4-10-2007

Symbolic Computations of Exact Solutions to Nonlinear Integrable Di®erential Equations

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Symbolic Computations of Exact Solutions to Nonlinear Integrable Differential Equations

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics
College of Arts and Sciences
University of South Florida

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Date of Approval:
April 10, 2007

Keywords: The tanh method, PDE, KdV, Solitary wave, Wave equation

Acknowledgment

It was my great honor to work with a person of the magnitude, kindness and enthusiasm of Dr. Wen-Xiu MA. Only with his overwhelming support and help, is how I managed to finish this thesis in a painfully short period of time. Great man with big heart...Thank you Dr. Ma.

I would also like to thank my master thesis committee members, Dr. Athanassios G. Kartsatos and Dr. Yuncheng You for their open support to me and for their belief in my bright future.

At last but not the least, I would like to thank my family: mom Lile, dad Viktor, myself, big brother Aleksandar, little brother Dimitar, majce Shana, dedo Srbe[rip], dedo Krume, baba Stojka[rip]...also: tetka Nade, tetin Gligor, tetka Vence, tetin Dimce, tetka Letka, tetin Bonac-gotch ya, tetka Lile[rip], tetin Krste, Sofija, Marko, Biljana, Slobodan, Keti, Zarko[rip], Slavco, Marija, Anita, Mirjana, Elena, Gligorce, Martina, Darko, Ivona, Natalija...my friends: Kreco Vladimir, Anatoli, Kreco Zoran, Zote, Pop(Goran), Upe[rip], Mishe, Zura, Orce, Mirjana R.,... my US friends: Sase, Mintie, Ferenc, Ana, Philip(of horse), Dijana, Enver...the people who had helped me in my life : Marija Popovska, prof. Katerina Zdravkova...if I have forgoten somebody, please forgive me.

Special thanks to the dog and friend Adi.

Very special, from the bottom of my heart thanks to my significant other Angela, without whom I would not have started, nor finished my graduate studies in mathematics.

Dedication

to

A*lways...***N****ever...***G****ood...***E****vil...***L****ove...***A****ngel*

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Symbolic Computations of Exact Solutions to Nonlinear Integrable Differential Equations

Vladimir Grupcev

ABSTRACT

In this thesis, first the tanh method, a method for obtaining exact traveling wave solutions to nonlinear differential equations, is introduced and described. Then the method is applied to two classes of Nonlinear Partial Differential Equations. The first one is a system of two $(1+1)$-dimensional nonlinear Korteweg-de Vries (KdV) type equations and it is of the form:

$$\begin{aligned}\frac{\partial\phi}{\partial t} - 6\phi\frac{\partial\phi}{\partial x} + \frac{\partial^3\phi}{\partial x^3} &= \alpha\frac{\partial\psi}{\partial x} + \beta\frac{\partial^3\psi}{\partial x^3},\\ \frac{\partial\psi}{\partial t} - 6\mu\psi\frac{\partial\psi}{\partial x} + \eta\frac{\partial\psi}{\partial x} + \delta\frac{\partial^3\psi}{\partial x^3} &= k\frac{\partial\phi}{\partial x} + \gamma\frac{\partial^3\phi}{\partial x^3},\end{aligned}$$

where α, β, γ, μ, η, δ and k are constants. The second one is a $(3+1)$-dimensional nonlinear wave equation of the form:

$$\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial x}(\nabla^2\phi) + \alpha\frac{\partial^5\phi}{\partial x^5} = 0,$$

where α is a constant.

At the end, a few graphic representations of the obtained solitary wave solutions are provided, in correspondence to different values of the parameters used in the equations.

1 Introduction

As the earth evolves around the sun and deepens the gap in between, the inhabitants of the blue planet are determined more than ever to represent things they are seeing happening around through laws and equations. The more they learn the more they see the equations "written" on every natural phenomenon as a seal of its nature, features and interaction with other phenomena. Of course, the simpler the equation the better, but I guess we are still too "young" to match the nature with our system of representation. We know the existence of really complex laws represented through "heavy" mathematical equations. And not always and under arbitrary circumstances those equations have a solution. At least not one that is (yet) known to the man. That is the second, probably harder problem the mankind is facing in his representation of the nature. Namely, one can come up with an equation representing a phenomenon and then the solution of that equation is not always intuitive as it seems to be. But the world is trying...

Differential equations are among the most used equations to represent different kind of phenomena. The need of Differential Equations arises in almost every scientific field, no matter what the subject is. Moreover, whenever a deterministic dependence between some quantities that are continuously changing (usually modeled by functions) and the rates of their changes (formulated as derivatives) is known or postulated we reach for the differential equation to represent that relationship. Many of the fundamental laws of physics, chemistry and other sciences are expressed by using differential equations. Their usage in biology and economics is reflexed through modeling the behavior of complex systems.

One type of differential equations are the partial differential equations (PDEs). A large number of the fundamental processes in the nature can be explained by PDEs. For instance, the Einstein equations are used to describe the curvature of space-time caused by its interaction with energy and matter. The dynamics of elastic solids and fluids are conducted by PDEs that started drifting up at the time of Euler and Cauchy. James Clerk Maxwell modeled the electro-magnetic waves including the propagation of light in various media with PDEs known as Maxwell's equations. The Black-Sholes PDE corroborates the mathematical understanding of options pricing model in financial mathematics. Reaction-diffusion models are used to represent the rate of change of the size of a population, known as a population dynamics. Also in the field of Neural Networks, the learning methods of the network are done using PDEs, i.e., the work of the natural neurons is represented through PDEs.

Furthermore, The Nonlinear Partial Differential Equations are of great importance in many scientific fields. Many phenomena from fluid dynamics, particle and continuum mechanics are expressed by nonlinear PDEs, and the nonlinearities play a crucial role in the realistic representation.

One famous nonlinear partial differential equation, in which we are interested in this thesis, is the so-called Korteweg-de Vries Equation or the KdV equation. Its general form is

$$u_t(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0$$

where the subscripts denote the partial derivatives:

$$\begin{aligned} u_t(x,t) &= \frac{\partial u(x,t)}{\partial t}, \quad u_x(x,t) = \frac{\partial u(x,t)}{\partial x}, \\ u_{xx}(x,t) &= \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u_{xxx}(x,t) = \frac{\partial^3 u(x,t)}{\partial x^3}. \end{aligned}$$

The coefficient of 6 in front of the nonlinear term is chosen just because it gives an easer way of expressing solutions. The KdV equation is known to model waves in a canal or shallow water waves in general.

Partial Differential Equations are considered from a few different angels, but in most of the cases scientists are concerned about whether or not they have solutions. One important prospective of a solution to a given PDE is that some properties of the solution may be determined without even finding its exact form. Sometimes when there is no explicit formula that expresses a solution, we can use computer based symbolic computational software, like Maple and Mathematica, to approximate the solution.

But in general, when we deal with nonlinear differential equations there is no general algorithm that leads to families of solutions. The only exception is when the nonlinear differential equation manifests symmetries. In many cases, linear differential equations serve as approximations to nonlinear equations, having in mind that these approximations can be considered as solutions under restricted conditions. Here we are interested in nonlinear wave phenomena which appear almost in all areas of the natural sciences like fluid dynamics, chemistry, mathematical biology, solid state physics, etc. Because of their popularity and in need of their exact solutions, several methods have been introduced that produce exact solutions to different classes of nonlinear PDEs. Among such kind of methods are Hirota's bilinear technique, inverse scattering transform, Painlevé analysis.

The method we are going to use is the so-called tanh method.We will apply it to a system of two (1+1)-dimensional nonlinear KdV type equations of the form:

$$\begin{aligned}
&\frac{\partial\phi}{\partial t} - 6\phi\frac{\partial\phi}{\partial x} + \frac{\partial^3\phi}{\partial x^3} = \alpha\frac{\partial\psi}{\partial x} + \beta\frac{\partial^3\psi}{\partial x^3},\\
&\frac{\partial\psi}{\partial t} - 6\mu\psi\frac{\partial\psi}{\partial x} + \eta\frac{\partial\psi}{\partial x} + \delta\frac{\partial^3\psi}{\partial x^3} = k\frac{\partial\phi}{\partial x} + \gamma\frac{\partial^3\phi}{\partial x^3},
\end{aligned}$$

where α, β, γ, δ, μ, η and k are arbitrary constants. We also apply the tanh method to a (3+1)-dimensional nonlinear equation of the following form:

$$\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial x}(\nabla^2\phi) + \alpha\frac{\partial^5\phi}{\partial x^5} = 0,$$

where α is a arbitrary constant.

The thesis is structured as follows:

In section 2 we are going to introduce and give an overview of the tanh method. Section 3 covers solitary wave solutions to a system of two $(1+1)$-dimensional nonlinear KdV type equations. In section 4 we provide solitary solutions to a system of two $(3+1)$-dimensional nonlinear wave equation. Section 5 gives a few graphical presentations to the solutions obtained in section 3 and section 4. Finally, in section 6 we conclude the results obtained throughout this thesis and we address a few problems for further research.

