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A Glance at Tropical Operations and Tropical Linear Algebra

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SEMERE TSEHAYE TESFAY

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Abstract

The tropical semiring is $\mathbb{R} \cup \{\infty\}$ with the operations $x \oplus y = \min\{x, y\}$, $x \oplus \infty = \infty \oplus x = x, x \odot y = x + y, x \odot \infty = \infty \odot y = \infty$. This paper explores how ideas from classical algebra and linear algebra over the real numbers such as polynomials, roots of polynomials, lines, matrices and matrix operations, determinants, eigen values and eigen vectors would appear in tropical mathematics. It uses numerous computed examples to illustrate these concepts and explores the relationship between certain tropical matrices and graph theory, using this to provide proofs of some tropical computations.

Acknowledgments

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Contents

Introduction

The main purpose of this thesis is to describe the arithmetic operations in the tropical semi-ring, generally known as tropical arithmetic, and then explore how many of the standard notions of algebra and linear algebra would look in this tropical world. To begin, we should address the question of just what tropical mathematics is and why it is so named. The first of these issues is the thread that runs throughout the thesis. The basic operations of tropical arithmetic and their applications to the tropical versions of polynomials and lines are described and explored in the next three sections. Tropical matrix algebra and its application to tropical linear algebra form the rest of the paper.

This leaves the question of why this entire area is referred to as tropical mathematics. One of the pioneers of the subject was Dr. Imre Simon – a Hungarian born mathematician who spent almost all of his academic life in Brazil. A group of French mathematicians who were following up on Simon's work seemed to simply view Brazil as "tropical" so began associating that term with this area of mathematics in his honor. It should be noted that a group of Russian mathematics worked on the mirror-image of tropical mathematics independently beginning in the 1990s.

Tropical Operations

Consider the real number system $(\mathbb{R}, +, \cdot)$ and replace the binary operations "+" by minimum of two numbers, denoted by "$\oplus$", and "$\cdot$" by ordinary plus, denoted by "$\odot$", i.e. redefine the basic arithmetic operations of addition and multiplication of real numbers as follows:

$$x \oplus y = \min\{x, y\}$$

$$x \odot y = x + y$$

.

Example: The tropical sum of 6 and 2 is 2, and the tropical product of 6 and 2 equals 8. That is

$$6 \oplus 2 = 2$$

$$6 \odot 2 = 8$$

One can simply prove that many of the familiar axioms of arithmetic remain valid in tropical mathematics. To see this, for any $x, y, z \in \mathbb{R}$

- Both addition and multiplication are associative.

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(x \odot y) \odot z = x \odot (y \odot z)$$

Associativity of tropical addition holds since $\min\{x, \min\{y, z\}\} = \min\{x, y, z\}$. Associativity of tropical multiplication is obvious since the usual addition is associative.

- Both addition and multiplication are commutative.

$$x \oplus y = y \oplus x$$

$$x \odot y = y \odot x$$

- The tropical multiplication is distributive over tropical addition.

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

To illustrate this property:

$$14 \odot (5 \oplus 9) = 14 \odot 5 = 19$$

$$(14 \odot 5) \oplus (14 \odot 9) = 19 \oplus 23 = 19$$

In fact, this follows from

$$\min\{x + y, x + z\} = x + \min\{y, z\}$$

- Note that 0 is the multiplicative identity, and for every $a \in \mathbb{R}$, the multiplicative inverse $-a \in \mathbb{R}$ exists. This is clear since tropical multiplication is the usual addition.

- The additive identity e, if it exists, must satisfy the statement below

$$\forall x \in \mathbb{R}, \qquad x \oplus e = e \oplus x = x$$

which is equivalent to

$$\forall x \in \mathbb{R}, \qquad x \oplus e = e \oplus x = \min\{x, e\} = x$$

Observe that there doesn't exist $e \in \mathbb{R}$ which satisfy this condition. However, if we extend our tropical set $\mathbb{R}$ to include ∞, ∞ will serve as the zero element. Let $\mathbb{T} = \mathbb{R} \cup \{\infty\}$, and define

$$\forall x \in \mathbb{R}, \qquad x \oplus \infty = \infty \oplus x = \min\{x, \infty\} = x$$

$$\forall x \in \mathbb{R}, \qquad x \odot \infty = \infty \odot x = x + \infty = \infty$$

Note that no $a \in \mathbb{R}$ has an additive inverse. So we cannot really talk about tropical subtraction.

Therefore, the algebraic structure $(\mathbb{T}, \oplus, \odot)$ is a semiring [3].

Tropical Polynomials

Consider functions of the form

$$p(x) = \bigoplus_{i=0}^{n} (a_i \odot x^i)$$

with $a_i \in \mathbb{T}$, in other words, tropical polynomials. Rewriting $p(x)$ in classical notation yields

$$p(x) = \min\{a_i + ix\}_{i=0,1,\ldots,n}$$

Notice that $p(x)$ is the minimum of collection of linear functions.

Example: Consider the tropical polynomial

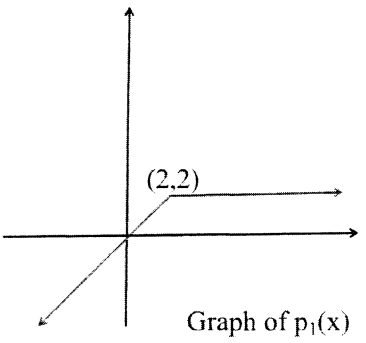
$$p_1(x) = 2 \oplus x = \min\{2, x\}$$

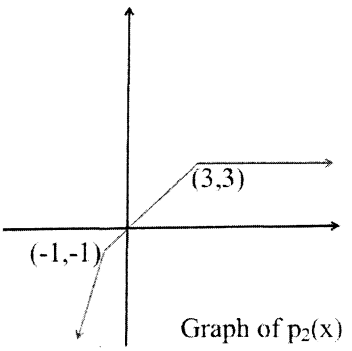
It is the minimum of constant function $f(x) = 2$ and the identity function $f(x) = x$. Similarly,

$$p_2(x) = 3 \oplus x \oplus (2 \odot x^3) = \min\{3, x, 3x + 2\}$$

is the minimum of the given three linear functions.

This tells us that a tropical polynomial is a piecewise linear function and each piece has an integer slope, and its graph is continuous and concave down.





It is natural to ask what the roots of these polynomials are. However, first we should examine what we mean by a root in the sense tropical algebra? In the usual

sense, x_0 is said to be a root of a polynomial $p(x)$ if $p(x_0)$ is equal to zero. However, this definition is not appropriate in tropical sense since the zero element is ∞ and there is no x for which $p(x) = \infty$ for most polynomials p.
To illustrate this, consider the polynomial

$$p_2(x) = 3 \oplus x \oplus (2 \odot x^3) = \min\{3, x, 3x + 2\}.$$

If we try to solve

$$p_2(x) = 3 \oplus x \oplus (2 \odot x^3) = \min\{3, x, 3x + 2\} = \infty$$

Note that $p_2(x)$ is at most 3 which implies the equation has no solution. In general, this tells any polynomial $p(x)$ has no roots if it has a constant term. In standard algebra, an equivalent definition of a root, in the classical sense, of a polynomial is an element $x_0 \in \mathbb{R}$ such that

$$p(x) = (x - x_0)q(x)$$

for some polynomial $q(x)$. So we can take the definition for tropical roots of the polynomial $p(x)$ to be: A real number x_0 is a tropical root of $p(x)$ if there exists a tropical polynomial $q(x)$ such that

$$p(x) = (x \oplus x_0) \odot q(x)$$

Consider tropical polynomial $P(x)$ of degree two defined by

$$P(x) = (x \oplus a) \odot (x \oplus b) = \min\{x, a\} + \min\{x, b\}$$

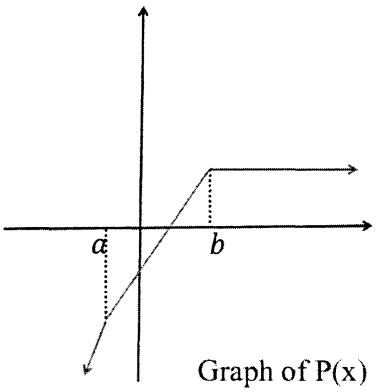
Without loss of generality, suppose $a < b$, then

$$\min\{x, a\} = \begin{cases} x & x \leq a \\ a & x > a \end{cases} \quad \text{and} \quad \min\{x, b\} = \begin{cases} x & x \leq b \\ b & x > b \end{cases}$$

which leads to

$$P(x) = \begin{cases} 2x & x \leq a \\ x + a & a < x \leq b \\ a + b & x > b \end{cases}$$

and its graph is



This leads to the fact that the roots of $p(x)$ are all points x_0 of $\mathbb{R}$ for which the graph of $p(x)$ has a corner at x_0.

