolute, or a semicircle centered on the absolute. The next concept that we need to verify is the very first axiom belonging to any geometry: *there exists a unique line through any two points.* We have to consider this statement as a theorem because of the unusual concept of a point and a line in this setting.

There are two cases to consider:

Case 1 : *A hyperbolic line passes through two points in $\mathbb{U}^2$ having the same x-coordinate.* These two points lie on the same vertical ray, which is a hyperbolic line. Hence, the hyperbolic line is unique in this case.

Case 2 : *Two points in $\mathbb{U}^2$ with different x-coordinates.*

These two points lie on the same semicircle. To see this, suppose that P and Q are two such points and draw the segment PQ. At the midpoint M of PQ erect the perpendicular line $PQ^{\perp}$. This line intersects the absolute at a point $C = PQ^{\perp} \cap l$, equidistant to both P and Q. Hence, C is the center of the circle through P and Q. Then the unique hyperbolic line containing the points P and Q is the semicircle, $c(C, QC)$, centered at the point $C \in l$ having radius QC.

From now on, we will call all hyperbolic lines in the upper half-plane u-lines for simplicity. Also, any geometric object in the upper half-plane considered as a hyperbolic object we will call a u-object. In the upper half-plane, two u-lines (as described above) intersect at either a point in the plane, or at a point on the absolute, or do not intersect in the plane at all. Two u-lines which intersect at a point on the absolute are called *asymptotically parallel.* There are three possibilities, two semicircles, or a semicircle and a vertical ray, or two vertical rays can be asymptotically parallel. On the other hand, two lines which do not intersect and are not both vertical rays are called *divergently parallel.* Later we will see that these correspond directly with asymptotically parallel and divergently parallel lines in the other models.

For any point P in the upper half-plane, we can draw a bundle of semicircles, passing through the point P.

7.1.3 Circles in $\mathbb{U}^2$

Let's draw a Euclidean circle c located completely in the upper half-plane. We show now that c is also a hyperbolic circle in the upper half-plane $\mathbb{U}^2$, but its hyperbolic center differs from its Euclidean center. First of all, c is a hyperbolic circle because c is the image of a circle on the sphere under streographic projection, and stereographic projection maps circles to circles. Now we need to determine the location of the hyperbolic center of the hyperbolic circle. If we find a point from which all of the points on the circle c are located at an equal hyperbolic distance, then this point will be the hyperbolic center of the circle c. In fact, the hyperbolic center of c turns out to be closer to the absolute than the Euclidean center. Here is a construction of the hyperbolic center of a given circle c in the upper half-plane model $\mathbb{U}^2$. Draw the vertical u-line p containing the Euclidean center $\mathcal{O}_1$. By symmetry, the hyperbolic center must lie on the line p. This u-line p intersects the absolute at a point, call it Q. Then construct the two Euclidean lines through point Q which are tangent to c, denoting the tangent points on the circle Σ and Ω. Draw the semicircle $c(Q, Q\Sigma)$ centered at Q with radius $Q\Sigma$. The resulting intersection of the semicircle and the u-line p is a point $\mathscr{O}_2$. We claim the point $\mathscr{O}_2$ is in fact the hyperbolic center of the hyperbolic circle c.

