THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Matrix product solution to a 2-species TASEP with open integrable boundaries

OPEN ACCESS

Matrix product solution to a 2-species TASEP
with open integrable boundaries

N. Crampe[a1], M. R. Evans[b2], K. Mallick[c3], E. Ragoucy[d4] and M. Vanicat[d5]

[a] Laboratoire Charles Coulomb (L2C), UMR 5221 CNRS-Université de Montpellier, Montpellier, F-France.

[b] SUPA, School of Physics and Astronomy, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh EH9 3FD, UK

[c] Institut de Physique Théorique, CEA Saclay, F-91191 Gif-sur-Yvette, France

[d] Laboratoire de Physique Théorique LAPTh, CNRS and Université de Savoie. 9 chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France.

Abstract

We present an explicit representation for the matrix product ansatz for some two-species TASEP with open boundary conditions. The construction relies on the integrability of the models, a property that constrains the possible rates at the boundaries. The realisation is built on a tensor product of copies of the DEHP algebras. Using this explicit construction, we are able to calculate the partition function of the models. The densities and currents in the stationary state are also computed. It leads to the phase diagram of the models. Depending on the values of the boundary rates, we obtain for each species shock waves, maximal current, or low/high densities phases.

[1] nicolas.crampe@umontpellier.fr

[2] mevans@staffmail.ed.ac.uk

[3] kirone.mallick@cea.fr

[4] eric.ragoucy@lapth.cnrs.fr

[5] matthieu.vanicat@lapth.cnrs.fr

1   Introduction

The totally asymmetric exclusion process (TASEP) has proved itself to be a paradigmatic model for non-equilibrium, current-carrying states [14, 10]. The model comprises particles hopping stochastically along a one-dimensional lattice with hard-core exclusion interactions. The version with open boundary conditions, where particles enter and leave at the boundaries, has revealed the existence of boundary induced phase transitions [35, 23, 42]. An exact solution of the stationary state is provided by a matrix product Ansatz [23, 10] which allows the stationary density profiles and all equal-time correlation functions to be computed. The combinatorial nature of this solution has been explored in various works [2, 11, 25, 31, 32]. Moreover, recent works have explored the integrability of the model and allowed the spectrum of the Markov matrix governing the dynamics to be established via the Algebraic Bethe Ansatz [19, 15].

The generalisation to several species of particles has been under intense study, initially motivated by the case of second-class particles. Second-class particles hop forward when there is a vacancy immediately in front of them but may be overtaken by a first-class particle immediately behind them, in which case they hop back one lattice site. Second-class particles have proven to be a useful tool in tracking the positions of microscopic shocks in the system [29, 30]. In the case of periodic boundary conditions the exact stationary state of a system of first and second class particles has been established in a matrix product formulation [24]. Inspired by work in the probabilistic literature which invoked a queueing interpretation [31, 32], the solution was later generalised to a multi-species hierarchy [26, 38, 39, 6]. The resulting solution is constructed from tensor products of the fundamental matrices used in the two-species solution. The Bethe integrability of the multi-species problem has been established through co-ordinate Bethe Ansatz for the two-species case [1, 22, 12] and by Algebraic Bethe Ansatz for the multi-species case [5, 7].

Various examples of open boundary conditions with two or more species of particles have been considered initially motivated by boundary-induced symmetry breaking transitions [27]. Matrix product solutions have been found in the two-species case where the stationary current of second-class particles vanishes, which occurs under certain symmetric boundary conditions [28, 21] or when the second-class particles are constrained to remain in the system through semi-permeable boundaries [3, 4, 44, 8, 9].

However, the case of general open boundary conditions for a system of first and second-class particles has remained an elusive problem. Recently progress was made in identifying classes of boundary conditions for which the system is integrable, in the sense of the Algebraic Bethe Ansatz [17]. In that work the algebraic structure of the stationary state was worked out in a particular case of integrable boundary conditions.

In the present work we consider the manifold of integrable boundary conditions established in [17] and show that these boundary conditions lead to a non-trivial phase diagram with phases that manifest non-zero currents of the second-class particles. We find an explicit solution for the stationary state in terms of tensor products of the fundamental matrices which appear in the two-species solution for periodic boundary conditions. Interestingly, the stationary state has various factorisation properties and these allow us to compute exactly the partition function for the system. The proof of the stationary state is simplified by the factorisation properties and allows us to make the connection with techniques used in the study of integrable systems such as Zamolodchikov-Faddeev and Ghoshal-Zamolodchikov equations.

The paper is organised as follows. In section 2 we review the integrable boundary conditions

established in [17] and derive the corresponding phase diagrams. We then present in section 3 the tensor product solution for the stationary state and present various factorisation properties, and use this solution to compute the partition function for various cases. In section 4 we prove that the tensor product state is stationary under the Markov matrix that generates the dynamics. In section 5, we elucidate the connection to techniques of Algebraic Bethe Ansatz. We conclude in section 6 with an overview of open problems. Finally, supplementary technical details are presented in appendices.

2 Definition of the models and Phase Diagrams

The two-species totally asymmetric exclusion process (2-TASEP) is a stochastic dynamical system, defined on a one-dimensional lattice with L sites in contact with two boundary reservoirs, where each site $i = 1, \ldots, L$ can be in one of three states $\sigma_i = 0$, 1 or 2. State 0 may be considered as an empty site or hole. State 1 corresponds to a first class particle, and state 2 corresponds to a second class particle. In the bulk, at each pair of nearest neighbor sites, the rates of exchange are

$$1\,0 \xrightarrow{1} 0\,1 \quad , \qquad 2\,0 \xrightarrow{1} 0\,2 \quad , \qquad 1\,2 \xrightarrow{1} 2\,1 \, . \tag{2.1}$$

(We remark that various other conventions for the labelling of the three particle states have been employed in the literature e.g. [27, 28].) The sites 1 and L are in contact with boundary reservoirs and particles are exchanged at different rates at the boundaries. For generic values of these boundary rates, the system is not integrable (in contrast with the 1-species TASEP, which is integrable for arbitrary boundary rates); finding an exact solution looks hopeless. However, in [17], using a systematic procedure, all possible boundary rates for the 2-species TASEP that preserve integrability were classified. Amongst such models, some had been studied earlier: the first open two-species matrix product solutions were derived in [28]; in [21] the boundary conditions for which the stationary state may be expressed using the matrices D, E, A of [23, 24] were deduced; in [8, 9] the restricted class of *semi-permeable* boundaries, in which second class particles can neither enter nor leave the system was studied. In all of these cases a matrix product representation of the stationary state was found involving the quadratic algebra used by Derrida, Evans, Hakim and Pasquier [23] in their exact solution of the 1-species exclusion process with open boundaries.

In the present work, we construct a matrix Ansatz for integrable 2-TASEP with open boundaries that allow all species of particles to enter and leave the system. The algebraic structures required will be much more involved than the fundamental quadratic algebra of [23].

We shall study two classes of 2-species TASEP models with the following boundary rates

$$
\begin{array}{lcl}
& \text{left boundary} & \text{right boundary} \\[4pt]
& 2 \xrightarrow{\alpha} 1 & 2 \xrightarrow{\beta} 0 \\[4pt]
(P_1): & 0 \xrightarrow{\alpha} 1 & 1 \xrightarrow{\beta} 0 \\[4pt]
& 0 \xrightarrow{1-\alpha} 2 & 1 \xrightarrow{1-\beta} 2
\end{array} \tag{2.2}
$$

or

$$(P_2): \quad \begin{array}{cc} \text{left boundary} & \text{right boundary} \\ 2 \xrightarrow{\alpha} 1 & 2 \xrightarrow{\beta} 0 \\ 0 \xrightarrow{\alpha} 1 & 1 \xrightarrow{\beta} 0 \\ 0 \xrightarrow{1-\alpha} 2 & \end{array} \tag{2.3}$$

Hereafter, the two different models will be denoted by (P_1) and (P_2). Note that in the classification of [17] the left boundary conditions were referred to as L_2 and the right hand boundary conditions for (P_1) or (P_2) were referred to as R_2 and R_3 respectively. It is a simple matter to translate our results for (P_2) to the case of right boundary R_2 and left boundary L_3. The final case of right boundary R_3 and left boundary L_3 leaves the stationary state devoid of holes and thus reduces to a one-species TASEP.

The physical interpretation of the boundary conditions is as follows. In both models (P_1), (P_2) the left-hand boundary conditions correspond to a boundary reservoir containing only first and second class particle with densities α and $1 - \alpha$ respectively, with no holes. In model (P_1) the right-hand boundary conditions correspond to a reservoir containing second-class particles and holes with densities $1 - \beta$ and β respectively, with no first-class particles. In model (P_2) the right-hand boundary conditions correspond to a reservoir containing first-class particles and holes with densities $1 - \beta$ and β respectively, with no second-class particles.

The 2-TASEP is a finite Markov process that reaches a unique steady-state in the long time limit, in which each configuration has the stationary probability (or weight) $P(\sigma_1, \sigma_2, \ldots, \sigma_L)$. The column-vector $\boldsymbol{P}$ of length 3^L, whose components are the probabilities $P(\sigma_1, \sigma_2, \ldots, \sigma_L)$, satisfies the stationary master equation

$$M^{(3)} \boldsymbol{P} = 0, \tag{2.4}$$

where $M^{(3)}$ is the $3^L \times 3^L$ Markov matrix for the 2-TASEP system. Finding the steady state thus amounts to solving a linear system that grows exponentially with the size of the system. Basically, the matrix product representation of the stationary weights, based on integrability, will allow us to reduce this exponential complexity to a polynomial computation.