2 The tanh method

2.1 Introduction

The tanh method or hyperbolic tangent method is a powerful facility for obtaining traveling wave solutions for classes of nonlinear wave equations (NLWEs) and nonlinear evolution equations (NLEEs). The tanh method was introduced many years ago and it has become a standard simple wave solution technique. In particular, this method plays an important role in solving the class of problems that exhibit dispersive effects and reaction-diffusion features. Using the tanh method, exact solutions as well as approximate solutions can be obtained for a vast class of nonlinear ordinary equations (ODEs) and partial differential equations (PDEs). All computations are not very "messy" but pretty much straightforward. One advantage of this method is that it bypasses annoying algebraic work. On top of that lies the fact that this method can provide solutions not obtainable by the other methods while determining solvable equations. So, all in all the tanh method serves as a really useful tool for obtaining solutions of NLWEs and NLEEs and the number of solvable equations using this method increases every day as scientists solve nonlinear PDEs this way.

2.2 Overview of the tanh method

Throughout the time, the tanh method has been modified and improved. There are several, slightly different forms of the method that are in use since the method has been proposed. In this subsection, we are going to present an overview of the tanh method that we are going to use in this thesis. More details on the method available in [1]. In this overview, the tanh method is described in 5 steps.

We start with a general nonlinear evolution equation:

$$u_t = K(u). \tag{2.1}$$

- ***Step 1***

Since we are looking for a traveling wave solution, we first introduce the wave variable

$$\zeta = c(x - \nu t) \quad for \quad (1+1) - \text{dimensional case},$$
$$\zeta = kx + my + nz - \omega t \quad for \quad (3+1) - \text{dimensional case}.$$

Moreover, we set:

$$u(x,t) = U(\zeta) \quad and \quad u(x,y,z,t) = U(\zeta). \tag{2.2}$$

The parameters k, m and n symbolize the wave numbers in x, y and z directions, respectively, and ω is the frequency, which is assumed to be a function of the wave numbers k, m and n. The introduction of the new variable ζ brings the following changes:

$$\begin{aligned}
&\frac{\partial}{\partial t} \to -c\nu\frac{d}{d\zeta}, \quad \frac{\partial}{\partial x} \to c\frac{d}{d\zeta}, \quad \frac{\partial}{\partial t} \to -c\nu\frac{d}{d\zeta} \quad in \quad (1+1) - \text{dimensional case},\\
&\frac{\partial}{\partial t} \to -\omega\frac{d}{d\zeta}, \quad \frac{\partial}{\partial x} \to k\frac{d}{d\zeta},\\
&\frac{\partial}{\partial y} \to m\frac{d}{d\zeta}, \quad \frac{\partial}{\partial z} \to n\frac{d}{d\zeta} \quad in \quad (1+3) - \text{dimensional case}.
\end{aligned} \tag{2.3}$$

Basically, in this step, we have transformed the starting PDE into an ODE.

- ***Step 2***

We integrate the newly obtained ODE as long as all of the terms in the equation contain derivatives in ζ. This step may be repeated so that a simpler equation is produced.

- ***Step 3***

In this step, we introduce a new independent variable $\tilde{y}$. This introduction yields the following changes of derivatives:

$$\begin{aligned}
&\frac{d}{d\zeta} \to (1-\tilde{y}^2)\frac{d}{d\tilde{y}},\\
&\frac{d^2}{d\zeta^2} \to (1-\tilde{y}^2)\Big[\frac{d}{d\tilde{y}} + (1-\tilde{y}^2)\frac{d}{d\tilde{y}}\Big],\\
&.\\
&.\\
&.
\end{aligned} \tag{2.4}$$

- ***Step 4***

  The following Sum is suggested as a solution:

$$\begin{aligned}
u(x,t) &= U(\zeta) = S(\tilde{y}) = \sum_{r=0}^{R} a_r \tilde{y}^r,\\
u(x,y,z,t) &= U(\zeta) = S(\tilde{y}) = \sum_{r=0}^{R} a_r \tilde{y}^r,
\end{aligned} \tag{2.5}$$

  where

$$\tilde{y} = tanh(\zeta) = \begin{cases} tanh[c(x-\nu t)], \\ tanh(kx+my+nz-\omega t). \end{cases}$$

  We should determine the degree of the linear term(s) of highest order and the degree of the nonlinear term(s) of highest order in the simplified equation. The parameter R is then obtained by balancing the linear term(s) with the nonlinear term(s) of highest order, i.e, by equating the degrees of these terms (every possible combination of them). We discard any non-positive solution for R just for the purpose of obtaining an analytical solution in closed form.

- ***Step 5***

  Put everything computed so far back into the original equation and apply the boundary conditions if any. In our case, we will only implement special boundary conditions for the suggested solutions, i.e. we are interested in solitary traveling wave solutions.

3 Exact solutions to a system of (1+1)-dimensional KdV type equations

Let us consider the coupled system of KdV type equations:

$$\begin{aligned} &\frac{\partial \phi}{\partial t} - 6\phi\frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = \alpha\frac{\partial \psi}{\partial x} + \beta\frac{\partial^3 \psi}{\partial x^3}, \\ &\frac{\partial \psi}{\partial t} - 6\mu\psi\frac{\partial \psi}{\partial x} + \eta\frac{\partial \psi}{\partial x} + \delta\frac{\partial^3 \psi}{\partial x^3} = k\frac{\partial \phi}{\partial x} + \gamma\frac{\partial^3 \phi}{\partial x^3}, \end{aligned} \tag{3.1}$$

where α , β , γ , δ , μ , η and k are real constants. We introduce

$$\zeta = c(x - \nu t), \quad \phi(x,t) = U(\zeta) \quad and \quad \psi(x,t) = W(\zeta). \tag{3.2}$$

Because of the introduction of the new variables, we observe the following changes

$$\begin{aligned} &\frac{\partial \phi}{\partial x} = c\frac{dU}{d\zeta}, \quad \frac{\partial^3 \phi}{\partial x^3} = c^3\frac{d^3 U}{d\zeta^3}, \quad \frac{\partial \phi}{\partial t} = -c\nu\frac{dU}{d\zeta}, \\ &\frac{\partial \psi}{\partial x} = c\frac{dW}{d\zeta}, \quad \frac{\partial^3 \psi}{\partial x^3} = c^3\frac{d^3 W}{d\zeta^3}, \quad \frac{\partial \psi}{\partial t} = -c\nu\frac{dW}{d\zeta}. \end{aligned} \tag{3.3}$$

If we put these formulas into the system (3.1), we get the following system:

$$\begin{aligned} &-c\nu\frac{dU}{d\zeta} - 6cU\frac{dU}{d\zeta} + c^3\frac{d^3 U}{d\zeta^3} = \alpha c\frac{dW}{d\zeta} + \beta c^3\frac{d^3 W}{d\zeta^3}, \\ &-c\nu\frac{dW}{d\zeta} - 6\mu cW\frac{dW}{d\zeta} + c\eta\frac{dW}{d\zeta} + \delta c^3\frac{d^3 W}{d\zeta^3} = kc\frac{dU}{d\zeta} + \gamma c^3\frac{d^3 W}{d\zeta^3}. \end{aligned} \tag{3.4}$$

Now, let us divide the two equations in (3.4) by c and integrate them with respect to

ζ once. As a result we get the system:

$$
\begin{aligned}
&-\nu U - 3U^2 + c^2\frac{d^2U}{d\zeta^2} = \alpha W + \beta c^2\frac{d^2W}{d\zeta^2},\\
&-\nu W - 3\mu W^2 + \eta W + \delta c^2\frac{d^2W}{d\zeta^2} = kU + \gamma c^2\frac{d^2U}{d\zeta^2}.
\end{aligned} \tag{3.5}
$$

Now, we introduce $\tilde{y} = tanh(\zeta)$ as a new independent variable. We also make the following substitutions and propose the following series expansion to be a solution for $\phi(x,t)$ and $\psi(x,t)$:

$$
\begin{aligned}
\phi(x,t) = U(\zeta) = F(\tilde{y}) = \sum_{r_1=0}^{R_1} a_{r_1}\tilde{y}^{r_1},\\
\psi(x,t) = W(\zeta) = G(\tilde{y}) = \sum_{r_2=0}^{R_2} b_{r_2}\tilde{y}^{r_2}.
\end{aligned} \tag{3.6}
$$

Because of the substitutions, we observe the following changes:

$$
\begin{aligned}
\frac{dU}{d\zeta} = \tilde{y}'\frac{dF}{d\tilde{y}}, \quad \frac{d^2U}{d\zeta^2} = \tilde{y}''\frac{dF}{d\tilde{y}} + \tilde{y}'^2\frac{d^2F}{d\tilde{y}^2},\\
\frac{dW}{d\zeta} = \tilde{y}'\frac{dG}{d\tilde{y}}, \quad \frac{d^2W}{d\zeta^2} = \tilde{y}''\frac{dG}{d\tilde{y}} + \tilde{y}'^2\frac{d^2G}{d\tilde{y}^2}.
\end{aligned} \tag{3.7}
$$

Since we are going to use $\tilde{y}'$ and $\tilde{y}''$, let us express them in terms of $\tilde{y}$:

$$
\tilde{y}' = [tanh(\tilde{y})]' = 1 - \tilde{y}^2, \quad \tilde{y}'' = -2\tilde{y}'\tilde{y} = -2\tilde{y}\tilde{y}' = -2\tilde{y}(1-\tilde{y}^2). \tag{3.8}
$$

