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A Comparison Of Ancient Mathematical And Calendrical Systems

Karen Schlauch
Eastern Illinois University

A COMPARISON OF ANCIENT MATHEMATICAL AND CALENDRICAL SYSTEMS

SCHLAUCH

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A Comparison of Ancient Mathematical and Calendrical Systems

(TITLE)

BY

Karen Schlauch

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

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YEAR

I HEREBY RECOMMEND THIS THESIS BE ACCEPTED AS FULFILLING
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July 18, 1991 — DATE | Robert E. Megginson — ADVISER

7/18/91 — DATE | Ira Rosenholtz — DEPARTMENT HEAD

7/18/91 — DATE | Claire E. Krukenberg — COMMITTEE MEMBER

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I would also like to show my appreciation to those in the mathematics department who have expressed interest in the development of this thesis.

I would also like to express my deepest thanks for all the wonderful support I received from my parents, Barbara and Wolfgang Schlauch.

ABSTRACT

The purpose of this paper is to analyze and contrast for five civilizations the extent of knowledge of mathematics and its incorporation into the life of the societies. The research is centered around the civilizations of the Central American Maya, the South American Inca, the Native North American Indians, the ancient Egyptians, and the ancient Greeks. Four major areas are studied and compared: the type of mathematical system developed by each culture and its efficiency, the range of mathematical skills of each culture and the extent of their use within daily activities, the incorporation of the system into the culture's ceremonial and religious life, and the existence of an accurate calendrical system. These and other related areas are discussed for each culture, and the areas in each culture are then compared. The Mayan, Incan, Egyptian and Greek cultures are then rated on a subjective scale, with the goal of forming a comparison, again subjective, of the relative mathematical achievements of the four civilizations. Due to the lack of resource material on the Native North American Indians, their mathematical advancements are not rated; instead, a brief overview of the mathematical achievements of this society is presented.

A number of conclusions are made based on the comparison of the four cultures. The ratings awarded to the four civilizations show that the Greek mathematical development was somewhat more thorough than that of the other four cultures.

However, not all portions of Greek mathematical knowledge were as extensive as that of their competitors.

The research and subjective comparison concludes that the cultures studied in this research have made contributions to international mathematical development. The Maya, Inca, Greek and Egyptian civilizations introduced both mathematical skills and calendrical development to the ancient world predating today's expansion of science and technology. The comparison of the mathematical abilities and the selection of a superior mathematical development does not carry as much importance as the recognition of each civilization's individual mathematical achievements. It is this recognition that the comparison and conclusion of this research portrays.

DEDICATIONS

This graduate thesis is dedicated to Dr. Robert Megginson for his assistance and endless patience he provided during the entire period of my graduate program.

TABLE OF CONTENTS

"The history of mathematics should not be detached from the general history of culture. Mathematics is a domain of intellectual activity, intimately related not only to astronomy and mechanics, but also to architecture and technology, to philosophy, and even to religion."

[Van der Waerden, p. 5]

INTRODUCTION

Although the primary focus of this research began with the study of the Mayan Indians and their mathematical knowledge, it has expanded to include the analysis of four other civilizations and their efforts to incorporate mathematics into their lives. The five cultures included in this research are those of the Central American Maya, the South American Inca, the Native North American Indians, the ancient Egyptians, and the ancient Greeks.

The purpose of this paper is to analyze and contrast the extent of each culture's knowledge of mathematics and the incorporation of this knowledge into the daily life of each society. For each culture, four major areas are studied and compared: the type of mathematical system developed by each culture and its efficiency, the range of mathematical skills of each culture and the extent of their use within daily activities, the incorporation of the system into the culture's ceremonial and religious lives, and the existence of an accurate calendrical system.

The Mayan mathematical achievements in each of the four areas mentioned above will be discussed first. Immediately following the Mayan discussion, each area will be discussed for the Incan Indians. After completing the Incan discussion, the two American Indian cultures are compared and given a subjective rating based on the author's scale. To complete the study of Native Western Hemisphere mathematics, a short discussion of Native North American mathematics is presented at this point. No attempt is made to rate the mathematical developments of the Native American Indian society, as was done in the Mayan and Incan societies, due to the lack of resource material focusing on this civilization. However, from the existing research, it is evident that some mathematical accomplishments of the Native North Americans are significant, and thus are mentioned in the discussion. Following the discussion of the Central, South, and North American Indians, the focus shifts from ancient American mathematics to ancient Mediterranean mathematics, and the mathematical developments of the ancient Egyptian and Greek cultures are compared and rated using the same scale as for the Maya and Inca. The areas of each culture are rated with the goal of forming a comparison, again subjective, of the relative mathematical achievements of the five civilizations.

RATING SCALE

The following scale was used in the comparison of the Mayan and Incan civilizations and in the comparison of the Egyptian and Greek civilizations.

A maximum of **two points** was awarded for the construction of an efficient numerical system and the efficiency in which it was employed. Factors considered include are the type of system, method of representing larger numbers, and ease in which it was incorporated into the society.

A maximum of **two points** was assigned to systems which could easily employ all four mathematical operations and the efficiency of this performance.

One point was awarded for the development of the concept of zero and the invention of a zero symbol.

One point was granted for an adequate fractional system which could be used within daily life.

A maximum of **one point** was awarded for the understanding of geometry in regard to areas, volumes, measures of circumference and the use of these within the societies' architectural structures.

A maximum of **one point** was awarded for a somewhat accurate approximation of pi.

A maximum of **two points** was awarded for a complete and accurate calendrical system and its ease of use by each society.

A maximum of **two points** was assigned for the completeness of the numerical system's incorporation into daily activities.

A maximum of **two points** was awarded to a mathematical system was evident within the cultures' ceremonial life.

The scale consists of a total of **fourteen points**.

THE ANCIENT MAYA

The ancient Mayan Indians emerged during the first or second century of the Christian Era, establishing their community within the realms of what is now Mexico. During the ten centuries of their independence, before their civilization was destroyed by Spanish conquerors in 1541, the Maya distinguished themselves as a tribe of highly intellectual and skilled people, developing an intricate hieroglyphical mathematical and calendrical system. The Maya were a simple and hard-working people, whose main occupation was the planting and harvesting of maize and whose interests included an obsession with the passage of time. Constant agricultural labor and a transcendent interest in the measurement of time led to their development of the first agricultural calendar which precisely recorded the seasons of planting and harvest. Without the use of advanced arithmetical methods, it was not possible to perform the required computations of the calendrical system.

MAYAN NUMERICAL SYSTEMS

Most of what is known of the Mayan mathematical system has been correlated to their ancient calendar and related computations. No existing evidence suggests conclusively the use of a Mayan numerical system to complete economic calculations: decisive evidence of real estate sales or commodity trades does not seem to exist. Instead, the Maya created well-developed numerical methods used mostly to complete calendrical computations which became significant in every aspect of Mayan life.

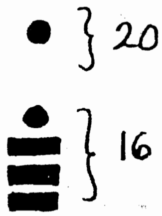
The Mayan Indians developed a <u>vigesimal</u>, or base twenty, numerical system, based on counting using ten fingers and ten toes, as opposed to a decimal system, based on counting with ten fingers. [Schele and Freidel, p. 78] Utilizing only two symbols, a closed or solid small circle (●) to represent one unit and a solid bar (▬) to represent a cluster of five units, the system was very efficient. In an alternative notational system, numerals were written in an elaborate manner, the inessential decoration used only as ornamentation. In other notations, red and black colors were added to distinguish the "day signs" from the "distance numbers" and "month coefficients". [Thompson, p. 130]

The simple, two-symbol system, was in a sense superior to the Roman notation, which employed at least five different symbols to portray all numerals. Mayan numbers were usually constructed from digits written vertically with lowest order digits on the bottom, in contrast to our modern method of horizontal placement with lowest-order digits on the right. [Closs (1986), p. 299]

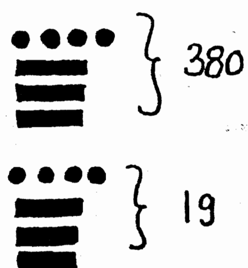
In the Mayan system, one unit in each position represented twenty units of the next lower order position, just as ten units equal one unit in the next higher position in the modern decimal system. The lowest order position was used to represent the units zero to nineteen. Entries in the second-order position contained zero through nineteen units of twenty; therefore, the first two lowest orders together could represent the numbers zero through 399. Similarly, adding the third level allowed for additional numbers between 400 and 7,999, and so forth through the higher levels. [Thompson]

Numbers were written as a combination of the bar and dot symbols; in each position or level, the bars were placed below the dots. Occasionally, numbers were written horizontally, placing the dots to the left of the bars. The Maya expressed the numbers one, two, three and four by one dot, two dots, three dots and four dots, respectively. The number five was expressed by one bar which represented five dots or units; six seven, eight and nine were written as one bar and one, two, three and four dots respectively. Numbers ten through fourteen were represented in the lowest order position by two bars and the corresponding number of dots; fifteen through nineteen would be written as three bars and the proper number of dots. The number twenty was represented by one dot in the second position and zero units in the lowest position, corresponding to the number ten written in the today's system as one ten and zero ones. The numbers twenty-one through thirty-nine were drawn as one dot in the second level and the appropriate number of bars and dots in the first order position. The same notation was used in each higher order position with each dot or bar representing twenty of one hundred units, respectively, of the next lower order position.

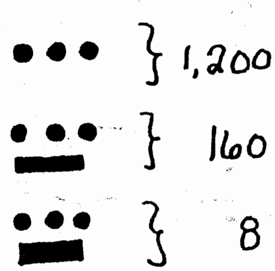
For example, the number thirty-six was written in the following manner:



The number 399 was represented this way:



and 1,368 was written like so:

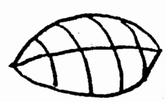


Using this method, no more than four dots were ever present in one order, because one bar was drawn for every group of five dots. Using the same rule, no more than three bars were ever drawn in one level, because four bars represented twenty units in one order, equivalent to one dot in the next higher digit position.

This Mayan numerical system was used primarily for calculations of elapsed time. Although George Sanchez, a historian of Mayan mathematics, believed that the system was never employed for economic purposes, others have suggested that such alternate uses did exist. According to Bishop Diego de Landa, the vigesimal method was used to keep track of trading by merchants, using counters such as cacao beans on flat surfaces. [Morley (1983), p. 547]

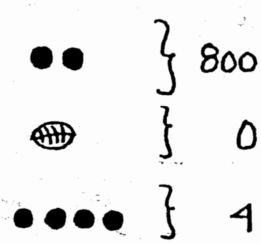
It is possible that only three cultures have established a symbol for zero in a numbering system: the Hindus, the Incan Empire and the Mayan civilization in Mexico. [McIntyre, p. 31] Although Mayan mathematics seemed to represent a basic and perhaps primitive system, the Maya were able to discover the importance of a zero symbol and incorporate it into their number system. One of their symbols for zero was a hieroglyph resembling a closed fist, which perhaps signified the count of zero fingers. At other times the zero was drawn as a stylized shell painted red. (In the literature consulted, one occasionally sees the Mayan symbol for zero written as an open circle slightly larger than that of their unit circle. This is remarkably similar to our own symbol 0; however, it is merely an invention of historians to simplify their own representation of the Mayan symbol for zero. It is interesting, though, to note that the Mayan closed fist does have a resemblance to our own symbol for zero.) The Mayan symbol was used in the same manner as in the modern decimal system: the symbol was placed in a position which contained no numerals. For example, the number forty was represented as two shaded dots in the second level and an oval zero symbol in the lowest position. Another form of zero developed for the calendrical system was used only to express the first position in each of the nineteen divisions of the year. [Closs, Gallenkamp, p. 76; Sanchez]

This is the Mayan closed-fist zero symbol:



The use of the zero symbol is shown in the symbol for forty:

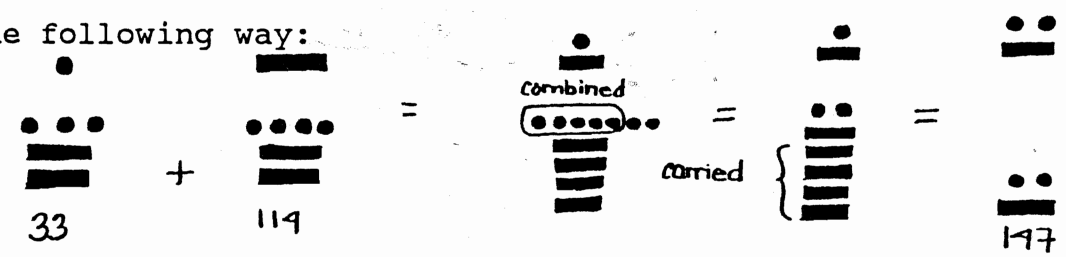
and also in the numeral 804:



MAYAN COMPUTATIONS

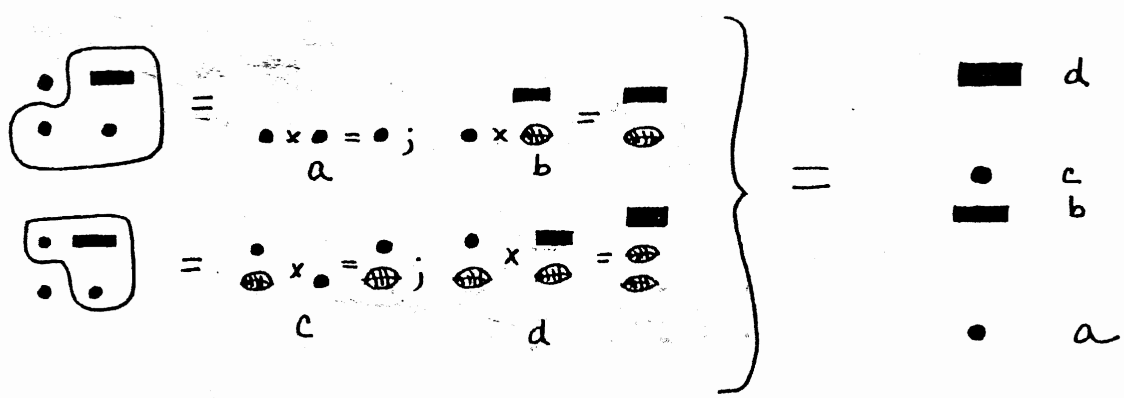
A great portion of the Mayan written records were destroyed during the Spanish conquest and therefore little is known of Mayan use of the four basic mathematical operations. Although the majority of the Mayan calculations focus on the calendrical counting system, words denoting "division" and "multiplication" existed in the Mayan language. Recent studies show that the Mayan system can be adapted to multiplication, division and even square-root extractions, although the extent of Mayan knowledge of these is unknown. [Gallenkamp, p. 80] Evidence of Mayan multiplication tables and calculations have been discovered; however, all focus on calendric computations. [Sanchez, p. 8]

The Mayan addition process was simple and analogous to ours. Numbers were added one level at a time. After the digits on a given level for all summands had been combined, the dots and bars were rearranged so that no more than four closed dots and three bars were placed in each level: each group of five dots would be converted into one bar and each group of four bars would be replaced by one dot in the next higher level. The numbers 33 and 114 would thus be added in the following way:

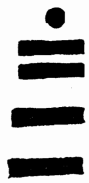


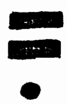
Subtraction was performed similarly: each level was subtracted separately. When subtracting a larger number from a smaller number in a specific level, one dot is carried as twenty units from the next higher level. For example, if eight were to be subtracted from twenty-four, it would be necessary that eight units would be subtracted from four units in the first level of the numbers. One unit of twenty from the second level of the larger number (24) would be carried down into its first level as twenty units. The smaller number's first level thus contains in effect twenty-four dots; the eight units are then easily subtracted from the twenty-four. The result amounts to sixteen dots in the first level expressed as three bars and one dot; the second level of both numbers is empty, the only dot being carried down earlier. The Mayan subtraction process thus differed from the present decimal method only by the borrowing of twenty units as opposed to ten.

Although evidence of multiplication within the Mayan numerical system is not solid, a possible process could be conducted following three rules: one dot multiplied by one dot results in one dot; one dot multiplied by one bar results in one bar; one bar multiplied by one bar results in five bars. Each level of the first number would be separately multiplied by each level of the second number. The first level of the first number would be multiplied by the first level of the second number and recorded, then by the second level of the second number, until all levels of the second number had been multiplied by the first level of the first number. The same process would be used on the rest of the levels of the first number. For example, the two numbers 21 and 101 could be multiplied by first multiplying the one dot of the first level of 21 by the one unit of 101 and would be recorded as one closed dot in the first level. The dot in the first level of 21 would then be multiplied by the bar in the second level of the 101 and recorded as one bar in the second level of the resulting number. After the completion of the first level, the dot in the second level of twenty-one would be first multiplied by the unit of 101 and recorded as one dot in the second level of the result; then the same dot would be multiplied by the bar of the second level of 101 and recorded as one bar in the third level of the result. The result contains one dot in the first level, one bar and one dot in the second level and one bar in the third level to represent the result of 21 x 101, equivalent to 2121.



Mayan division calculations could possibly have been performed by a "reversal" of the multiplication process analogous to our own long-hand division process. A quotient could be found by dividing the highest number of the divisor into the highest or next-highest number if necessary, of the dividend. For example, when dividing the number



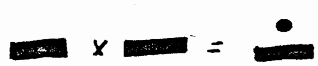
by the number  ,

the highest "digit" of the divisor

would be divided into the highest and next-highest "digits" of the dividend

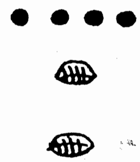
 .

Using the fact that

 ,

the first "digit" of the quotient would thus be  .
Multiplying the divisor by one bar results in

which is subtracted from the dividend and resulted in a remainder of

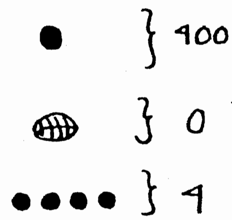


[Sanchez, pp. 34-36]

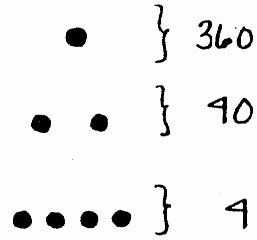
BASIC CALENDRICAL COMPUTATIONS

The Mayan vigesimal system was modified somewhat to accommodate their chronological computations. One type of Mayan calendar year consisted of eighteen months of twenty days each; to adapt to the calendar year, the vigesimal system was adjusted by allowing the second level to contain only eighteen units, representing the eighteen months of every year. The new system otherwise remained base twenty. The first two levels now represented a calendrical Mayan year of 360 days: the twenty Mayan days in each month were represented by the first level of the new system, and the eighteen monthly divisions of the year were contained in the second level. The third level would contain twenty positions, as would each remaining level. [Sanchez]

In this new system, the first level recorded entries between zero and nineteen days; a day count ranging between zero and 359 days was recorded by the combination of the first two levels; the first three levels were used to record time periods involving between zero and 7,199 days. With this slightly modified vigesimal system, calendrical computations could be done more efficiently than today. The difference between the true vigesimal and calendric systems can be seen in the two distinct representations of the number 404:



True Vigesimal Representation



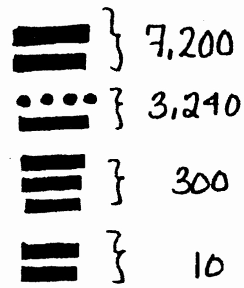
Calendric Representation.