Claim: *$\mathscr{O}_2$ is the hyperbolic center of circle c.*

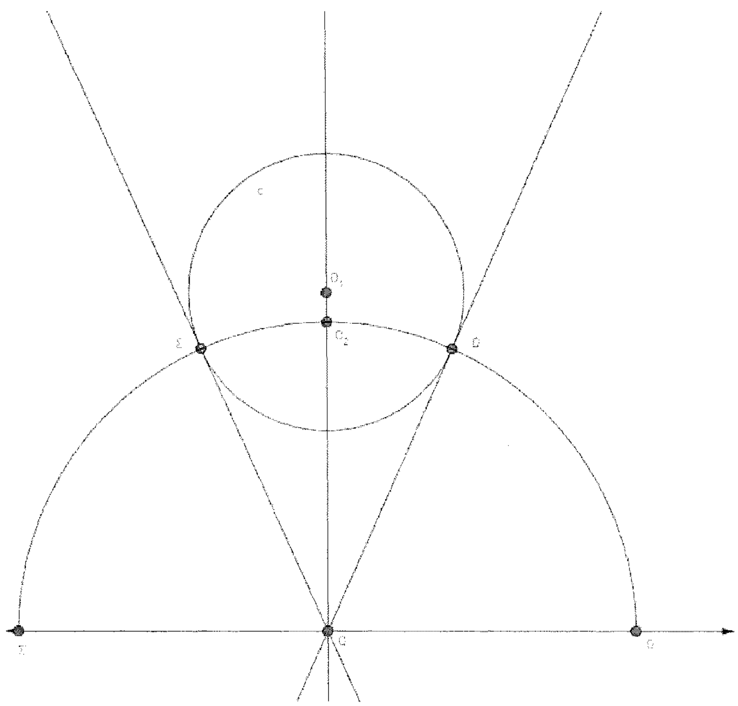

Figure 7.1: A circle contained entirely in the upper half-plane model, $\mathbb{U}^2$.

Proof. Recall the fact that two diameters intersect at the center of a circle. Also, we know that a stereographic projection preserves angles, so this means that diameters will be mapped to diameters. In order to show that $\mathscr{O}_2$ is the hyperbolic center of circle c, we need to show that the arc $\widehat{\Sigma\Omega}$ is a diameter. From the construction above, recall that the Euclidean line containing the segment $Q\Omega$ is tangent to the circle c at Ω. Moreover, the radius $\mathscr{O}_1\Omega$ is tangent to the circle $c(Q, Q\Omega)$, centered at point Q having radius $Q\Omega$, at the point Ω. For any circle we know that a radius and a tangent meeting at a point on a circle are orthogonal. Then by symmetry of intersecting circles we conclude that the arc $\widehat{\Sigma\Omega}$ is orthogonal to the circle c. Thus, $\widehat{\Sigma\Omega}$ is a diameter of the hyperbolic circle c; hence, the intersection point of the two diameters $\Sigma_1\Omega_1 \bigcap \widehat{\Sigma\Omega} = \mathscr{O}_2$ is the hyperbolic center of the hyperbolic circle. $\square$

We have shown a method of determining the hyperbolic center of a hyperbolic circle located completely in the upper half-plane $\mathbb{U}^2$.

7.2 Isomorphism between the two Poincaré Models

The isomorphism between the Poincaré upper half-plane model and the Poincaré disk model follows from the composition of two stereographic projections. This procedure is described in two steps.

Step 1:

Consider the Poincaré disk model ω with center $\mathcal{O}$ in the plane Π. Without loss of generality, we assume that the radius of the disk ω equals 2. Then we place the sphere of unit radius tangent to the plane Π so that the south pole of the sphere, S, and the center of the disk $\mathcal{O}$ coincide. Then we project stereographically the Poincaré disk model onto the southern hemisphere (the boundary of the disk, $\partial\omega$, maps to the equator) and its complement including the point at infinity onto the other hemisphere. Recall that in the Poincaré disk $\mathbb{D}^2$, a p-line is either a diameter or a circular arc. And circular arcs are the portions of the circle $c(P, P\Sigma)$ centered at the pole $P = P(\Sigma\Omega)$ having radius $P\Sigma$. Then the complement of the circular arc is the portion of this circle exterior to the disk ω. We can view a diameter in $\mathbb{D}^2$ as the portion of a circle of infinite radius inside ω. So, we have via stereographic projection the Poincaré disk model and its complement mapped on the unit sphere.

Step 2:

Project stereographically onto any vertical plane tangent to the equator of the sphere, say at point Q, from the antipodal point of point Q. Then the equator projects onto an infinite line (which we denote the x-axis). And the northern hemisphere projects to points lying above the x-axis. Indeed, this projection gives the upper half-plane. Recall that the absolute of the Poincaré disk $\mathbb{D}^2$ mapped to the equator on the sphere, so this infinite line is the absolute in our upper half-plane model.

In this procedure we made use of two facts which we will not prove here: the stereographic projection and its inverse are conformal maps; the composition of two stereographic projections is still a conformal map.