After substituting (3.7) and (3.8) into (3.5), we get the following system:

$$
\begin{aligned}
&-\nu F - 3F^2 + c^2(1-\tilde{y}^2)[-2\tilde{y}\frac{dF}{d\tilde{y}} + (1-\tilde{y}^2)\frac{d^2F}{d\tilde{y}^2}] = \\
&\alpha G + \beta c^2(1-\tilde{y}^2)[-2\tilde{y}\frac{dG}{d\tilde{y}} + (1-\tilde{y}^2)\frac{d^2G}{d\tilde{y}^2}], \\
&-\nu G - 3\mu G^2 + \eta G + \delta c^2(1-\tilde{y}^2)[-2\tilde{y}\frac{dG}{d\tilde{y}} + (1-\tilde{y}^2)\frac{d^2G}{d\tilde{y}^2}] = \\
&kF + \gamma c^2(1-\tilde{y}^2)[-2\tilde{y}\frac{dF}{d\tilde{y}} + (1-\tilde{y}^2)\frac{d^2F}{d\tilde{y}^2}].
\end{aligned}
\tag{3.9}
$$

The nonlinear term of highest order in the first equation of (3.9) is $-3F^2$, therefore its degree, remembering (3.6), is $2R_1$. There are a few linear terms that can be considered of highest order. The first one is $2c^2\tilde{y}^3\frac{dF}{d\tilde{y}}$ which yields the degree of R_1+2. The second one is $2\beta c^2\tilde{y}^3\frac{dG}{d\tilde{y}}$ which yields degree of R_2+2. Now let us consider the second equation of (3.9). The nonlinear term of highest order is $-3\mu G^2$ and its degree is $2R_2$. There are several linear terms that can be considered of highest order. The first one is $2\delta c^2\tilde{y}^3\frac{dG}{d\tilde{y}}$ and its degree is R_2+2. The second one is $2\gamma c^2\tilde{y}^3\frac{dF}{d\tilde{y}}$ and its degree is R_1+2.

Now, let us balance the linear term of highest order with the nonlinear one in both equations of (3.9) as we can see we have two alternatives: 1. $R_1 > R_2$ or 2. $R_2 > R_1$. Both of them give the values for $R_1 = 2$ and $R_2 = 2$. So now knowing this, (3.6) becomes:

$$
\begin{aligned}
\phi(x,t) &= U(\zeta) = F(\tilde{y}) = \sum_{r_1=0}^{2} a_{r_1}\tilde{y}^{r_1} = a_2\tilde{y}^2 + a_1\tilde{y} + a_0, \\
\psi(x,t) &= W(\zeta) = G(\tilde{y}) = \sum_{r_2=0}^{2} b_{r_2}\tilde{y}^{r_2} = b_2\tilde{y}^2 + b_1\tilde{y} + b_0.
\end{aligned}
\tag{3.10}
$$

Substituting (3.10) into (3.9) and organazing it in order of the powers of $\tilde{y}$, we can have the following computations.

From the first equation in (3.9):

$$(6c^2a_2 - 3a_2^2)\tilde{y}^4 + (2c^2a_1 - 6a_1a_2)\tilde{y}^3 + (-8c^2a_2 - \nu a_2 - 6a_0a_2 - 3a_1^2)\tilde{y}^2 +$$
$$+ (-2c^2a_1 - \nu a_1 - 6a_0a_1)\tilde{y} + 2c^2a_2 - \nu a_0 - 3a_0^2 =$$
$$6c^2\beta b_2\tilde{y}^4 + 2c^2\beta b_1\tilde{y}^3 + (-8c^2\beta b_2 + \alpha b_2)\tilde{y}^2 + (-2c^2\beta b_1 + \alpha b_1)\tilde{y} + \alpha b_0 + 2c^2\beta b_2. \tag{3.11}$$

From the second equation in (3.9):

$$(-3\mu b_2^2 + 6c^2\delta b_2)\tilde{y}^4 + (-6\mu b_1b_2 + 2c^2\delta b_1)\tilde{y}^3 + ((\eta - \nu)b_2 - 6\mu b_0b_2 - 3\mu b_1^2 -$$
$$- 8c^2\delta b_2)\tilde{y}^2 + ((\eta - \nu)b_1 - 6\mu b_0b_1 - 2c^2\delta b_1)\tilde{y} + ((\eta - \nu)b_0 - 3\mu b_0^2 + 2c^2\delta b_2) =$$
$$6c^2\gamma a_2\tilde{y}^4 + 2c^2\gamma a_1\tilde{y}^3 + (ka_2 - 8c^2\gamma a_2)\tilde{y}^2 + (ka_1 - 2c^2\gamma a_1)\tilde{y} + (ka_0 + 2c^2\gamma a_2). \tag{3.12}$$

Comparing the coefficients of each power of $\tilde{y}$ in (3.11), we get:

$$\begin{aligned}
&6a_2c^2 - 3a_2^2 = 6c^2\beta b_2,\\
&2c^2a_1 - 6a_1a_2 = 2c^2\beta b_1,\\
&- \nu a_2 - 6a_0a_2 - 3a_1^2 - 8c^2a_2 = \alpha b_2 - 8c^2\beta b_2,\\
&- \nu a_1 - 2c^2a_1 - 6a_0a_1 = -2c^2\beta b_1 + \alpha b_1,\\
&- 3a_0^2 - \nu a_0 + 2c^2a_2 = 2c^2\beta b_2 + \alpha b_0.
\end{aligned} \tag{3.13}$$

Comparing the coefficients of each power of $\tilde{y}$, in (3.12) we get:

$$\begin{aligned}
&- 3\mu b_2^2 + 6c^2\delta b_2 = 6c^2\gamma a_2,\\
&- 6\mu b_1b_2 + 2c^2\delta b_1 = 2c^2\gamma a_1,\\
&- \nu b_2 - 6\mu b_0b_2 - 3\mu b_1^2 + \eta b_2 - 8c^2\delta b_2 = ka_2 - 8c^2\gamma a_2,\\
&- \nu b_1 - 6\mu b_0b_1 + \eta b_1 - 2c^2\delta b_1 = ka_1 - 2c^2\gamma a_1,\\
&- \nu b_0 - 3\mu b_0^2 + \eta b_0 + 2c^2\delta b_2 = ka_0 + 2c^2\gamma a_2.
\end{aligned} \tag{3.14}$$

As we can see, in (3.13) and (3.14), we have a system of 10 equations and 8 variables,

a_0, a_1, a_2, b_0, b_1, b_2, ν and c. The solutions of this system are quite long. Here, we are going to discuss the way in which this system can be solved. We are going to use the expressions of the variables in special cases to represent the solutions to some equations that we are going to consider.

If we express c^2 from the first equations of (3.13) and (3.14), we get:

$$c^2 = \frac{3a_2^2}{6(a_2 - \beta b_2)}, \quad c^2 = \frac{3\mu b_2^2}{6(\delta b_2 - \gamma a_2)}, \tag{3.15}$$

and then, if we equate the right hand sides of the two representations of c^2 in (3.15), we get:

$$\frac{3a_2^2}{6(a_2 - \beta b_2)} = \frac{3\mu b_2^2}{6(\delta b_2 - \gamma a_2)}. \tag{3.16}$$

If we cross multiply the expressions in (3.16) and divide the result in equation by b_2, we get the following equation:

$$-\gamma(\frac{a_2}{b_2})^3 + \delta(\frac{a_2}{b_2})^2 - \mu(\frac{a_2}{b_2}) + \mu\beta = 0. \tag{3.17}$$

Now, let denote

$$\frac{a_2}{b_2} = x_2.$$

The exact solution for x_2 can be obtained using Maple but here, we are going to just present x_2, having in mind that it can be computed by Maple:

$$x_2 = \sqrt[3]{p} - \sqrt[3]{q} + \frac{\delta}{3\gamma}, \tag{3.18}$$

where

$$p = \frac{-3q_1\sqrt{3} \pm \sqrt{4p_1^3 + 27q_1^2}}{6\sqrt{3}} \quad and \quad q = \frac{3q_1\sqrt{3} \pm \sqrt{4p_1^3 + 27q_1^2}}{6\sqrt{3}},$$

where

$$p_1 = \frac{\delta^2\gamma - 2\delta^2 + 3\mu\gamma}{3\gamma^2} \quad and \quad q_1 = \frac{3\mu\gamma\delta - \delta^3 - 9\mu\gamma^2\beta}{9\gamma^3}.$$

Now, since we know x_2, we can use the following substitution $a_2 = x_2 b_2$ and plug it into the first equation in (3.13) (or (3.14)). We then get:

$$2x_2 b_2 c^2 - x_2^2 b_2^2 = 3c^2 \beta b_2.$$

If we divide this by b_2 and solve for b_2 we get the following:

$$b_2 = \frac{2c^2(x_2 - \beta)}{x_2^2}. \tag{3.19}$$

Going back to $a_2 = x_2 b_2$, we get:

$$a_2 = \frac{2c^2(x_2 - \beta)}{x_2}. \tag{3.20}$$

Now, from the second equations of (3.13) and (3.14) we express c^2, equate the right hand sides of the two representation for c^2, cross multiply, divide by b_2 and then divide by b_1. As a result of these operations, we get:

$$-\gamma x_2 x_1^2 + (\delta x_2 - \mu)x_1 + \mu\beta = 0, \tag{3.21}$$

where

$$\frac{a_1}{b_1} = x_1.$$

Equation (3.21) can be solved using the quadratic formula

$$x_1 = \frac{\gamma - \delta x_2 \pm \sqrt{(\delta x_2 - \gamma)^2 + 4\mu\beta\gamma x_2}}{-2\gamma x_2}. \tag{3.22}$$