2.1 Phase diagrams

The stationary state of the exclusion process can exhibit different qualitative features and different analytical expressions for macroscopic quantities in the infinite size limit, $L \to \infty$. The system is said to exhibit various phases, that depend on the values of the boundary exchange rates. These different phases can be discriminated by the values of the currents and by the shapes of the density profiles. More refined features, such as correlations length or even dynamical behaviour, can even lead us to define subphases (see [10] for details and references). The phase diagram of the one-species TASEP has been well-known for a long time; first determined using a mean-field approximation [35, 20], it was rigorously established and precisely investigated after the finding of the exact solution [23, 42, 10]. We recall that the dynamical rules of the one-species TASEP are given by

$$\begin{array}{ccc} \text{left boundary} & \text{bulk} & \text{right boundary} \\ 0 \xrightarrow{\alpha} 1 & 1\,0 \xrightarrow{1} 0\,1 & 1 \xrightarrow{\beta} 0 \end{array} \tag{2.5}$$

3

The phase diagram is determined by the behavior of the stationary current J and bulk density of particles in the limit $L \to \infty$ [10]. The different phases are detailed in table (2.6).

Region	Phase	Current J	Bulk density	
$\alpha < \beta,\ \alpha < \frac{1}{2}$	Low-density (LD)	$\alpha(1-\alpha)$	α	(2.6)
$\beta < \alpha,\ \beta < \frac{1}{2}$	High-density (HD)	$\beta(1-\beta)$	$1-\beta$	
$\alpha > \frac{1}{2},\ \beta > \frac{1}{2}$	Maximal current (MC)	$\frac{1}{4}$	$\frac{1}{2}$	

The phase diagrams of the (P_1) and (P_2) models can be determined rigorously without having to compute exactly the steady-state probabilities. Indeed, the various phases of these two species models can extracted from the knowledge of the one-species TASEP phase diagram, by using an identification procedure formalised in [9]. Then, 2-TASEP models (P_1) and (P_2) can both be mapped to the one-species TASEP model using two possible identifications:

1. One can identify holes and species 2 to get a one-species TASEP model for which the phase diagram is given in table (2.6). The boundary conditions read

$$
\begin{array}{ccc}
 & \text{left boundary} & \text{right boundary} \\
(P_1): & (0,2) \xrightarrow{\ \alpha\ } 1 & 1 \xrightarrow{\ 1\ } (0,2) \\
(P_2): & (0,2) \xrightarrow{\ \alpha\ } 1 & 1 \xrightarrow{\ \beta\ } (0,2)
\end{array}
\tag{2.7}
$$

2. One can identify species 1 and 2 to get another version of the one-species TASEP model. In that case the two models (P_1) and (P_2) produce the same boundary conditions:

$$
\begin{array}{ccc}
 & \text{left boundary} & \text{right boundary} \\
(P_1)\,\&\,(P_2): & 0 \xrightarrow{\ 1\ } (1,2) & (1,2) \xrightarrow{\ \beta\ } 0
\end{array}
\tag{2.8}
$$

For the two-species TASEP, we denote by J_1, J_2 the particle currents in the stationary state for the particles of species 1 and 2 respectively. The currents are counted positively when particles flow from the left to the right. In the same way, ρ_1 and ρ_2 denote the densities of particles of species 1 and 2 respectively.

Identification 1. allows us to compute the current J_1 and the density ρ_1, while identification 2. yields the current $J_1 + J_2 \equiv -J_0$ and the density $\rho_1 + \rho_2$. Gathering these results, we obtain the phase diagrams depicted in figure 1.

2.1.1   Phase diagram of the (P_1) model

The phase diagram of the (P_1) model is displayed in Figure 1(a). It comprises four phases. Using the identification procedure, we observe that ρ_1 behaves as the density for the one-species TASEP with boundary rates $(\alpha, 1)$, while $\rho_1 + \rho_2$ behaves as the density for the one-species TASEP with boundary rates $(1, \beta)$. The values of the currents in each phase (see fig 1(a)) are readily found by this identification. The behaviour of the density profile $(\rho(j))$ in each phase can be found in [10].

4

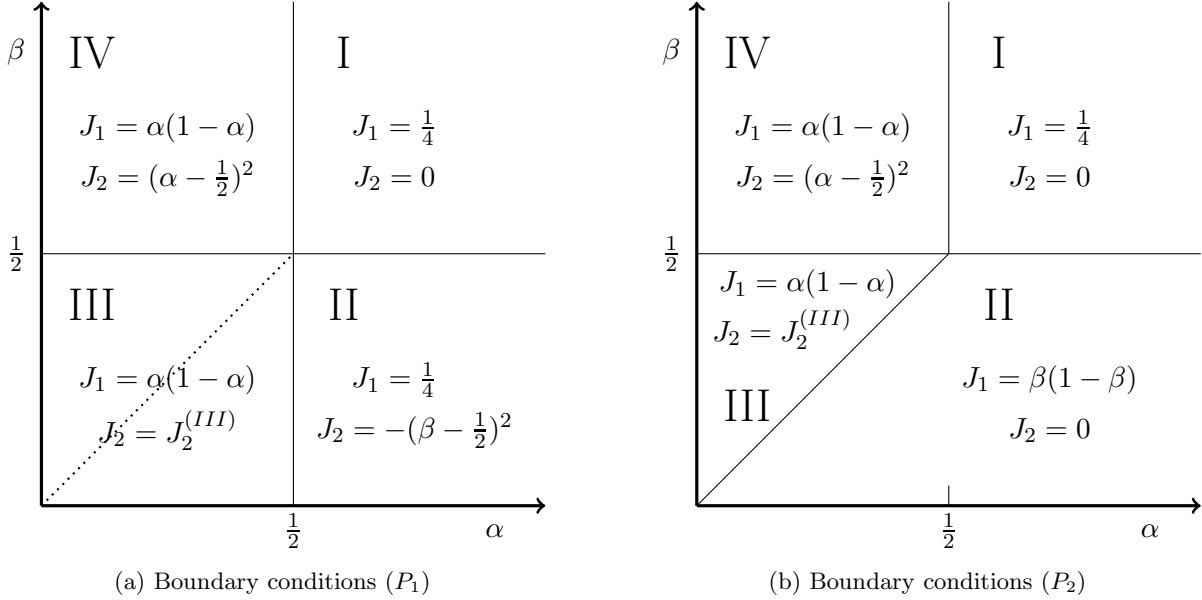(a) Boundary conditions (P_1) (b) Boundary conditions (P_2)

Figure 1: Phase diagrams of the 2-TASEP with open boundaries for the boundary conditions (P_1) and (P_2) (we have used the notation $J_2^{(III)} = (\beta - \alpha)(1 - \alpha - \beta)$).

Phase I: For $\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$, first-class particles exhibit a maximal current, whereas the current of the second-class particles vanishes. The bulk density of first-class particles and of holes is equal to $1/2$, while the number of second class particles in the bulk is vanishingly small. The density profiles of first and second class particles are characterised by power law decays to the bulk values:

$$\rho_1 = \frac{1}{2} + \frac{1}{2\sqrt{\pi j}} + \mathcal{O}\left(\frac{1}{j^{3/2}}\right) \quad \text{and} \quad \rho_2 = \mathcal{O}\left(\frac{1}{j^{3/2}}\right), \tag{2.9}$$

where j is the site position on the lattice. The system is similar to the one-species TASEP in its maximal current phase.

Phase II: For $\alpha > \frac{1}{2}$ and $\beta < \frac{1}{2}$, none of the currents J_1, J_2 and J_0 vanishes. This is a genuine 2-TASEP phase with boundaries permeable to all the species. The two species and the holes coexist in the bulk with non-zero bulk densities and density profiles characterised by power-law decays:

$$\rho_1 = \frac{1}{2} + \frac{1}{2\sqrt{\pi j}} + \mathcal{O}\left(\frac{1}{j^{3/2}}\right) \quad \text{and} \quad \rho_2 = \frac{1}{2} - \beta - \frac{1}{2\sqrt{\pi j}} + \mathcal{O}\left(\frac{1}{j^{3/2}}\right). \tag{2.10}$$

First-class particles are in their maximal current phase. Boundary effects are long-range for species 1 and 2.

Phase III: For $\alpha < \frac{1}{2}$ and $\beta < \frac{1}{2}$, we obtain a 'massive' phase in which boundary effects are

5

localized: after a finite correlation length, the system reaches its bulk behaviour,

$$\rho_1 = \alpha + \mathcal{O}\left(\frac{1}{j^{3/2}}\exp\left(-\frac{j}{\xi}\right)\right) \quad \text{and} \quad \rho_2 = 1 - \alpha - \beta + \mathcal{O}\left(\frac{1}{j^{3/2}}\exp\left(-\frac{j}{\xi}\right)\right). \quad (2.11)$$

The current of second-class particles J_2 vanishes along the line $\alpha = \beta < \frac{1}{2}$ and changes its sign across this line.

Phase IV: This phase, obtained for $\alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$, is massive for first-class particles but 'massless' (exhibiting long-range correlations characterised by power laws) for second-class particles and holes. Here again, the two species and the holes coexist in the bulk:

$$\rho_1 = \alpha + \mathcal{O}\left(\frac{1}{j^{3/2}}\exp\left(-\frac{j}{\xi}\right)\right) \quad \text{and} \quad \rho_2 = \frac{1}{2} - \alpha - \frac{1}{2\sqrt{\pi j}} + \mathcal{O}\left(\frac{1}{j^{3/2}}\right). \quad (2.12)$$

Holes are in their maximal current phase $J_0 = -1/4$.