7.3 One-to-One Correspondence between Hyperbolic Lines of the Four Models of Hyperbolic Geometry

Having established the isomorphisms between the four models of hyperbolic geometry (Klein disk $\mathbb{K}^2$, Poincaré disk $\mathbb{D}^2$, Poincaré sphere $\mathbb{S}^2$, and the Poincaré upper half-plane $\mathbb{U}^2$), the one-to-one correspondence between lines of the various models is explored. In Chapter 4, when discussing the Klein model, we discovered that there are three types of lines in hyperbolic geometry, namely regular, asymptotically parallel, and divergently parallel lines. And in each subsequent model, we described what lines looked like. Table 7.1 reviews the lines in each model.

Table 7.1: Hyperbolic lines in the models

Model	Notation	Line
Klein disk	$\mathbb{K}^2$	Euclidean chords
Poincaré disk	$\mathbb{D}^2$	Euclidean diameters and circular arcs
Poincaré sphere	$\mathbb{S}^2$	semicircles
Poincaré upper half-plane	$\mathbb{U}^2$	vertical rays and semicircles

Let's consider the three types of lines (regular, asymptotic, and divergent) in each of the four models and understand their connection to one another. One move that we will use throughout is the stereographic projection of the northern hemisphere of the Poincaré sphere onto a vertical plane tangent to the sphere at a point located on the equator. Once a point on the equator is chosen as the point from which the stereographic projection of the northern hemisphere will be performed, then this points antipodal point (also located on the equator) is the point where the vertical plane onto which the stereographic projection occurs is tangent to the sphere.

7.3.1 Regular Hyperbolic Lines

In the Klein model, we said that two lines were regular if they intersected inside of the disk ω at a regular point. Suppose that we have two such lines in ω, then under the radial isomorphism between $\mathbb{K}^2$ and $\mathbb{P}^2$, the corresponding p-lines also intersect at a point inside ω. Note that a diameter in the model $\mathbb{K}^2$ is also a diameter in the model $\mathbb{D}^2$, and a non-diameter in $\mathbb{K}^2$ relates to a circular arc in $\mathbb{D}^2$. Then under the stereographic projection, regular p-lines in $\mathbb{D}^2$ together with their complements map to circles on the Poincaré sphere $\mathbb{S}^2$ having two points of intersection for each pair of circles. Observe that on the sphere the images of the two regular lines have 4 ends located at the equatorial circle. Since we must project stereographically the northern hemisphere from a point on the equator to obtain the upper half-plane, there are two cases to consider.

Case 1: (Non end point)

In the first case, we perform a stereographic projection from a point different from one of those 4 ends. This results in the intersection of two semicircles in the upper half-plane.

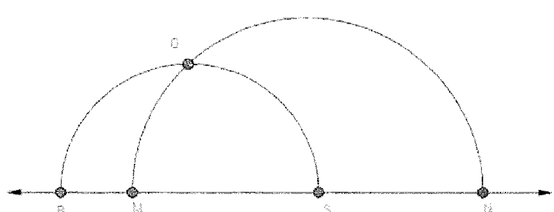

Figure 7.2: Case 1: The intersection of two regular semicircles in $\mathbb{U}^2$.

Case 2: (End point)

On the other hand, if we project from one of the four ends, then the point we project from is sent to infinity. Thus, we obtain the intersection of a semicircle and a vertical ray in the upper half-plane, case 2. These two cases describe the two types of regular u-lines which can occur in $\mathbb{U}^2$.

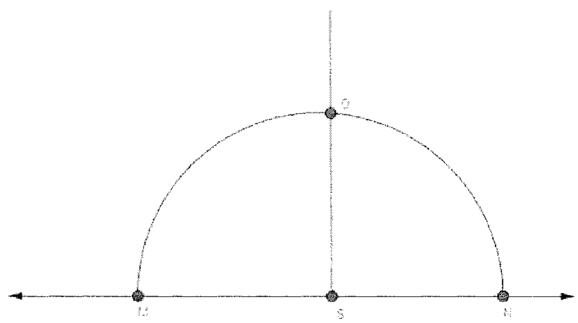

Figure 7.3: Case 2: The intersection of a vertical ray and a semicircle in $\mathbb{U}^2$.

