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POWER ASPECTS OF ANALYSIS OF VARIANCE IN VARIOUS MODELS

by

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POWER ASPECTS OF ANALYSIS OF VARIANCE IN VARIOUS MODELS

Summary

The object of the present work is to study the robustness of the power in Analysis of Variance in relation to the departures from the in-built assumptions (i) equality of variance of the errors, (ii) statistical independence of the errors, and (iii) normality of the errors in fixed and random effects models. It is difficult if not impossible, to conduct an exhaustive study of the problem, because the above assumptions can be violated in many ways. However, a general model and some important particular models have been used to obtain fairly conclusive evidence regarding the robustness of the power in Analysis of Variance.

In order to obtain the power value in relation to the departure from the usual test assumptions, the general linear hypothesis model is considered. The power values when the assumptions of equality of variances and independence of errors are violated, are obtained and presented in Table IA and IB. The result suggests that in the above model, for tests regarding the inference about means, the power value is greatly affected by the inequality of error variances but only slightly affected by the serially correlated error variables. By using the permutation theory an approximate method is developed to study the effect of non-normality of the errors on the probability of type two errors in the above situation.

Having studied the most general case in Analysis of Variance some particular models are discussed to investigate certain important aspects of the problem that are generated by these models.

First of all fixed model one-way classification is considered to investigate whether it could show a different picture for unequal replication. The results so obtained are presented in Table IIA and IIB. They indicate that the power value is greatly affected by the inequality of error variances and unequal group sizes. This procedure is easily modified to handle the random model.

Another particular case of the general linear model, that is fixed effect model two-way classification, is discussed. The results so obtained are presented in Table IIIA and IIIB. They indicate that in two-way classification for the between Column test, the power value is greatly affected by the inequality of column variances but only slightly affected by the serially correlated within rows error variables. Again this procedure is easily modified to handle the random model.

The use of simulation methods for calculating the power values in the case of non-normal errors is discussed. One and two-way classifications are considered for the fixed effect model. The Erlangian and contaminated normal distribution are taken as examples of a non-normal error distribution. The results obtained by these methods are given in Table IVA and IVB which indicate that for the inference concerning means, the power calculated under normal theory is only slightly affected by the non-normality of the errors.

Finally, the effect of non-normality on the power in analysis of variance for a random effect model is also discussed by a simulation method. One and two-way classification are considered for this model and the Erlangian and contaminated normal distributions are taken as examples of non-normality. The results obtained by these methods are given in Tables VA and VB which indicate that non-normality has little effect on the power of the test.

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CONTENTS

1. INTRODUCTION

1.1 Historical Background

The assumptions usually associated with analysis of variance are that the errors in the measurements (i) have equal variances, (ii) are statistically independent and (iii) are normally distributed.

Box (1953) introduced the term 'Robust' to denote a statistical procedure which is insensitive to departures from assumptions underlying the model on which it is based. Such procedures are in common use, and several studies of robustness have been carried out in the field of 'Analysis of Variance'.

Numerous attempts have been made to study the effects of departures from the usual test assumptions on Analysis of Variance techniques. For example, the effect of departure from normality in the distribution of the error term was studied for a one-way layout by Pearson (1931), Geary (1947) and Gayen (1950). David and Johnson (1951) considered the extent to which the non-normality of the error distribution affects the F test. The test in general has been found very insensitive to non-normality of errors. Welch (1938) studied the effect of unequal group variances on the 't' test. His results indicate that when the groups are of equal size the effect is small, but this effect becomes larger when the groups are of unequal size. Hsu (1938(a)) also attempted to find the exact probability for this case. Gronow (1951) carried out the investigation using a different approximating method. Both of their investigations supported Welch's finding. Horsnell (1953) brought David & Johnson's work a step further, and considered the effect of unequal group variances on the power of the test for a special case of the one-way layout. The method used by David & Johnson is only approximate.

Box (1954(a),(b)) discussed the effect on tests of the null-hypothesis in Analysis of Variance of departures from the assumptions that errors (i) have equal variances and (ii) are statistically independent. The result he obtained for the one-way layout shows that if the group variances are unequal, and group sizes are equal, then the test is not seriously affected. In the two-way layout, when the error variances are unequal from column to column, then there is an increased chance of exceeding the significance level for the test that column means are equal. For the corresponding test on row means, the chance of exceeding the significance level is decreased. For small differences in the variances neither effect is large. First order serial correlation within rows affects the between rows comparison more than the between columns comparison.

Ito and Schull (1964) investigated the robustness of the T_O^2 test in multivariate analysis of variance when variance and co-variance matrices are not equal. They showed that, for large samples of equal size and moderate inequality of variance and co-variance matrices, the test is not seriously affected but that for unequal size the effects are quite large. Murphy (1967) used a simulation method for his study of the two sample test when the variances are unequal. His investigation indicates that the permutation test and 't' test are virtually identical in practice and are fairly robust to inequality of variances as long as sample sizes are equal.

The statistically important problem of the distribution of homogeneous positive quadratic forms has been discussed in detail by Robbins (1948), Robbins and Pitman (1949), and Hotelling (1948).

The more difficult distributions of non-homogeneous quadratic forms have been investigated by Solomon (1961). Ruben (1962) has obtained a very general result, expressing the distribution of both homogeneous and non-homogeneous quadratic forms as an infinite linear combination of chi-square distributions with arbitrary scale parameters. He has also expressed the non-homogeneous quadratic form as an infinite linear combination of non-central chi-square distributions with arbitrary scale parameters.

Box (1954) discussed the effect on tests of null-hypothesis in analysis of variance when the in-built assumptions other than the normality of errors are violated. He has ennunciated certain theorems concerning the distribution of relevant quadratic forms and applied his results to determine the effect of inequality of group variances in one way layout.

The permutation theory which provides a method for deriving robust criteria was first discussed by Box and Andersen (1955). When the errors are non-normal, Box and Watson (1962) developed an approximate method for studying the robustness of the regression test in the null-hypothesis case. Through an approximation to the permutation test, they adjusted for non-normality by modifying the degress of freedom of the usual F-test in Analysis of variance. The extent of the adjustment provided a means of assessing the effect of non-normality though little work has appeared on how the test's power is affected.

1.2 Relationship of this thesis to earlier work

In this thesis with the help of certain theorems due to Ruben (1962), a distribution of the ratio of two independent quadratic forms is obtained and has been referred to as a generalised incomplete beta distribution. It is then applied to investigate the effect of unequal error variances and serially correlated errors on the power in the general linear model, in one-way and two-way layout analysis of variance for fixed and random effect models.

This thesis differs from most other works in this field in that it is concerned with the direct approach and is more accurate than those of previous authors. In particular, it is shown that Tang's (1938) result for the power of the test can be easily obtained as a special case.

Using permutation theory and the generalised incomplete Beta distribution introduced earlier, a convenient method is devised to calculate the power values for the general linear hypothesis model. Unlike others this method provides power values for a desired non-centrality parameter and degrees of freedom to study the robustness in analysis of variance. In particular it is shown that the Welch (1938 page 152) result for the variance of E^2 for a limited population can be easily obtained as a special case.

In this thesis, unlike the previous authors (i.e. Geary, Gayen, David and Johnson) a simulation method is used to investigate the sensitivity of the power of the test for the non-normality of the error distribution in one and two-way layout analysis of variance. Both fixed and random effect models are considered and the Erlangian and contaminated normal distributions are used for non-normal distributions.

1.3 Note

Some of the results presented in this thesis have already appeared in various journals. Copies of the relevant papers are included at the end of the thesis.

2.1 Estimation of the parameters

The general linear model of full rank can be written as

$$\underset{\sim}{y} = \underset{\sim}{x}\underset{\sim}{\beta} + \underset{\sim}{e} \qquad (2.1.1)$$

where $\underset{\sim}{y}$ is a (n x 1) vector of observations, $\underset{\sim}{x}$ is a (n x p) matrix of known coefficients ($p \leqslant n$), $\underset{\sim}{\beta}$ is a (p x 1) vector of parameters and $\underset{\sim}{e}$ is a (n x 1) vector of 'error' random variables.

An assumption which is made on the $\underset{\sim}{e}$ vector of random variables is that $\underset{\sim}{e}$ is distributed as $N(o,\sigma^2\underset{\sim}{I})$ where $\underset{\sim}{I}$ is a (n x n) unit matrix and σ^2 is unknown.

In order to investigate the effect of a departure from the usual test assumptions on the power in Analysis of Variance, we will consider the vector $\underset{\sim}{e}$ such that $\underset{\sim}{e}$ is distributed as $N(O,\sigma^2\underset{\sim}{\delta})$ where $\underset{\sim}{\delta}$ is an (n x n) unknown positive definite symmetric matrix and σ^2 a scale factor. This will allow for both heteroskedasticity (differing diagonal elements of $\underset{\sim}{\delta}$) and interdependence (non zero off diagonal elements of $\underset{\sim}{\delta}$) of the errors. Since the errors are normally distributed with expectation zero and variance covariance matrix of $\sigma^2\underset{\sim}{\delta}$, the sum of squares that would appear in the exponent of the likelihood function is

$$\frac{1}{2\sigma^2}\{(\underset{\sim}{y}-\underset{\sim}{x}\underset{\sim}{\beta})'\ \underset{\sim}{\delta}^{-1}\ (\underset{\sim}{y}-\underset{\sim}{x}\underset{\sim}{\beta})\}$$

This exponent will have to be minimized in order to maximize the likelihood function.

The likelihood equation is given by

$$f(\underset{\sim}{e},\underset{\sim}{\beta}\ \sigma^2\underset{\sim}{\delta}) = \frac{|\underset{\sim}{\delta}|^{-\frac{n}{2}}}{(2\pi\sigma^2)^{\frac{n}{2}}}\ \text{Exp}\left[-\frac{(\underset{\sim}{t}\underset{\sim}{y} - \underset{\sim}{t}\underset{\sim}{x}\underset{\sim}{\beta})'(\underset{\sim}{t}\underset{\sim}{y} - \underset{\sim}{t}\underset{\sim}{x}\underset{\sim}{\beta})}{2\sigma^2}\right] \qquad (2.1.2)$$

When $\underset{\sim}{\delta}^{-1} = \underset{\sim}{t}'\underset{\sim}{t}$, since any symmetric matrix can be split up into the product of triangular matrices, the maximum likelihood estimates of $\underset{\sim}{\beta}$ and σ^2 are

$$\hat{\underset{\sim}{\beta}} = (\underset{\sim}{x}'\underset{\sim}{\delta}^{-1}\underset{\sim}{x})^{-1}\underset{\sim}{x}'\underset{\sim}{\delta}^{-1}\underset{\sim}{y} \tag{2.1.3}$$

and

$$\hat{\sigma}^2 = \frac{(\underset{\sim}{t}\underset{\sim}{y}-\underset{\sim}{t}\underset{\sim}{x}\underset{\sim}{\beta})'(\underset{\sim}{t}\underset{\sim}{y}-\underset{\sim}{t}\underset{\sim}{x}\underset{\sim}{\beta})}{n} \tag{2.1.4}$$

since $E(\hat{\underset{\sim}{\beta}}) = \underset{\sim}{\beta}$, then $\underset{\sim}{\beta}$ is an unbiased estimate of $\underset{\sim}{\beta}$. It can also be proved that $E(\hat{\sigma}^2) = \frac{n-p}{n}\sigma^2$ and therefore $\hat{\sigma}^2$ is a biased estimate of σ^2. But

$$\tilde{\sigma}^2 = \frac{n}{n-p}\,\hat{\sigma}^2 = \frac{(\underset{\sim}{t}\underset{\sim}{y}-\underset{\sim}{t}\underset{\sim}{x}\hat{\underset{\sim}{\beta}})'(\underset{\sim}{t}\underset{\sim}{y}-\underset{\sim}{t}\underset{\sim}{x}\hat{\underset{\sim}{\beta}})}{n-p} \tag{2.1.5}$$

is an unbiased estimate of σ^2.

2.2 Test of Hypothesis

Testing the hypothesis $\underset{\sim}{\beta} = \underset{\sim}{\beta}^*$ in the model (2.1.1) is equivalent to testing simultaneously that each β_i equals a given constant β_i^*. In testing the hypothesis H_o: $\underset{\sim}{\beta} = \underset{\sim}{\beta}^*$ it is essential to devise a test function. For the evaluation of the power of the test, it is also necessary to know the distribution of the test function when the alternative hypothesis H_1: $\underset{\sim}{\beta} \neq \underset{\sim}{\beta}^*$ is true. Also we can test any sub-hypothesis $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}^*$ where the elements of $\underset{\sim}{\gamma}$ constitute a subset of the parameters and those of the $\underset{\sim}{\gamma}^*$ are given constant, (see, for example, Graybill (1961)p.135). This can be seen in a following chapter which will examine one-way and two-way layouts.

The likelihood ratio criterion that has been used to test the hypothesis can be expressed as

$$L = \left[\frac{o}{V_o + V_E}\right] = \left[\frac{1}{1 + \frac{V_E}{V_o}}\right] \qquad (2.2.1)$$

where $V_o = (ty - tx\hat{\beta})'(ty - tx\hat{\beta})$, $V_E = (tx\hat{\beta} - tx\beta^*)'(tx\hat{\beta} - tx\beta^*)$

Let,

$$\tau = \frac{V_E}{V_o} = \frac{(tx\hat{\beta} - tx\beta^*)'(tx\hat{\beta} - tx\beta^*)}{(ty - tx\hat{\beta})'(ty - tx\hat{\beta})} \qquad (2.2.2)$$

Since $M_1 = \left[tx(x't'tx)^{-1}\, x't'\right]$ is an idempotent matrix.

We therefore have

$$\tau = \frac{(ty - tx\beta^*)' M_1 \, (ty - tx\beta^*)}{(ty - tx\beta^*)'(I - M_1)(ty - tx\beta^*)} \qquad (2.2.3)$$

Let us denote the numerator and denominator of τ as q_1 and q_2 the two quadratic forms. In order to determine the rank of the matrix M_1 which is also the rank of the quadratic form q_1 we proceed as follows

$$\text{trace} \quad (M_1) = \text{tr. } tx(x't'tx)^{-1}\, x't' = p$$

Therefore the rank of M_1 is p and similarly the rank of $(I-M_1)$ is n-p, and hence q_1 and q_2 are positive semidefinite quadratic forms.

Since we are interested in knowing whether the two quadratic forms q_1 and q_2 are independent, we will express q_1 and q_2 as

$$q_1 = (z - \mu)' M_1 (z - \mu) \text{ and } q_2 = (z - \mu)'(I - M_1)(z - \mu)$$

where z is an n-dimensional vector distributed as multivariate normal distribution with expectation zero and variance covariance matrix v and M_1 is a positive semidefinite matrix, μ being a given vector.