2.1.2 Phase diagram of the (P_2) model

The phase diagram of the (P_2) model also comprises four phases, displayed in Figure 1(b). The diagram is qualitatively different from that of the (P_1) model. Here, ρ_1 behaves as the density for the one-species TASEP with boundary rates (α, β), while $\rho_1 + \rho_2$ behaves as the density for the one-species TASEP with boundary rates $(1, \beta)$.

Phase I: For $\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$, first-class particles exhibit a maximal current. This phase is similar to Phase I of model (P_1).

Phase II: This phase is obtained for $\beta < \alpha < \frac{1}{2}$. First-class particles are in their high density phase. The bulk density of second-class particles in the bulk vanishes; moreover, the probability to find a second class particle at a distance larger than the correlation length ξ away from the boundaries, is exponentially small:

$$\rho_1 = 1 - \beta + \mathcal{O}\left(\frac{1}{j^{3/2}}\exp\left(-\frac{j}{\xi}\right)\right) \quad \text{and} \quad \rho_2 = \mathcal{O}\left(\frac{1}{j^{3/2}}\exp\left(-\frac{j}{\xi}\right)\right). \quad (2.13)$$

Phase III: For $\alpha < \beta < \frac{1}{2}$, the two species and the holes are simultaneously present with non-vanishing currents. The current of second-class particles J_2 is strictly positive. This phase is massive for the two classes of particles and the holes:

$$\rho_1 = \alpha + \mathcal{O}\left(\exp\left(-\frac{j}{\xi}\right)\right) \quad \text{and} \quad \rho_2 = 1 - \alpha - \beta + \mathcal{O}\left(\exp\left(-\frac{j}{\xi}\right)\right). \quad (2.14)$$

Shock Line: This line corresponds to $\alpha = \beta < \frac{1}{2}$. The density profiles ρ_1 and ρ_2 display a linear behaviour that reflect a coexistence between a low density and a high density regions:

$$\rho_1 = \alpha + \frac{j}{L}(1 - 2\alpha) \quad \text{and} \quad \rho_2 = (1 - \alpha)(1 - \frac{j}{L}). \quad (2.15)$$

The density profile of first-class particles takes the values α and $1 - \alpha$ with a discontinuous shock between the two regions. The second-class particles have a plateau density of $1 - \alpha$ to the left of the shock and zero density to the right shock. This means effectively that in the stationary state in the infinite system limit only the left reservoir is active as far as second-class particles are concerned.

6

Phase IV This phase, obtained for $\alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$, is similar to Phase IV of the (P_1) model.

3   Summary of matrix product solutions

In this section, the stationary state of the 2-TASEP with open boundaries is represented as a matrix state, for the models (P_1) and (P_2). We show that the steady-state probability of finding a given configuration $(\sigma_1, \sigma_2, ..., \sigma_L)$ can be written as a contraction on two vectors $\langle W|$ and $|V\rangle$ over a suitable algebra

$$P(\sigma_1, \sigma_2, ..., \sigma_L) = \frac{1}{Z_L} \langle W | X_{\sigma_1} X_{\sigma_2} \cdots X_{\sigma_L} | V \rangle , \tag{3.1}$$

where Z_L is a normalisation constant ensuring that $P(\sigma_1, \sigma_2, ..., \sigma_L)$ is a probability.

In the following, we give explicit formulas for the operators X_σ that generate the algebra and for the boundary vectors $\langle W|$ and $|V\rangle$. The operators X_σ and the left vector $\langle W|$ are the same for both models (P_1) and (P_2); only the right vectors $|V\rangle$ differ. We shall also present an important factorisation property of the Matrix Ansatz that will allow us to derive explicit expressions for the normalisation constant Z_L. The proofs of these results will be given in section 4.

3.1   Explicit representation of the matrices

The Matrix Ansatz for the 2-TASEP will be constructed in terms of tensor products of the fundamental operators A, δ and ε that appear in the solution of the one-species exclusion process [23]. These operators A, δ and ε define a quadratic algebra and satisfy

$$\delta \varepsilon = 1 , \quad A = 1 - \varepsilon \delta , \quad \delta A = 0 , \quad A \varepsilon = 0 . \tag{3.2}$$

The relation with the operators D and E of [23] is $\delta = D - 1$, $\varepsilon = E - 1$ and $A = DE - ED$.

We also define the following parameters

$$a = \frac{1 - \alpha}{\alpha} \quad \text{and} \quad b = \frac{1 - \beta}{\beta} . \tag{3.3}$$

Finally, we shall need four commuting copies of the algebra (3.2), $(\varepsilon_n, \delta_n, A_n)$, $n = 1, 2, 3, 4$. A simple way to achieve this is to make four-fold tensor products:

$$\begin{aligned}
\varepsilon_1 &= \varepsilon \otimes 1 \otimes 1 \otimes 1, \\
\varepsilon_2 &= 1 \otimes \varepsilon \otimes 1 \otimes 1, \\
\varepsilon_3 &= 1 \otimes 1 \otimes \varepsilon \otimes 1, \\
\varepsilon_4 &= 1 \otimes 1 \otimes 1 \otimes \varepsilon,
\end{aligned} \tag{3.4}$$

and similarly for δ_n and A_n.

We are now in a position to present explicit matrices for the 2-TASEP with open boundaries

$$X_0 = \left(1 + aA_1A_2 + \varepsilon_2\delta_3\right)(1 + \varepsilon_4) + \left(\varepsilon_2 + \varepsilon_3 + aA_1A_2\varepsilon_3 + \varepsilon_1A_3\right)(1 + \delta_4), \qquad (3.5)$$

$$X_2 = a\delta_1A_2\left(1 + \varepsilon_4\right) + \left(a\delta_1A_2\varepsilon_3 + aA_2A_3\right)(1 + \delta_4), \qquad (3.6)$$

$$X_1 = \left(\delta_2 + \delta_3\right)(1 + \varepsilon_4) + \left(1 + \delta_2\varepsilon_3 + \varepsilon_1\delta_2A_3\right)(1 + \delta_4). \qquad (3.7)$$

N. B. It is important to realise that the integer suffices on the left and right hand sides of (3.5)–(3.7) are unrelated: on the left $\sigma = 0, 1, 2$ corresponds to particle species whereas on the right $n = 1, 2, 3, 4$ labels the tensor product as in the example (3.4).

3.2 Expressions of the boundary vectors

To construct the vectors $\langle W|$ and $|V\rangle$, we first define the elementary vectors $\langle x|$ and $|x\rangle$ that obey

$$\langle x|\,\varepsilon = x\,\langle x| \quad \text{and} \quad \delta\,|x\rangle = x\,|x\rangle. \qquad (3.8)$$

It is known [23] that explicit representations of such elementary vectors exit. Here we use a representation where

$$\langle x|y\rangle = \frac{1}{1 - xy}. \qquad (3.9)$$

The left boundary vector reads:

$$\langle W|_{1234} = \langle 1|_1\langle 0|_2\langle 0|_3\langle 0|_4, \qquad (3.10)$$

where the indices indicate again which copy of the (A, δ, ε) algebra acts on the vector. To make the notation less cluttered, we shall simply write $\langle W|$ instead of $\langle W|_{1234}$. Note that the left vector is the same for the models (P_1) and (P_2).

The right boundary vector depends on the choice of the dynamics at the right boundary (i.e. on the choice of the model (P_1) or (P_2)). We have

$$|V(P_1)\rangle_{1234} = |\frac{b}{a}\rangle_1|0\rangle_2|1\rangle_3|b\rangle_4 \quad \text{for} \quad (P_1) \qquad (3.11)$$

$$\text{and} \quad |V(P_2)\rangle_{1234} = |0\rangle_1|b\rangle_2|1\rangle_3|b\rangle_4 \quad \text{for} \quad (P_2). \qquad (3.12)$$

We shall simply write $|V\rangle$ for the right vector, without specifying the indices and which model we consider. This should be unambiguous from the context.

3.3 A factorisation property of the matrix ansatz

The expressions (3.5)–(3.7) for the X_σ's can be written in a factorized form which will be useful to compute the normalisation and for the proof of the matrix ansatz in section 4.

Let us consider the following matrices

$$\mathbb{L}^{(3)} = \begin{pmatrix} 1 + \lambda A_1A_2 + \varepsilon_2\delta_3 & \varepsilon_2 + \varepsilon_3 + \lambda A_1A_2\varepsilon_3 + \varepsilon_1A_3 \\ \lambda\delta_1A_2 & \lambda\delta_1A_2\varepsilon_3 + \lambda A_2A_3 \\ \delta_2 + \delta_3 & 1 + \delta_2\varepsilon_3 + \varepsilon_1\delta_2A_3 \end{pmatrix}, \quad \mathbb{L}^{(2)} = \begin{pmatrix} 1 + \varepsilon_4 \\ 1 + \delta_4 \end{pmatrix}, \qquad (3.13)$$

where $\mathbb{L}^{(3)}$ is a 3×2 matrix which contains a parameter λ, while $\mathbb{L}^{(2)}$ is a 2×1 matrix. Then, for $\lambda = a$, the following, important, relation is satisfied

$$\begin{pmatrix} X_0 \\ X_2 \\ X_1 \end{pmatrix} = \mathbb{L}^{(3)} \mathbb{L}^{(2)} \tag{3.14}$$

where X_0, X_2 and X_1 are the operators that perform the matrix Ansatz for the open 2-TASEP. This identity can readily be checked using equations (3.5)–(3.7).

