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A Grassmann Matrix Approach for the Computation of Degenerate Solutions for Output Feedback Laws

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Abstract: The paper is concerned with the improvement of the overall sensitivity properties of a method to design feedback laws for multivariable linear systems which can be applied to the whole family of determinantal type frequency assignment problems, expressed by a unified description, the so-called Determinantal Assignment Problem (DAP). By using the exterior algebra/algebraic geometry framework, DAP is reduced to a linear problem (zero assignment of polynomial combinants) and a standard problem of multilinear algebra (decomposability of multivectors) which is characterized by the set of Quadratic Plücker Relations (QPR) that define the Grassmann variety of $\mathcal{P}$. This design method is based on the notion of degenerate compensator, which are the solutions that indicate the boundaries of the control design and they provide the means for linearising asymptotically the nonlinear nature of the problems and hence are used as the starting points to generate linearized feedback laws. A new algorithmic approach is introduced for the computation and the selection of degenerate solutions (decomposable vectors) which allows the computation of static and dynamic feedback laws with reduced sensitivity (and hence more robust solutions). This approach is based on alternative, linear algebra type criterion for decomposability of multivectors to that defined by the QPRs, in terms of the properties of structured matrices, referred to as Grassmann Matrices. The overall problem is transformed to a nonlinear maximization problem where the objective function is expressed via the Grassmann Matrices and the first order conditions for optimality are reduced to a nonlinear eigenvalue-eigenvector problem. Hence, an iterative method similar to the power method for finding the largest modulus eigenvalue and the corresponding eigenvector is proposed as a solution for the above problem.

Keywords: Linear multivariable systems; Output feedback control (linear case); Linear systems; Frequency assignment.

1. INTRODUCTION

The Determinantal Assignment Problem (DAP) approach unifies the study of frequency assignment problems of multivariable systems under constant, dynamic centralised, or decentralised control schemes (Karcanias and Giannakopoulos, 1984),(Giannakopoulos and Karcanias, 1985). The multilinear nature of DAP suggests that the natural framework for its study is that of exterior algebra (Marcus, 1973). By using exterior algebra-algebraic geometry tools, DAP may be reduced to a linear problem of zero assignment of polynomial combinants and a standard problem of multilinear algebra, the decomposability of multivectors (Marcus, 1973). The solution of the linear sub-problem, whenever it exists, defines a linear space in a projective space $\mathcal{P}$, whereas decomposability is characterised by the set of Quadratic Plücker Relations (QPR) that define the Grassmann variety of $\mathcal{P}$ (Hodge and Pedoe, 1952). Thus, solvability of DAP may be seen as a problem of finding real intersections between the linear variety and the Grassmann variety of $\mathcal{P}$. The use of the algebraic geometry methods began with (Brockett and Byrnes, 1981) which applied exterior algebra tools to the pole placement problem. They also introduced the concept of degenerate solutions, as the compensation solutions where the feedback configuration vanishes. Although such solutions are prohibited from the practical control viewpoint, they have the significant property that linearize asymptotically the multilinear nature of DAP and thus they become key instruments in the computation of solutions.

In the contrast, other design approaches such as LMI's and Lyapunov functions, for the development of output feedback stabilizing controllers were applied in (Cao et al., 1998), (Vesel, 2001); and the more recent advances in (Palacios-Quionero et al., 2014), (Bluhmthaler and Oberst, 2012).

A computational procedure based on the notion of degeneracy, known as Global Linearisation, has been introduced in (Leventides and Karcanias, 1995) and has been used recently in (Leventides et al., 2014a,b) to develop numerical methods to design feedback laws for DAP applications. This exterior algebra-algebraic geometry method, has provided new invariants (Plücker Matrices and the Grassmann Matrices) for the characterisation of rational vector spaces, solvability of control problems, ability to discuss both generic and non-generic cases and it is flexible as far as handling dynamic schemes, as well as structurally constrained compensation schemes.

This paper exploits the previous results on the parametrisation of the family of degenerate solutions (Karcanias et al., 2013, 2016) and introduces an alternative method for searching for degenerate solutions with improved sensitivity properties in the family of linear solutions $\mathcal{K}(\underline{\alpha})$ (i.e. inside the kernel of the Plücker matrix associated with the particular DAP). This new approach is algorithmic, thus can be programmed using numerical algebra tools, and is also much faster than the previous methods presented in (Leventides and Karcanias, 1995),(Karcanias et al., 2013).