The revised vigesimal levels represented the structure of Mayan time periods of their calendar. The first level represented the twenty days in each Mayan month: twenty 'kins' composed one 'uinal', denoted by one unit in the second level. The second level represented the eighteen 'uinals'. Eighteen months or 'uinals' completed one year; one 'tun' was equivalent to one unit in the third level of the calendric system. This position was calculated by multiplying one unit of twenty by one unit of eighteen to designate eighteen months of twenty days each. A period of twenty 'tuns', labeled a 'katun', or 360 x 20 days, was represented by each unit in the fourth level of the calendrical system. The fifth level represented the periods of one through nineteen 'baktuns'; one baktun was commonly labeled one **cycle** by historians. [Morley (1946), p. 276] Further levels were indicated in this summary:

1	KIN	=	1 DAY							
20	KINS	=	1 UINAL	=	20	DAYS				
18	UINALS	=	1 TUN	=	360	DAYS				
20	TUNS	=	1 KATUN	=	7,200	DAYS				
20	KATUNS	=	1 BAKTUN	=	144,000	DAYS	=	400	TUNS	
20	BAKTUNS	=	1 PICTUN	=	2,880,000	DAYS	=	8,000	TUNS	

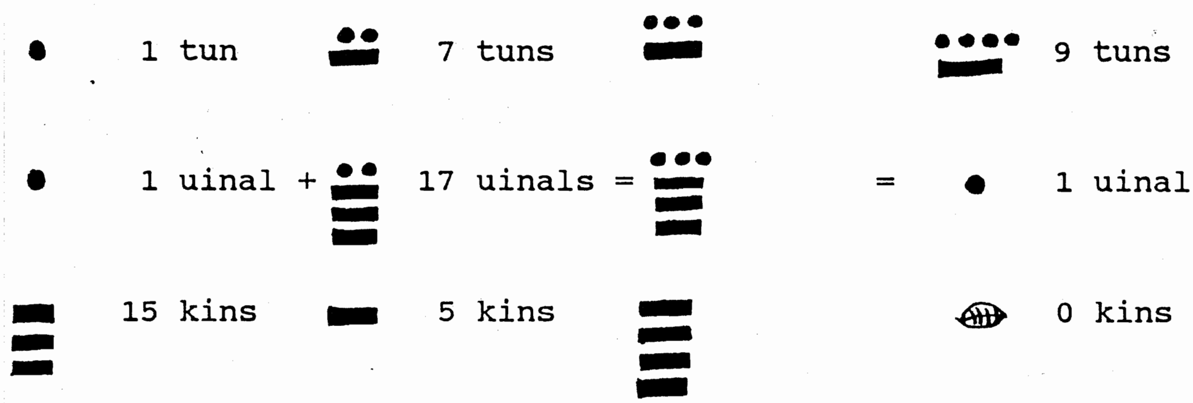
Recording higher day counts was resolved by continuing to multiply each level by twenty. [Morely (1983), p. 548]. As an example, a day count of three hundred was recorded as

in which the fifteen units in the second level denoted fifteen months of twenty days each and the shell in the first level denoted zero days. A count of 75,550 days was recorded as ten 'katuns' equivalent to 72,000 days; nine 'tuns' equalling 3,240 days; fifteen 'uinals' or months representing 300 days and ten 'kins' to complete the total 75,550 days:



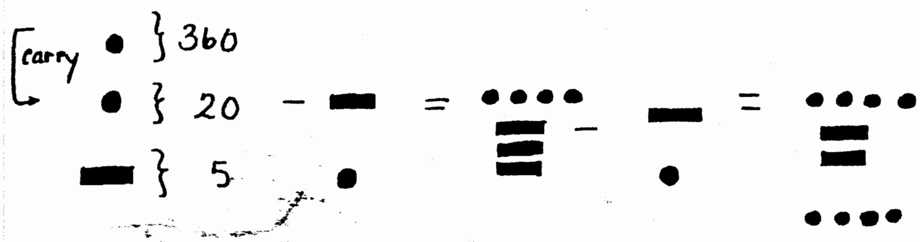
The Maya used their calendric system to record counts of days, as well as absolute dates expressed as the time elapsed since a certain fixed starting date (analogous to writing 8-16-1980 to represent 1980 years, eight months, and sixteen days elapsed since a certain fixed starting date for our calendar.) More will be said about the Mayan "starting date" later. Just as we compute future dates by adding a number of days to a current date, and compute elapsed time by subtracting one date from another, so could the Maya. However, their modified vigesimal calendrical numbering system greatly simplified the computations. [Sanchez, p. 14-15]

Calendrical additions were done as in their true vigesimal system but compensating for the maximum of only 18 units in the second level; chronological calculations recorded one dot in the third level for each unit of 360, as opposed to 400. An example of calendric addition portrays this modification:



[Sanchez, pp. 45-46]

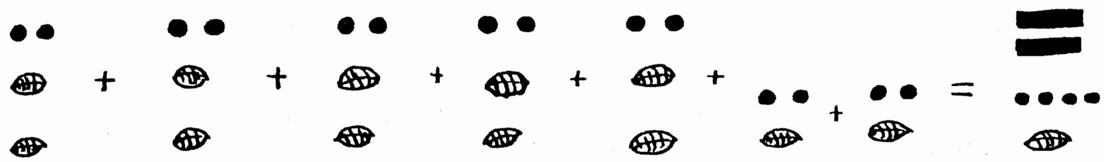
Subtracting dates from each other was done by borrowing one unit of <u>360</u> from the third level into the second level when required:



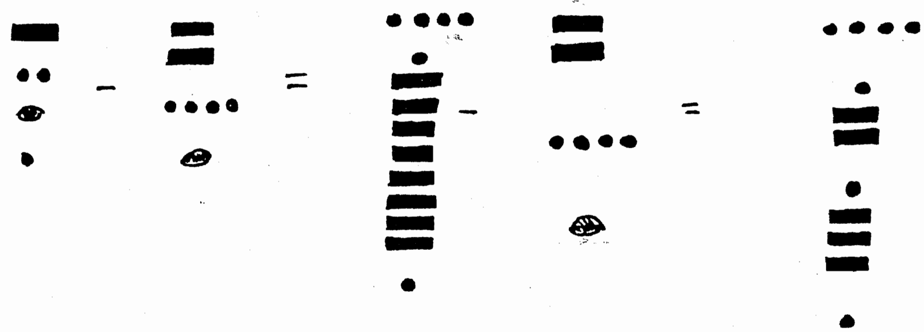
Conversion from the calendrical system to the true vigesimal system was easy. A calendrical number could be converted to a true vigesimal number by treating it as the representation of a true vigesimal number, then subtracting conversion factors from it. For example, the calendrical number

represents 5 x 7,200 + 2 x 360 + 0 x 20 + 1 x 1. Treating it as a true vigesimal number, representing 5 x 8,000

+ 2 x 400 + 0 x 20 + 1 x 1, results in an "overrepresentation" by 5 x 800 + 2 x 40 units, so subtracting

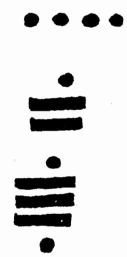


effects the correction:



; i.e.,

Calendrical



True Vigesimal.

It proved more effective to use the true vigesimal system than the calendrical system in multiplication and division processes, so computations of day counts would have very likely been converted into vigesimal representations before completing the operations. The result could then have been converted back into the calendrical representation.

Operations involving multiplication or division of calendrical dates were generally not performed; more commonly, Mayan priests would determine the day count or "distance number" which linked two recorded dates together. Two common day-count computations included the determination of the Calendar Round date reached when counting through a given interval of time in a forward or backward manner. Another calculation determined the interval of time between two specific Mayan dates. [Closs (1986), p. 306]

Mayan dates were recorded as the time elapsed since a particular point in time. The given point could be explicitly stated; if it was not, it was assumed to be the date treated by the Maya as the starting point of time, falling approximately in 3114 B.C. [Morley (1983), p.556] (Note that the Mayan calendrical starting point is very close to that of the Hebrew Calendar.) The dates were written as a series of numbers, separated by dots, written in a horizontally right-to-left manner, indicating the amount of 'kins', 'uinals', 'tuns', 'baktuns' and all higher periods of the date. For example, the count of 3.15.6. represented six 'kins', fifteen 'uinals' and three 'tuns' of elapsed time. Determining whether the date was to be counted forward or backward from the reference point was the next step. Although the majority of such calculations involved passing from an earlier date to a later date, a backward count also existed and was signified by a special minus symbol and a red circle around the date's lowest term.

The Maya had "day prefixes" that repeated in a thirteen-day cycle as our modern weekday names repeat in a seven-day cycle. Each day of the twenty-day Mayan month had a "day name" associated with it. Also, the Maya recognized the importance of the 365-day "Vague Year", though their Calendar Round recognized a formal 360 day year, and would wish to know the number of the day of a given date within a 365 day "year" cycle. The Maya could have computed these from a given date number by a process involving division and saving of remainder; that is, the process we call modular division. For example, suppose that the "day prefix", "day name", and day position within the "Vague Year" were desired for the date 9.12.2.0.16, stated relative to the beginning point of Mayan time. Since the date is 1,383,136 days after that starting point, and since

$$1,383,135 \bmod 13 = 1$$

$$1,383,136 \bmod 20 = 16$$

$$1,383,136 \bmod 365 = 151 \quad ,$$

the given date has a "day prefix" of 1, a "day name" of 16, and the 151st day position within the 365 day year, relative to that of the Mayan starting date. The Mayan starting date was labeled by the fourth "day prefix", the twentieth "day name", and the eighth day of the eighteenth month of the year; the corresponding labels for the 9.12.2.0.16 would have to be adjusted to account for that.

While this completes the discussion of most of the mathematics involved in the Mayan calendrical system, a further discussion of the system placed within a cultural context, will be presented in a later section.

OTHER NUMERICAL REPRESENTATIONS

Another form of Mayan numbering incorporated the use of certain pictorially intricate hieroglyphical symbols: the head variant numerals. Although not as commonly used as the bar and dot notation, these numerals were efficient and simple to use. Each number from zero to twelve was depicted by a unique hieroglyph, which almost always represented a portrait of a deity. The detailed features of each portrait distinguished which number it represented. The numbers thirteen through nineteen were commonly written as the specific head hieroglyph representing the number ten and the head hieroglyph representing the numbers three through nine, respectively. Although the number thirteen could also be represented by a unique hieroglyph, it was usually denoted as a combination of the portraits for ten and three. This system is not all that different from the Arabic system, in which one distinct symbol denotes the first ten units. [Closs (1986), pp. 334-336; Morley (1983), pp. 547-548] These head variant numerals were found in both records of Calendar Round dates and distance numbers. Most head variant numerals have been found in "Initial Series" representations, which indicated the first date of the inscription process of hieroglyphical panels.

As did many other ancient civilizations, the Maya developed an abacus. Their abacus had seven rows, each containing seven beads partitioned into two groups: one group of four beads on the left side of each row represented the four possible dots at that level, while a second group of three beads on the right side of the row represented the three possible bars at that level. Just as four bars represented one dot in the next higher level in the Mayan vigesimal system, when four "bar" beads were needed in one abacus row, a "dot" bead in the next larger row was used. [Sanchez]

DEVELOPMENT OF FRACTIONS

The Mayan civilization had little need for fractions, and therefore did not develop a complete fractional system. [Thompson (1966), p. 169] A few simple fractions with a denominator of one were known to exist in the Mayan system. When a calendrical cycle was broken down into several smaller parts, the Maya considered each section as a fraction of the whole, but no evidence proves that they treated these fractions as numbers, with one exception. They did have a hieroglyph to represent one-half of a period. This is the only Mayan hieroglyph of a fraction that has been found. [Thompson, pp. 139-140]

GEOMETRICAL DEVELOPMENT

The Maya developed some understanding of geometry. Their perception of the four world directions, North, South, East and West, may imply their knowledge of right angles, often used in their architecture. Mayan architecture included both temples and pyramids; other buildings were constructed having a flat top. [Thompson, p. 6] Some isosceles triangles and angles measuring thirty degrees were found in other Mayan architecture, as were hexagonal structures, signifying more than just a scant knowledge of geometrical structures. [Aveni, p. 125]

THE CULTURAL IMPORTANCE OF THE MAYAN CALENDRICAL SYSTEM

Mayan life was deeply involved with the harvest of maize, religious ceremonial events, and the focus of superstition and mythology. Von Hagen states that the harvesting of corn was actually the main part of the people's life. [Von Hagen] Observing the stars, as well as the seasonal risings of the planets and the recordings of any indication of rain or sun to aid in the upcoming harvest of the Mayan crop, led to the development of their calendrical system.

The Maya certainly had no greater intellectual capacity than other ancient tribes, yet their civilization became the most sophisticated in the New World. [Von Hagen] The people's life was ruled by their calendar and the recording of past milestones and their predictions of future events.

> "No other people in history were so obsessed with the passage of time, and they labored tirelessly to understand its mysteries and control its awesome influences.
> ... time was a supernatural phenomenon involving omnipotent forces of creation and destruction, ... directly influenced by evil or benevolent gods." [Gallenkamp, p. 74]

"The Mayan numbers, time and the cosmos were

ruled by supernatural forces. By discovering and recording regularities in these forces, therefore, the Maya believed they were in a position to better understand and even predict events. Of course, the calendrical system was also used to record the events of history, the reign of rulers, their conquests and achievements, and other earthly matters."
[Morley (1983), p. 545]

The passing of Mayan time was regarded as a series of burdens carried by bearers in relays. When one period of time was completed, the next time-bearer or god would carry the next burden. Each day was thought of as belonging to a deity whose individual character was present during the time period in which he ruled. Thus each day portrayed a different character or attitude, depending on that of the current time-bearer. Days falling on the same date in a calendrical cycle portrayed similar characteristics; dates occurring on the same day in more than one cycle portrayed even more similar characteristics. [Schele and Freidel, p. 252] Mayan days became living, personified objects for the people. Each day held a certain amount of luck; the Maya so strongly believed in these deities and their powers that each day was lived according to its fortune. The luck of the Gods, imparted to the days, influenced every daily activity. Indeed, when a specific day included no good fortune, important Mayan activities such as farming would not be performed. [Thompson, p. 66]

The completion of the sun's cycle from sunrise to sunset denoted the basic division of time upon which the Maya relied. Twenty consecutive days completed one basic "luck" cycle, each day being depicted by a portrait of a God. The twenty Mayan days in such a cycle were named:

1	IMIX	11	CHUEN
2	IK	12	EB
3	AKBAL	13	BEN
4	KAN	14	IX
5	CHICCHAN	15	MEN
6	CIMI	16	CIB
7	MANIK	17	CABAN
8	LAMAK	18	EZNAB
9	MULUC	19	CAUAC
10	OC	20	AHAU

These twenty names were used to number the first twenty consecutive days of the 360-day year; the cycle then repeated. "Day names" were always prefixed by numbers between one and thirteen, called "day prefixes", similar to the modern seven days of the week. Together, the "day prefix" and "day name" would form a unit; neither was ever used by itself. The Mayan language indicated the importance of the day prefix: the day 3 IMIX was written as 'oxil IMIX ', meaning "day prefix oxil (3) <u>belonging to</u> IMIX". [Thompson, p. 66]

The combination of the thirteen numbers and the twenty names formed the centerpiece of the Mayan calendar. Each day name could be joined with one of the thirteen prefixes, resulting in 260 different combinations. These 260 different combinations of the twenty "day names" and the thirteen "day prefixes" formed another important cycle, no combination would be repeated until 260 days had been completed. Beginning with the first day of the day name cycle, IMIX, and the first number of the prefix cycle, 1, the first day of the 260-day cycle was established as 1 IMIX. Following the first day would be the combination of the second day name IK, combined with the second number became 2 IK. The fourteenth day would be represented by the fourteenth day name and the first number prefix: 1 IX; the twentieth day was written as 7 AHAU. This 260-day cycle formed the Mayan Sacred Year, 'Tzolkin", or the "Tonalamatl', meaning "the book of days". [Morley (1946), pp. 265-267]

This process represents a good example of an important characteristic called <u>absolute continuity</u>, present throughout the entire Mayan calendar system. Every cycle would be counted through from beginning to end, then would begin anew. [Thompson, pp. 66-67] The different cycles of prefixes, day names and months proceeded independently of each other. Their combination determined the complete structure of the Mayan calendar.

Although the 260-day cycle was not used to count time, but often only to record and anticipate the time for their maize harvest, it became a significant part of Mayan society. The people's ceremonial life was determined from the Sacred Year; the God of each day was a guardian to those born on a particular day. The 'Tonalamatl' was the only part of the Mayan calendar system with which the common folk were familiar. [Thompson, p. 103]

Another important component of the Mayan calendar was yet another year, the 365-day year, or 'Haab', also called the <u>Vague Year</u>. It was constructed of eighteen periods or "months" of twenty days each and one shorter period of five extra days to complete the 365-day cycle. These five extra days were named the 'xma kaba kin', meaning "days without a name", and were considered to be highly unlucky. The Mayan Indians realized that the solar year lasted approximately six hours longer than 365 days. They did not, however, add a leap day every four years, knowing that this would complicate their calendrical computations. [Gann, pp. 127, 208; Thompson, p. 121] The eighteen primary twenty-day divisions or "months" of the Mayan <u>Vague Year</u> were labeled in the following cycle:

1	POP	10	YAX
2	UO	11	ZAC
3	ZIP	12	CEH
4	ZOTZ	13	MAC
5	TZEC	14	KANKIN
6	XUL	15	MUAN
7	YAXKIN	16	PAX
8	MOL	17	KAYAB
9	CHEN	18	CUMHU

To complete the 365-day Vague Year, a nineteenth five-day "month" was added, labeled as UAYEB, which represented the 'xma kaba kin', or "days without a name". It must be emphasized that these "months" were not closely related to the twenty-day cycles within the Sacred Year. To emphasize this distinction, each day was denoted both by its "day prefix, day name" designation within the Sacred Year and by another designation within the Vague Year more closely related to our modern "month, day of the month" system. The first day of the 365-day Mayan 'Haab' was the first day of the first "month" POP. The Maya considered the day of the month to represent the number of days already completed in that month, so the first day of a month was numbered by a zero. Thus, 0 POP designated the Mayan New Year. Each 365-day Vague Year would contain eighteen complete twenty-day cycles from the 260-day Sacred Year, plus five additional days. As a result, the day name of New Year's Day would advance by five days each Vague Year. At the end of four years, then, the day count advanced by twenty days, and the New Year's Day would again have the same day name as it had four years before. Notice that only four different day names would coincide with the first day of each month. These four day names: IK, MANIK, EB, and CABAN, in combination with the thirteen day prefixes, became the Mayan New Year's days. [Thompson, p. 123]

Combining the 365-day year or 'Haab' with the 260-day 'Tonalamatl', resulted in the Mayan Calendar Round which became the ultimate basis of Mayan chronology. This Calendar Round was pictorially recorded as two cog wheels; one containing the 365 positions of the year and the other the 260 possible day names. When the two wheels were intermeshed, each day-name-unit was combined with one of the 365 positions of the year at each connected cog, beginning with the start of both cycles, 2 IK 0 POP. The least common multiple of 260 and 365 gave the 18,980 possible combinations of Mayan dates. The 'Haab' wheel would therefore make fifty-two revolutions before returning to 2 IK 0 POP. Each of these fifty-two 'Haabs' was given a name corresponding to the four previously-mentioned "day names" of the New Year's Days and the thirteen possible "day prefixes" that were affixed to these "day names". A person, then, living a life longer than fifty-two years would see repeated New Year's day names. The fifty-two year Calendar Round period was used widely throughout Mesoamerica, both as a calendar and as an almanac, as early as the fifth century B.C. It became an important part both of the Mayan calendar and daily life; Mayan festivals had revolved around this Calendar Round until recently, when the ancient timing was altered to correspond to the Christian Calendar. [Gallenkamp, p.76; Morley, pp. 543-545; Schele and Freidel, p.45]

The Mayan people were one of the first to establish a fixed starting point in their calendar, as a reference point for the beginning of time. A specific date in their calendar, 13.0.0.0.0 4 AHAU 8 CUMHU, was referred to as the "zero date", and was considered by the Maya as the day of the world's creation. Although this date has been found to be more of a hypothetical starting point, the concept of the day is analogous to the Birth of Christ in the Gregorian calendar. The date itself occurred more than three millennia before the earliest known Maya inscription, therefore strongly suggesting that it must have been an invention of the Maya, rather than corresponding to a true historical event. [Gallenkamp, p. 77; Morley (1983), pp. 555-556]

The Maya devised three systems to count periods of time from their calendar. The Long Count, considered to be the most accurate ancient-world calendar ever developed [Gallenkamp, p. 76], fixed the chronological position of all dates with respect to the calendar's starting point. The Long Count was a number which designated the chronological count, or elapsed number of days since the starting date. The Long Count would therefore record the first day following the "zero date" 4 AHAU 8 CUMHU by the number 1. On some of their monument inscriptions, the Maya included a date which indicated the first day of the inscription process. It was recorded in an "Initial Series" hieroglyph at the beginning of the text. [Closs (1986), p. 317; Schele and Freidel, p. 81]. A typical date using the Long Count procedure may have been 8.14.10.13.15 7 AHAU 3 XUL. This represented an elapsed amount of time since the initial date of eight baktuns, fourteen katuns, ten tuns, thirteen uinals and fifteen kins, or 1,256,660 days, denoting the day 7 AHAU of the 'Tzolkin' and the third day of the month XUL in the Mayan 'Haab'. [Gallenkamp, p. 77]

Expressing only one day in the Long Count required ten different glyphs. The "Secondary Series" occurring later in the hieroglyph would record further dates in a much abbreviated form. "Secondary series dates consist of the number of days (distance numbers) to be counted forward or backward from the base date to arrive at the new calendar-round position." [Morley (1983), p. 558]. The distance numbers of the secondary series were recorded in increasing order, from left to right, instead of right-to-left order, as done in the Long Count. For example, if the long-count date 9.16.0.0.0 was followed by a forward count of 11.8 representing eleven 'kins' and eight 'uinals', the two counts were combined to form the long-count date 9.16.0.8.11. The long count method later was simplified by a "short count" method, which designated a specific time period and the date upon which it would end. [Morley (1983), pp. 557-559]

Not fully satisfied with this collection of time cycles, the Maya also devised a "moon count", or lunar calendar. The "moon count" began with a new moon; the moon was described by its shape on the first evening after the beginning moon, the second evening, and each day thereafter of the moon's cycle. The cycle ended on the last night before the new moon, called the "the end of the moon (month) day". While the moon was waxing, certain restrictions were put upon the Mayan lifestyle: crops were not harvested until a later phase of the moon, charcoal was not burned and people became lazy. [Aveni, pp. 80-1] Another count gave the age and position of the new moon and the length of the particular lunar month of either twenty-nine or thirty days. [Gann, p. 216] Although many observations were made to complete such a detailed moon calendar, it was not as commonly used as the other Mayan calendrical systems.