So, so far, we know the values of x_1 and x_2. Now we plug $a_1 = x_1 b_1$ into the fourth equation in (3.13), we get:

$$-\nu x_1 b_1 - 2c^2 x_1 b_1 - 6a_0 x_1 b_1 = -2c^2\beta b_1 + \alpha b_1,$$

and then dividing it by b_1 and solving it for a_0 we get:

$$a_0 = -\frac{1}{6}\frac{(\nu x_1 + 2c^2 x_1 - 2\beta c^2 + \alpha)}{x_1}. \tag{3.23}$$

If we now plug $a_1 = x_1 b_1$ into the fourth equation in (3.14), we get:

$$-\nu b_1 - 6\mu b_0 b_1 + \eta b_1 - 2c^2\delta b_1 = kx_1 b_1 - 2c^2\gamma x_1 b_1,$$

and the dividing it by b_1 and solving it for b_0 we get:

$$b_0 = \frac{1}{6}\frac{(-\nu + \eta - 2\delta c^2 - kx_1 + 2\gamma c^2 x_1)}{\mu}. \tag{3.24}$$

By now, we have a_0, b_0, a_2 and b_2 as functions of ν and c^2. Plugging these values for a_0, b_0, a_2 and b_2 into the third equation of (3.13) and solve that for a_1, we get a_1 as a function of ν and c^2:

$$a_1 = \sqrt{-\frac{2}{3}\frac{c^2 u_1}{x_2^2 x_1}}, \tag{3.25}$$

where

$$u_1 = (x_2 - \beta)[(8\beta c^2 - \alpha)x_1 + (\alpha - 2\beta c^2)x_2 - 6c^2 x_1 x_2].$$

By plugging the computed values for a_0, b_0, a_2 and b_2 in the third equation in (3.14) and solving it for b_1, we get:

$$b_1 = \sqrt{-\frac{2}{3}\frac{c^2 u_2}{x_2^2 x_1^2}}, \tag{3.26}$$

where

$$u_2 = (x_2 - \beta)[(k - 2\gamma c^2)x_1 + (8\gamma c^2 - k)x_2 - 6c^2\delta].$$

Now, we have a_0, b_0, a_1, b_1, a_2 and b_2 expressed as functions of ν and c. We plug these values into the fifth equations in (3.13) and (3.14) and we solve them, first for ν and then for c^2. The equation that we are getting for c^2 is of fourth order polynomial and so, additional polynomial solving techniques are required to solve this polynomial and we skip this step, leaving it as a future problem on our list.

Let us now impose specific boundary condition to the proposed solutions in (3.10). We are going to consider the following condition:

$$a_1 = 0, \quad b_1 = 0, \quad a_0 = -a2, \quad b_0 = -b_2. \tag{3.27}$$

We are going to use the previously computed values of x_2, a_2 and b_2 presented in (3.18), (3.20) and (3.19) respectively and plug them into the third equations in (3.13) and (3.14). This is what we get as a result of that:

from (3.13):

$$\frac{2}{(-d^{\frac{2}{3}} + 12\mu\gamma - 4\delta^2 - 2\delta d^{\frac{1}{3}})^2}(-d^{\frac{2}{3}} + 12\mu\gamma - 4\delta^2 - 2\delta d^{\frac{1}{3}} + 6\beta\gamma d^{\frac{1}{3}}(\nu d^{\frac{2}{3}} - 12\nu\mu\gamma + \\ + 4\nu\delta^2 + 2\nu\delta d^{\frac{1}{3}} - 4c^2 d^{\frac{2}{3}} + 48c^2\mu\gamma - 16c^2\delta^2 - 8c^2\delta d^{\frac{1}{3}} + 24\beta c^2\gamma d^{\frac{1}{3}} + 6\alpha\gamma d^{\frac{1}{3}})) = 0, \tag{3.28}$$

and from (3.14):

$$
\begin{aligned}
&-\frac{144}{(d^{\frac{2}{3}}-12\mu\gamma+4\delta^2+2\delta d^{\frac{1}{3}})^4}(d^{\frac{2}{3}}-12\mu\gamma+4\delta^2+2\delta d^{\frac{1}{3}}-6\beta\gamma d^{\frac{1}{3}})\times\\
&\times(-60k\mu\gamma\delta^4-8\eta\gamma\delta^5+4k\delta^5 d^{\frac{1}{3}}+144\gamma^3c^2\mu^2\delta^2+8\nu\gamma\delta^5-32\gamma^2c^2\mu\delta^4+\\
&+2kd^{\frac{2}{3}}\delta^4+4\nu\gamma d^{\frac{1}{3}}\delta^4+12\nu\gamma^3d^13\mu^2-12\eta\gamma^3d^{\frac{1}{3}}\mu^2+24\mu^2c^2\gamma^3d^{\frac{2}{3}}+2\nu\gamma d^{\frac{2}{3}}\delta^3-\\
&-4\eta\gamma d^{\frac{1}{3}}\delta^4+108k\mu^2\gamma^2\delta^2-2\eta\gamma d^{\frac{2}{3}}\delta^3+72\nu\gamma^3\mu^2\delta+1296\gamma^5c^2\mu^2\beta^2+\\
&+324k\mu^2\beta^2\gamma^4+6kd^{\frac{2}{3}}\mu^2\gamma^2-52\nu\gamma^2\mu\delta^3-216\nu\gamma^4\mu^2\beta+216\eta\gamma^4\mu^2\beta-\\
&-72\eta\gamma^3\mu^2\delta+52\eta\gamma^2\mu\delta^3+8k\delta^6-24\nu\gamma^3\mu h+12\nu\gamma^2\delta^2h+24\eta\gamma^3\mu h-\\
&-12\eta\gamma^2\delta^2h+72\delta c^2\gamma^3d^{\frac{1}{3}}\mu^2-16\delta^3c^2\gamma^2d^{\frac{1}{3}}\mu+36kd^{\frac{1}{3}}\delta^2\mu\beta\gamma^2+9kd^{\frac{2}{3}}\delta\mu\beta\gamma^2+\\
&+9\nu\gamma^3d^{\frac{2}{3}}\mu\beta-9\eta\gamma^3d^{\frac{2}{3}}\mu\beta+36\nu\gamma^3d^{\frac{1}{3}}\delta\mu\beta-216\mu^2c^2\gamma^4d^{\frac{1}{3}}\beta-36\eta\gamma^3d^{\frac{1}{3}}\delta\mu\beta-\\
&-54kd^{\frac{1}{3}}\mu^2\gamma^3\beta+4kd^{\frac{1}{3}}\delta^2h\gamma+4\nu\gamma^2d^{\frac{1}{3}}\delta h-4\eta\gamma^2d^{\frac{1}{3}}\delta h-6kd^{\frac{1}{3}}\mu\gamma^2h-\\
&-20\nu\gamma^2d^{\frac{1}{3}}\mu\delta^2-24k\mu\gamma\delta^3d^{\frac{1}{3}}-432k\mu^2\delta\gamma^3\beta+132k\mu\beta\gamma^2\delta^3-864\gamma^4c^2\mu^2\delta\beta+\\
&+96\gamma^3c^2\mu\beta\delta^3+108\nu\gamma^3\delta^2\mu\beta-108\eta\gamma^3\delta^2\mu\beta-36k\mu\delta\gamma^2h+36k\mu\beta\gamma^3h+\\
&+12k\delta^3h\gamma-48\gamma^3c^2\mu\delta h+144\gamma^4c^2\mu\beta h+kd^{\frac{2}{3}}\delta h\gamma+\nu\gamma^2d^{\frac{2}{3}}h-\eta\gamma^2d^{\frac{2}{3}}h-\\
&-24\mu c^2\gamma^3d^{\frac{1}{3}}h-7\nu\gamma^2d^{\frac{2}{3}}\mu\delta-8\mu c^2\gamma^2d^{\frac{2}{3}}\delta^2+20\eta\gamma^2d^{\frac{1}{3}}\mu\delta^2+7\eta\gamma^2d^{\frac{2}{3}}\mu\delta-\\
&-9kd^{\frac{2}{3}}\mu\gamma\delta^2+30k\mu^2\gamma^2\delta d^{\frac{1}{3}})=0. \qquad (3.29)
\end{aligned}
$$

where

$$
\begin{aligned}
d&=-36\mu\delta\gamma+108\mu\beta\gamma^2+8\delta^3+12h\gamma,\\
h&=\sqrt{3}\sqrt{\mu(4\mu^2\gamma-\mu\delta^2-18\mu\delta\gamma\beta+27\mu\beta^2\gamma^2+4\beta\delta^3)}.
\end{aligned}
$$

These two equation are linear with regards to c^2 and ν so we solve them for c^2 and ν using Maple. Since the solutions for c^2 and ν are quite long, we are not going to present them here, but we advice that we have found them using Maple. Now, we plug those values for c^2 and ν into (3.20) and (3.19) and we get solutions for a_2 and b_2. Having in mind the condition (3.27) we actually computed all the variables.