Let ξ be the orthogonal matrix such that:

$$\xi' M_1 \xi = \Lambda$$

Now $(\underset{\sim}{z} - \underset{\sim}{\mu})'\underset{\sim}{I}(\underset{\sim}{z} - \underset{\sim}{\mu}) = (\underset{\sim}{z} - \underset{\sim}{\mu})'\underset{\sim}{M}_1(\underset{\sim}{z} - \underset{\sim}{\mu}) + (\underset{\sim}{z} - \underset{\sim}{\mu})'(\underset{\sim}{I} - \underset{\sim}{M}_1)(\underset{\sim}{z} - \underset{\sim}{\mu})$

let $\underset{\sim}{H} = \underset{\sim}{\xi}\underset{\sim}{z}$, $\underset{\sim}{\mu} = \underset{\sim}{\xi}\underset{\sim}{\eta}$

then $(\underset{\sim}{H} - \underset{\sim}{\eta})'(\underset{\sim}{H} - \underset{\sim}{\eta}) = (\underset{\sim}{H} - \underset{\sim}{\eta})'\underset{\sim}{\Lambda}(\underset{\sim}{H} - \underset{\sim}{\eta}) + (\underset{\sim}{H} - \underset{\sim}{\eta})'(\underset{\sim}{I} - \underset{\sim}{\Lambda})(\underset{\sim}{H} - \underset{\sim}{\eta})$

$$= \sum_{i=1}^{p} (H_i - \eta_i)^2 + \sum_{i=p+1}^{n} (H_i - \eta_i)^2 \qquad (2.2.4)$$

so that the quadratic form $(\underset{\sim}{z} - \underset{\sim}{\mu})'\underset{\sim}{M}_1(\underset{\sim}{z} - \underset{\sim}{\mu})$ and $(\underset{\sim}{z} - \underset{\sim}{\mu})'(\underset{\sim}{I} - \underset{\sim}{M}_1)(\underset{\sim}{z} - \underset{\sim}{\mu})$ are independent, i.e. the numerator q_1 and denominator q_2 of τ are mutually independent with rank p and (n - p) respectively.

2.3 Distribution of the quadratic forms

We now apply a theorem due to Ruben (1962) concerning the distribution of the quadratic form to find the distribution of q_1 and q_2. Now q_1 can be expressed as

$$q_1 = (\underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta}^*)'\underset{\sim}{M}_1^*(\underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta}^*) \qquad (2.3.1)$$

where

$$\underset{\sim}{M}_1^* = \underset{\sim}{\delta}^{-1}\underset{\sim}{x}(\underset{\sim}{x}'\underset{\sim}{\delta}^{-1}\underset{\sim}{x})^{-1}\underset{\sim}{x}'\underset{\sim}{\delta}^{-1}$$

since the $\underset{\sim}{y}$'s are distributed as $N(\underset{\sim}{x}\underset{\sim}{\beta}, \underset{\sim}{V})$ we therefore have that $\underset{\sim}{\Psi}^*$'s are distributed as $N(O,\underset{\sim}{V})$ where $\underset{\sim}{\Psi}^* = \underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta}$.
Hence substituting the value of $\underset{\sim}{y}$ in (2.3.1) we have

$$q_1 = (\underset{\sim}{\Psi}^* - \underset{\sim}{\mu}^*)'\underset{\sim}{M}_1^*(\underset{\sim}{\Psi}^* - \underset{\sim}{\mu}^*) \qquad (2.3.2)$$

where $\underset{\sim}{\mu}^* = (\underset{\sim}{x}\underset{\sim}{\beta} - \underset{\sim}{x}\underset{\sim}{\beta}^*)$

To achieve the required quadratic form for the application of Ruben (1962) theorem 1, we find that the linear transformation

$$\underset{\sim}{\Psi}^* = \underset{\sim}{N}\underset{\sim}{K}\underset{\sim}{x} \qquad \underset{\sim}{\mu}^* = \underset{\sim}{N}\underset{\sim}{K}\underset{\sim}{b}$$

changes the quadratic form q_1 to the canonical form given by $(\underset{\sim}{x} - \underset{\sim}{b})'\underset{\sim}{A}(\underset{\sim}{x} - \underset{\sim}{b})$. Where $\underset{\sim}{x}$'s are $N(0,\underset{\sim}{I})$ and $\underset{\sim}{N}$ is the lower triangular matrix defined by $\underset{\sim}{\delta}^{-1} = \underset{\sim}{V}^{-1} = \underset{\sim}{N}\underset{\sim}{N}'$ and $\underset{\sim}{K}$ is the orthogonal matrix of the eigen vectors of $\underset{\sim}{N}'\underset{\sim}{M}^*\underset{\sim}{N}$. The a_i's are the diagonal elements of the matrix $\underset{\sim}{A} = \underset{\sim}{K}'\underset{\sim}{N}'\underset{\sim}{M}_1^*\underset{\sim}{N}\underset{\sim}{K}$ and also the eigen values of $\underset{\sim}{N}'\underset{\sim}{M}_1^*\underset{\sim}{N}$ and $\underset{\sim}{b}$ is a fixed n dimensional vector. Since q_1 is a nonhomogenous quadratic form we can apply Ruben's (1962) theorem 1 (Appendix A) and we see that

$$H_{n',A,b,}(\alpha) = P[q_1 \leqslant \alpha] = \sum_{j=o}^{\infty} C_j \chi^2_{n'+2j}(\alpha/g) \quad (2.3.3)$$

where $n' = p$ is the rank of matrix $\underset{\sim}{M}_1^*$, g is an arbitrary constant and χ^2_{n+2j} (.) is a chi-square distribution. C_j can be calculated by the recursion relation given in the theorem. In equation (2.3.3) the expression $H_{n',A,b}(\alpha)$ is represented for $\underset{\sim}{b} \neq 0$ as a linear combination of central χ^2-distribution function. The noncentrality parameter (say λ) which specifies the alternative hypothesis can be obtained by using the vector $\underset{\sim}{b}$.

We now proceed to derive the distribution of the quadratic form q_2; we have

$$q_2 = (\underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta}^*)'\underset{\sim}{M}_2^*(\underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta}^*) \quad (2.3.4)$$

where

$$\underset{\sim}{M}_2^* = \underset{\sim}{\delta}^{-1} - \underset{\sim}{M}_1^*$$

Proceeding as for q_1, we find that we can apply the Ruben (1962) theorem 1 to find the distribution function of the quadratic form q_2. But in this case the noncentrality parameter λ is zero, and we therefore have $\underset{\sim}{b} = 0$ and hence q_2 is a homogeneous quadratic form. Applying theorem 2, we find that the distribution of q_2 is

$$H_{n',A,O}(\alpha) = P[q_2 \leqslant \alpha] = \sum_{j=o}^{\infty} d_j \chi^2_{n'+2j}(\alpha/g) \quad (2.3.5)$$

where $n' = n - p$, the rank of the matrix $\underset{\sim}{M}_2^*$.

2.4 Noncentrality Parameter

It is always desirable to express the noncentrality parameter λ in terms of $\underset{\sim}{\mu}^*$ and $\underset{\sim}{V}$. Therefore we proceed to relate the $\underset{\sim}{b}$'s in terms of $\underset{\sim}{\mu}^*$ and $\underset{\sim}{V}$ where $\underset{\sim}{\mu}^*$'s and $\underset{\sim}{V}$'s are as before. From the equations

$$\underset{\sim}{\Psi}^* = \underset{\sim}{N}\underset{\sim}{K}'\underset{\sim}{x} \qquad \underset{\sim}{\mu}^* = \underset{\sim}{N}\underset{\sim}{K}\underset{\sim}{b}$$

we have $\underset{\sim}{b} = \underset{\sim}{K}^{-1}\underset{\sim}{N}^{-1}\underset{\sim}{\mu}^* = \underset{\sim}{K}'\underset{\sim}{N}^{-1}\underset{\sim}{\mu}^*$ (2.4.1)

where $\underset{\sim}{K}$ is orthogonal matrix. Again we have

$$\underset{\sim}{V}^{-1} = \underset{\sim}{N}\underset{\sim}{N}' \quad \text{or} \quad \underset{\sim}{N}^{-1} = \underset{\sim}{N}'\underset{\sim}{V}$$

and hence

$$\underset{\sim}{b} = \underset{\sim}{K}'\underset{\sim}{N}^{-1}\underset{\sim}{\mu}^* = \underset{\sim}{K}'\underset{\sim}{N}'\underset{\sim}{V}\,\underset{\sim}{\mu}^*$$

Now $\lambda^2 = \frac{1}{2}(\underset{\sim}{b}'\underset{\sim}{b}) = \frac{1}{2}\Sigma bi^2$

$$\underset{\sim}{b}'\underset{\sim}{b} = \underset{\sim}{\mu}^{*\prime}\underset{\sim}{V}'\ \underset{\sim}{N}\underset{\sim}{K}\underset{\sim}{K}'\underset{\sim}{N}'\underset{\sim}{V}\ \underset{\sim}{\mu}^* = \underset{\sim}{\mu}^{*\prime}\underset{\sim}{V}'\ \underset{\sim}{N}\underset{\sim}{N}'\ \underset{\sim}{V}\ \underset{\sim}{\mu}^*$$

$$\underset{\sim}{b}'\underset{\sim}{b} = \underset{\sim}{\mu}^{*\prime}\underset{\sim}{V}'\ \underset{\sim}{\mu}^*$$

We therefore obtain $\lambda^2 = \frac{1}{2}\underset{\sim}{b}'\underset{\sim}{b} = \frac{1}{2}\underset{\sim}{\mu}^{*\prime}\underset{\sim}{V}'\underset{\sim}{\mu}^*$

2.5 Distribution of the ratio of quadratic forms

The distribution of q_1 and q_2 having been obtained in the preceding section, we require the distribution of the ratio of q_1 to q_2 i.e. distribution of τ. Since the g's in the equation (2.3.3) and (2.3.5) are arbitrary scale parameters, we can take value of g equal to unity in all cases.

It can also be noted that q_1 and q_2 are independently distributed as mixtures of central χ^2's so that the ratio q_1/q_2 is distributed as a mixture of ratios of central χ^2's. (See Appendix B).

Thus

$$P(\tau = q_1/q_2 \leqslant \alpha) = \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i \; F_{p+2j,n-p+2i}(\frac{n-p+2i}{p+2j}\alpha) \qquad (2.5.1)$$

and $F\nu,\tau(.)$ is an F distribution.

2.6 Power of the test

As it is easier to compute the incomplete Beta function than the F distribution, we express the series in (2.5.1) in terms of incomplete Beta function with the help of the identity

$$F_{m,n}(x) = I_{\frac{x}{1+x}}(\tfrac{1}{2}m, \tfrac{1}{2}n)$$

where

$I_x(.)$ is the incomplete Beta function.

The series (2.5.1) then can be written as

$$\rho(\tau = q_1/q_2 \leqslant \alpha) = \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i I_{\frac{\alpha}{1+\alpha}}(\frac{p+2j}{2}, \frac{n-p+2i}{2}) \qquad (2.6.1)$$

where I_α (p,q) is an incomplete Beta function.

Let $G = 1 - L = \dfrac{1}{1 + \frac{1}{\tau}} = \dfrac{\tau}{1 + \tau}$

then $P(G \leqslant D) = P(\frac{\tau}{1+\tau} \leqslant D) = P(\tau \leqslant \frac{D}{1-D})$

i.e.

$$P(G \leqslant \frac{\alpha}{1 + \alpha}) = P(\tau \leqslant \alpha) \text{ if we put } \frac{D}{1 - D} = \alpha$$

Hence,

$$P(G \leqslant \frac{\alpha}{1 + \alpha}) = \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i I_{\frac{\alpha}{1+\alpha}}(\frac{p+2j}{2}, \frac{n-p+2i}{2}) \qquad (2.6.2)$$

Let P_{11} be the type two error. Hence

$$P_{11} = P(\tau = q_1/q_2 \leqslant u_0 \,|\, \lambda \neq o)$$

$$p_{11} = \sum_{j=o} \sum_{i=o} C_j d_i I_{\frac{u_o}{1+u_o}} \left(\frac{p+2j}{2}, \frac{n-p+2i}{2}\right) \qquad (2.6.3)$$

is a generalised incomplete beta distribution,

where

$u_o = \frac{p}{n-p} F_\varepsilon$, and where ε is the level of significance.

Therefore the power of the test is given by

$$\beta(\lambda) = 1 - \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i I_{\frac{u_o}{1+u_o}} \left(\frac{p+2j}{2}, \frac{n-p+2i}{2}\right) \qquad (2.6.4)$$

The manner in which the calculation of p_{11} values is carried out is given in Appendix E.

3.1 Assumptions and Test Criterion

The general linear model of full rank can be written

$$\underset{\sim}{Y} = \underset{\sim}{x}\underset{\sim}{\beta} + \underset{\sim}{e} \tag{3.1.1}$$

where $\underset{\sim}{Y}$ is a (nx1) vector of observation, $\underset{\sim}{x}$ is a (nxp) matrix of known coefficients ($p \leq n$), $\underset{\sim}{\beta}$ is a (px1) vector of parameters and $\underset{\sim}{e}$ is a (nx1) vector of error random variables.

An assumption which is made on the $\underset{\sim}{e}$ vector of random variables is that $\underset{\sim}{e}$ is distributed as $N(\underset{\sim}{O},\underset{\sim}{V})$ where $\underset{\sim}{V} = \sigma^2\underset{\sim}{I}$, $\underset{\sim}{I}$ is a (nxn) unit matrix, and σ^2 is unknown. The estimate $\hat{\underset{\sim}{\beta}}$ of $\underset{\sim}{\beta}$ is then given by $\hat{\underset{\sim}{\beta}} = (\underset{\sim}{x}'\underset{\sim}{x})^{-1}\underset{\sim}{x}'\underset{\sim}{Y}$.

In testing the hypothesis $\underset{\sim}{\beta} = \underset{\sim}{\beta}^*$ in the model (3.1.1) we shall use the likelihood ratio criterion

$$L = \left[\frac{1}{1 + \frac{V_E}{V_o}}\right]^{\frac{n}{2}} \tag{3.1.2}$$

where $V_o = (\underset{\sim}{Y} - \underset{\sim}{x}\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - \underset{\sim}{x}\hat{\underset{\sim}{\beta}})$, $V_E = (\underset{\sim}{x}\hat{\underset{\sim}{\beta}} - \underset{\sim}{x}\underset{\sim}{\beta}^*)'(\underset{\sim}{x}\hat{\underset{\sim}{\beta}} - \underset{\sim}{x}\underset{\sim}{\beta}^*)$.

Let

$$T = \frac{V_E}{V_o} = \frac{(\underset{\sim}{x}\hat{\underset{\sim}{\beta}} - \underset{\sim}{x}\underset{\sim}{\beta}^*)'(\underset{\sim}{x}\hat{\underset{\sim}{\beta}} - \underset{\sim}{x}\underset{\sim}{\beta}^*)}{(\underset{\sim}{Y} - \underset{\sim}{x}\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - \underset{\sim}{x}\hat{\underset{\sim}{\beta}})} \tag{3.1.3}$$

which after simplification can be written as

$$T = \frac{(\underset{\sim}{Y} - \underset{\sim}{x}\underset{\sim}{\beta}^*)'\underset{\sim}{M}(\underset{\sim}{Y} - \underset{\sim}{x}\underset{\sim}{\beta}^*)}{(\underset{\sim}{Y} - \underset{\sim}{x}\underset{\sim}{\beta}^*)'(\underset{\sim}{I}-\underset{\sim}{M})(\underset{\sim}{Y} - \underset{\sim}{x}\underset{\sim}{\beta}^*)} \tag{3.1.4}$$

where $\underset{\sim}{M} = \underset{\sim}{x}(\underset{\sim}{x}'\underset{\sim}{x})^{-1}\underset{\sim}{x}'$ is a symmetric idempotent matrix.