MATHEMATICAL APPLICATIONS IN DAILY LIFE

Not every Mayan Indian had been educated in the uses of the numerical systems, nor of the calendrical systems. Priests of high rank and powerful positions in Mayan society were, for the most part, the employers of the Mayan mathematical and chronological computations. Almost all knowledge of mathematics, astronomy, hieroglyphic writing and calendars was restricted to the upper classes, while most of the peasants remained illiterate. [Gallenkamp, pp. 108-113] The farmers and peasants, however, did employ the 260-day 'Tonalmatl', which specified the harvesting time and designated the amount of luck each day contained. An almanac for planting was constructed by the Mayan astronomers, which listed good and bad days, lucky and unlucky days, as well as rainy or dry days. A typical entry in the almanac could have been "the month and day of 9 CABAN was a good and lucky day, with heavy rains, and good for planting everything". [Von Hagen] All Mayan people had at least <u>some</u> knowledge and use of some part of the calendrical system, although many knew little or perhaps nothing of the mathematical system or computations of any chronological dates.

MATHEMATICAL APPLICATIONS IN CEREMONIAL LIFE

Mayan ceremonial life revolved around the 260-day calendar year which indicated both the daily luckiness and the daily ruling deity. This portion of the Mayan calendar also specified which God was the ruling power of each birthday. The chronological computations made by the higher Mayan priests designated the ceremonial life of the Mayan people.

THE ANCIENT INCA

"The Incan civilization is noteworthy as being the highest type found on the American continent, (its only rivals being the Maya of Yucatan and the Aztec)..." [Locke, p. 9] These South American Indians established their society during the twelfth century, acquiring much power and conquering many neighboring civilizations in the next three hundred years. The technology and architecture of these people were highly developed and their economy depended mostly on the harvest of maize, as did that of the Mayan Indians.

INCAN MATHEMATICAL DEVELOPMENT

The Incan people were known to have a great knowledge of mathematics. The Incan society had no use for money as all economic transactions were completed by barter and trade. Although the Inca knew nothing of renting, buying or selling, mathematical calculations were common in many facets of their everyday living. [Locke, p. 40]. A highly developed socialistic government required each Incan district to furnish information on matters such as revenues, taxes, census, and records of crops and herds. Detailed accounts of the matters of all the Incan provinces were kept, including records of citizens, llamas, soldiers, and so forth. To service demands from the government, the Inca developed not one but two mechanical counting devices: an abacus and the Incan quipu, a method of record-keeping involving knots tied in ropes.

The Incan knotted quipus originated with the Amazon Indians. Men of the Amazon Indian societies knotted ropes to denote the numbers of days of an absence. After the passing of each day, wives would loosen one knot until all knots in the rope were untied; an unknotted rope foretold their husbands' return. [Flornoy, p. 13].

Four hundred quipus have been excavated from Incan graves, the only place they have been found. The device differed from other ancient computational devices, such as papyri inscriptions, in that the quipus recorded only results and no intermediate computations; no evidence of intermediate stages of actual mathematical operations is contained in the surviving quipus. Nor did the Incan quipumakers employ two-dimensional recording material such as clay or papyrus as in the ancient Sumerican and Egyptian cultures. Tying the knots was not done in a methodical left to right or right to left manner; instead, a group of cords was connected and knotted using a method based on the position and color of the cords and knots. The quipus were quite advanced compared to other early mathematical devices, because they indicated the existence of a general recording system and mathematical concepts involving both verbal communication and craftiness. [Ascher and Ascher. p. 78] As practical as these quipus seemed, they were probably neither employed as calculating nor writing devices; quipus were not well adapted for calculations and were used instead to record the results of computations done with pebbles and grains. [Day, p. 13; Locke pp. 32, 37]

Quipus commonly recorded such information as population censuses, size of the military, counts of animals and cities, production of gold and accounts of goods stored in storehouses such as silver, clothing and maize. [Ascher and Ascher, p. 10; Day, p. 1; Locke, pp. 31-32] Some quipus recorded traditional songs and customs, stories, genealogies, dates and religious laws, by knotting the ropes in specific pattern in which both placement and color were important. [Day, p. 1] Various quipus were constructed with colored cords which distinguished the ropes from one another. Information recorded on uncolored quipus was arranged according to the objects' importance, beginning with that of most consequence and proceeding in order to the most insignificant. [Locke, pp. 40; Von Hagen, pp. 536-537, 561-564].

Although the quipus seemed of simple construction, knot-translations could become somewhat complicated and required a verbal translation by a commentator, a 'quipu-camayoc'. These interpreters, or 'quipu-camayu', meaning "he who has charge of the accounts", were responsible for the construction and translation of the quipu. Each quipumaker developed his own style of knotting the quipus. The most trustworthy and honest Incan citizens were selected for these positions due to the importance of the work. Each quipumaker was responsible for correctly recording all accounts of the province in which he resided. The villages commonly selected four interpreters to manage the accounts; more populated villages employed a larger number of camayocs. [Locke, p. 40].

Pedro de Cieza de Leon talked to some of the old "rememberers" in 1549, who explained that

> "The Quipucamayos, which were the officers and intendants kept the account of all with their strings and knottes, without failing, setting downe what every one had paied, even to a hen or a burthen of wood, and in a moment they

did see by diverse registers what everyone ought to pay" [Locke, p. 38]

"They added up, and multiplied by these knots, and to know what portions referred to each village, they divided the strings by grains of maize or small stones, so that their calculation might be without confusion" [Locke, p. 39].

"knots counted from one to ten and ten to a hundred, and from a hundred to a thousand. Each ruler of a province was provided with accountants, and by these knots they kept account of what tribute was to be paid... and with such accuracy that not so much as a pair of sandals would be missing" [Von Hagen, p. 562],

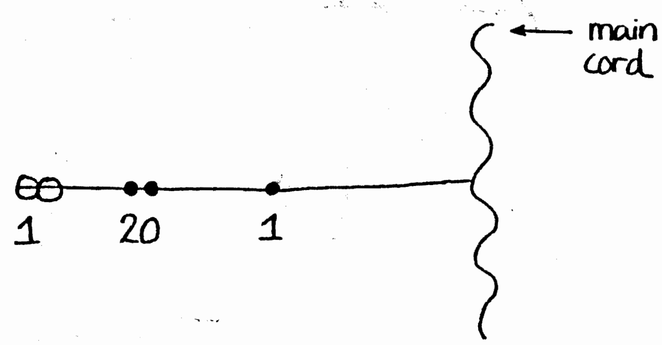
These passages portray both the simplicity and accuracy of the Incan quipus.

The quipu was constructed from one main, thick grayish-white cotton or wool cord to which smaller, thinner, sometimes colored <u>pendant</u> cords were attached. Each of the smaller, attached cords were knotted at intervals by clusters of knots called interval knots. Even smaller knotted <u>subsidary</u> cords were hung from the pendant cords. [Ascher and Ascher, p. 56; Closs (1986), pp. 266-267; Von Hagen, pp. 561-564] Data was stored on a quipu either as a single number, multiple numbers or number labels; most quipus were of the first kind. The recorded information was separated such that each quipu cord represented one item. Each knotted "cluster" on a cord had zero to nine knots, and recorded one digit of each number being recorded on that particular string; each digit denoted one higher power of ten. Thus, numbers were recorded in decimal notation. [Closs (1986), pp. 268-272] The clusters were separated by spaces of unknotted rope to distinguish knots in one cluster from those in the next. Leland Locke was able to show that knots placed at the lower ends of the cords of a particular quipu represented units, knots in the cords' middle denoted tens, and at the top of the cords, at their attached ends, the knots represented hundreds. [Day, pp. 15-17]. Thus the values of the clusters increased by one higher power of ten when translating the cord from its dangling end to its end attached to the main cord. Values greater than one thousand were rarely if ever recorded due to the limited accounts of each village. The system, however, included the ability to express <u>any</u> number.

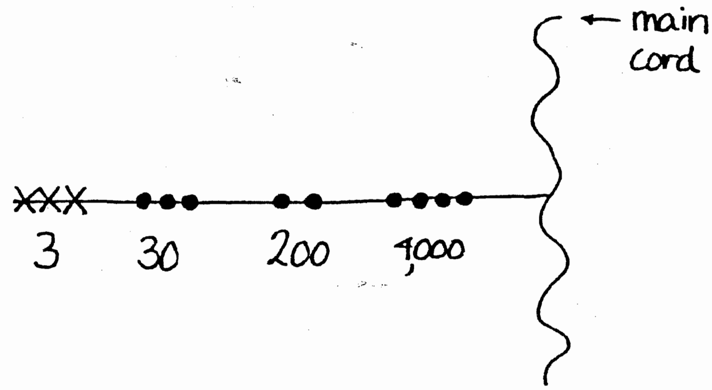
[Ascher and Ascher, p. 29; Locke, pp. 39-40]

Only three types of knots were used to record numbers in the Incan decimal system: multiple long knots, a special figure-eight knot and overhand single knots. Multiple long knots were constructed by two or more turns in the rope and represented only the units two through nine. Each turn of these knots, treated as one knot in itself, was easily distinguished and represented one unit in the numbers' units position. Because a long knot could not be constructed from only one turn in the rope, the number one was represented by a special figure-eight knot only in the units' position. [Closs (1986), pp. 268-272; Day, p. 17] Single knots recorded all higher powers of ten; each single knot represented one of the higher powers of ten.

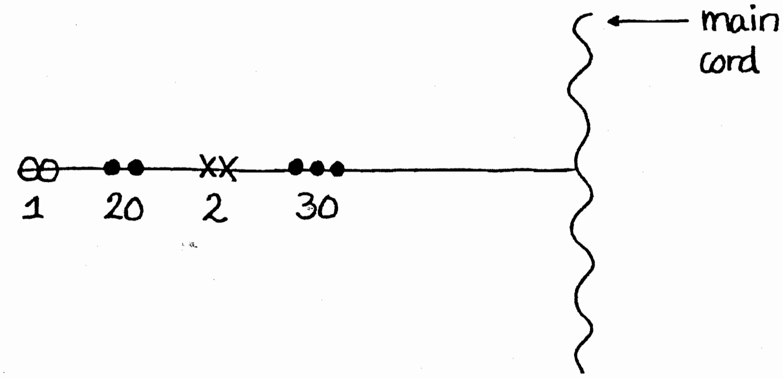
The first of two special features of this Incan recording method was the space placed between each knot cluster to separate the powers of ten. The number 121 would be recorded as one figure-eight knot at the dangling end of the pendant cord; a space; two single knots in the tens position denoting twenty; a space; and one single knot at the attached end of the cord representing one hundred:



The second feature was the use of no more than nine knots in any cluster, making the system a true decimal system, [Locke, p. 15] As another example, a surviving quipu records the number 4,233 as four knot clusters representing each of the four powers of ten like so: a three-turn long knot on the dangling end of the cord portraying the three units; a space; a cluster with three single knots representing thirty; a space; two single knots one position higher representing two hundred; a space; and four single knots at the attached end of the pendant cord denoting the four thousands.



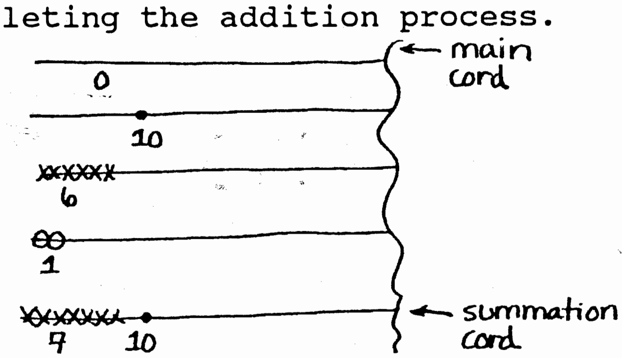
Quipus could record a group of numbers on one single cord, though this required a slightly modified method. When one quipu cord contained more than one number, cords required a symbol to separate each recorded number. The problem was resolved by separating the multiple numbers by the characteristic turns of the long knot or the special figure-eight knot occurring only in the units position. Because these two knots were the only knots placed in this position, they indicated the boundary between multiple numbers. If one of the numbers had no knots in the units position, the separation of the multiple numbers was indicated by a slightly larger space than the spaces separating the single knots representing consecutive powers of ten. A cord holding three single knots at its connected end; a space; two long knot turns; a space; two single knots; a space; and one figure-eight knot represented the numbers thirty-two and twenty-one, separated by the first number's units position (the two turns of the long knot):



[Closs (1986), pp. 272-274]

The Aschers concluded from their research and inspection of the specific patterns of the quipus that the Incan quipus indicate the extent of Incan mathematical knowledge, including, probably, addition, division into equal parts, division into simple unequal fractions, division into proportional parts, multiplication of integers by integers, and multiplication of integers by fractions. These conclusions are derived only from the comparison of patterns and hypotheses about the quipus, and cannot, in fact, prove the mathematical advancement of the Incan Indians. [Ascher and Ascher, pp. 151-152]

A summation process has been indicated on approximately one-fourth of the surviving Incan quipus. These large and complicated quipus are separated into groups of cords, each group containing a **summation cord** which records the sums of the numbers in each group. Knots were aligned as are the figures that a modern-day accountant records, column by column as in a ledger: "the knots of each number and each thread were placed in a line with each other in the same way a good accountant places his figures to make a long addition sum " [Locke, p. 40]. When the cords were spread out so that each lay horizontally and all knots were aligned in columns of powers of ten, the summation cord would be at the bottom, similar to the layout of an addition problem done on paper. Numbers in each column from the cords in the group were added and the result knotted into the summation cord corresponding to each power of ten. As an example, one quipu includes a group of knotted strings containing four cords. The first cord is blank: its value is zero; the second cord has one overhand knot placed in its middle; cord three has a six-fold long knot on its dangling end; the last cord carries one figure-eight knot on its dangling end. Cord two has a value of ten; cord three a value of six; and cord four a value of one. The summation cord contains seven long-knot turns in its units position and one knot in the tens position. The sum of the four cords is equivalent to the number knotted into the **summation cord**, thus completing the addition process.



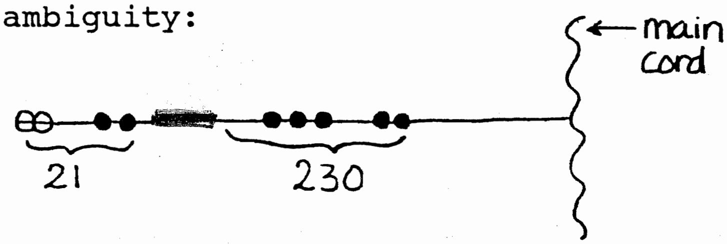
[Day, pp. 15-17] Note: it must be emphasized that one cannot clearly identify the method of the addition process using the Incan quipus because no actual addition process has been observed. [Ascher and Ascher, pp. 93-94]

The Incan Indians are one of only three cultures to have developed a symbol for zero within their numerical system. The Aschers divide the concept of zero into:

> "...first, the understanding that positions containing nothing contribute to the overall value of a number; second, that there must be a way of representing nothing; and third, that when the representation of nothing stands by itself, it is also a number. "
> [Ascher and Ascher, p. 30]

The Inca displayed a zero simply by a space: the absence of a knot in a quipu cord represented the absence of a number or zero units in a specific position. Using this system, the Inca were able to express not only the absence of a number, but also the existence of the number zero.
[Ascher and Ascher, p. 30; Locke, pp. 17-18]

A complication arose when no units occurred in one of two multiple numbers recorded on the same cord. Did the space denote a void in the units' position or was this just a space placed to separate multiple numbers? The following quipu cord which represents the two multiple numbers 230 and 21 portrays this ambiguity:



The knots could be translated as either the number 2,321, or as the two multiple numbers 230 and 21. The ambiguity is resolved by a comparison of the controversial cord with other pendant cords in the quipu. Hopefully, another cord in this quipu would have a knot in the position between the two numbers, (marked in red ink for simplicity) and the space would be easily recognized as a zero. If, however, all cords in the quipu contained an empty space in the same position, the problem was more serious (and probably unlikely). Spaces separating multiple numbers not containing the characteristic long knots or figure-eight knots were made somewhat larger to indicate a boundary between the numbers.

Incan knots were carefully aligned, and each cluster was knotted in the same relative location and an identical distance apart on each cord, which reduced any ambiguity, if not eliminating it entirely by spacing size; inter-number spaces were larger than inter-digit spaces. The efficiency of the Incan system lies in the fact that the units position is identifiable from the other positions by the type of knot used to represent the units.

Some quipus recorded multiples of one or two specific numbers and have repeated them, indicating an emphasis of some sort. For example, one such quipu is dominated by the numbers thirteen and twenty-six; all but three cords record either thirteen or its double. Another quipu is possibly related to some sort of calendrical system. Its emphasis of the number nineteen and its multiples may have associated this quipu with the alignment of cycles based on the sun. Seven pendant cords at the end of the quipu display a combination of nineteen knots, suggests a possible Incan nineteen-year calendrical cycle made up of seven parts.

Another feature of the Incan knot-system which extended its versatility was the use of colored cords. The different colors represented each cord's association with other cords [Closs (1986), p. 267].

> "... the quipumaker alone had to recognize and recall color differences and use them to his advantage. His color vocabulary was large; it was not simply red, green, white and so on, but various reds, greens and whites. ... his task was to choose, combine, and arrange colors in varied patterns to express the relationships in whatever it was that he was recording." [Ascher and Ascher, p. 61]

Each different color attached a specific meaning to the cord.

For example, the color black represented time; a black knotted cord recorded a date. A red knotted cord depicted the current Incan army or king; knots on a green cord denoted the number of Incan enemies. Yellow cords represented gold; knots on these colored cords recorded the amounts of gold in the province. Similarly, white knotted ropes displayed the quantities of silver. [Flornoy, pp. 115-117] Other quipu colors included carmine to represent the Inca themselves; brown to portray another tribe; gray, to denote the Incan provinces; variegated to represent the government, and blue, yellow and white to represent religion. [Locke, p. 16]

One colored quipu was constructed of a thick black rope to indicated time; thousands of little knots on this colored rope indicated emptiness. A red rope connected to the black was knotted four times to denote the fourth year of the Incan reign; to this last knot a brown thread with ten smaller knots was attached, indicating ten little provinces established during this year. To each of these ten knots a green thread was attached which contained thousands of new and very tiny knots to represent the number of enemies killed in battle. [Flornoy, pp. 115-117]

The Incan mathematical device employed by citizens in everyday calculations was not the quipu-knot method, for knot-tying took much time and thus became inconvenient in daily operations. Rather, a simple abacus was used consisting of a counting board and pebbles or maize kernels.