Therefore, we can state the following theorem as a roundup to the solutions of (3.1).

Theorem 3.1 *The system of two (1+1)-dimensional KdV type equations in* (3.1) *has the following solutions*

$$\phi(x,t) = a_2 tanh^2(\zeta) - a_2,$$
$$\psi(x,t) = b_2 tanh^2(\zeta) - b_2,$$

where c*,* ν*,* a_2*,* b_2*,* a_0*,* b_0*,* a_1*,* b_1 *are defined by* (3.28)*,* (3.29)*,* (3.20)*,* (3.19)*,* (3.27) *respectively.*

In this thesis, in order to present the solution to (3.1) graphically, we are going to consider a special case of the general solutions to for certain values of the constants α, β, γ, δ, μ and η. This will be presented in section 5 for several different sets of values for the constants.

4 Exact solutions to a (3+1)-dimensional nonlinear wave equation

Consider the (3+1)-dimensional nonlinear wave equation

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}(\nabla^2 \phi) + \alpha \frac{\partial^5 \phi}{\partial x^5} = 0, \tag{4.1}$$

where $\nabla^2 = (\frac{\partial^2}{\partial x^2}) + (\frac{\partial^2}{\partial y^2}) + (\frac{\partial^2}{\partial z^2})$. To solve this equation, we introduce

$$\zeta = kx + my + nz - \omega t, \quad \phi(x, y, z, t) = U(\zeta). \tag{4.2}$$

Because of the previous introduction, we observe the following changes

$$\begin{aligned}
&\frac{\partial \phi}{\partial t} = -\omega \frac{dU}{d\zeta}, \quad \frac{\partial \phi}{\partial x} = k \frac{dU}{d\zeta}, \quad \frac{\partial \phi}{\partial y} = m \frac{dU}{d\zeta}, \quad \frac{\partial \phi}{\partial z} = n \frac{dU}{d\zeta}, \\
&\frac{\partial^2 \phi}{\partial x^2} = k^2 \frac{d^2 U}{d\zeta^2}, \quad \frac{\partial^2 \phi}{\partial y^2} = m^2 \frac{d^2 U}{d\zeta^2}, \quad \frac{\partial^2 \phi}{\partial z^2} = n^2 \frac{d^2 U}{d\zeta^2}, \\
&\frac{\partial^3 \phi}{\partial x^3} = k^3 \frac{d^3 U}{d\zeta^3}, \quad \frac{\partial^3 \phi}{\partial x \partial y^2} = km^2 \frac{d^3 U}{d\zeta^3}, \quad \frac{\partial^3 \phi}{\partial x \partial z^2} = kn^2 \frac{d^3 U}{d\zeta^3}, \\
&\frac{\partial^5 \phi}{\partial x^5} = k^5 \frac{d^5 U}{d\zeta^5}.
\end{aligned} \tag{4.3}$$

So the equation (4.1) becomes

$$-\omega \frac{dU}{d\zeta} + kU \frac{dU}{d\zeta} + k(k^2 + m^2 + n^2) \frac{d^3 U}{d\zeta^3} + \alpha k^5 \frac{d^5 U}{d\zeta^5} = 0. \tag{4.4}$$

Now we integrate once the whole equation and we get

$$-\omega U + \frac{1}{2} k U^2 + k(k^2 + m^2 + n^2) \frac{d^2 U}{d\zeta^2} + \alpha k^5 \frac{d^4 U}{d\zeta^4} = 0. \tag{4.5}$$

Then, we introduce the tanh function $\tilde{y} = tanh(\zeta)$ as a new independent variable and we propose the following series expansion as a solution:

$$\phi(x, y, z, t) = U(\zeta) = S(\tilde{y}) = \sum_{r=0}^{R} a_r \tilde{y}^r. \tag{4.6}$$

This new introduction yields the following necessary changes:

$$\begin{aligned}
&\frac{dU}{d\zeta} = \tilde{y}'\frac{dS}{d\tilde{y}}, \quad \frac{d^2U}{d\zeta^2} = \tilde{y}''\frac{d^S}{d\tilde{y}} + \tilde{y}'^2\frac{d^2S}{d\tilde{y}^2},\\
&\frac{d^3U}{d\zeta^3} = \tilde{y}'''\frac{dS}{d\tilde{y}} + 3\tilde{y}''\tilde{y}'\frac{d^2S}{d\tilde{y}^2} + \tilde{y}'^3\frac{d^3S}{d\tilde{y}^3},\\
&\frac{d^4U}{d\zeta^4} = \tilde{y}^{(4)}\frac{dS}{d\tilde{y}} + 4\tilde{y}'\tilde{y}'''\frac{d^2S}{d\tilde{y}^2} + 6\tilde{y}'^2\tilde{y}''\frac{d^3S}{d\tilde{y}^3} + \tilde{y}'^4\frac{d^4S}{d\tilde{y}^4}.
\end{aligned} \tag{4.7}$$

Substituting (4.7) and (4.6) in (4.5) we get

$$\begin{aligned}
&-\omega S + \frac{1}{2}kS^2 + [k(k^2 + m^2 + n^2)\tilde{y}'' + \alpha k^5\tilde{y}^{iv}]\frac{dS}{d\tilde{y}}\\
&+ [k(k^2 + m^2 + n^2)\tilde{y}'^2 + \alpha k^5(4\tilde{y}'\tilde{y}''' + 3\tilde{y}''^2)]\frac{d^2S}{d\tilde{y}^2}\\
&+ 6\alpha k^5\tilde{y}'^2\tilde{y}''\frac{d^3S}{d\tilde{y}^3}\\
&+ \alpha k^5\tilde{y}'^4\frac{d^4S}{d\tilde{y}^4}\\
&= 0.
\end{aligned} \tag{4.8}$$

Let us now compute the derivatives of $\tilde{y}$ up to the $2nd$ power

$$\tilde{y}' = 1 - \tilde{y}^2, \quad \tilde{y}'' = -2\tilde{y}(1 - \tilde{y}^2). \tag{4.9}$$

Now, substituting (4.9) in (4.8), we observe the following equation

$$
\begin{aligned}
&-\omega S+\frac{1}{2}kS^2+\\
&+[24\alpha k^5\tilde{y}^5+(2km^2+2k^3-40\alpha k^5+2kn^2)\tilde{y}^3+(-2km^2-2kn^2-2k^3+\\
&+16\alpha k^5)\tilde{y}]\frac{dS}{d\tilde{y}}+[36\alpha k^5\tilde{y}^6+(km^2+k^3+kn^2-80\alpha k^5)\tilde{y}^4+(52\alpha k^5-2km^2-\\
&-2kn^2-2k^3)\tilde{y}^2-8\alpha k^5+km^2+k^3+kn^2]\frac{d^2S}{d\tilde{y}^2}+\\
&+[-12\alpha k^5\tilde{y}+36\alpha k^5\tilde{y}^3-36\alpha k^5\tilde{y}^5+12\alpha k^5\tilde{y}^7]\frac{d^3S}{d\tilde{y}^3}+\\
&+[-4\alpha k^5\tilde{y}^2+6\alpha k^5\tilde{y}^4-4\alpha k^5\tilde{y}^6+\alpha k^5\tilde{y}^8+\alpha k^5]\frac{d^4S}{d\tilde{y}^4}\\
&=0. \qquad (4.10)
\end{aligned}
$$

Let us take a look at the linear and nonlinear term of highest order in (4.10). We can see that there are more than one linear term that can be considered of highest order but all of them yield the same degree. Consider one of them, say $24\alpha k^5\tilde{y}^5\frac{dS}{d\tilde{y}}$. From the proposed solution for $S(\tilde{y})$ in (4.6) it follows that the term of highest order in $S(\tilde{y})$ has a degree of R, thus $\frac{dS}{d\tilde{y}}$ has a degree of the highest term of $R-1$. Therefore the above chosen linear term of highest order in (4.10) has a degree of $R+4$. The Nonlinear term of highest order (and the only nonlinear term) in (4.10) is $\frac{1}{2}kS^2$. Knowing the proposed solution for $S(\tilde{y})$ in (4.6), it is obvious that the degree of the nonlinear term is $2R$. Balancing the Linear term of highest order with the Nonlinear term of highest order, we get $2R=R+4$ which yields $R=4$. So now, by substituting this value for R in the equation (4.6), we get the following for the solution of $S(\tilde{y})$:

$$U(\zeta)=S(\tilde{y})=\sum_{r=0}^{4}a_r\tilde{y}^r=a_4\tilde{y}^4+a_3\tilde{y}^3+a_2\tilde{y}^2+a_1\tilde{y}+a_0. \qquad (4.11)$$

Now we find the derivatives of S with regards to $\tilde{y}$

$$
\begin{aligned}
&\frac{dS}{d\tilde{y}}=4a_4\tilde{y}^3+3a_3\tilde{y}^2+2a_2\tilde{y}+a_1, &&\frac{d^2S}{d\tilde{y}^2}=12a_4\tilde{y}^2+6a_3\tilde{y}+2a_2\\
&\frac{d^3S}{d\tilde{y}^3}=24a_4\tilde{y}+6a_3, &&\frac{d^4S}{d\tilde{y}^4}=24a_4. \qquad (4.12)
\end{aligned}
$$

