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Analysis of a Multiple Year Gas Sales Agreement with Make-up, Carry-forward and Indexation

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Abstract

A typical gas sales agreement, also called a gas swing contract, is an agreement between a supplier and a purchaser for the delivery of variable daily quantities of gas, between specified minimum and maximum daily limits, over a certain number of years at a strike price. The main constraint of such an agreement that makes them difficult to value is that there is a minimum volume of gas (termed the take-or-pay or minimum bill) for which the buyer will be charged at the end of the year (or penalty date), regardless of the actual quantity of gas taken. We propose a framework for pricing such swing contracts where both the gas price and strike price (an index) are stochastic processes. With the help of a two-dimensional trinomial tree, we are able to price such swing contracts with both so-called make-up and carry-forward provisions; find optimal daily decisions and optimal yearly usage of both the make-up bank and the carry-forward bank. With the help of a number of numerical examples, we also provide a detailed analysis, not only of the different features these contracts have, but also how different model parameters will affect both the optimal value and the optimal decisions

Keywords: gas sales agreement, swing contract, indexation, make-up, carry-forward, forward price curve, index price curve.

1. Introduction

A gas sales agreement (GSA, also known as the gas swing option) is an American-style option with daily exercise opportunities. The delivery of daily quantities is constrained between pre-determined minimum and maximum daily limits. Under a GSA, the gas supplier encourages the buyer to purchase more gas when the market gas price is low, however, the buyer prefers to postpone the purchase until the market gas price is high. To protect the interests of gas suppliers, a minimum volume of gas, which is called the minimum bill, appears in the GSA. If the actual quantity of gas taken at each year end does not meet the minimum bill, the buyer will face a penalty. Typically, there is also a maximum annual quantity that can be taken. There are also two common and important features in multi-year GSA contracts: the make-up bank and the carry-forward bank. In a year where the

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GSA contract is out-of-the-money on most days, due to the existence of the minimum bill, the buyer is at risk of facing great losses under the contract. These losses can be from the delivery of gas on those out-of-the-money days, the year-end penalties or both. In this case, the buyer needs ways to reduce losses, and the so-called make-up bank is introduced. With the make-up bank, the buyer can take some quantities of gas which are less than the minimum bill and pay penalties in an out-of-the-money year, and the shortfall between the actual gas taken and the minimum bill is then added to the make-up bank. In later years, where the gas taken is greater than some reference level, the extra gas purchased can be withdrawn from the make-up bank and a refund paid. Whereas the make-up bank encourages the buyer to pay penalties, the so-called carry-forward bank gives the buyer the right to reduce the minimum bill and hence reduce possible penalties. With the carry-forward bank, if the buyer anticipates that the contract will be out-of-the-money in future years, he/she can take more gas in the current year. If the gas taken is greater than some reference level, the excess gas can be added to the carry-forward bank. In later years, where the buyer is under pressure to pay penalties, the gas in the carry-forward bank can be withdrawn to reduces the minimum bill in those years. Both the make-up bank and the carry-forward bank offer the buyer opportunities to apply flexible strategies for the sake of loss minimization, hence how they affect the contract values and the trading strategies should be investigated carefully.

A number of papers have discussed the valuation of general swing options (GSAs without the make-up and carry-forward banks). The first treatment of gas swing valuations appeared in [25], where the author applies a binomial tree method to take-or-pay gas contracts. In [19], the authors extend the idea of lattice methods by considering a mean-reverting spot price model and using a trinomial tree. In [12, 21, 24], the Least Squares Monte Carlo method (see [20]) has been used, while a dual-pricing approach has been used in [21] to get the upper bound of the swing options when a take-or-pay provision is not included. Another simulation-based approach can be found in [17], where the authors apply the technique in [18] by finding optimal exercise prices before pricing swing options. In [4], the authors propose and summarize several methods using both simulations and dynamic programming techniques. A so-called optimal quantization method is also built in [3] for mean-reverting spot prices.

When the strike price of a GSA is a constant, the valuation of the GSA is a classical dynamic programming problem. In real contracts, however, the strike price is set based on the indexation principle under which the strike price is called the index. In each month, the value of the index is determined by the weighted average price of some energy substitutions in the previous month (see [1] for details). This feature links the valuation of the GSAs to the moving average problem. So far, however, no effective method has been derived in the literature to value GSAs embedded with moving average features. In order to perform a detailed analysis of GSAs in terms of values, decisions, make-up usage, carry-forward usage, etc., we follow the assumption in [13] that the evaluation of the index follows a mean-reverting Markov process. For the moving average pricing, we refer interested readers to [8] (a Least Square Monte Carlo approach), [11] (a binomial tree based approach), [26] (a willow tree approach) and [5] (a finite-dimensional approximation approach).

In cases where there are make-up and carry-forward banks the GSA becomes more complicated to evaluate. In [6], the authors explained basic features of both the make-up bank and the carry-forward bank. In [14], an algorithm is proposed using the Least Squares Monte Carlo method to evaluate GSAs with a carry-forward bank, although the authors only price the contract values and do not extract optimal decisions. The evaluation of GSAs is not only about the contract value, however, but also about finding the optimal daily decisions based on various gas prices and index prices. Once one starts to introduce a volume taken dimension, however, the Monte Carlo approach does not work very well. The first formally quantitative treatment of GSAs containing the make-up provision appears in [13]. The authors modelled one type of make-up clause and performed a sensitivity analysis of the contract value with respect to various parameters. The GSA in [13] uses hard constraints in both the volume taken dimension and the make-up bank dimension. That is, their GSAs do not have penalties and the volume taken in each year must be within a pre-specified range. Also, the volume of gas in the make-up bank must be zero at the end of the contract. In addition, however, there are GSA contracts in the market, that offer more flexibility (see contracts in Subsection 4.2). In [13], the GSAs contain either the make-up clause or the carry-forward clause, but not both. The second treatment of make-up and carry-forward banks appears in [9], where the authors evaluate GSAs with both make-up and carry-forward banks in a regime-switching forward price curve model. The authors, however, only evaluate GSAs with constant strike prices and do not take the indexation into account. In addition, while the optimal daily decisions and optimal usage of both make-up and carry-forward banks are given in [9], there is no detailed analysis of parameters and indexation. Furthermore, in both [9] and [13], the effect of the make-up and carry-forward banks on the volume taken dimension is not given. The main contribution of the paper presented here, therefore, is to evaluate multiple year GSAs with both make-up and carry-forward banks, as in [6], but with stochastic strike prices (indexation), while also providing a detailed analysis of how the make-up bank, the carry-forward bank, the indexation and the different parameter settings affect both values and decisions. With the help of our numerical examples in Section 5, traders can have a better understanding of the impact of various important contract features on both the contract value and the trading strategies, such as recovery limits, penalties, the minimal bill, etc..

When it comes to the evaluation of GSAs, the so-called bang-bang consumption is usually observed numerically. The bang-bang consumption means that the optimal decision on each day within a year is either the minimum daily limit or the maximum daily limit. In [2, 4], the authors prove that, without the make-up and carry-forward banks, GSAs have the bang-bang consumption. Following the result in [2], The authors in [13] shows that GSAs with both hard constraints and the make-up banks have the bang-bang consumption. In Section 4, we follow the result in [4] and show that, when the penalty functions are smooth enough (see Remark 4.2), our GSAs (with penalties, make-up banks and carry-forward banks) have the bang-bang consumption.

This paper is organized as follows: Sections 2 and 3 propose a mean-reverting model for the gas price and the index, and build a two-dimensional trinomial tree to approximate the gas price process and the index process. Section 4 formulates both the cases of GSAs that do not have make-up and carry-forward banks and those that do. Section 5 compares GSAs with indexation with those with constant strike prices, and also analyzes the effect of make-up and carry-forward clauses and how various parameters affect the contract value. At the end of Section 5 the ways in which the indexation affects decisions about weekly takes, make-up takes and carry-forward takes are analyzed. We draw conclusions and propose future research in Section 6.

2. Modelling the gas price and the index

This paper considers the one factor forward curve model built in [10].[1] We assume the forward prices of both gas and index follow the stochastic differential equations below:

$$\frac{dF^S(t,T)}{F^S(t,T)} = \sigma_S e^{-\alpha_S(T-t)} dB^S(t), \tag{1}$$

$$\frac{dF^I(t,T)}{F^I(t,T)} = \sigma_I e^{-\alpha_I(T-t)} dB^I(t), \tag{2}$$

where $F^S(t,T)$ and $F^I(t,T)$ are the forward prices of gas and index at time $t \in [0,T]$ with the maturity T, respectively. $B^S(t)$ and $B^I(t)$ are standard Brownian motions with correlation ρ, and α_S, σ_S, α_I and σ_I are constants.

Denote the spot prices of gas and index at time t by $S(t)$ and $I(t)$, respectively. The gas price and index processes can be written as:

$$S(t) = F^S(0,t)\exp\left[-\frac{1}{2}\int_0^t \sigma_S^2 e^{-2\alpha_S(t-u)}du + \int_0^t \sigma_S e^{-\alpha_S(t-u)} dB^S(u)\right],$$

$$I(t) = F^I(0,t)\exp\left[-\frac{1}{2}\int_0^t \sigma_I^2 e^{-2\alpha_I(t-u)}du + \int_0^t \sigma_I e^{-\alpha_I(t-u)} dB^I(u)\right].$$

By letting the log prices of both gas and index be $X(t) = \ln S(t)$ and $Y(t) = \ln I(t)$, respectively, we have a two-dimensional Markov process (X,Y) which is given by:

$$dX(t) = \left[\theta^S(t) - \alpha_S X(t)\right]dt + \sigma_S dB^S(t), \tag{3}$$

$$dY(t) = \left[\theta^I(t) - \alpha_I Y(t)\right]dt + \sigma_I dB^I(t), \tag{4}$$

where

$$\theta^S(t) = \frac{\partial \ln F^S(0,t)}{\partial t} + \alpha_S \ln F^S(0,t) + \frac{\sigma_S^2}{4}(1-e^{-2\alpha_S t}) - \frac{1}{2}\sigma_S^2,$$

$$\theta^I(t) = \frac{\partial \ln F^I(0,t)}{\partial t} + \alpha_I \ln F^I(0,t) + \frac{\sigma_I^2}{4}(1-e^{-2\alpha_I t}) - \frac{1}{2}\sigma_I^2.$$

We show the derivation of the spot and log prices along with the proof of the Markov property in Appendix A.

[1] To model the whole forward price curve, a multi-factor model is usually required, but since we focus on computing the price and optimal decisions of the GSA, the overall volatility level matters more than the individual factors.

3. Construction of the two-dimensional trinomial tree

In this paper, we evaluate the price of the gas sales agreement on a two-dimensional trinomial tree. To take advantage of being Markovian, we first build a simplified two-dimensional tree for a two-dimensional Markov process[2] (x, y) below, which is obtained by assuming $\theta^S(t) = \theta^I(t) = 0$ in (3) and (4):

$$dx(t) = -\,\alpha_S x(t)dt + \sigma_S dB^S(t) \tag{5}$$

$$dy(t) = -\,\alpha_I y(t)dt + \sigma_I dB^I(t) \tag{6}$$

Then we shift the nodes on the simplified tree by adding the proper drift in order to be consistent with the observed forward curve. In the rest of this section, we summarize the tree building procedures in [10, 15, 16].

Step one. We build two separate one-dimensional trees for both x and y. Taking the log-normal price x as an example, the trinomial tree for y will be built in a very similar manner.

We discretize the time value t into N equally spaced time steps. That is, the time values are denoted by $t_i = i\triangle t$, where $i = 0, 1, \ldots, N$ and $\triangle t$ is the time step. Similarly, the values of x at time t_i are referenced by $x_{i,j} = j\triangle x$, where j is an integer representing the level index and $\triangle x$ is the space step. This means that any node in the tree can be referenced by a pair of integers (i, j). In addition, due to convergence and stability considerations, it is suggested in [15] that $\triangle x = \sigma_S\sqrt{3\triangle t}$.

Because of the mean-reverting nature of this model, the trinomial tree will reach its maximum level $j_{\max}$ at some point (the minimal level $j_{\min}$ is also reached at the same time, $j_{\min} = -j_{\max}$). In the trinomial tree, the nodes emanating from node (i, j) are $(i+1, k+1)$, $(i+1, k)$ and $(i+1, k-1)$, where k is chosen so that the value of x reached by the middle branch is as close as possible to the expected value of x at time t_{i+1}. Indeed,

$$k = \begin{cases} j-1 & j = j_{\max} \\ j & j_{\min} < j < j_{\max} \\ j+1 & j = j_{\min} \end{cases}.$$

According to (5), we have

$$\mathbb{E}(\triangle x(t)) = -\alpha_S x(t)\triangle t,$$

which means that the expected value of $x_{i,j}$ at time $i + 1$ is

$$x_{i,j} + \mathbb{E}(\triangle x_{i,j}) = x_{i,j} - \alpha_S x_{i,j}\triangle t.$$

Let $p_{u,i,j}$, $p_{m,i,j}$ and $p_{d,i,j}$ be the probabilities associated with the upper, middle and lower branches emanating from node (i, j) respectively. Again, according to (5), we can get

$$\begin{aligned}\mathbb{E}(\triangle x_{i,j}) &= -\alpha_S x_{i,j}\triangle t \\ &= p_{u,i,j}((k+1) - j)\triangle x + p_{m,i,j}(k - j)\triangle x + p_{d,i,j}((k-1) - j)\triangle x,\end{aligned}$$

[2] The Markov property of this two-dimensional process can be checked by applying Theorem 7.1.2 in [22]

and

$$\begin{aligned}\mathbb{E}(\triangle x_{i,j}^2) &= \sigma_S^2\triangle t + (\alpha_S x_{i,j}\triangle t)^2 \\ &= p_{u,i,j}((k+1)-j)^2\triangle x^2 + p_{m,i,j}(k-j)^2\triangle x^2 + p_{d,i,j}((k-1)-j)^2\triangle x^2.\end{aligned}$$

Together with $p_{u,i,j} + p_{m,i,j} + p_{d,i,j} = 1$, we obtain

$$p_{u,i,j} = \frac{1}{2}\left[\frac{\sigma_S^2\triangle t + (\alpha_S x_{i,j}\triangle t)^2}{\triangle x^2} + (k-j)^2 - \frac{\alpha_S x_{i,j}\triangle t}{\triangle x}(1-2(k-j)) - (k-j)\right], \quad (7)$$

$$p_{d,i,j} = \frac{1}{2}\left[\frac{\sigma_S^2\triangle t + (\alpha_S x_{i,j}\triangle t)^2}{\triangle x^2} + (k-j)^2 + \frac{\alpha_S x_{i,j}\triangle t}{\triangle x}(1+2(k-j)) + (k-j)\right], \quad (8)$$

$$p_{m,i,j} = 1 - p_{u,i,j} - p_{d,i,j}. \quad (9)$$

It is shown in [15] that, in order to make all of equations (7), (8) and (9) be positive, $j_{\max}$ should be an integer between $\frac{0.184}{\alpha_S\triangle t}$ and $\frac{0.816}{\alpha_S\triangle t}$. To achieve the best efficiency, it is also suggested to set $j_{\max}$ at the smallest integer greater than $\frac{0.184}{\alpha_S\triangle t}$.