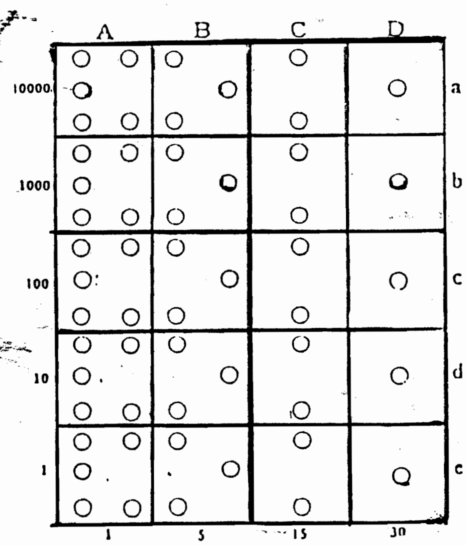
"...they do draw so many grains from one side, and adde so many to another, with a thousand other inventions."

"These Indians will take their Graines and place five of one side, three of another, and eight of another and will change one graine of one side, and three of another. So as they finish a certaine account, without erring in any poynt and they sooner submitte themselves to reason by these Quippos, what everyone ought to pay, then we can do with the penne."

[Locke, pp. 37-38].

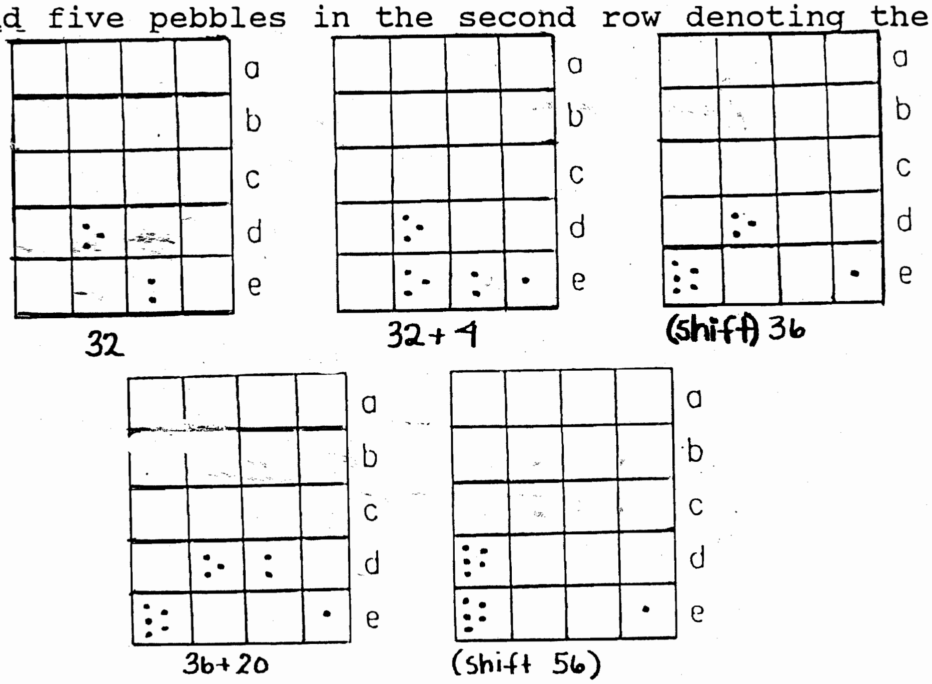
Father Jose de Acosta discovered that performing calculations on the Incan counting board proved more efficient than the modern-day use of ink and paper for that purpose. [Day, p. 32] The Incan abacus was a rectangle divided into twenty squares each containing a number of small circles and dots: a counting board on which computations and results were figured. The small circles in each square were empty holes to be filled with pebbles or grains of maize as calculations were performed. Pebbles and grains were easily placed and moved on the counting board to complete calculations quickly.

The rectangular abacus was constructed as a grid with horizontal rows and vertical columns. The vertical columns represented the quipu decimal notation in which each column corresponded to an individual quipu cord. The squares of each column represented clusters of each quipu cord, beginning with the units position on the bottom, continuing to the highest power of ten at the top. The horizontal coordinates represented multiples of each power of ten. Pebbles placed on the bottom row represented units occurring in the number; those placed on the fifth level represented ten thousands. The number of holes in each of the twenty squares varied: each square of the first column contained five holes, those of the second column contained three holes, squares in the third column contained two holes, and those of the last column on the right each contained only one hole. [Day, pp. 13, 31-37]



[Day, pp. 35-36]

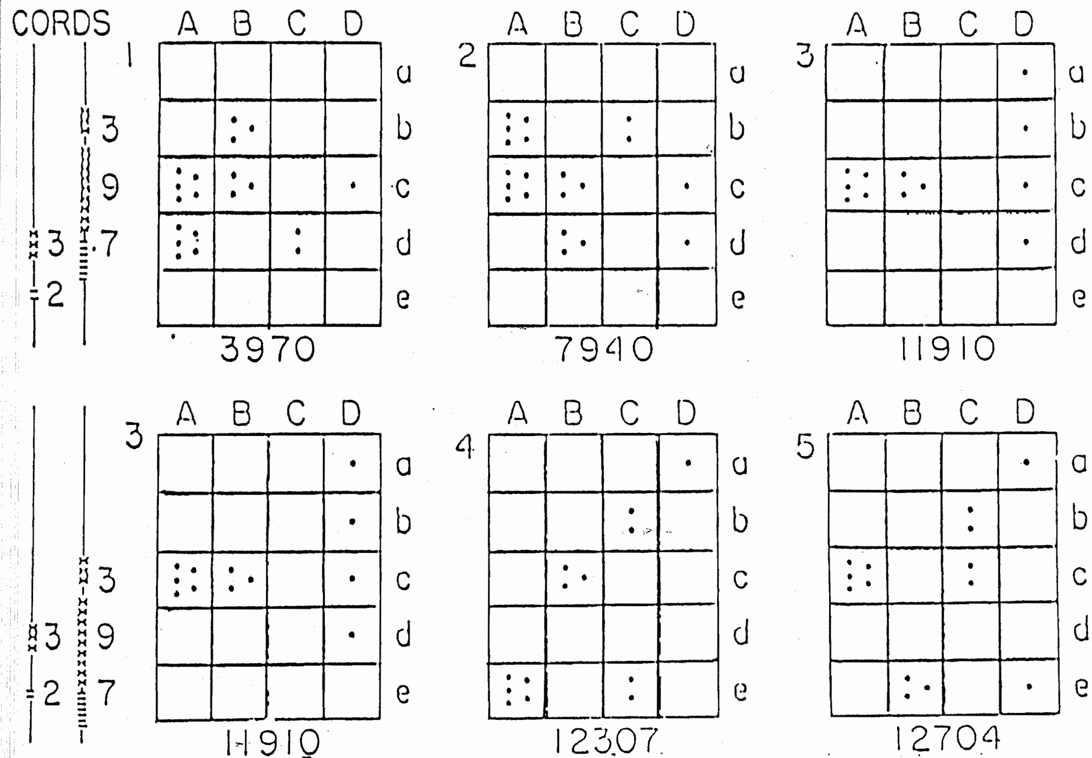
Addition on these counting boards was done by placing the counters on the designated holes in each square. Each square was only allowed a designated number of pebbles; when one square was filled, the next square to the right would be used. The first number in each addition was transmitted from the quipu onto the counting board. To represent the number thirty-two, for example, the third square in the bottom-most row would be filled with two pebbles to denote two units. Three pebbles were placed in the second square in the second row to record thirty. To add twenty-four to thirty-two, the four units were added to the two units already represented by the first row by filling two more holes. The six pebbles were shifted into the five holes of the left-most square and the single hole of the right-most square in the first row. To combine the twenty units with the thirty units within the second row, the five positions in the left-most square would be filled to represent fifty. Thus the result of fifty-six was represented as six pebbles in the first row denoting six units and five pebbles in the second row denoting the five tens:



No histories consulted recorded any evidence of either knot-method or abacus subtraction.

Incan multiplication was more complicated than their addition process. Two quipu cords representing the two numbers to be multiplied were placed vertically next to the counting board so that the units positions of the cords and the counting board were both located on the bottom. In this manner it was not difficult to compare the abacus and the numbers in the quipu.

To multiply two numbers together, the larger number was multiplied by the highest power of ten occurring in the smaller number. For example, when multiplying 24 and 456, 456 would be multiplied by ten and this product would be recorded on the abacus. It can be assumed that the product of any number and any power of ten was easily calculated and could be recorded without recording intermediate calculations. After the initial intermediate product was recorded, this number was added to itself as many times as the highest power of ten occurred in the smaller number. Therefore, in the previous example, 456 x 10 = 4560 would be added to itself only once, because the highest power of ten occurred only twice in the smaller number. The same process was then done with the larger number and the second highest power of ten of the smaller number. In the previous example, the product of 456 and one would be added to itself three times. This process was continued until all powers of ten of the smaller number had been exhausted. The intermediate products were added to get the final product. [Day, p. 38]



[Day, pp. 35-37]

The correspondence between the knotted segments of the quipus and the digit positions on the counting boards helped minimize errors when using this method. [Day, p. 38]

DEVELOPMENT OF FRACTIONS

The Incan quipus supply no evidence of an Incan fractional system. What does exist, however, are records of even division into parts and the use of common ratios. As in the summation quipus, no calculation methods have actually been discovered, but some quipus do portray division into parts. One quipu recorded a total value of two hundred, consisting of two groups each of which represent one-half of two hundred. One of the two pendant groups itself contains two groups each representing fifty, half of one hundred. The other pendant group is divided into six equal cords representing the value of one hundred divided into six roughly even parts: 16,16,17,17,17,17. Another includes summation cords containing multiples of eight and pendant cords each recording a value of one-eighth of the sum.

GEOMETRICAL DEVELOPMENT

Geometrical applications in the Incan lifestyle were found within their architecture, as well as on their pottery. Temples were constructed of rectangular one room units placed around a rectangularly-enclosed courtyard. A trapezoidal opening within the walls supplied a passageway. Stones used in construction were cut into rectangular blocks such that each fit snugly with the block surrounding it. Incan pottery was decorated by geometrical repetitions and symmetric designs, using rotation and reflection, double reflection, and vertical reflection. Mirrored reflections were also present in Incan pottery decorations, storehouses, designs on fabric and temple walls [Ascher and Ascher, pp. 50-7]

INCAN CALENDRICAL SYSTEMS

Although little is known of Incan calendric systems, an Incan calendar consisting of twelve months was discovered, thought to have been based on the observations of the sun and moon. The Inca may have added one or two days to certain months in order to catch up with the solar year. An Incan textile calendar was found consisting of ten rows and thirty-six circles each, possibly representing a 360-day year. [Aveni, pp. 219-226]

MATHEMATICAL APPLICATIONS IN DAILY LIFE

The Inca developed an accurate knowledge of the movements of the sun, moon and a number of the planets. Sophisticated astronomical observations were performed in order to fix dates for agricultural and ritualistic purposes. Astrologers secured the dates for sowing and harvesting the crops and for celebrating the festivals of the sun. One quipu was discovered to record accurate calculations of the orbital periods of Mercury, Venus, Jupiter and the moon. This may suggest that the Incan astronomers were able to predict the date of lunar eclipses, similar to the accomplishment of the Mayan astronomers. [Day, pp. 19-31]

The quipus also recorded counts of the lower ranking citizens of each village and province. The first cord showed a census of men over sixty years of age; the second recorded those between fifty and sixty years of age; the next cord represented the next lower age bracket, until every baby was counted. Different sized and different colored cords represented specific traits in the citizens, such as being married or single. Resources such as crops, agricultural produce, herds of animals, stores of wool and cotton, weapons and military supplies were also recorded on the quipus. Some quipus also "recorded" laws, rites, treaties, speeches and history of the Inca by a specific placement of colored knots. Quipus were also used in a type of messenger service. Trained runners were stationed in pairs at interval of about one mile along their highways, running at top speed and handing their quipus on, as in a relay. In this way, they could transmit a message two or three hundred miles in twenty-four hours. [Day, pp. 38-40]

By combining the various knot-tying methods and the specific patterns of colored cords, the Incas were able to record numerical accounts and records, as well as songs, peace negotiations, calendars and Incan histories. The Incan Empire wanted to be known as <u>the</u> "culture bearers"; the history they recorded ignored all undesirable information of their past, and thus passed on only positive information about the former Incan cultures. [Day, pp. 29, 39; Von Hagen, p. 564] It is therefore no wonder that the quipu knot system became so important to the Incan people: the counting device not only kept all accounts and records for each province, but also carried the Inca name into the years ahead.

MATHEMATICAL APPLICATIONS IN CEREMONIAL LIFE

The knotted quipus often played an important part of the Incan cultural ceremonies and their superstitious lifestyle. Women were known to knot the number of their lovers on strings and throw them into the fire during their husbands' absence; this practice supposedly purified the women. Quipus buried in graves contained no information about the living: this was believed to give the dead power over those still alive. These grave-quipus were thought to contain magical numbers. Nordenskioeld showed that the a number of the grave-quipus were calendars indicating days and like the Mayan codices were "nothing but books of divination and prophecy" [Day, pp. 9, 19].

Quipus supposedly were used as a magical device to create and manipulate a variety of spells. "The act of <u>tying</u> a knot", says Dilling, "implies something 'bound' and hence the action becomes a spell towards hindering the actions of other persons and things." [Day, pp. 42-43] In the same manner, loosening a knot removed the spell caused by the knot. The knots tied with a magic spell were sometimes blown or spit on; this was done to increase the power of the spell. These Magic Knots were thought to have power over weather, disease and death, sex, and spirits; the Inca believed that by chanting, tying and untying a knot, sicknesses could be healed. Some colored quipus were used as amulets: white cords indicated health; black cords portrayed disease and death; red represented the blood of life. Cords made out of wool supposedly possessed great magic power. [Day, pp. 42-68]

COMPARISON OF MAYAN AND INCAN MATHEMATICAL SYSTEMS

In this section, the development of Mayan and Incan mathematics is compared using the subjective numerical scale mentioned in the introduction, taking into account each of the significant topics discussed concerning these systems.

The Mayan <u>vigesimal</u> numbering system was both uncomplicated and efficient to use; only two symbols were needed to record all numbers employed by the Mayas. Both the abacus and the bar and dot method enabled the Mayan Indians to record and compute large numbers: the bar and dot system was observed to record numbers as large as 1,872,000, and in theory could record those of any size. The abacus was known to compute easily up to one billion. The head-variant numerals were also established as an efficient although not as commonly-used numerical recording system. The calendrical method was developed into a detailed system for the computations for dates and elapsed periods of time. Using this method, the Mayan priests were able to record and calculate time to any extent needed. For the constructions of the efficient bar and dot system, the ease of the calendrical conversion and effective applications of the abacus, the Maya receive **one and three-fourth points**, deducting only for the complication of deciphering large numbers.

The Inca developed a decimal system, the efficiency of which was perhaps hindered by the complexity of the quipus.

Deciphering the knotted quipus was not a practical nor time-efficient method of recording numbers; an interpreter was needed to translate each quipu. Another factor complicating the Incan method was the coloring of certain quipu cords; it was necessary to decipher each knot's position and color to extract the message. Recording large numbers required no extra work, although, again, the numbers were difficult to interpret. The Incan Indians receive **one point** for the versatility of their quipu knot-tying method; the score was reduced for the complications involved in the deciphering of the knotted cords.

The Mayan method facilitated by all four mathematical operations, although most calculations represented chronological counts. Addition was accomplishable by hand and by using the Mayan abacus. Two simple rules were used to easily complete the addition and subtraction process in the bar and dot system. Multiplication was also an efficient, simple process if three simple rules were followed. A conversion factor was used for every chronological computation: the vigesimal and calendrical systems differed by two units in the second level, and it was necessary to compensate for this. This conversion became somewhat complicated when using large chronological dates. Only one consulted source discovered the existence of a Mayan division process and although it was not frequently used, division was shown to be of the same efficiency as multiplication. The total score awarded to the Mayan mathematical development is **one and three-fourth points**, the deduction being due to the vagueness of the division process. The Incan people were able to record statistical accounts as well as their histories on the quipus. Both the quipu and their abacus were employed in the addition process. The quipu addition method was simple and efficient, using a summation cord to tally results. Addition on the abacus by shifting pebbles into its holes was discovered to be more efficient than using ink and paper, although the lengthy multiplication calculations performed on the counting board became tedious and inefficient. Both subtraction and division were mentioned in Incan research; however, both processes have been interpreted from the quipus, supplying no concrete evidence of Incan knowledge. This culture receives a total of **one point** for the ease of their addition process on both the quipu and abacus and the existence of the multiplication method upon the abacus.

The Maya receive the maximum of **one point** for the understanding of the concept of zero, as well as the development of its distinct symbols and efficient placement within the Mayan numerical system. The Inca receive **three-fourth of a point** for an invention of a zero symbol. The placement of the Incan symbol as a blank space in a cord was theoretically ambiguous, and in some cases became difficult to interpret.

The Maya did not incorporate fractions into their numerical system, although one symbol representing one-half of a period was discovered as a part of the hieroglyphic system. Other evidence of Mayan knowledge of a fractional system has not been discovered. The Mayan numerical system receives **zero points** for the almost nonexistent fractional system. Evidence of an Incan fractional system was not apparent in the use of quipus; they receive a score of **zero points** for this.

Geometry was not a significant aspect of the Mayan civilization. Some of the architecture does reflect some knowledge and understanding of the four worldly directions, as well as the construction of both right and isosceles triangles. The score for the geometrical knowledge of the Mayas is **one-half point.** Incan knowledge of geometry was limited to their architecture and pottery; the Incan culture also receives **one-half point.**

The Maya developed a highly effective method of recording dates and counting elapsed amounts of time. The calendrical system was at the core of Mayan society; it determined the timing of farming, religious and ceremonial events. The 360-day cycle was developed as an effective calendar system analogous to the modern-day calendar. The Maya were also one of the first cultures to develop a starting point in their calendar, which is similar to the zero year in the Christian calendar. A score of **two points** is awarded to the effective and intricate Mayan calendrical system and its use within the society.

The Inca developed a twelve-month year similar to the one used today, but little is known of the efficiency of date-keeping. It has been discovered, however, that one or two days were added to the year to correlate to the Incan year with the solar year. The Incan calendrical system receives **one point.**

The Mayan mathematical system was used mostly by the elite priests who were the only citizens to perform the day counts and other chronological calculations. The extent of the lower-ranking citizens' use of the numerical system involved only the employment of the 260-day cycle which determined the days of Mayan harvest. For the centralized use of the mathematical system, the Maya receive **one and one-half point.** The Inca were able to incorporate their mathematical system into their lives with ease. Although only a very small percentage of the population was able to "read" the quipus, the records they kept were still of value to the entire community of each Incan province. Many of the daily accounts were recorded on the knot-string devices; the quipus were thus employed by a number of trade merchants. For the ease and wide range of the system's use, the Inca receive **two points.**

Ceremonial applications of Mayan mathematics included calendrical computations used to record or predict religious or ceremonial events. For this application, the Mayan system is awarded **one and one-half point** for the incorporation of the mathematical system. The Incan ceremonial application of their mathematical system included their 'magical knots' and the magic spells. The Incan astrologers were able to record the event of any new agricultural seasons which often developed into ritualistic festivals. The four most important Incan ceremonies followed the two solstices and equinoxes. These dates were apparently observed by the astrologers. The Incan use of mathematics within the ceremonial lifestyle was awarded **one and one-half point.**

The total score received by the Mayan culture is **ten points.**

The total score received by the Incan culture is **seven and three-fourths points.**

In this author's opinion, the Maya were the more mathematically adept of the two cultures.

NORTH AMERICAN INDIANS

The study of Native North American mathematics is quite brief. Unfortunately, little research has been done in this direction compared to the developed studies of the mathematical systems of Mesoamerica and South American Indians. However, including a discussion of the North American Indians at this point expands the comparison of Indian cultures by supplying an enlarged outlook on the mathematical achievements of Indian civilizations.