Substituting (4.12) and (4.11) into (4.10), we get:

$$
\begin{aligned}
&(840\alpha k^5 a_4 + \frac{1}{2}ka_4^2)\tilde{y}^8\\
&+ (ka_4a_3 + 360\alpha k^5 a_3)\tilde{y}^7\\
&+ (\frac{1}{2}ka_3^2 + 20km^2a_4 + 20k^3a_4 + ka_4a_2 - 2080\alpha k^5 a_4 + 20kn^2a_4 + 120\alpha k^5 a_2)\tilde{y}^6\\
&+ (12kn^2a_3 + 24\alpha k^5 a_1 + ka_4a_1 + ka_3a_2 + 12km^2a_3 + 12k^3a_3 - 816\alpha k^5 a_3)\tilde{y}^5\\
&+ (\frac{1}{2}ka_2^2 + 1696\alpha k^5 a_4 - 32k^3a_4 + 6km^2a_2 - 240\alpha k^5 a_2 - \omega a_4 + 6k^3a_2 -\\
&- 32kn^2a_4 + ka_4a_0 + 6kn^2a_2 + ka_3a_1 - 32km^2a_4)\tilde{y}^4\\
&+ (-40\alpha k^5 a_1 + 2km^2a_1 - 18kn^2a_3 + 576\alpha k^5 a_3 - 18km^2a_3 + ka_3a_0 + ka_2a_1 +\\
&+ 2kn^2a_1 + 2k^3a_1 - 18k^3a_3 - \omega a3)\tilde{y}^3\\
&+ (ka_2a_0 - 480\alpha k^5 a_4 + \frac{1}{2}ka_1^2 + 12kn^2a_4 - 8km^2a_2 + 12km^2a_4 + 12k^3a_4 +\\
&+ 136\alpha k^5 a_2 - 8k^3a_2 - 8kn^2a_2 - \omega a_2)\tilde{y}^2\\
&+ (6kn^2a_3 + 6k^3a_3 + ka_1a_0 - \omega a_1 - 2k^3a_1 + 16\alpha k^5 a_1 - 2km^2a_1 - 120\alpha k^5 a_3 +\\
&+ 6km^2a_3 - 2kn^2a_1)\tilde{y}\\
&+ 2km^2a_2 + 2kn^2a_2 + 2k^3a_2 - 16\alpha k^5 a_2 + 24\alpha k^5 a_4 - \omega a_0 + \frac{1}{2}ka_0^2\\
&= 0.
\end{aligned}
\tag{4.13}
$$

Comparing the coefficients of each power of $\tilde{y}$ in both sides, we can get the following result:
From $\tilde{y}^8$

$$840\alpha k^5 a_4 + \frac{1}{2}ka_4^2 = 0 \tag{4.14}$$

which gives us the solution for a_4:

$$a_4 = -1680\alpha k^4 \qquad or \qquad a_4 = 0.$$

We first consider the first one. From $\tilde{y}^7$

$$ka_4a_3 + 360\alpha k^5 a_3 = 0, \tag{4.15}$$

which, considering the previously computed value for a_4, gives us the solution for a_3:

$$a_3 = 0.$$

From $\tilde{y}^6$

$$\frac{1}{2}ka_3^2 + 20km^2a_4 + 20k^3a_4 + ka_4a_2 - 2080\alpha k^5a_4 + 20kn^2a_4 + 120\alpha k^5a_2 = 0, \quad (4.16)$$

and knowing the values for a_4 and a_3, we get the value of a_2:

$$a_2 = \frac{-280}{13}m^2 - \frac{280}{13}k^2 + 2240\alpha k^4 - \frac{280}{13}n^2.$$

From $\tilde{y}^5$

$$12kn^2a_3 + 24\alpha k^5a_1 + ka_4a_1 + ka_3a_2 + 12km^2a_3 + 12k^3a_3 - 816\alpha k^5a_3 = 0. \quad (4.17)$$

From this, the value of a_1 is

$$a_1 = 0.$$

From $\tilde{y}^4$

$$\begin{aligned}&\frac{1}{2}ka_2^2 + 1696\alpha k^5a_4 - 32k^3a_4 + 6km^2a_2 - 240\alpha k^5a_2 - \omega a_4 + 6k^3a_2 - \\ &- 32kn^2a_4 + ka_4a_0 + 6kn^2a_2 + ka_3a_1 - 32km^2a_4 = 0, \quad (4.18)\end{aligned}$$

and we then express a_0 as a function of ω:

$$\begin{aligned}a_0 = &\frac{-\frac{1}{507}(-507\omega\alpha k^3 - 7280\alpha k^4n^2 - 7280m^2\alpha k^4 - 31k^4 - 31m^4}{\alpha k^4} + \\ &+ \frac{-62m^2k^2 - 62k^2n^2 - 31n^4 - 62m^2n^2 + 264992\alpha^2k^8 - 7280k^6\alpha}{\alpha k^4}. \quad (4.19)\end{aligned}$$

From $\tilde{y}^3$

we get $0 = 0$ which is Tautology.

From $\tilde{y}^2$

$$12km^2a_4 + 12k^3a_4 + ka_2a_0 - 480\alpha k^5a_4 + 12kn^2a_4 - 8k^3a_2 -$$
$$- 8km^2a_2 - \omega a_2 + 136\alpha k^5a_2 - 8kn^2a_2 + \frac{1}{2}ka_1^2 = 0. \tag{4.20}$$

When we use the previously computed values for a_4, a_3, a_2, a_1 and a_0, then ω can cancel out. So this equation leads to some restrictions for the constant k. Factoring the resulting equation for k and plugging the previously computed values for a_4, a_3, a_2, a_1 and a_0, we set all factors as follows:

$$\begin{aligned}
&\frac{-280}{6591\alpha k^3},\\
&(52\alpha k^4 + n^2 + k^2 + m^2),\\
&(27040\alpha^2k^8 - 1612k^6\alpha - 1612\alpha\, k^4n^2 - 1612m^2\alpha k^4 + 31k^4 +\\
&+ 62m^2k^2 + 62k^2n^2 + 62m^2n^2 + 31m^4 + 31n^4)
\end{aligned} \tag{4.21}$$

We are going to use the second factor to obtain real solutions for k. After solving the following equation for k:

$$52\alpha k^4 + n^2 + k^2 + m^2 = 0,$$

we get the four solutions for k (as a functions of m, n and α):

$$\begin{aligned}
k_1 &= \frac{\sqrt{26}}{52\alpha}\sqrt{\alpha(-1 + \sqrt{1 - 208\alpha n^2 - 208\alpha m^2})},\\
k_2 &= -\frac{\sqrt{26}}{52\alpha}\sqrt{\alpha(-1 + \sqrt{1 - 208\alpha n^2 - 208\alpha m^2})}, && (4.22)\\
k_3 &= \frac{1}{52\alpha}\sqrt{-26\alpha(1 + \sqrt{1 - 208\alpha n^2 - 208\alpha m^2})}, && (4.23)\\
k_4 &= -\frac{1}{52\alpha}\sqrt{-26\alpha(1 + \sqrt{1 - 208\alpha n^2 - 208\alpha m^2})}.
\end{aligned}$$

We use the first solution for k here. So, we have

$$k = \frac{\sqrt{26}}{52\alpha}\sqrt{\alpha(-1 + \sqrt{1 - 208\alpha n^2 - 208\alpha m^2})}. \tag{4.24}$$

From now on we are going to use k instead of the above solution just for spacial purposes. Nevertheless its value is exactly the one stated above.
From $\tilde{y}$
we get $0 = 0$ which is Tautology.
From the constant term, we get

$$2km^2a_2 + 2kn^2a_2 + 2k^3a_2 - 16\alpha k^5a_2 + 24\alpha k^5a_4 - \omega a_0 + \frac{1}{2}ka_0^2 = 0. \tag{4.25}$$

Now, every variable in this equation is either known or is a function of ω, and thus in this equation the only unknown variable is ω. Since (4.25) is a quadratic equation of ω, we get two solutions for ω:

$$\omega = \frac{1}{507\alpha k^3}\sqrt{h_1}, \tag{4.26}$$

where

$$\begin{aligned} h_1 = & -1377958400k^{12}n^2\alpha^3 + 451360m^6\alpha k^4 + 14423136k^8m^4\alpha^2 + 11532m^2k^2n^4 + \\ & + 14423136k^8n^4\alpha^2 + 28846272k^{10}m^2\alpha^2 + 1354080\alpha k^6n^4 + 1354080m^2\alpha k^8 + \\ & + 1354080\alpha k^4n^4m^2 + 28846272k^8m^2\alpha^2n^2 + 1354080\alpha k^4n^2m^4 + \\ & + 2708160\alpha k^6n^2m^2 + 451360\alpha k^4n^6 + 11532k^4m^2n^2 + 1354080\alpha k^8n^2 + \\ & + 1354080m^4\alpha k^6 - 1377958400k^{12}m^2\alpha^3 + 28846272k^{10}n^2\alpha^2 + \\ & + 11532m^4k^2n^2 + 961m^8 + 961n^8 + 31067056384\alpha^4k^16 + 961k^8 + \\ & + 14423136k^{12}\alpha^2 - 1377958400k^{14}\alpha^3 + 5766k^4m^4 + 3844k^6m^2 + 3844k^6n^2 + \\ & + 5766k^4n^4 + 451360k^{10}\alpha + 3844m^6k^2 + 5766m^4n^4 + 3844m^6n^2 + \\ & + 3844k^2n^6 + 3844n^6m^2, \end{aligned}$$