Furthermore, the operator $\mathbb{L}^{(3)}$, defined above, can be factorized into the product of a 3×3 matrix by a 3×2 matrix, as follows

$$\mathbb{L}^{(3)} = L^{(3)} \widetilde{L}^{(3)} \tag{3.15}$$

with

$$L^{(3)} = \begin{pmatrix} 1 + \lambda A_1 A_2 & \varepsilon_1 & \varepsilon_2 \\ \lambda \delta_1 A_2 & \lambda A_2 & 0 \\ \delta_2 & \varepsilon_1 \delta_2 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{L}^{(3)} = \begin{pmatrix} 1 & \varepsilon_3 \\ 0 & A_3 \\ \delta_3 & 1 \end{pmatrix}. \tag{3.16}$$

A similar type of factorisation holds for $\mathbb{L}^{(2)}$:

$$\mathbb{L}^{(2)} = L^{(2)} \widetilde{L}^{(2)} \tag{3.17}$$

with

$$L^{(2)} = \begin{pmatrix} 1 & \varepsilon_4 \\ \delta_4 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{L}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.18}$$

The origin of these factorisations will be clarified in Section 4. Note that the factorisations are valid for arbitrary values of the parameter λ. We present also in Appendix A a relation between $\widetilde{L}$ and L.

3.4 Factorisation property of the steady-state probabilities

A compact way of writing the steady-state probabilities is to define a vector :

$$\boldsymbol{X} = \begin{pmatrix} X_0 \\ X_2 \\ X_1 \end{pmatrix}. \tag{3.19}$$

Then, the matrix Ansatz for the stationary probability vector reads

$$\boldsymbol{P} = \frac{1}{Z_L} \langle W | \boldsymbol{X} \otimes \boldsymbol{X} \otimes \cdots \otimes \boldsymbol{X} | V \rangle. \tag{3.20}$$

The factorisation (3.14) leads to

$$\boldsymbol{P} = \frac{1}{Z_L} \mathbb{P}^{(3)} \mathbb{P}^{(2)}, \tag{3.21}$$

where

$$\mathbb{P}^{(3)} = \langle W |_{123} \, \mathbb{L}^{(3)} \otimes \cdots \otimes \mathbb{L}^{(3)} \, |V\rangle_{123} \quad \text{and} \quad \mathbb{P}^{(2)} = \langle W |_4 \, \mathbb{L}^{(2)} \otimes \cdots \otimes \mathbb{L}^{(2)} \, |V\rangle_4 \,. \tag{3.22}$$

9

Here, $\mathbb{P}^{(3)}$ is a $3^L \times 2^L$ matrix and $\mathbb{P}^{(2)}$ is a 2^L-component vector so that $\boldsymbol{P}$ is a 3^L-component vector as expected. We also remark that $\mathbb{P}^{(2)}$ (up to a normalisation) is identical to the steady-state vector of the one species TASEP with open boundaries. Therefore, we have

$$M^{(2)} \mathbb{P}^{(2)} = 0 , \qquad (3.23)$$

where $M^{(2)}$ is the Markov matrix of the one-species TASEP.

Finally, thanks to the factorisation properties of (3.15)–(3.18) of $\mathbb{L}^{(3)}$ and $\mathbb{L}^{(2)}$, the stationary state can be further decomposed as

$$\boldsymbol{P} = \frac{1}{Z_L} \mathbb{P}^{(3)} \, \mathbb{P}^{(2)} = \frac{1}{Z_L} P^{(3)} \, \widetilde{P}^{(3)} \, P^{(2)} \, \widetilde{P}^{(2)} , \qquad (3.24)$$

with

$$P^{(3)} = \langle W|_{12} \, L^{(3)} \otimes L^{(3)} \otimes \cdots \otimes L^{(3)} \, |V\rangle_{12} \qquad (3.25)$$

$$\widetilde{P}^{(3)} = \langle W|_3 \, \widetilde{L}^{(3)} \otimes \widetilde{L}^{(3)} \otimes \cdots \otimes \widetilde{L}^{(3)} \, |V\rangle_3 \qquad (3.26)$$

$$P^{(2)} = \langle W|_4 \, L^{(2)} \otimes L^{(2)} \otimes \cdots \otimes L^{(2)} \, |V\rangle_4 \qquad (3.27)$$

$$\widetilde{P}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \qquad (3.28)$$

Let us note that $P^{(3)}$ is a $3^L \times 3^L$ matrix, $\widetilde{P}^{(3)}$ is a $3^L \times 2^L$ matrix, $P^{(2)}$ is a $2^L \times 2^L$ matrix and $\widetilde{P}^{(2)}$ is a 2^L-component vector with constant components.

3.5   Calculation of the normalisation

We may now use the factorisation properties of the previous subsection to calculate the normalisation Z_L of the stationary probabilities (3.1). The results we obtain are

$$Z_L = \frac{a}{a - b} \, \mathcal{Z}_L(\alpha, 1) \, \mathcal{Z}_L(1, \beta) \qquad \text{for} \quad (P_1)$$

$$Z_L = (1 - ab) \, \mathcal{Z}_L(\alpha, \beta) \, \mathcal{Z}_L(1, \beta) \quad \text{for} \quad (P_2) \qquad (3.29)$$

where $\mathcal{Z}_L(\alpha, \beta)$ is the partition function of the open one-species TASEP with injection rate α and extraction rate β. Its exact expression [10] is given by

$$\mathcal{Z}_L(\alpha, \beta) = \langle a|(2 + \varepsilon + \delta)^L|b\rangle = \langle a|(D + E)^L|b\rangle = \sum_{p=0}^{L} \frac{p \, (2L - p - 1)!}{L!(L - p)!} \frac{\left(\frac{1}{\alpha}\right)^{p+1} - \left(\frac{1}{\beta}\right)^{p+1}}{\frac{1}{\alpha} - \frac{1}{\beta}} \langle a|b\rangle . \qquad (3.30)$$

From the matrix Ansatz, we know that

$$Z_L = \langle W| \, (X_0 + X_1 + X_2)^L \, |V\rangle. \qquad (3.31)$$

Using the factorisations (3.14) and (3.15), we obtain

$$X_0 + X_1 + X_2 = (1, 1, 1) \cdot \begin{pmatrix} X_0 \\ X_2 \\ X_1 \end{pmatrix} = (1, 1, 1) L^{(3)} \, \widetilde{L}^{(3)} \mathbb{L}^{(2)}. \qquad (3.32)$$

10

We first compute

$$(1, 1, 1) \cdot L^{(3)} = \Big(1 + a(A_1 + \delta_1)A_2 + \delta_2 ,\ \varepsilon_1(1 + \delta_2) + aA_2,\ 1 + \varepsilon_2\Big). \tag{3.33}$$

Then, from the relations $\langle 1|(A + \delta) = \langle 1|$ and $\langle 1|\varepsilon = \langle 1|$, we deduce

$$\langle 1|_1 (1, 1, 1) \cdot L^{(3)} = \Big(1 + aA_2 + \delta_2 ,\ 1 + aA_2 + \delta_2,\ 1 + \varepsilon_2\Big)\langle 1|_1 \tag{3.34}$$

This implies that the space 1 drops out (because neither $\widetilde{L}^{(3)}$ nor $\mathbb{L}^{(2)}$ act on it). Remarking that

$$
\begin{aligned}
\Big(1 + aA_2 + \delta_2 ,\ 1 + aA_2 + \delta_2,\ 1 + \varepsilon_2\Big)\widetilde{L}^{(3)} &= \Big(1 + aA_2 + \delta_2 ,\ 1 + \varepsilon_2\Big) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \widetilde{L}^{(3)} \\
&= \Big(1 + aA_2 + \delta_2 ,\ 1 + \varepsilon_2\Big) \begin{pmatrix} 1 & A_3 + \varepsilon_3 \\ \delta_3 & 1 \end{pmatrix} \quad (3.35)
\end{aligned}
$$

and using $(A + \varepsilon)|1\rangle = |1\rangle$ and $\delta|1\rangle = |1\rangle$, we have

$$\begin{pmatrix} 1 & A_3 + \varepsilon_3 \\ \delta_3 & 1 \end{pmatrix} |1\rangle_3 = |1\rangle_3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = |1\rangle_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1, 1) \tag{3.36}$$

so that space 3 also drops out. Gathering the different results, we obtain

$$\begin{cases} Z_L = \langle 1|b/a\rangle_1\ \langle 0|(2 + aA_2 + \delta_2 + \varepsilon_2)^L|0\rangle_2\ \langle 0|1\rangle_3\ \langle 0|(2 + \varepsilon_4 + \delta_4)^L|b\rangle_4 & \text{for } (P_1) \\ Z_L = \langle 1|0\rangle_1\ \langle 0|(2 + aA_2 + \delta_2 + \varepsilon_2)^L|b\rangle_2\ \langle 0|1\rangle_3\ \langle 0|(2 + \varepsilon_4 + \delta_4)^L|b\rangle_4 & \text{for } (P_2). \end{cases} \tag{3.37}$$

We conclude the derivation of (3.29) by using (3.30) and by observing that

$$\langle 0|(2 + aA + \delta + \varepsilon)^L|b\rangle = \frac{\langle 0|b\rangle}{\langle a|b\rangle} \mathcal{Z}_L(\alpha, \beta) \tag{3.38}$$

because the operators $\widetilde{\varepsilon} = aA + \varepsilon$ and δ obeys the same algebraic rules as ε and δ, but now $\langle 0|$ is a left eigenvector of $\widetilde{\varepsilon}$ with eigenvalue a.