Step two. Once we have built two simplified trinomial trees for both x (the gas tree) and y (the index tree), we combine these two trees together. Let node (i, h) be the node on the index tree at time t_i where the index price level is h. Any node on our new two-dimensional tree can be referenced by a triplet of integers (i, j, h), which indicates that when the time index is i, the level of the gas price on the gas tree is j and the level of the index price on the index tree is h. There are three possible movements (an up movement, a middle movement and a down movement) at each node on both the gas tree and the index tree, which gives a total of nine possible movements at each node on the two-dimensional tree. We let $\{\mathbf{m_S}, \mathbf{m_I}\}$ represent the movement on the two-dimensional tree which is the combinaton of the $\mathbf{m_S}$ movement on the gas tree and the $\mathbf{m_I}$ movement on the index tree. This means that we have the following nine movements with their corresponding probabilities:

$\{up, up\}$ with p_{uu}	$\{up, middle\}$ with p_{um}	$\{up, down\}$ with p_{ud}
$\{middle, up\}$ with p_{mu}	$\{middle, middle\}$ with p_{mm}	$\{middle, down\}$ with p_{md}
$\{down, up\}$ with p_{du}	$\{down, middle\}$ with p_{dm}	$\{down, down\}$ with p_{dd}

Denote the probabilities associated with the upper, middle and lower branches on the gas tree by p_u, p_m and p_d, respectively. Similarly, denote the probabilities associated with the upper, middle and lower branches on the index tree by q_u, q_m and q_d. By assuming the correlation $\rho = 0$, the probabilities on the two-dimensional tree are the product of the corresponding probabilities associated with the branches on the gas tree and the index tree. The probability matrix is given by

$$\Pi_{\rho=0} = \begin{pmatrix} p_{uu} & p_{um} & p_{ud} \\ p_{mu} & p_{mm} & p_{md} \\ p_{du} & p_{dm} & p_{dd} \end{pmatrix} = \begin{pmatrix} p_u q_u & p_u q_m & p_u q_d \\ p_m q_u & p_m q_m & p_m q_d \\ p_d q_u & p_d q_m & p_d q_d \end{pmatrix} \quad (10)$$

When the correlation is non-zero, according to [7], each probability is shifted to maintain the marginal distribution and the covariance structure of x and y. The probabilities are given by

$$\Pi_{\rho>0} = \begin{pmatrix} p_{uu} & p_{um} & p_{ud} \\ p_{mu} & p_{mm} & p_{md} \\ p_{du} & p_{dm} & p_{dd} \end{pmatrix} = \begin{pmatrix} p_u q_u + 5\varepsilon & p_u q_m - 4\varepsilon & p_u q_d - \varepsilon \\ p_m q_u - 4\varepsilon & p_m q_m + 8\varepsilon & p_m q_d - 4\varepsilon \\ p_d q_u - \varepsilon & p_d q_m - 4\varepsilon & p_d q_d + 5\varepsilon \end{pmatrix} \tag{11}$$

and

$$\Pi_{\rho<0} = \begin{pmatrix} p_{uu} & p_{um} & p_{ud} \\ p_{mu} & p_{mm} & p_{md} \\ p_{du} & p_{dm} & p_{dd} \end{pmatrix} = \begin{pmatrix} p_u q_u - 1\varepsilon & p_u q_m - 4\varepsilon & p_u q_d + 5\varepsilon \\ p_m q_u - 4\varepsilon & p_m q_m + 8\varepsilon & p_m q_d - 4\varepsilon \\ p_d q_u + 5\varepsilon & p_d q_m - 4\varepsilon & p_d q_d - \varepsilon \end{pmatrix}, \tag{12}$$

where $\varepsilon = \frac{\rho}{36}$ if $\rho > 0$ and $\varepsilon = -\frac{\rho}{36}$ if $\rho < 0$.

Step three. Returning to the notations in Step one, we now make both the gas tree and the index tree to be consistent with the observed forward curve by adding an amount a_i at each node of each time step t_i. Again, we use the gas tree as an example, but the index tree can be adjusted in a very similar manner.

We define the state price $G_{i,j}$ as the value of a security that pays 1 if the node (i, j) is reached, and zero otherwise, at time 0. Then the state price at each node can be obtained by forward induction:

$$G_{i+1,j} = \sum_{j'} G_{i,j'} p_{j',j} e^{-r\triangle t},$$

where (i, j') represents the node at time $i\triangle t$ that has a branch leading to node $(i+1, j)$ and $p_{j',j}$ is the probability of moving from node (i, j') to node $(i + 1, j)$, r is the risk-free rate. Thus, the price at time 0 of any European claim with payoff function $C(S)$ at time step i in the tree is given by

$$C(0) = \sum_j G_{i,j} C(S), \tag{13}$$

where the summation takes place through all nodes (i, j) at time t_i. Recall that S is the gas price. Now consider a case where $C(S_{i,j}) = S_{i,j}$, then by (13) we have

$$e^{-rt_i} F^S(0, t_i) = \sum_j G_{i,j} S_{i,j}. \tag{14}$$

Let $X_{i,j} = x_{i,j} + a_i$. By the fact that $S_{i,j} = e^{X_{i,j}} = e^{x_{i,j}+a_i}$ and through equation (14), we obtain

$$e^{-rt_i} F^S(0, t_i) = \sum_j G_{i,j} e^{x_{i,j}+a_i}.$$

It follows that

$$a_i = \ln\left(\frac{e^{-rt_i} F^S(0, t_i)}{\sum_j G_{i,j} e^{x_{i,j}}}\right).$$

Therefore, by adding a_i at each node at each time t_i, we obtain the trinomial tree we need.

The right panel of Figure 1 demonstrates some examples of trinomial trees that have been constructed to be consistent with the forward curves shown in the left panel of Figure 1.

4. Gas sales agreements

As introduced in Section 1, the GSA is an American-style option between a gas supplier and a gas buyer for the delivery of daily quantities of gas at variable strike prices over several years. In the absence of make-up and carry-forward banks, the main constraint in a GSA is the minimum bill. In each year, a penalty is applied if the actual gas taken is below the minimum bill. When it comes to the make-up and carry-forward banks, however, the contract becomes more complicated. In the years where the gas taken is less than the minimum bill, the shortfall is added to a make-up bank. In years where the gas taken is greater than some reference level (this level is called the carry-forward base), the excess gas is added to the carry-forward bank. In later years, under some pre-specified conditions, the gas in the make-up bank can be withdrawn to get a refund, while the gas in the carry-forward bank can be withdrawn to reduce the minimum bill. Since both the make-up and carry-forward banks offer the buyer opportunities to reduce possible losses, some pre-specified limits are imposed to protect the interests of the gas supplier. For example, in each year, the gas withdrawn from these banks cannot exceed some pre-specified limits: the carry-forward bank recovery limit and the make-up bank recovery limit. These recovery limits will play large roles in the contract value. Also, since both the make-up and carry-forward banks are related to the minimum bill (and hence related to the penalty), how the penalty is calculated will also greatly affect the contract value. In addition, in each year, the evaluation of the make-up and carry-forward banks should be formulated carefully.

In this section, we provide an analytical representation of GSA contracts with the above features. GSA contracts without make-up and carry-forward banks are presented in Subsection 4.1. GSA contracts with make-up and carry-forward banks are formulated in Subsection 4.2.

4.1. Gas sales agreements in the absence of make-up and carry-forward banks

A gas sales agreement with indexation between the gas provider and the gas user could have many specific features to satisfy the needs of both parties to the agreement but these contracts usually have a number of common features as set out below:

- The contract lasts L years and there are J time periods (typically days) in each year, L and J are positive integers. Let T_i be the end of each year i, for $i = 1, \dots, L$. Obviously, T_L is the maturity of this contract.
- Let the time interval be $\triangle t = 1/J$, then the contract duration is equally spaced into $J \cdot L$ periods. Denote the j-th period of the i-th year by $t_{i,j}$, $i = 1, \dots, L$ and

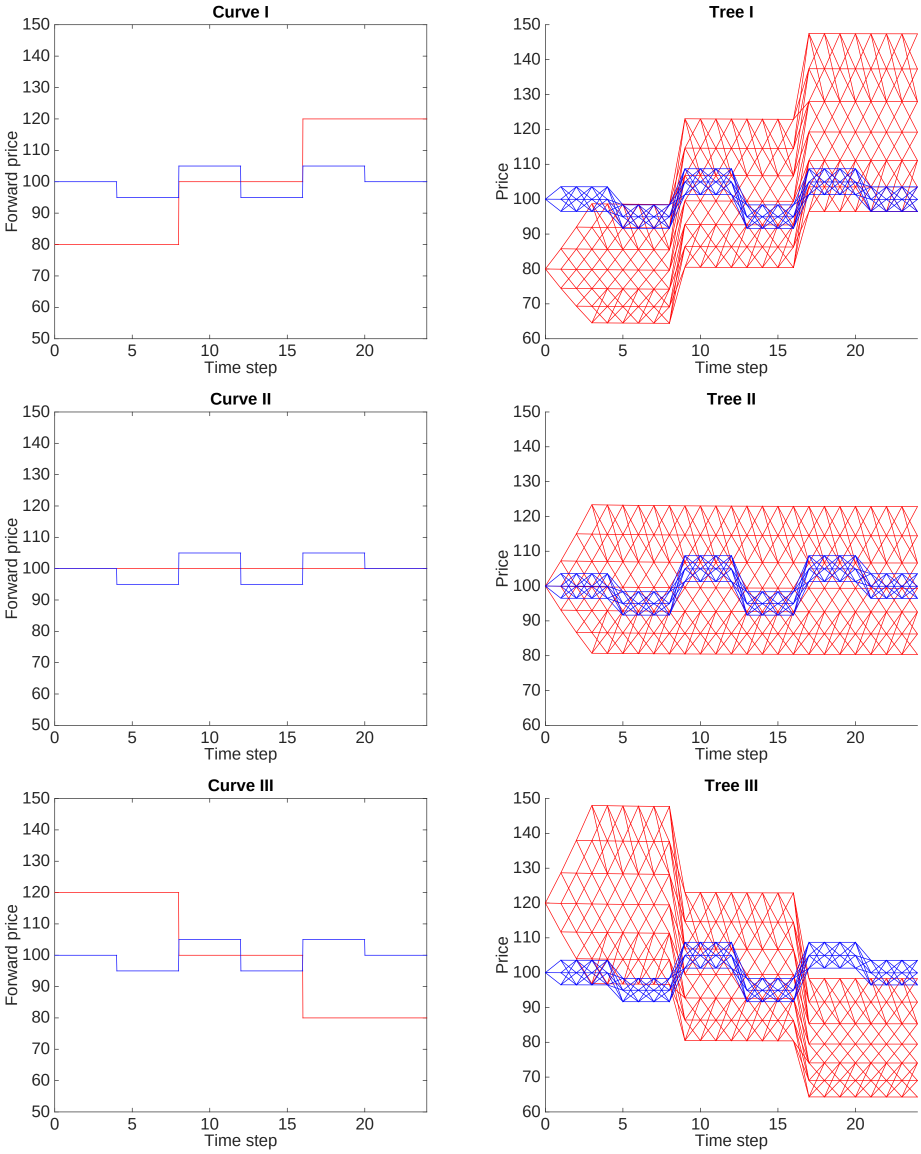

Figure 1: Trinomial tree examples. The red plots refer to the natural gas price and the blue plots refer to the index. The parameters used are as follows: $T = 1$, $N = 24$, $\alpha_S = 1.5$, $\alpha_I = 5$, $\sigma_S = 0.2$, $\sigma_I = 0.1$ and $r = 0.05$.

$j = 0, \ldots, J$. Hence we have

$$0 = t_{1,0} < t_{1,1} < \ldots < t_{1,J} = T_1 = t_{2,0} < t_{2,1} < \ldots$$
$$\ldots < t_{L-1,J} = T_{L-1} = t_{L,0} < t_{L,1} < \ldots < t_{L,J} = T_L.$$

We assume that the holder of a GSA has exactly one exercise opportunity at each time $t_{i,j}$ in each gas year where $i = 1, \ldots, L$ and $j = 1, \ldots, J$, which amounts to J rights for a whole year. Typical GSAs can usually be exercised daily ($J = 365$) or weekly ($J = 52$).

- Let $q_{t_{i,j}}$ be the amount of gas taken (exercise decision) at time $t_{i,j}$, which is constrained by the minimum daily quantity $q_{\min}$ and the maximum daily quantity $q_{\max}$, that is $q_{\min} \leq q_{t_{i,j}} \leq q_{\max}$. Such an admissible policy is denoted by $\mathbf{q} = \{q_{t_{i,j}}\}$. Let $Q_{t_{i,j}}$ be the cumulative amount of gas taken up to time $t_{i,j}$ in year i (also known as the period to date). It is straightforward to see that

$$Q_{t_{i,j}} = \sum_{k=1}^{j-1} q_{t_{i,k}}.$$

  In addition, we let Q_{T_i} be the total amount of gas taken during the year i. That is, $Q_{T_i} = Q_{t_{i,J}} + q_{t_{i,J}}$.

- Let $S_{i,j}$ and $I_{i,j}$ be the gas price and the index at time $t_{i,j}$, respectively. Then, upon taking the volume $q_{t_{i,j}}$, the payoff from the buyer's point of view at time $t_{i,j}$ is

$$q_{t_{i,j}} \left(S_{i,j} - I_{i,j}\right).$$

- In each year, there is a maximum quantity of gas the buyer can take, which is called the annual contract quantity. Denote the annual contract quantity in year i by ACQ_i. Similarly, there is a minimum quantity of gas the buyer has to take in each year, which is called the minimum bill. Denote the minimum bill in year i by MB_i. In each year, the amount of gas the buyer has taken cannot exceed the annual contract quantity. That is, $Q_{T_i} \leq ACQ_i$ for each year i. The total gas taken in each year can be below the minimum bill, however, in which case the buyer has to pay penalties at the end of that year. More precisely, if $Q_{T_i} < MB_i$ in year i, at the end of year i, there is an out cash flow generated by the penalty, in addition to the cash flow generated by the exercise decision.

- Usually, the penalty rate is a percentage of the index price, and hence the buyer would prefer to take enough gas to avoid penalty. In this paper, we let the penalty be given by

$$\eta \cdot I_{i,J} \cdot \min\left\{Q_{T_i} - MB_i, 0\right\}$$

  for each year i, where $\eta \in [0, 1]$ is a constant, which is called the penalty coefficient.

- The risk-free rate is denoted by r.

From the buyer's point of view, the goal is to maximize the total expected discounted payoff of the contract, including the penalty. That is, to find the value V_0 of this contract at time 0 which is given by

$$V_0 = \sup_{\mathbf{q}} \mathbb{E}\left[\sum_{i=1}^{L}\left(\sum_{j=1}^{J} e^{-rt_{i,j}} q_{t_{i,j}}(S_{i,j} - I_{i,j}) + e^{-rt_{i,J}} \cdot \eta \cdot I_{i,J} \cdot \min\{Q_{T_i} - MB_i, 0\}\right)\right]. \quad (15)$$

Remark 4.1. In [13], the author claims that the non-trivial constraints below are extremely important:

$$q_{\min} \cdot J < MB_i \leq Q_{T_i} \leq ACQ_i < q_{\max} \cdot J \text{ for each } i.$$

Since under normal circumstances, however, the gas supplier would encourage the buyer to purchase more gas, it is natural to have $ACQ_i = q_{\max} \cdot J$, namely

$$q_{\min} \cdot J < MB_i \leq Q_{T_i} \leq ACQ_i = q_{\max} \cdot J \text{ for each } i.$$

In a GSA contract, the user and the supplier can specify an existing volume of gas which has been taken at the beginning of each gas year. We denote this volume of gas by $\widetilde{Q_i}$. In this paper, we assume $\widetilde{Q_i} = 0, \forall i$ but it is straightforward to relax this assumption as long as we know the exact value specified in a particular contract.