The Dakota Indians were able to develop an effective numbering system. The Dakota, as well as the Algonquian and Iroquoian Indians, used a decimal numbering system. The Dakota could count past one million with their system, as could the Aztec, Maya and Inca. The Dakota's numerical system was developed around the positive integers used within the addition and multiplication operations. [Closs (1977), pp. 13-15] Both the Dakota and the Ojibwa Indians carved single strokes on grave posts to record numbers; in some cases different types of strokes were used to denote different objects being counted. [Closs (1977), pp. 13-15]

The Inuit tribe of Alaska developed a well-structured numbering system able to count above one hundred and that had the capability of being extended. The system was developed around groups of twenty, similar to the Mayan system, and was based on man's twenty fingers and toes, as opposed to the more common decimal systems that were based on the fingers only. [Closs (1977), pp. 15-16]. The Inuits divided the body into four groups: the upper and lower digits and the left and right digits. The word for the number five was related to "arm"; "ten" was related to "top" and referred to the upper ten digits on the hands. The word for twenty was related to "limbs" and referred to counting on all four appendages of the body. The numbers six through ten were related to the right hand; eleven through fifteen were based on the left foot and sixteen through twenty on the right foot. [Closs, pp. 135-139] One person would equal the number twenty, five people would constitute one hundred, equivalent to one **bundle.** [Closs (1977), pp. 58-60]

The Inuit society had no great use for large numbers, and thus the system was developed for numbers less than one hundred, with the capability of expansion where necessary. Most school-boys easily counted to one hundred and above; some men could count up to four hundred. [Closs (1977), p. 62] The Inuit people, however, developed a negative attitude towards large numbers. There exists an old Copper Eskimo tale of two Indian hunters who starved to death while arguing whose animal had the most hairs; this represents their belief in the nonexistent need for such great quantities. [Closs (1986), pp. 15-16]

The Ojibwa Indians also developed an effective numbering system containing few restrictions, which enabled them to count into the millions. Their own language includes a word for one billion. [Closs (1986), pp. 13-14] The language also includes words used as multipliers indicating the number of times a process is to be completed. Distributive numbers also existed to express the number of things given to each of a group of people. The numbering system combines base five and base ten, constructing numbers from groups of five and ten:

6:	1 + 5	100:	1 x 100	1000:	10 x 100
7:	2 + 5	200:	2 x 100	2000:	2 x 1000
8:	3 + 5	300:	3 x 100		
		400:	4 x 100		
		500:	5 x 100	1000000:	1000 x 1000
11:	10 + 1				
12:	10 + 2				
20:	2 x 10				
30:	3 x 10				
40:	4 x 10				
50:	5 x 10				

The Coahuiltecan, a Texan tribe, developed a numbering system which was also constructed with multiplicative and additive principles:

3: 2 + 1

6: (2 + 1) x 2

7: 4 + 2 + 1

8: 4 x 2

9: 4 + 5

10: 5 x 2

11: 10 + 1

12: 4 x 3

13: 12 + 1

14: 12 + 2

15: 5 x 3

16: 15 + 1

17: 15 + 2

18: 6 x 3

19: 18 + 1

30: 20 + 10

40: 20 x 2

50: 40 + 10

The overall ability of all Northern American Indian numbering systems to be used in calculations was very low. The addition, subtraction and multiplication operations could be done only with the aid of fingers, pebbles, sticks, kernels of maize or other available counters. The Native Americans had little need of mathematical operations, and developed no idea of mental arithmetic. [Closs (1977), p. 16] The traditional method used for the few needed calculations was the method *of placing counters on a counting board similar to* a primitive abacus. The counting board is considered to be a "technological improvement" which grouped pebbles into identical piles representing identical values. A surface divided into strips or columns which represented the place values in a given numbering system was used to group together all those items of each numerical value.

Although most calculations were done upon the counting board, a few other computing and counting methods were used. In some cases, hand signs were used to count; each number was represented by a unique sign, based on twenty fingers and toes. [Closs (1977), p. 11] Some societies used beans and sticks as counters: the Nevada Paiute would use ten tally sticks inserted into the ground to represent ten fingers and indicated numbers on them with a movable counter. The California Yokuts used the same process with twenty-five sticks in a row. [Closs (1977), p. 13] Another group would count the numbers by moving a stick along a circle of forty stones that had larger openings between every ten stones. Notches

cut into sticks were commonly used to record numbers in the Western United States, where sometimes the tenth notch was marked differently. [Closs (1977), pp. 13-15]

The use of <u>fractions</u> was found in some Indian cultures. The words for simpler fractions were common, but recorded only rarely; the only fractions found recorded were unit fractions. The best system was developed by the Iroquoian Tribe of the Onondaga Indians, containing:

1/2

1/3: "three divided"

1/4: "four times divided"

The Algonquian Fox tribe also developed a fractional system:

1/2: "one half"

2/2: "two halves"

1/4: "one fourth"

[Closs (1977), p. 4]

<u>Development of geometry</u> and its use by the Native North American Indian societies was slight or simply not well-known. Evidence of their knowledge exists within the architecture of the Omaha; the use of the circle has been found in their construction of buildings. The Chavante of Brazil also used the circle in their architecture. [Closs (1977), pp. 21-22]

<u>Mathematical applications in the daily life</u> of the Northern American Indians seems to have been limited to the distribution of food and supplies and counting games. These number games included games involving categorizing groups of

sticks into certain numerical divisions (by the Fox Indians), choosing certain combinations of bone and brass that denote a specified score depending on their colors (by the Ojibway), and dice games that were played by many North American Indians. [Closs (1977), pp. 9-10]

The mathematics of the Native Americans is often overlooked as a contribution to modern mathematical development. These cultures, however, established not only a number of adequate numerical systems, but also original methods of calculations. Because evidence of development of geometry, calendrical systems and the incorporation of mathematics into the daily activities is vague or nonexistent, the scores of the North American Indians when compared to the Maya and the Inca are relatively low, and thus are not included in the competition. This author, however, found it necessary to state the accomplishments of these people and to consider them as contributors to the general development of mathematics.

To the largest extent, this section of the Native American research was developed from the A Survey of Mathematical Development in the New World and Native American Mathematics, both by Michael Closs.

THE ANCIENT EGYPTIANS

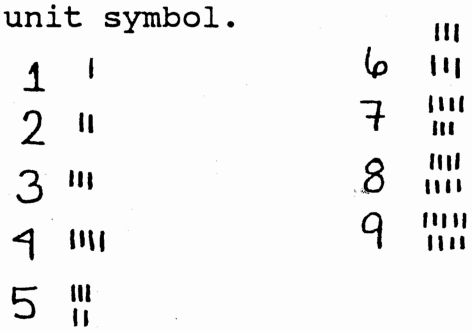
The Egyptian civilization established itself as one of the earliest literate societies of the ancient Near East over five thousand years ago in the Nile Valley of central Egypt. During the first four thousand years of Egyptian history, the swamps along the Nile valley were cleared and settled by people from Africa and possibly Asia, intermingling to form the ancient Egyptian culture. After developing a number of small and sparse kingdoms and centers of civilization, the Egyptian people divided themselves into Upper and Lower Egypt. Upper Egypt was located within the Nile valley including the area leading down to the Delta, and was influenced by western Asia and Libya; Lower Egypt was located in the Delta itself and was influenced by the inhabitants of Africa. Lower Egypt developed rapidly and it was this society which invented the first documented calendar of 365 days in the earliest fixed dated in history of 4241 B.C. [Breasted, p. 15] The two cultures were united by the Pharaoh Menes in 3100 B.C., beginning the first dynasty of Egypt and a period of great cultural growth. After a period of Egyptian prosperity and independence, Egypt was conquered in 526 B.C., when the civilization was overpowered first by the Persians and later by the Greeks and Romans.

EGYPTIAN NUMERICAL SYSTEMS

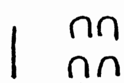
The ancient Egyptian civilization developed for integers a decimal system which did not include a decimal place holder. Two numerical systems were constructed; one was represented in hieroglyphic writing, another used the cursive script of the hieratic system. The numerical symbols of both systems were most frequently written from *right* to *left*, but were also discovered written in a vertical pattern read from top to bottom. [Gillings, p. 1]. The decimal system, employed more frequently than the hieratic system, was constructed similarly to the modern-day numerical system; however, only each *power* of ten was represented by an Egyptian symbol, whereas today's Arabic system includes a symbol for each *multiple* of ten. The Egyptian numerical symbols were:

	1		10,000
	10		100,000
	100		1,000,000
	1,000		

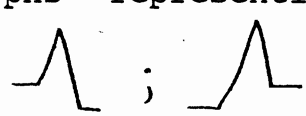
[Chace, p. 3; Freebury, p. 21; Heath, p. 64; Van der Waerden, pp. 5, 17] Each Egyptian numeral was constructed only from powers of ten; numbers were written using the hieroglyphic symbols which corresponded to the powers of ten occurring in each number. The numbers one through nine were written as the corresponding number of vertical slashes, which was the Egyptian unit symbol.



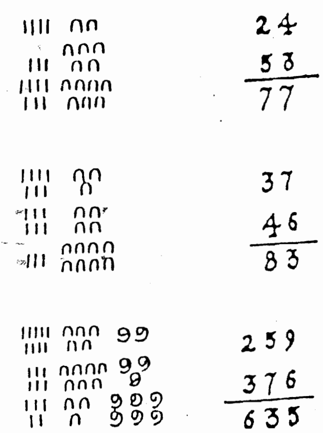
It was not required to separate the group of symbols of each power, because each power of ten was represented by a unique hieroglyph. The numeral forty-one, for example, was constructed of one ten to the power zero and four tens of the first power and was thus portrayed by one unit symbol and four ten-symbols:



Evidence of Egyptian addition and subtraction methods is vague; these simple operations were done by some method and only the solutions were transferred to the papyri.

Both addition and subtraction calculations were indicated by special hieroglyphs representing the addition and subtraction process: 

[Gillings, p. 6] It seems probable that the use of addition and subtraction tables was central in Egyptian computations, yet no evidence of these exists. [Gillings, p. 10] Evidence of the addition process occurred throughout Egyptian hieroglyphics. [Van der Waerden. p. 30] Only one rule was crucial: ten symbols of one order were written as one symbol of the next higher order. The process was completed by combining the symbols of each power of ten and converting each group of ten to a symbol of the next higher power. Calculating the sum of the numbers seventy-nine and twenty-two would be completed by first combining all symbols representing the units; combining nine and two would result in one unit symbol one symbol representing one ten. This ten was then combined with all other symbols representing tens, resulting in ten symbols of ten; these were converted into one symbol for one hundred. The final sum was written as one symbol denoting one hundred and one unit symbol to represent the result of one hundred and one. [Gillings, p. 1; Van der Waerden, p. 18] Here are some examples of Egyptian addition using the hieroglyphic system:



[Van der Waerden, p. 12]

The subtraction process was closely related to the addition method. If a number contained a greater amount of symbols in a specific order than the number from which it was subtracted, one symbol was borrowed from a higher order, similar to the Mayan process. One symbol from a higher order would be conveyed into ten units of the next lower order.

The Egyptian multiplication method became complicated, as it was constructed of two or more processes. Van der Waerden described the Egyptian multiplication process as a written operation with slow development. [Van der Waerden, pp. 18,30] Only two actual multiplicative subcomputations were used by Egyptian mathematicians to effect all multiplication: multiplying a number by two or by ten. A product was found by successively doubling the multiplicand and recording these results; multiples of ten would also be included where they aided the multiplication process.

For example, the product of thirteen and seven was computed by first designating one of the two numbers as the multiplicand or <u>base number</u>. Using thirteen as the <u>base number</u>, the first multiple was recorded as 1 13. Thirteen would then be doubled and the result recorded as 2 26, the two denoting the second multiple of thirteen; twenty-six represented the result of the second multiple. The second multiple was once again doubled and recorded as 4 52. This process was continued until the number in the left column exceeded the original multiplier; in the previous case, the process would cease at the fourth multiple; doubling this again would result in the eighth multiple, surpassing the number seven. Multiples from the left column were selected so that their sum produced the multiplier; the numbers chosen were labeled by the symbol " / ". The sum of these marked multiples resulted in the final solution to the multiplication.

13 x 7:

/	1	13
/	2	26
/	4	52
Total:	7	91

[Chase, p. 3; Gillings, p. 15]

Another example of the Egyptian multiplication process portrays the multiplication of twelve and twelve:

	1	x	12	12
	2	x	12	24
/	4	x	12	48
/	8	x	12	96
Total:	12			144

Division computations were completed by denoting the divisor as the base number and recording its multiples and if necessary, powers of ten, as in the multiplication process. The following example of dividing 1120 by 80 was found in the Rhind papyrus, and translated as literally "add beginning with 80 until 1120 is produced":

	1	80
/	10	800
	2	160
/	4	320
sum of "/"	14	1120

The result of dividing 1120 by eighty was the sum of the marked multiples. We would write this process as "1120 divided by eighty equals fourteen". [Van der Waerden, p. 22]

A symbol for zero had not been yet developed during the time of the ancient Egyptians, and therefore could not be recorded by their scribes or clerks. In the Egyptian mathematical papyri, a blank space on the paper did sometimes indicate a zero. In a sense, then, the Egyptians were able to understand and incorporate the concept of a zero value into their calculations.

DEVELOPMENT OF FRACTIONS

The Egyptians became obsessed with developing tables of fractions, and although they employed only the natural fractions occurring in their daily activities, some of these fractional calculations were so obscure that it retarded the process of Egyptian mathematical development. The Egyptian mathematicians developed simple decompositions of fractions, a process which Hogben stated as taking "... extraordinary pains to split up fractions like 2/43 into a sum of unit fractions... A procedure as useless as it was ambiguous." [Gillings, pp. 48, 71] The profound interest of accurate calculations with fractions originated from practical problems such as the division of food or supplies among families or troops; as with the Mayan computations, none of the Egyptian calculations involved currency or money. [Gillings, p. 105] The Rhind papyrus included tables in which fractions were reduced into only the unit fractions and those with a numerator of two. One of these tables dating back to 400 A.D. contains the numbers one through ten and each number's multiples of ten, one hundred and one thousand broken up into unit fractional parts up to tenths. Important fractions were given their own name and then broken down into unit fractions; Egyptian fractions were always written as the sum of integers and unit fractions. Unit fractions were represented by an open oval written above the number whose reciprocal the fraction represented. The oval was also the hieroglyphical symbol for and open mouth. One-twelfth, for example, was written as:

The calculations involving fractions caused the Egyptians much work and anxiety. [Freebury, pp. 22-23]

Only one specific fraction with a numerator other than one was found in any of the Egyptian papyri: the fraction two-thirds. Instead of being separated into unit fractions, this fraction was given its own hieroglyph:

The name given to two-thirds was "the two parts" and the name for one-third was "the third part". The Egyptians considered only two parts of a unit divided into n pieces: the first part of n-1 pieces and the remaining part. For example, a unit broken into five pieces would be considered to have "the two parts" consisting of four-fifths and one-fifth of the original unit. A fraction such as two-fifths was not directly representable. [Van der Waerden, pp. 19-22]

Evidence of employing fractions within mathematical operations was found in the Rhind Papyrus. One table displays the duplication of fractions; another portrays a method of duplicating the unit fractions 1/n by dividing two by n. Also found was a large table of calculations displaying a division process performed with fractions. Tables computing the products of fractions were included in the Rhind papyrus, although Chace stated that fractions were only multiplied by 2/3, 1/2 or 1/10. [Chace, p. 4] By using the multiplicative doubling process and recording the multiples of simple fractions, the Egyptians were able to complete fractional multiplications. These lengthy computations were stated to have been an everyday process of the Egyptian scribes of that time. [Gillings, p. 40]

The Recto Table, the most extensive arithmetical Table found in Egyptian papyri, contained a table of the odd numbers one through one hundred and one, all divided by two. The answer to each division was recorded first and then proven correct by the multiplication of the fractions involved in each answer. The Rhind papyrus also displays one calculation recording a fraction of 8 + 2/3 + 1/10 + 1/2190 ro (a unit of measure); fractional computation methods similar to these were used by the Greeks 2,200 years later. [Gillings, pp. 45-48]

The use of fractions in division was also noted in the papyri. To find one-third of a number, the Egyptians first found two-thirds of it and then halved this product. [Gillings, pp. 2, 22]

Other operations with fractions included the division process using a remainder. The Egyptians completed the division of nineteen by eight by using eight as the <u>base number</u> to produce nineteen. The <u>base number</u> was doubled as in the multiplication method, but was also halved to result in the correct solution.

		19 :	8	
multiplication portion		1	8	
	/	2	16	
halving portion		$\bar{2}$	4	one-half of eight = 4
	/	$\bar{4}$	2	one-fourth of eight = 2
	/	$\bar{8}$	1	one-eighth of eight = 1

Result: 2 4 8

from multiplication from halving

The division of nineteen by eight was computed by combining the <u>second</u> multiple of eight, <u>one-fourth</u> of eight and <u>one-eighth</u> of eight, resulting in nineteen.

[Van der Waerden, p. 23]

GEOMETRICAL DEVELOPMENT

Herodotus once stated that geometry originated during the annual flooding of the Nile; after every seasonal flood, the Egyptian surveyors redefined the boundaries of the land that had not been destroyed. [Johnston, p. 81] The Egyptians developed a geometry which was analogous to an applied arithmetic, consisting not of calculations of quantities of beer and bread but of areas and volumes. The Egyptian mathematicians computed the areas of rectangles and squares by multiplying their length times the breadth. As early as 2000 B.C., the Egyptians were able to calculate the area of a triangle by halving the base "in order to make the triangle square" and then multiplied the halved base by the height of the triangle. [Van der Waerden, p. 32] The area of a circle was recorded on the Rhind papyrus by using the square of eight-ninths of the diameter, resulting in a close approximation of pi: 3.1605. Gillings states that the Egyptians recorded the circumference of a semi-circle as the product of one-half, pi, and the diameter, using 256/81 as their approximation for pi. A formula for the surface area of a hemisphere was also discovered by the Egyptians this early: the product of two, pi and the square of the radius of the sphere, which was unknown to the Greeks until Archimedes' time of 250 B.C. The formula for finding the volume of a cylinder was also developed by these early mathematicians: the area of the circular base was recorded first and then multiplied by the height of the cylinder, the same process used today. The slope of the sides of a pyramid, as well as its volume were also computed by Egyptian mathematicians. The Egyptian geometrical development also included calculations of the frustum of a square pyramid and also the cubical content of a hemisphere, a value which was not rediscovered until over 3,000 years later. [Freebury, pp. 23-25; Gillings, pp. 137, 146, 185, 197 ; Johnston, p. 66; Van der Waerden, pp. 31-33]

EGYPTIAN CALENDRICAL SYSTEMS

The Egyptians developed two types of calendrical systems to regulate both their daily lifestyle and agricultural cycles. Egyptian astronomers found a correlation between the annual flooding of the Nile and certain celestial movements, and thus based their primary calendar upon the moon and the star Sirius, with a year corresponding closely to the true solar year, being only twelve minutes shorter. This combined lunar and sidereal calendar was designed to regulate religious ceremonies and everyday activities. Each day was assigned a specific name, some of which were derived from the various phases of the moon's cycle.

The Egyptians also developed a civil year of twelve months of thirty days; five "year-end" or epagomenal days were added to the end of the year to complete a year of three hundred and sixty-five days. These five extra days were dedicated to the Gods Osiris, Horus, Seth, Isis and Nephthys. The twelve months were divided into three seasons of four months each named in accordance with the planting cycle: the sowing period, the "coming forth" or growing period and the summer or harvest period. The twelve months were not labeled by names but were numbered with respect to the season in which they existed; the days other than the epagomenal days were labeled by numbers within their months. The years themselves were counted from the beginning of a King's reign, starting new with each King. [Breasted, p. 46] A date was listed by the name of the presently-reigning King, the year of his reign, the season in which the date fell, the number of the month, and the specific day of the month, respectively. This system, using both civil calendar and the agricultural seasons, was later adopted by the Greek and Roman civilizations.

Egyptian astronomers named the fixed constellations and divided those along the Zodiac into thirty-six "decans", for use as a star clock to calculate the time at night. A day was divided into twelve night hours and twelve day hours, a division still employed today. Compensating for the variation of the ratio of these two divisions between the summer and winter seasons, the Egyptians altered the length of the hours and constructed devices such as the sundial and water clocks designating the varying lengths of hours during the seasons of the year. A standard length of the hour was never developed in ancient Egypt.

Both the civil and lunar/sidereal calendars were already in existence during the time of the first Pharaoh of Upper and Lower Egypt and were noted as "the most scientific organization of calendars which had yet been used by man", by J.W. S. Sewell. [Gillings, p. 235] Otto Neugebauer labeled the Egyptian calendrical system as forming the "only intelligent calendar". [Chace, p. 43; Gillings, pp. 235-236; Moffat, pp. 66-67]

MATHEMATICAL APPLICATIONS IN DAILY LIFE

The Egyptian daily lifestyle used much of their mathematical knowledge. The Rhind papyrus displays distributions of wages, calculations of amounts of grain, conversion of various measures for grain and calculations of areas and volumes. Uses of mathematics were also found in Egyptian architecture. The Egyptian pyramids were built using precise mathematical measurements and calculations; the structures had a side-to-height ratio of 11:7, in which the ratio of one-half the perimeter to the height was 3 1/7. [Freebury, pp. 23-25; Van der Waerden, pp. 16, 29]

The Egyptian mathematical system employed approximation: if the result could not be computed, it was approximated and later refined if additional information became available. The Egyptians were problem-solvers, taking particular cases of calculations and generalizing from them. [Chace, pp. 38-40] Although the Egyptian mathematicians had a vigorously-developed mathematical knowledge, some researchers believed that the Egyptian system was not fully developed. Struik stated that "all available texts point to an Egyptian mathematics of rather primitive standards". [Gillings, p. 232] Early Egyptian mathematical calculations focused on the abstract part of mathematics; the mathematicians were not as concerned with the intermediate proofs or derivations of solutions as with the final results [Moffat, p. 30].