Since the $\underset{\sim}{Y}$'s are distributed as $N(\underset{\sim}{x}\underset{\sim}{\beta}, \underset{\sim}{V})$, we therefore have that the $\underset{\sim}{D}$'s are distributed as $N(\underset{\sim}{O},\underset{\sim}{V})$ where $\underset{\sim}{D} = \underset{\sim}{Y} - \underset{\sim}{x}\underset{\sim}{\beta}$. Substituting the value of $\underset{\sim}{Y}$ in (3.1.4), we have

$$V_E = (\underset{\sim}{D} - \underset{\sim}{\mu}^*)'\underset{\sim}{M}(\underset{\sim}{D} - \underset{\sim}{\mu}^*)$$

where $\quad \mu^* = \underset{\sim}{x}(\underset{\sim}{\beta} - \underset{\sim}{\beta}^*)$.

When the null-hypothesis is true the equation (3.1.4) is given by

$$T = \frac{\underset{\sim}{D}'\underset{\sim}{M}\,\underset{\sim}{D}}{\underset{\sim}{D}'(\underset{\sim}{I}-\underset{\sim}{M})\underset{\sim}{D}} \tag{3.1.5}$$

Since the elements of $\underset{\sim}{D}$ are the deviations from the mean we therefore have $\underset{\sim}{D}'\underset{\sim}{1} = 0$, where $\underset{\sim}{1}$ is the (nxl) vector, all of whose elements are unity.

To study the shape of the distribution of the test criterion $Z = 1 - L$, and the power of the test when the errors are not normally distributed we proceed as follows. We will assume like Box and Watson (1962) that the vector $\underset{\sim}{e}$ in (3.1.1) are symmetrically distributed and hence the moments of the test criterion can be obtained using permutation theory. We will also assume that $x_{1i} = 1$ for all the values of i.

3.2 Moments of the Test Criterion

Expressing the D's in terms of power sums

i.e. $\sum_{u=1}^{n} D_u^{\,r} = V_r$

we have $V_1 = 0$, $V_2 = \underset{\sim}{D}'\underset{\sim}{D} = V_o + V_E$ and $Z = \frac{V_E}{V_2}$.

To obtain the expectation of Z in the above situation we will first of all establish the required condition $\underset{\sim}{M}\,\underset{\sim}{1} = \underset{\sim}{1}$, in the following theorem.

Theorem

If $\underset{\sim}{X} = (\underset{\sim}{a}|\underset{\sim}{b})$ is a partitioned matrix where $\underset{\sim}{a}$ is a (nx1) matrix, $\underset{\sim}{b}$ is a [nx(p-1)] matrix and $\underset{\sim}{M} = \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'$ is a symmetric idempotent matrix then the product $\underset{\sim}{M}\underset{\sim}{a} = \underset{\sim}{a}$.

Proof

The Matrix $\underset{\sim}{M}$ can be written as

$$\underset{\sim}{M} = (\underset{\sim}{a}|\underset{\sim}{b})\left[\begin{pmatrix}\underset{\sim}{a}'\\ \hline \underset{\sim}{b}'\end{pmatrix}(\underset{\sim}{a}|\underset{\sim}{b})\right]^{-1}\begin{pmatrix}\underset{\sim}{a}'\\ \hline \underset{\sim}{b}'\end{pmatrix}$$

$$\underset{\sim}{M} = (\underset{\sim}{a}|\underset{\sim}{b})\left[\left(\begin{array}{c|c}N_1 & \underset{\sim}{a}'\underset{\sim}{b}\\ \hline \underset{\sim}{b}'\underset{\sim}{a} & \underset{\sim}{b}'\underset{\sim}{b}\end{array}\right)\right]^{-1}\begin{pmatrix}\underset{\sim}{a}'\\ \hline \underset{\sim}{b}'\end{pmatrix} \qquad \left[N_1 = \text{constant} = \underset{\sim}{a}'\underset{\sim}{a}\right.$$

$$\underset{\sim}{M}\underset{\sim}{a} = (\underset{\sim}{a}|\underset{\sim}{b})\left[\left(\begin{array}{c|c}N_1 & \underset{\sim}{a}'\underset{\sim}{b}\\ \hline \underset{\sim}{b}'\underset{\sim}{a} & \underset{\sim}{b}'\underset{\sim}{b}\end{array}\right)\right]^{-1}\begin{pmatrix}\underset{\sim}{a}'\\ \hline \underset{\sim}{b}'\end{pmatrix}(\underset{\sim}{a})$$

$$= (\underset{\sim}{a}|\underset{\sim}{b})\left[\left(\begin{array}{c|c}N_1 & \underset{\sim}{a}'\underset{\sim}{b}\\ \hline \underset{\sim}{b}'\underset{\sim}{a} & \underset{\sim}{b}'\underset{\sim}{b}\end{array}\right)\right]^{-1}\begin{pmatrix}N_1\\ \hline \underset{\sim}{b}'\underset{\sim}{a}\end{pmatrix}$$

Since

$$\left(\begin{array}{c|c}N_1 & \underset{\sim}{a}'\underset{\sim}{b}\\ \hline \underset{\sim}{b}'\underset{\sim}{a} & \underset{\sim}{b}'\underset{\sim}{b}\end{array}\right)\begin{pmatrix}1\\ \hline \underset{\sim}{0}\end{pmatrix} = \begin{pmatrix}N_1\\ \hline \underset{\sim}{b}'\underset{\sim}{a}\end{pmatrix}$$

Then

$$\begin{pmatrix}1\\ \hline \underset{\sim}{0}\end{pmatrix} = \left(\begin{array}{c|c}N_1 & \underset{\sim}{a}'\underset{\sim}{b}\\ \hline \underset{\sim}{b}'\underset{\sim}{a} & \underset{\sim}{b}'\underset{\sim}{b}\end{array}\right)^{-1}\begin{pmatrix}N_1\\ \hline \underset{\sim}{b}'\underset{\sim}{a}\end{pmatrix}$$

when $\underset{\sim}{0}$ is a null vector.

Therefore

$$\underset{\sim}{M}\underset{\sim}{a} = (\underset{\sim}{a}|\underset{\sim}{b})\begin{pmatrix}1\\ \hline \underset{\sim}{0}\end{pmatrix} = \underset{\sim}{a}$$

Since we assumed that $x_{1i} = 1$ for all values of i, we therefore have $\underset{\sim}{a} = \underset{\sim}{1}$ and hence $\underset{\sim}{M}\,\underset{\sim}{1} = \underset{\sim}{1}$.

Now using the relation $\underset{\sim}{M}\,\underset{\sim}{1} = \underset{\sim}{1}$, we obtain (see Appendix C) the permutation mean as

$$E(Z) = \frac{(p-1)}{(n-1)} \qquad (3.2.1)$$

Also using David and Kendall's table (1949) and writing V_2 and V_4 in terms of Fisher K Statistics we obtain (see Appendix C) the permutation variance as

$$V(Z) = \frac{2(p-1)(n-p)}{(n-1)\ (n+1)} + \frac{K_4/K_2{}^2}{(n-1)^2}\left[m - \frac{p^2}{n} - \frac{2(p-1)(n-p)}{n(n+1)}\right] \qquad (3.2.2)$$

where m is the sum of squares of the diagonal elements of $\underset{\sim}{M}$. The result obtained in (3.2.2) has great similarity to the result given by Box and Watson (1962). Also with proper grouping and substituting $m = \Sigma \frac{1}{n_i}$ in (3.2.2) for one-way layout analysis of variance we can easily obtain the value $V(E^2)$ by Welch (1938).

3.3 Approximate Distribution of Test Criterion

We know that when the elements of the error vector are normally distributed, the test criterion Z is distributed as a Beta distribution. We have assumed earlier that $x_{1i} = 1$ for all the values of i. Therefore the model (3.1.1) can be written as

$$\underset{\sim}{Y} = \underset{\sim}{1}\,\underset{\sim}{R}_1 + \underset{\sim}{X}_2\underset{\sim}{R}_2 + \underset{\sim}{e} \qquad (3.3.1)$$

given the following partition matrices,

$$\underset{\sim}{\beta} = \begin{pmatrix} \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \end{pmatrix} \text{ and } \underset{\sim}{X} = (\underset{\sim}{1} | \underset{\sim}{X}_2)$$

Hence a test of the hypothesis $\underset{\sim}{R}_2 = \underset{\sim}{R}_2^*$ is required in the model (3.3.1) where $\underset{\sim}{R}_2^*$ is known. Following the method for the testing the sub-hypothesis, we find that when the errors are normally distributed and the null-hypothesis is true, the test criterion Z is distributed as a Beta distribution with (p-1) and (n-p) degrees of freedom, comparing the normal theory moments of Z with the permutation moments, we find that the mean is the same for both cases, whereas the variance differs. Pitman (1937) has shown that the third and fourth moments of the permutation distribution of Z agree closely with those of the Beta distribution. Hence the permutation distribution of Z could be approximated to a Beta distribution by adjusting the degrees of freedom. It could be readily shown that the approximating distribution has degrees of freedom d(p-1) and d(n-p) where

$$d = \left[\frac{2\{n(n-1)^2 - (n-3)S_1S_2\}}{(n-1)\{2n(n-1) + (n-3)S_1S_2\}}\right]$$

$$S_1 = \frac{K_4}{K_2^{\,2}} ,$$

$$S_2 = \frac{n(n-1)(n+1)}{(p-1)(n-p)(n-3)}\left[m - \frac{p^2}{n} - \frac{2(p-1)(n-p)}{n(n+1)}\right]$$

Equivalently the permutation distribution of T could be approximated by an F distribution with degrees of freedom d(p-1) and d(n-p). The numerical value of d for a special case i.e. one-way layout Analysis of Variance could be easily obtained [see Johnson and Leone (1964) p.21].

3.4 Power of the Test

In order to obtain the power of the test in the case of the non-normality of errors, we proceed as in Section 2.6 and find that the test criterion is given by

$$Z = 1-L = \frac{1}{1+\frac{1}{T}} = \frac{T}{1+T}$$

and

$$P(Z \leqslant \theta) = P\left[\frac{T}{1+T} \leqslant \theta\right] = P\left[T \leqslant \frac{\theta}{1-\theta}\right]$$

or

$$P(Z \leqslant \frac{\alpha}{1+\alpha}) = P(T \leqslant \alpha) \text{ if we put } \frac{\theta}{1-\theta} = \alpha$$

When the errors are normally distributed then from the equation (2.6.2) we have the test criterion Z as a Beta distribution with p-1 and N-p degrees of freedom. With the help of the theory developed in the preceding sections of this chapter we can now approximate the distribution of Z (when the errors are not normally distributed) by Beta distribution, by adjusting the degrees of freedom in the normal theory case.

Let P_{11} be the probability of type two errors. Hence

$$P_{11} = P(T \leqslant \alpha | \lambda \neq 0)$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_j d_i \, I_{\frac{\alpha}{1+\alpha}} \left(\frac{d(p-1) + 2j}{2}, \frac{d(n-p) + 2i}{2}\right) \qquad (3.4.1)$$

where $I_\alpha(.)$ is an incomplete Beta distribution, $\alpha = \frac{p-1}{n-p} F_\varepsilon$ and where ε is the level of significance. Therefore the power of the test is given by $\beta(\lambda) = 1 - P_{11}$. The non-centrality parameter λ and P_{11} in (3.4.1) can be easily obtained by following previous chapter. We have thus found a practical method to study the effect of non-normality on the probability of type two errors in the Analysis of Variance.

POWER ASPECTS IN FIXED AND RANDOM EFFECT MODELS

4.1 Fixed model: one-way classification

In certain circumstances, the group to group heterogeneity of variances may be obtained while testing the group to group homogeneity of means in the one-way analysis of variance classification.

Suppose we have n_i observations in group i, $i = 1,2,\ldots,k$. Denote by y_{ij} the j^{th} observation in group i, by $\bar{y}_i$ the i^{th} group mean and $\bar{y}_{..}$ the grand mean. Suppose there are N observations allocated. Usually we assume,

$$y_{ij} = \xi + \gamma_i + e_{ij} \tag{4.1.1}$$

where $\xi + \gamma_i$ is the population mean from the i^{th} group, $\Sigma n_i \gamma_i = 0$, and e_{ij} are errors distributed normally and independently about zero with the same variance σ^2. We retain the assumption of normality and independence but now assume variances

$$\sigma_1{}^2,\ \sigma_2{}^2,\ \ldots,\ \sigma_k{}^2, \text{ for each group.}$$

The sum of squares for the fixed effect model can be expressed as Q_1 and Q_2 where Q_1 is the within groups and Q_2 the between groups sum of squares.

We will first consider the distribution of Q_1 and Q_2. Here Q_2 is a quadratic form in $\bar{y}_{1.}\ \bar{y}_{2.}\ \ldots\ \bar{y}_{k.}$ and the matrix of the quadratic form is

$$\underset{\sim}{B} = \{n_i\delta_{ij} - \frac{n_i n_j}{N}\} \tag{4.1.2}$$

where δ_{ij} is the Kronecker delta.

We may write the quadratic form $Q_2 = \underset{\sim}{Y}'\underset{\sim}{B}\underset{\sim}{Y}$, where $\underset{\sim}{Y}$ is the vector of normally distributed variables $\bar{y}_{i\cdot}$, with expectation $\underset{\sim}{\zeta}^*$ and diagonal covariance matrix $\underset{\sim}{V} = \{\frac{\sigma_i^2}{n_i}\}$

Setting $\underset{\sim}{Z} = \underset{\sim}{Y} - \underset{\sim}{\zeta}^*$

we may write this in the form $Q_2 = (\underset{\sim}{Z} + \underset{\sim}{\zeta}^*)'\underset{\sim}{B}(\underset{\sim}{Z} + \underset{\sim}{\zeta}^*)$.

Since the elements of $\underset{\sim}{\zeta}^*$ are deviations from the mean, the elements of $\underset{\sim}{Z}$ are distributed with mean zero and variance $\underset{\sim}{V}$.