Note that the above discussion is when $a < b$, how about when $a = b$? If we look at our polynomial $P(x)$ will be reduced to

$$P(x) = \begin{cases} 2x & x \leq a \\ 2a & x > a \end{cases}$$

and the graph of $P(x)$ will have only one corner. The difference in the slopes of the two pieces adjacent to a corner gives the order of the corresponding root (which is 2 for this particular polynomial $P(x)$).

This is a good time to explore one of the interesting consequences of tropical arithmetic. One can simply notice that, when $a = b$, the reduction of $P(x)$ gives

$$\Rightarrow P(x) = (x \oplus a) \odot (x \oplus a) = \min\{x, a\} + \min\{x, a\}$$

$$= 2\min\{x, a\} = \min\{2x, 2a\} = x^2 \oplus a^2$$

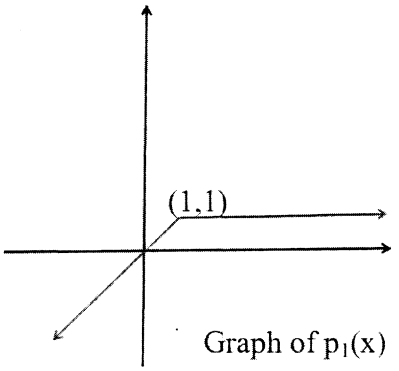
$$\Rightarrow (x \oplus a)^2 = x^2 \oplus a^2$$

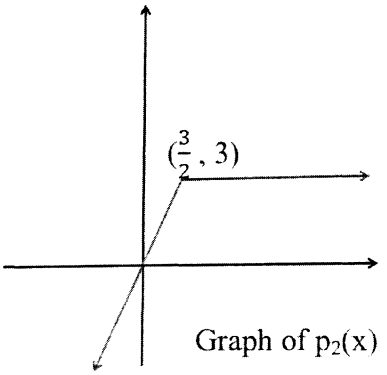
This fact can be generalized to

$$(x \oplus a)^n = x^n \oplus a^n$$

Additionally, x_0 is a tropical root of at least order k of $p(x)$ if there exists a tropical polynomial $q(x)$ such that $p(x) = (x \oplus x_0)^k \odot q(x)$.

Example: The polynomial $p_1(x) = 1 \oplus x$ has a simple root at $x_0 = 1$ and the polynomial $p_2(x) = 3 \oplus x^2 = \min\{3, 2x\}$ has a double root at $\frac{3}{2}$. From the graph depicted below, one can observe that the order of the root is the difference in the slopes of the two pieces adjacent to a corner.





Example: The polynomial $p_3(x) = (x \oplus 2) \odot (x \oplus 3)^2$ has a simple root at $x_0 = 2$ and a double root at $x_0 = 3$.

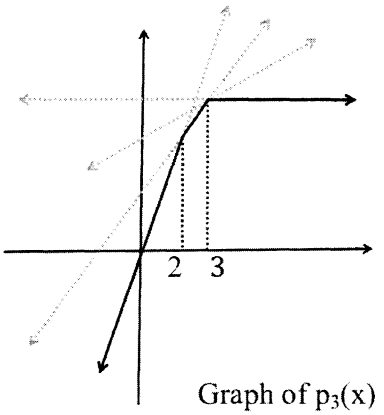
To see this,

$$\begin{aligned} p_3(x) &= (x \oplus 2) \odot (x \oplus 3)^2 = (x \oplus 2) \odot (x \oplus 3) \odot (x \oplus 3) \\ &= [(x^2) \oplus (x \odot 2) \oplus (x \odot 3) \oplus (2 \odot 3)] \odot (x \oplus 3) \\ &= [(x^2) \oplus (x \odot 2) \oplus 5] \odot (x \oplus 3) \\ &= (x^3) \oplus (x^2 \odot 3) \oplus (2 \odot x^2) \oplus (2 \odot x \odot 3) \oplus (5 \odot x) \oplus (5 \odot 3) \\ &= (x^3) \oplus (2 \odot x^2) \oplus (5 \odot x) \oplus 8 \end{aligned}$$

Now, note that the above expression is equivalent to the minimum of four linear functions given by

$$\min\{3x, 2x + 2, x + 5, 8\}$$

and its graph is given below



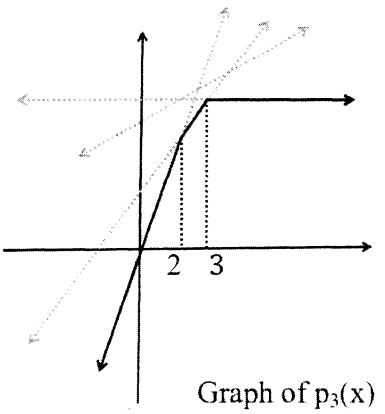
Graph of $p_3(x)$

On the other hand,

$$\begin{aligned} p_3(x) &= (x \oplus 2) \odot (x \oplus 3)^2 = (x \oplus 2) \odot (x^2 \oplus 3^2) \\ &= (x \oplus 2) \odot (x^2 \oplus 6) \\ &= (x^3) \oplus (2 \odot x^2) \oplus (6 \odot x) \oplus (2 \odot 6) \\ &= (x^3) \oplus (2 \odot x^2) \oplus (6 \odot x) \oplus 8 \end{aligned}$$

which is equivalent to

$$\min\{3x, 2x + 2, x + 6, 8\}$$



Graph of $p_3(x)$

Notice that, the graphs are identical, and have a single root at $x = 2$ and double root at $x = 3$.

Tropical Curves

Definition: A tropical polynomial in two variables is written

$$p(x,y) = \bigoplus_{i,j} a_{i,j} \odot x^i \odot y^j = \min_{i,j}\{a_{i,j} + ix + jy\}$$

Here our tropical polynomial is again a piecewise linear function, and the tropical curve defined by $p(x,y)$ is the corner locus of this function. That is, a tropical curve consists of all points $(x_0, y_0) \in \mathbb{T}^2$ for which the minimum of $p(x,y)$ is obtained at least twice at (x_0, y_0) (i.e. at every solution of the two piecewise-linear equations, precisely two of the linear forms attain the minimum value in each of the two equations).

Example: Consider the tropical line defined by the polynomial

$$p(x,y) = 1 \oplus (-2) \odot x \oplus 3 \odot y$$

We must find the points (x_0, y_0) in $\mathbb{R}^2$ that satisfy the following three systems of equations:

$$1 = x_0 + (-2) \leq y_0 + 3$$

$$y_0 + 3 = 1 \leq x_0 + (-2)$$

$$x_0 + (-2) = y_0 + 3 \leq 1$$

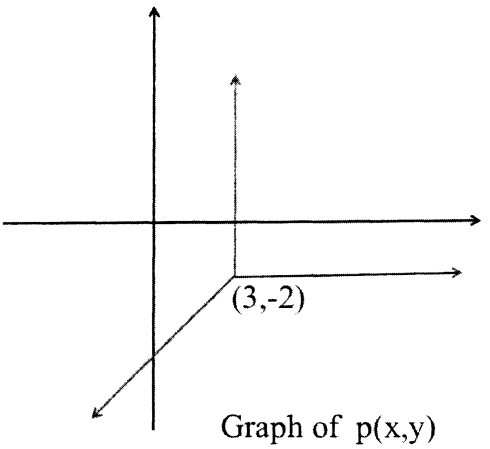
As depicted on the figure below, the curve is made up of three standard half-lines given by:

$$\{(3,y) : y \geq -2\},$$

$$\{(x,-2) : x \geq 3\}$$

and

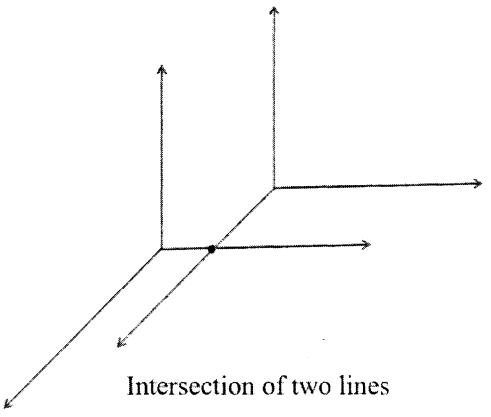
$$\{(x, x-5) : x \leq 3\}$$



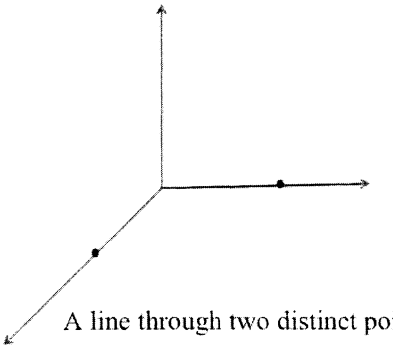
Graph of p(x,y)

This curve is known as tropical line and it has several common properties with the usual line in Euclidean plane. A tropical line is made up of three half-lines emanating from a point in the direction of $(1,0)$, $(0,1)$ and $(-1,-1)$. As depicted in the figure below:

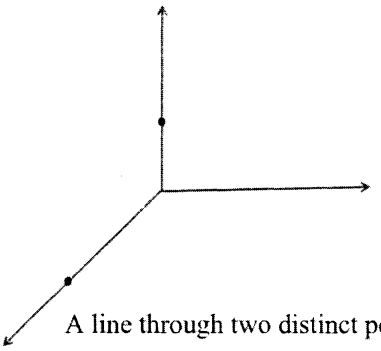
- Tropical lines are given by an equation of the form $(a \odot x) \oplus (b \odot y) \oplus c$.
- Most pairs of tropical lines intersect in a single point; and
- For most choices of pairs of points in the plane, there is a unique tropical line passing through the two points.