and

$$\omega = -\frac{1}{507\alpha k^3}\sqrt{h_2},$$

where

$$
\begin{aligned}
h_2 = &- 1377958400k^{12}n^2\alpha^3 + 451360m^6\alpha k^4 + 14423136k^8m^4\alpha^2 + 11532m^2k^2n^4+\\
&+ 14423136k^8n^4\alpha^2 + 28846272k^{10}m^2\alpha^2 + 1354080\alpha k^6n^4 + 1354080m^2\alpha k^8+\\
&+ 1354080\alpha k^4n^4m^2 + 28846272k^8m^2\alpha^2n^2 + 1354080\alpha k^4n^2m^4+\\
&+ 2708160\alpha k^6n^2m^2 + 451360\alpha k^4n^6 + 11532k^4m^2n^2 + 1354080\alpha k^8n^2+\\
&+ 1354080m^4\alpha k^6 - 1377958400k^{12}m^2\alpha^3 + 28846272k^{10}n^2\alpha^2+\\
&+ 11532m^4k^2n^2 + 961m^8 + 961n^8 + 31067056384\alpha^4k^16 + 961k^8+\\
&+ 14423136k^{12}\alpha^2 - 1377958400k^{14}\alpha^3 + 5766k^4m^4 + 3844k^6m^2 + 3844k^6n^2+\\
&+ 5766k^4n^4 + 451360k^{10}\alpha + 3844m^6k^2 + 5766m^4n^4 + 3844m^6n^2+\\
&+ 3844k^2n^6 + 3844n^6m^2.
\end{aligned}
$$

Now, plugging the first value for ω back into (4.19), we get result for a_0:

$$
\begin{aligned}
a_0 =& \frac{1}{k}\omega - \frac{1}{507\alpha k^4}(-7280\alpha k^4n^2 - 7280m^2\alpha k^4 - 31k^4 - 31m^4 - 62m^2k^2 - 62k^2n^2)\\
&- \frac{1}{507\alpha k^4}(-31n^4 - 62m^2n^2 + 264992\alpha^2k^8 - 7280k^6\alpha),
\end{aligned}
\tag{4.27}
$$

where ω and k have the values previously computed.

So, now by substituting the previously computed values for all variables, we get the solution for (4.1):

$$
\begin{aligned}
\phi(x,y,z,t) =& - 1680\alpha k^4 tanh^4(kx + my + nz - \omega t)+\\
&+ \Big[-\frac{280}{13}(k^2 + m^2 + n^2) + 2240\alpha k^4 \Big] tanh^2(kx + my + nz - \omega t)+\\
&+ a_0,
\end{aligned}
\tag{4.28}
$$

where a_0, ω and k have the previously computed values.

If we use the other value for a_4, which is 0, then we get the following result for the other variables from compering the coefficients in front of each power of $\tilde{y}$ in (4.13):

$$
a_3 = 0, \qquad a_2 = 0, \qquad a_1 = 0.
$$

Only from the constant term, we get something:

$$a_0(\omega + \frac{1}{2}ka_0) = 0,$$

which yields

$$a_0 = 0, \qquad or \qquad \omega = -\frac{1}{2}ka_0.$$

This yields a constant value solution for $\phi(x, y, z, t)$.

So we propose the following theorem as a roundup for the solutions of (4.1):

Theorem 4.1 *Let k be given by on of the k_1, k_2, k_3 or k_4 presented in* (4.24). *Then the (3+1)-dimensional nonlinear wave equation* (4.1) *has the following solution:*

$$\begin{aligned}\phi(x, y, z, t) =& -1680\alpha k^4 tanh^4(kx + my + nz - \omega t)+ \\ &+ \Big[-\frac{280}{13}(k^2 + m^2 + n^2) + 2240\alpha k^4\Big] tanh^2(kx + my + nz - \omega t) + a_0,\end{aligned}$$

where ω and a_0 are determined in (4.26) *and* (4.27), *respectively, m and n are arbitrary constants and $\alpha < 0$.*

5 Solution graphic representations

5.1 Graphic representation to the solution of the system of (1+1)-dimensional KdV type equations

Here, we are going to present few graphic presentation to the solution of (3.1), obtained for different values of the constants.
First two figures are presentation of the first equation in (3.1) where as the third figure is presentation of the second equation in (3.1). The last three figures show the actual movement of the solitary wave in time. The captions of all the figures state the conditions for the constants or/and the variables.

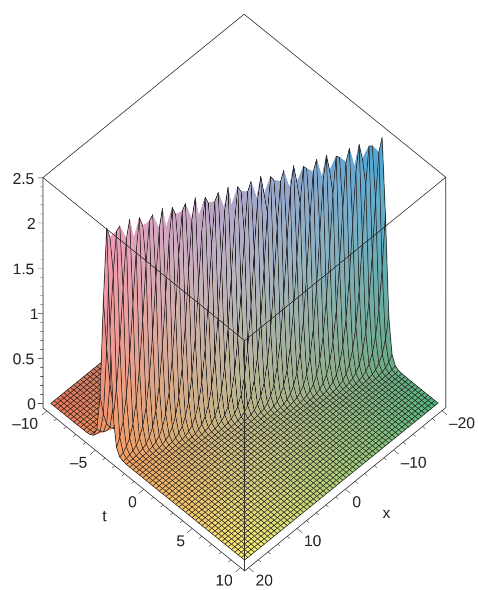

Figure 1: $\gamma = 1, \alpha = -1, \beta = 30, \delta = 2, \mu = 2, \eta = 2, k = 1$

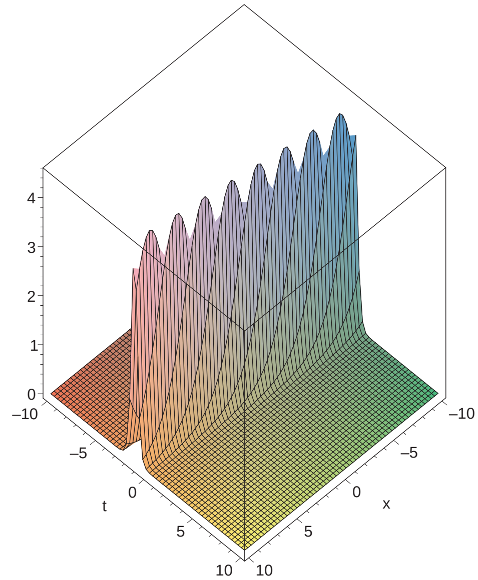

Figure 2: $\gamma = 5, \alpha = -5, \beta = 50, \delta = 2, \mu = 2, \eta = 2, k = 1$

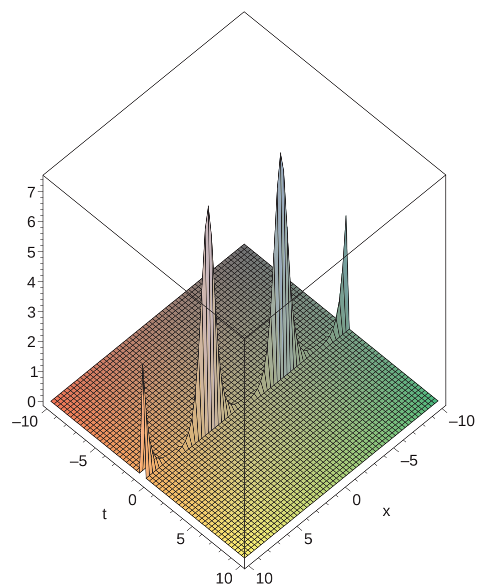

Figure 3: $\gamma = 5, \alpha = -5, \beta = 10, \delta = 2, \mu = 2, \eta = 10, k = 1$

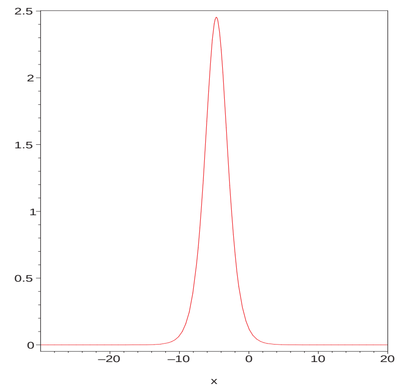

Figure 4: $\gamma = 5, \alpha = -5, \beta = 10, \delta = -10, \mu = 2, \eta = 20, k = 1,$ **t=1**

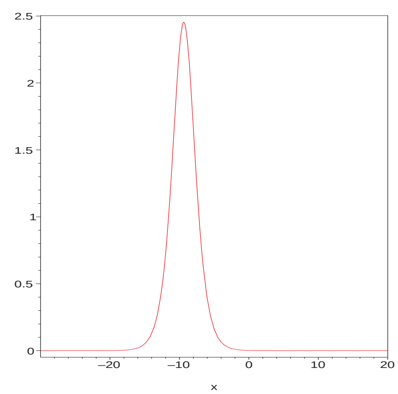

Figure 5: $\gamma = 5, \alpha = -5, \beta = 10, \delta = -10, \mu = 2, \eta = 20, k = 1,$ **t=2**

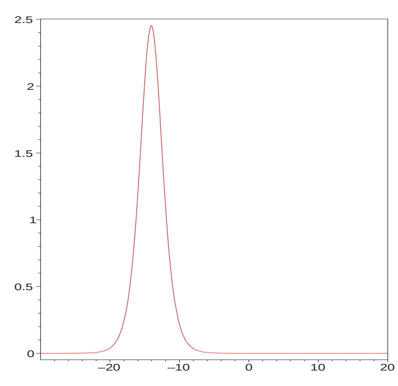

Figure 6: $\gamma = 5, \alpha = -5, \beta = 10, \delta = -10, \mu = 2, \eta = 20, k = 1,$ **t=3**

5.2 Graphic representation to the solution of (3+1)-dimensional nonlinear wave equation

Now, we are going to present a few graphic presentations to the solution of (4.1), obtained for different values of the constants.