We shall transform the quadratic form to

$$Q_2 = (\underset{\sim}{x}-\underset{\sim}{b})'\underset{\sim}{A}(\underset{\sim}{x}-\underset{\sim}{b}).$$

The transformation used is

$$\underset{\sim}{Z} = \underset{\sim}{N}\,\underset{\sim}{K}\,\underset{\sim}{x} \qquad \underset{\sim}{\zeta} = -\underset{\sim}{N}\,\underset{\sim}{K}\,\underset{\sim}{b} \qquad (4.1.3)$$

and the elements of $\underset{\sim}{x}$ are now normally distributed with zero mean and unit variance. $\underset{\sim}{A}$ is a diagonal matrix of the form $\underset{\sim}{A} = \underset{\sim}{K}'\underset{\sim}{N}'\underset{\sim}{B}\,\underset{\sim}{N}\,\underset{\sim}{K}$, where $\underset{\sim}{K}$ is the orthogonal matrix of eigenvectors of $\underset{\sim}{N}'\underset{\sim}{B}\,\underset{\sim}{N}$ and a_i's the diagonal elements of $\underset{\sim}{A}$ are the eigen values of $\underset{\sim}{N}'\underset{\sim}{B}\,\underset{\sim}{N}$. $\underset{\sim}{N}$ is the lower triangular matrix $\underset{\sim}{V}^{-1} = \underset{\sim}{N}\,\underset{\sim}{N}'$. Thus the quadratic form Q_2 or the between groups Sum of Squares can be expressed as a non-homogeneous quadratic form. The distribution of Q_2 (see Section 2.3) is given by

$$P(Q_2 \leqslant \alpha) = \sum_{j=0}^{\infty} d_j \chi^2_{p'+2j}\left(\frac{\alpha}{g}\right) \qquad (4.1.4)$$

where $p' = k-1$ is the rank of the positive semidefinite quadratic form $\underset{\sim}{B}$, g is an arbitrary constant and $\chi^2_{p'+2j}(\cdot)$ is a chi-square distribution. d_j can be calculated by the recursion relation

$$d_j = (2j)^{-1} \sum_{r=0}^{j-1} h_{j-r} d_r \qquad j = 1,2,\ldots$$

$$d_0 = e^{-\frac{1}{2}\sum_{i=1}^{p'} b_i^2} \prod_{i=1}^{p'} (g/a_i)^{\frac{1}{2}}$$

where

$$h_m = \sum_{i=1}^{p'} (1-g/a_i)^m + mg \sum_{i=1}^{p'} \left(\frac{b_i^2}{a_i}\right)(1-g/a_i)^{m-1} \qquad m = 1,2,\ldots$$

Similarly, Q_1 can be written as a quadratic form $\underset{\sim}{Z}'\underset{\sim}{D}\,\underset{\sim}{Z}$ where $\underset{\sim}{Z}$ is normally distributed with mean zero and variance $\underset{\sim}{V}$. By the transformation $\underset{\sim}{Z} = \underset{\sim}{N}\,\underset{\sim}{K}\,\underset{\sim}{Z}$ the quadratic form Q_1 is reduced to $\underset{\sim}{X}'\underset{\sim}{A}\,\underset{\sim}{X}$ where $\underset{\sim}{X}$'s are normally distributed with zero mean and unit variance and the a_i's are the latent roots of the matrix $\underset{\sim}{N}'\underset{\sim}{D}\,\underset{\sim}{N}$, where $\underset{\sim}{N}$ and $\underset{\sim}{K}$ are defined earlier and $a_i = \sigma^2{}_i$.

The distribution of Q_1 is given by

$$P(Q_1 \leqslant \alpha) = \sum_{i=0}^{\infty} C_i \chi^2_{N-k+2i}\left(\frac{\alpha}{g}\right), \qquad (4.1.5)$$

where C_i can be obtained by the recursion relation given by

$$C_i = (2i)^{-1} \sum_{r=0}^{i-1} h_{i-r} C_r \qquad i = 1,2,\ldots$$

$$C_o = \prod_{j=1} (g/a_j)^{\frac{1}{2}}$$

where $$h_n = \sum_{j=1} (1-g/a_j)^n \qquad n = 1,2,\ldots$$

The quadratic forms Q_1 and Q_2 are statistically independent, since for each i, $\bar{Y}_{i\cdot}$ and

$\sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\cdot})$ are independently distributed.

The non-centrality parameter is given by

$$\lambda = (\tfrac{1}{2}\underset{\sim}{b}'\underset{\sim}{b})^{\frac{1}{2}} = (\tfrac{1}{2}\Sigma b_i^2)^{\frac{1}{2}} \qquad (4.1.6)$$

where $\underset{\sim}{b} = \underset{\sim}{K}'\underset{\sim}{N}^{-1}\underset{\sim}{\zeta}^*$

Proceeding as in Section 2.5 the distribution of the test criterion u is given by

$$P(u = \frac{Q_2}{Q_1} \leq \alpha) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} d_j C_i F_{p'+2j, r'+2i}(\frac{r'+2i}{p'+2j}\alpha) \qquad (4.1.7)$$

where $p' = k - 1$, $r' = N-k$ and $F_{p',r'}(.)$ is the central F distribution function, or

$$P(u \leq \alpha) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} d_j C_i I_{\frac{\alpha}{1+\alpha}}(\frac{p'+2j}{2}, \frac{r'+2i}{2}) \qquad (4.1.8)$$

a generalised incomplete beta distribution, where

$\alpha = p'/r'F_{\varepsilon}$, where ε is the chosen level of significance.

Thus the probability of a type II error of magnitude $p(u \leq \alpha/\lambda \neq 0)$ can be calculated from the equation (4.1.8) and given by

$$P_{II} = \sum_{j=0}^{\infty} \sum_{i=o}^{\infty} d_j C_i I_{\frac{\alpha}{1+\alpha}}(\frac{p'+2j}{2}, \frac{r'+2i}{2}) \qquad (4.1.9)$$

where the power of the test is $\beta(\lambda) = 1 - P_{II}$.

4.2 Random Model: One Way Classification

The situation may arise where a sample of K populations is drawn from a large set of populations. If we then consider that the K populations are randomly drawn from the large (possibly infinite) set of populations, then the model described by (4.1.1) changes to the Random effect model outlined below.

For example consider the determination of the effect of certain treatments on the nitrogen content of the tree leaves in an orchard. We select at random a group of trees, and then choose a set of leaves at random from each selected tree. Let y_{ij} be the observed nitrogen content of the j^{th} leaf from the i^{th} tree then the structure of the model is given by

$$y_{ij} = \xi + \gamma_i + e_{ij} \tag{4.2.1}$$

The general procedure for testing a hypothesis, and for estimation with the random effects model is the same as with the fixed effect model. Scheffé (1959) has discussed the power of the test when the error variances are equal and the lay out is balanced. We will now discuss the power of the test in the Random effects model when the error variances are unequal and the layout is not necessarily balanced.

In the model (4.2.1) we shall assume that γ_i and e_{ij} are independent random variables, each with expectation zero and with variances σ_γ^2 and σ_i^2 respectively. The σ_i^2 $(i = 1,2,\ldots K)$ are not necessarily equal. We shall also assume that the γ_i and e_{ij} are normally distributed.

Now the sums of squares that are involved are

$$Q_1 = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 \tag{4.2.2}$$

and

$$Q_2 = \sum_{i=1}^{K} n_i (\bar{y}_i - \bar{y}_{..})^2 \tag{4.2.3}$$

Under the present model, the quadratic form

$$Q_1 = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (e_{ij} - e_i)^2 \tag{4.2.4}$$

is the same as that for the fixed effect model, and thus the distribution of Q_1 is also the same.

As before, we find that Q_2 can be expressed as a quadratic form in $\bar{y}_1. \bar{y}_2. \ldots \bar{y}_K.$ Namely $Q_2 = \underset{\sim}{Y}'\underset{\sim}{B}\,\underset{\sim}{Y}$, where $\underset{\sim}{B}$ is given in (4.1.2). Since the y_{ij} are now distributed as $N(\underset{\sim}{\xi}, \sigma_\gamma{}^2\underset{\sim}{J} + \sigma_i{}^2\underset{\sim}{I})$, we have that the $\bar{y}_i$ are distributed as $N(\underset{\sim}{\xi}, \sigma_\gamma{}^2\underset{\sim}{J} + \sigma_i{}^2/n_i\underset{\sim}{I})$. The $\underset{\sim}{\xi}$'s are the same as in the fixed effect model, but the $\underset{\sim}{Y}$'s are now distributed as $N(\underset{\sim}{\xi}, \sigma_\gamma{}^2\underset{\sim}{J} + \sigma_i{}^2/n_i\underset{\sim}{I})$. When $n_i = n$, $\underset{\sim}{J}$ is an $n \times n$ matrix each element of which is equal to unity.

Setting $\underset{\sim}{Z} = \underset{\sim}{Y} - \underset{\sim}{\xi}$, we may express Q_2 in the form $Q_2 = (\underset{\sim}{Z} - \underset{\sim}{\xi})'\underset{\sim}{B}(\underset{\sim}{Z} - \underset{\sim}{\xi})$ where the Z's are distributed as $N(0,\underset{\sim}{V})$ the variance-covariance matrix being $\underset{\sim}{V} = (\sigma_\gamma{}^2\underset{\sim}{J} + \sigma_i{}^2/n_i\underset{\sim}{I})$.

We have seen in Section 4.1 that without loss of generality the quadratic form Q_2 can be reduced to the form Q_2' where $Q_2' = (\underset{\sim}{x} - \underset{\sim}{b})'\underset{\sim}{A}(\underset{\sim}{x} - \underset{\sim}{b})$. The elements of $\underset{\sim}{x}$ are standard normal variates. The distribution of Q_2' can then be obtained easily and is given by

$$P[Q_2' \leqslant \alpha] = \sum_{j=0}^{\infty} d_j \, \chi^2_{p'+2j} \, (\alpha/g) \qquad (4.2.5)$$

where p' is the rank of the positive semidefinite quadratic form $\underset{\sim}{B}$.

To test the hypothesis of equal treatment effects, we must choose between the null and alternative hypothesis.

$$\begin{aligned} H_0 &: \alpha_\gamma{}^2 = 0 \\ H_1 &: \sigma_\gamma{}^2 \neq 0 \end{aligned} \qquad (4.2.6)$$

The power of the test for the situation (4.2.6) is then given by $1 - P_{II}$, where

$$P_{II} = P[Q'_2/Q_1 \leqslant \alpha] = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} d_j' \, C_i \, I_{\frac{\alpha}{1+\alpha}}\left(\frac{p' + 2j}{2}, \frac{\gamma' + 2i}{2}\right)$$

(4.2.7)

where $p' = K - 1$, $\gamma' = N - K$ and $\alpha = \frac{p'}{\gamma'} F_{\varepsilon}$. ε is the chosen level of significance and $d_j' \neq d_j$

4.3 Fixed Model - Two-way classification

Sometimes, when testing the k treatments in the n blocks in the two-way layout, circumstances arise where the variances of the k treatments differ from treatment to treatment. Similarly, when the experimental material is not homogeneous in mean from block to block, changes in variance may also occur from block to block.

The data given by Fisher (1958) on the frequency of rainfall at different hours in different months of the year, can be classified as a two-way layout. Fisher has mentioned the strong serial correlation of the errors within months, since rainfall which continues for more than one hour is recorded in successive hours. The method of randomisation cannot be applied in this case. Fisher has also remarked on the non-validity of the 'between months' comparison, due to the serial correlation between hours within months.

Let us consider a set of S values of the variate Y arranged in k columns and n rows where y_{ti} represents the value of the member belonging to the t^{th} column and the i^{th} row. We accept the usual assumptions that y_{ti} may be represented by a linear model

$$y_{ti} = \xi + \psi_i + \gamma_t + e_{ti} \qquad (4.3.1)$$

where $\Sigma\psi_i = 0$, $\Sigma\gamma_t = 0$; our assumptions concerning the e_{ti} will be given later.

We shall represent the model in (2.1) for all elements of the t^{th} column of the table ($t = 1,2,...k$) by

$$\underset{\sim}{y}_{t.} = \xi \underset{\sim}{1}_n + \underset{\sim}{\psi} + \gamma_t \underset{\sim}{1}_n + \underset{\sim}{e}_{t.} \qquad (4.3.2)$$

where $\underset{\sim}{y}_{t.}$ is the n x 1 vector, $\underset{\sim}{1}_n$ is a n x 1 column vector all of whose elements are unity, $\underset{\sim}{e}_{t.}$ is the vector of errors and $\underset{\sim}{\psi}$ is a n x 1 vector of row constants $\psi_1, \psi_2, \ldots \psi_n$. We shall also use the notation $\underset{\sim}{y}_{.i}$ and $\underset{\sim}{e}_{.i}$ where $\underset{\sim}{y}_{.i}$ and $\underset{\sim}{e}_{.i}$ are respectively k x 1 vectors of observations and errors in the i^{th} row of the table.

Instead of making the usual assumptions concerning the e_{ti}, namely that they have the same variance and are statistically independent, we shall also assume like Box (1954) that the $\underset{\sim}{e}_{.i}$ are normally distributed with

$$E(\underset{\sim}{e}_{.i}) = 0$$
$$E(\underset{\sim}{e}_{.i} \underset{\sim}{e}'_{.i}) = \underset{\sim}{V}$$

The $\underset{\sim}{e}_{.i}$ being mutually independent for $i = 1,2,...,n$. Thus the k variances and $\frac{1}{2}k(k - 1)$ covariances are the same for every row. This assumption permits us to study the effect of column to column inequality of variances and within row correlation of errors.

Box (1954) has shown with the help of the orthogonal transformation

$$\underset{\sim}{H}_{t.} = \underset{\sim}{P}\underset{\sim}{y}_{t.} = \xi \underset{\sim}{\Lambda} + \underset{\sim}{\Psi} + \gamma_t \underset{\sim}{\Lambda} + \underset{\sim}{E}_{t.}$$

(where P is the n x n orthogonal matrix, $\underset{\sim}{\Lambda} = \underset{\sim}{P}\underset{\sim}{1}_n$, $\underset{\sim}{\Psi} = \underset{\sim}{P}\underset{\sim}{\psi}$ $\underset{\sim}{E}_{t.} = \underset{\sim}{P}\underset{\sim}{e}_{t.}$) that the original two-way table can be changed and the e_{ti} and E_{ti} are distributed in the same manner and $\underset{\sim}{E}_{\cdot i}$ has the variance covariance matrix $\underset{\sim}{V}$.

Now the sums of squares involved in the analysis of variance two-way lay-out are

$$Q_C = n \sum_{t=1}^{k} (\bar{y}_{t.} - \bar{y}_{..})^2$$

$$Q_R = k \sum_{i=1}^{n} (\bar{y}_{.i} - \bar{y}_{..})^2$$

and

$$Q_E = \sum_{i=1}^{n} \sum_{t=1}^{k} (y_{ti} - \bar{y}_{t.} - \bar{y}_{.i} + \bar{y}_{..})^2$$

where Q_C, Q_R and Q_E are the Between columns, Between rows and Error sums of squares respectively.