7.3.2 Asymptotically Parallel Lines

Recall that in $\mathbb{K}^2$ we said that two lines were asymptotically parallel if they intersected at a point on $\partial\omega$. This extends to two circular arcs or a circular arc and a diameter meeting at a point on $\partial\omega$ in the Poincaré disk model. Then in the Poincaré sphere model two asymptotically parallel lines are circular arcs which intersect at a point on the equator. This can be seen after applying a stereographic projection of two asymptotically parallel lines in $\mathbb{D}^2$. Now, these two circular arcs meet the equator at three distinct points (two points of degree 1 and one point of degree 2). As a result, there are three potential ways for the asymptotically parallel lines to project onto the upper half-plane.

Case 1: (Non end point)

Choose a point on the equator which is not an end of any of the semicircles. Then projecting stereographically the northern hemisphere from this point will produce two semicircles which intersect at some common point on the absolute of $\mathbb{U}^2$.

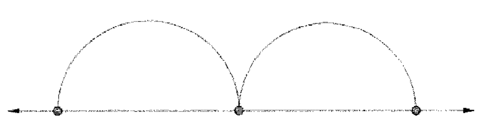

Figure 7.4: Case 1: First possible orientation of two asymptotically parallel semicircles.

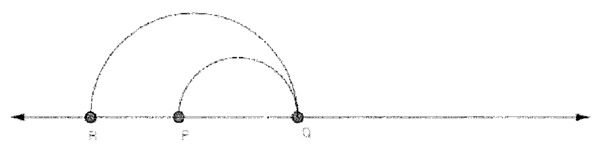

Figure 7.5: Case 1: Second possible orientation of two asymptotically parallel semicircles.

Case 2: (Non-shared end point)

Choose one of the two ends of the semicircles. Then project stereographically the northern hemisphere from this point. This will yield the upper half-plane with one vertical ray and one semicircle meeting at a point on the absolute.

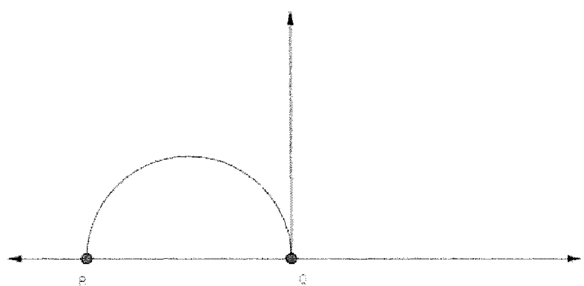

Figure 7.6: Case 2: A vertical ray and a semicircle are asymptotically parallel.

Case 3: (Shared end point)

Choose the point on the equator where both semicircles meet. Projecting stereographically the northern hemisphere from this point will yield the upper half-plane with two vertical rays which meet at infinity.

Therefore in the Poincaré upper half-plane model $\mathbb{U}^2$ there are three types of asymptotically parallel lines.

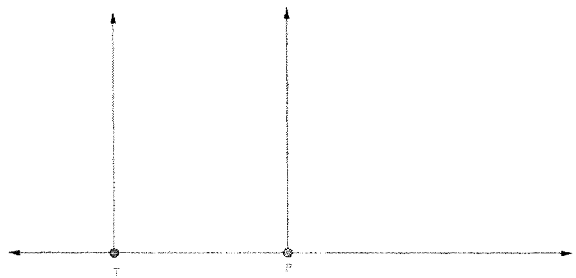

Figure 7.7: Case 3: Two vertical rays are asymptotically parallel.

7.3.3 Divergently Parallel Lines

Divergently parallel lines in the Klein disk model $\mathbb{K}^2$ are two lines which intersect at a point exterior to the disk ω. From the radial isomorphism, we see that in $\mathbb{D}^2$ divergently parallel lines are the circular arcs that do not intersect in ω and the pairs of diameters and circular arcs which do not intersect. Then using the stereographic projection onto the sphere, we see that divergently parallel lines on the Poincaré sphere are the pairs of circular arcs which do not intersect on the sphere. As in the case of regular lines, these two circular arcs intersect the equator at 4 distinct points, meaning that there are two scenarios we have to consider when we stereographic project the northern hemisphere.