In addition, we further assume that, once the annual contract quantity is met, the daily minimum quantity can be violated, since the amount of gas the buyer has taken cannot exceed the annual contract quantity. That is, we let $q_{t_{i,j}} = 0$ as long as $Q_{t_{i,j}} = ACQ_i$, and we omit the discussion of this scenario in the rest of this paper.

Dynamic Programming. On the last day of the contract $t_{L,J}$, we have the following three scenarios: if $S_{L,J} > I_{L,J}$, that is the payoff is increasing in the volume purchased then we should take as much gas as we can; if $S_{L,J} \leq I_{L,J}$ and $I_{L,J} - S_{L,J} < \eta \cdot I_{L,J}$, we can reduce the total loss by purchasing a quantity up to that required to avoid the penalty, or the maximum possible, whichever is smaller; if $S_{L,J} \leq I_{L,J}$ and $I_{L,J} - S_{L,J} \geq \eta \cdot I_{L,J}$, that is the loss on the purchase of the gas is not compensated by the reduction in the penalty payment, we take as little gas as we can.

More precisely, let $q^*(S, I, Q, t)$ be the optimal take decision where the gas price equals S, the index equals I and the period to date equals Q at time t. In particular, on the last day $t_{L,J}$ of the contract, we have:

$$q^*(S, I, Q, t_{L,J}) = \begin{cases} q_{\max}, & S > I, \\ \min\{\max\{MB_L - Q, q_{\min}\}, q_{\max}\}, & S \leq I \text{ and } (1-\eta)I < S, \\ q_{\min}, & S \leq I \text{ and } (1-\eta)I \geq S, \end{cases}$$

On the final day, the buyer still has to face the penalty if the minimum bill is not reached. After the last decision $q^* = (S, I, Q, t_{L,J})$, the total amount of gas taken in the last year is known. Then the penalty becomes:

$$\eta \cdot I_{L,J} \cdot \min\{Q_{t_{L,J}} + q^*(S, I, Q, t_{L,J}) - MB_L, 0\}.$$

Denote the value of the contract by $V(S,I,Q,t)$ where the gas price equals S, the index equals I and the period to date equals Q at time t. The terminal payoff $V(S,I,Q,t_{L,J})$ is determined by the instant strike payoff and the penalty, which leads to the following expression:

$$V(S,I,Q,t_{L,J}) = q^*(S,I,Q,t_{L,J}) \cdot (S-I) + \eta \cdot I \cdot \min\{0, Q + q^*(S,I,Q,t_{L,J}) - MB_L\}.$$

Now that we have the terminal payoff, we can work backwards in time to find the optimal take and optimal value through the life of the contract. In fact, at each day $t_{i,j}$ within a year, we should choose the optimal exercise decision according to

$$\begin{aligned} q^*(S,I,Q,t_{i,j}) =& \mathrm{argmax}_{q_{t_{i,j}} \in [q_{\min}, q_{\max}]} \left\{ q_{t_{i,j}}(S-I) + \right. \\ & \left. \mathbb{E}\left[e^{-r\triangle t} V(S_{i,j+1}, I_{i,j+1}, Q + q_{t_{i,j}}, t_{i,j+1}) \mid S_{i,j} = S, I_{i,j} = I \right] \right\}, \end{aligned}$$

and the contract value is

$$\begin{aligned} V(S,I,Q,t_{i,j}) =& q^*(S,I,Q,t_{i,j}) \cdot (S-I) + \\ & \mathbb{E}\left[e^{-r\triangle t} V(S_{i,j+1}, I_{i,j+1}, Q_{t_{i,j}} + q^*(S,I,Q,t_{i,j}), t_{i,j+1}) \mid S_{i,j} = S, I_{i,j} = I \right]. \end{aligned}$$

On the last day of each year (except the final year of the contract), we choose the optimal exercise decision according to

$$\begin{aligned} & q^*(S,I,Q,t_{i,J}) \\ =& \mathrm{argmax}_{q_{t_{i,J}} \in [q_{\min}, q_{\max}]} \left\{ q_{t_{i,J}}(S-I) + \eta \cdot I \cdot \min\{0, Q + q_{t_{i,j}} - MB_i\} \right. \\ & \left. \mathbb{E}\left[e^{-r\triangle t} V(S_{i+1,1}, I_{i+1,1}, \widetilde{Q}, t_{i+1,1}) \mid S_{i,J} = S, I_{i,J} = I \right] \right\}, \end{aligned} \tag{16}$$

Then the contract value is obtained by

$$\begin{aligned} & V(S,I,Q,t_{i,J}) \\ =& q^*(S,I,Q,t_{i,J}) \cdot (S-I) + \eta \cdot I \cdot \min\{0, Q + q^*(S,I,Q,t_{i,J}) - MB_i\} \\ & \mathbb{E}\left[e^{-r\triangle t} V(S_{i+1,1}, I_{i+1,1}, \widetilde{Q}, t_{i+1,1}) \mid S_{i,J} = S, I_{i,J} = I \right]. \end{aligned} \tag{17}$$

4.2. Multi-year gas sales agreements with make-up and carry-forward banks

Turning now to the make-up and carry-forward features of multiple year GSA contracts, which were previewed in Section 1 of this paper, we now formulate these two banks based on the notations in Subsection 4.1 above:

- Denote the amount of gas available in the carry-forward bank in year i by C_i, $i = 1, \ldots, L$. Let CRL_i be the carry-forward bank recovery limit and CB_i be the carry-forward base in year i. In addition, denote the usage of gas in the carry-forward bank by c_i in year i. Denote the amount of gas available in the make-up bank in year i by M_i, $i = 1, \ldots, L$. Let MRL_i be the make-up bank recovery limit and m_i be the usage of gas in the make-up bank in year i.

- In year i, when the total gas taken is $Q_{T_i} > \max\{MB_i+m_i, CB_i\}$, then the excess gas is added to the carry-forward bank, which gives $C_{i+1} = C_i+(Q_{T_i}-\max\{MB_i+m_i, CB_i\})$. In year i, if the total gas taken is $Q_{T_i} < MB_i$, the buyer can use the gas in the carry-forward bank to reduce the minimum bill, which adjusts the minimum bill in year i to $MB_i - c_i$. In addition, the possible year end penalty becomes $\eta \cdot I_{i,J} \cdot \min\{Q_{T_i} - (MB_i - c_i), 0\}$. The usage c_i, however, is constrained by the amount of gas in the carry-forward bank and the recovery limit, that is, $c_i \leq \min\{C_i, CRL_i\}$. Based on the above formulation, we obtain the evolution of the carry-forward bank, which is given by
$$C_{i+1} = (C_i - c_i) + \max\{Q_{T_i} - \max\{MB_i + m_i, CB_i\}, 0\}.$$
- In year i, when the gas taken is less than the adjusted minimum bill, that is $Q_{T_i} < MB_i-c_i$, the shortfall is added to the make-up bank and $M_{i+1} = M_i+(MB_i - c_i - Q_{T_i})$. In year i, when the gas taken is $Q_{T_i} > MB_i + C_i$, a refund will be given based on the excess gas. Then at the end of those years, in addition to the instant exercise payoff, an income cashflow is generated from the refund, which is given by $I_{i,J} \cdot m_i$; noting, of course, that $m_i \leq \min\{M_i, MRL_i, \max\{Q_{T_i} - (MB_i + C_i), 0\}\}$. We can also obtain the evolution of the make-up bank by
$$M_{i+1} = (M_i - m_i) + \max\{MB_i - c_i - Q_{T_i}, 0\}.$$

The objective function. With both make-up and carry-forward banks, the payoff at the end of each year not only depends on the instant exercise and the possible penalty, but also the possible income from the refund. That is, at the end of each year i, we have the following cash flows:

- the instant exercise payoff $q_{t_{i,J}}\,(S_{i,J} - I_{i,J})$.
- the possible penalty if the total gas taken is less than the adjusted minimum bill $MB_i - c_i$:
$$\eta \cdot I_{i,J} \cdot \min\{Q_{t_{i,J}} + q_{t_{i,J}} - (MB_i - c_i), 0\}.$$
- the income generated by the possible refund when the total gas taken exceeds MB_i+C_i:
$$I_{i,J} \cdot \min\{m_i, \max\{Q_{t_{i,J}} + q_{t_{i,J}} - (MB_i + C_i), 0\}\}.$$

The payoff for each day within a year is the same as that with GSAs in the absence of make-up and carry-forward banks. Thus, from the perspective of the buyer, the goal is to find the initial contract value V_0 that is given by

$$\begin{aligned} V_0 = \sup_{q_{t_{i,j}}, m_i, c_i} \mathbb{E} \Bigg[\sum_{i=1}^{L} \Bigg(& \sum_{j=1}^{J} e^{-rt_{i,j}} q_{t_{i,j}} (S_{i,j} - I_{i,j}) \\ & + e^{-rt_{i,J}} \cdot \eta \cdot I_{i,J} \cdot \min\{Q_{t_{i,J}} + q_{t_{i,J}} - (MB_i - c_i), 0\} \\ & + e^{-rt_{i,J}} \cdot I_{i,J} \cdot \min\{m_i, \max\{Q_{t_{i,J}} + q_{t_{i,J}} - (MB_i + C_i), 0\}\} \Bigg) \Bigg]. \end{aligned}$$

The terminal condition. With both make-up and carry-forward banks, the decision to be made on the last day is much more complicated. On the last day of the contract, we need to use as much of the balance in both the make-up and carry-forward banks as possible. Let $q^*(S, I, Q, C, M, t)$ and $V(S, I, Q, C, M, t)$ be the optimal exercise decision and the contract value respectively where the gas price equals S, the index equals I, the period to date equals Q, the amount of gas in the carry-forward bank equals C and the amount of gas in the make-up bank equals M, at time t. Next, let us find $q^*(S, I, Q, C, M, t_{L,J})$. On the last day, if $S > I$, the decision is obviously $q^*(S, I, Q, C, M, t_{L,J}) = q_{\max}$. If $S \leq I$, however, we have following scenarios:

- If there is no gas available in the make-up bank, or the buyer has to take more gas to meet the minimum bill, that is $M = 0$ or $Q < MB_L$.
  - If $(1 - \eta) I < S$, that is the loss on the purchase of the gas is compensated by the reduction in the penalty payment. The optimal choice is to purchase a quantity up to the amount required to avoid the penalty, or the maximum possible, whichever is smaller. Then
  $$q^*(S, I, Q, C, M, t_{L,J}) = \min\{\max\{MB_L - C - Q, q_{\min}\}, q_{\max}\}.$$
  - If $(1 - \eta) I \geq S$, that is the loss on the purchase of the gas is not compensated by the reduction in the penalty payment. Then
  $$q^*(S, I, Q, C, M, t_{L,J}) = q_{\min}.$$
- If there is gas available in the make-up bank, that is $M > 0$. And $Q \geq MB_L + C$, this means that the buyer can get a refund from the make-up bank.
  - If there is gas available in the make-up bank, that is $0 \leq Q - (MB_L + C) < M_T$, the buyer should buy a quantity of gas up to the gas available in the make-up bank, or the maximum possible:
  $$q^*(S, I, Q, C, M, t_{L,J}) = \max\{\min\{M - (Q - (MB_L + C)), q_{\max}\}, q_{\min}\}.$$
  - If the buyer cannot take more refund because of a shortage of gas in the make-up bank, that is $Q - (MB_L + C) \geq M$, then they should take the daily minimum so as to avoid the loss:
  $$q^*(S, I, Q, C, M, t_{L,J}) = q_{\min}.$$
- If $M > 0$, but the period to date is not yet large enough to get a refund, that is $MB_L \leq Q < (MB_L + C)$.
  - If $(MB_L + C) - Q < q_{\max}$, the buyer can get some refund by taking $q_{\max}$. In this scenario, if the buyer decides to get a refund, they should take as much as they can, that is $q_{\max}$. Then the buyer makes a payment congruent with the gas purchased to meet the condition of refund taking. This payment equals $((MB_L + C) - Q) \cdot I$. At the same time, the buyer receives gas which has a value of $q_{\max} \cdot S$.

* If $((MB_L + C) - Q) \cdot I > q_{\max} \cdot S$, that is the buyer loses money through this transaction, then

$$q^*\left(S, I, Q, C, M, t_{L,J}\right) = q_{\min}.$$

* If $((MB_L + C) - Q) \cdot I \leq q_{\max} \cdot S$, that is the buyer makes a profit through this transaction, then

$$q^*\left(S, I, Q, C, M, t_{L,J}\right) = q_{\max}.$$

- If $(MB_L + C) - Q \geq q_{\max}$, the buyer cannot get a refund, even if the daily maximum quantity of gas has been purchased, and they should therefore take the daily minimum.

$$q^*\left(S, I, Q, C, M, t_{L,J}\right) \quad = \quad q_{\min}.$$

Dynamic programming. Once we have the terminal optimal decision $q^*_{t_{L,J}} = q^*(S, I, Q, C, M, t_{L,J})$, we can obtain the terminal value of the contract

$$\begin{aligned} V(S, I, Q, C, M, t_{I,J}) =& q^*_{t_{L,J}} \cdot (S - I) \\ &+ \eta \cdot I \cdot \min\{Q + q^*_{t_{L,J}} - (MB_L - C), 0\} \\ &+ I \cdot \min\{M, \max\{Q + q^*_{t_{L,J}} - (MB_L + C), 0\}\}. \end{aligned}$$

Then we work backwards in time. On the last day of each year i (except the final year) the buyer should choose the optimal exercise quantity, the optimal usage of the carry-forward bank and the optimal usage of the make-up bank $(q^*, c^*, m^*)(S, I, Q, C, M, t_{i,J})$ according to

$$\begin{aligned} &(q^*, c^*, m^*)(S, I, Q, C, M, t_{i,J}) \\ =& \text{argmax}_{q_{t_{i,J}}, c_i, m_i} \Big\{ q_{t_{i,J}} \left(S - I\right) \\ &+ \eta \cdot I \cdot \min\{Q + q_{t_{i,J}} - (MB_i - c_i), 0\} \\ &+ I \cdot \min\{m_i, \max\{Q + q_{t_{i,J}} - (MB_i + C), 0\}\} \\ &+ \mathbb{E}\left[e^{-r\triangle t} V(S_{i+1,1}, I_{i+1,1}, \widetilde{Q}, C_{i+1}, M_{i+1}, t_{i+1,1}) \mid S_{i,J} = S, I_{i,J} = I\right] \Big\}. \end{aligned}$$

Once the optimal decisions $(q^*_{t_{i,J}}, c^*_i, m^*_i) = (q^*, c^*, m^*)(S, I, Q, C, M, t_{i,J})$ are determined, we have the contract value

$$\begin{aligned} &V(S, I, Q, C, M, t_{i,J}) \\ =& q^*_{t_{i,J}}(S - I) + \eta \cdot I \cdot \min\{Q + q^*_{t_{i,J}} - (MB_i - c^*_i), 0\} \\ &+ I \cdot \min\{m^*_i, \max\{Q + q^*_{t_{i,J}} - (MB_i + C), 0\}\} \\ &+ \mathbb{E}\left[e^{-r\triangle t} V(S_{i+1,1}, I_{i+1,1}, \widetilde{Q}, C_{i+1}, M_{i+1}, t_{i+1,1}) \mid S_{i,J} = S, I_{i,J} = I\right] \end{aligned}$$

On each day within a gas year, the buyer should choose the optimal quantity $q^*(S, I, Q, C, M, t_{i,j})$ according to

$$
\begin{aligned}
& q^*(S, I, Q, C, M, t_{i,j}) \\
=& \text{argmax}_{q_{t_{i,j}}} \{ q_{t_{i,j}}(S - I) + \\
& \mathbb{E}\left[e^{-r\triangle t} V(S_{i,j+1}, I_{i,j+1}, Q + q_{t_{i,j}}, C, M, t_{i,j+1}) \mid S_{i,j} = S, I_{i,j} = I \right] \}
\end{aligned}
$$

Again, once we have the optimal quantity $q^*_{t_{i,j}} = q^*(S, I, Q, C, M, t_{i,j})$, we have the contract value

$$
\begin{aligned}
& V(S, I, Q, C, M, t_{i,j}) \\
=& q^*_{t_{i,j}}(S - I) + \\
& \mathbb{E}\left[e^{-r\triangle t} V(S_{i,j+1}, I_{i,j+1}, Q + q^*_{t_{i,j}}, C, M, t_{i,j+1}) \mid S_{i,j} = S, I_{i,j} = I \right].
\end{aligned}
$$

The bang-bang consumption.