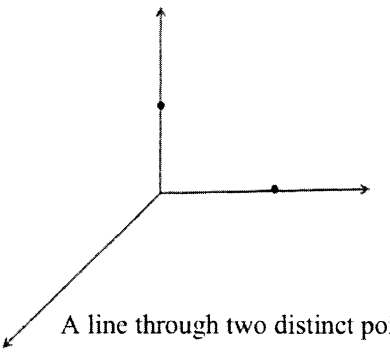
Intersection of two lines



A line through two distinct points



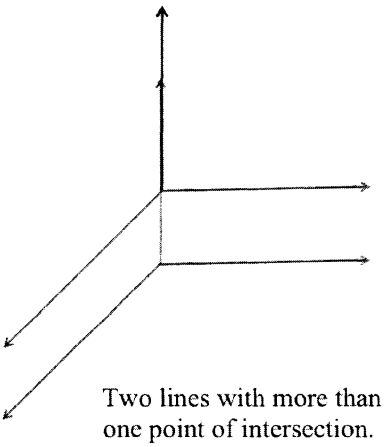
A line through two distinct points



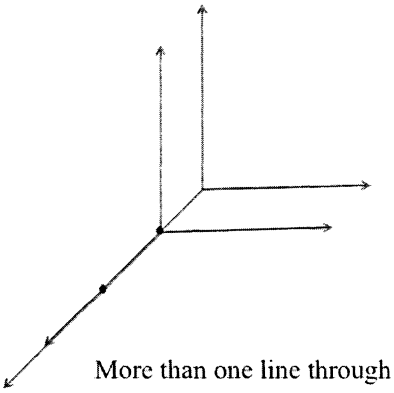
A line through two distinct points

Note that we have:

- some pairs of tropical lines that intersect at more that one point, and
- for some choices of pairs of points in the plane, there are infinitely many tropical line passing through these two points.



Two lines with more than one point of intersection.



More than one line through two distinct points

Matrix Operations

This notion can be adopted to define these tropical vector and matrix operations in $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$.

Definition: Let $A = [A_{ij}], B = [B_{ij}] \in \mathbb{R}^{m\times n}$. The tropical matrix sum, $A \oplus B$, is then obtained by evaluating the tropical sum of the corresponding entries. That is,

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \min\{A_{ij}, B_{ij}\}$$

Definition: Let $A = [A_{ij}] \in \mathbb{R}^{m\times n}$ and c be any scalar. The product $c \odot A$ (scalar multiplication) is obtained by adding c to each entry in A. That is,

$$c \odot A = [c \odot A_{ij}] = [c + A_{ij}]$$

Definition: Let $A = [A_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [B_{ij}] \in \mathbb{R}^{n\times p}$. The tropical matrix product, $C = A \odot B \in \mathbb{R}^{m\times p}$, is the given by the matrix $C = [C_{ij}]$ with entries

$$C_{ij} = \bigoplus_{k=1}^{n} A_{ik} \odot B_{kj} = \min\{A_{ik} + B_{kj}\}_{k=1,\ldots,n}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, p$

Example: In $\mathbb{R}^4$ tropical sum and tropical scalar multiplication can be performed as follows:

$$\begin{pmatrix} 4 \\ -1 \\ 2 \\ 8 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 5 \\ 0 \\ 12 \end{pmatrix} = \begin{pmatrix} \min\{4,3\} \\ \min\{-1,5\} \\ \min\{2,0\} \\ \min\{8,12\} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 8 \end{pmatrix}$$

$$3 \odot \begin{pmatrix} 4 \\ -1 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 3+4 \\ 3-1 \\ 3+2 \\ 3+8 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 5 \\ 11 \end{pmatrix}$$

Example: In $\mathbb{R}^{2\times 2}$ tropical sum and tropical multiplication of 2×2 matrices can be performed as follows:

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 \oplus 1 & -1 \oplus -7 \\ 2 \oplus 3 & 0 \oplus -6 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 2 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 \odot 1 \oplus -1 \odot 3 & 1 \odot 7 \oplus -1 \odot -6 \\ 2 \odot 1 \oplus 0 \odot 3 & 2 \odot 7 \oplus 0 \odot -6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \oplus 2 & 8 \oplus -7 \\ 3 \oplus 3 & 9 \oplus -6 \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 3 & -6 \end{pmatrix}$$

Definition: Let $A = [A_{ij}]$ be an $n \times n$ matrix. Define the tropical determinant $\text{tropdet}(A)$ by

$$\begin{aligned} \text{tropdet}(A) &= \bigoplus_{\sigma \in S_n} \{A_{1\sigma(1)} \odot A_{2\sigma(2)} \odot \ldots \odot A_{n\sigma(n)}\} \\ &= \min_{\sigma \in S_n} \{A_{1\sigma(1)} + A_{2\sigma(2)} + \ldots + A_{n\sigma(n)}\} \end{aligned}$$

S_n is a group of permutations of $\{1, 2, \ldots, n\}$

Example: Find the tropical determinants of A and B where

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 3 & 4 \\ 8 & 7 \end{pmatrix}.$$

$$\text{tropdet}(A) = (3 \odot 6) \oplus (2 \odot 4) = 9 \oplus 6 = 6$$

$$\text{tropdet}(B) = (3 \odot 7) \oplus (8 \odot 4) = 10 \oplus 12 = 10$$

$$A \odot B = \begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix} \odot \begin{pmatrix} 3 & 4 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix}$$

Note that,

$$\text{tropdet}(A \odot B) = 12 \neq 6 \odot 10 = \text{tropdet}(A) \odot \text{tropdet}(B)$$

The evaluation of the tropical determinant is the classical assignment problem of combinatorial optimization. For instance, suppose a company needs to assign n jobs to n workers in such a way that each worker can perform only one job and each job needs be assigned to exactly one of the workers. Let a_{ij} be the cost of assigning job i to worker j. For $\sigma \in S_n$, $A_{1\sigma(1)} + A_{2\sigma(2)} + ... + A_{n\sigma(n)}$ is the cost of assigning job i to worker $\sigma(i)$ then

$$\min_{\sigma \in S_n} \{A_{1\sigma(1)} + A_{2\sigma(2)} + ... + A_{n\sigma(n)}\}$$

is the cheapest assignment and is also the tropical determinant of the $n \times n$ matrix $A = (A_{ij})$.

The assignment problems in linear programming like minimizing total cost, minimizing total time, maximizing performance etc. can be reduced to the evaluation of the tropical determinant.

Definition: A permutation matrix is a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n.

Note that every row and column of a permutation matrix contains precisely a single 1 with $0's$ everywhere else, and every permutation corresponds to a unique permutation matrix. There are, therefore, $n!$ permutation matrices of size $n \times n$.

One way of solving assignment problems is a brute force algorithm which leads to generate all n! permutations of $\{1, 2, ..., n\}$ for $n \times n$ matrix. In practice, this is not a helpful method as it is not efficient for large n. However, in optimization theory, there is a well known polynomial time algorithm, known as the Hungarian Method, which can compute the tropical determinant.

Before discussing Hungarian Method, let $A = [A_{ij}]$ be any $n \times n$ matrix in which A_{ij} is the cost of assigning worker i to job j. Let $S = [s_{ij}]$ be the $n \times n$ matrix where

$$s_{ij} = \begin{cases} 1 & \text{if worker i is assigned to job j} \\ 0 & \text{otherwise} \end{cases}$$

In other words, S is the permutation for a particular worker-job assignment.

The assignment problem can then be expressed in terms of a function z as:

$$\text{minimize} \quad z(S) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}s_{ij} \qquad \text{over all } S$$

Any matrix S realizing this minimum is called a solution and corresponds to a permutation σ of N obtained by setting $\sigma(i) = j$ if and only if $s_{ij} = 1$. Furthermore, if S is a solution corresponding to σ, then

$$\sum_{j=1}^{n} A_{ij}s_{ij} = A_{i\sigma(i)}$$

Summing over i from 1 to n, we obtain

$$\sum_{i=1}^{n} A_{i\sigma(i)} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}s_{ij}$$

Thus, any solution S on which $z(S)$ is minimum is called an optimal solution.

The Hungarian Method: Consider a matrix $A = \{a_1, ..., a_n\} \in \mathbb{R}^{n\times n}$.

Step 1: For each row, subtract its minimum value from each entry in the row. This corresponds to $a_i = -(\oplus_j a_{ij}) \odot a_i$ and the resulting matrix is non-negative.