Case 1: (Non end point)

Choose a point on the equator that does not coincide with either of the two ends of each semicircle. From this point project stereographically the northern hemisphere onto the vertical plane tangent to the sphere at the antipodal point. Then we obtain the upper half-plane $\mathbb{U}^2$ with two disjoint semicircles.

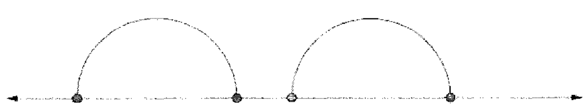

Figure 7.8: Case 1: Two non-meeting semicircles are divergently parallel.

Case 2: (End point)

Choose one of the four ends of the semicircles located on the equator. We project stereographically the northern hemisphere from this point. Then the semicircle which contains the ends from which we performed the stereographic projection is mapped to a vertical ray and the other semicircle is mapped to a semicircle so that the intersection of both u-lines is empty.

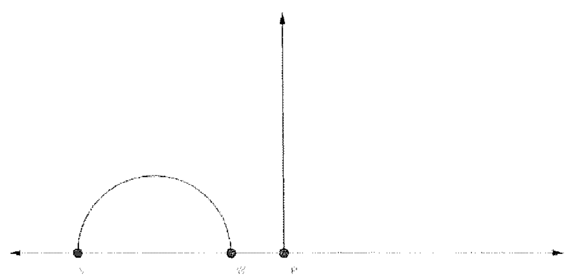

Figure 7.9: Case 2: A non-meeting vertical ray and semicircle are divergently parallel.

Via the isomorphisms between the four models, we have established the one-to-one correspondence between the various types of lines in each of the models.

Chapter 8

Equidistant Curves and Horocycles

In Euclidean geometry the idea of an equidistant curve to a line is a curve whose points have the same orthogonal distance to a line. One might realize that such a curve is in fact a parallel line. We look to develop a similar notion of equidistant curves in the setting of hyperbolic geometry.

Definition 8.1. An **equidistant curve** is a curve whose points have the same orthogonal distance from a given line.

Let's consider a p-line $\Sigma\Omega$ in $\mathbb{D}^2$ and a point $P \notin \Sigma\Omega$. We want to construct the equidistant curve to the line $\Sigma\Omega$ containing the point P. As it turns out, this curve is the portion of the Euclidean circle containing points Σ, Ω, and P lying in the disk ω. The construction of the circle follows from the straight-edge and compass construction of a circle containing three given points. Join two pairs of the three points, say $P\Sigma$ and $P\Omega$. Find the median of each of the resulting segments. Erect the lines perpendicular to each of the segments at the median. These perpendicular lines intersect at a point, say Q, which is the center of the circle. Draw the circle $c(Q, QP)$ centered at Q of radius QP. The portion of the circle inside the disk ω is the equidistant curve to the line $\Sigma\Omega$ containing the point P.

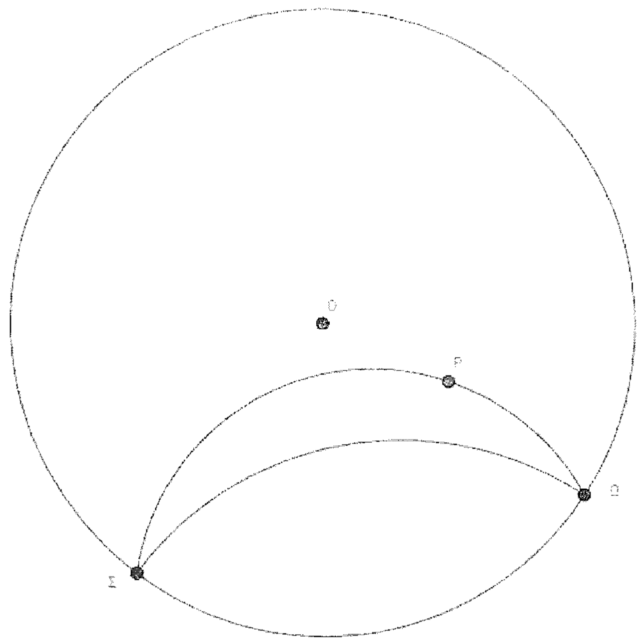

Figure 8.1: The circular arc is the equidistant curve passing through point P of the p-line $\Sigma\Omega$.