Theorem 4.1. *Suppose $\mathcal{P}_i(Q, M, C)$ is the function which gives the cash flow generated by the possible penalty and the possible refund at the end of year i. If $\mathcal{P}_i(Q, M, C)$ is continuously differentiable and $P[\kappa_{i,j} = 0] = 0$, where*

$$
\kappa_{i,j} = \left[e^{-rt_{i,j}}(S_{i,j} - I_{i,j}) + \mathbb{E}\left(\sum_{d=i}^{L} e^{-rt_{d,J}} \cdot I_{d,J} \cdot \mathcal{P}'_d(Q^*_{T_d}, M^*_d, C^*_d)) \mid I_{i,j}, Q^*_{t_{i,j}}, M^*_i, C^*_i \right) \right].
$$

then the optimal consumption at time $t_{i,j}$ is necessarily of a bang-bang type and is given by

$$
q^*(S_{i,j}, I_{i,j}, Q^*_{t_{i,j}}, M^*_i, C^*_i, t_{i,j}) = q_{\max}\mathbf{1}_{\kappa_{i,j}>0} + q_{\min}\mathbf{1}_{\kappa_{i,j}<0}.
$$

where $Q^*_{t_{i,j}}$, $Q^*_{T_i}$, M^*_i and C^*_i are the period to date, yearly take, make-up bank allowance and carry-forward bank allowance obtained through an optimal trading strategy.

Remark 4.2. The above condition involves an unknown optimal strategy which makes it hard to check. As reported in [3, 4], however, it can be expected if one can show that $\kappa_{i,j}$ is absolutely continuous. In our contract, the function

$$
\mathcal{P}_i(Q, M, C) = \eta \cdot \min\{Q - (MB_i - c_i), 0\} + \min\{m_i, \max\{Q - (MB_i + C), 0\}\},
$$

where $c_i \leq \min\{C, CRL_i\}$ and $m_i \leq \min\{M, MRL_i, \max\{Q_{T_i} - (MB_i + C), 0\}\}$, is non-differentiable at some points. That is, Theorem 4.1 only applies to contracts in which the function $\mathcal{P}_i(Q, M, C)$ is a bit smoother than our contract. The bang-bang consumption, in our contract, however, is observed numerically through the dynamic programming we present in this section. Hence, we assume throughout our paper that GSAs have such a bang-bang consumption.

Contract **A** with a constant strike price $K = 100$				
$\alpha_S = 5$	$\sigma_S = 0.5$	$r = 0.05$	$q_{\min} = 0$	$q_{\max} = 1$
$MB_1 = 273$	$ACQ_1 = 365$	$\eta = 1$	$F^S(0,t) = 100$, for $0 \le t \le L$	
Contract **B** with indexation				
$\alpha_I = 15$	$\sigma_I = 0.2$	$\rho = 0.5$	$F^I(0,t) = 100$, for $0 \le t \le L$	

Table 1: Parameters set for Contract **A** and Contract **B**.

5. Numerical analysis

5.1. Comparison with the constant strike GSA

In this subsection, we provide two contracts, one with a constant strike price and one with indexation. These contracts last one year with daily exercise opportunities. That is, $L = 1$ and $J = 365$. We show the features of the daily decision surface and the difference between these two contracts. The parameter inputs of those two contracts are shown in Table 1.

Contract **A** is a normal one-year contract with daily exercise opportunities and an at-the-money constant strike, $K = 100$. Since we focus on the strike price, we use a flat gas price forward curve to minimize the effect of the gas price. Since we have assumed a bang-bang consumption, without losing generality, we let the daily take constraints be $q_{\min} = 0$ and $q_{\max} = 1$. Contract **B** uses the same parameter inputs as Contract **A**, except for the constant strike K. Since the index is driven by (4), we have those inputs in Table 1. For the index process, we choose a relatively large mean-reverting speed and small volatility since the index is determined by the weighted average prices of some other energy products in order to reduce uncertainty. Also, to offer a relatively fair comparison, we let the index forward curve be $F^I(0,t) = K = 100$, for $0 \le t \le L$.

Decision surfaces. We define the exercise threshold as the gas price equal to or above which the buyer would be better off taking the daily maximum $q_{\max}$. A low exercise threshold means that the buyer is willing to buy gas under the contract, while a high exercise threshold means the buyer is reluctant to purchase gas. For the buyers, the most important thing is to make the optimal daily exercise decision at the beginning of each gas day, based on the current gas price and the index price. In Figure 2 we present some decision surfaces for Contract **B**. Note that the buyer and supplier can specify an existing volume of gas which has been taken before entering a GSA contract. That is why the "Day 1" surface also provides decisions for the period-to-date greater than zero. As we can observe in Figure 2, Contract **B** captures almost all the key features of the decision surfaces of a typical GSA with a constant strike (for the analysis of decision surfaces of constant strike GSAs, we refer interested readers to [6]). In these plots, the exercise threshold is below the index price when the period to date is low, which would cause an immediate loss. This is due to the minimum bill condition. If the buyers try to avoid any loss at this point then they would have to face more losses in order to meet the minimum bill, or even face penalties, in the future. In addition, as is shown in the "Day 120", "Day 240" and "Day 365" plots, as the time approaches the year

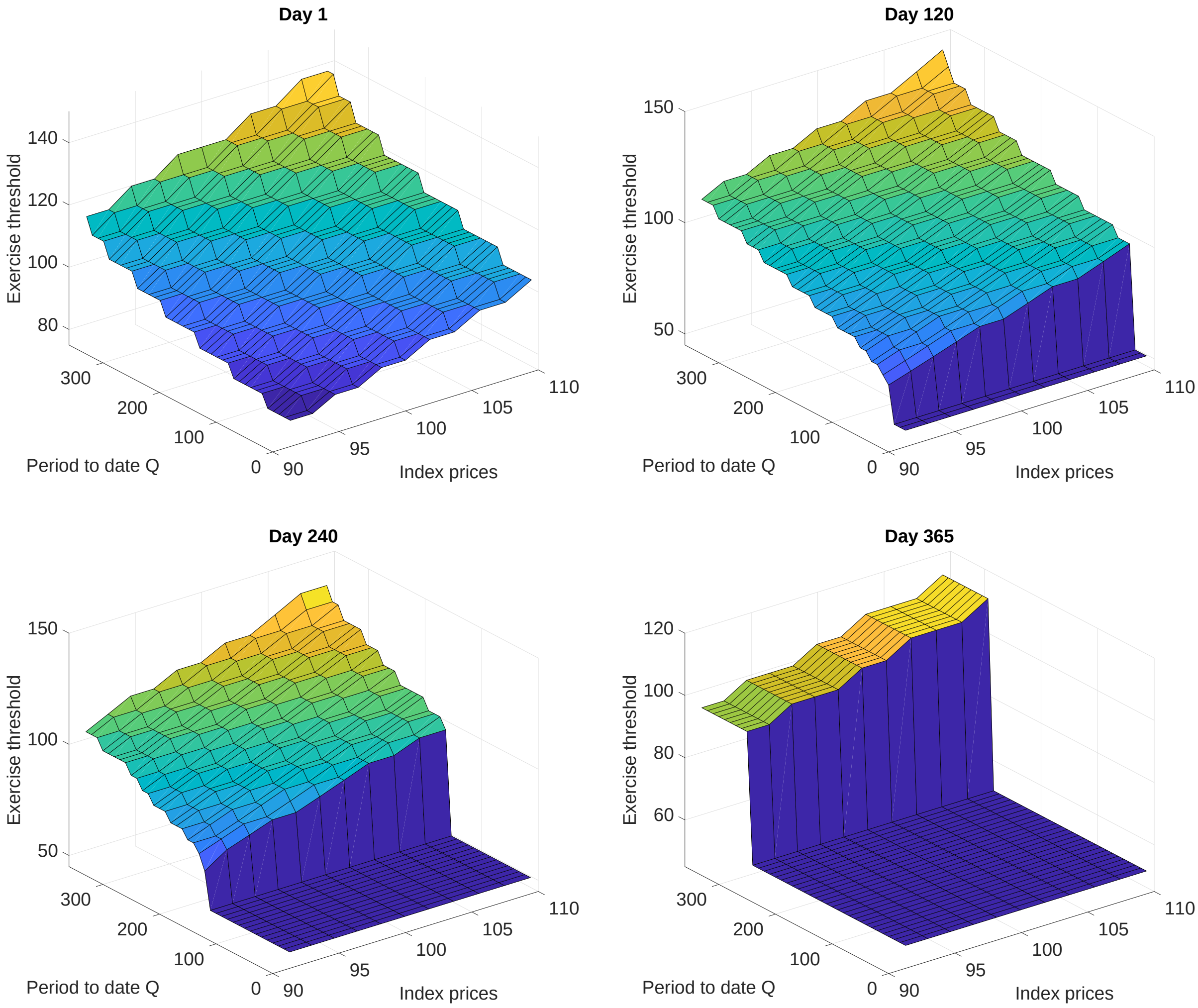

Figure 2: Decision surfaces for Contract **B**.

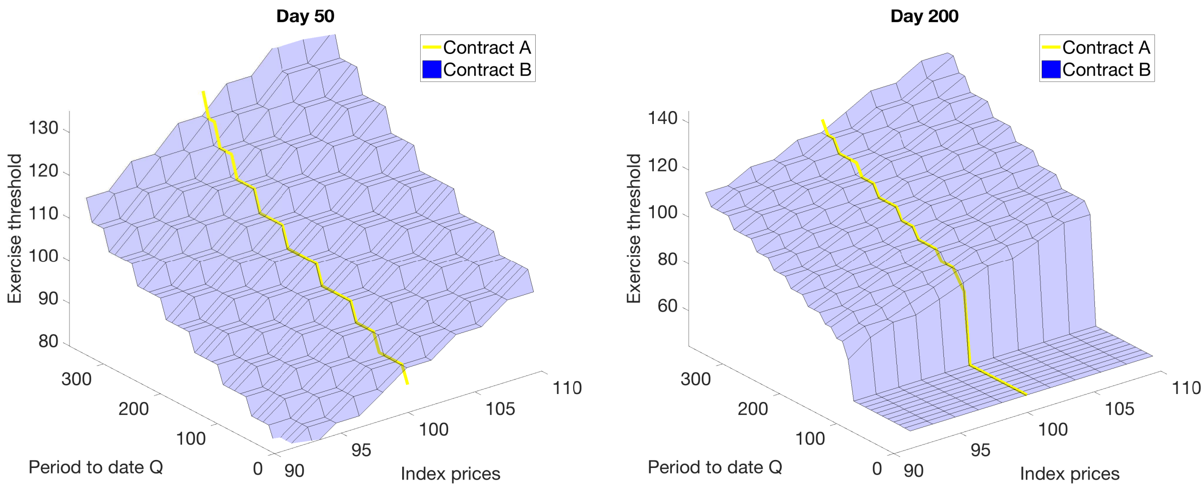

Figure 3: Comparison of decision surfaces between Contract **A** and Contract **B**

end, there will be a quantity of the period to date below which the exercise threshold is the minimum gas price obtained from the tree, regardless of the index price. This is due to the need to meet the minimum bill or reduce the possible penalty. We call this quantity the minimum bill constrained quantity.

Next, we present the difference in decision surfaces between Contract **A** and Contract **B**. As we can see in Figure 3, the exercise threshold of Contract **A** (the yellow line) is above the decision surface of Contract **B** when the period to date is high and beneath the surface when the period to date is low. This means that, as long as the period to date exceeds the minimum bill constrained quantity, the Contract **B** holder would purchase more gas when the period to date is high and purchase less gas when the period to date is low. With a high period to date, the possibility of failing to meet the minimum bill, or facing a penalty, is low. In this scenario, the holder of Contract **B** buys more gas, indicating that the indexation increases the uncertainty of future cashflow. In contrast, with a small period to date, the possibility of failing to meet the minimum bill, or facing a penalty, is high. In this scenario, exercising less gas indicates that the holder of Contract **B** benefits from the indexation.

Contract values. The left-hand plot in Figure 4 shows the value surface at day 0 of Contract **B**. The right-hand one is obtained by using the value of Contract **B** minus the value of Contract **A** at day 0. As we can see, the value of Contract **A** dominates the value of Contract **B** at each point, which means that the GSA contract holder may not benefit from the indexation. Furthermore, it can be seen that the difference between the values of the two contracts first becomes larger as the period to date increases. This is because, with an initially small period to date, the possible need to meet the minimum bill reduces as the period to date increases. The difference then becomes smaller as the period to date increases. This is due to the fact that the exercise strategies of both contracts become less flexible as the number of possible exercise opportunities decreases. Less flexibility means

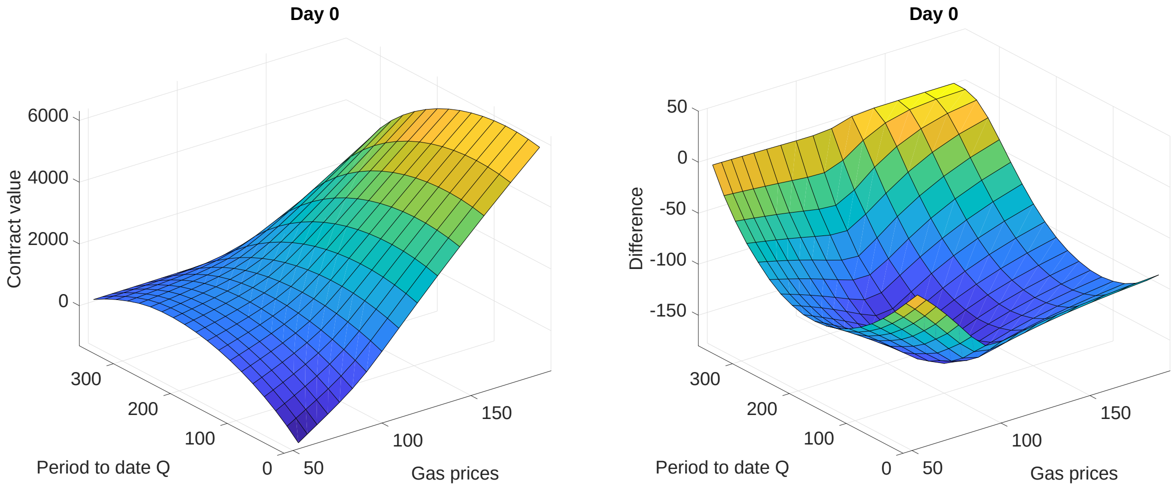

Figure 4: The left plot shows the value surface of Contract **B**. The right plot shows the difference of values between Contract **A** and Contract **B**.

more similarity, that is, the exercise strategies of both contracts converge to each other as the period to date increases.