Step 2: For each column, subtract its minimum value from each entry in the column. This is just applying Step 1 to the transpose of A and then transposing back.

Step 3: Select rows and columns in a minimal way such that each 0 in the matrix is in one of the selected rows or columns. If the number of rows and columns chosen is n, then for the resulting matrix B, there exists $\sigma \in S_n$ such that $z(S) = 0$ when S is the permutation matrix for σ. Otherwise, continue to Step 4.

Step 4: Let a be the smallest entry in A that lies in none of the chosen rows and columns. Then subtract a from each element not in the chosen rows and columns, and add a to each element that is in both a chosen row and column. This corresponds to tropically scaling each chosen row and column by a and then scaling every other entry by $-a$. Go back to Step 3.

The idea behind the Hungarian method is to try to transform a given assignment problem specified by matrix $A = [A_{ij}]$ into another one specified by a matrix $A' = [A'_{ij}]$, such that each $A'_{ij} \geq 0$, where both problems have the same set of optimal solutions; and then find a solution S' for which

$$z(S') = \sum_{i=1}^{n}\sum_{j=1}^{n} A'_{ij}s'_{ij} = 0$$

Since $A'_{ij} \geq 0$, S' must be an optimal solution to the problem specified by A', and hence must also be an optimal solution to the one specified by A.

Theorem: [4] A solution S is an optimal solution for

$$z(S) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}s_{ij} = 0$$

if and only if it is an optimal solution for

$$z'(S) = \sum_{i=1}^{n}\sum_{j=1}^{n} A'_{ij}s_{ij} = 0$$

where $A'_{ij} = A_{ij} - u_i - v_j$ for any choice of $(u_1, ..., u_n)$ and $(v_1, ..., v_n)$ where u_i and v_j are real numbers for all i and j.

Proof. It is sufficient to show that the functions $z(S)$ and $z'(S)$ differ by a constant.

$$\begin{aligned}
z'(S) &= \sum_{i=1}^{n}\sum_{j=1}^{n} A'_{ij}s_{ij} = 0 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (A_{ij} - u_i - v_j)s_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}s_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} u_i s_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} v_j s_{ij} \\
&= z(S) - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} s_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{n} s_{ij} \\
&= z(S) - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j
\end{aligned}$$

$$\Longrightarrow z(S) - z'(S) = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

This implies z(S) and z'(S) differ only by the constant $\sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$ which completes the proof.

□

This fact describes how we can transform a matrix into another one which has the same set of optimal solutions.

Definition: Given any $n \times n$ matrix $A = [A_{ij}]$, let

$$u_i = \min\{A_{ij}\}_{j=1,\ldots,n} \quad \text{and} \quad v_j = \min\{A_{ij} - u_i\}_{i=1,\ldots,n}$$

The $n \times n$ matrix $A' = [A'_{ij}]$ given by $A'_{ij} = A_{ij} - u_i - v_j$ for all pairs i and j is called the reduced matrix for A.

A' is nothing more than the result of first two steps of Hungarian Method. After performing the first two steps of the Hungarian Method, all entries in the reduced matrix are non-negative.

If we have an independent set of zeros in a matrix of non-negative numbers the minimum for that matrix is zero and realized by the permutation determined by these zeros.

Step 4 is performed only when no independent set of n zeros exists. This step is equivalent to first subtracting the smallest entry k from every entry in the matrix, and then adding k to every entry covered by a line. Subtracting k from every entry is the transformation $u_i = k$, for all i, and $v_j = 0$, for all j. This does not change the set of optimal solutions but the sum of all entries will decrease by at least k.

Eventually, if the sum of all entries is zero, then all entries in the matrix are zero and an independent set of n zeros exists. Note that the algorithm terminates, because if that is not the case the sums of all matrix entries would give an infinite decreasing sequence of positive integers, which is impossible.

Example: Find the tropical determinant of the matrix A, where

$$A = \begin{pmatrix} 8 & 23 & 13 & 18 \\ 13 & 28 & 3 & 13 \\ 33 & 18 & 10 & 22 \\ 15 & 23 & 11 & 18 \end{pmatrix}.$$

To find the determinant, for each row, subtract its minimum value from each entry in the row. Then we have,

$$\begin{pmatrix} 0 & 15 & 5 & 10 \\ 10 & 25 & 0 & 10 \\ 23 & 8 & 0 & 12 \\ 0 & 8 & 7 & 3 \end{pmatrix}.$$

For each column, subtract its minimum value from each entry in the column.

$$\begin{pmatrix} 0 & 7 & 5 & 7 \\ 10 & 17 & 0 & 7 \\ 23 & 0 & 0 & 9 \\ 0 & 0 & 7 & 0 \end{pmatrix}.$$

Select rows and columns in a minimal way such that each 0 in the matrix is in one of the selected rows or columns.

$$\begin{pmatrix} 0 & 7 & 5 & 7 \\ 10 & 17 & 0 & 7 \\ 23 & 0 & 0 & 9 \\ 0 & 0 & 7 & 0 \end{pmatrix}.$$

The number of rows and columns chosen is 4 which is the order of the matrix, which corresponds to the fact that each row subscript i is assigned to exactly one column subscript and vice versa.

$$\begin{pmatrix} \mathbf{0} & 7 & 5 & 7 \\ 10 & 17 & \mathbf{0} & 7 \\ 23 & \mathbf{0} & \mathit{0} & 9 \\ \mathit{0} & \mathit{0} & 7 & \mathbf{0} \end{pmatrix}.$$

Hence, the minimum is attained and it corresponds to the permutation transposition $\sigma = (2, 3)$, (set of zeros marked in bold face).

Therefore, the tropical determinant is the sum of the corresponding entries of the original matrix. That is $\text{tropdet}(A) = 8 + 3 + 18 + 18 = 47$

Example: Suppose four workers must be assigned to four jobs, and the matrix A, given below, indicates the cost of training each worker for each job. How should the workers be assigned to jobs so that each worker is assigned one job, each job is assigned one worker, and the total training cost is minimized?

$$A = \begin{pmatrix} 4 & 5 & 6 & 3 \\ 7 & 2 & 19 & 16 \\ 4 & 12 & 5 & 11 \\ 6 & 3 & 13 & 9 \end{pmatrix}.$$

Solving this problem is equivalent to finding the tropical determinant of the matrix A. Similar to the previous example, for each row, subtract its minimum value from each entry in the row. Then we have,

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 5 & 0 & 17 & 14 \\ 0 & 8 & 1 & 7 \\ 3 & 0 & 10 & 6 \end{pmatrix}.$$

For each column, subtract its minimum value from each entry in the column.

$$\begin{pmatrix} 1 & 2 & 2 & 0 \\ 5 & 0 & 16 & 14 \\ 0 & 8 & 0 & 7 \\ 3 & 0 & 9 & 6 \end{pmatrix}.$$

Select rows and columns in a minimal way such that each 0 in the matrix is in one of the selected rows or columns.

$$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{2} & \cancel{0} \\ 5 & \cancel{0} & 16 & 14 \\ \cancel{0} & \cancel{8} & \cancel{0} & \cancel{7} \\ 3 & \cancel{0} & 9 & 6 \end{pmatrix}.$$

The number of rows and columns chosen is 3 which is less than 4, hence we precede to step 4. Observe that 3 is the smallest entry that lie in none of the chosen rows and columns, Subtracting 3 from each element not in the chosen rows and columns and adding 3 to each element that is in both a chosen row and column yields

$$\begin{pmatrix} \cancel{1} & \cancel{5} & \cancel{2} & \cancel{0} \\ 2 & \cancel{0} & 13 & 11 \\ \cancel{0} & \cancel{11} & \cancel{0} & \cancel{7} \\ \cancel{0} & \cancel{0} & \cancel{6} & \cancel{3} \end{pmatrix}.$$

with a corresponding independent set of zeros marked in bold face.

$$\begin{pmatrix} 1 & 5 & 2 & \mathbf{0} \\ 2 & \mathbf{0} & 13 & 11 \\ \mathit{0} & 11 & \mathbf{0} & 7 \\ \mathbf{0} & \mathit{0} & 6 & 3 \end{pmatrix}.$$

Hence, the minimum assignment corresponds to the permutation transposition $\sigma = (1, 4)$.

Therefore, $\text{tropdet}(A) = 3 + 2 + 5 + 6 = 16$

Powers of a Matrix

In graph theory, a directed graph (or digraph) is a graph, or set of nodes (vertices) connected by edges, where the edges have a direction associated with them.

A weighted graph is a digraph which has values attached to the directed edges. These values represent the cost of traveling from one node to the next. The figure below is an example of directed graph.