Grassmann matrices provide a new explicit matrix representation of abstract results on skew-symmetric tensors relating to decomposability of multivectors (Karcanias and Leventides, 2015). The decomposability of the multivector $\underline{z} \in \wedge^m \mathcal{U}$, where $\mathcal{U}$ is a vector space, is equivalent to the solvability of the exterior equation

$$\underline{v}_1 \wedge \underline{v}_2 \wedge \ldots \wedge \underline{v}_m = \underline{z} \qquad (1)$$

with $\underline{v}_i \in \mathcal{U}$. The conditions for decomposability are given by the set of QPRs (Hodge and Pedoe, 1952). For every multivector $\underline{z} \in \wedge^m \mathcal{U}$ with Plücker Coordinates (PC) $\{a_\omega,\ \omega \in Q_{m,n}\}$ is associated a structured Grassmann Matrix (GM) denoted by $\Phi_n^m(\underline{z})$. It has been shown (Karcanias and Leventides, 2015) that $rank\{\Phi_n^m(\underline{z})\} \geqslant n-m$ for all $\underline{z} \neq 0$, and $\underline{z}$ is decomposable if and only if

$$rank\{\Phi_n^m(\underline{z})\} = n-m.$$

Then, the solution space is defined by $\mathcal{V}_{\underline{z}} = \mathcal{N}_r\{\Phi_n^m(\underline{z})\}$ (Giannakopoulos et al., 1985). As we will see later the rank based test for decomposability is much easier to handle than the QPRs.

The paper describes the DAP framework, the concept of degeneracy and presents the background definitions on decomposability of multivectors. The properties of the structured Grassmann Matrix relating to decomposability are given as well. The algorithmic procedure is demonstrated by a numerical example on the well-known output feedback pole assignment problem.

2. THE GENERAL DETERMINANTAL ASSIGNMENT PROBLEM AND DEGENERACY

The study of Determinantal Assignment Problems has emerged as the abstract unifying description of all the linear (pole, zero) frequency assignment problems of control theory (Karcanias and Giannakopoulos, 1984), (Giannakopoulos and Karcanias, 1985) and the natural framework for its study is that of exterior algebra (Marcus, 1973) and algebraic geometry.

Let $M(s) \in \mathbb{R}^{l\times m}[s]$, $m < l$, $\mathrm{rank}_{\mathbb{R}[s]}(M(s)) = m$ and consider the set of matrices $\mathcal{H} = \{H(s) \in \mathbb{R}^{m\times l}[s], \mathrm{rank}(H(s)) = m\}$ and its subset $\mathcal{H}_{\mathbb{R}}$ which contain all the constant matrices $H \in \mathbb{R}^{m\times l}$. The general DAP is associated with the solution of the determinantal equation

$$f(s,H) = \det\{H(s)\cdot M(s)\} = \alpha(s) \qquad (2)$$

where, we seek to find a matrix $H(s) \in \mathcal{H}$ such that the polynomial $\alpha(s)$ has assigned zeros.

Remark 1. If we seek to find a static matrix $H \in \mathcal{H}_{\mathbb{R}}$, then the corresponding problem is defined as the constant DAP ($\mathbb{R}-$DAP) (Karcanias and Giannakopoulos, 1984). It has been shown that each dynamic DAP may be transformed to an equivalent static problem of augmented dimensions, hence, we focus only to the static case in this study.

□

Let $\underline{h}_i^t$, $\underline{m}_i(s)$, $i \in 1,2,\ldots,m$ be the rows of $H \in \mathcal{H}$ and columns of $M(s)$ respectively. Then, we define

$$C_m(H) = \underline{h}_1{}^t \wedge \underline{h}_2{}^t \wedge \ldots \wedge \underline{h}_m{}^t = \underline{h}^t \wedge \in \mathbb{R}^{m\times q}[s], q = \binom{l}{m}$$

$$C_m(M(s)) = \underline{m}_1(s) \wedge \underline{m}_2(s) \wedge \ldots \wedge \underline{m}_m(s) = \underline{m}(s)\wedge \in \mathbb{R}^q[s]$$

and by the Binet-Cauchy theorem (Marcus and Minc, 1964) we have

$$f(s,H) = C_m(H)\cdot C_m(M(s)) = \\ = \langle \underline{h}\wedge, \underline{m}(s)\wedge\rangle = \sum_{\omega\in Q_{m,n}} h_\omega m_\omega(s)$$

where, $\langle\cdot\,,\cdot\rangle$ denotes the scalar product, $\omega = (\underline{i}_1,\ldots,\underline{i}_l) \in Q_{l,m}$ and $\underline{h}_\omega, \underline{m}_\omega(s)$ are the entries in $\underline{h}\wedge, \underline{m}(s)\wedge$, respectively. Note, that $\underline{h}_\omega$ is the $m\times m$ minor of H, which corresponds to the ω set of rows of H and thus is a multilinear alternating function of the $h_{i,j}$ entries of H. DAP may be reduced to a linear and a multilinear subproblem (Karcanias and Giannakopoulos, 1984) as shown below:

Linear sub-problem: Let define $\underline{m}(s)\wedge = \underline{p}(s) \in \mathbb{R}^q[s]$. Investigate the existence of $\underline{k}(s) \in \mathbb{R}^q[s]$, such that for some given $\alpha(s) \in \mathbb{R}[s]$, $d = \deg\alpha(s)$, we have

$$f_{\underline{p}}(s,\underline{k}) = \underline{k}(s)^t\cdot\underline{p}(s) = \sum_{i=1}^{q} k_i(s)\cdot p_i(s) = \alpha(s) \qquad (3)$$
$$= \underline{\alpha}^t\cdot\underline{e}_d(s)$$

where, $\underline{e}_d(s) = [1, s, \ldots, s^d]^t$ i.e. the standard basis vector.

Multilinear sub-problem: Assume that for the given $\alpha(s)$ the linear sub-problem is solvable and let denote the family of solutions by $\mathcal{K}(\underline{\alpha})$. Determine whether there exists $H \in \mathcal{H}$, $H^t = [\underline{h}_1,\ldots,\underline{h}_m]$ such that

$$\underline{h}_1 \wedge \ldots \wedge \underline{h}_m = \underline{k},\ \ \underline{k} \in \mathcal{K}(\alpha) \qquad (4)$$

□

Clearly, $M(s)$ is the matrix defined by the system and the particular control problem and its invariant structure is essential to the solution of the corresponding DAP.

2.1 The Frequency Assignment Map and its Differential

The Frequency Assignment Map associated with the (constant) problem is the map assigning H to the coefficient vector $\underline{\alpha}$ of $\alpha(s)$ i.e.,

$$\mathrm{F}:\mathbb{R}^{m\times l}\to\mathbb{R}^{d+1} : \mathrm{F}(H) = \underline{\alpha}$$

and the solution of DAP requires to find H such that $\mathrm{F}(H) = \underline{\alpha}$ for a given $\underline{\alpha}$. It is well known, that a system has the arbitrary zero assignment property if and only if the map F is onto (Wang, 1992).

The differential of the map F, denoted as $DF = D(\mathrm{F}(H))$, is crucial for the solvability of the design problem and its properties are related with the sensitivity of the generated solutions.

An important family of compensators for the solvability of the design problem are the so-called *degenerate* solutions which are the progenitors of the feedback design laws.

2.2 Degenerate Solutions

Degenerate solutions can be viewed as the points of the Grassmannian where the Frequency Assignment Map cannot be continuously extended. In fact, degenerate points possess a very important property, that is, they scatter the sequences of gains approaching them; this implies that we may have two sequences of gains converging to a degenerate point and yet the corresponding sequences of polynomials converge into two different limits.

Definition 2. For a composite system matrix $M(s)$ a generalized gain, $H_d = rowspan\,[A, K] \in \mathcal{H}_{\mathbb{R}}$, is called *degenerate*, if and only if, satisfies $\mathrm{F}(H_d) \equiv 0$, or equivalently, the following (multilinear) equation

$$\det\{[A,\ K]\cdot M(s)\} \equiv 0$$

□

Conditions of their existence and the standard theoretical procedure for constructing degenerate solutions is described in (Leventides and Karcanias, 1995), (Karcanias et al., 2013); whereas, the parametrization of the families of static and dynamic degenerate compensators was examined in (Karcanias et al., 2016).

Here, we propose an improved systematic algorithmic approach that is much faster and allows the searching for degenerate solutions in the family of linear solutions $\mathcal{K}(\underline{\alpha})$, i.e. inside the kernel of the Plücker matrix associated with the particular DAP. The following result shows the importance of degenerate compensators:

Lemma 3. If there exists a degenerate matrix $H_d \in \mathcal{H}_{\mathbb{R}}$ such that the differential $D(\mathrm{F})_{H_d}$ is onto, then any polynomial of $d-$th degree can be assigned via some static compensator.