4 Proof of the matrix ansatz

In this section, we give an algebraic proof that the matrix Ansatz given in the previous section is indeed a representation of the steady-state probabilities of the models (P_1) and (P_2). We shall use the method of auxiliary matrices [10, 33, 40] that set out a general cancellation scheme and led Krebs and Sandow [34] to a general proof (albeit not constructive) of the matrix-product form for a general class of stochastic processes.

4.1 Local update operators

The evolution rules of the exclusion process are local: a particle moves to one of its neighbouring sites. Hence, the Markov matrix of the process can be written as the sum of local operators. For the one-species TASEP with open boundaries, we have

$$M^{(2)} = B_1^{(2)} + \sum_{\ell=1}^{L-1} m_{\ell,\ell+1}^{(2)} + \overline{B}_L^{(2)}, \tag{4.1}$$

11

where the local bulk Markov matrix between site ℓ and $\ell+1$ and the boundary matrices are given by[6]

$$m^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ; \quad B^{(2)} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \; ; \quad \overline{B}^{(2)} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix} . \tag{4.2}$$

Similarly, the dynamics of the 2-species TASEP is governed by the Markov matrix $M^{(3)}$. It can be decomposed as

$$M^{(3)} = B_1 + \sum_{\ell=1}^{L-1} m^{(3)}_{\ell,\ell+1} + \overline{B}_L \,, \tag{4.3}$$

with the local bulk update operator acting on nearest neighbour sites

$$m^{(3)} = \begin{pmatrix} . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . \\ . & . & . & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & -1 & . \\ . & . & . & . & . & . & . & . \end{pmatrix} \tag{4.4}$$

where the points in the matrix stand for vanishing entries. The boundary operators read

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 1-\alpha & -\alpha & 0 \\ \alpha & \alpha & 0 \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} 0 & \beta & \beta \\ 0 & -\beta & 1-\beta \\ 0 & 0 & -1 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} 0 & \beta & \beta \\ 0 & -\beta & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \tag{4.5}$$

These operators are written in the local state basis $(0, 2, 1)$ which is the natural choice corresponding to increasing order of priority in the update rules. In equation (4.3), the subscripts indicate on which sites of the lattice the local operators act non-trivially, and the right boundary matrix $\overline{B}$ corresponds to $\widehat{B}$ for the processes (P_1) and $\widetilde{B}$ for (P_2). As a rule, the superscripts in (4.2) and (4.3) indicate the number of possible states at a site, i.e. the number of species plus one. However, to lighten the notation we do not put a superscript (3) in the boundary matrices for the 2-TASEP, defined in (4.5).

4.2 Auxiliary matrices

We want to prove that (3.20) is a representation of the stationary vector of the 2-TASEP with open boundaries, i.e. that the master equation (2.4) is satisfied. The matrix Ansatz has a straightforward algebraic proof [33, 34, 40]: if one can find auxiliary operators $\boldsymbol{X'} = \begin{pmatrix} X_0' \\ X_2' \\ X_1' \end{pmatrix}$ such

---

[6] We present here only the boundary matrix $B^{(2)}$ with an injection rate $\alpha = 1$ that is needed for our purposes, see relations (4.21) and (4.22). Remark that the matrices $B^{(2)}$ and $\overline{B}^{(2)}$ correspond to identification (2.8).

that $\boldsymbol{X}$ and $\boldsymbol{X'}$ satisfy

$$m^{(3)} \boldsymbol{X} \otimes \boldsymbol{X} = \boldsymbol{X'} \otimes \boldsymbol{X} - \boldsymbol{X} \otimes \boldsymbol{X'} \tag{4.6}$$

$$\langle W | B \, \boldsymbol{X} = -\langle W | \boldsymbol{X'} \quad \text{and} \quad \overline{B} \boldsymbol{X} | V \rangle = \boldsymbol{X'} | V \rangle \tag{4.7}$$

where we recall that $\overline{B}$ is either $\widehat{B}$ or $\widetilde{B}$, then the stationary master equation (2.4) is satisfied for for the stationary probability vector given by (3.20).

Before giving an explicit realisation of these new operators X_0', X_1' and X_2', we want to explain the notations. The auxiliary generators X_0', X_2' and X_1' are often denoted by a hat or a bar in the literature (and nicknamed 'hat-matrices'). However, we have purposely written them with a prime, because we shall show in section 5 that the $\boldsymbol{X'}$ can be constructed by taking the derivative with respect to a spectral parameter of the Zamolodchikov-Faddeev relation. Recalling that the tensor product in equation (4.6) is given by

$$\boldsymbol{X} \otimes \boldsymbol{X'} = \begin{pmatrix} X_0 X_0' \\ X_0 X_2' \\ X_0 X_1' \\ X_2 X_0' \\ X_2 X_2' \\ X_2 X_1' \\ X_1 X_0' \\ X_1 X_2' \\ X_1 X_1' \end{pmatrix} , \tag{4.8}$$

we can spell out the formula (4.6) to obtain the quadratic relations that couple X_0, X_2 and X_1 with the auxiliary matrices X_0', X_2' and X_1'.

$$[X_i \, , \, X_i'] = 0 \, , \quad i = 0, 1, 2 \tag{4.9}$$

$$X_1 \, X_0 = X_0' X_1 - X_0 X_1' = X_1 X_0' - X_1' X_0 \, , \tag{4.10}$$

$$X_2 \, X_0 = X_0' X_2 - X_0 X_2' = X_2 X_0' - X_2' X_0 \, , \tag{4.11}$$

$$X_1 \, X_2 = X_2' X_1 - X_2 X_1' = X_1 X_2' - X_1' X_2 \, . \tag{4.12}$$

In Appendix B, we give the connection between the bases presented here and the ones introduced in [17].

4.3 Explicit formulas for the auxiliary matrices

As was done for the matrices X_i, we wish to express the auxiliary matrices X_i' in terms of tensor products of the fundamental operators A, δ and ε satisfying the defining relations (3.2). We have seen, in (3.14), that $\boldsymbol{X} = \mathbb{L}^{(3)} \, \mathbb{L}^{(2)}$. A similar representation for $\boldsymbol{X'}$ is given by

$$\boldsymbol{X'} = \mathbb{L}^{(3)'} \, \mathbb{L}^{(2)} + \mathbb{L}^{(3)} \, \mathbb{L}^{(2)'} \tag{4.13}$$

where $\mathbb{L}^{(3)}$ and $\mathbb{L}^{(2)}$ have been defined in (3.13) and

$$\mathbb{L}^{(3)'} = \begin{pmatrix} 1 & \varepsilon_1 A_3 + \varepsilon_3 \\ 0 & 0 \\ -\delta_3 & -1 \end{pmatrix} \quad \text{and} \quad \mathbb{L}^{(2)'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{4.14}$$

13

As in (3.4), the generators $(\varepsilon_n, \delta_n, A_n)$, $n = 1, 2, 3, 4$ are commuting copies of the algebra (3.2). At first sight, the sum expression (4.13) for $\boldsymbol{X'}$ may seems a bit arbitrary but this form that is reminiscent of the derivative of a product will appear natural in section 5. Now, using (3.14) and (4.13), one can verify by a direct but lengthy calculation that the relations (4.6) and (4.7) are satisfied and thus prove the matrix Ansatz. However, the calculation can be simplified using some factorisations, as explained in the next subsection.

4.4 Synthetic proof of the auxiliary algebra

We have shown in Section 3.3 that the matrices $\mathbb{L}^{(3)}$ and $\mathbb{L}^{(2)}$ can be factorized, see equations (3.15)–(3.18). We have a corresponding property that holds for $\mathbb{L}^{(3)'}$:

$$\mathbb{L}^{(3)'} = L^{(3)'}\widetilde{L}^{(3)} + L^{(3)}\widetilde{L}^{(3)'} \tag{4.15}$$

with

$$L^{(3)'} = \begin{pmatrix} 1 & \varepsilon_1 & \varepsilon_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{L}^{(3)'} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\delta_3 & -1 \end{pmatrix} \tag{4.16}$$

and where $L^{(3)}$ and $\widetilde{L}^{(3)}$ are given in (3.16).

Using the explicit form of these operators, one can verify the following relations

$$m^{(3)}L^{(3)} \otimes L^{(3)} - L^{(3)} \otimes L^{(3)}m^{(3)} = L^{(3)'} \otimes L^{(3)} - L^{(3)} \otimes L^{(3)'} \tag{4.17}$$

$$m^{(3)}\widetilde{L}^{(3)} \otimes \widetilde{L}^{(3)} - \widetilde{L}^{(3)} \otimes \widetilde{L}^{(3)}m^{(2)} = \widetilde{L}^{(3)'} \otimes \widetilde{L}^{(3)} - \widetilde{L}^{(3)} \otimes \widetilde{L}^{(3)'} \tag{4.18}$$

where $m^{(2)}$ is the local operator for the one-species TASEP, given in (4.2). Recalling that $\mathbb{L}^{(3)} = L^{(3)}\widetilde{L}^{(3)}$ and $\mathbb{L}^{(3)'} = L^{(3)'}\widetilde{L}^{(3)} + L^{(3)}\widetilde{L}^{(3)'}$, we conclude that

$$m^{(3)}\mathbb{L}^{(3)} \otimes \mathbb{L}^{(3)} - \mathbb{L}^{(3)} \otimes \mathbb{L}^{(3)}m^{(2)} = \mathbb{L}^{(3)'} \otimes \mathbb{L}^{(3)} - \mathbb{L}^{(3)} \otimes \mathbb{L}^{(3)'} . \tag{4.19}$$

Similarly, from the factorisation of $\mathbb{L}^{(2)}$, (3.17) and (3.18), we can check that

$$m^{(2)}\mathbb{L}^{(2)} \otimes \mathbb{L}^{(2)} = \mathbb{L}^{(2)'} \otimes \mathbb{L}^{(2)} - \mathbb{L}^{(2)} \otimes \mathbb{L}^{(2)'} , \tag{4.20}$$

where $\mathbb{L}^{(2)'}$ has been defined in (4.14).