5.2. The effect of the make-up and carry-forward banks

In GSA contracts, make-up and carry-forward banks offer buyers the opportunity to reduce the potential risk. These rights are constrained, however, by the make-up and carry-forward recovery limits. In this subsection, we investigate the effects of these limits. All contracts appearing in this subsection share the parameters of Contract **B** in Table 1. Since the make-up and carry-forward banks appear in multi-year contracts, we let $L = 3$, $J = 365$. In addition, we have the same minimum bill, carry-forward base and annual contract quantity in all three years: $MB_i = 273$, $CB_i = 292$ and $ACQ_i = 365$ for $i = 1, 2, 3$.

The effect of the make-up recovery limit. In order to minimize the effect of the carry-forward bank, we let $CRL_i = 0$, $i = 1, 2, 3$. That is, there is no carry-forward clause in the contract. Figure 5 shows the difference between Contract $\mathbf{M}^{36}$ with $MRL_i = 36$ and Contract $\mathbf{M}^{73}$ with $MRL_i = 73$, $i = 1, 2, 3$. In (a) and (b), we can see that the value difference increases rapidly if the period to date is less than 51 (i.e. $Q_{t_{1,180}} < 51$). This coincides with our expectations. Intuitively speaking, at the beginning of day 180, once $Q_{t_{1,180}} < 87$, the buyer cannot meet the minimum bill even if he/she takes the maximum daily gas every day for the rest of the current year ($Q_{t_{1,180}} + 1 + (365 - 180) < 273$). It is at this point that the make-up bank starts to work; in other words, the shortfall is added to the make-up bank. Once $Q_{t_{1,180}} < 87 - 36 = 51$, however, the quantity of gas in the make-up bank of Contract $\mathbf{M}^{36}$ will reach the make-up recovery limit, even if the buyer takes the maximum daily gas every day for the rest of the current year. That is, the holder of Contract $\mathbf{M}^{36}$ faces penalties at the year end due to the shortfall, and the extra gas cannot be added to the make-up bank

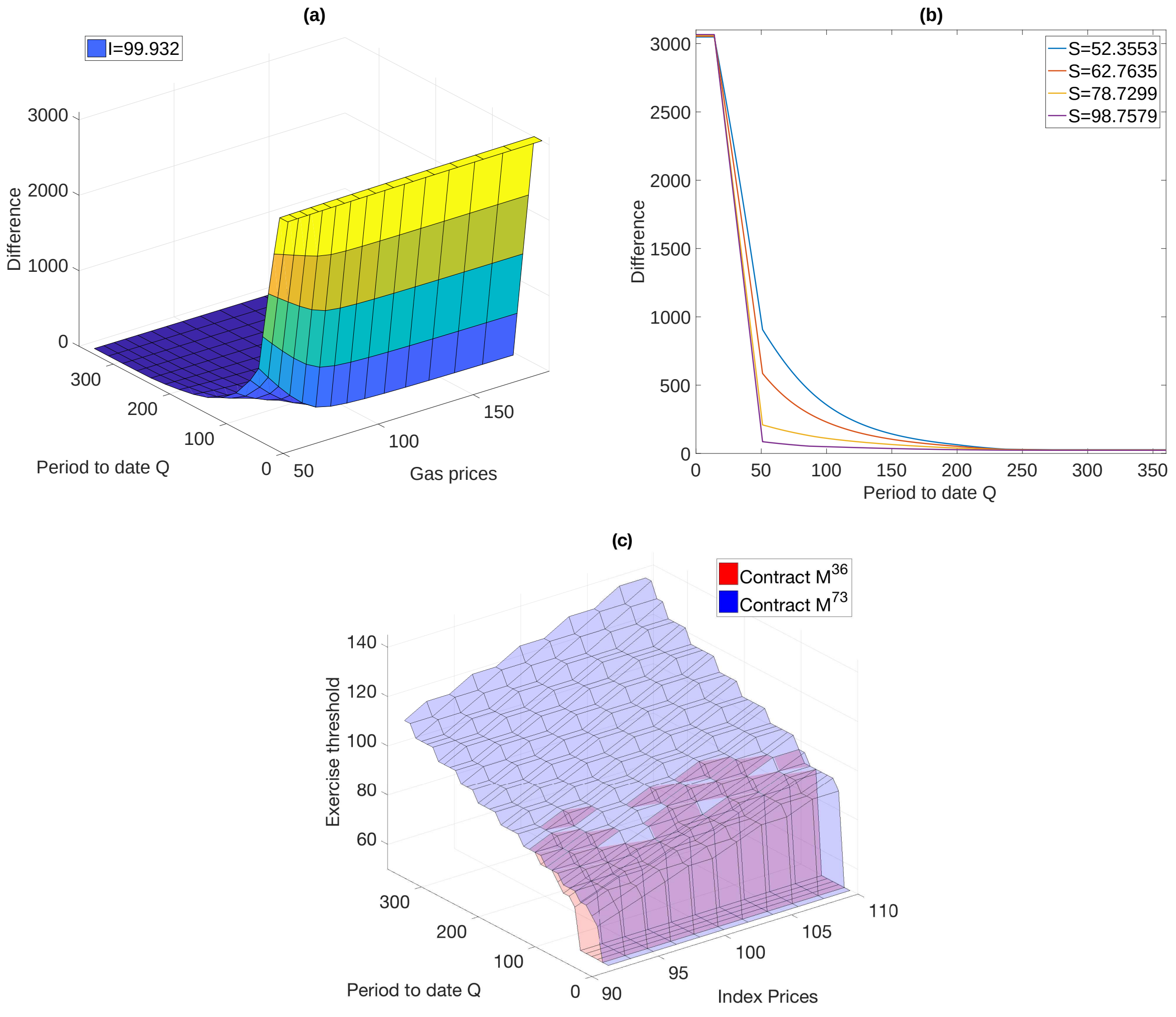

Figure 5: Plots at day 180. (a) is obtained by using the value of Contract $\mathbf{M}^{73}$ minus the value of Contract $\mathbf{M}^{36}$. By selecting some specified gas prices in (a), we get (b). (c) shows both the decision surfaces for Contracts $\mathbf{M}^{73}$ and $\mathbf{M}^{36}$.

and cannot be refunded in the future. At the same time, the shortfall is still being added to the make-up bank of Contract $\mathbf{M}^{73}$, and can be used in future years. The difference then becomes stable when the period to date is less than 14 ($Q_{t_{1,180}} < 51 - (73 - 36) = 14$). At this stage, the gas in the make-up bank of Contract $\mathbf{M}^{73}$ also reaches its recovery limit.

In addition, as we can see in (b) that when the period to date is larger than 51, the difference increases faster in a lower gas price regime than in a higher gas price regime, since the period to date decreases. This means that when the gas price is low, the buyer with more make-up rights has the opportunity to gain more profit by taking less gas in the current year and getting a refund in future years. As we can see in the decision surfaces (Figure 5(c)), however, the exercise threshold of Contract $\mathbf{M}^{73}$ is still low, and the buyer with more make-up rights only takes less gas when the period to date is very small. This means that, even with more make-up rights, paying a penalty at year end is still a risky action.

The effect of the carry-forward recovery limit. In order to minimize the effect of the make-up bank, we let $MRL_i = 0$,$i = 1, 2, 3$. That is, there is no make-up clause in the contract. Figure 6 shows the difference between Contract $\mathbf{C}^{54}$ with $CRL_i = 54$ and Contract $\mathbf{C}^{36}$ with $CRL_i = 36$, $i = 1, 2, 3$. In (a) and (b), we can see that the difference starts to increase when the period to date is larger than 106 ($Q_{t_{1,180}} > 106$) in the high gas price regime. This is because the buyer can meet the carry-forward base if he/she takes the maximum daily gas every day for the rest of the current year ($Q_{t_{1,180}} + 1 + (365 - 180) > 292$). At this stage, both holders of Contract $\mathbf{C}^{54}$ and $\mathbf{C}^{36}$ could have carry-forward gas at the year end, but the difference starts to increase. This indicates that the buyer with more carry-forward rights has the chance to buy more gas when the period to date is relatively low. Once the period to date is larger than 142 ($Q_{t_{1,180}} > 106 + 36 = 142$), the difference starts to increase rapidly in the high gas price regime. This is because, once $Q_{t_{1,180}} > 142$, the quantity of gas in the carry-forward bank of Contract $\mathbf{C}^{36}$ will reach its carry-forward recovery limit if the buyer takes the maximum daily gas every day for the rest of the current year. That is, the extra gas cannot be added to the carry-forward bank of Contract $\mathbf{C}^{36}$ and cannot be used to reduce the minimum bill in future years. At the same time, the extra gas is added to the carry-forward bank of Contract $\mathbf{C}^{54}$, and can be used in future years. The difference then becomes stable when the period to date is larger than 346. At this stage, for both contracts, the gas in the carry-forward bank has already reached the respective recovery limits, regardless of future purchases.

In addition, as we can see in (b), when the carry-forward bank comes into play, the difference is less sensitive with respect to the period to date in the low gas price regime than in the high gas price regime. This is because, with a lower gas price, the possibility of low gas prices in the future is high, and thus the buyer may not buy enough gas to meet the carry-forward base. As we can see in Figure 6(c), which shows the decision surfaces for these two contracts, the buyer with more carry-forward rights even takes more gas when the contract is out of the money (i.e. when the exercise threshold is smaller than its corresponding index price). This means that, the possible penalty has a big influence on the daily decisions. The buyer intends to do whatever they can to avoid possible penalties. Also, compared with Figure 5, it seems that the effect of the carry-forward recovery limit is much less significant

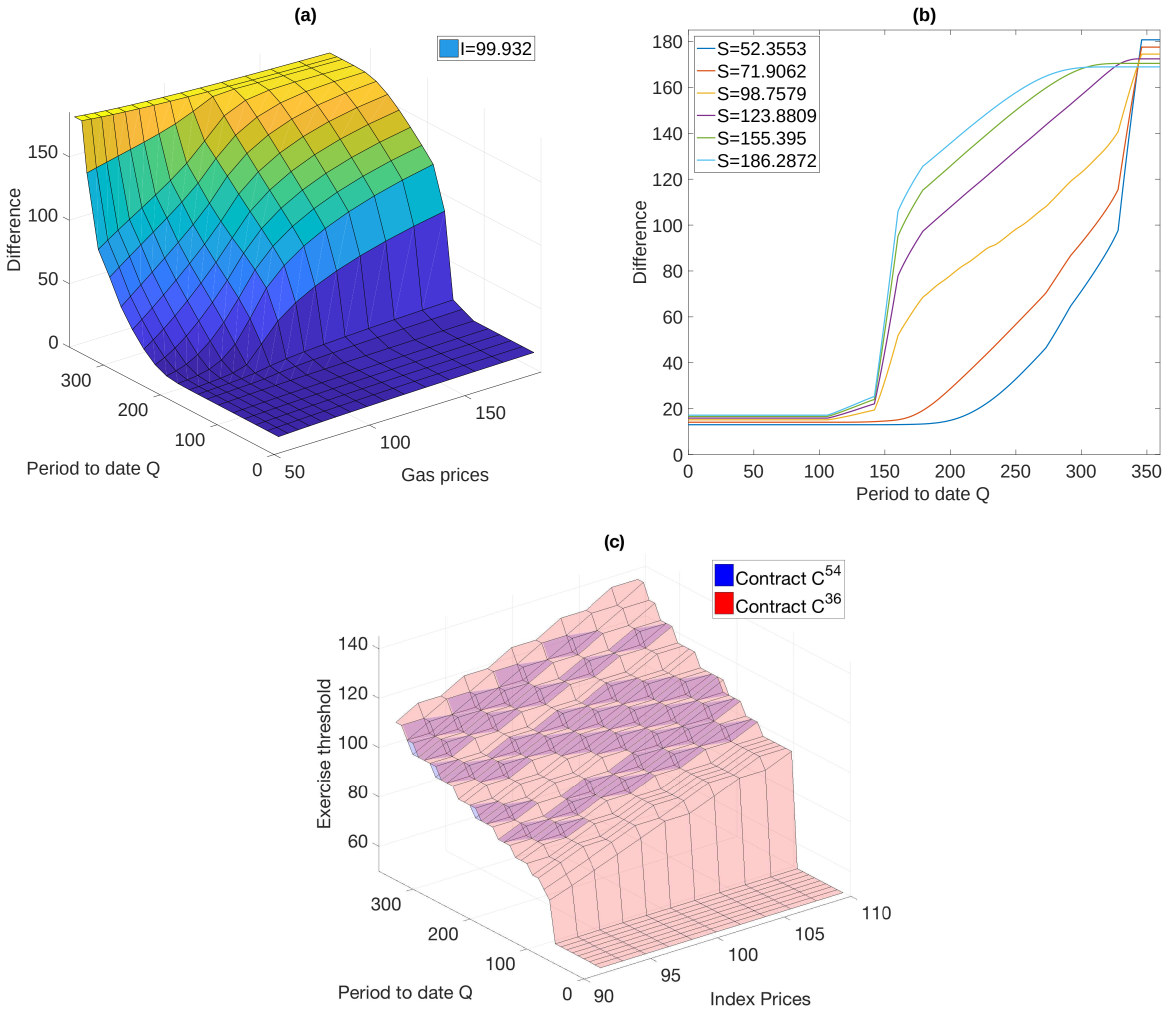

Figure 6: Plots are at day 180. (a) is obtained by using the value of Contract $\mathbf{C}^{54}$ minus the value of Contract $\mathbf{C}^{36}$. By selecting some specified gas prices in (a), we get (b). (c) shows the decision surfaces of both Contracts $\mathbf{C}^{54}$ and $\mathbf{C}^{36}$.

$\alpha_S = 2$	$\sigma_S = 0.5$	$\alpha_I = 3$	$\sigma_I = 0.4$	$r = 0.05$	$\rho = 0.5$
$q_{\min} = 0$	$q_{\max} = 1$	$MB = 39$	$CB = 42$	$ACQ = 52$	$\eta = 1$

Table 2: Parameter values.

than the effect of the make-up recovery limit. This is because the adding of make-up gas would mean an immediate penalty at the current year end, and in future years, the buyer will try to make full use of the make-up gas in order to get refunds. Adding gas to the carry-forward bank, however, usually comes without a loss in the current year, and the buyer may not make full use of the carry-forward gas in the future.

5.3. The effect of various parameters

In this subsection, we provide a detailed analysis of how various parameters affect the contract values.