MATHEMATICAL APPLICATIONS IN CEREMONIAL LIFE

The Egyptian people incorporated a great number of ceremonial activities into their life. The change of the seasons was celebrated annually in accordance with their civil calendar. A new king was honored as was the transformation of a dead person into the Afterworld. All festivals were celebrated according to the Egyptian calendar systems, which recorded and regulated the various festivals and ceremonies.

An important goal of every Egyptian was to attain the honor of reaching the Afterworld. Much of Egyptian ceremonial life centered on this transformation. The Egyptian pyramids were constructed for this ceremonial transformation process.

Without the use of Egyptian mathematical skills, the building of the pyramids would not have been possible. The tombs were developed from approximately six and one quarter million tons of stone, each block averaging two and one half tons. The stones were fitted with a tolerance of one-fiftieth of an inch, a dimension similar to what might be used by a jeweler. The margin of error of the squareness of each side was very slight, portraying "an almost superhuman fidelity and devotion to the physical task at hand." [Wilson, pp. 54-55]

THE ANCIENT GREEKS

The Greeks developed their culture within a short time, settling first in the southern part of the Balkan Peninsula and islands in the Aegean Sea and quickly spreading into Asia Minor, Lower Italy and the African littoral. The peak of the Greek civilization was reached during the fifth century B.C., which saw a quick decline of Greek influence at the time of the rise of that of Rome. Within these few short centuries, much of what is called the "cradle of our culture" was developed and its influence distributed throughout the Ancient World. This period also saw the birth of Greek mathematics, later to become the foundation of many future sciences developed by modern civilizations. [Dantzig, pp. 15-17]

GREEK MATHEMATICAL SYSTEMS

Although European mathematics is often considered to have originated in the ancient Greek world, a large part of the Greek mathematical development began more than 2000 years after the Egyptians had completed their mathematical Rhind papyrus. The Greeks developed several numerical systems which were based on different units such as the Greek drachma or letters of the Greek alphabet; each notation was based on a decimal system. When dealing with monetary calculations, the Greeks incorporated a numbering system which used a drachma (the ancient Greek currency) as its primary unit. Other units such as units of weights or measure were also employed. [Tod, pp. 4-6]

The first Greek numeric notation, labeled as Herodianic or Attic, was found to be less advanced than that of the Babylonians. The Greeks used symbols similar to those of the Romans to represent the numbers of their system:

Ι	1	𐅅	500
Γ	5	Χ	1,000
Δ	10	𐅆	5,000
𐅄	50	Μ	10,000
Η	100	𐅇	50,000

Compared to the ancient Egyptian system, the Greek was more versatile; the Greeks developed symbols not only for the powers of ten, but also for intermediate multiples of five. Numbers were portrayed by these symbols written from left to right , with the highest denomination placed in the left-most position. Examples of Greek numerals:

ΓΙ 6 ΔΙΙΙΙ 14 ΗΓ 105

The Greeks developed a second numerical system whose twenty-seven "digits" were based on the Phoenician and Greek alphabets. These twenty-seven letters were divided into three groups of nine numbers each:

1 — 9 $\alpha, \beta, \gamma, \delta, \varepsilon, \varsigma, \zeta, \eta, \vartheta$
10 — 90 $\iota, \kappa, \lambda, \mu, \nu, \xi, o, \pi, ϙ$
100 — 900 $\varrho, \sigma, \tau, \upsilon, \varphi, \chi, \psi, \omega,$ ϡ
1000 — 9000 $,\alpha, ,\beta,$ etc.

[Van der Waerden, p. 46].

Using these letters, this system could directly represent the numbers one through 999; for numbers greater than 999, a stroke was written next to the unit symbols one through nine. For example, the symbol α represented the number one, whereas $,\alpha$ denoted 1,000. By combining the Herodianic and alphabetic notations, the Greeks could also represent numbers 10,000 to 90,000, by borrowing the symbol " M " from the Herodianic notation to represent numbers greater than 10,000. For example, 20,000 was represented by : the symbol for two from the alphabetic notation and an " M " ; this numeral was written as:

$$\beta^{M} \quad , \quad {}^{M}\beta \quad , \quad \overset{\bullet\bullet}{M} \quad \text{or} \quad \underset{M}{\beta} \quad .$$

To distinguish between letters and numerals, the Greeks either placed the numerals between vertical columns of two or three dots, left a space on each side of the letter, or placed a horizontal line above the letter. Using both letters and Herodianic numbers as numerical symbols created ambiguity in deciphering the numerals, and left only few letters to represent any unknowns or variables. This confusion may have hindered the development of Greek algebra. [Heath, pp. 11-19; Tod, pp. 30, 42, 127; Van der Waerden, p. 45] Addition was performed similarly to the way it is done today; numbers to be added were written above one another, with all numbers in each denomination written in the same column. Each column was tallied and the result was written below the last number in the addition.

	$,\alpha\ \upsilon\ \kappa\ \delta$	1,424
	$\rho\ \ \gamma$	103
	$\overset{\alpha}{M},\beta\ \sigma\ \pi\ \alpha$	12,281
	$\overset{\gamma}{M}\ \ \lambda$	+ 30,030
Total:	$\overset{\delta}{M},\gamma\ \omega\ \lambda\ \eta$	43,838

Subtraction was done in the same manner, carrying one unit from a higher level as ten units into a lower level when required.

	$\overset{\theta}{M}\,,\gamma\chi\lambda\varsigma$	93,636
	$M\,,\gamma\upsilon\;\theta$	- 23,409
Difference:	$\overset{\zeta}{M}\,\theta\kappa\zeta$	76,227

[Heath, p. 28-30]

Greek methods of multiplication were generally tedious. When multiplying two numbers written as only one Greek numeral or letter, the product of the two radices was multiplied by the highest power of ten in each of the two terms. Computing the product of 200 and 3,000, for example, was completed by multiplying together the two radices, two and three, and multiplying this result by the product of 10^2 and 10^3, the highest powers of ten in each of the two terms.

More complex multiplication problems generally required the use of multiplication tables. In these, the multiplicand was written above the multiplier separated by the Greek words for "upon" or "by". Each number was broken up into smaller numbers that could be represented by one symbol alone; the smaller numbers were then multiplied together and each of these products were added together. For example, if 265 was multiplied by 270, the first number was broken down into 200, 60 and 5, the second into 200 and 70. Each portion of the first number was multiplied by each part of the second number; the six products were together produced the final result.

```
         265                200, 60, 5
     X   270                200, 70
    ----------
    200  x  200  =  40,000
    200  x   70  =  14,000
     60  x  200  =  12,000
     60  x   70  =   4,200
      5  x  200  =   4,000
      5  x   70  =     350
------------------------------
Total:              74,550
```

Greek division was performed by the same method used today.

A Greek abacus also existed and was used like the Mayan abacus for everyday calculations. The number in each order was denoted by pebbles, buttons or pegs placed the column representing that order denomination. [Heath, pp. 28-30]

DEVELOPMENT OF FRACTIONS

The fractional system of the Greeks surpassed that of the Egyptians, and resembled more closely the one of today. In the third century B.C., fractions were written with the positions of the numerator and the denominator reversed; it was not until the first Century A. D. that the Greek fractions were written with the numerator on the top and the denominator on the bottom position. [Heath, p. 20; Van der Waerden, pp. 49-50]

Although fractions were also reduced into unit fractions as done by the Egyptians, the Greeks did not restrict themselves to such computations. Unit fractions were expressed by the numeral in the denominator and a slash written immediately to the left of it. Other notations for general fractions placed the denominator to the right of the numerator; some fractions were written with the numerator below a line and the denominator written above the line. Here are some examples:

$\frac{\iota\varsigma}{\rho\kappa\alpha}$	121/16
$\iota\varepsilon^{\varsigma}$	15/4
$\alpha\gamma^{\chi}$	1 1/3
$\tau o \iota' \iota \varsigma^{\chi}$	370 + 1/2 + 1/16

Special symbols were developed for one-half and one-third, as in the Egyptian system, again denoting the importance of these fractions. Fractions of base sixty, originating in Babylonia, were used by the Greeks in astronomical calculations, although they were much less convenient than the other types of fractions. The units were written first and were followed by a number marked by one accent for the sixtieths or minutes, and a number marked with two accents denoting the three-hundred-sixtieths or seconds, like the present day method. This special system included the only evidence of the Greek use of the zero symbol. A round open circle was used to denote zero units in cases involving no units. [Heath, p. 23]

The Greek mathematicians were able to do many types of computations with fractions. They were able to reduce fractions to lowest terms, as well as find common denominators. The development of fractions did not deter the

Greek expansion of mathematics, as happened in the Egyptian mathematical system. [Heath, p. 23, Van der Waerden, p. 48]

GEOMETRICAL DEVELOPMENT

Although most researchers agree that European geometry originated in Egypt, this subject is thought to have been the Greeks' greatest achievement. The word itself comes from the Greek language: 'geo' meaning earth and 'metria' meaning measurement. Heath states that Thales travelled to Egypt and brought geometry back to Greece. [Heath, p. 75] Some of the geometrical accomplishments of Thales included the bisection of a circle by its diameter and the proof of the equality of the base angles in an isosceles triangle. Archimedes was able to isolate the value of pi between 3 10/71 and 3 1/7 by constructing regular polygons in and about a circle, calculating the perimeters and assuming that the value of the circumference of the circle fell between the two values. [Freebury, pp. 34-53] The Pythagorean order also contributed to the growth of Greek geometry. These mathematicians developed the properties of parallels and used them to prove that the sum of the angles of any triangle is equal to the measure of two right angles. They also discovered the theorems about the sums of the exterior and interior angles of any polygon. The Pythagoreans discovered three of the regular solids, approximated the length of the diagonal of a square, and knew that the earth is a sphere. Euclid also contributed greatly to the advancement of Greek mathematics with the incorporation of proofs and theorems in geometry into his Elements. [Heath, pp. 82, 110; Hooper, p. 41]

Other ancient Greek developments included the solution to quadratic equations and the discovery of irrational numbers. Ptolemy was able to extract square roots by using the sexagesimal fractions. Thales predicted an eclipse as early as 585 B.C. Eratosthenes calculated the circumference of the earth in 250 B.C., missing the correct value by only fifty miles. [Freebury, p. 34; Van der Waerden, pp. 123,125]

GREEK CALENDRICAL SYSTEMS

The earliest evidence of a Greek calendrical system exists in the writings of Homer and Hesiod discovered on stone tablets in the thirteenth century B.C. Early Greek systems employed lunar months and astronomical observations; the agricultural cycles were found to be correlated with the rising of a specific star group. A lunar year of three hundred and fifty four days was constructed and reconciled with the solar year by adding an "extra" month every second year.

The Greeks developed a civil year similar to the Egyptian year. It consisted of twelve months, each given a specific name and consisting of twenty-nine or thirty days each. Approximately one half of the months had their twenty-ninth day deleted to get a better correspondence with the solar year. A device with movable pegs was constructed to relate the civil year to solar cycles. Each month was divided into three decades of ten days each. The days were labeled in accordance with the decade in which they occurred. A date was recorded by its month, the decade it followed and the specific day on which it fell after the end of the decade. Days falling in the first decade were recorded as the first nine days with no reference to the decade, because the Greeks had no symbol representing zero. For example, the seventeenth day of the month was recorded as the seventh day after the first decade of the specific month.

MATHEMATICAL APPLICATIONS IN DAILY LIFE

Greek alphabetic numbers were used to indicate values such as the length of time of military service or priesthood, monetary values, distances from towns, and days of the month. The Greek mathematicians discovered the golden mean, a ratio of $(1+\sqrt{5})/2$, which is often found in nature. Music, for example, is considered by some to be most pleasurable to the ear when the notes are played at an interval corresponding to the golden mean. [Huntley, pp. 52-55] Many Greek buildings were constructed using the golden mean; architecture of this type has been found to be visually more pleasurable than that built with other measurements.

MATHEMATICAL APPLICATIONS WITHIN CEREMONIAL LIFE

The Pythagorean Order developed many superstitions and ceremonial applications which were related to their mathematical works. Gorman finds that this society's philosophy was not truly one of mathematics or science. Instead it was religious and philosophical, based on loyalty and simplicity; purification for the mind and body was most important to life in the society. The society developed a devout belief in God, expressed in mathematics. Boyer writes that mathematics <u>was</u> life for the Pythagorean Order, and Van der Waerden states that mathematics was part of their religion. [Boyer, pp. 52-62] The society worshipped numbers as Gods; numbers were in a free and pure form comparable to the Gods. The Pythagoreans believed that all things consisted of numbers; numbers formed the universe and had a life independent of the minds of men. The Pythagoreans believed that God was all numbers; when the Pythagoreans thought about numbers, they were communing with the Gods in prayer. [Gorman, pp. 134-136; Van der Waerden pp. 88,93] Mathematics was a way for the Pythagoreans to elevate their soul and form a union with God, who had ordered the universe by means of numbers. [Van der Waerden, pp. 93-94] It is said that Thales, although not a member of the Pythagoreans, ceremonially sacrificed a bull to honor his construction of a circle around a right triangle.

COMPARISON OF EGYPTIAN AND GREEK MATHEMATICAL SYSTEMS

The scale constructed in the Mayan and Incan comparison will now be used to compare the Egyptian and Greek mathematics.

Both the Egyptian and Greek cultures developed decimal mathematical systems; evidence of counting by fingers exists in the Greek language. The Egyptians developed a hieroglyphic numerical system in which only powers of ten were represented. "All available texts point to an Egyptian mathematics of a rather primitive culture." [Gillings, p. 232]. The Greeks employed a numerical system based upon their alphabet which included a symbol for each power of ten as well as for the intermediate numbers 5, 50, 500, 5,000 50,000. The system could result in confusion due to the similarity of the Greek numerals and letters; using the same symbols to represent both letters and numerals left little room for the development of algebraic variables or constants. Both numerical systems had the ability to record large numbers using an efficient method. The Egyptians receive a score of **one point**, reduced because of the limited development of the hieroglyphic symbols; the Greek score was reduced to **one and one-fourth point** due to the problem of distinguishing numerals from letters.

Egyptian addition was an elementary process which combined common symbols in each order: ten units of one order was equivalent to one symbol of the next higher order. The Greeks lined up all numbers of each denomination above one another and then added all symbols of each denomination; the subtraction process was done similarly. The Egyptians left few if any records of subtraction; scribes were expected to perform addition and subtraction, recording only the result and showing no calculations. The Egyptians completed multiplication calculations by using the multiplier as a base number, doubling and multiplying by ten to find a sum equivalent to the multiplicand. The Egyptian division method also used this process of doubling and addition. The Greeks developed a multiplication system similar to the one used today. The Greek division process was also similar to the method used today. The Egyptian score is reduced to **one and one-fourth point**, due to the difficulty and length of both multiplication and division processes; for the ease and similarity to the system used today, the Greek numerical system and its incorporation of the four mathematical operations deserves a perfect score of **two points.**

Neither society invented a designated position or symbol for zero throughout their mathematical system. Gillings stated that "the Egyptians had no symbol for zero", although in some of the scribes' calculations a blank space designated an empty position. [Gillings, p. 15]. The Greeks, specifically Ptolemy, developed a symbol for zero which was apparently used in only one context: by Ptolemy in a sexagesimal fractional system. Both the Egyptian and Greek system were awarded **one-half point** for their limited recognition of the concept of zero.

Both societies developed a fractional system. The Egyptians used only natural fractions, or those occurring in their daily life; all other fractions besides 2/3 were reduced to unit fractions. Reducing fractions to unit fractions was very time-consuming and allowed less time for the development of other Egyptian mathematics. The Greeks developed a more sophisticated fractional system which included all fractions. The Egyptians receive **three-fourth point** for the development of an intricate yet somewhat useless system of fractions. The Greeks were awarded a maximum score of **one point** for their construction of an effective fractional system.

All sources consulted agree that modern European geometry originated in Egypt, yet Egyptian geometrical knowledge was not as extensive as in the Greek society. The Egyptians developed a fairly accurate measure of areas of squares, triangles, trapezoids and circles, the formulas for some volumes and the surface area of a hemisphere, which the Greeks did not rediscover until later. The Egyptians did not develop a demonstrative geometry; they were not interested in proofs, whereas this was of major concern to the Greeks. The Greeks, namely the Pythagoreans, discovered that the angles of a triangle equalled the measurement of two right angles; had knowledge of volumes of regular solids; and developed some theorems about parallel lines. The Egyptians receive **three-quarters point** for their geometrical advancements and its incorporation into their lives; the Greeks are awarded the maximum score of **one point** for their well-developed knowledge of geometry and its applications into their lives.

Both cultures developed an approximation of pi. The Egyptians used the fraction 256/81 as pi, derived from their formula $(8d/9)^2$ for the area of a circle. The Greek mathematician Archimedes found that the value of pi lay between 3 10/71 and 3 1/7, roughly, 3.1408 < pi < 3.14286. Again, both cultures receive the maximum score of **one point** for approximations of pi.

Neither the Greeks nor the Egyptians valued the existence of a calendrical system as much as did the Mayan or Incan cultures. The Egyptians developed a calendrical system consisting of a twelve-month year. It involved three seasons of four months each: a sowing, growing and harvest period. The Greeks developed an elastic <u>civil</u> year similar to that of the Egyptians, which varied in length to parallel lunar phases. For their development of an accurate calendrical year and its incorporation into the ceremonial and daily lives, both the Egyptians and Greeks receive a score of **one and one-half point.**

Both cultures were able to incorporate the numerical systems into their daily life. The golden mean ratio can be found in a number of Greek structures as well as in their music. Eratosthenes developed a scientific measurement of the earth, and Greek astronomers were able to predict an eclipse. An abacus of pebbles was used for simple every-day calculations. Numerical notations were found that dealt with monetary matters, the length of tenure of office and priesthood, and distances from town. The Egyptians developed the pyramids; used the paths of the planets to calculate time, and developed calculations of measures of grain, distribution of wages, and conversions of different measures of grains. The Egyptian calculations did not deal with money or currency. Both cultures were awarded a maximum score of **two points** for the extent of incorporation of their mathematical skills into their lives.

The incorporation of mathematics into Greek ceremonial life was epitomized by the Pythagorean order, whose mathematicians perceived a strong relationship between mathematics and religion. Mathematical applications within the Egyptian ceremonial life centered around the building of tombs to honor those on the path to the Afterlife. The calendar recording Egyptian seasons designated ceremonies and important activities and was maintained by astronomical observations. The Egyptian score for incorporating their mathematical knowledge into the ceremonial life is **one point.** The Greek ceremonial applications were awarded **one and one-half point.**

The total score of the Egyptian culture is **nine and three-fourth points.**
The total score received by the Greek culture is **eleven and three-quarter points.**

In this author's opinion, the Greeks were the more mathematically adept of the two cultures.