Combining equations (4.19) and (4.20) with expressions (4.13) ends the proof of the bulk relation (4.6).

The following identities hold for $\mathbb{L}^{(3)}$

$$\langle W|_{123}\Big(B\mathbb{L}^{(3)} - \mathbb{L}^{(3)}B^{(2)}\Big) = -\langle W|_{123}\mathbb{L}^{(3)'} \tag{4.21}$$

$$\Big(\overline{B}\mathbb{L}^{(3)} - \mathbb{L}^{(3)}\overline{B}^{(2)}\Big)|V\rangle_{123} = \mathbb{L}^{(3)'}|V\rangle_{123} . \tag{4.22}$$

Similar relations also exist for $\mathbb{L}^{(2)}$

$$\langle 0|_4 B^{(2)}\mathbb{L}^{(2)} = -\langle 0|_4 \mathbb{L}^{(2)'} \tag{4.23}$$

$$\overline{B}^{(2)}\mathbb{L}^{(2)}|b\rangle_4 = \mathbb{L}^{(2)'}|b\rangle_4 . \tag{4.24}$$

Combining equations (4.21)-(4.24) leads us to the boundary relations (4.7).

Finally, from the identities (4.19), (4.21) and (4.22), we deduce that $M^{(3)}\mathbb{P}^{(3)} = \mathbb{P}^{(3)}M^{(2)}$, so that we obtain

$$M^{(3)}\boldsymbol{P} = M^{(3)}\mathbb{P}^{(3)}\mathbb{P}^{(2)} = \mathbb{P}^{(3)}M^{(2)}\mathbb{P}^{(2)} = 0 \,, \tag{4.25}$$

where we have used (3.23). This concludes the proof that $\boldsymbol{P} = \frac{1}{Z_L}\mathbb{P}^{(3)} \, \mathbb{P}^{(2)}$ is the stationary state of the 2-TASEP with open boundaries.

5 Zamolodchikov-Faddeev algebra and Ghoshal-Zamolodchikov relation

In this section, we show that the relations used in section 4.4 can be obtained from a more general framework. The relations introduced in this section will depend on an additional parameter, called the spectral parameter. The relations of the previous section will be recovered by setting this parameter to a specific value.

The main objects necessary in this section are similar to those used to prove the integrability of the Markov matrix $M^{(3)}$ (see [43] for historical paper or [17] for the case treated here). We need the R-matrix encoding the bulk dynamics and the K-matrices encoding the boundaries rates.

For the 2-species TASEP, the braided R-matrix reads $\check{R}^{(3)}(x) = 1 + (1-x)\,m^{(3)}$ with the property $-\check{R}^{(3)'}(1) = m^{(3)}$. The K-matrix for the left boundary is

$$K(x) = \begin{pmatrix} x^2 & 0 & 0 \\ -\frac{ax(x^2-1)}{xa+1} & \frac{x(a+x)}{xa+1} & 0 \\ -\frac{x^2-1}{xa+1} & -\frac{x^2-1}{xa+1} & 1 \end{pmatrix} \tag{5.1}$$

and the ones for the two choices of right boundary are

$$\widehat{K}(x) = \begin{pmatrix} 1 & \frac{x^2-1}{(b+x)x} & \frac{x^2-1}{(b+x)x} \\ 0 & \frac{bx+1}{(b+x)x} & \frac{b(x^2-1)}{(b+x)x^2} \\ 0 & 0 & x^{-2} \end{pmatrix} \quad \text{and} \quad \widetilde{K}(x) = \begin{pmatrix} 1 & \frac{x^2-1}{(b+x)x} & \frac{x^2-1}{(b+x)x} \\ 0 & \frac{bx+1}{(b+x)x} & 0 \\ 0 & 0 & \frac{bx+1}{(b+x)x} \end{pmatrix}. \tag{5.2}$$

One obtains

$$-\frac{1}{2}K'(1) = B \quad , \quad \frac{1}{2}\widehat{K}'(1) = \widehat{B} \quad \text{and} \quad \frac{1}{2}\widetilde{K}'(1) = \widetilde{B}. \tag{5.3}$$

We also define the Lax operators by $\mathbb{L}^{(3)}(x) = L^{(3)}(x)\widetilde{L}^{(3)}(x)$ where

$$L^{(3)}(x) = \begin{pmatrix} x + \lambda A_1 A_2 & x\varepsilon_1 & x\varepsilon_2 \\ \lambda\delta_1 A_2 & \lambda A_2 & 0 \\ \delta_2 & \varepsilon_1\delta_2 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{L}^{(3)}(x) = \begin{pmatrix} 1 & & \varepsilon_3 \\ 0 & A_3 & \\ \delta_3/x & 1/x \end{pmatrix}, \tag{5.4}$$

and $\mathbb{L}^{(2)}(x) = L^{(2)}(x)\widetilde{L}^{(2)}(x)$ where

$$L^{(2)}(x) = \begin{pmatrix} x & x\varepsilon_4 \\ \delta_4 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{L}^{(2)}(x) = \begin{pmatrix} 1 & \\ & 1/x \end{pmatrix}. \tag{5.5}$$

15

Finally, we introduce the 3-component vector

$$\boldsymbol{X}(x) = \mathbb{L}^{(3)}(x)\mathbb{L}^{(2)}(x). \tag{5.6}$$

The matrices introduced in this section depend on the supplementary parameter x (the spectral parameter). They lead to the matrices used previously to construct the matrix ansatz by remarking that

$$\boldsymbol{X} = \boldsymbol{X}(1) \quad, \quad \boldsymbol{X'} = \frac{d}{dx}\boldsymbol{X}(x)\Big|_{x=1} \quad, \quad L^{(3)} = L^{(3)}(1) \quad, \quad L^{(3)'} = \frac{d}{dx}L^{(3)}(x)\Big|_{x=1},\dots \tag{5.7}$$

As mentioned previously, the introduction of this spectral parameter makes some of our previous definitions now appear natural. For example, the factorized form of $\mathbb{L}^{(3)}(x) = L^{(3)}(x)\widetilde{L}^{(3)}(x)$ implies

$$\mathbb{L}^{(3)'} = \frac{d}{dx}\mathbb{L}^{(3)}(x)\Big|_{x=1} = L^{(3)'}\widetilde{L}^{(3)} + L^{(3)}\widetilde{L}^{(3)'} \tag{5.8}$$

and reproduces relation (4.15).

The results obtained in section 4 can be deduced from the two main relations (proved in appendix C):

$$\check{R}^{(3)}(x_1/x_2)\boldsymbol{X}(x_1) \otimes \boldsymbol{X}(x_2) = \boldsymbol{X}(x_2) \otimes \boldsymbol{X}(x_1), \tag{5.9}$$

$$\langle W|K(x)\,\boldsymbol{X}(1/x) = \langle W|\boldsymbol{X}(x) \quad \text{and} \quad \overline{K}(x)\boldsymbol{X}(1/x)|V\rangle = \boldsymbol{X}(x)|V\rangle, \tag{5.10}$$

where $\overline{K}(x)$ is $\widehat{K}(x)$ or $\widetilde{K}(x)$ depending on the right boundary considered. Taking the derivative of these relations w.r.t. x_1 and setting $x_1 = x_2$, we recover the main relations (4.6) and (4.7) used to prove the matrix ansatz. In the context of integrable quantum field theory, an equation such as (5.9) is usually called a Zamolodchikov-Faddeev (ZF) relation and equations (5.10) are called Ghoshal-Zamolodchikov (GZ) relations. For more details about the use of these relations in the context of Markov chains see [41, 18].

6   Conclusion

In this paper, the stationary state of the open 2-species TASEP in the case of integrable boundary conditions (2.2), (2.3) is computed using a matrix product ansatz. We find that the 'matrices' are in fact four-fold tensor products whose generators are expressed in terms of more fundamental generators introduced to study the 1-species TASEP. However in the stationary state there are various factorisation properties which reduce the complexity of the calculations. We also show the utility of these expressions by computing exactly the normalisation factor of the stationary state.

The two-species models we have considered are distinguished by the the integrability of the boundary conditions [17]. Here, we have further shown that the proof of the matrix product state involves relations that can be recovered from equations of the Zamolodchikov-Faddeev and Ghoshal-Zamolodchikov types.