3. DECOMPOSABILITY OF MULTIVECTORS AND THE GRASSMANN MATRIX

Let $\mathcal{U}$ be a vector space over a field $\mathcal{F}$ and let us denote the Grassmannian by $\mathcal{G}(m,\mathcal{U})$, i.e. the set of all $m-$dimensional subspaces of $\mathcal{U}$. For every $\mathcal{V} \in \mathcal{G}(m,\mathcal{U})$, the injection map

$$\wedge^p f : \wedge^m\mathcal{V} \to \wedge^p\mathcal{U} \tag{5}$$

is well defined and if $p = m$, then $\wedge^m\mathcal{V}$ is a 1-dimensional subspace of $\wedge^m\mathcal{U}$, if $\{\underline{v}_i, i \in \tilde{m}\}$ is a basis of $\mathcal{V}$ then $\wedge^m\mathcal{V}$ is spanned by $\underline{v}_1 \wedge \ldots \wedge \underline{v}_m$.

A vector $\underline{z} \in \wedge^m\mathcal{U}$ is called *decomposable* if there exists $\underline{v}_i \in \mathcal{V},\ i \in \tilde{m}$, such that

$$\underline{v}_1 \wedge \underline{v}_2 \wedge \ldots \wedge \underline{v}_m = \underline{z} \tag{6}$$

The vector space $\mathcal{V}_{\underline{z}} = \mathrm{span}_{\mathbf{F}}\{\underline{v}_i, i \in \tilde{m}\}$ is called the *generating space* of $\underline{z}$. It is well known that if $\underline{z}$ is nonzero and decomposable is called a *Grassmann Representative* (GR) of $\mathcal{V}_{\underline{z}}$ and is equivalent to $\mathcal{V}_{\underline{z}} \equiv \mathcal{V}_{\underline{z}} \in \mathcal{G}(m,\mathcal{U})$.

The coordinates $\{\underline{\alpha}_\omega; \omega \in Q_{m,n}\}$ of a decomposable vector $\underline{z} \in \wedge^m\mathcal{U}$ are known as the PC of $\mathcal{V}_{\underline{z}}$.

3.1 The Grassmann Matrix

The Grassmann Matrix (GM) of $\underline{z} \in \wedge^m\mathcal{U}$ has been introduced in (Karcanias and Giannakopoulos, 1988) and it will be used here as an alternative test for decomposability of $\underline{z}$. We state the following definition and result adopted from (Karcanias and Giannakopoulos, 1988),(Karcanias and Leventides, 2015):

Definition 4. Let $\{\underline{\alpha}_\omega; \omega \in Q_{m,n}\}$ be the coordinates of $\underline{z} \in \wedge^m\mathcal{U}$ with respect to a basis B_U^m of $\wedge^m\mathcal{U}$, $m+1 \leqslant n$, $\gamma = (\underline{j}_1, \ldots, \underline{j}_k, \underline{j}_{m+1})\omega \in Q_{m+1,n}$. We define the function:

$$\phi : \{i : i = 1, \ldots, n\} \times \{\gamma : \gamma \in Q_{m+1,n}\} \to \mathcal{F}$$

with $\rho_\gamma[\underline{\widehat{j_k}}] = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{m+1}) \in Q^\gamma_{m,m+1}$ by

$$\begin{cases} \varphi^i_\gamma = \phi_\gamma(i) = 0, \text{ if } i \neq \gamma \\ \phi^i_\gamma = \phi_\gamma(i) = sign(j_k : \rho_\gamma[\underline{\widehat{j_k}}])a_{\rho_\gamma[\underline{\widehat{j_k}}]}, \text{if } i = j_k \in \gamma \end{cases}$$

where $sign(j_k : \rho_\gamma[\underline{\widehat{j_k}}]) = (j_k, j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{m+1})$.