A couple of items to note. The equidistant curve is not orthogonal to the boundary $\partial\omega$ of the disk. Additionally, if P, Σ, Ω are collinear (in the Euclidean sense), then the resulting equidistant curve is the Euclidean chord between Σ and Ω. So, in the Poincaré disk model, equidistant curves can either be Euclidean chords, or circular arcs.

In the Poincaré upper half-plane we determine equidistant curves. Since there are two types of lines in $\mathbb{U}^2$ (vertical rays and semicircles), we treat the equidistant curve in each case separately.

Case 1: (semicircle)

Suppose that we have a semicircle with points P and Q on the absolute, and a point R not on the semicircle. To find the equidistant curve to the semicircle that contains the point R, we construct the Euclidean circle which contains the points P, Q, and R.

Case 2: (vertical ray)

Suppose that we have a vertical ray, l, meeting the absolute at the point Q, and a point P not contained in this vertical ray. Then the equidistant curve to the line l through point P is the ray emanating from point Q passing through the point P.

In the Poincaré disk model, a "circle" is a Euclidean circle. We call "circle" a hyperbolic, or a p-circle and its hyperbolic center by p-center. The only difference being that the p-center of the circle is not the Euclidean center. In fact, the hyperbolic center is closer to the absolute than the Euclidean center. Symmetry allows us to remove some of the ambiguity of the location of the hyperbolic center of the circle since the p-center must lay on the radius containing the Euclidean center. Now suppose that we move this circle as a rigid Euclidean body along the radius containing the disk's center towards the absolute. Then as the boundary of the circle approaches the absolute, the p-center gets closer and closer to the absolute and to the boundary of the moving circle. At the moment when the circle is tangent to the disk at a point T on the absolute, the center of the circle is on the absolute as well. More importantly, the center of the p-circle is the tangent point T. Now our p-circle is no longer a circle in $\mathbb{D}^2$, yet it has another significance. This limiting Euclidean circle, as it will follow from the Statement below, is a horocycle (a curve satisfying Definition 8.2).

Definition 8.2. A **horocycle** is a curve in $\mathbb{D}^2$ such that every geodesic through point T is orthogonal to the horocycle.

Statement: *The limiting Euclidean circle is a horocycle.*

Proof. Take any p-line l passing through T and limiting Euclidean circle γ at T. We have to prove that $l \perp \gamma$ at the intersection point $M = l \bigcap \gamma$. Together p-line l with its complement forms a Euclidean circle $c(P, PT)$ centered at point P having radius PT. Since PT is tangent to the limiting Euclidean circle γ at T, then the Euclidean radius OT is perpendicular to PT at T. Moreover, by symmetry, the two angles formed by the intersection of two circles are equal. Therefore, the line l is perpendicular to γ at M. $\square$