In the case of the periodic boundary conditions, a similar tensor-product construction to that employed here has been used for the totally asymmetric and partially asymmetric simple exclusion

process with N-species [26, 39, 6, 13]. The present paper takes a first step to generalizing these results to the case of integrable open boundary conditions (see [16] for integrable boundaries of the N-species ASEP). In particular, we believe that the factorisation scheme proposed in this paper, using $\mathbb{L}$ and K matrices of decreasing sizes, see (3.14) and (C.8), will remain valid in the N-species case and allow the matrix ansatz for an N-species model to be constructed from that for an (N-1)-species model. Let us also mention that the queueing interpretation [31, 32, 26, 39, 6] provides a clear understanding of the role of the different spaces used for the fundamental generators in the case of the periodic boundary conditions. It would be of interest to see whether a similar analysis may also be carried out for the open boundary case presented here, which would then clarify the raison d'être of each of the spaces needed in our construction.

Besides, the matrix ansatz for the one-species ASEP was used in [37] to obtain the Baxter's Q-operator of the model. We believe that the matrix ansatz found here may be useful to construct the Q-operator for the multi-species TASEP.

Finally, another interpretation of the matrix ansatz was proposed recently for the multi-species TASEP with periodic boundary conditions [36], leading to new relations between the tetrahedron equation, the stochastic R-matrix and the matrix ansatz. It would be of interest to see whether the construction of [36] may be generalized to the cases with integrable boundaries. In turn this might reveal a 3D integrability in the matrix ansatz proposed here.

A   Relation between L and $\widetilde{L}$

In this section, we show how $\widetilde{L}$ can be obtained from L.

We define the transposition in the space of generators as follows

$$\varepsilon^t = \delta \quad , \quad \delta^t = \varepsilon \quad , \quad A^t = A \quad \text{and} \quad \langle x|^t = |x\rangle \tag{A.1}$$

Let us remark that starting from an $L^{(3)}$ solution to the relation (4.17), the matrix

$$\overline{L}^{(3)} = U L^{(3)^t} U \quad \text{where} \quad U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{A.2}$$

is also a solution of (4.17). We have used the following property of the local operator m

$$U_1 U_2 m_{21} U_1 U_2 = m . \tag{A.3}$$

Starting from the realisation (5.4) for $L^{(3)}$, one gets

$$\overline{L}^{(3)} = \begin{pmatrix} 1 & \delta_1\varepsilon_2 & \varepsilon_2 \\ 0 & \lambda A_2 & \lambda\varepsilon_1 A_2 \\ \delta_2 & \delta_1 & 1 + \lambda A_1 A_2 \end{pmatrix} . \tag{A.4}$$

The trivial representation for the ε, δ, A algebra is defined as $\varepsilon = \delta = 1$ and $A = 0$. These values are consistent with the relation (3.2) and the definition of A. In the $\overline{L}^{(3)}$ matrix, we may

17

choose the trivial representation for the generators in the space 1 (*i.e.* $\varepsilon_1 = \delta_1 = 1$ and $A_1 = 0$). Changing the name of space 2 to space 3 and putting $\lambda = 1$, one establishes a link with the matrix $\widetilde{L}^{(3)}$:

$$\overline{L}^{(3)}\Big|_{\varepsilon_1=\delta_1=1, A_1=0, \lambda=1} = \begin{pmatrix} 1 & \varepsilon_3 & \varepsilon_3 \\ 0 & A_3 & A_3 \\ \delta_3 & 1 & 1 \end{pmatrix} = \widetilde{L}^{(3)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} . \tag{A.5}$$

The procedure to choose the trivial representation to get a simpler matrix has been used previously for the periodic case in [13].

B Relation with the algebra found in [17]

In [17], the stationary state of the model (P_2) with $\alpha = \frac{1}{2}$ and $\beta = 1$ was constructed using an algebra based on 9 generators G_i (i=1,2,...,9). The algebra generated by the G_i's was shown to be well defined. However, no explicit realisation of that algebra was given. Using the matrix Ansatz found here, we give, in this appendix, an explicit representation of the G_i's. Using the following identification [17]

$$\boldsymbol{X}(x) = \begin{pmatrix} x^2 + G_9 x + G_8 + G_7/x \\ G_6 x + G_5 + G_4/x \\ G_3 x + G_2 + G_1/x + 1/x^2 \end{pmatrix} , \tag{B.1}$$

the generators X_0, X_1 and X_2 are given by

$$X_0 = 1 + G_7 + G_8 + G_9 \tag{B.2}$$
$$X_2 = G_4 + G_5 + G_6 \tag{B.3}$$
$$X_1 = 1 + G_1 + G_2 + G_3 . \tag{B.4}$$

We get the following realisation for the 9 generators G

$$G_1 = \delta_3\varepsilon_4 + \varepsilon_1\delta_2 A_3 + \delta_2\varepsilon_3 + \delta_4 ; \quad G_2 = \delta_3 + \delta_2\varepsilon_4 + \delta_2\varepsilon_3\delta_4 + \varepsilon_1\delta_2 A_3\delta_4 ; \quad G_3 = \delta_2 \tag{B.5}$$
$$G_4 = \lambda(\delta_1 A_2\varepsilon_3 + A_2 A_3) ; \quad G_5 = \lambda(\delta_1 A_2\varepsilon_4 + \delta_1 A_2\varepsilon_3\delta_4 + A_2 A_3\delta_4) ; \quad G_6 = \lambda\delta_1 A_2 \tag{B.6}$$
$$G_7 = \lambda A_1 A_2\varepsilon_3 + \varepsilon_2 \quad ; \quad G_8 = \lambda A_1 A_2\varepsilon_4 + \varepsilon_2\delta_3\varepsilon_4 + \lambda A_1 A_2\varepsilon_3\delta_4 + \varepsilon_2\delta_4 + \varepsilon_1 A_3 + \varepsilon_3 \tag{B.7}$$
$$G_9 = \lambda A_1 A_2 + \varepsilon_2\delta_3 + \varepsilon_4 + \varepsilon_1 A_3\delta_4 + \varepsilon_3\delta_4 \tag{B.8}$$

We checked using a symbolic calculation program [45] that the representation presented here indeed obeys the commutation relations given in [17].

C Algebraic proof of relations (5.9) and (5.10)

We can prove relations (5.9) and (5.10) by direct computations. However, using the factorisation (5.6), we can split the proof of these relations into simpler ones.

C.1 ZF relations

One can show that the following relations hold

$$\check{R}^{(3)}(x_1/x_2)L^{(3)}(x_1) \otimes L^{(3)}(x_2) = L^{(3)}(x_2) \otimes L^{(3)}(x_1)\check{R}^{(3)}(x_1/x_2), \tag{C.1}$$

$$\check{R}^{(2)}(x_1/x_2)L^{(2)}(x_1) \otimes L^{(2)}(x_2) = L^{(2)}(x_2) \otimes L^{(2)}(x_1)\check{R}^{(2)}(x_1/x_2), \tag{C.2}$$

$$\check{R}^{(3)}(x_1/x_2)\widetilde{L}^{(3)}(x_1) \otimes \widetilde{L}^{(3)}(x_2) = \widetilde{L}^{(3)}(x_2) \otimes \widetilde{L}^{(3)}(x_1)\check{R}^{(2)}(x_1/x_2), \tag{C.3}$$

$$\check{R}^{(2)}(x_1/x_2)\widetilde{L}^{(2)}(x_1) \otimes \widetilde{L}^{(2)}(x_2) = \widetilde{L}^{(2)}(x_2) \otimes \widetilde{L}^{(2)}(x_1), \tag{C.4}$$

where we used the braided R-matrix for the single-species TASEP, built on the local operator $m^{(2)}$ (see (4.2)):

$$\check{R}^{(2)}(x) = 1 + (1 - x)\, m^{(2)}. \tag{C.5}$$

These identities imply

$$\check{R}^{(3)}(x_1/x_2)\mathbb{L}^{(3)}(x_1) \otimes \mathbb{L}^{(3)}(x_2) = \mathbb{L}^{(3)}(x_2) \otimes \mathbb{L}^{(3)}(x_1)\check{R}^{(2)}(x_1/x_2), \tag{C.6}$$

$$\check{R}^{(2)}(x_1/x_2)\mathbb{L}^{(2)}(x_1) \otimes \mathbb{L}^{(2)}(x_2) = \mathbb{L}^{(2)}(x_2) \otimes \mathbb{L}^{(2)}(x_1). \tag{C.7}$$

Let us remark that taking the derivative of these relations w.r.t. x_1 and setting $x_1 = x_2$, we recover the relations we used previously. For instance, (C.1) implies (4.17), and (C.6) implies (4.19).

Finally, using (C.6) and (C.7), we prove (5.9).

C.2 GZ relations

The following relations hold

$$\langle W|_{123}\, K(x)\mathbb{L}^{(3)}(1/x) = \langle W|_{123}\, \mathbb{L}^{(3)}(x)K^{(2)}(x) \quad ; \quad \langle W|_4\, K^{(2)}(x)\mathbb{L}^{(2)}(1/x) = \langle W|_4\, \mathbb{L}^{(2)}(x)$$

$$\overline{K}(x)\mathbb{L}^{(3)}(1/x)\, |V\rangle_{123} = \mathbb{L}^{(3)}(x)\overline{K}^{(2)}(x)\, |V\rangle_{123} \quad ; \quad \overline{K}^{(2)}(x)\mathbb{L}^{(2)}(1/x)\, |V\rangle_4 = \mathbb{L}^{(2)}(x)\, |V\rangle_4\,, \tag{C.8}$$

where we have introduced the K-matrices for the single-species open TASEP

$$K^{(2)}(x) = \begin{pmatrix} x^2 & 0 \\ 1 - x^2 & 1 \end{pmatrix} \quad \text{and} \quad \overline{K}^{(2)}(x) = \begin{pmatrix} 1 & \frac{x^2-1}{(b+x)x} \\ 0 & \frac{bx+1}{(b+x)x} \end{pmatrix}. \tag{C.9}$$

These reflection matrices are related to the boundary matrices through

$$-\frac{1}{2}\frac{d}{dx}K^{(2)}(x)\bigg|_{x=1} = B^{(2)} \quad , \quad \frac{1}{2}\frac{d}{dx}\overline{K}^{(2)}(x)\bigg|_{x=1} = \overline{B}^{(2)}. \tag{C.10}$$

Let us remark that taking the derivative of relations (C.8) w.r.t. x and setting $x = 1$, we recover the relations (4.21)-(4.24).