CONCLUSIONS

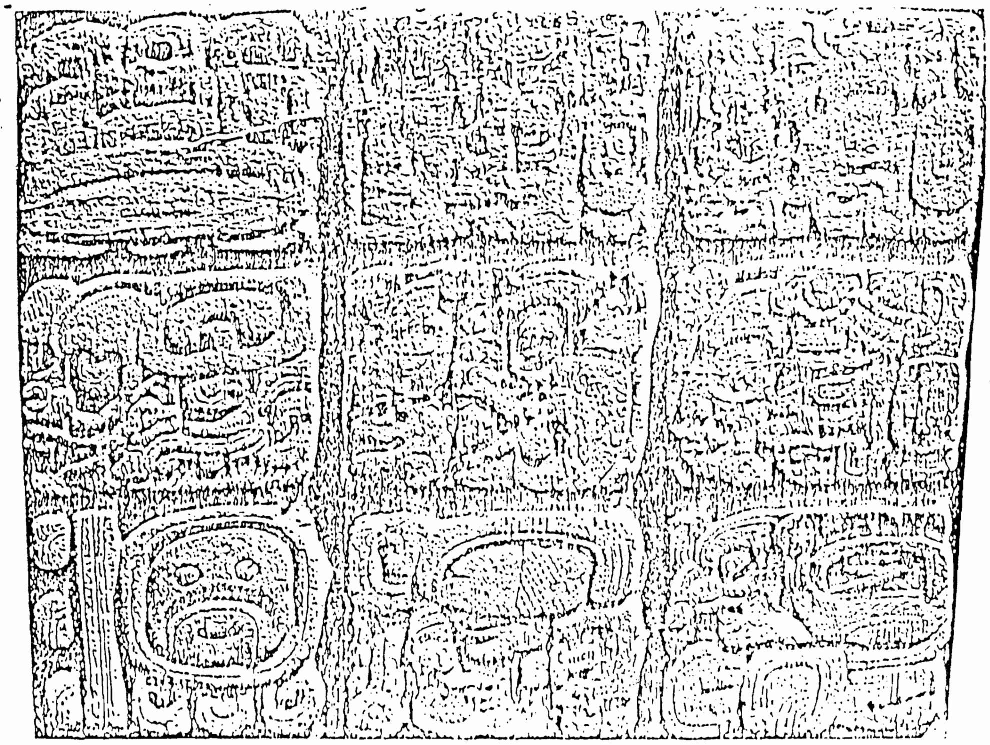
After completing the comparison of the five cultures included in this research, a number of observations become evident. The results of the comparison would indicate that the extent of Greek mathematical knowledge was superior to that of the Egyptian, Mayan and Incan cultures. It is crucial, however, to understand that **all** results in the latter portion of this research are based upon the opinions of the author, and were not derived by quantitative objective methods. Although the Greek civilization was awarded the greatest numerical score, not all portions of Greek mathematical knowledge were as extensive as that of their competitors. The Mayan system represented the concept of zero most effectively. Not only was this culture able to represent a lack of a number in any denomination, but was also able to portray the zero digit when needed within a numerical symbol. It is also apparent that the detailed Mayan calendrical system was a much more efficient and effective system than that of the Incan, Greek or Egyptian civilizations. No other calendrical system with such involved mathematical calculative ability and having the ability to deal with both ceremonial and practical situations has been discovered in other ancient civilizations.

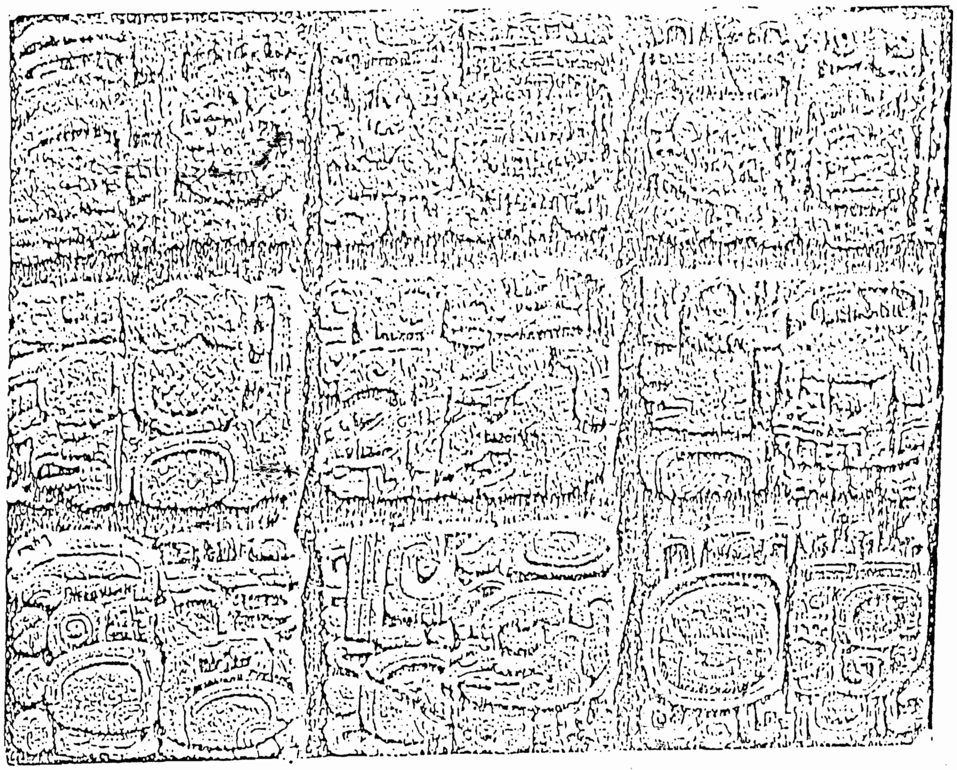
The ancient Inca developed a mathematical system involving the most intricate and creative methods of recording statistical references and completing mathematical calculations. By also recording both Incan calendars and selected histories, the quipus were able to immortalize the Incan civilization.

Mathematical discoveries more advanced than those of the Greek culture were also prevalent in the Egyptian civilization. The ancient Egyptians constructed a number of detailed buildings using their mathematical knowledge. Egyptian mathematicians and their scribes were able to calculate and record a large number of intricate mathematical computations on papyri which are still in existence today.

It can be concluded, then, that the cultures studied in this research have not only contributed to the international development of mathematics, but have helped shaped the path of mathematical development throughout the past and into the future. Each of the Mayan, Incan, Egyptian and Greek civilizations introduced mathematical skills and calendrical development to the ancient world predating today's expansion of science and technology. The comparison of the mathematical abilities and the selection of a superior mathematical development does not carry as much importance as the recognition of each civilization's individual mathematical achievements. It is this recognition that the comparison and conclusion of this research portrays.

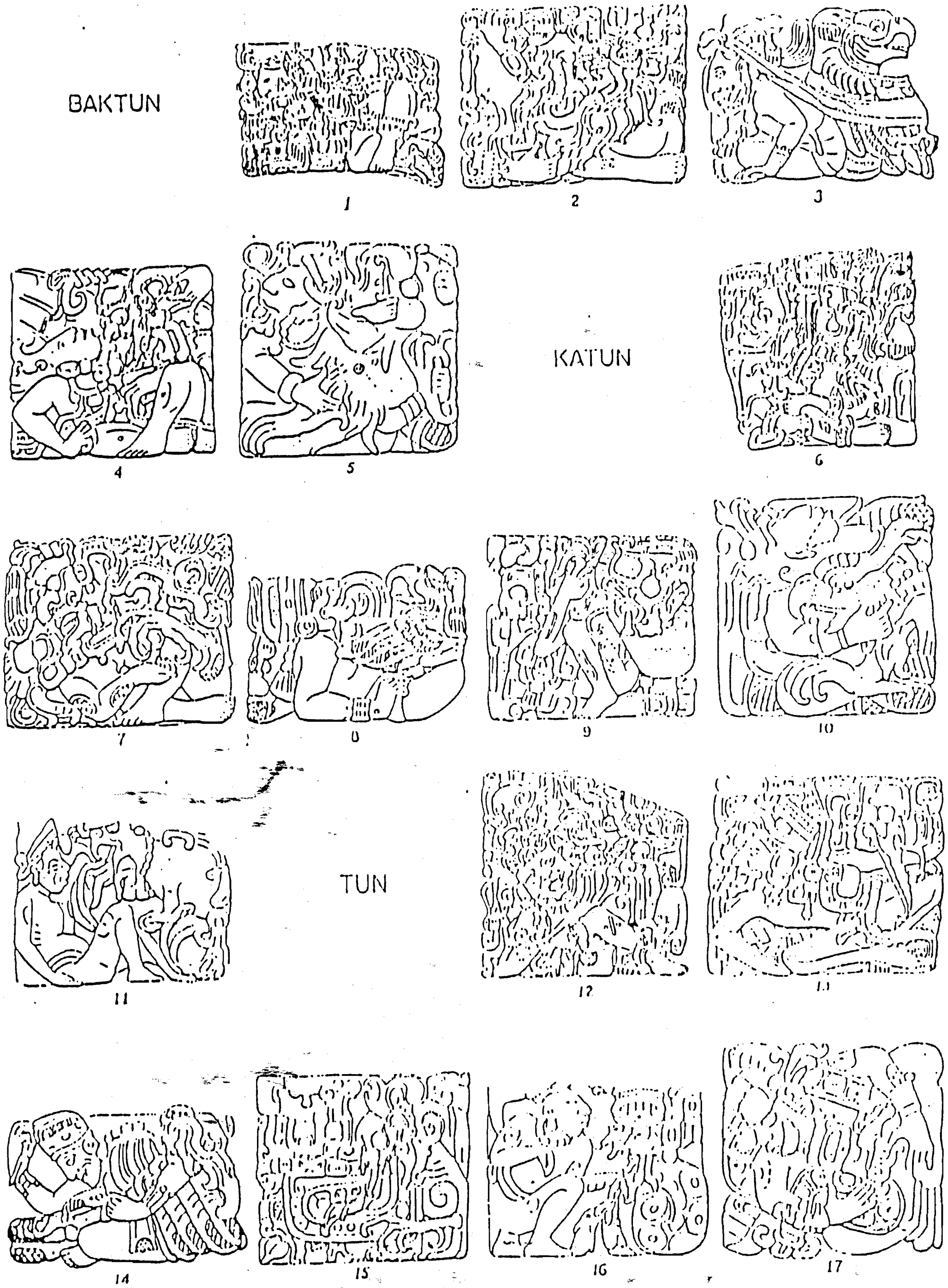
APPENDIX



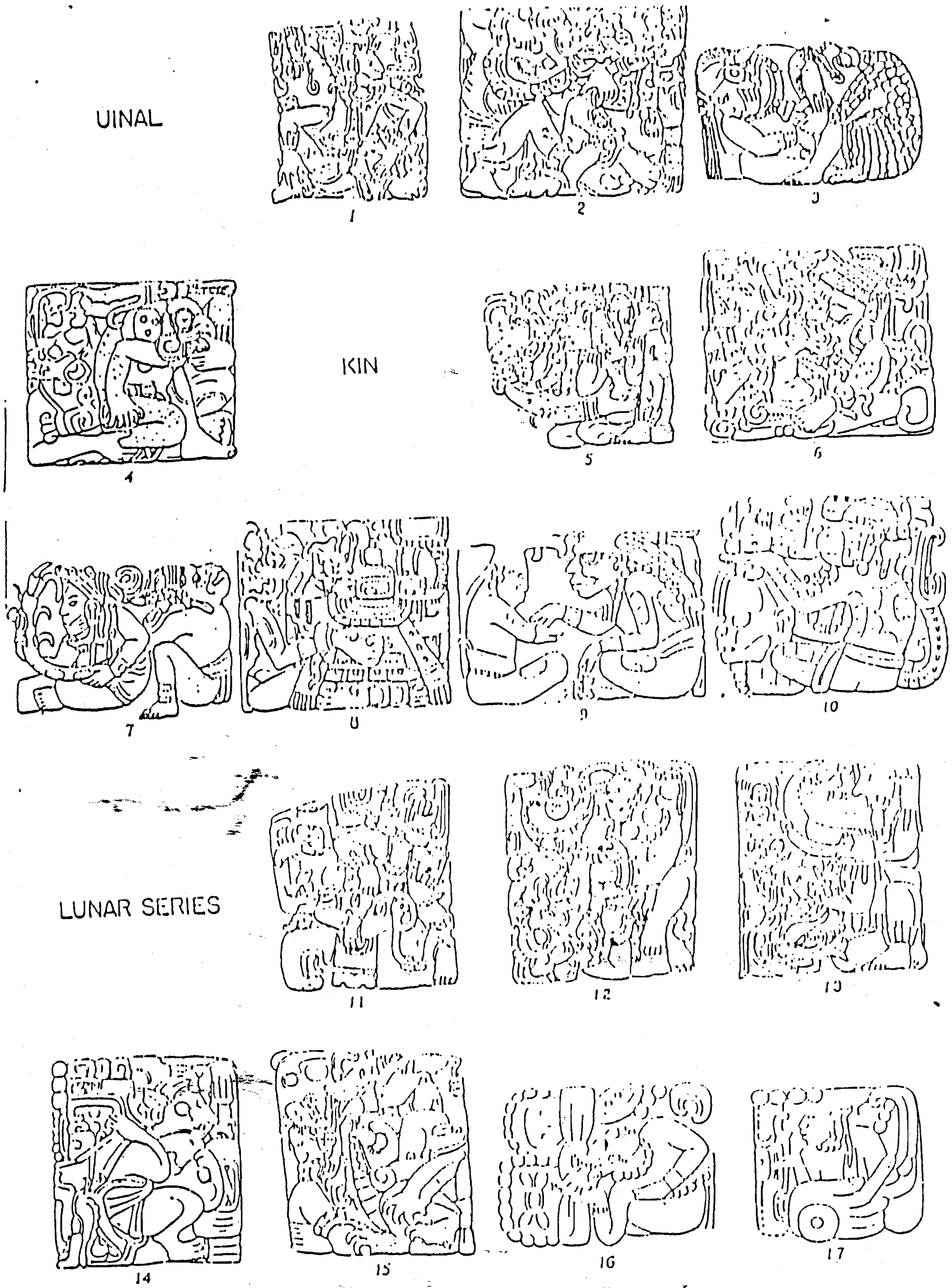


Part of the Hieroglypic Stairway, Naranjo [Thompson]

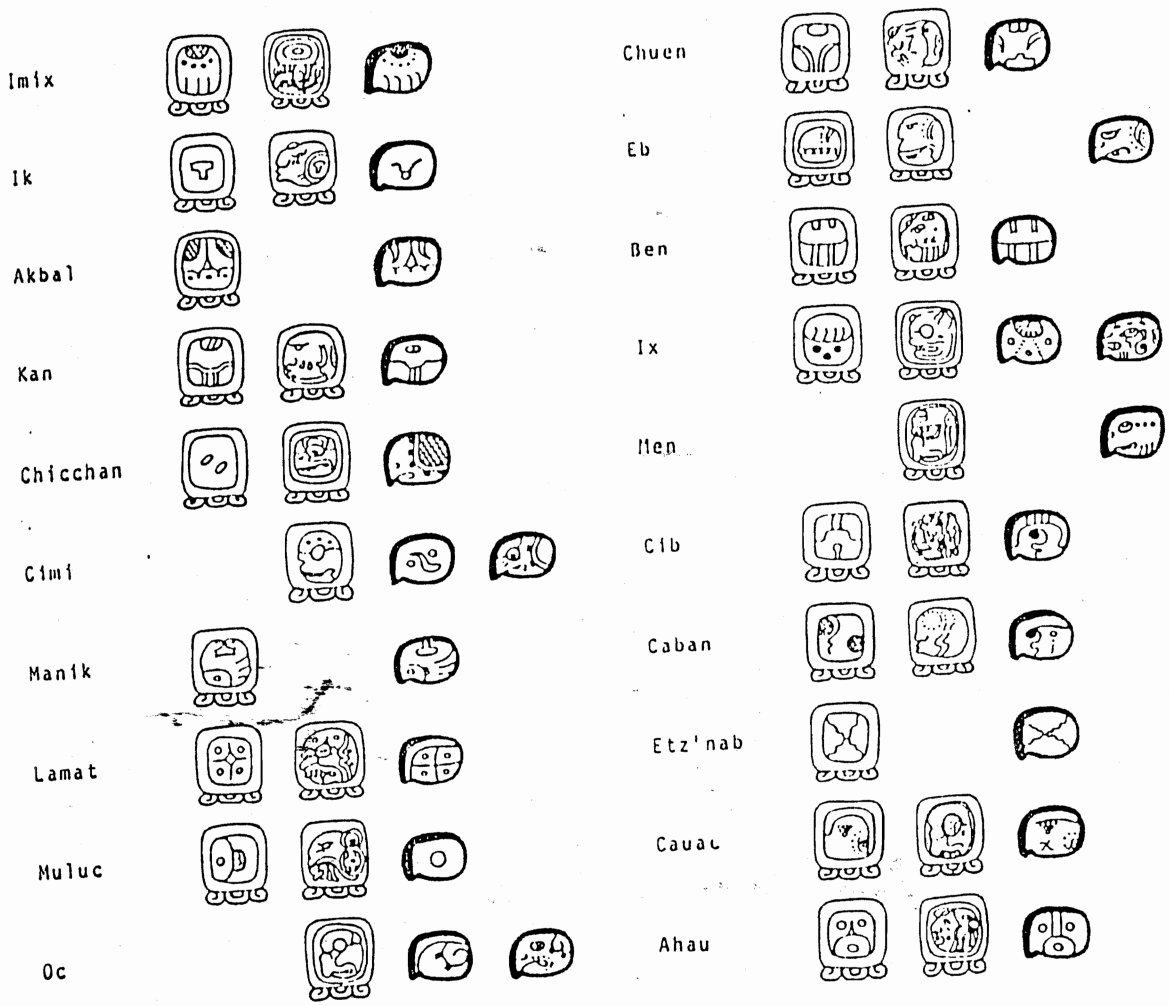
Head Variant Numerals [Closs (1986)]



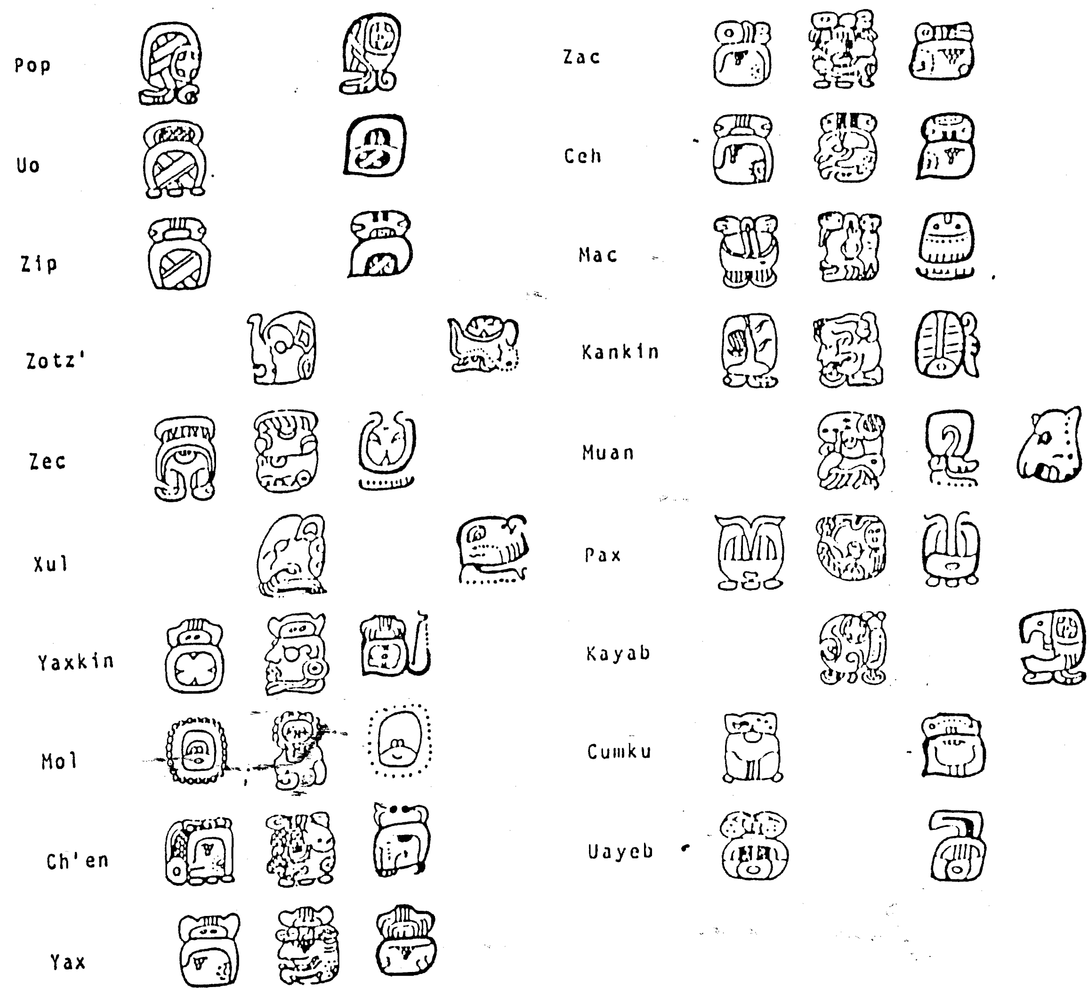
Full-Figure glyphs of time periods [Thompson]