Parameter settings. We choose a three-year GSA contract with weekly exercise opportunities. That is $L = 3$, $J = 52$. When the parameters are not variable, we use those values in Table 2. By the nature of make-up and carry-forward banks, for such a three-year example, the make-up bank will play a large role if the contract is out-of-the-money in the first year and at-the-money or in-the-money in the future years. Similarly, the carry-forward bank would contribute more if the contract is in-the-money in the first year and at-the-money or out-of-the-money in the future years. Thus one cannot have a fair analysis without taking the forward curves into consideration, since the forward curves can control the price movement in some sense. Although we can adjust either the gas forward curve or the index forward curve, since the index is obtained by the average of other energy substitutes, and is therefore more stable than the gas price, we choose the gas forward curve so as to model a changeable situation. We use a flat index forward curve, that is $F^I(0,t) = 100$, for $0 \le t \le L$. Let the gas forward curve be given as follows:

$$F^S(0,t) = \begin{cases} 100 + d, & 0 \le t \le 1, \text{ Year } 1, \\ 100, & 1 < t \le 2, \text{ Year } 2, \\ 100 - d, & 2 < t \le 3, \text{ Year } 3, \end{cases}$$

where $d \in (-100, 100)$ is a constant. We call d the decoupling level. When $d < 0$, then the contract is likely to be out-of-the money in the first year, at-the-money in the second year and in-the-money in the third year. In this situation, we say the gas forward curve is negatively decoupled. Curve I and Tree I in Figure 1 present an example of a negatively decoupled gas forward curve and its corresponding trinomial tree. When $d = 0$, then the contract is likely to be at-the-money in all three years. In this situation, we say that the gas forward curve is not decoupled. Curve II and Tree II in Figure 1 present an example of a gas forward curve that is not decoupled, along with its corresponding trinomial tree. When $d > 0$, then the contract is likely to be in-the-money in the first year, at-the-money in the second year and out-of-the-money in the third year. In this situation, we say that the gas

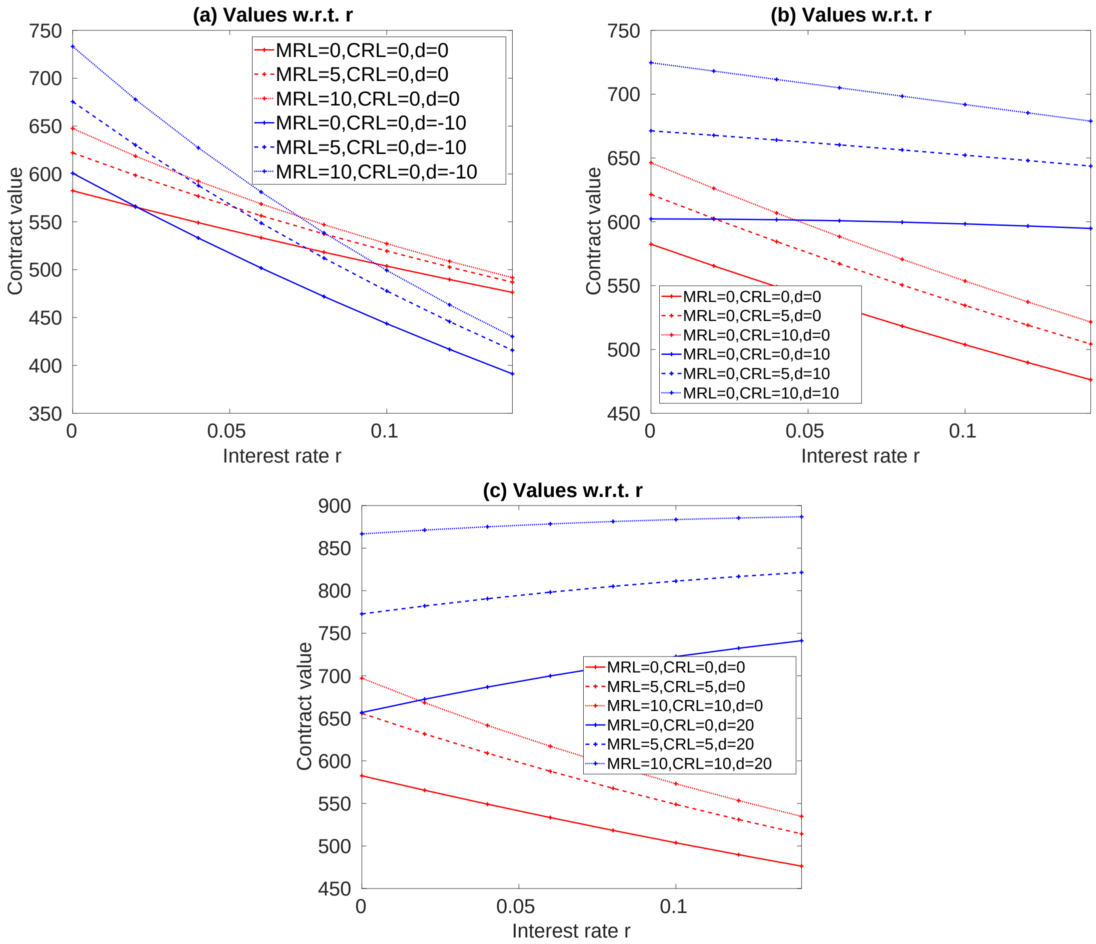

Figure 7: The contract value w.r.t. r

forward curve is positively decoupled. Curve III and Tree III in Figure 1 present an example of a positively decoupled gas forward curve, along with its corresponding trinomial tree.

The risk-free rate. By the nature of the make-up bank, the addition of make-up gas in some years comes with penalties at the year end. The make-up gas is refunded in future years, which means that there is a time gap between the penalty and the refund. Thus the benefit of the make-up bank is affected by the risk-free rate. Intuitively speaking, the higher the risk-free rate is, the less contribution the make-up bank makes. Recall that the make-up and carry-forward rights are measured by the make-up and carry-forward recovery limits. The larger these limits are, the more make-up and carry-forward rights the holder has. Furthermore, a contract that has larger make-up and carry-forward recovery limits should also have a larger contract value. Figure 7(a) evidences our intuitions. The benefit of the greater make-up rights peaks when the risk-free rate is 0, and reduces as the risk-free rate increases. For the carry-forward bank, although there is no penalty when the holder gains carry-forward gas, this gas carried forward could be used to reduce the minimum

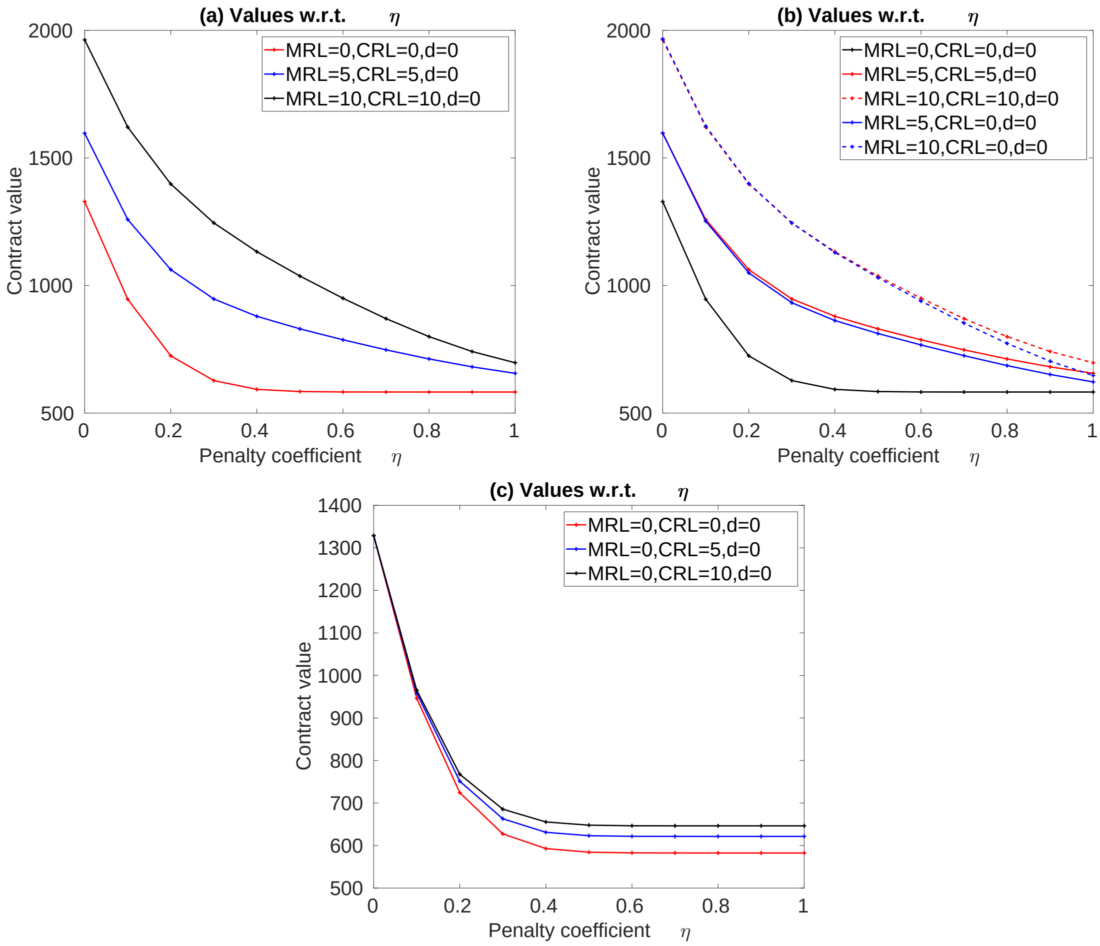

Figure 8: The contract value w.r.t. η

bill in the future years, and when the carry-forward gas is withdrawn, the possible loss is reduced. There is also a time gap between the addition and withdrawal of carry-forward gas. Thus, the benefit of the carry-forward bank should also be reduced by a higher risk-free rate. Figure 7(b) shows these intuitions. As we can see, the benefit of more carry-forward becomes less significant when the risk-free rate increases. Also, compared with the positively decoupled gas forward curve ($d = 10$), these phenomena are less significant when the forward curve is not decoupled ($d = 0$). This is because the holder may have fewer opportunities to withdraw carry-forward gas with $d = 0$. As we can see in Figure 7(c), the benefits of both make-up and carry-forward banks are reduced by a higher risk-free rate. In addition, when the forward curve is highly positively decoupled ($d = 20$), the contract value increases as the risk-free rate increases. This is due to the fact that the contract is highly in-the-money in the first year and highly out-of-the-money in the third year. A higher risk-free rate, therefore, amplifies the benefit of the in-the-money period.

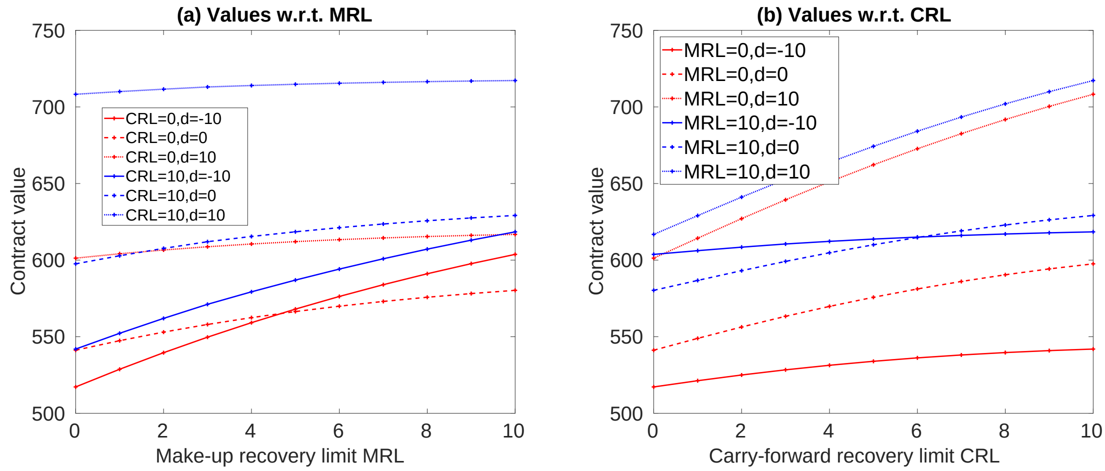

Figure 9: The contract value w.r.t. MRL and CRL.

The penalty coefficient. To minimize the time value of money, we let the risk-free rate be $r = 0$. The adding of make-up gas in some years comes with penalties at the year end, which means the penalty coefficient should have a significant effect on the make-up clause. A smaller penalty coefficient leads to smaller possible penalties, and thus to buyers potentially accruing more benefit from their make-up rights. Intuitively speaking, the greatest potential benefit arising from make-up rights is probably when η equals zero, since at that level buyers can store make-up gas without paying penalties. As we can see in plot Figure 8(a), however, the benefit arising from having more make-up rights is at its greatest when the penalty coefficient is around 0.3. Furthermore, as we can see in Figure 8(b) and (c), the penalty coefficient also affects the carry-forward bank. When $\eta = 0$, the contract values are the same regardless of how many carry-forward rights the holder has (Figure 8(c)). This is because the carry-forward bank loses its function when $\eta = 0$; there is no need to reduce the minimum bill by using the carry-forward gas if there is no penalty. The gap between these lines becomes stable when the penalty coefficient is larger. This is because, at this point, the penalty is large enough to maximize the carry-forward bank's functionality. It also suggests that, without a make-up bank, when η is large enough, the buyer will do whatever they can to avoid possible penalties.

The make-up and carry-forward recovery limits. Figure 9(a) shows that the contract value increases as the make-up recovery limit increases. It also evidences that the increasing speed is faster with a negatively decoupled forward curve than a positively decoupled curve, or a curve that is not decoupled. Figure 9(b) shows that the contract value increases as the carry-forward recovery limit increases. It also shows that the increasing speed is faster with a positively decoupled forward curve.