Using the transformation $H_{tn} = n^{\frac{1}{2}}\bar{y}_{t.}$ and $H_{.n} = n^{\frac{1}{2}}\bar{y}_{..}$ the sum of squares Q_c can be written as a quadratic form in H_{tn}. We can therefore express the quadratic form Q_c as $\underset{\sim}{H}'\underset{\sim}{C}\,\underset{\sim}{H}$ where $\underset{\sim}{H}$'s are $N(\underset{\sim}{\gamma}, \underset{\sim}{V})$, $\underset{\sim}{C} = (\underset{\sim}{I}_k - \frac{\underset{\sim}{1}_k \underset{\sim}{1}_k'}{k})$ and $\underset{\sim}{\gamma} = \{\xi_i - \xi\}$. Now setting $\underset{\sim}{Y} = \underset{\sim}{H} - \underset{\sim}{\gamma}$ we will have that $Q_c = (\underset{\sim}{Y} + \underset{\sim}{\gamma})'\underset{\sim}{C}(\underset{\sim}{Y} + \underset{\sim}{\gamma})$ where the $\underset{\sim}{Y}$'s are $N(\underset{\sim}{O}, \underset{\sim}{V})$. The Q_c can be then transformed to the form where $Q_c = (\underset{\sim}{x} - \underset{\sim}{b})'\underset{\sim}{A}(\underset{\sim}{x} - \underset{\sim}{b})$ in which the $\underset{\sim}{x}$'s are normally distributed with expectation zero and unit variance co-variance matrix. $\underset{\sim}{A}$ is a diagonal matrix of the form $\underset{\sim}{A} = \underset{\sim}{K}'\underset{\sim}{M}'\underset{\sim}{C}\underset{\sim}{M}\underset{\sim}{K}$, where $\underset{\sim}{K}$ is the orthogonal matrix of eigenvectors of $\underset{\sim}{M}'\underset{\sim}{C}\underset{\sim}{M}$ and $\underset{\sim}{M}$ is the lower triangular matrix given by $\underset{\sim}{V}^{-1} = \underset{\sim}{M}\,\underset{\sim}{M}'$. The distribution of Q_c can then be obtained (see Section 2.3) and is given by

$$P(Q_c \leqslant \alpha) = \sum_{j=o}^{\infty} d_j \, \chi^2_{p+2j} \left(\frac{\alpha}{g}\right) \qquad (4.3.3)$$

where $p = k - 1$ is the rank of the positive semi-definite quadratic form $\underset{\sim}{C}$, g is an arbitrary constant and $\chi^2_{p+2j}(\cdot)$ is a chi-square distribution. d_j can be calculated by the recursion relation given by

$$d_j = (2j)^{-1} \sum_{r=o}^{j-1} h_{j-r}\, d_r \qquad j = 1,2,...$$

$$d_o = e^{-\frac{1}{2} \sum_{i=1}^{p} b_i^2} \prod_{i=1}^{p} (g/a_i)^{\frac{1}{2}}$$

where $h_m = \sum_{i=1}^{p} (1 - g/a_i)^m + mg \sum_{i=1}^{p} (\frac{b_i^2}{a_i}) (1 - g/a_i)^{m-1} \qquad m = 1,2,...$

and a_i's are the latent roots of the matrix $\underset{\sim}{M}' \underset{\sim}{C} \underset{\sim}{M}$. The non-centrality parameter λ is equal to

$$\lambda = (\tfrac{1}{2}\, \underset{\sim}{b}'\, \underset{\sim}{b})^{\frac{1}{2}} = (\tfrac{1}{2} \Sigma\, b_i^2)^{\frac{1}{2}}.$$

where $\underset{\sim}{b} = \underset{\sim}{K}'\underset{\sim}{M}^{-1}\, \gamma$.

We can write Q_E as the quadratic form in E_{ti} and in the matrix notation express it as $\underset{\sim}{Z}'\underset{\sim}{C}\, \underset{\sim}{Z}$ where $\underset{\sim}{C}$ is the matrix given by $\underset{\sim}{C} = (\underset{\sim}{I}_k - \frac{\underset{\sim}{1}_k \underset{\sim}{1}_k'}{k})$. The $\underset{\sim}{E}_{\cdot i}$'s are distributed as $N(0, \underset{\sim}{V})$ and Z's are similarly distributed with expectation zero and variance covariance matrix $\underset{\sim}{V}$.

In order to obtain the distribution of $\underset{\sim}{Z}'\underset{\sim}{C}\, \underset{\sim}{Z}$ we reduce this to the canonical form given by $\underset{\sim}{x}'\underset{\sim}{A}\, \underset{\sim}{x}$; the transformation used is $\underset{\sim}{Z} = \underset{\sim}{M}\, \underset{\sim}{K}\, \underset{\sim}{x}$, where the $\underset{\sim}{x}$'s are normally distributed with zero mean and unit variance, and a_i's are the latent roots of the matrix $\underset{\sim}{M}'\, \underset{\sim}{C}\, \underset{\sim}{M}$. (The matrix $\underset{\sim}{M}$ and $\underset{\sim}{K}$ were defined earlier.) The distribution of the homogeneous quadratic form Q_E is then given by

$$P(Q_E \leqslant \alpha) = \sum_{i=0}^{\infty} C_i \chi^2_{(k-1)(n-1)+2i} (\tfrac{\alpha}{g}) \qquad (4.3.4)$$

where $\underset{\sim}{C}$ is a positive semi-definite matrix of rank k-1. C_i in equation (4.3.4) can be calculated by the recursion relation given by

$$C_i = (2i)^{-1} \sum_{r=0}^{i-1} f_{i-r} C_r \qquad i=1,2,\ldots$$

$$C_o = \prod_{j=1} (g/a_j)^{\frac{1}{2}}$$

where $$f_m = \sum_{j=1} (1-g/a_j)^m. \qquad m=1,2,\ldots$$

Box (1954) has proved that the quadratic form Q_c and Q_E are mutually independent whereas Q_R and Q_E are not statistically independent.

In order to find the distribution of the Between columns test criterion, we proceed as follows, Between column test criterion is given by $u = \frac{Q_c}{Q_E}$. Since the distributions of Q_c and Q_E are known, the distribution of u is found to be (see section 2.6).

$$P(u = \frac{Q_c}{Q_E} \leqslant \alpha) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} d_j C_i I_{\frac{\alpha}{1+\alpha}} \left(\frac{k-1+2j}{2}, \frac{(n-1)(k-1)+2i}{2}\right) \tag{4.3.5}$$

Thus the Type II error of magnitude $P(u \leqslant \alpha/\lambda \neq 0)$ can be calculated from the equation (4.3.5) and given by

$$P_{II} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} d_j C_i I_{\frac{\alpha}{1+\alpha}} \left(\frac{k-1+2j}{2}, \frac{(n-1)(k-1)+2i}{2}\right) \tag{4.3.6}$$

where the power of the test is $\beta(\lambda) = 1 - P_{II}$.

If we now consider the $\underset{\sim}{V}$ matrix as diagonal, with unequal diagonal elements, then we have the case where variances change from column to column. The errors remain statistically independent. Again, if the errors $e_{1i}, e_{2i}, \ldots e_{ki}$ in the i^{th} row are normally distributed but not independently, then their variance covariance matrix is $\underset{\sim}{V} = \sigma^2 \underset{\sim}{\delta}$ where $\underset{\sim}{\delta} = (\rho_{ts})$ is a k x k positive definite matrix with unit diagonal elements and off-diagonal elements ρ_{ts} is the coefficient of correlation between e_{ti} and e_{si}.

We will consider the serial correlation which arises when the observations within columns or rows are made at equally spaced intervals of time or space.

4.4 Random Model: Two way classification

In the preceding section we have confined ourselves to the fixed effects model. We now consider a situation where the treatments and Blocks are also random samples, from the population of treatments and Blocks respectively. This is the random effects model.

Consider the analysis of variance in the Two way layout of our Random effect model. The error variances may be unequal and errors are not necessarily uncorrelated. We will assume a model similar to that of the fixed effect case, namely

$$y_{ti} = \xi + \psi_i + \gamma_t + e_{ti} \tag{4.4.1}$$

But unlike for the fixed effect model we assume that ψ_i, γ_t, e_{ti} are three independent random variables. Further, ψ_i and γ_t are taken to be normally distributed with zero expectations and variances $\sigma_\psi^2 I$ and $\sigma_\gamma^2 I$ respectively.

Using the same notation in this model for all the elements of the t^{th} column of the table $(t = 1,2,\ldots k)$ as for the fixed effects model, we have

$$\underset{\sim}{Y}_{t.} = \xi \underset{\sim}{1}_n + \underset{\sim}{\psi} + \gamma_t \underset{\sim}{1}_n + \underset{\sim}{e}_{t.} \qquad (4.4.2)$$

Here the $\underset{\sim}{e}_{.i}$ are random normal variables with variance covariance matrix $E(\underset{\sim}{e}_{.i}\ \underset{\sim}{e}_{.i}') = \underset{\sim}{V}$. We also assume that the $\underset{\sim}{e}_{.j}$ $(j = 1,2,\ldots n)$ follow the same distribution independently of the $\underset{\sim}{e}_{.i}$. Let us choose an n x n orthogonal matrix $\underset{\sim}{p}$, such that all the elements in the last row are $n^{-\frac{1}{2}}$; transforming the $\underset{\sim}{Y}_{t.}$ into $\underset{\sim}{H}_{t.}'$ we find that $\underset{\sim}{H}_{tn} = \sqrt{n}\ \bar{Y}_{t}$.

Then

$$\underset{\sim}{H}_t = \underset{\sim}{p}\ \underset{\sim}{Y}_{t.} = \xi \underset{\sim}{\Lambda} + \underset{\sim}{\Psi} + \gamma_t \underset{\sim}{\Lambda} + \underset{\sim}{E}_{t.} \qquad (4.4.3)$$

where $\underset{\sim}{\Lambda}$, $\underset{\sim}{\Psi}$ and $\underset{\sim}{E}_{t.}$ are the same as in the fixed effect model.

We have seen in the earlier section that owing to the nature of the orthogonal matrix $\underset{\sim}{p}$, We can obtain the transformed columns of the original two way table.

In the random effect model, unlike for the fixed effects model, γ.does not vanish. The error sum of squares remains the same as in the fixed effect model case, i.e. the distribution of Q_E in the Random effect model is the same as in the fixed effect model. The form of Q_c is then given by

$$Q_c = n \sum_{t=1}^{K} (\bar{y}_{t.} - \bar{y}_{..})^2$$

$$= \sum_{t=1} (\underset{\sim}{H}_{tn} - \bar{H}_{.n})^2$$

The sum of squares Q_c can then be written in matrix notation as a quadratic form in $\underset{\sim}{H}_{tn}$. The H's are distributed as $N(\underset{\sim}{\gamma}, \underset{\sim}{V})$ where $\underset{\sim}{V} = \{\sigma_\gamma^2 \underset{\sim}{I} + \sigma^2 \underset{\sim}{\delta}\}$, and the $\underset{\sim}{\delta}$ matrix in $\underset{\sim}{V}$ is the positive definite matrix which introduces the inequality of error variances and the correlation of errors.

Setting $\underset{\sim}{Y} = \underset{\sim}{H} - \underset{\sim}{\gamma}$ in the Quadratic form $Q_c = \underset{\sim}{H}'\underset{\sim}{C}\,\underset{\sim}{H}$ where $\underset{\sim}{C} = \{\underset{\sim}{I}_K - \frac{\underset{\sim}{I}_K\underset{\sim}{I}_K'}{K}\}$, we have $Q_c = (\underset{\sim}{Y} + \underset{\sim}{\gamma})'\underset{\sim}{C}(\underset{\sim}{Y} + \underset{\sim}{\gamma})$ where the $\underset{\sim}{Y}$'s are distributed as $N(0, \underset{\sim}{V})$. Again, with the help of an orthogonal transformation, we can always transform the quadratic form Q_c into the form $(\underset{\sim}{x} - \underset{\sim}{b})'\underset{\sim}{A}(\underset{\sim}{x} - \underset{\sim}{b})$ where the x's are $N(0, I)$. $\underset{\sim}{A}$ is the diagonal matrix whose elements a_i are the latent roots of the matrix $\underset{\sim}{N}'\underset{\sim}{C}\,\underset{\sim}{N}$, where $\underset{\sim}{V} = \underset{\sim}{N}\,\underset{\sim}{N}'$. The transformation used is

$$\underset{\sim}{Y} = \underset{\sim}{N}\,\underset{\sim}{K}\,\underset{\sim}{x}, \qquad \underset{\sim}{\gamma} = \underset{\sim}{N}\,\underset{\sim}{K}\,\underset{\sim}{b}$$

where $\underset{\sim}{K}$ is the orthogonal matrix of the eigenvectors of $\underset{\sim}{N}'\underset{\sim}{C}\,\underset{\sim}{N}$. The distribution of the quadratic form Q_c is then given by

$$P[Q_c \leqslant \alpha] = \sum_{j=o}^{\infty} d_j' \chi_{k-1+2j} (\alpha/g) \qquad (4.4.4)$$

The d_j' s in (4.4.4) are not the same as those in (4.3.3) since the $\underset{\sim}{V}$ matrix has changed.

To test the hypothesis of equal treatment effects we choose between the null and alternative hypothesis

$$H_o: \sigma_\gamma^2 = 0 \qquad (4.4.5)$$

$$H_1: \sigma_\gamma^2 \neq 0$$

The power of the test for the situation (3.4.5) is then given by $1 - P_{II}$, where

$$P_{II} = P[Q_c/Q_E \leqslant \alpha] = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} d_j' C_i I_{\frac{\alpha}{1+\alpha}} \left(\frac{p'+2j}{2}, \frac{\gamma'+2i}{2}\right)$$

(4.4.6)

with $p' = k-1$, $\gamma' = (n-1)(k-1)$, and α is the same as fixed model case.

EFFECT OF NON-NORMALITY ON THE POWER: A SIMULATION STUDY

5.1 Simulation method and Non-normal Distributions

The effect of departure from normality in the distribution of the error term was studied for a one-way classification by Pearson (1931), Geary (1947) and Gayen (1950). David and Johnson (1951) discussed the effects on the F-test as a result of the non-normality of the error distribution. The test in general was found to be very insensitive to non-normality of errors.

In this chapter, unlike the previous authors, a simulation method is used to investigate the sensitivity of the power of the test for the non-normality of the error distribution in one and two-way layout analysis of variance.

Numerically, analysis of variance can be regarded as an algebraic decomposition of variation into different components. More specifically, it is concerned with observed data and the sum of squares of deviations of individual observations from their mean. The decomposition of this sum of squares takes account of various criteria of classification into which the data has been grouped.

The method of simulation does not give us the polished analytic results of mathematical theory but it helps us to duplicate the observations resulting from a particular mathematical model, without first questioning the exact realism of the model used to fit data. Modern computing facilities have taken a leading role in overcoming the tedious work involved in carrying out such simulation and it is justifiable to believe that concentrated research on simulation method will improve the reliability and usefulness of these techniques.

The calculation of power values in this thesis is carried out by the simulation method on an electronic computer by first generating independent random variables uniformly distributed on (0,1), and then allowing them to take the shape of the standard normal, the Erlangian and the contaminated normal distribution.