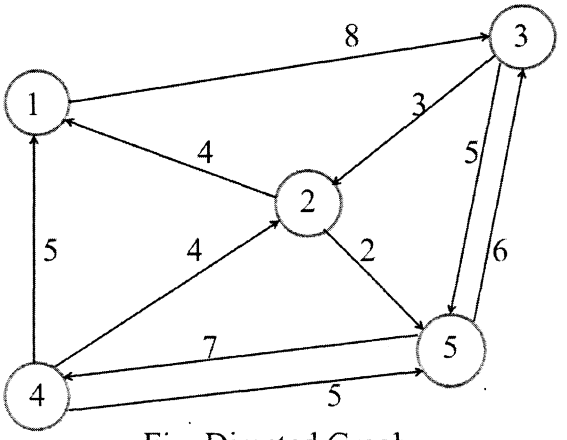


Fig. Directed Gragh

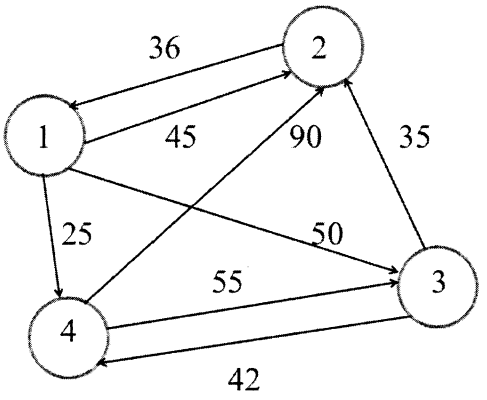
The graph may represents flight distance between terminals, the dollar cost of a plane ticket between the hubs, the time spent to travel form one city to the other etc. A weighted graph can be represented by a matrix.

Suppose that one has a directed graph with vertices $V = \{1, ..., n\}$ and weighted edges E. Then we can form the transition cost matrix $C \in R^{n \times n}$ where C_{ij} is the weight given to the edge (i, j). By convention $C_{ii} = 0$ and if edge (i, j) does not exists then $C_{ij} = \infty$. If we interpret the weight of an edge (i, j) as the cost of moving from vertex i to vertex j, then we will show the tropical power $(C^m)_{ij}$ will be the minimum cost of moving from vertex i to vertex j in at most m steps.

Definition: A weighted adjacency matrix is a matrix representation A of a weighted graph. A_{ij} is the weight of the edge (if any) from vertex i to vertex j.

In the next example, the shortest path problem will be used to illustrate the power of tropical matrix multiplication.

Example: Consider the weighted graph, given below, which represents the cost of flight between pair of cities of the given four cities



Ticket Cost

The corresponding adjacency (transition) matrix is given by

$$C = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}.$$

Notice that

$$C^2 = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

Observe that $(3,1)$ entry, 71, corresponds to the path length from vertex 3 to vertex 1, $\boxed{3} \longrightarrow \boxed{2} \longrightarrow \boxed{1}$, and $(1,3)$ entry, 50, corresponds to the path length

from vertex 1 to vertex 3, $\boxed{1} \longrightarrow \boxed{3}$. In fact, each entry $(C^2)_{ij}$ is the minimum length of all possible paths from vertex i to j using at most in 2 steps.

$$C^3 = C^2 \odot C = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

$$C^4 = C^2 \odot C^2 = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

Moreover, for any $m \in \mathbb{N}$ and $m \geq 3$, we will have

$$C^m = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

In fact, for any $n \times n$ adjacency matrix C, the $(ij)^{th}$ entry of the matrix C^{n-1} is the length of a shortest path from vertex i to vertex j in the corresponding graph.

Notice that, from the definition of tropical matrix multiplication and the associativity of the tropical operations, for $n \geq 2$, we can use the recursive formula given

below to evaluate the $(ij)^{th}$ entry of C^{n-1}

$$C_{ij}^{n} = \min\{C_{ik}^{n-1} + C_{kj} : k = 1, 2, ..., n\}$$

Theorem: Let G be a weighted directed graph on n nodes with $n \times n$ adjacency matrix D_G. The entry of the matrix D_G^{n-1}(tropical matrix power) in row i and column j equals the length of a shortest path from node i to node j in G.

Proof. Let $d_{ij}^{(r)}$ denote the minimum length of any path from node i to node j which uses at most r edges in G. We have $d_{ij}^{(1)} = d_{ij}$ for any two nodes i and j.

Now, any shortest path in the directed graph G uses at most $n - 1$ directed edges. Hence the length of a shortest path from i to j equals $d_{ij}^{(n-1)}$.

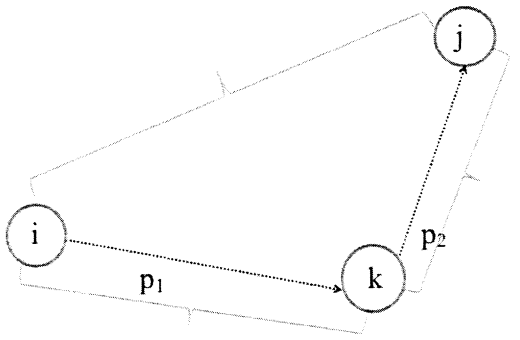


Fig. Shortest path from i to j

For $r \geq 2$ we have a recursive formula for the length of a shortest path. As shown in the figure above, if path p is a shortest path (of length r) from vertex i to vertex j, and k is next to last vertex of p. Then path p is the shortest path p_1 (of length $r - 1$), the portion of path p from vertex i to vertex k, plus path p_2 (of length 1) from vertex k to vertex j.

This gives

$$d_{ij}^{(r)} = \min\{d_{ik}^{(r-1)} + d_{kj} : k = 1, 2, ..., n\}$$

Using tropical arithmetic, this formula can be rewritten as follows:

$$d_{ij}^{(r)} = d_{i1}^{(r-1)} \odot d_{1j} \oplus d_{i2}^{(r-1)} \odot d_{2j} \oplus ... \oplus d_{in}^{(r-1)} \odot d_{nj}$$

$$= (d_{i1}^{(r-1)}, d_{i2}^{(r-1)}, ..., d_{in}^{(r-1)}) \odot (d_{1j}, d_{2j}, ..., d_{nj})$$

From this it follows, by induction on r, that $d_{ij}^{(r)}$ coincides with the entry in row i and column j of the tropical product of row i of D_G^{r-1} and column j of D_G, which is the (i, j) entry of D_G^r. This proves $d_{ij}^{(r)} = D_G^r$, for $r \leq n-1$. In particular, $d_{ij}^{(n-1)} = D_G^{(n-1)}$, since the edge weights d_{ij} were assumed to be non-negative, a shortest path from node i to node j visits each node of G at most once. So $D_G^{(n-1)}$ actually forms shortest path from i to j. This proves the result.

□

This recursive formula is essentially the *Dijkstra's* algorithm. The iteration will be more interesting if we replace the diagonal entries by $C_{ii} = \infty$, i.e. when we restrict ourselves of no possibility of having a loop in our graph. In fact, the iterations of the form A^k, where $k \in \mathbb{N}$ will reveal some important features of the graph $G(A)$.

Consider, the corresponding transition cost matrix with diagonal entries set to ∞:

$$C = \begin{pmatrix} \infty & 45 & 50 & 25 \\ 36 & \infty & \infty & \infty \\ \infty & 35 & \infty & 42 \\ \infty & 90 & 55 & \infty \end{pmatrix}$$

Notice that

$$C^2 = \begin{pmatrix} \infty & 45 & 50 & 25 \\ 36 & \infty & \infty & \infty \\ \infty & 35 & \infty & 42 \\ \infty & 90 & 55 & \infty \end{pmatrix} \odot \begin{pmatrix} \infty & 45 & 50 & 25 \\ 36 & \infty & \infty & \infty \\ \infty & 35 & \infty & 42 \\ \infty & 90 & 55 & \infty \end{pmatrix}$$

$$= \begin{pmatrix} 81 & 85 & 80 & 92 \\ \infty & 81 & 86 & 61 \\ 71 & 132 & 97 & \infty \\ 126 & 90 & \infty & 97 \end{pmatrix}$$

$$C^3 = C^2 \odot C = \begin{pmatrix} 81 & 85 & 80 & 92 \\ \infty & 81 & 86 & 61 \\ 71 & 132 & 97 & \infty \\ 126 & 90 & \infty & 97 \end{pmatrix} \odot \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 121 & 115 & 131 & 106 \\ 117 & 121 & 116 & 128 \\ 168 & 116 & 121 & 96 \\ 126 & 171 & 152 & 151 \end{pmatrix}$$

$$C^4 = C^2 \odot C^2 = \begin{pmatrix} 81 & 85 & 80 & 92 \\ \infty & 81 & 86 & 61 \\ 71 & 132 & 97 & \infty \\ 126 & 90 & \infty & 97 \end{pmatrix} \odot \begin{pmatrix} 81 & 85 & 80 & 92 \\ \infty & 81 & 86 & 61 \\ 71 & 132 & 97 & \infty \\ 126 & 90 & \infty & 97 \end{pmatrix}$$

$$= \begin{pmatrix} 151 & 166 & 166 & 146 \\ 157 & 151 & 166 & 142 \\ 152 & 156 & 151 & 163 \\ 207 & 171 & 176 & 151 \end{pmatrix}$$

Notice that the matrix entry $[A^k]_{ij}$ gives the path of least weight of length k from vertex i to vertex j in $G(A)$. In particular, each diagonal entry $[A^k]_{ii}$ gives the cycle of least weight of length k beginning at vertex i in $G(A)$.