□

Proposition 5. Let $B_U = \{\underline{v}_i, i \in \tilde{n}\}$, $B_U^m = \{\underline{v}_{\omega i}, \omega \in Q_{m,n}\}$ be bases of $\mathcal{U}$, $\wedge^m\mathcal{U}$ respectively, where $\underline{v} = \sum_{i=1}^n c_i\underline{u}_i \in U$, $\underline{v} \neq 0$, and $\underline{z} = \sum_{\omega\in Q_{m.,n}} a_\omega \underline{u}_\omega \wedge \in \wedge^m\mathcal{U}$, $\underline{z} \neq \underline{0} \cdot \underline{v} \wedge \underline{z} = \underline{0}$, if and only if

$$\sum_{i=1}^n \underline{\phi}^i_\gamma c_i = 0, \text{ for all } \gamma \in Q_{m+1,n} \tag{7}$$

If we denote by γ_i the elements of $Q_{m+1,n}$ (lexicographically ordered), $t = 1, 2, \ldots, \binom{n}{m+1} = \tau$, then (7) may be written in a matrix form as

$$\underbrace{\begin{bmatrix} \phi^1_{\gamma_1} & \phi^2_{\gamma_1} & \cdots & \phi^i_{\gamma_1} & \cdots & \phi^n_{\gamma_1} \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi^1_{\gamma_t} & \phi^2_{\gamma_t} & \cdots & \phi^i_{\gamma_t} & \cdots & \phi^n_{\gamma_t} \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi^1_{\gamma_\tau} & \phi^2_{\gamma_\tau} & \cdots & \phi^i_{\gamma_\tau} & \cdots & \phi^n_{\gamma_\tau} \end{bmatrix}}_{\Phi^m_n(\underline{z})} \cdot \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{bmatrix}}_{\underline{c}} = \underline{0} \tag{8}$$

The matrix $\Phi^m_n(\underline{z})$ is a structured matrix (has zeros in fixed positions), it is defined by the pair m, n and the PC of $\underline{z} \in \wedge^m\mathcal{U}$ and it was originally introduced in (Karcanias and Giannakopoulos, 1988).

3.2 The Grassmann Matrix Construction Procedure

Given $n, m,\ \tau = \binom{n}{m+1}$, we form a $\tau \times n$ matrix, where the rows are indexed by the sequences $\gamma \in Q_{m+1,n}$ and the columns by $i = 1, 2, \ldots, n$. The elements of the $\gamma-$indexed row are defined for every $i \in \tilde{n}$ as follows:

1) If $i \notin (j_1, \ldots, j_{m+1})$, then $\phi^i_\gamma = 0$
2) If $i = j_k \in (j_1, \ldots, j_{m+1})$, then we define as $\omega = (j_k, j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{m+1}) \in Q_{m,n}$ and $\phi^i_\gamma = sign(j_k : \omega)\alpha_\omega$

3) The procedure is repeated for all $i = 1, 2, \ldots, n$ and for all $\gamma \in Q_{m+1,n}$ indexed rows.

A procedure for deriving the QPRs using the Grassmann Matrices has been given in (Karcanias and Leventides, 2015). The decomposability property of the GM is related to its singular values as demonstrated bellow:

Corollary 6. The vector $\underline{z} \in \wedge^m \mathbb{R}^n$ is decomposable, if and only if, the matrix $\Phi_n^m(\underline{z})$ has m singular values equal to 0 and $n-m$ singular values equal to $\|\underline{z}\|$.

4. A SYSTEMATIC ALGORITHMIC METHOD FOR THE COMPUTATION OF DEGENERATE SOLUTIONS

This section formulates the output feedback pole assignment problem in the DAP setup and gives a clear connection on how the proposed algorithmic approach based on the Grassmann Matrix described previously, can be applied to the output feedback design problem.

4.1 Output Feedback Pole Assignment Problem

We consider linear (proper) multivariable systems, $\dot{x} = Ax + Bu$, $y = Cx$, that is applied an output feedback law $u = Ky$. The closed-loop characteristic polynomial can be expressed as

$$\begin{aligned}\det(sI - A - BKC) &= \det(D(s) - KN(s)) \\ &= \det([I, K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}) = a(s)\end{aligned} \tag{9}$$

where, $G(s) = N(s)D(s)^{-1}$, $G(s) \in \mathbb{R}^{m\times p}$, is the right coprime (polynomial) Matrix Fraction Description (MFD) of the $p-$input, $m-$output open loop transfer function $G(s) = C(sI - A)^{-1}B$ and $a(s)$ the $d-$th degree target closed-loop pole polynomial. Hence, for the output feedback design problem we seek to construct a suitable static compensator $K \in \mathbb{R}^{p\times m}$ such that (9) is fulfilled and the characteristic polynomial of the multilinear equation $\det(D(s) - KN(s))$ would coincide with the desired pole polynomial.

