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Commentary on Grigori Mints' "Classical and Intuitionistic Geometric Logic"

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This paper studies a *Glivenko sequent class*, i.e. a class of sequents where classical derivability entails intuitionistic derivability; more specifically, the paper is about "geometric sequents". The main old result in this topic is a direct consequence [10] of Barr's theorem[1]. As background, Mints sketches an old deductive proof (from [6]) and an old model-theoretic proof, as in Exercise 2.6.14 of [8]; but, his interest being in complexity of proof transformations, he gives a third proof, of a result both more and less general.

A modern reconstruction [5] of Orevkov's proof [6, Theorem 4.1, part (1)] relies on what we would now call the "cut-free **G3c** calculus" [9], in which *Cut* and other structural rules are admissible and all the logical rules are invertible (indeed, height-preserving invertible). His result is that the list (or "σ-class") $[\to^+, \neg^+, \forall^+]$ is a "completely Glivenko class"; in other words, he shows that if a sequent with a single succedent has no positive occurrences of $\to$, $\neg$ or $\forall$ then its classical derivability implies its intuitionistic derivability. In modern terminology, this means just that if a sequent $\Gamma \Rightarrow A$ (where Γ consists of geometric implications and A is a positive formula) is derivable in cut-free **G3c**, then it is already derivable in the intuitionistic calculus **m-G3i** (also from [9]). The proof method actually shows the stronger result, that the cut-free **G3c** derivation is already a **m-G3i** derivation. The weaker result extends to the case where A is a geometric implication by using the invertibility in cut-free **G3c** of the succedent rules for the three mentioned connectives. Other work,

[1] "Let $\mathcal{E}$ be a Grothendieck topos. Then there is a complete Boolean algebra $\mathcal{B}$ and an exact cotripleable functor $\mathcal{E} \to \mathcal{F}\mathcal{B}$", $\mathcal{F}\mathcal{B}$ being the topos of sheaves over $\mathcal{B}$ [1].

such as [3], related to the deductive proof of this result, is cited in the bibliographies of [4] and [5]. The usefulness of cut-free **G3** calculi in the study of Glivenko classes has been further demonstrated in [5], with direct proofs of generalisations of results in [6].

Mints' interest, however, in this paper is in derivations in **G3c** with *Cut*. One can apply standard cut-elimination transformations, and then those corresponding to the inversions; but this leads to a "super-exponential blow-up", as can be seen in a similar context in [9, Section 5.2]. How can this be avoided? One solution is just to start with a cut-free derivation. One can go even further, using the cut-free calculi introduced in [4], where the axioms Γ are replaced by inference rules: this avoids proof transformations entirely (since, in such calculi, classical proofs of a geometric implication A are already intuitionistic proofs). But, Mints would insist that **G3c** with *Cut* is a traditional (i.e. respectable) starting point.

The question then arises: can the transformation be changed so that there is an at most polynomial expansion of the derivation? Clearly it should not begin with cut elimination, so a trick is needed to handle instances of the *Cut* rule rather than eliminating them. The trick is attributed to Orevkov [6]; one might also attribute it to Skolem, who pioneered in [7] the use of what [2] should have called "relational Skolemisation", i.e. the replacement, by introduction of new relation symbols, of complex formulae by atomic formulae. When this is sufficiently thorough to ensure that every formula is equivalent to an atomic formula, it is called "atomisation" or "Morleyisation"; this paper doesn't go so far.

The novel result of this paper is now the result (both weaker and stronger) that, if d is a classical proof of a geometric sequent, then it can be polynomially transformed into an intuitionistic proof of the sequent conservatively extended by extra antecedent formulae that are geometric implications. These extra implications are generated by relational Skolemisation of the subformulae of the cut formulae in d. The result is weaker by virtue of having these extra implications; it is stronger by virtue of the complexity reduction.

There are the following points at which the paper is incorrect:

1. Mints' (9) should be $\forall \mathbf{x}(P_{\exists y G}(\mathbf{x}) \rightarrow \exists y P_G(\mathbf{x}, y))$ rather than $\forall \mathbf{x}(P_{\exists y P_G}(\mathbf{x}) \rightarrow \exists y P_G(\mathbf{x}, y))$;

2. His (10) should be $\forall \mathbf{x} \forall y(P_G(\mathbf{x}, y) \rightarrow P_{\exists y G}(\mathbf{x}))$ rather than $\forall \mathbf{x} \forall y(P_G(\mathbf{x}, y) \rightarrow P_{\exists y P_G}(\mathbf{x}))$;

3. His (11) should be $\forall \mathbf{x} \forall y(P_{\forall y G}(\mathbf{x}) \rightarrow P_G(\mathbf{x}, y))$ rather than $\forall \mathbf{x} \forall y(P_{\forall y P_G}(\mathbf{x}) \rightarrow P_G(\mathbf{x}, y))$;

4. His (12) should be $\forall\mathbf{x}(\forall y P_G(\mathbf{x}, y) \to P_{\forall y G}(\mathbf{x}))$ rather than $\forall\mathbf{x}(\forall y P_G(\mathbf{x}, y) \to P_{\forall y P_G}(\mathbf{x}))$;
5. His (19) (replacing (16)) is not a geometric implication;
6. His (17) (replacing (12)) is not a geometric implication.