The decoupling level. As can be seen in the previous figures, with make-up and carry-forward clauses, the decoupling level has a huge impact on the contract. Figure 10(a) and (b) shows

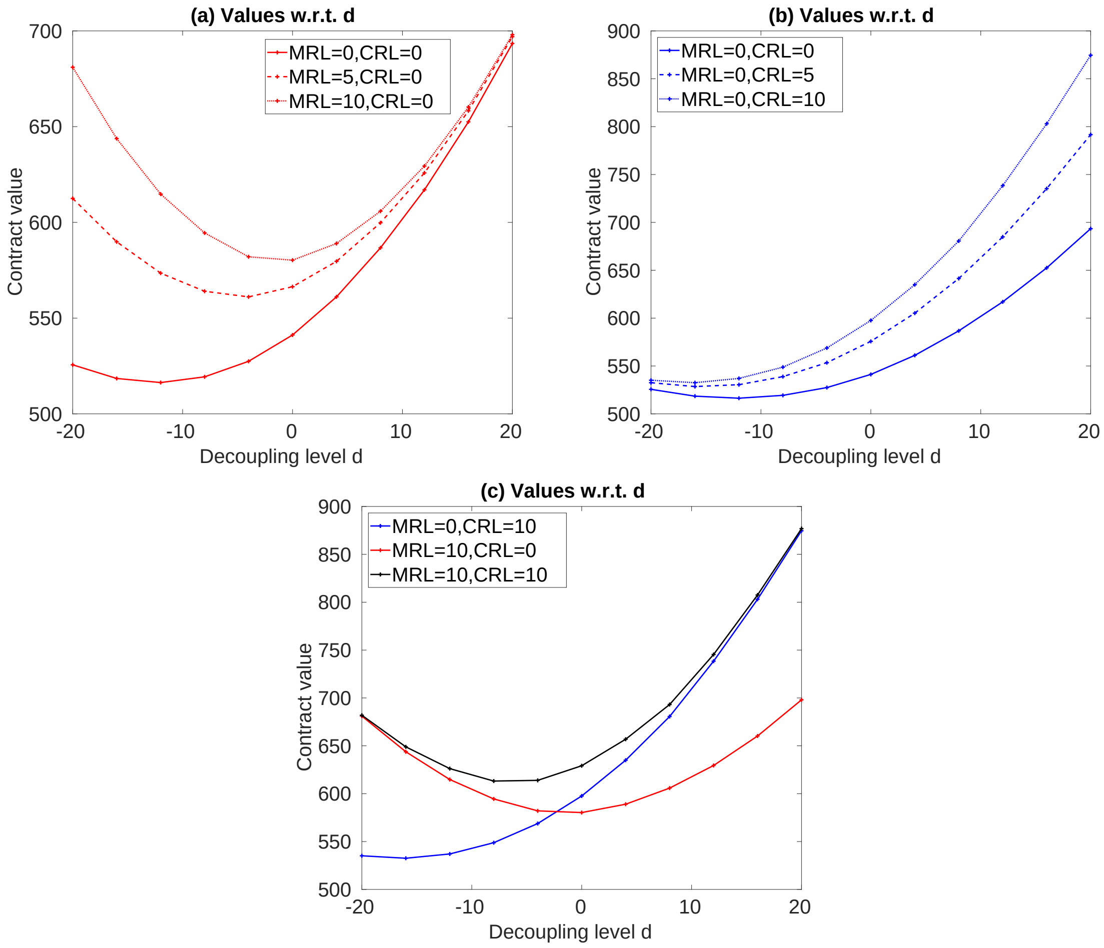

Figure 10: The contract value w.r.t. d

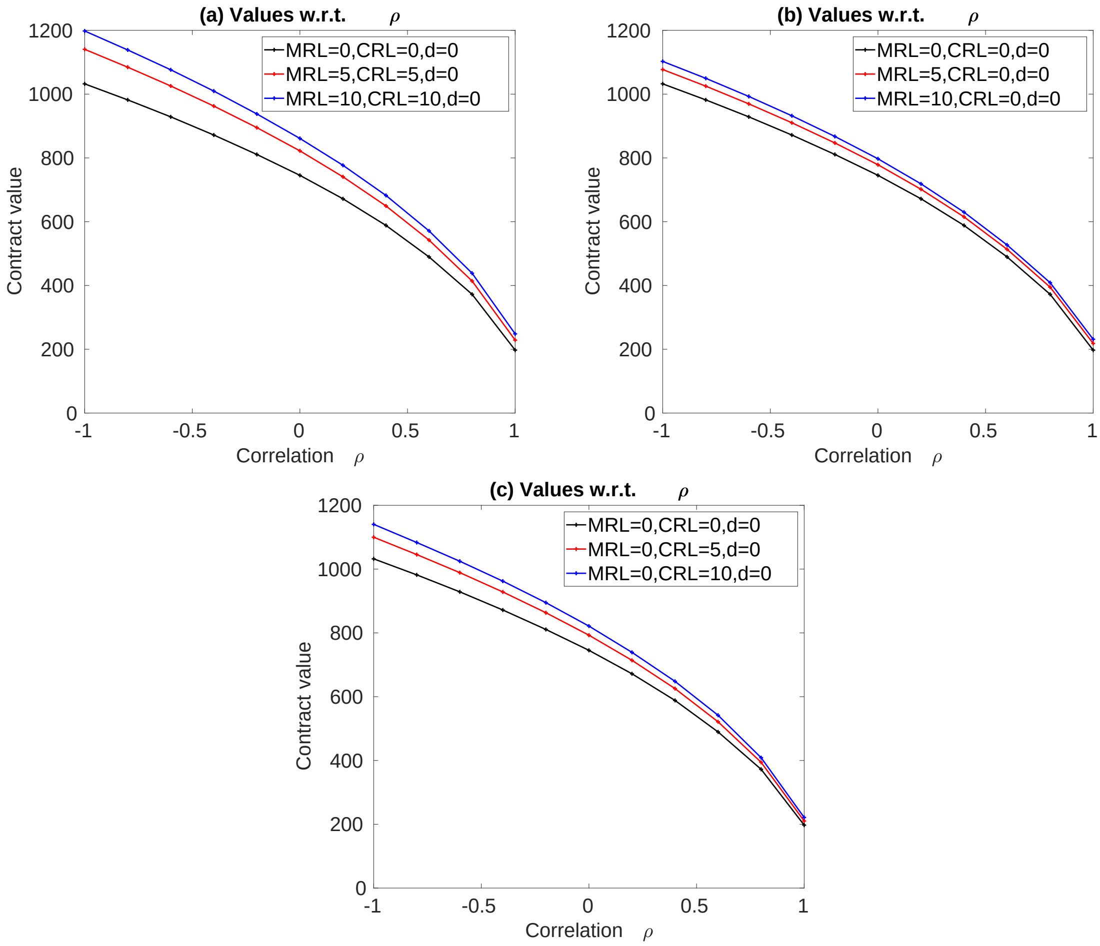

Figure 11: The contract value w.r.t. ρ.

that the benefit of more make-up rights increases as the decoupling level d decreases, while the benefit of more carry-forward increases as the decoupling level increases. The more the forward curve is negatively decoupled, the more make-up gas the holder can store in the first year and withdraw in the third year when the contract is highly in-the-money. When the forward curve is highly positively decoupled, meanwhile, the holder can buy more gas and gain more carry-forward gas, since when the contract is highly out-of-the-money in the third year, the holder will have more carry-forward gas to use to reduce the minimum bill. Figure 10(c) also evidences that, when the forward curves are not decoupled, the carry-forward bank contributes more to the contract value than the make-up bank.

The correlation. The effect of the correlation ρ is presented in Figure 11. It can be observed that the contract value decreases when the correlation increases. This means that the buyer can benefit more when the gas price and the index are negatively correlated. Figure 11(b) and (c) also show that a positive correlation weakens the function of both the make-up and carry-forward banks.

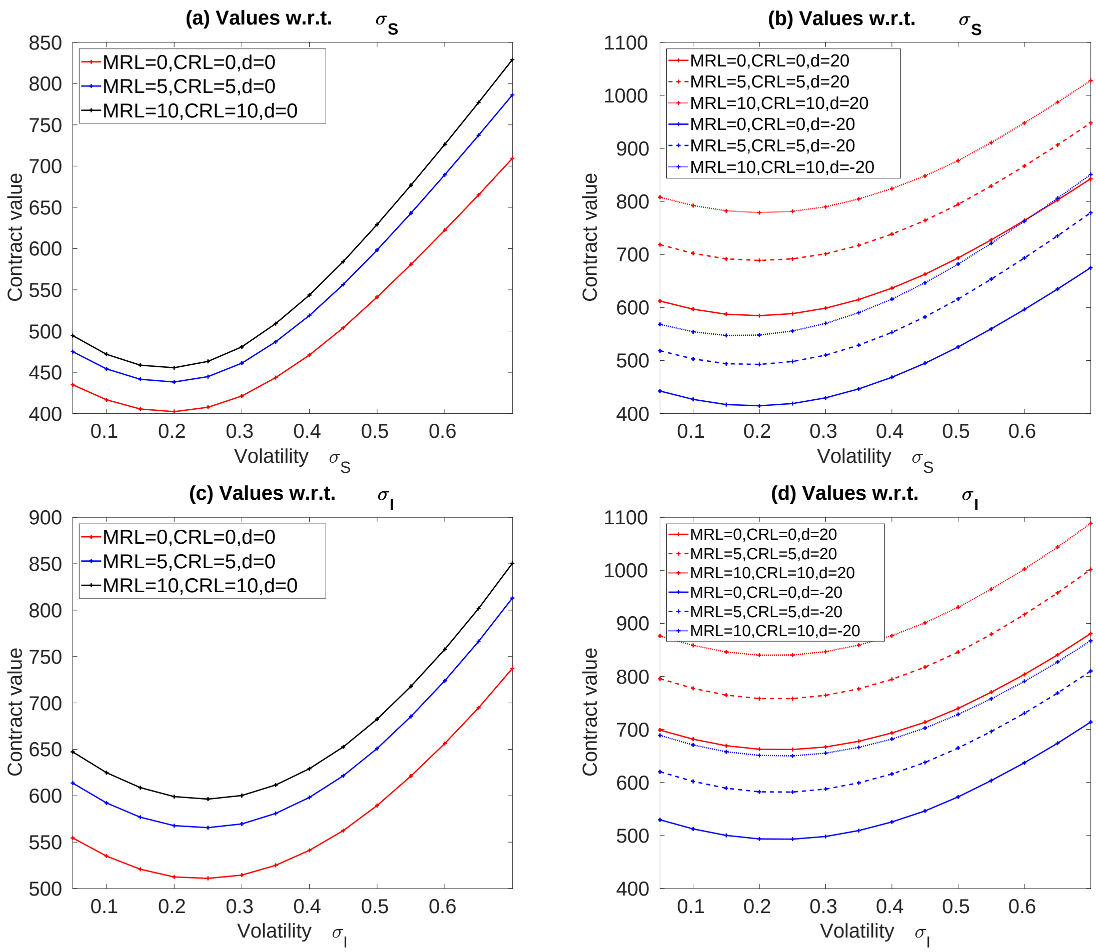

Figure 12: The contract value w.r.t. σ_S and σ_I

The volatility. In respect to the effect of for the volatility of both the gas price and the index, Figure 12 shows that the contract value decreases as the volatility increases in a lower volatility regime, but then increases in a higher volatility regime. In addition, the contract value peaks at the largest value of the volatility. This is because the buyer could implement a more flexible trading strategy when the gas price or the index price is fluctuating more. Figure 12(a) also shows that, when the forward curve is less decoupled (the absolute value of d is small) the holder with more make-up and carry-forward rights would benefit more from higher gas price volatility, while Figure 12(c) shows that one would not appreciate this feature when it comes to the volatility of the index. As we can see in Figure 12(b), when the forward curve is highly decoupled (the absolute value of d is large), the holder with more make-up and carry-forward rights would not gain any significant benefit from higher volatility. This is because the highly decoupled curve has already made sure the make-up or carry-forward bank works to its full capacity.

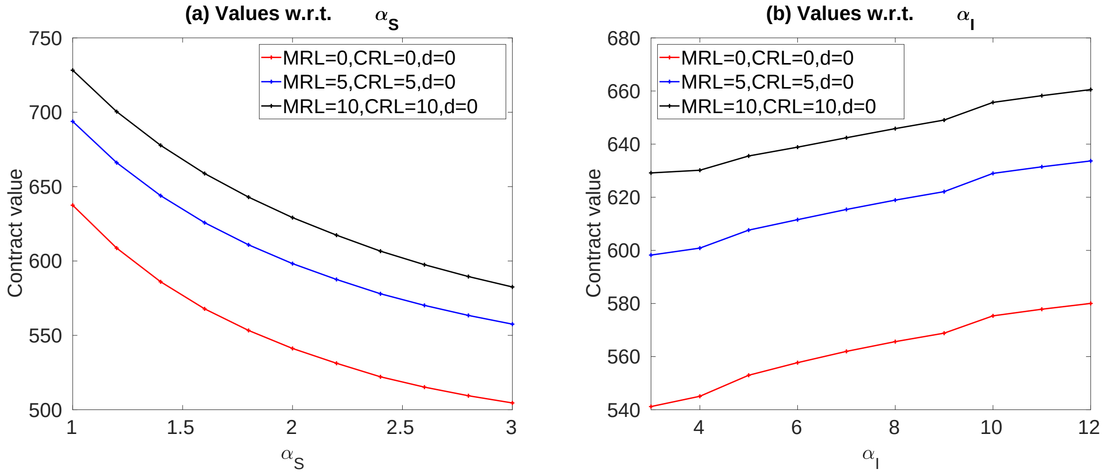

Figure 13: The contract value w.r.t. α_S and α_I.

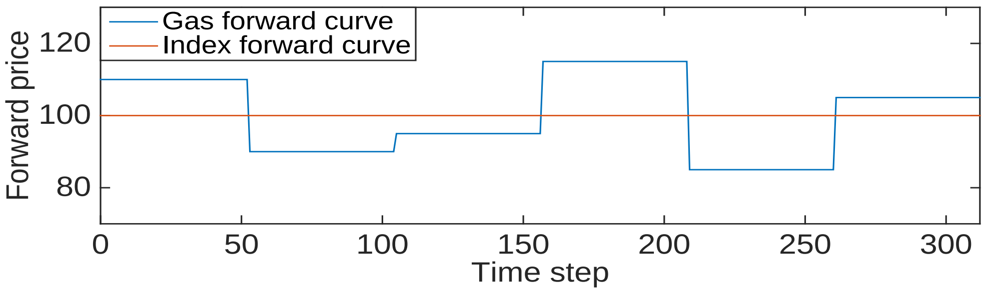

Figure 14: Forward curves

The mean-reverting speed. Figure 13 shows that the contract value is decreasing in α_S and increasing in α_I. This also shows that the speed of reversion to the mean for both gas price and index does not have a significant impact on the benefit of having more make-up and carry-forward rights.

5.4. How the indexation affects the decisions

In this section, we show how the indexation affects the decisions on weekly exercise, carry-forward bank and make-up bank, and also the influences on the volume taken. We use a six-year GSA contract with weekly exercise opportunities ($L = 6$, $J = 52$) as an example. The forward curves for both gas price and index are given in Figure 14. Using the parameters in Table 2, we let $MRL_i = 10$ and $CRL_i = 10$ for $i = 1, \ldots, 6$. We simulate a path of gas spot prices and two paths of index at the same time. Then we make decisions based on the optimal decision surface we calculated by using the dynamic programming technique in Section 4. Figures 15 and 16 demonstrate how decisions change when the realizations of the index are different.

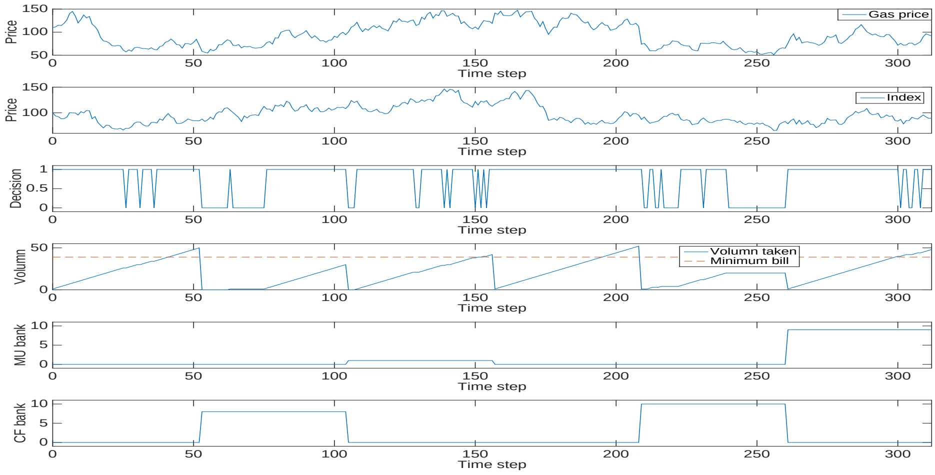

Figure 15: One realization of the index and the corresponding optimal takes, volume taken, and the evolution of both make-up and carry-forward banks.