4.2 Algorithmic Procedure for Constructing Degenerate Solutions

An iterative method resembling the power method (Kolda and Mayo, 2011) is applied here to the problem of finding degenerate solutions that allows to design feedback laws for determinantal frequency assignment problems and fulfill desired sensitivity properties. Recall that the general DAP can be reduced to the following two sub-problems:

Linear problem: In the Plücker space (3) can be expressed as:

$$\underline{k}^t P = \underline{a}^t$$

where $\underline{k}^t$ is an unknown $q-$vector, $q = \binom{l}{m}$, $P \in \mathbb{R}^{q\times(d+1)}$ is the Plücker matrix of the problem and $\underline{a}$ is the $(d+1)-$coefficient vector of the target polynomial $a(s)$ (i.e. the set of desired frequencies to be assigned).

Multilinear problem: It is given by (4), which expresses the fact that the vector $\underline{k}^t$ is decomposable.

For finding degenerate solutions we require that

$$\underline{k}^t P = \underline{0}. \tag{10}$$

If V is an orthonormal basis matrix for the left kernel of P, then $\underline{k}^t = \underline{x}^t V$, $V \in \mathbb{R}^{n_\ell \times q}$ (n_ℓ is the dimension of the left kernel of P). Thus, for $\underline{k}^t$ to be a degenerate compensator (decomposable vector) we require one of the following conditions to hold (Karcanias and Leventides, 2015):

(a) The QPRs are exactly zero, i.e.
$$\Phi_l^m(\underline{k}) \cdot \Phi_{l-m}^m(\underline{k}^*)^T = \underline{0}$$
(b) The square norms of the QPRs is minimum, i.e.
$$\min \left\| \Phi_l^m(\underline{k}) \cdot \Phi_{l-m}^m(\underline{k}^*)^T \right\|$$

Thus, we have to solve the following optimization problem:

$$\min \left\| \Phi_l^m(\underline{k}) \cdot \Phi_{l-m}^m(\underline{k}^*)^T \right\| \text{ s.t. } \underline{k}^t = \underline{x}^t V \text{ and } \|\underline{x}\| = 1$$

which can be reduced to a maximization problem:

$$\max \ \mathrm{tr}\left(\Phi_l^m(\underline{x}^t V)^T \cdot \Phi_l^m(\underline{x}^t V)\right)^2 \text{ s.t. } \|\underline{x}\| = 1 \tag{11}$$

The objective function of the maximization problem is a homogeneous polynomial in n_ℓ variables, $\underline{x} = (x_1, x_2, \ldots, x_{n_\ell})$, under the constraint $\|\underline{x}\| = 1$. It is a nonlinear maximization problem which can be tackled using standard optimization methods.

Here, we propose an iterative method to develop the degenerate solutions and hence design output feedback laws. First, we define the matrix

$$\Phi = \Phi_l^m(\underline{x}^t V)^T \cdot \Phi_l^m(\underline{x}^t V) = \begin{pmatrix} & \vdots & \\ \cdots & \phi_{ij}(\underline{x}) & \cdots \\ & \vdots & \end{pmatrix} \tag{12}$$

where $\phi_{ij}(\underline{x}) = \underline{x}^t A_{ij} \underline{x}$ is a quadratic function in $\underline{x}$. Then, the new objective function becomes

$$tr(\Phi)^2 = \sum_{i,j=1}^{m} \phi_{ij}^2(\underline{x})$$

and the Lagrangian of the problem is given by

$$\mathcal{L}(\underline{x}, \lambda) = \sum_{i,j=1}^{m} \phi_{ij}^2(\underline{x}) - \lambda\left(\|\underline{x}\|^2 - 1\right)$$

The First Order Optimality Conditions are

$$4\sum_{i,j=1}^{m} \phi_{ij}(\underline{x}) A_{ij}\underline{x} - 2\lambda\underline{x} = 0$$

and if we define by $A(\underline{x})$ the $n_\ell \times n_\ell$ matrix

$$A(\underline{x}) = \sum_{i,j=1}^{m} \phi_{ij}(\underline{x}) A_{ij}$$

then the first-order conditions can be rewritten as a nonlinear eigenvalue problem, i.e.

$$A(\underline{x}) \cdot \underline{x} = \frac{\lambda}{2} \cdot \underline{x}$$

The solution of the problem is that $\underline{x}$ that corresponds to the maximum eigenvalues of the above matrix. Hence, it may be found by applying the following iterative method which stems from the power method

$$\underline{x}_{n+1} = A(\underline{x}_n) \cdot \underline{x}_n / \|A(\underline{x}_n) \cdot \underline{x}_n\| \tag{13}$$

Algorithm terminates when $\left\|\underline{x}_{n+1} - \underline{x}_n\right\| \leqslant etol = 1 * 10^{-6}$ and converges to an exact solution whenever the objective function takes the value $l-m$. The computational procedure is summarized next.