Finally, relations (C.8) imply equations (5.10).

References

[1] F.C. Alcaraz and R.Z. Bariev, *Exact solution of the asymmetric exclusion model with particles of arbitrary size,* Phys. Rev. E **60**, (1999) 79.

[2] O. Angel, *The stationary measure of a 2-type totally asymmetric exclusion process*, J. Comb. Th. A **113**, (2006) 625.

[3] C. Arita, *Exact analysis of two-species asymmetric exclusion process with open boundary conditions,* J. Phys. Soc. Jpn. **75**, (2006) 065003.

[4] C. Arita, *Phase transitions in the two-species totally asymmetric exclusion process with open boundaries,* J. Stat. Mech. (2006) P12008.

[5] C. Arita, A. Kuniba, K. Sakai and T. Sawabe, *Spectrum of a multi-species asymmetric simple exclusion process on a ring*, J. Phys. A: Math. Theor. **42**, (2009) 345002.

[6] C. Arita, A. Ayyer, K. Mallick and S. Prolhac, *Recursive structures in the multi-species TASEP*, J. Phys. A: Math. Theor. **44**, (2011) 335004.

[7] C. Arita, A. Ayyer, K. Mallick and S. Prolhac, *Generalized matrix Ansatz in the multi-species exclusion process - the partially asymmetric case*, J. Phys. A: Math. Theor. **45**, (2012) 195001.

[8] A. Ayyer, J.L. Lebowitz and E.R. Speer, *On the Two Species Asymmetric Exclusion Process with Semi-Permeable Boundaries,* J. Stat. Phys. **135**, (2009) 1009.

[9] A. Ayyer, J.L. Lebowitz and E.R. Speer, *On Some Classes of Open Two-Species Exclusion Processes,* Markov Processes and Related Fields **18**, (2012) 157.

[10] R.A. Blythe and M.R. Evans, *Nonequilibrium steady states of matrix-product form: a solver's guide*, J. Phys. A: Math. Theor. **40**, (2007) R333.

[11] R. Brak, S. Corteel, J. Essam, R. Parviainen and A. Rechnitzer, *A combinatorial derivation of the PASEP stationary state*, Electron. J. Comb. **13**, (2006) R108.

[12] L. Cantini, *Algebraic Bethe Ansatz for the two species ASEP with different hopping rates*, J. Phys. A: Math. Theor. **41**, (2009) 095001.

[13] L. Cantini, J. de Gier and M. Wheeler, *Matrix product formula for Macdonald polynomials*, J. Phys. A: Math. Theor. **48**, (2015) 384001.

[14] T. Chou, K. Mallick and R.K.P. Zia, *Non-equilibrium statistical mechanics: From a paradigmatic model to biological transport*, Rep. Prog. Phys. **74**, (2011) 116601.

[15] N. Crampe, *Algebraic Bethe Ansatz for the totally asymmetric simple exclusion process with boundaries*, J. Phys. A: Math. Theor. **48**, (2015) 08FT01.

[16] N. Crampe, C. Finn, E. Ragoucy and M. Vanicat, *Integrable boundary conditions for multi-species ASEP,* `arXiv:1606.01018`.

[17] N. Crampe, K. Mallick, E. Ragoucy and M. Vanicat, *Open two-species exclusion processes with integrable boundaries,* J. Phys. A: Math. Theor. **48**, (2015) 175002.

[18] N. Crampe, E. Ragoucy and M. Vanicat, *Integrable approach to simple exclusion processes with boundaries. Review and progress*, J. Stat. Mech. (2014) P11032.

[19] J. de Gier and F. Essler, *Bethe Ansatz Solution of the Asymmetric Exclusion Process with Open Boundaries,* Phys. Rev. Lett. **95**, (2005) 240601;
Exact spectral gaps of the asymmetric exclusion process with open boundaries, J. Stat. Mech. (2006) P12011.

[20] B. Derrida, E. Domany and D. Mukamel, *An exact solution of a one-dimensional asymmetric exclusion model with open boundaries*, J. Stat. Phys. **69**, (1992) 667.

[21] B. Derrida and M.R. Evans, Chapter 14 in *Nonequilibrium Statistical Mechanics in One Dimension*, V. Privman Ed. (Cambridge Univ. Press, 1997).

[22] B. Derrida and M.R. Evans, *Bethe Ansatz Solution for a Defect Particle in the Asymmetric Exclusion Process*, J. Phys. A: Math. Gen. **32**, (1999) 4833.

[23] B. Derrida, M.R. Evans, V. Hakim and V. Pasquier, *Exact solution of a 1d asymmetric exclusion model using a matrix formulation*, J. Phys. A: Math. Gen. **26**, (1993) 1493.

[24] B. Derrida, S.A. Janowsky, J.L. Lebowitz and E.R. Speer, *Exact solution of the totally asymmetric exclusion process: shock profiles*, J. Stat. Phys. **73**, (1993) 813.

[25] E. Duchi and G. Schaeffer, *A combinatorial approach to jumping particles*, J. Combin. Theory A **110**, (2005) 1.

[26] M.R. Evans, P.A. Ferrari and K. Mallick, *Matrix Representation of the Stationary Measure for the multi-species TASEP*, J. Stat. Phys. **135**, (2009) 217.

[27] M.R. Evans, D.P. Foster, C. Godrèche and D. Mukamel, *Spontaneous Symmetry Breaking in a One Dimensional Driven Diffusive System*, Phys. Rev. Lett. **74**, (1995) 208.

[28] M.R. Evans, D.P. Foster, C. Godrèche and D. Mukamel, *Asymmetric exclusion model with two species: spontaneous symmetry-breaking*, J. Stat. Phys **80**, (1995) 69.

[29] P.A. Ferrari, *Microscopic shocks in one dimensional driven system*, Ann. Inst. Henri. Poinc. **55**, (1991) 637.

[30] P.A. Ferrari, C. Kipnis and E. Saada, *Microscopic structure of travelling waves in the asymmetric simple exclusion process*, Ann. Prob. **19**, (1991) 226.

[31] P.A. Ferrari and J.B. Martin, *Multiclass processes, dual points and M/M/1 queues*, Markov Processes Rel. Fields **12**, (2006) 175.

[32] P.A. Ferrari and J.B. Martin, *Stationary distributions of multi-type totally asymmetric exclusion processes*, Ann. Prob. **35**, (2007) 807.

[33] H. Hinrichsen, S. Sandow and I. Peschel, *On matrix product ground states for reaction - diffusion models,* J. Phys. A: Math. Gen. **29**, (1996) 2643.

[34] K. Krebs and S. Sandow, *Matrix Product Eigenstates for One-Dimensional Stochastic Models and Quantum Spin Chains,* J. Phys. A: Math. Gen. **30**, (1997) 3165.

[35] J. Krug, *Boundary-induced phase transitions in driven diffusive systems*, Phys. Rev. Lett. **67**, (1991) 1882.

[36] A. Kuniba, S. Maruyama and M. Okado, *Multispecies TASEP and the tetrahedron equation,* J. Phys. A: Math. Theor. **49**, (2016) 114001;
Multispecies TASEP and combinatorial R, J. Phys. A: Math. Theor. **48**, (2015) 34FT02.

[37] A. Lazarescu and V. Pasquier, *Bethe Ansatz and Q-operator for the open ASEP,* J. Phys. A: Math. Theor. **47**, (2014) 295202 .

[38] K. Mallick, S. Mallick and N. Rajewsky, *Exact solution of an exclusion process with three classes of particles and vacancies*, J. Phys. A: Math. Gen. **32**, (1999) 8399.

[39] S. Prolhac, M. R. Evans and K. Mallick, *Matrix product solution of the multi-species partially asymmetric exclusion process*, J. Phys. A: Math. Theor. **42**, (2009) 165004.

[40] N. Rajewsky, L. Santen, A. Schadschneider and M. Schreckenberg, *The asymmetric exclusion process: Comparison of update procedures,* J. Stat. Phys. **92**, (1998) 151.

[41] T. Sasamoto and M. Wadati, *Stationary state of integrable systems in matrix product form,* J. Phys. Soc. Japan **66**, (1997) 2618.

[42] G. Schütz and E. Domany, *Phase transitions in an exactly soluble one-dimensional exclusion process,* J. Stat. Phys. **72**, (1993) 277.

[43] E.K. Sklyanin, *Boundary conditions for integrable quantum systems,* J. Phys. A.: Math. Gen. **21**, (1988) 2375.

[44] M. Uchiyama, *Two-Species Asymmetric Simple Exclusion Process with Open Boundaries,* Chaos, Solitons & Fractals **35**, (2008) 398.

[45] J. A. M. Vermaseren, *New features of FORM*, `arXiv:math-ph/0010025`.