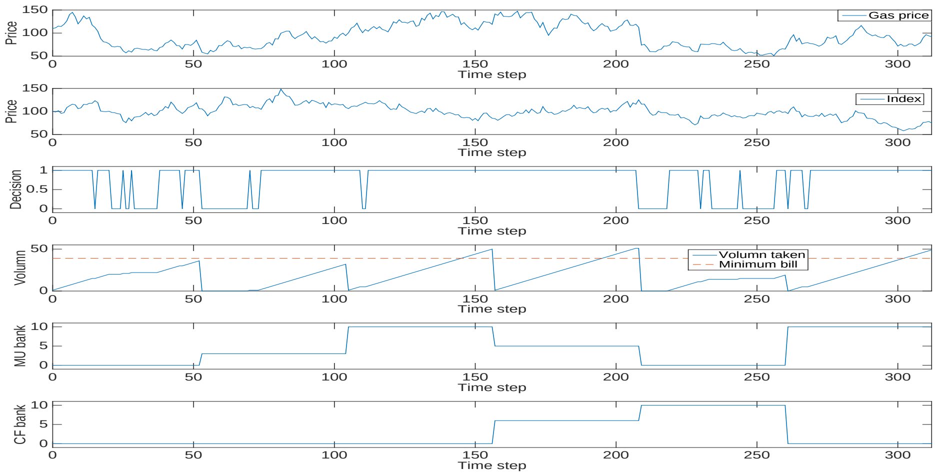

Figure 16: Another realization of the index and the corresponding optimal takes, volume taken, and the evolution of both make-up and carry-forward banks.

In the first year of the example the gas forward curve is above the index forward curve (i.e. in-the-money). Assuming that the spot price and the index both follow their forward curves (we call this the intrinsic strategy), the optimal strategy would then be to take the maximum possible ($Q_{T_1} = 52$) and create ten units of carry-forward ($C_2 = 52-42 = 10$). In Figure 15, however, the simulated take is 50 ($Q_{T_1} = 50$), which creates eight units of carry-forward gas ($C_2 = Q_{T_1} - 42 = 8$). While in Figure 16, the simulated take is 36 ($Q_{T_1} = 36$), which creates three units of make-up gas ($M_2 = 39 - Q_{T_1} = 3$).

In the second year, the gas forward curve is below the index curve (i.e. out-of-the-money) and so the optimal intrinsic strategy would be to use all the carry-forward gas ($c_2 = 10$) so as to reduce the minimum bill to 29 ($39 - c_2 = 29$) and then take 19 ($Q_{T_2} = 19$) to create a make-up bank of ten units ($M_3 = 39-c_2-Q_{T_2} = 10$, the maximum that can be recovered in the coming year). In Figure 15, the simulated strategy uses all the carry-forward gas ($c_2 = 8$) to reduce the minimum bill to 31 ($39-c_2 = 31$) and taking 30 ($Q_{T_2} = 30$), which also creates a make-up bank of one unit ($M_3 = 31 - Q_{T_2} = 1$). In the case of Figure 16, the simulated take is 32 ($Q_{T_2} = 32$), increasing the make-up bank to ten units ($M_3 = M_2+(39-32) = 10$).

The third year is also out-of-the-money, so the intrinsic strategy is to take the minimum bill plus the amount of gas in the make-up bank ($m_3 = 10$) that will be refunded, giving a take of 49 ($Q_{T_3} = MB + m_3 = 49$). In Figure 15, the simulated strategy is to take 42 ($Q_{T_3} = 42$) and use one unit of make-up gas ($m_3 = 1$), which makes the make-up bank next year to be 0 ($M_4 = M_3 - m_3 = 0$). The simulated take in Figure 16 is $Q_{T_3} = 50$ but we only use five units of gas ($m_3 = 5$) in the make-up bank, thus reducing the make-up bank to 5 ($M_4 = M_3 - m_3 = 5$). We also create six units of carry-forward gas ($C_4 = Q_{T_3} - \max\{39 + m_3, 42\} = 6$).

In the fourth year the contract is in-the-money, so the optimal intrinsic strategy is to take the maximum quantity of gas ($Q_{T_4} = 52$) and create another ten units of carry-forward ($C_5 = 10$). The simulated strategy in Figure 15 is to take 52 ($Q_{T_4} = 52$), which also creates ten units of carry-forward gas ($Q_{T_4}-42 = 10$). The simulated strategy in Figure 16 is also to take 51 ($Q_{T_4} = 51$) and use all the make-up gas ($m_4 = M_4 = 5$), and thus create a carry-forward bank of ten units for the next year ($C_5 = \max\{C_4+Q_{T_4}-\max\{39 + m_4, 42\}, CRL_5\} = 10$).

In the fifth year the contract is out-of-the-money, so the intrinsic strategy would be the same as that seen in year two. The simulated strategy in Figure 15 is to use all the carry-forward gas ($c_5 = 10$) to reduce the minimum bill to 39-10=29 and take 20 units of gas ($Q_{T_5} = 20$) which makes the make-up bank for the next year to be nine units ($M_6 = 29 - Q_{T_5} = 9$). The simulated strategy in Figure 16, meanwhile, is to use ten units of the carry-forward gas ($c_5 = 10$) to reduce the minimum bill to 39-10=29 and take 19 units of gas ($Q_{T_5} = 19$), which makes the make-up bank next year to be ten units ($M_6 = 29-Q_{T_5} = 10$).

In the final year the contract is in-the-money, so the intrinsic strategy would be to take the maximum possible, with ten units being taken from the make-up bank. The simulated strategy in Figure 15 is to take 52 ($Q_{T_6} = 52$) and use all the make-up gas, which leaves no gas left in either the make-up or carry-forward banks. The simulated strategy in Figure 16 is to take 50 ($Q_{T_6} = 50$), which also leaves no gas in either the make-up or carry-forward banks.

6. Conclusion

In this paper, we have proposed a two-dimensional trinomial tree framework for pricing multiple year GSAs with make-up, carry-forward and indexation, given knowledge of forward price dynamics of both gas and index. GSAs are complicated to evaluate both because the buyers can exercise their rights in a daily manner while making decisions on the make-up bank and carry-forward bank on a yearly basis, and because the strike prices are able to move stochastically. Hence, in the evaluation, we need to keep track of multiple variables on a daily basis over a number of years. These complexities require efficient numerical procedures to value these contracts, and herein lies the main contribution of this paper.

With the help of a two-dimensional trinomial tree, we are able efficiently to evaluate the prices of the contracts so as to find both the optimal daily decisions and the optimal yearly use of both the make-up bank and carry-forward bank. We also demonstrate various features of this complex contract with the help of a number of numerical studies. For example, a buyer with more make-up rights only intends to take less gas when the period to date is very small; whereas a buyer with more carry-forward rights intends to take more gas even when the contract is out-of-the-money. A high interest rate weakens the functionality of both the make-up and carry-forward banks, while a large penalty coefficient increases the value of the carry-forward bank and decreases the value of the make-up bank. The values of both make-up and carry-forward banks are most affected by the decoupling level; and when the forward curves are not decoupled, the carry-forward bank contributes more to the contract value than the make-up bank.

It should be noted that the definition and properties of the index have been simplified in this paper. The index in the current month in a real contract is determined by the weighted average of some other energy prices in the previous month, which links the valuation to the moving average problem. These non-Markovian and non-continuous properties make the evaluation of such a real contract much harder than the scenario envisaged in the current paper, but we leave this problem to future research.

Appendix A. The Markov Property

We proceed to get the spot and log prices in analogy with [10]. Given (1), we have the following solution:

$$F^S(t,T) = F^S(0,T)\exp\left[-\frac{1}{2}\sigma_S^2\int_0^t e^{-2\alpha_S(T-u)}du + \sigma_S\int_0^t e^{-\alpha_S(T-u)}dB^S(u)\right].$$

Then the spot price is obtained by setting $T = t$:

$$S(t) = F^S(0,t)\exp\left[-\frac{1}{2}\sigma_S^2\int_0^t e^{-2\alpha_S(t-u)}du + \sigma_S\int_0^t e^{-\alpha_S(t-u)}dB^S(u)\right]. \qquad \text{(A.1)}$$

It follows by differentiating $S(t)$:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F^S(0,t)}{\partial t} + \alpha_S\sigma_S^2\int_0^t e^{-2\alpha_S(t-u)}du - \alpha_S\sigma_S\int_0^t e^{\alpha_S(t-u)}dB^S(u)\right]dt + \sigma_S dB^S(t).$$

By (A.1), we have

$$\ln S(t) = \ln F^S(0,t) - \frac{1}{2}\sigma_S^2 \int_0^t e^{-2\alpha_S(t-u)}du + \sigma_S \int_0^t e^{\alpha_S(t-u)}dB^S(t),$$

which gives

$$\alpha_S\sigma_S \int_0^t e^{\alpha_S(t-u)}dB^S(u) = \alpha_S \left[\ln S(t) - \ln F(0,t) + \frac{1}{2}\sigma_S^2 \int_0^t e^{-2\alpha_S(t-u)}du\right].$$

By the fact of

$$\int_0^t e^{-2\alpha_S(t-u)}du = \frac{1-e^{-2\alpha_S t}}{2\alpha_S},$$

we have

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F^S(0,t)}{\partial t} + \alpha_S(\ln F(0,t) - \ln S(t)) + \frac{\sigma_S^2(1-e^{-2\alpha_S t})}{4}\right] dt + \sigma_S dB^S(t).$$

After applying Ito's formula with $X(t) = \ln S(t)$, we get

$$dX(t) = \phi^S(t, X(t))dt + \sigma_S dB^S(t), \tag{A.2}$$

where

$$\phi^S(t,x) = \frac{\partial \ln F^S(0,t)}{\partial t} + \alpha_S(\ln F^S(0,t) - x) + \frac{\sigma_S^2}{4}(1-e^{-2\alpha_S t}) - \frac{1}{2}\sigma_S^2.$$

Through similar calculations, with $Y(t) = \ln I(t)$, for (2) we have

$$dY(t) = \phi^I(t, Y(t))dt + \sigma_I dB^I(t),$$

where

$$\phi^I(t,y) = \frac{\partial \ln F^I(0,t)}{\partial t} + \alpha_I(\ln F^I(0,t) - y) + \frac{\sigma_I^2}{4}(1-e^{-2\alpha_I t}) - \frac{1}{2}\sigma_I^2.$$

To check the Markov property of the two-dimensional process (X, Y), we rewirte X and Y. The Brownian motions B^S and B^I can be expressed in terms of two independent Brownian motions B_1 and B_2, where $B^S = B_1$ and $B^I = \rho B_1 + \sqrt{1-\rho^2}B_2$. Then (X, Y) can be written as follows:

$$d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = A_{2\times 3}\begin{pmatrix} dt \\ dB_1 \\ dB_2 \end{pmatrix},$$

where

$$A_{2\times 3} = \begin{pmatrix} \phi^S(t, X(t)) & \sigma_S & 0 \\ \phi^I(t, Y(t)) & \sigma_I\rho & \sigma_I\sqrt{1-\rho^2} \end{pmatrix}.$$

Being each entry of matrix $A_{2\times 3}$ a Lipschitz function and $(t, B_1(t), B_2(t))$ a set of independent Levy processes, one can apply Theorem 32 of Chapter V in [23] to confirm the Markov property of (X, Y).

Appendix B. The Bang-bang Consumption

We proceed with the proof in analogy with [4]. Let $\mathbf{q}^* = \{q^*_{t_{i,k}}\}$ be an optimal strategy and $\tilde{\mathbf{q}} = \{\tilde{q}_{t_{i,k}}\}$ be any admissible strategy, $i = 1, 2, \ldots, L$ and $k = 1, 2, \ldots, J$. That is $\tilde{q}_{t_{i,k}} \in [q_{\min}, q_{\max}]$ for all i, k. For any $\epsilon \in [0, 1]$, we define another admissible strategy $\mathbf{q}^\epsilon = \{q^\epsilon_{t_{i,k}}\}$

$$q^\epsilon_{t_{i,k}} = \begin{cases} q^*_{t_{i,h}} + \epsilon(\tilde{q}_{t_{i,h}} - q^*_{t_{i,h}}) & k = h \\ q^*_{t_{i,k}} & k \neq h \end{cases},$$

where h can be any day in year i. The year i value function with respect to $\mathbf{q}^\epsilon$ is given by

$$V(\mathbf{q}^\epsilon) = \mathbb{E}\left[\sum_{i=1}^{L}\left(\sum_{k=1}^{J} e^{-rt_{i,k}} q^\epsilon_{t_{i,k}}(S_{i,k} - I_{i,k}) + e^{-rt_{i,J}} \cdot I_{i,J} \cdot \mathcal{P}_i(Q^\epsilon_{T_i}, M^\epsilon_i, C^\epsilon_i)\right)\right].$$

Since $\mathbf{q}^*$ is the optimal strategy, it follows that $V(\mathbf{q}^\epsilon) - V(\mathbf{q}^*) \leq 0$. Given $\epsilon \in [0, 1]$, we have

$$\frac{\partial V(\mathbf{q}^\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} \leq 0.$$

That is

$$\begin{aligned}
\frac{\partial V(\mathbf{q}^\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} &= \lim_{\epsilon \to 0} \frac{V(\mathbf{q}^\epsilon) - V(\mathbf{q}^*)}{\epsilon} \\
&= \mathbb{E}\left\{(\tilde{q}_{t_{i,h}} - q^*_{t_{i,h}})\left[e^{-rt_{i,h}}(S_{i,h} - I_{i,h}) + \sum_{d=i}^{L} e^{-rt_{d,J}} \cdot I_{d,J} \cdot \mathcal{P}'_d(Q^*_{T_d}, M^*_d, C^*_d)\right]\right\} \\
&= \mathbb{E}\Bigg\{(\tilde{q}_{t_{i,h}} - q^*_{t_{i,h}})\Bigg[e^{-rt_{i,h}}(S_{i,h} - I_{i,h}) \\
&\quad + \mathbb{E}\left(\sum_{d=i}^{L} e^{-rt_{d,J}} \cdot I_{d,J} \cdot \mathcal{P}'_d(Q^*_{T_d}, M^*_d, C^*_d)) \mid I_{i,h}, Q^*_{t_{i,h}}, M^*_i, C^*_i\right)\Bigg]\Bigg\} \\
&\leq 0.
\end{aligned}$$

The above inequality holds true for all admissible strategies $\tilde{\mathbf{q}}$. Now, for any fixed decision $q \in [q_{\min}, q_{\max}]$, let

$$\tilde{q}_{t_{i,k}} = q^*_{t_{i,k}} + (q - q^*_{t_{i,h}})\mathbf{1}_{\lambda_{i,h}>0}\mathbf{1}_{k=h},$$

where

$$\begin{aligned}
\lambda_{i,h} =& (q - q^*_{t_{i,h}})\Bigg[e^{-rt_{i,h}}(S_{i,h} - I_{i,h}) \\
&+ \mathbb{E}\left(\sum_{d=i}^{L} e^{-rt_{d,J}} \cdot I_{d,J} \cdot \mathcal{P}'_d(Q^*_{T_d}, M^*_d, C^*_d)) \mid I_{i,h}, Q^*_{t_{i,h}}, M^*_i, C^*_i\right)\Bigg]\Bigg\}.
\end{aligned}$$

It follows that

$$\mathbb{E}(\mathbf{1}_{\lambda_{i,h}>0}\lambda_{i,h}) \leq 0,$$

which gives

$$(q - q^*_{t_{i,h}})\bigg[e^{-rt_{i,h}}(S_{i,h} - I_{i,h}) \\ + \mathbb{E}\left(\sum_{d=i}^{L} e^{-rt_{d,J}} \cdot I_{d,J} \cdot \mathcal{P}'_d(Q^*_{T_d}, M^*_d, C^*_d)) \mid I_{i,h}, Q^*_{t_{i,h}}, M^*_i, C^*_i\right)\bigg]\bigg\} \le 0.$$

Since the above inequality holds for any $q \in [q_{\min}, q_{\max}]$, then we complete the proof.

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