The Erlangian random variable, X, which we shall consider here is defined as the sum of k independent negative exponential random variables each with parameter θ. Its distribution is of the form

$$g(x)dX = e^{-x} \frac{(\theta x)^{k-1}}{(k-1)!} \theta dX \qquad (k \geq 1, \text{ integer}) \qquad (5.1.1)$$

with the known mean and variance $\frac{k}{\theta}$ and $\frac{k}{\theta^2}$ respectively.

The contaminated normal distribution which we shall consider is obtained as follows. Suppose we have two normal populations with the same mean, the first having h times the standard deviation of the second; if we mix populations by adding small amounts of the second to the first then we obtain a contaminated normal distribution. The probability density of such a distribution of contamination, c, and ratio of standard deviations, h, of the component normal distributions is given by

$$N_{c,h}(z)dZ = (1-c) \frac{1}{2\pi} e^{-z^2/2} dZ + c \frac{1}{h2\pi} e^{-z^2/2h^2} dZ \qquad (5.1.2)$$

If c = 0, then (1) reduces to the standard normal distribution. The standard deviation of the distribution (1) is given by $\sqrt{ch^2-c+1}$.

5.2 Fixed Model: one-way Classification

Let N observations be classified into s groups, the j^{th} observation in the i^{th} group being y_{ij}. Instead of the usual assumption of normality for the errors, we will assume that the error follows (a) the Erlangian distribution, and (b) the contaminated normal distribution. The sum of squares in the one-way layout when the group sizes are equal (i.e. n) are given by $q = q_1 + q_2$, where q is the total sum of squares, q_1 the within sum of squares and q_2 the between sum of squares. The hypothesis and the test criterion concerning the inference about the mean in the above situations are given by

$$H_o: \gamma_i = 0, \quad H_1: \gamma_i \neq 0 \qquad (i = 1,2,\ldots S)$$

and

$$U = \frac{q_2/s-1}{q_1/N-s} \qquad (5.2.1)$$

When the y_{ij}'s are normally distributed and the null hypothesis H_o is true, we have U distributed as a central F distribution with s-1 and N-s degrees of freedom. But when the alternative hypothesis H_1 is true then the power value $\beta(\lambda)$ is given by

$$\beta(\lambda) = p\left[U' > \frac{s-1}{N-s} F_\varepsilon\right] \qquad (5.2.2)$$

where U' is a non-central F distribution, F_ε is the value of F at ε per cent level of significance, and λ denotes the non-centrality parameter.

The process of generating the random variables and finally the calculation of the ratio U' is repeated 2000 times. The power value $\beta(\lambda)$ is obtained by counting the number of times U' is greater than $\frac{s-1}{N-s} F_\varepsilon$ and dividing the number by the total number of repetitions. When the y_{ij} follow the Erlangian distribution, or the contaminated normal distribution, then the U' will no longer be distributed as a non-central F. The ratio U' may, however, still be computed by the same method as for the normal theory, and the power value can thus be obtained.

5.3 Fixed model: two-way classification

Let us consider ns values of the variate y_{ij} arranged in s columns and n rows where y_{ij} represents the value of the member belonging to the i^{th} column and j^{th} row. The sum of squares involved in the two-way layout analysis of variance are given by $q = q_C + q_R + q_E$ where q, q_C, q_R and q_E are the total, the between columns, the between rows and the error sum of squares respectively.

In testing the hypothesis of equal treatment effects, we will consider the hypothesis and the test criterion as follows:

$$H_o : \gamma_j = 0, \quad H_1 : \gamma_j \neq 0 \quad (j = 1,2,\ldots n)$$

and

$$U = \frac{q_C/s-1}{q_E/(n-1)(s-1)} \tag{5.3.1}$$

The power of the test for this situation when the errors are normally distributed is given by

$$\beta(\lambda) = P\left[U' > \frac{s-1}{(n-1)(s-1)} F_\varepsilon\right] \tag{5.3.2}$$

where U' is the non-central F distribution.

In order to obtain the power value $\beta(\lambda)$ for the non-normal errors, we allow the random variables to take the shape of Erlangian and contaminated normal distribution and follow the same procedure as for the one-way layout.

5.4 Random Model: One-way classification

The general procedure for estimation and testing of the hypothesis in the case of the random effects model is the same as for the fixed effects model.

The structure we shall assume for the one-way layout random effects model is given by

$$y_{ij} = \xi + \gamma_i + \ell_{ij} \tag{5.4.1}$$

where γ_i and ℓ_{ij} are independent random variables each with expectation zero and variance α_γ^2 and σ_ℓ^2 respectively. Here y_{ij} is a linear function of two variables. In order to obtain the power of the test in the case where the errors are not normally distributed, we shall consider the random variables γ_i as normally distributed. The random variable γ_{ij} will be allowed to follow different non-normal distributions and the power value will be obtained in different cases of non-normality of error, in particular for the Erlangian and contaminated normal distributions of errors.

Now the sums of squares involved are

$$q_1 = \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij}-\bar{y}_{i}.)^2 \text{ and } q_2 = \sum_{i=1}^{k} n(\bar{y}_i-\bar{y}..)^2$$

where $\bar{y}_{i.}$ is the mean of the i^{th} treatment effects and $\bar{y}..$ is the grand mean.

To test the hypothesis of equal treatment effects we must choose between the null and alternative hypothesis

$$H_o:\sigma_\gamma^2 = 0, \quad H_1:\sigma_\gamma^2 \neq 0 \tag{5.4.2}$$

The test criterion for the above model is given by

$$U = \frac{q_2/K-1}{q_1/N-K} \tag{5.4.3}$$

when the ℓ_{ij} are normally distributed and the null hypothesis H_o is true, then U follows the central F distribution. But when the alternative hypothesis is true then the power of the test is given by

$$\beta(\lambda) = P[U > \frac{K-1}{N-K} F_\varepsilon] \tag{5.4.4}$$

where $\beta(\lambda)$ is a function of $\lambda = \sigma_\gamma^2/\sigma_\ell^2$ and U is again an F-distribution. The method of calculation of U is the same as for the fixed effects model, but here y_{ij} is a linear function of two random variables. When the ℓ_{ij} are non-normal, the U will no longer be an F-distribution; but we can still compute the value of U and $\beta(\lambda)$ by the method of simulation. The detailed procedure for calculation of the power is given in section 5.2.

5.5 Random Model: Two-way classification

We shall now consider the random effect model in a two-way layout. The linear model we accept for the present case is given by

$$y_{ij} = \xi + \psi_i + \gamma_j + \ell_{ij} \qquad (5.5.1)$$

where ψ_i and γ_j are independent random normal variables with mean zero, and variance α_ψ^2 and σ_γ^2 respectively. The random variable ℓ_{ij} will not necessarily follow the normal distribution. In order to obtain the power of the test in the case of non-normal errors, we allow the error random variable to assume Erlangian and contaminated normal distributions.

In the random effects model, to test the hypothesis of equal treatment effects, we must choose between the null and alternative hypothesis

$$H_o:\sigma_\gamma^2 = 0 \qquad H_1:\sigma_\gamma^2 \neq 0 \qquad (5.5.2)$$

The test criterion for the between column test in the above situation is given by

$$U = \frac{\sum_{i=1}^{K} n(\bar{y}_{i.} - \bar{y}_{..})^2 / K-1}{\sum_{i=1}^{K} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2 / (n-1)(K-1)} \qquad (5.5.3)$$

where $\bar{y}_{i.}$ is the mean of the i^{th} column, $\bar{y}_{.j}$ is the mean of the j^{th} row and $\bar{y}_{..}$ is the ground mean.

When the error random variables are normally distributed, then in the null-hypothesis case U will be distributed as a central F-distribution. Again if the null-hypothesis is not true, the ratio U will be distributed as a central F-distribution and the power of the test is given by

$$\beta(\lambda) = P[U > \frac{K-1}{(n-1)(K-1)} F_\varepsilon] \qquad (5.5.4)$$

where $\lambda = \frac{\sigma_\gamma^2}{\sigma_\ell^2}$, F_ε is the table value of F at the ε percent level of significance and (K-1) and (n-1)(K-1) degrees of freedom. For the method of calculation of $\beta(\lambda)$ the reader is referred to section 5.2.

DISCUSSION OF THE RESULTS AND CONCLUSIONS

6.1 Power of the Test in General Linear Model

The results of the Tables in 1A give us values of P_{11} (i.e. type two error) at the 5% and 1% level of significance for normally distributed errors with unequal variance in the general linear hypothesis model. For comparison, the values of P_{11} for equal error variances are given in the first row of table 1A. It is seen from both fig.1 (one of the table values from 1A) and the tables in 1A that the power value is seriously affected when normally and independently distributed error variables have unequal error variances. Wherever error variances are unequal, the power value is greater than for equal error variances; the largest effect on the type two errors is observed where one of the error variances is much greater than the rest. The values of P_{11} for equal error variances given in the first row of table 1A can also be obtained by Tang's methods (1938).

The values of P_{11} at the 5% and 1% level of significance in table 1B show the effect on the power value due to the largest serial correlation among the normally distributed error variables. The first row of table 1B gives the values of P_{11} for uncorrelated error variables. Tables in 1B and figure 2 (one of the tables values from 1B) show that when the error variables are normally distributed and the errors are serially correlated then the type two errors are neither much greater nor much smaller than for uncorrelated error variables. Hence it can be inferred from the results obtained that the power of the test is little affected by the serial correlation of normally distributed error variables.

Table 1C indicates the accuracy of the results for equal error variances obtained by the present method compared with Tang's (1938) results.

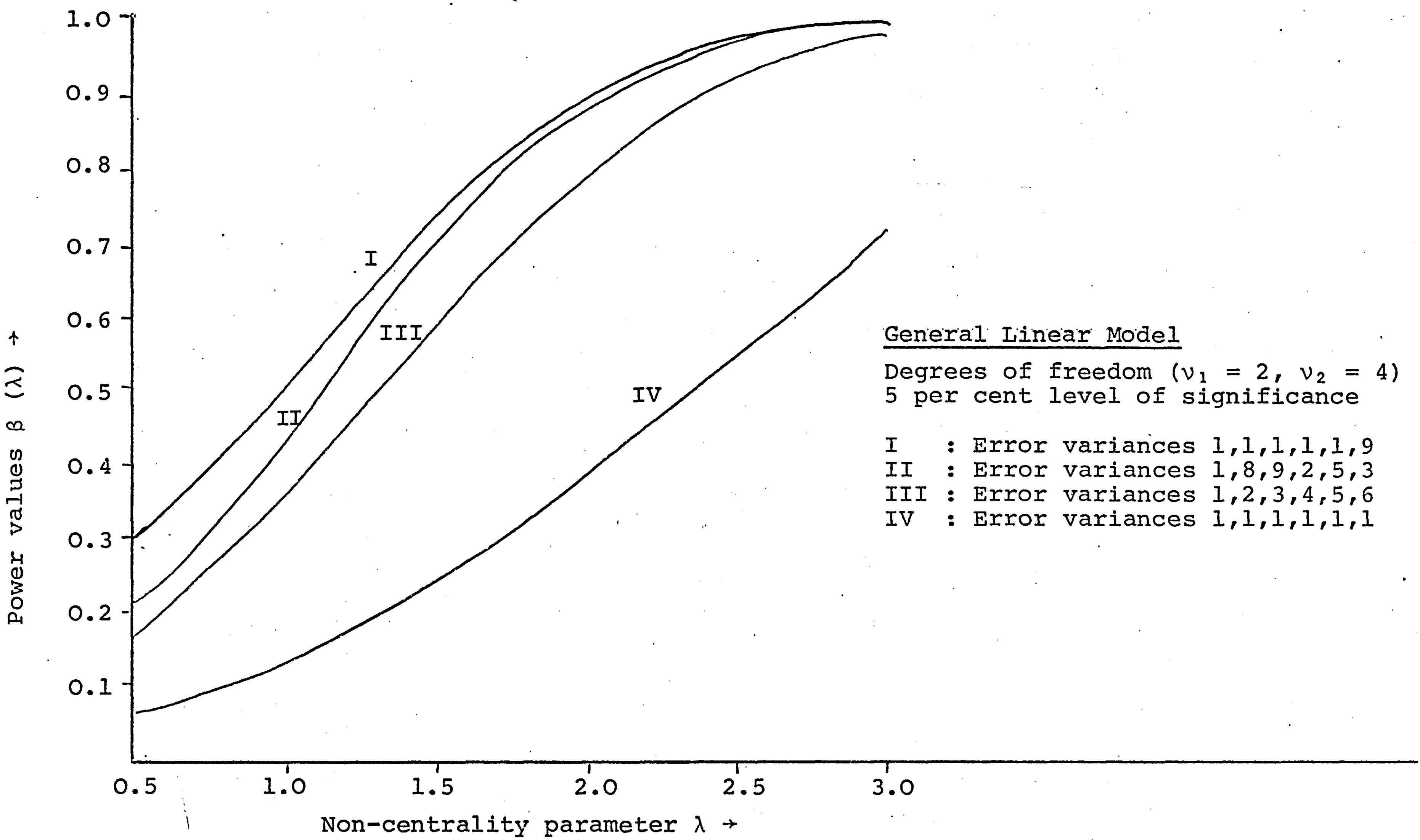


Fig.1 Effect of heteroskedasticity of the errors in Analysis of Variance on power of the test.