Eigenvalues and Eigenvectors

Definition: Let A be an $n \times n$-matrix with entries in the tropical semiring $(\mathbb{T}, \oplus, \odot)$. An eigenvalue of A is a real number λ such that

$$A \odot v = \lambda \odot v$$

for some $v \in \mathbb{R}^n$. The vector v is called an eigenvector of the tropical matrix A.

Example: Consider a matrix A and a vector v such that

$$A = \begin{pmatrix} 7 & 6 \\ 6 & 4 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$A \odot v = \begin{pmatrix} 7 & 6 \\ 6 & 4 \end{pmatrix} \odot \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \odot 5 \oplus 6 \odot 3 \\ 6 \odot 5 \oplus 4 \odot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \oplus 9 \\ 11 \oplus 7 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} = 4 \odot \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Therefore, $\lambda = 4$ is an eigenvalue of the matrix A and v is eigenvector in the corresponding eigenspace.

Definition: Let G be a weighted digraph. The graph G is said to be strongly connected if there is a path from any vertex to any other vertex.

Definition: A tropical matrix A is irreducible if and only if its corresponding graph, $G(A)$, is strongly connected.

Definition: The normalized length of a directed path $i_0, i_1, ..., i_k$ in $G(A)$ is the sum (in classical arithmetic) of the lengths of the edges divided by the number of edges of the path. Thus the normalized length is

$$\frac{(a_{i_0 i_1} + a_{i_1 i_2} + ... + a_{i_{k-1} i_k})}{k}$$

If $i_k = i_0$ then the path is a directed cycle and we refer to this quantity as the normalized length of the cycle.

Theorem: [1] Let A be a tropical $n \times n$-matrix whose graph G(A) is strongly connected. Then A has precisely one eigenvalue $\lambda(A)$. That eigenvalue equals the minimal normalized length of any directed cycle in $G(A)$.

Proof. Let $\lambda = \lambda(A)$ be the minimum of the normalized lengths over all directed cycles in $G(A)$. We first prove that $\lambda(A)$ is the only possibility for an eigenvalue. Suppose that $z \in \mathbb{R}^n$ is any eigenvector of A, and let γ be the corresponding eigenvalue. For any cycle $(i_1, i_2, ..., i_k, i_1) \in G(A)$ we have

$$a_{i_1 i_2} + z_{i_2} \geq \gamma + z_{i_1}, a_{i_2 i_3} + z_{i_3} \geq \gamma + z_{i_2}$$

$$a_{i_3 i_4} + z_{i_3} \geq \gamma + z_{i_1}, ..., a_{i_k i_1} + z_{i_1} \geq \gamma + z_{i_k}$$

Adding the left hand sides and the right hand sides, we find that the normalized length of the cycle is greater than or equal to γ. In particular, we have $\lambda(A) \geq \gamma$. For the reverse inequality, start with any index i_1. Since z is an eigenvector with eigenvalue γ, there exists i_2 such that $a_{i_1 i_2} + z_{i2} = \gamma + z_{i_1}$. Likewise, there exists i_3 such that $a_{i_2 i_3} + z_{i3} = \gamma + z_{i_2}$. We continue in this manner until we reach an index i_l which was already in the sequence, say, $i_k = i_l$ for $k < l$. By adding the equations along this cycle, we find that

$$(a_{i_k i_{k+1}} + z_{i_{k+1}}) + (a_{i_(k+1) i_{k+2}} + z_{i_{k+2}}) + ... + (a_{i_{l-1} i_l} + z_{i_l})$$

$$= (\gamma + z_{i_k}) + (\gamma + z_{i_{k+1}}) + ... + (\gamma + z_{i_l})$$

We conclude that the normalized length of the cycle $(i_k, i_{k+1}, ..., i_l = i_k)$ in $G(A)$ is equal to γ. In particular, $\gamma \geq \lambda(A)$. This proves that $\gamma = \lambda(A)$. It remains to prove the existence of an eigenvector. Let B be the matrix obtained from A by (classically) subtracting $\lambda(A)$ from every entry in A. All cycles in the weighted graph $G(B)$ have non-negative length, and there exists a cycle of length zero. Using tropical matrix operations we define

$$B^* = B \oplus B^2 \oplus B^3 \oplus ... \oplus B^n$$

The entry B^*_{ij} in row i and column j of the matrix B^* is the length of a shortest path from node i to node j in the weighted directed graph $G(B)$. Since the graph is strongly connected, we have $B^*_{ij} < \infty$. Moreover,

$$(Id \oplus B) \odot B^* = B^* \qquad (**)$$

Here $Id = B^0$ is the tropical identity matrix whose diagonal entries are 0 and off-diagonal entries are ∞. Fix any node j that lies on a zero length cycle of $G(B)$, and let $x = B^*_j$ denote the j^{th} column vector of the matrix B^*. We have $x_j = B^*_{ij} = 0$. This property together with (**) implies

$$x = (Id \oplus B) \odot x = x \oplus B \odot x = B \odot x$$

This is because

- i^{th} entry of $B \odot x = \min\{b_{i1} + x_1, b_{i2} + x_2, \ldots, b_{in} + x_n\}$, but $x_j = 0$ so the j^{th} term in this is just b_{ij}. This means that the i^{th} entry of $B \odot x$ is less than or equal to b_{ij}.

- $B \odot x = B \odot (j^{th}$ col of $B^*) = j^{th}$col of $B \odot B^* = j^{th}$ col of $(B^2 \oplus \ldots \oplus B^{n+1})$

- i^{th} entry of $B \odot x$ is $\min\{B^2_{ij}, B^3_{ij}, \ldots, B^{n+1}_{ij}\}$
  $= \min\{B_{ij}, B^2_{ij}, B^3_{ij}, \ldots, B^{n+1}_{ij}\}$ from the first point
  $\leq \min\{B_{ij}, B^2_{ij}, B^3_{ij}, \ldots, B^n_{ij}\}$
  $= B^*_{ij} = x_i$

This means $x \oplus B \odot x = B \odot x$. Hence, we conclude that x is an eigenvector with eigenvalue λ of our matrix A:

$$A \odot x = (\lambda \odot B) \odot x = \lambda \odot (B \odot x) = \lambda \odot x$$

This completes the proof of Theorem □

Definition: The eigenspace of the matrix A, is the set

$$Eig(A) = \{v \in \mathbb{R}^n : A \odot v = \lambda(A) \odot v\}$$

One can simply observe that $Eig(A)$ is closed under tropical scalar multiplication, that is, if $v \in Eig(A)$ and $c \in \mathbb{R}$ then $c \odot v$ is also in $Eig(A)$.

This leads to the fact every eigenvector of the matrix A is also an eigenvector of the matrix $B = (-\lambda(A)) \odot A$ and vice versa. Hence the eigenspace

$$Eig(B) = \{v \in \mathbb{R}^n : B \odot v = v\}$$

For instance, consider the directed graph G given below. It is strongly connected, and its adjacency matrix A is irreducible.

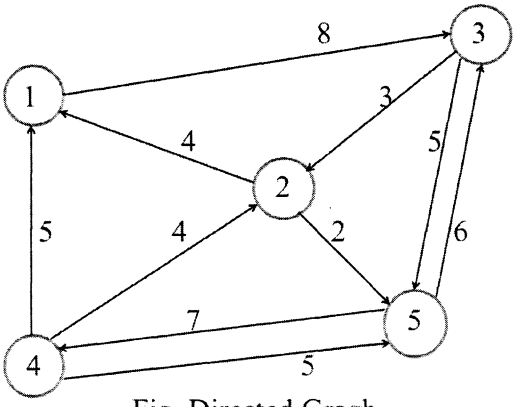


Fig. Directed Gragh

The adjacency matrix A is given by

$$A = \begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \infty & 11 & \infty & \infty & 13 \\ \infty & \infty & 8 & 9 & \infty \\ 7 & \infty & 11 & 12 & 5 \\ 8 & \infty & 11 & 12 & 6 \\ 12 & 9 & \infty & \infty & 11 \end{pmatrix}$$

$$A^3 = A^2 \odot A = \begin{pmatrix} \infty & 11 & \infty & \infty & 13 \\ \infty & \infty & 8 & 9 & \infty \\ 7 & \infty & 11 & 12 & 5 \\ 8 & \infty & 11 & 12 & 6 \\ 12 & 9 & \infty & \infty & 11 \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 15 & \infty & 19 & 20 & 13 \\ 14 & 11 & \infty & \infty & 13 \\ 17 & 14 & 11 & 12 & 16 \\ 17 & 14 & 12 & 13 & 16 \\ 15 & \infty & 17 & 18 & 11 \end{pmatrix}$$