First triple of figures are presentation of the solution to the given equation in (4.1) with the condition that in $\zeta = kx + my + nz - \omega t$, z is a constant and x and y are variables . The first figure in the triplet shows the solution at time $t = 0$, the second figure shows the solution at time $t = 3$ and the third figure shows the solution at time $t = 5$, so we can actually see the movement of the wave in time. The second set of three figures shows the same thing, but with condition y is a constant and x and z are variables. The third triplet of figures presents the solutions when x is a constant and z and y are variables. The captions of all the figures state the conditions for the constants or/and the variables.

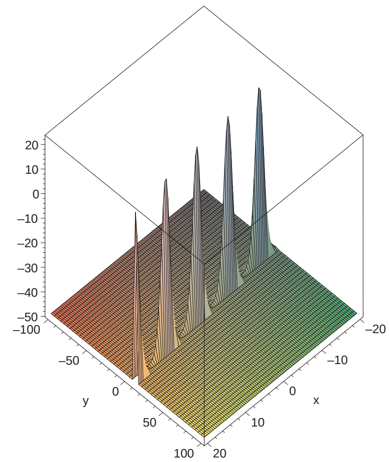

Figure 7: $z = 1, \alpha = -1, m = 1, n = -1,$ **t=0**

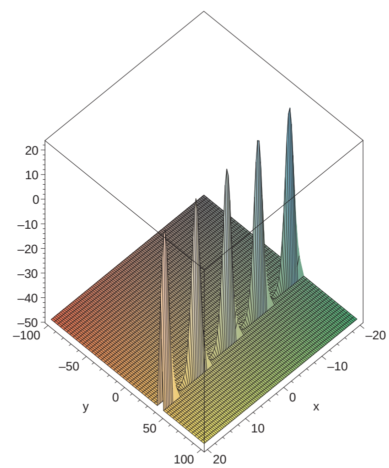

Figure 8: $z = 1, \alpha = -1, m = 1, n = -1,$ **t=3**

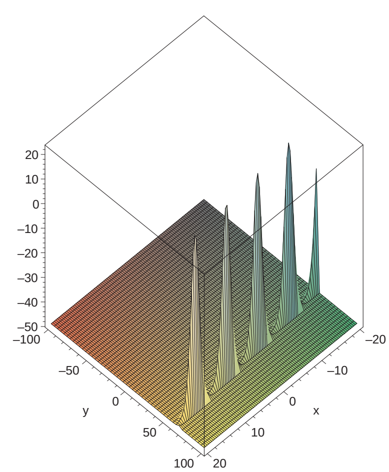

Figure 9: $z = 1, \alpha = -1, m = 1, n = -1,$ **t=5**

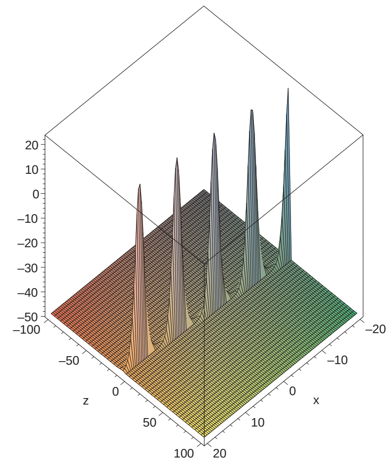

Figure 10: $y = 1, \alpha = -1, m = 1, n = -1,$ **t=0**

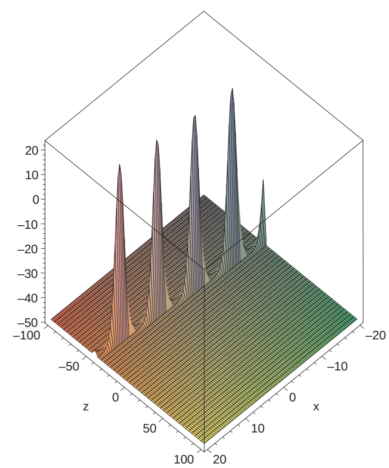

Figure 11: $y = 1, \alpha = -1, m = 1, n = -1,$ **t=3**

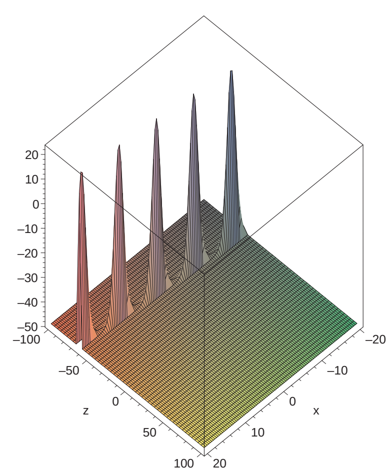

Figure 12: $y = 1, \alpha = -1, m = 1, n = -1,$ **t=5**

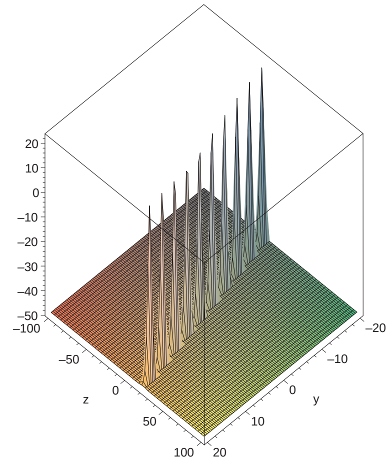

Figure 13: $x = -1, \alpha = -1, m = 1, n = -1, \mathbf{t{=}0}$

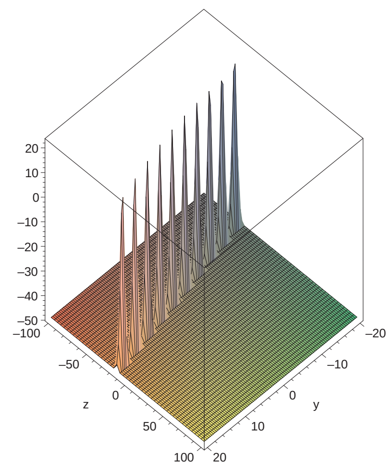

Figure 14: $x = -1, \alpha = -1, m = 1, n = -1, \mathbf{t{=}3}$

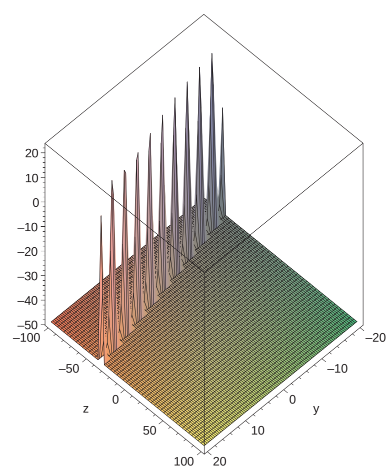

Figure 15: $x = -1, \alpha = -1, m = 1, n = -1, \mathbf{t{=}5}$

6 Conclusion

In this thesis, we have described the tanh method, which is a powerful tool for obtaining solutions to nonlinear wave equation, and applied it to the two particular KdV type equations. We discussed and got solitary wave solutions to a system of (1+1)-dimensional KdV type equations (not general solutions yet), and to a (3+1)-dimensional nonlinear wave equation. The ease of use of the tanh method was slightly compromised by the fact that we did not impose general boundary conditions on the proposed solutions. One can set zero boundary conditions for the proposed solutions and that way makes the method more easier to use, avoiding some algebra of the polynomial solving techniques.

As we mentioned before, we left the part of finding exact solutions to the system of equation in (3.13) and (3.14) as a future problem. Also, finding the exact solutions to the following fifth order KdV type equation:

$$\frac{\partial \phi}{\partial t} - 6\phi\frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = \alpha\frac{\partial \psi}{\partial x} + \beta\frac{\partial^3 \psi}{\partial x^3} + \delta\frac{\partial^5 \psi}{\partial x^5},$$
$$\frac{\partial \psi}{\partial t} - 6\mu\psi\frac{\partial \psi}{\partial x} + \eta\frac{\partial \psi}{\partial x} + \delta\frac{\partial^3 \psi}{\partial x^3} = k\frac{\partial \phi}{\partial x} + \gamma\frac{\partial^3 \phi}{\partial x^3} + \lambda\frac{\partial^5 \phi}{\partial x^5},$$

can be carried out by the tanh method. The solutions to this equation will probably involve even more algebra on polynomials, but this is much more general system than the systems considered in [1] and have in this thesis. We finally point out that a more general solution process can be pursued using the general idea proposed in [7]. If doing so, two types of exact solutions involving either exponential functions or trigonometric functions can be generated.

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