4.3 Algorithm

Algorithm 1 GM-Degenerate Compensator

Input: m, l, $M(s)$, $etol$ and $maxiter$
Output: The degenerate compensator K_d

1: Calculate the Plücker matrix of the problem and find a basis, V, for its left kernel $N_\ell(P)$
2: Perform an optimal scaling on V which minimizes the (uniform) condition number // (optional step)
3: Calculate the solution of the linear problem (10) parameterised in the form $\underline{x}^t V$
4: Calculate the parameterised GM $\Phi_l^m(\underline{x}^t V)$ and the matrix Φ;
5: Calculate the matrix: $A(\underline{x}) = \sum_{i,j=1}^{n} \phi_{ij}(\underline{x}) A_{ij}$
6: Apply the iteration (13) until the stopping criteria are met. The vector $\underline{x}_n$ of the last iteration gives rise to the decomposable multivector $k^t = \underline{x}_n V$
7: Calculate the decomposable vector and hence the desired degenerate compensator K_d.
8: Save K_d and evaluate the sensitivity measures.
9: Repeat the algorithmic procedure several times and select the degenerate solution that minimizes the sensitivity measures.

This degenerate compensator K_d can be used as a starting point to develop (linearized) feedback laws with low sensitivity by applying Newton-type iterative methods or predictor-corrector schemes as in (Leventides et al., 2014a,b).

The sensitivity properties are strongly related with the differential of the mapping F (that contains the pole placement equations of the design problem). For sensitivity measures we consider the following:

(i) The norm of the differential $\left\| D(F)_{K_d} \right\|$ of the pole assignment map F evaluated at the degenerate compensator;
(ii) The condition number, γ, of the differential of the Pole Assignment Map, F, evaluated at the degenerate solution;
(iii) Maximum singular values of the sensitivity function $S_i(j\omega) = [I + K_i G(j\omega)]^{-1}$ for $\omega = 0.01, \ldots, 10$ and $i \in \{1, 2, \ldots\}$ different compensators.

The computational procedure and all the necessary subroutines have been implemented in Mathematica® and tested by using several multivariable systems of various dimensions. Next, we present briefly some of the results.

5. NUMERICAL EXAMPLE

Let consider a system transfer matrix $G(s) = N(s)D(s)^{-1}$ with 2-inputs, 3-outputs and McMillan degree $n = 5$, represented by its polynomial Matrix Fraction Description

$$G(s) = \begin{pmatrix} s & s+2 \\ 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (s-1)^3 & 0 \\ s^2 & (s-1)^2 \end{pmatrix}^{-1}$$

and the pole assignment problem by static output feedback, expressed as

$$\mathrm{F}(s, K): \ \det\left([I_2, K] \cdot \begin{pmatrix} D(s) \\ N(s) \end{pmatrix} \right) = a(s)$$

In order to design a static output feedback compensator, $K \in \mathbb{R}^{2\times 3}$ that places a desired stable characteristic polynomial, we need first to derive a degenerate compensator by solving

$$\underline{k}^t \cdot P = \underline{0}$$

with $\underline{k}^t \in \mathbb{R}^{10}$, $P \in \mathbb{R}^{10\times 6}$. Note that, the solvability of the problem requires that $\underline{k}^t$ must be decomposable.

The solution of the above linear problem is parametrised as

$$\underline{k}^t = \underline{x}^t \cdot V, \text{ where } \underline{x}^t \in \mathbb{R}^4 \text{ and } V \in \mathbb{R}^{4\times 10}$$

The objective function of the maximization problem (11) is a $4^{th}-$order homogeneous polynomial in $n_\ell = 4$ variables, i.e. $\underline{x} = (x_1, x_2, x_3, x_4)$. By selecting a random initial vector $\underline{x}_0 \in \mathbb{R}^4$ we apply the iteration

$$\underline{x}_{n+1} = A(\underline{x}_n) \cdot \underline{x}_n / \| A(\underline{x}_n) \cdot \underline{x}_n \|$$

for $n = 0, 1, \ldots, N$ and after $N = 700$ iterations (10.52 sec.) the algorithm has terminated when the objective function becomes $l - m = 5 - 2 = 3$ in which case we have decomposability.