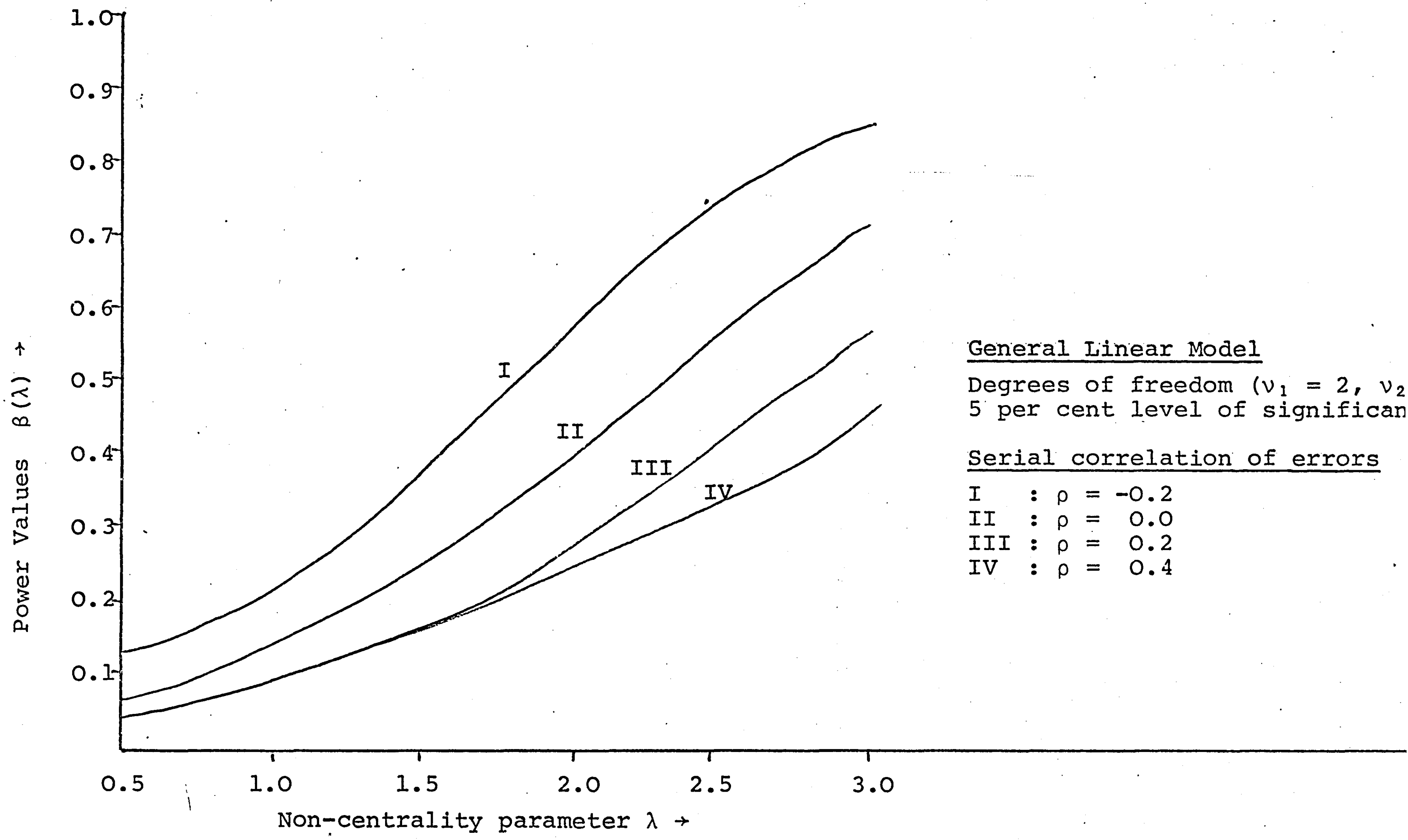


Fig.2 Effect of Interdependence of the errors in Analysis of Variance on Power of the test.

6.2 Power of the Test in One-way Classification

The results given in Table IIA are the values of p_{11} (i.e. type two error) at the 1% and 5% level of significance for the normally distributed errors with unequal group variances for our model. Table IIB shows the effect of unequal group sizes on the power value when the group variances are unequal.

It is obvious from Table IIA and figure 3 that the power of the test when the group variances are not equal is larger than when they are equal. Table IIB and figure 4 indicates that the group sizes do not greatly affect the power calculations, the allocation of 15 observations, 7,5,3 to groups gives greater power than 5,5,5.

However, it is obvious from the results that the P_{11} values are greatly affected when the variances are in the ratio 1:6:3 and group sizes are $n_1 = 7$, $n_2 = 5$, $n_3 = 3$. Hence it may be concluded that for the fixed effect one-way layout the power will be affected if the group variances and group sizes are greatly unequal.

6.3 Power of the Test in Two-way Classification

Table IIIA gives us a clear picture of the effect of unequal column variances on the power of the between-column test in the two-way layout Analysis of Variance. From Table IIIA, it is seen that the power of the between column test is greatly affected by the unequal column variances. The first row of Table IIIA shows the value of P_{II} when the column variances are equal.

Table IIIB gives us values of P_{II} when the variables within rows in the two-way layout Analysis of Variance are serially correlated. It appears that serial correlation within rows has

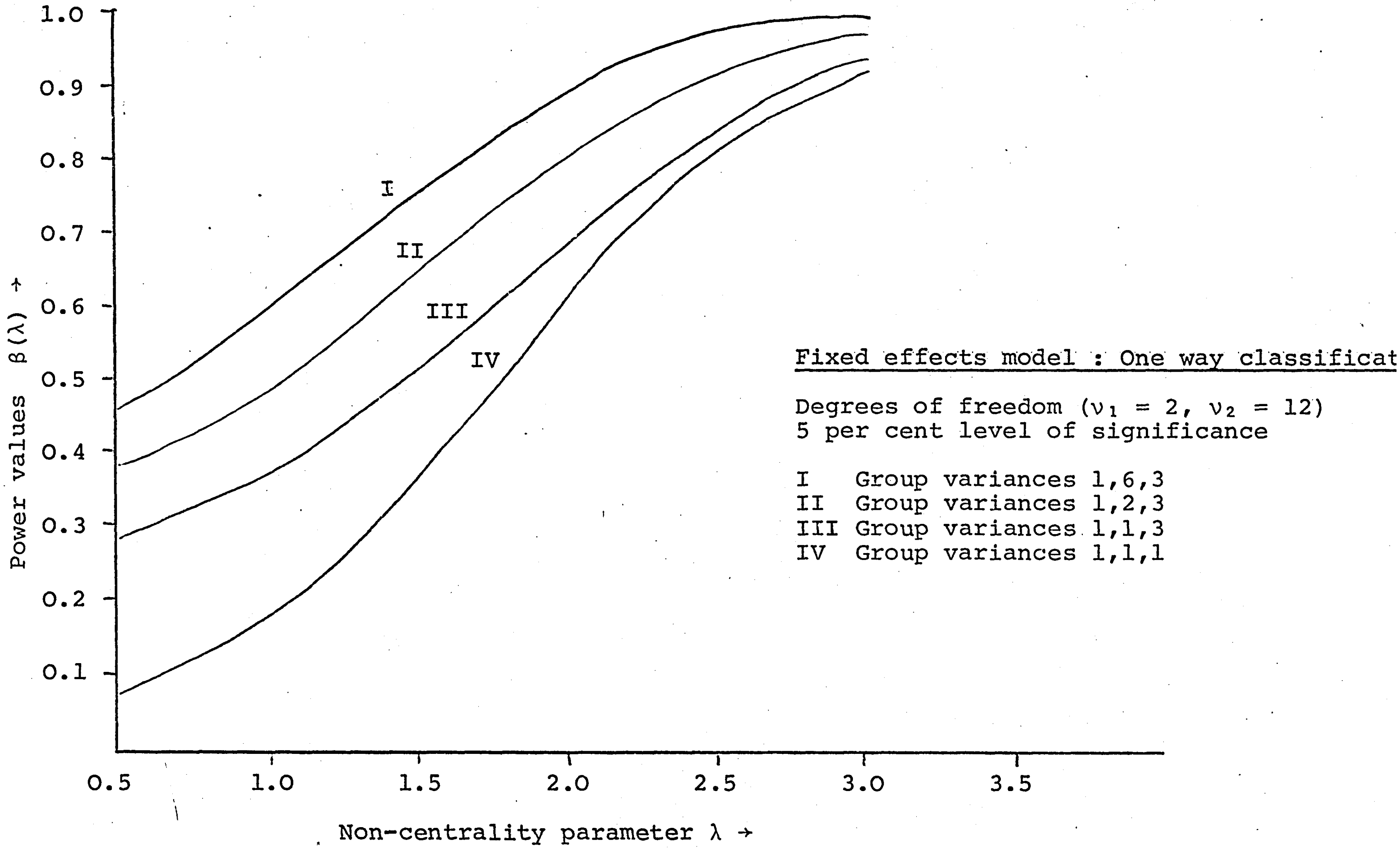


Fig.3 . Effect of heteroskedasticity of the errors in Analysis of Variance on Power of the test.

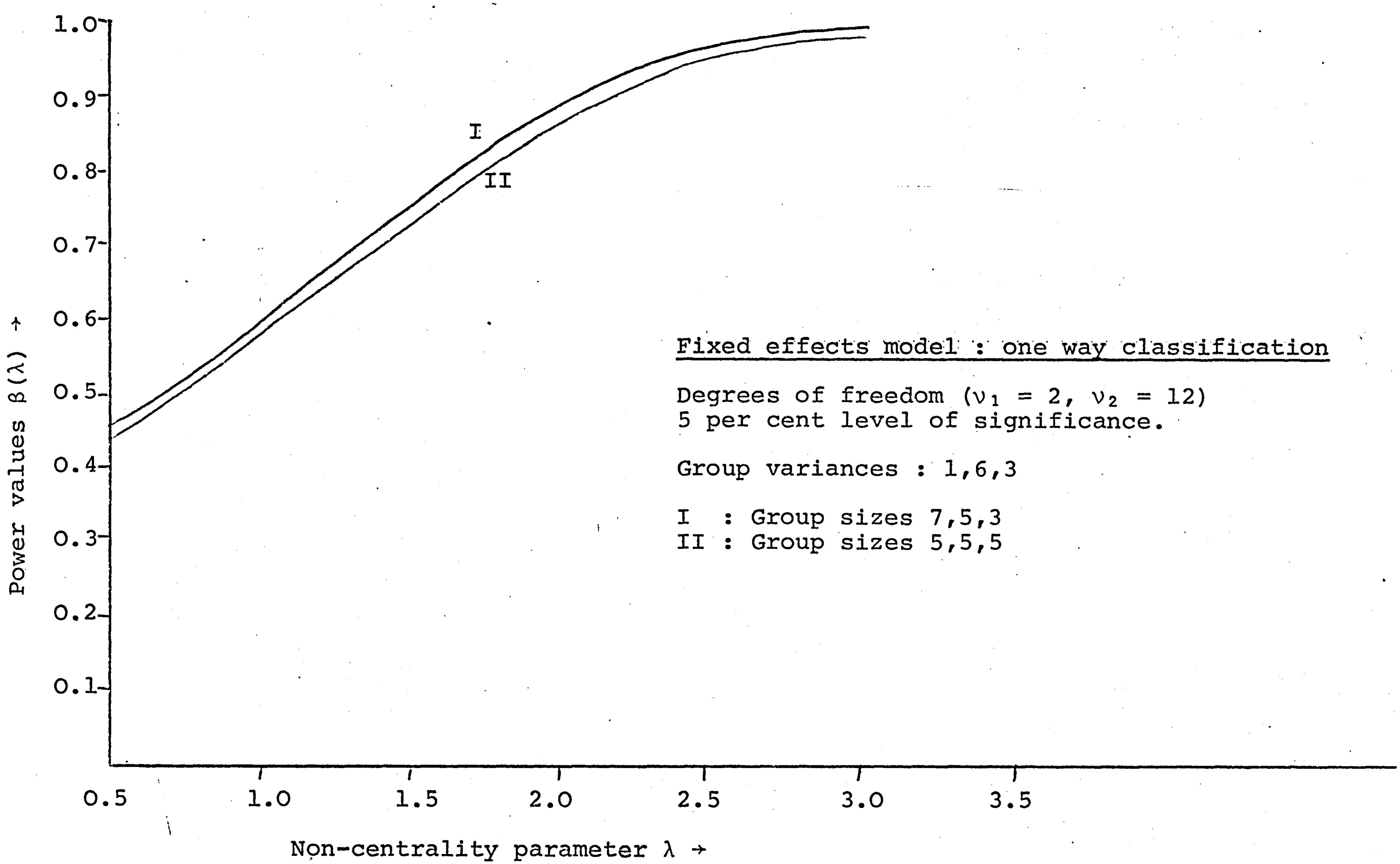


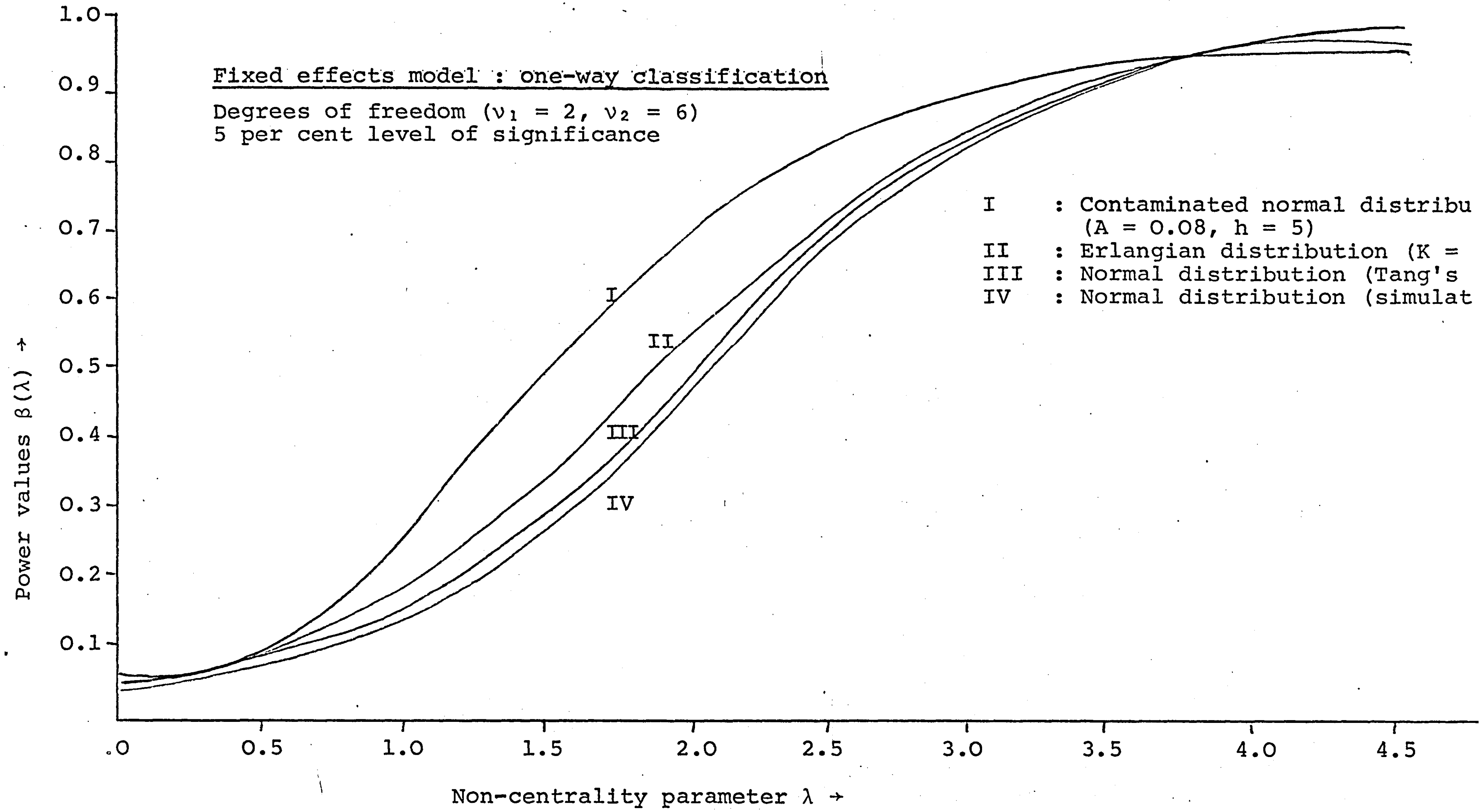
Fig.4 . Effect of heteroskedasticity of the errors in Analysis of Variance on Power of the test.

little effect on the power value for the between column comparison of homogeneity of means. The power values obtained for correlation coefficients $\rho = 0$ are those for independently distributed error variables within rows.