$$A^4 = A^2 \odot A^2 = \begin{pmatrix} \infty & 11 & \infty & \infty & 13 \\ \infty & \infty & 8 & 9 & \infty \\ 7 & \infty & 11 & 12 & 5 \\ 8 & \infty & 11 & 12 & 6 \\ 12 & 9 & \infty & \infty & 11 \end{pmatrix} \odot \begin{pmatrix} \infty & 11 & \infty & \infty & 13 \\ \infty & \infty & 8 & 9 & \infty \\ 7 & \infty & 11 & 12 & 5 \\ 8 & \infty & 11 & 12 & 6 \\ 12 & 9 & \infty & \infty & 11 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 25 & 22 & 19 & 20 & 24 \\ 15 & \infty & 19 & 20 & 13 \\ 17 & 14 & 22 & 23 & 16 \\ 18 & 15 & 22 & 23 & 16 \\ 23 & 20 & 17 & 18 & 22 \end{pmatrix}$$

$$A^5 = A^4 \odot A = \begin{pmatrix} 25 & 22 & 19 & 20 & 24 \\ 15 & \infty & 19 & 20 & 13 \\ 17 & 14 & 22 & 23 & 16 \\ 18 & 15 & 22 & 23 & 16 \\ 23 & 20 & 17 & 18 & 22 \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 25 & 22 & 30 & 31 & 24 \\ 25 & 22 & 19 & 20 & 24 \\ 18 & 25 & 22 & 23 & 16 \\ 19 & 25 & 22 & 23 & 17 \\ 23 & 22 & 28 & 29 & 22 \end{pmatrix}$$

Recall that the matrix entry $[A^k]_{ij}$ gives the path (or cycle) of least weight of length k from vertex i to vertex j in $G(A)$, and hence to find the shortest path (or cycle), we need to evaluate the tropical sum these power matrices A^*:

$$A^* = A \oplus A^2 \oplus A^3 \oplus A^4 \oplus A^5$$

$$\Longrightarrow A^* = \begin{pmatrix} 15 & 11 & 8 & 20 & 13 \\ 4 & 11 & 8 & 9 & 2 \\ 7 & 3 & 11 & 12 & 5 \\ 5 & 4 & 11 & 12 & 5 \\ 12 & 9 & 6 & 7 & 11 \end{pmatrix}$$

Note that the minimum diagonal entries of A^* is the minimal length of the cycles in the graph of G. That is

$$\min\{15, 11, 11, 12, 11\} = 11$$

However, the eigenvalue $\lambda(A)$ is the minimal normalized length which is

$$\lambda(A) = \min\{\frac{15}{3}, \frac{11}{3}, \frac{11}{3}, \frac{13}{3}, \frac{11}{3}\} = \frac{11}{3}$$

Notice that

$$\begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix} \odot \begin{pmatrix} 2 \\ -\frac{5}{3} \\ -\frac{7}{3} \\ -\frac{4}{3} \\ 0 \end{pmatrix} = \frac{11}{3} \odot \begin{pmatrix} 2 \\ -\frac{5}{3} \\ -\frac{7}{3} \\ -\frac{4}{3} \\ 0 \end{pmatrix}$$

which is in the form of $A \odot v = \lambda v$

In fact, we can use the proof proceeding theorem to find this eigenvector v. To see this consider the given adjacency matrix A and define martix B by subtracting the eigenvalue λ from each entry,

$$B = \begin{pmatrix} \infty & \infty & \frac{13}{3} & \infty & \infty \\ \frac{1}{3} & \infty & \infty & \infty & -\frac{5}{3} \\ \infty & -\frac{2}{3} & \infty & \infty & \frac{4}{3} \\ \frac{4}{3} & \frac{1}{3} & \infty & \infty & \frac{4}{3} \\ \infty & \infty & \frac{7}{3} & \frac{10}{3} & \infty \end{pmatrix} = -\frac{5}{3} \odot \begin{pmatrix} \infty & \infty & 6 & \infty & \infty \\ 2 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 3 \\ 3 & 2 & \infty & \infty & 3 \\ \infty & \infty & 4 & 5 & \infty \end{pmatrix}$$

$$B^2 = -\frac{10}{3} \odot \begin{pmatrix} \infty & \infty & 6 & \infty & \infty \\ 2 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 3 \\ 3 & 2 & \infty & \infty & 3 \\ \infty & \infty & 4 & 5 & \infty \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 6 & \infty & \infty \\ 2 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 3 \\ 3 & 2 & \infty & \infty & 3 \\ \infty & \infty & 4 & 5 & \infty \end{pmatrix}$$

$$= -\frac{10}{3} \odot \begin{pmatrix} \infty & 7 & \infty & \infty & 9 \\ \infty & \infty & 4 & 5 & \infty \\ 3 & \infty & 7 & 8 & 1 \\ 4 & \infty & 7 & 8 & 2 \\ 8 & 5 & \infty & \infty & 8 \end{pmatrix} = \begin{pmatrix} \infty & \frac{11}{3} & \infty & \infty & \frac{17}{3} \\ \infty & \infty & \frac{2}{3} & \frac{5}{3} & \infty \\ -\frac{1}{3} & \infty & \frac{11}{3} & \frac{14}{3} & -\frac{7}{3} \\ \frac{2}{3} & \infty & \frac{11}{3} & \frac{14}{3} & -\frac{4}{3} \\ \frac{14}{3} & \frac{5}{3} & \infty & \infty & \frac{14}{3} \end{pmatrix}$$

$$B^3 = -\frac{15}{3} \odot \begin{pmatrix} \infty & 7 & \infty & \infty & 9 \\ \infty & \infty & 4 & 5 & \infty \\ 3 & \infty & 7 & 8 & 1 \\ 4 & \infty & 7 & 8 & 2 \\ 8 & 5 & \infty & \infty & 8 \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 6 & \infty & \infty \\ 2 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 3 \\ 3 & 2 & \infty & \infty & 3 \\ \infty & \infty & 4 & 5 & \infty \end{pmatrix}$$

$$= -\frac{15}{3} \begin{pmatrix} 9 & \infty & 13 & 14 & 7 \\ 8 & 5 & \infty & \infty & 7 \\ 11 & 8 & 5 & 6 & 10 \\ 11 & 8 & 6 & 7 & 10 \\ 7 & \infty & 12 & 13 & 5 \end{pmatrix} = \begin{pmatrix} \frac{12}{3} & \infty & \frac{24}{3} & \frac{27}{3} & \frac{6}{3} \\ \frac{9}{3} & 0 & \infty & \infty & \frac{6}{3} \\ \frac{18}{3} & \frac{9}{3} & 0 & \frac{3}{3} & \frac{15}{3} \\ \frac{18}{3} & \frac{9}{3} & \frac{3}{3} & \frac{6}{3} & \frac{15}{3} \\ \frac{6}{3} & \infty & \frac{21}{3} & \frac{24}{3} & 0 \end{pmatrix}$$

$$B^4 = -\frac{20}{3} \odot \begin{pmatrix} \infty & 7 & \infty & \infty & 9 \\ \infty & \infty & 4 & 5 & \infty \\ 3 & \infty & 7 & 8 & 1 \\ 4 & \infty & 7 & 8 & 2 \\ 8 & 5 & \infty & \infty & 8 \end{pmatrix} \odot \begin{pmatrix} \infty & 7 & \infty & \infty & 9 \\ \infty & \infty & 4 & 5 & \infty \\ 3 & \infty & 7 & 8 & 1 \\ 4 & \infty & 7 & 8 & 2 \\ 8 & 5 & \infty & \infty & 8 \end{pmatrix}$$

$$= -\frac{20}{3} \odot \begin{pmatrix} 17 & 14 & 11 & 12 & 17 \\ 7 & \infty & 11 & 12 & 5 \\ 9 & 6 & 14 & 15 & 8 \\ 10 & 7 & 14 & 15 & 8 \\ 16 & 13 & 9 & 10 & 16 \end{pmatrix} = \begin{pmatrix} \frac{31}{3} & \frac{22}{3} & \frac{13}{3} & \frac{16}{3} & \frac{31}{3} \\ \frac{1}{3} & \infty & \frac{13}{3} & \frac{16}{3} & -\frac{5}{3} \\ \frac{7}{3} & -\frac{2}{3} & \frac{22}{3} & \frac{25}{3} & \frac{4}{3} \\ \frac{10}{3} & \frac{1}{3} & \frac{22}{3} & \frac{25}{3} & \frac{4}{3} \\ \frac{28}{3} & \frac{19}{3} & \frac{7}{3} & \frac{10}{3} & \frac{28}{3} \end{pmatrix}$$

$$B^5 = -\frac{25}{3} \odot \begin{pmatrix} 17 & 14 & 11 & 12 & 17 \\ 7 & \infty & 11 & 12 & 5 \\ 9 & 6 & 14 & 15 & 8 \\ 10 & 7 & 14 & 15 & 8 \\ 16 & 13 & 9 & 10 & 16 \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 6 & \infty & \infty \\ 2 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 3 \\ 3 & 2 & \infty & \infty & 3 \\ \infty & \infty & 4 & 5 & \infty \end{pmatrix}$$

$$= -\frac{25}{3} \odot \begin{pmatrix} 15 & 12 & 21 & 22 & 14 \\ 15 & 12 & 9 & 10 & 14 \\ 8 & 15 & 12 & 13 & 6 \\ 9 & 15 & 12 & 13 & 7 \\ 13 & 10 & 20 & 21 & 12 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} & \frac{11}{3} & \frac{38}{3} & \frac{41}{3} & \frac{17}{3} \\ \frac{20}{3} & \frac{11}{3} & \frac{2}{3} & \frac{5}{3} & \frac{17}{3} \\ -\frac{1}{3} & \frac{20}{3} & \frac{11}{3} & \frac{14}{3} & -\frac{7}{3} \\ \frac{2}{3} & \frac{20}{3} & \frac{11}{3} & \frac{14}{3} & -\frac{4}{3} \\ \frac{14}{3} & \frac{5}{3} & \frac{35}{3} & \frac{38}{3} & \frac{11}{3} \end{pmatrix}$$