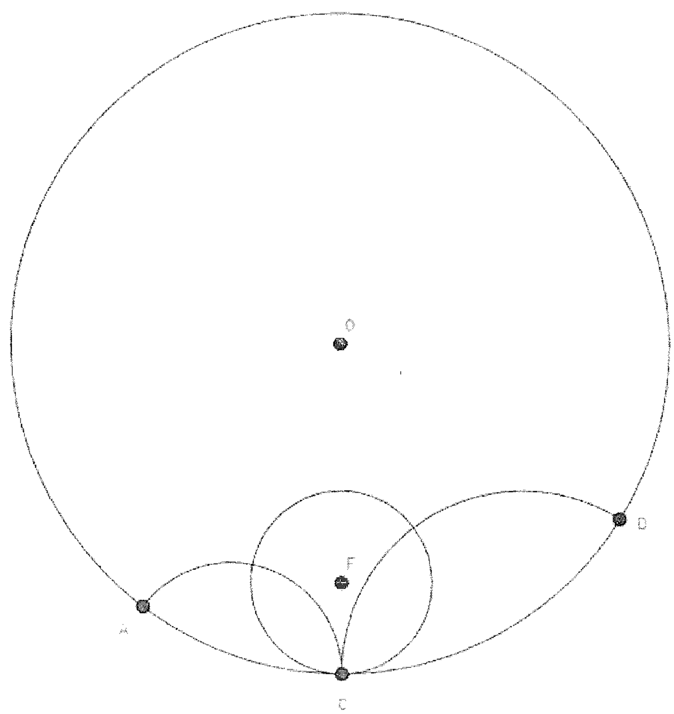

Figure 8.2: The horocycle to the disk ω at boundary point C with center F.

As for the Poincaré upper half-plane $\mathbb{U}^2$, there are two types of horocycles. One horocycle is a "circle" which is tangent to the absolute (x-axis) at a point, T. The other possibility is a Euclidean line parallel to the absolute (x-axis). These two cases can be seen by composing two stereographic projections taking a horocycle from the Poincaré disk $\mathbb{D}^2$ to the Poincaré upper half-plane $\mathbb{U}^2$. The first stereographic projection maps the horocycle in $\mathbb{D}^2$ to a circle on the Poincaré sphere $\mathbb{S}^2$ tangent to the equator at a point, T. As we discovered above in showing the one-to-one correspondence between lines in the models, there are two possible points from which we can perform the second stereographic projection taking the Poincaré sphere $\mathbb{S}^2$ to the Poincaré upper half-plane $\mathbb{U}^2$.

Case 1 : (non-tangent point)

Choose a point on the equator different from the tangent point T and perform a stereographic projection of the northern hemisphere onto the upper half-plane. This results in taking the circle tangent to the equator at T to a curve in the upper half-plane tangent to the absolute at the projection of point T, for simplicity we also denote it T. All u-lines passing through point T, including the unique vertical ray, are all orthogonal to this curve.

Case 2: (tangent point T)

From point T project stereographically the northern hemisphere of $\mathbb{S}^2$ onto the upper half-plane $\mathbb{U}^2$. Under this projection, the point T is mapped to infinity. As a result, the circle on $\mathbb{S}^2$ is projected onto a Euclidean line in $\mathbb{U}^2$ that is parallel to the absolute. The only u-lines passing through the projection of point T are vertical rays, which we know are orthogonal to the absolute; hence, they are parallel to any Euclidean line parallel to the absolute.

Chapter 9

Unifying the Models of Hyperbolic Geometry

Up to this point, we have seen four different models of the hyperbolic plane: the Klein disk model $\mathbb{K}^2$; the Poincaré disk model $\mathbb{D}^2$; the Poincaré spherical model $\mathbb{S}^2$; and finally, the Poincaré upper half-plane model $\mathbb{U}^2$. These models allowed us to visualize different properties of the hyperbolic plane in the familiar Euclidean plane. In Chapters 7 and 8, we constructed isomorphisms connecting these four models. Here we discuss a fifth model, called the *Minkowski model,* of hyperbolic geometry from which we will construct another isomorphism between the Minkowski model, the Klein disk model, and the Poincaré disk model. The Minkowski model is closely related to Einstein's Special Theory of Relativity.

The Minkowski model of $\mathbb{H}^2$ is the upper sheet of the two-sheeted hyperboloid given by the equation

$$x^2 + y^2 - z^2 = -R^2$$

We arrive at this equation from the standard equation of a 3-sphere, $x^2 + y^2 + z^2 = R^2$ by making the substitution $z \mapsto iz$ and $R \mapsto iR$ which yields $x^2 + y^2 + (iz)^2 = (iR)^2$. The hyperboloid has a north pole N at $(0,0,1)$ and a south pole S at $(0,0,-1)$. So, we see that points that lie on the sheet above the plane $z = 0$ satisfy the conditions

$$\begin{cases} x^2+y^2-z^2=-R^2, \\ z>0, \end{cases}$$

while points that lie on the hyperboloid sheet below the plane $z=0$ satisfy the conditions