We run the algorithm 100 times and we select the degenerate solution K_d that minimizes the norm of the differential, DF_{K_d}, of the Pole Assignment Map F. Such degenerate solution $\underline{k}^t = \underline{x}_N^t \cdot V$ in the Plücker space is

$$\underline{k}^t = (0, 0, 0, 0, 0.196116, 0, \\ -0.392232, -0.392232, -0.196116, -0.784465)$$

which gives rise to the degenerate compensator

$$K_d = \begin{pmatrix} 0 & -0.43209 & -0.249192 & -0.86418 & 0.0662931 \\ 0 & 0.0748728 & -0.410698 & 0.149746 & 0.896269 \end{pmatrix}$$

Using this as starting point and by applying the numerical Newton method developed in (Leventides et al., 2014b) one can design the final output feedback compensator, $(K_1 = K)$, that places a given polynomial. For instance, the following static compensator $[I_2, K_f]$

$$K_f = \begin{pmatrix} -105.174 & -112.895 & -43.0945 \\ 9.14343 & 10.8566 & 3.81789 \end{pmatrix}$$

assigns the closed-loop pole polynomial of the feedback system at $p_\epsilon(s) = (s+1)(s+2)\ldots(s+5)$.

The sensitivity properties can be evaluated using the metrics (i)-(iii) related with the differential of F calculated at the degenerate solution found above. We compare the results with a degenerate compensator

$$K_{d,2} = \begin{pmatrix} 0 & 0 & -1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which was found by using the standard method adopted in (Leventides and Karcanias, 1995), (Karcanias et al., 2013). The results are:

$$\|DF(K_d)\| = 5.09 < 186.486 = \|DF(K_{d,2})\|$$
$$\gamma\left(DF(K_d)\right) = 98.3 < 640.8 = \gamma\left(DF(K_{d,2})\right)$$
$$\sigma_{max}(S_1(j\omega)) = 10.68 < 1826.21 = \sigma_{max}(S_2(j\omega))$$

which clearly shows an improved performance of the overall sensitivity properties.

Furthermore, if we also consider the norm of the difference of the closed loop characteristic polynomial (with the resulting K_f as compensator) and the desired target pole polynomial $a(s)$ we can create a parametric plot (Fig. 1) against the norm of the differential for all the compensators calculated by the above described numerical method.

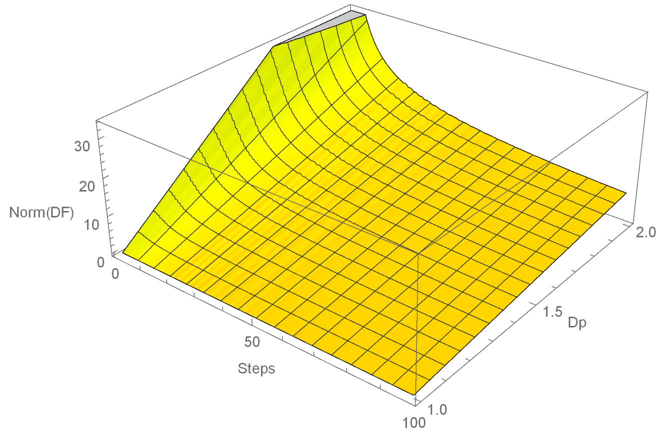

Fig. 1. Parametric plot of DF(K) against Δ_p for all the compensators calculated by our method.

6. CONCLUSION

A new algorithmic method that calculates degenerate solutions for the output feedback design problem has been presented based on the rank properties of structured Grassmann Matrices. With this approach the problem is transformed to an optimization problem which can be further reduced to a nonlinear eigenvalue-eigenvector problem which can be tackled using appropriate numerical methods. The systematic algorithmic approach to find degenerate compensators is much faster than the previous methods described in (Karcanias et al., 2013) and allows a systematic search for degenerate compensators that will generate output feedback laws with improved sensitivity properties. The further development of the method involves two different directions: the first concerns the selection of the best degenerate points where it is required to obtain results linking the selection of degenerate compensators with the singular values and the condition number of the differential of the Pole Assignment Map, that will allows to achieve the requirements for minimum sensitivity solutions (for given distance between the desired and the actual set of poles). The second task involves the characterization of the convergence properties and modifications to speed-up the algorithm.

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