Then

$$B^* = B \odot B^2 \odot B^3 \odot B^4 \odot B^5 = \begin{pmatrix} 4 & \frac{11}{3} & \frac{13}{3} & \frac{16}{3} & 2 \\ \frac{1}{3} & 0 & \frac{2}{3} & \frac{5}{3} & -\frac{5}{3} \\ -\frac{1}{3} & -\frac{2}{3} & 0 & 1 & -\frac{7}{3} \\ \frac{2}{3} & \frac{1}{3} & 1 & 2 & -\frac{4}{3} \\ 2 & \frac{5}{3} & \frac{7}{3} & \frac{10}{3} & 0 \end{pmatrix}$$

The proof of the previous theorem tells us that any column vector of the matrix B^*, for which $B^*_{ii} = 0$ is an eigenvector with eigenvalue $\lambda = \frac{11}{3}$. For this particular tropical matrix A, column vectors B^*_{i2}, B^*_{i3} and B^*_{i5} are the eigenvectors.

In addition, notice that column vectors B^*_{i1} and B^*_{i4} are also eigenvectors.

To see this,

$$B^*_{i1} = \begin{pmatrix} 4 \\ \frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \\ 2 \end{pmatrix} = 4 \odot \begin{pmatrix} 0 \\ -\frac{11}{3} \\ -\frac{13}{3} \\ -\frac{10}{3} \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix} \odot \begin{pmatrix} 0 \\ -\frac{11}{3} \\ -\frac{13}{3} \\ -\frac{10}{3} \\ -2 \end{pmatrix} = \frac{11}{3} \odot \begin{pmatrix} 0 \\ -\frac{11}{3} \\ -\frac{13}{3} \\ -\frac{10}{3} \\ -2 \end{pmatrix}$$

Similarly,

$$B^*_{i4} = \begin{pmatrix} \frac{16}{3} \\ \frac{5}{3} \\ 1 \\ 2 \\ \frac{10}{3} \end{pmatrix} = 2 \odot \begin{pmatrix} \frac{10}{3} \\ -\frac{1}{3} \\ -1 \\ 0 \\ \frac{4}{3} \end{pmatrix}$$

$$\begin{pmatrix} \infty & \infty & 8 & \infty & \infty \\ 4 & \infty & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty & 5 \\ 5 & 4 & \infty & \infty & 5 \\ \infty & \infty & 6 & 7 & \infty \end{pmatrix} \odot \begin{pmatrix} \frac{10}{3} \\ -\frac{1}{3} \\ -1 \\ 0 \\ \frac{4}{3} \end{pmatrix} = \frac{11}{3} \odot \begin{pmatrix} \frac{10}{3} \\ -\frac{1}{3} \\ -1 \\ 0 \\ \frac{4}{3} \end{pmatrix}$$

This is not surprising since every column in B^* is a tropical scalar multiplication of any other. Notice that, column vectors B^*_{i1} and B^*_{i4} are in the eigenspace of A, $Eig(A)$.

In fact, if B^*_0 be the submatrix of B^* given by the columns whose diagonal entry B^*_{jj} is zero. The image of this matrix (with respect to tropical multiplication of vectors on the right) is equal to the desired eigenspace.

$$Eig(A) = Eig(B) = Image(B^*_0)$$

In our example, the eigenspace is

$$Eig(A) = Image(B^*_0) = \begin{pmatrix} 2 \\ -\frac{5}{3} \\ -\frac{7}{3} \\ -\frac{4}{3} \\ 0 \end{pmatrix}$$

However, this is not always the case. For example, if we consider a tropical matrix

$$A = \begin{pmatrix} 1 & 4 & 4 & 6 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 6 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 4 & 4 & 6 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 6 & 4 \end{pmatrix} \odot \begin{pmatrix} 1 & 4 & 4 & 6 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 5 & 6 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 5 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 5 & 5 & 6 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 5 \end{pmatrix} \odot \begin{pmatrix} 1 & 4 & 4 & 6 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 6 & 7 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 6 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 2 & 5 & 5 & 6 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 5 \end{pmatrix} \odot \begin{pmatrix} 2 & 5 & 5 & 6 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 7 & 7 & 8 \\ 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 6 \\ 6 & 6 & 6 & 7 \end{pmatrix}$$

Then,

$$A^* = A \oplus A^2 \oplus A^3 \oplus A^4 = \begin{pmatrix} 1 & 4 & 4 & 6 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & 3 \\ 4 & 3 & 4 & 4 \end{pmatrix}$$

Hence, the eigenvalue $\lambda(A)$, the minimal normalized length, is

$$\lambda(A) = \min\{1, 1, 1, 4\} = 1$$

To find the eigenspace,

$$B = \begin{pmatrix} 0 & 3 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 5 & 2 & 5 & 3 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & 3 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 5 & 2 & 5 & 3 \end{pmatrix} \odot \begin{pmatrix} 0 & 3 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 5 & 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

$$B^3 = B^2 \odot B = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \odot \begin{pmatrix} 0 & 3 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 5 & 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

$$B^4 = B^2 \odot B^2 = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \odot \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

$$B^* = B \oplus B^2 \oplus B^3 \oplus B^4 = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

Here, we have a sub-matrix with all diagonal entries equal to zero. In addition, observe that the column vectors B^*_{i1}, B^*_{i2} and B^*_{i3} are eigenvectors and, unlike the column vectors of the previous example, they are linearly independent in tropical sense. Note that B^*_{i4} is a scalar multiple of B^*_{i2}. Therefore, the eigenspace $Eig(A)$ is

$$Eig(A) = Image \begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix} = Image \begin{pmatrix} -2 & 1 & 1 \\ -2 & -2 & -2 \\ -1 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that

$$\begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix} = 3 \odot \begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}$$

so the fourth column of B^* is in this eigenspace but not independent of the other three columns.

Conclusion

In this thesis, after defining the basic operations in tropical arithmetic, we have explored tropical versions of several well-known ideas in classical algebra and linear algebra: roots of polynomials, graphs of linear polynomials (tropical lines), matrix operations, determinants, eigenvalues and eigenspaces. It has been intriguing to see how similar and yet different many of the results are. For example, in the classical sense there are several equivalent definitions for the root of a polynomial. In the tropical world some of these don't even make sense and others have pretty straightforward translations into an appropriate and workable definition. Tropical lines have no notion of slope but in most cases two points do determine a unique line and in most cases two lines do intersect in a single point. Of course the word 'most' helps highlight the fact that there are still interesting differences.

Most of this paper deals with tropical versions of matrix arithmetic, determinants, eigenvalues and eigenvectors. While the tropical identity matrix looks much different it is just composed of the multiplicative identity (0) down the main diagonal and the additive identity (∞) in all other entries. The standard definition of the determinant as the sum of all the signed products of n-elements one from each row and column carries over – without the signs. We showed that, for square matrices with non-negative entries, there was a wonderful interpretation of the tropical determinant as the solution to a minimization problem. This give great practical uses for this computation. Moreover, we showed how this optimization representation provides a very efficient method for finding tropical determinants. A major improvement over the computation of ordinary determinants.

We also exploited the connection between matrices with non-negative tropical entries and weighted directed graphs. Here an entry of ∞ in the i, j spot corresponds to the absence of a directed edge from vertex i to vertex j. This led to a direct relationship between the powers of such an adjacency matrix and the shortest paths between vertices, which in turn provided the key elements of the surprising proof that such matrices have a unique eigenvalue which is directly related to the cycles

in the associated graph.

Along with the relevant proofs, the paper spent a great deal of time with the detailed computations of several illustrative examples for each of the concepts discussed. It is the hope that this effort will help provide both experience with the tropical operations and a more concrete basis for understanding how the tropical versions of classical concepts differ in sometimes strange and sometimes very profitable ways.

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