$$\begin{cases} x^2+y^2-z^2=-R^2, \\ z<0. \end{cases}$$

Now taking $R=0$, we arrive at a double cone, or two cones with their apexes meeting at the origin. In physics, this resulting cone figure is called the *light cone*: $x^2+y^2-z^2=0$. Note that the light cone is a sphere of radius 0 in Minkowski space. Placing the observer at the origin, the light cone is used to distinguish events, both future and past, which can be reached by the observer when traveling at speeds less than the speed of light. Points that occur on the light cone are called *light-like*, in physics, they correspond to moving photons. In order to reach such an event, the observer would have to travel at the speed of light. Points in the interior to the light cone are called *time-like*, meaning that the observer can reach such an event in time traveling at a speed less than the speed of light. These correspond to the events which you can reach. On the other hand, points lying in the exterior of the light cone are called *space-like*, meaning that the observer is unable to reach such an event traveling at any speed less than the speed of light.

Returning to the task at hand, let's understand what hyperbolic lines look like in the Minkowski model. In this model, hyperbolic lines are the geodesics on the hyperboloid formed from the intersection of the upper sheet of the hyperboloid with a plane passing through the origin; for brevity we denote such lines *m-lines*. We now construct the isomorphism between the Minkowski model $\mathbb{M}^2$ and the Klein disk model $\mathbb{K}^2$. Then we will construct the isomorphism between the Minkowski model $\mathbb{M}^2$ and the Poincaré disk model $\mathbb{D}^2$.

Take the plane $z = 1$ and consider its intersection with the light cone $x^2 + y^2 - z^2 = 0$. The intersection is a disk centered at the north pole, $N = (0,0,1)$, in the plane $z = 1$ whose boundary is a circle on the light cone. In fact, this disk is the disk, ω, of the Klein disk model $\mathbb{K}^2$. To see that this is the case, recall that an m-line resulted from the intersection of a plane passing through the origin $\mathcal{O}$ with the hyperboloid. For a given m-line, the unique plane containing it intersects the disk ω in a chord, a k-line. Readily, we see that there is a bijection between points, and m-lines are mapped to k-lines. Thus, we have an isomorphism between the Minkowski model and the Klein disk model.

If one were to make a stereographic projection of the Minkowski model $\mathbb{M}^2$ (considering the upper sheet of the hyperboloid as a unit (northern) hemisphere in Minkowski space $\mathbb{M}^3$ with the center $\mathcal{O}$) from the south pole $S = (0,0,-1)$ onto the (x, y)-plane $z = 0$, we obtain the Poincaré disk model $\mathbb{D}^2$ in the disk $x^2 + y^2 = 1$ in the (x, y)-plane. There is a bijection between points, m-lines are mapped onto p-lines, and stereographic projection is a conformal mapping. This mapping also preserves hyperbolic angles, a fact which we will not prove here. Thus, the isomorphism between the Minkowski hyperboloid model $\mathbb{M}^2$ and the Poincaré disk model $\mathbb{D}^2$ is established.

9.1 Conclusion

In the first part of this dissertation we considered hyperbolic geometry without models following the works of Saccheri, Lambert, Legendre, Bolyai, Lobachevsky, and Gauss. We build hyperbolic geometry from Euclid's first four postulates and from the negation of the parallel postulate. Then in an effort to visualize hyperbolic geometry, we considered the models of Klein, Poincaré, and Minkowski. Throughout the dissertation, we placed greater emphasis on the geometric presentation of the information rather than algebraic. As a result, we presented pictorially the geometries through numerous figures which allowed us to visualize the strange behavior of straight lines both with and without models. In terms of the models themselves, we gathered the five models (the Klein disk model $\mathbb{K}^2$, the Poincaré disk model $\mathbb{D}^2$, the Poincaré spherical model $\mathbb{S}^2$, the Poincaré upper half-plane model $\mathbb{U}^2$, and the Minkowski hyperboloid model $\mathbb{M}^2$) of the hyperbolic plane $\mathbb{H}^2$ and established the isomorphisms between them. The theorems, such as the hyperbolic Pythagorean theorem, were proved in the Klein disk model $\mathbb{K}^2$ in easier ways than in the Poincaré disk model $\mathbb{D}^2$.

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