The first four of these problems are minor: note that in Mints' (9) the suffix $\exists y P_G$ is not a subformula of one of the cut-formulae, and similarly for (10), (11) and (12). The penultimate problem can be fixed by distributing $\forall\mathbf{x}$ across the conjunction, thus obtaining two geometric implications: $\forall\mathbf{x}(P_H(\mathbf{z}) \to P_{G\to H}(\mathbf{x}))$ and $\forall\mathbf{x}(P_G(\mathbf{y}) \lor P_{G\to H}(\mathbf{x}))$. [It has already been made clear that $\mathbf{y}$ and $\mathbf{z}$ are subsets of the set $\mathbf{x}$ of variables.]

The final problem is not so easily fixed: the paper wrongly claims that the formula $\forall\mathbf{x}\exists y(P_G(\mathbf{x}, y) \to P_{\forall y P_G}(\mathbf{x}))$ is a geometric implication. This is not fixed by changing (12) (as proposed above) to $\forall\mathbf{x}(\forall y P_G(\mathbf{x}, y) \to P_{\forall y G}(\mathbf{x}))$ and then obtaining $\forall\mathbf{x}\exists y(P_G(\mathbf{x}, y) \to P_{\forall y G}(\mathbf{x}))$; this is still not geometric, because of the implication within the scope of the existential quantifier.

A partial solution may be had by changing this formula to the geometric implication

$$\forall\mathbf{x}(\exists y P_{\neg G}(\mathbf{x}, y) \lor P_{\forall y G}(\mathbf{x})) \tag{17}$$

but this introduces a new relation symbol $P_{\neg G}$, where $\neg G$ may not be a subformula of one of the cut formulae. To fix this problem, the relational Skolemisation needs to be applied not just to all such subformulae but also to **all** their negations.

With these changes, the application of the extra formulae (i.e. members of DEF_d) to deal with the special formulae of the derivation is unchanged for implication. We show (for example) the effects of improving (9) on the treatment of an antecedent $\exists$-inference and of correcting the treatment of universal quantification.

The improved version of (9) is $\forall\mathbf{x}(P_{\exists y G}(\mathbf{x}) \to \exists y P_G(\mathbf{x}, y))$. The step

$$\frac{G(b), \Gamma \Rightarrow \Delta}{\exists y G(y), \Gamma \Rightarrow \Delta}$$

is transformed to

$$\cfrac{\cfrac{P_{G(y)}(\mathbf{t}, b), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta}{\exists y P_{G(y)}(\mathbf{t}, y)), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta}\ \exists\Rightarrow}{\mathrm{DEF}_d, P_{\exists y G(y)}(\mathbf{t}), \Gamma \Rightarrow \Delta}.$$

Using the improved version of (17), the step

$$\frac{\Gamma \Rightarrow \Delta, G(\mathbf{t}, b)}{\Gamma \Rightarrow \Delta, \forall y G(\mathbf{t}, y)}$$

is transformed (with some implicit weakenings to save space and aid readability) to

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{G(\mathbf{x},y)}(\mathbf{t}, b)}{P_{\neg G(\mathbf{x},y)}(\mathbf{t}, b), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{G(\mathbf{x},y)}(\mathbf{t}, b)}\,Wkn}{P_{\neg G(\mathbf{x},y)}(\mathbf{t}, b), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{\neg G(\mathbf{x},y)}(\mathbf{t}, b) \wedge P_{G(\mathbf{x},y)}(\mathbf{t}, b)}\,{\Rightarrow}{\wedge}, axiom}{P_{\neg G(\mathbf{x},y)}(\mathbf{t}, b), \neg(P_{\neg G(\mathbf{x},y)}(\mathbf{t}, b) \wedge P_{G(\mathbf{x},y)}(\mathbf{t}, b)), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta}\,\neg{\Rightarrow}}{P_{\neg G(\mathbf{x},y)}(\mathbf{t}, b), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta}}{\exists y P_{\neg G(\mathbf{x},y)}(\mathbf{t}, y), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta}\,\exists{\Rightarrow} \qquad \dfrac{}{P_{\forall y G(\mathbf{x},y)}(\mathbf{t}), \Gamma \Rightarrow \Delta, P_{\forall y G(\mathbf{x},y)}(\mathbf{t})}\,axiom}{\exists y P_{\neg G(\mathbf{x},y)}(\mathbf{t}, y) \vee P_{\forall y G(\mathbf{x},y)}(\mathbf{t}), \mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{\forall y G(\mathbf{x},y)}(\mathbf{t})}\,\vee{\Rightarrow}}{\mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{\forall y G(\mathbf{x},y)}(\mathbf{t})} \quad .$$

Note the importance of having $P_{\forall y G(\mathbf{x},y)}(\mathbf{t})$ (rather than, from the succedent of the old (17), Mints' $P_{\forall y P_G}(\mathbf{t})$) in the antecedent of the lowest axiom step. It is not the case that $\forall y P_{G(\mathbf{x},y)}$ (i.e. Mints' $\forall y P_G$) is a subformula of one of the cut formulae; the presence of the fresh predicate symbol $P_{G(\mathbf{x},y)}$ forbids this.

Note also the use of the Weakening rule *Wkn*; either this rule should be included in the **m-G3i** calculus or the admissibility of the rule exploited once the derivation has been fully transformed.

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Received \jreceived