Full-Figure glyphs of time periods and lunar series [Thompson]



Names and hieroglyphs for the days of the Sacred Round [Closs (1986)]



Names and hieroglyphs for the months and residue of the Vague Year [Closs(1986)]

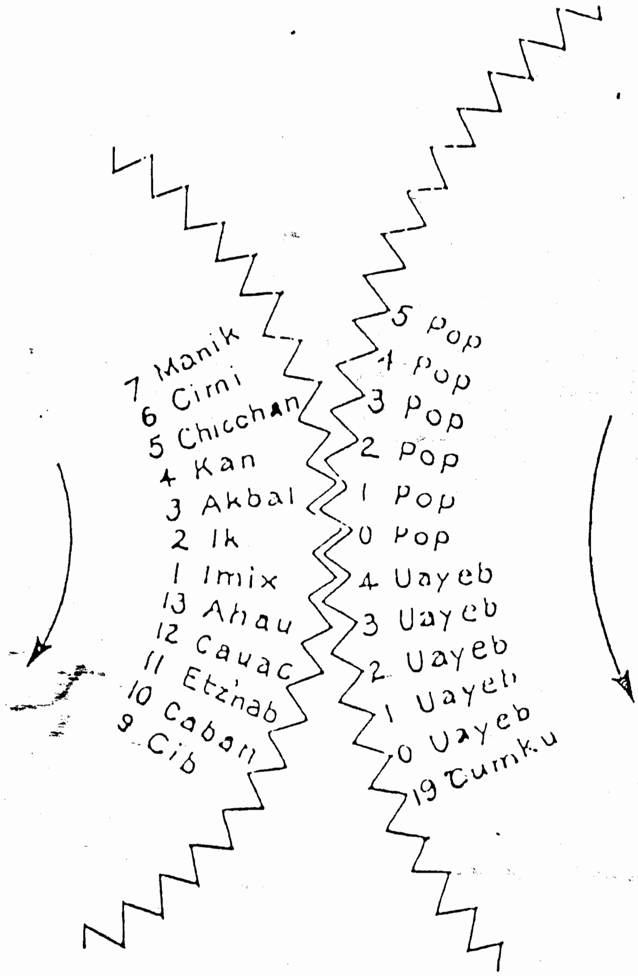
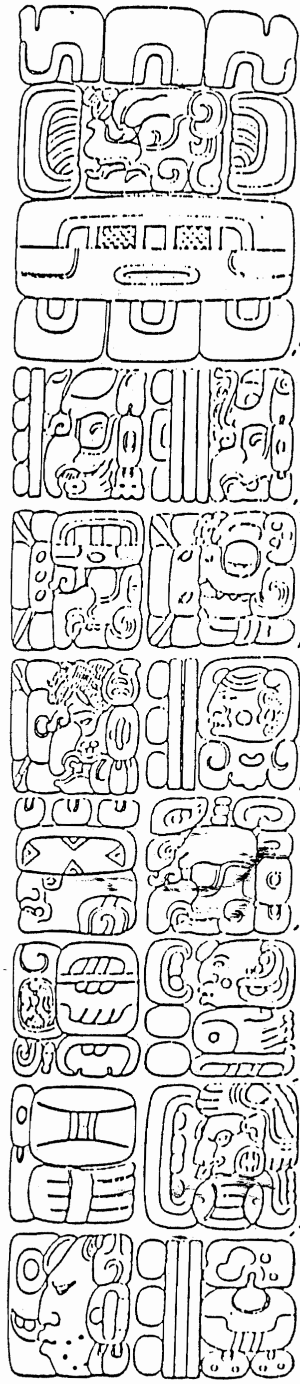


Diagram of the cog wheel illustrating the meshing of the 260-day almanac (left) with the 365-day year (right) [Morley (1983)]



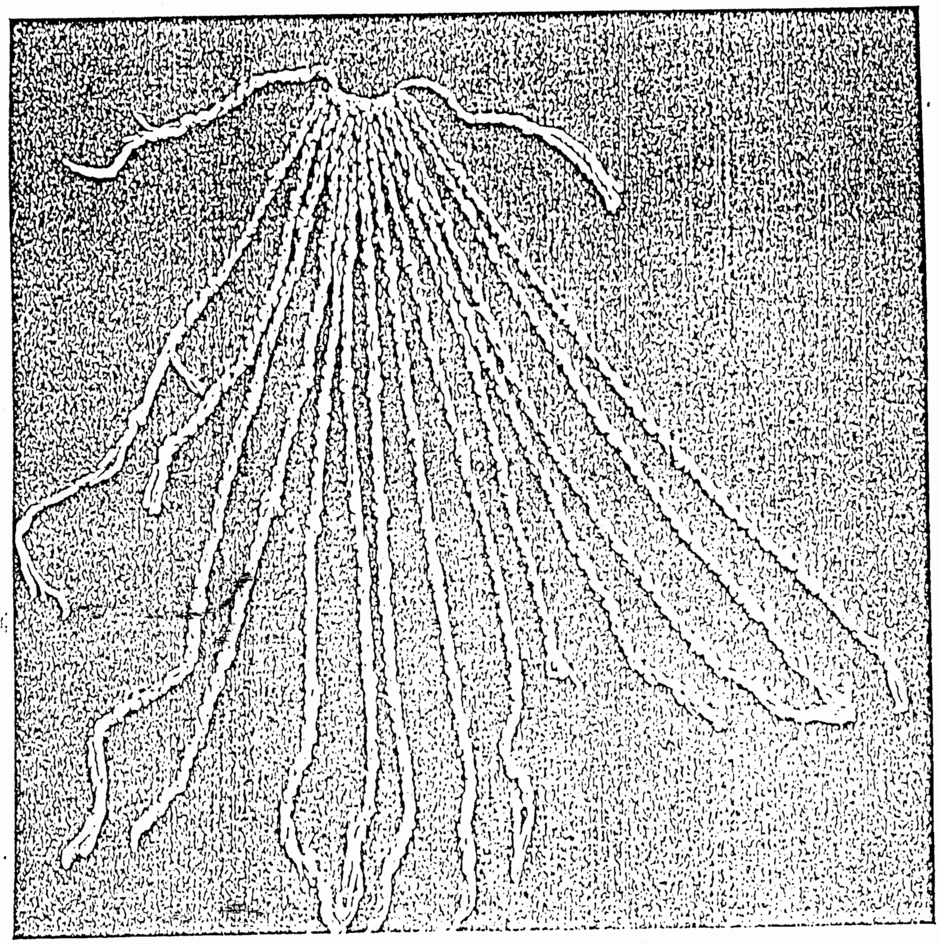
Long-Count Introducing Glyph: the head in the center is the only variable element of this sign. This is the name glyph of the deity who is patron of the month (here Cumku) in which the long-count terminal date falls.

9 baktuns (9 × 144,000 days = 1,296,000 days)	17 katuns (17 × 7,200 days = 122,400 days)
0 tuns (0 × 360 days = 0 days)	0 uinals (0 × 20 days = 0 days)
0 kins (0 × 1 day = 0 days)	13 Ahau (day reached by counting forward above total of days from starting point of Maya era)
Glyph G_9: name glyph of the deity who is patron of the ninth day in the nine-day series (the Nine Gods of the Lower World)	Glyph F: meaning unknown
Glyphs E and D: glyphs denoting the moon age of the long-count terminal date, here "new moon"	Glyph C: glyph denoting position of current lunar month in lunar half-year period, here the second position
Glyph X_3: meaning unknown	Glyph B: meaning unknown
Glyph A_9: current lunar month, here 29 days in length. Last glyph of the lunar series.	18 Cumku (month reached by counting forward above total of days from starting point of Maya era). Last glyph of the long count.

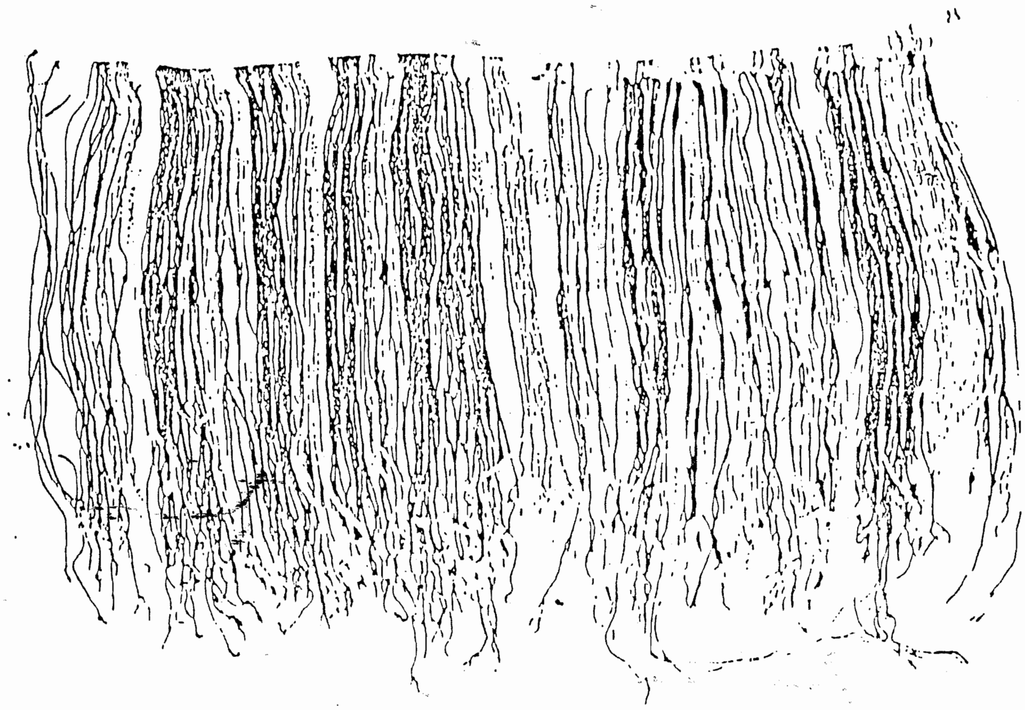
Long Count

Lunar Series

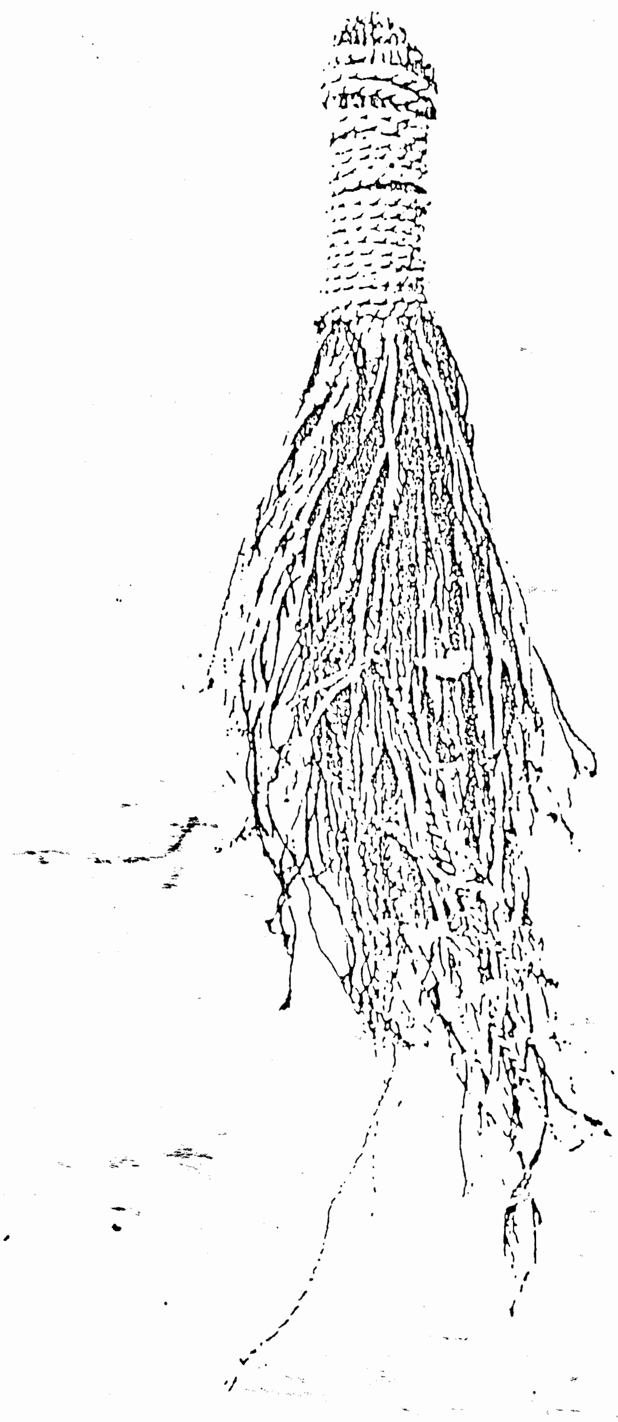
Example of a long-count date, from the inscription on the east side of a Monument Quirigua, Guatemala [Morley (1983)]



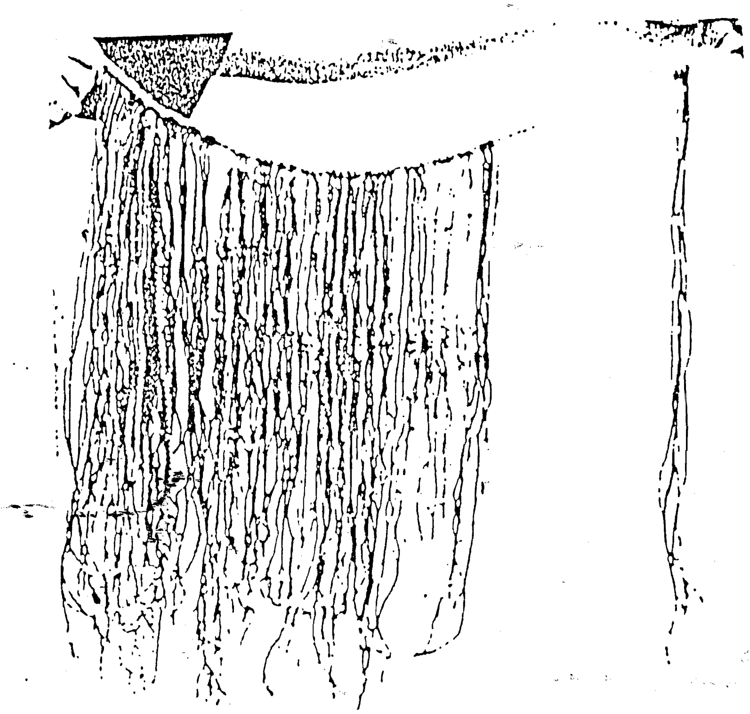
Quipu Number 21, from Cajamarquilla [Locke]



An imcomplete quipu [Ascher and Ascher]



A quipu that has been completed and rolled [Ascher and Ascher]



The same quipu unrolled [Ascher and Ascher]

BIBLIOGRAPHY

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1. Ascher, Marcia and Robert. The Code of the Quipu. A Study in Media, Mathematics and Culture. The University of Michigan Press, Ann Arbor, 1981.

2. Aveni, Anthony F. Native American Astronomy. University of Texas Press, Austin, London, 1977.

3. Boyer, Carl. B. History of Mathematics. Princeton University Press, Princeton, New Jersey, 1968.

4. Breasted, James Henry, Ph.D. A History of the Ancient Egyptians. Charles Scribner's Sons, New York, 1905, 1908.

5. Cajori, Florian. The Controversy of the Origin of Our Numerals. The Scientific Monthly. 1919, p. 458-464.

6. Callahan, Vera Alma. Mathematics in the Mayan, Aztec and Inca Cultures. A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Arts, The Graduate School, University of Maine, Orono, January 1969.

13. Dantzig, Tobias. The Bequest of the Greeks. New York, Charles Scribner's Sons. 1955.

14. Day, Cyrus Lawrence. Quipus and Witches' Knots. The Role of the Knot in Primitive and Ancient Cultures. The University of Kansas Press, Lawrence, 1977.

15. Flornoy, Bertrand. The World of the Inca. The Vanguard Press, New York, 1956.

16. Freebury, H.A. A History of Mathematics. The Macmillan Company, New York, 158, 1961.

17. Fries, Judith E. The American Indian in Higher Education (1975-1976 and 1984-1985). Center for Education Statistics, March 1987.

18. Gallenkamp, Charles. Maya. The Riddle and Rediscovery of an Ancient Civilization. Viking Press, New York, 1959, 1976, 1981, 1985.

19. Gann, Thomas and Thompson, Eric. The History of the Maya From the Earliest Times to the Present Day. Charles Scribner's Sons, New York, 1937.

20. Garcilaso de la Vega, El Inca. Royal Commentaries of the Incas and General History of Peru. University of Texas Press, Austin and London, 1966.

21. Gillings, Richard J. Mathematics in the Time of the Pharaohs. The MIT Press, Cambridge, Massachusetts, London, England, 1972.

22. Gilman, Robert, Trent, John. Math Achievement of Native Americans in Nevada. Journal of American Indian Education, 24 (1), January 1985, 39-45.

23. Heath, Sir Thomas L. A Manual of Greek Mathematics Dover Publications, Inc., New York, 1931, 1963.

24. Hooper, Alfred. Makers of Mathematics. Random House, New York, 1948

25. Hurlbert, Gade, and McLaughlin. Teaching Attitudes and Study Attitudes of Indian Education Students. Journal of American Indian Education, 29 (3), May 1990, 12-19.

26. Johnston, Alan. The Emergence of Greece. Elsevier Phaidon, United States, 1976.

27. Lin, Ruey-Lin. Perception of Family Background and Personal Characteristics Among Indian College Students. Journal of American Indian Education, 29 (3), May 1990, 19-29.

28. Locke, L. Leland. The Ancient Quipu or Peruvian Knot Record. The American of Natural History, 1923.

29. McIntyre, Loren. The Incredible Incas and Their Timeless Land. The National Geographic Society, 1975.

30. Moffat, Michael; Linn, Charles F., editor. The Ages of Mathematics Volume I The Origins. Doubleday and Company, Inc., Garden City, New York, 1977.

31. Morley, Sylvanus G. The Ancient Maya. Stanford University Press, Stanford University, California, Oxford University Press, London, First and Second Edition, 1946, 1947.

32. Morley, Sylvanus G. The Ancient Maya. Stanford University Press, Stanford University, California, Fourth Edition, 1983.

33. Morley, Sylvanus G. An Introduction to the Study of Maya Hieroglyphs. Smithsonian Institution, Bureau of American Ethnology. Bulletin 57. Washington, D.C., 1915.

34. Noyes, Ernest. Notes On the Maya Day-Count. Maya Research. 2:1935, p. 383-385.

35. Rhodes, Robert. Measurements of Navajo and Hopi Brain Dominance and Learning Styles. Journal of American Indian Education, 29 (3), May 1990, 29-39.

36. Sanchez, George I. Arithmetic in Maya. George Sanchez, Austin, Texas, 1961.

37. Schele, Linda and Freidel, David. A Forest of Kings The Untold Story of the Ancient Maya. William Morrow and Company, Inc., 1990.

38. Scott, Patrick B. Mathematics Achievement Test scores of American Indian and Anglo Students: a Comparison. Journal of American Indian Education, 22 (3), 1983, 17-19.

39. Stierlin, Henri. The Cultural History of Greece. Aurum Press Ltd., U.K. and Commonwealth, 1984.

40. Thompson, J. Eric S. Maya Hieroglyphic Writing: An Introduction. University of Oklahoma Press, U.S.A, 1960.

41. Thompson, J. Eric S. The Rise and Fall of Maya Civilization. Second Edition, University of Oklahoma Press, Norman, 1966.

42. Tod, Marcus Niebvhr. Ancient Greek Numerical Systems. Aries Publishers Inc, Chicago, Il., 1979.

43. Van der Waerden, B.L. Science Awakening. Oxford University Press, New York, 1971.

44. Von Hagen, Victor Wolfgang. The Ancient Sun Kingdoms of the Americas. The World Publishing Company, Cleveland and New York, 1961.

45. Waddell, Jack O. The American Indian in Urban Society. Little, Brown and Company (Inc.), Boston, 1971.

46. Wilson, John A. The Culture of Ancient Egypt. Phoenix Books, The University of Chicago Press, 1951.