6.4 Power of the Test by a Simulation Method in Fixed Model

The results obtained by simulation methods for the one and two-way classification analysis of variance when the errors are normally distributed are given in Tables IVA and IVB respectively. To check the accuracy of the results obtained by the simulation methods, the power values are also calculated by Tang's (1938) method. The power values obtained by Tang's method are given in the second column of Tables IVA and IVB, while the simulation results in the normal theory case are given in the third column. Columns four and five of these tables represent the power values obtained by simulation methods for the Erlangian distribution of errors. The values of k=1 and k=4 indicate the one and four stage Erlangian distribution. The power values concerning the contaminated normal distribution of errors are given in columns six and seven with the probability of contamination, A, and the standard deviation of the wider normal distribution, h.

It is quite clear from the results obtained in Table IVA and figure 5 that an Erlangian distribution for the errors has little effect on the power of the test. In the case of the contaminated normal distribution for the error variables, the power value is slightly more affected than for the Erlangian distribution. A similar conclusion can be drawn from Table IVB. In general the result indicates that the power value is not greatly affected by the non-normality of the errors.



<u>Fig.5</u> Effect of non-normality of the errors in Analysis of Variance on Power of the test.

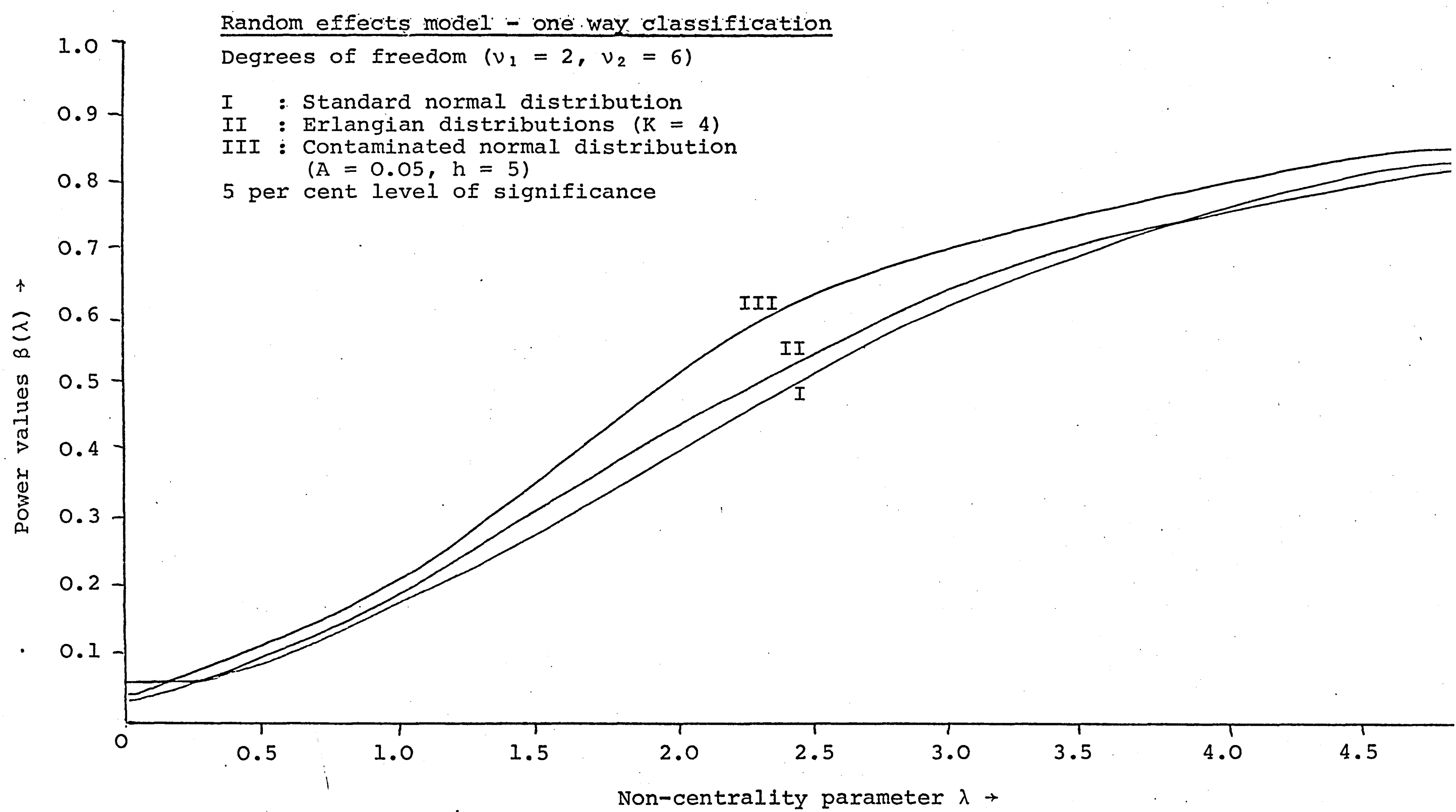


Fig.6 Effect of non-normality of the errors in Analysis of Variance on Power of the test.

6.5 Power of the Test by A Simulation Method in Random Model

The results obtained by simulation methods for the one and two-way layouts for this model when the errors are both normally and non-normally distributed are given in tables VA and VB respectively.

It is quite clear from the results obtained in table VA and figure 6 that both Erlangian and contaminated normal distributions for the errors have little effect on the power of the test. In general for the contaminated normal distribution the power value is slightly more affected than for the Erlangian distribution.

6.6 Discussion of Results

(i) One of the more interesting features of the results described above is the way in which the power of the test increases whenever the error variances are unequal. This is most important since not only is the case of unequal error variances the one which occurs most frequently in practice, but also statisticians have spent much time and ingenuity devising transformations to ensure equality of such variances. In the future, provided a computer is available, or more probably, suitable published tables, the applied statistician can not only save himself some work but use a more powerful test as well.

(ii) Serial correlation of the errors affects the power in an unexpected way. A negative correlation increases the power while a positive correlation decreases it (Fig.2). In view of the first result, one would have expected a positive correlation to produce a reduction in the power since this is moving towards equality of variances. However, a negative correlation by inserting more "unequalness" in the variances produces an increase in the power. This result may be useful in those practical experiments in which results are taken sequentially and it is impossible to eliminate a time effect.

(iii) Box (1954) has shown that the true significance level in the case of unequal group variances is different from the nominal significance level. If the curves in Fig.1, approach $\lambda = 0$ in a reasonably smooth way it will indicate that in the case of unequal group variances the true levels of significance are different from the nominal significance level supporting Box's finding. It is clear from the results that the power curves for unequal error variances do lie above the standard situation, possibly owing to the increased values of the true significance levels. Therefore, for different group variances one should use corresponding true level of significance because the evidence indicates that it will provide the most powerful test.

6.7 Areas for further research

The development and application of generalised incomplete Beta distribution suggests areas that lend themselves to further study. These are

(i) Robustness of power in mixed model.

(ii) Robustness of power when the assumption of additivity of the model is violated.

(iii) Relationship of the power with the cost function of the experimental design.

(iv) Effect on power values in analysis of variance when the experimental design is non-orthogonal.

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APPENDIX A

Theorem 1. Ruben (1962)

(i)

1. $H_{n'};\ A;\ b(\alpha) = \sum_{j=o}^{\infty} C_j\ \chi^2_{n+2j}\ (\alpha/g)$

where g is an arbitrary positive constant.

2. $C_j \equiv C_{j},\ n',\ A,\ b\ (g)$

$$= A^{-\frac{1}{2}}\ e^{-\frac{1}{2}\sum_{i=1}^{n'} b^2{}_i}\ g^{\frac{1}{2}n'+j}\ E[Q^j\ H_{2j}\ (L/Q^{\frac{1}{2}})]/(2j)!$$

3. where H_{2j} is the Hermite polynomial of degree 2j and L and Q are defined by

$$L \equiv L(x) = \Sigma(b_i/a_i^{\frac{1}{2}})x_i, \quad Q \equiv Q(x) = \Sigma(\frac{1}{a_i} - \frac{1}{g})x_i{}^2$$

and the x_i are independent normal variables with zero means and unit variances. Further, the series in (1) converges uniformly on every finite interval of α.

(ii)

4. $$\text{Exp}\left[-\tfrac{1}{2}\Sigma b_i{}^2\ \frac{1-Z^*}{1-(1-g/a_i)Z^*}\right]\ \pi(g/a_i)^{\frac{1}{2}}\ \left[1-(1-g/a_i)Z^*\right]^{-\frac{1}{2}}$$

$$|Z^*| < \min_i|1 - g/a_i|^{-1}$$

$$= \Sigma c_j Z^{*j}$$

(iii) The C_j satisfy the recursion relationship.

5. $$C_o = e^{-\frac{1}{2}\sum_{i=1}^{n'} bi^{\frac{1}{2}}}\ \prod_{i=1}^{n'} (g/a_i)^{\frac{1}{2}}$$

$$C_j = (2j)^{-1}\sum_{r=o}^{j-1} h_{j-\gamma}\ C_\gamma \qquad j = 1,2,...$$

where

$$h_m = \sum_{i=1}^{n'} (1 - g/a_i)^m + mg\sum_{i=1}^{n'} (b^2{}_i/a_i)(1 - g/a_i)^{m-1}$$

$$m = 1,2,...$$

(i)

6. $$H_{n', A, O}(\alpha) = \sum_{j=o}^{\infty} d_j \chi^2_{n'+2j}(\alpha/g)$$

where g is an arbitrary positive constant.

7. $$d_j \equiv d_{j, n', A, O}(g) = A^{-\frac{1}{2}} g^{\frac{1}{2}n'+j} E[(-Q)^j]/(2^j j!)$$

and Q is defined in (3). Further the series in (6) converges uniformly on every finite interval of α.

(ii)

8. $$\sum_{i=1}^{n'} \{(g/a_i)^{\frac{1}{2}} [1 - (1 - g/a_i)Z^*]^{\frac{1}{2}}\} = \Sigma d_j Z^{*j} \qquad (|Z^*| < \min_i |1-g/a_i|^{-1})$$

The d_j satisfy the recursion relationship.

(iii)

9. $$d_j = (Zj)^{-1} \sum_{=o}^{j-1} h_{j-\gamma} d_\gamma$$

$$d_o = \prod_{i=1}^{n'} (g/a_i)^{\frac{1}{2}} \qquad j = 1, 2, \ldots$$

where

$$h_m = \sum_{i=1}^{n'} (1 - g/a_i) \qquad m = 1, 2, \ldots$$

Distribution of the ratio of two quadratic forms

Since the g's in the equations (2.3.3) and (2.3.5) in chapter 2 are arbitrary scale parameters, we can take the value of g equal to unity in all cases. We have from (2.3.3) and (2.3.5)

$$p[q_1 \leqslant \alpha] = \sum_{i=o}^{\infty} c_j \, \chi^2_{p+2j}(\alpha) \qquad (1)$$

and

$$p[q_2 \leqslant \alpha] = \sum_{i=o}^{\infty} d_i \, \chi^2_{n-p+2i}(\alpha) \qquad (2)$$

where $\chi^2_p(.)$ is the central χ^2 distribution with p.d.f.

With the help of conditional probabilities we find that

$$p[q_1/q_2 \leqslant \alpha] = \int_0^\infty p(q_1 \leqslant \alpha \, q_2/q_2) f(q_2) dq_2 \qquad (3)$$

where $f(q_2)$ is the probability density function of q_2

$$= \int_0^\infty \{ \sum_{j=o}^{\infty} C_j \, \chi^2_{p+2j} \; (\alpha q_2/q_2) \sum_{i=o}^{\infty} d_i . \frac{e^{-q_2/2} \, q_2^{\frac{n-p+2i}{2}-1}}{2^{n-p+2i} \, \Gamma(\frac{n-p+2i}{2})} dq_2 \}$$

since the two series are uniformly convergent on every finite interval of α, we have

$$p[q_1/q_2 \leqslant \alpha] = \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i \int_0^\infty \frac{\chi^2_{p+2j}(\alpha q_2/q_2)}{2^{n-p+2i} \, \Gamma(\frac{n-p+2i}{2})} e^{-q_2/2} \, q_2^{\frac{n-p+2i}{2}-1} dq_2$$

$$= \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i \int_0^\alpha h_{p+2j,n-p+2i} \; (u = q_1/q_2) du \qquad (4)$$

where $h_{p+2j,n-p+2i}(u)$ denotes the probability distribution function of the ratio of two independent chi-square variates (central) with d.f $p+2j$ and $n-p+2i$ in the numerator and denominator respectively.

Consider now,

$P(u = \frac{u_1}{u_2} \leqslant \alpha)$ where u_1 and u_2 are two independent χ^2 (central) variate with d.f ν and γ respectively. Then

$$P(u \leqslant \alpha) = \int_0^{\alpha} h_{\nu,\gamma}(u) \quad du$$

as just defined.

$$\text{But } P(u \leqslant \alpha) = P\left(\frac{u_1/\nu}{u_2/\gamma} \leqslant \frac{\gamma}{\nu}\alpha\right) = F_{\nu,\gamma}\left(\frac{\gamma}{\nu}\alpha\right)$$

where $F_{\nu,\gamma}(.)$ denote the cumulative distribution of Fisher's variance ratio (central F).

Hence returning to equation (4), we have

$$P(u = q_1/q_2 \leqslant \alpha) = \sum_{j=o}^{\infty} \sum_{i=o}^{\infty} C_j d_i \, F_{p+2j,n-p+2i}\left(\frac{n-p+2i}{p+2j}\alpha\right)$$

Expectation and Variance of Test Criterion Z

$$E_p(Z) = E_p(\frac{V_E}{V_2})$$

or

$$V_2 E_p(Z) = E_p (V_E) \qquad (1)$$

Now,

$$E_p(V_E) = E_p(D'M D) = E_p(M_1 DD')$$

$$= \text{trace } \{M E_p (DD')\}$$

$$= \frac{V_2}{N(N-1)} \text{ trace } \{M (NI - 1_n 1_n')\}$$

$$= \frac{V_2}{N(N-1)} [\text{trace } (M NI) - \text{trace } (M 1_n 1_n')]$$

$$E_p(V_E) = \frac{V_2}{N(N-1)} [Np - N] \qquad \begin{array}{|l} M 1_n = 1_n \\ \text{trace } (M) = p \end{array}$$

$$E_p(V_E) = \frac{V_2 (p-1)}{N-1}$$

Substituting the value of $E_p(V_E)$ in (1) we obtain

$$E_p(Z) = \frac{p-1}{N-1} \qquad (2)$$

The variance of Z is given by

$$V_p(Z) = E_p\{\frac{V_E^2}{V_2^2}\} - \{E_p(Z)\}^2 \qquad (3)$$

Now, since

$$V_E = D'MD = \sum_{i=1}^{N} D_i^2 M_{ii} + \sum_i \sum_{j \neq i} D_i D_j M_{ij}$$

it follows that

$$V_E^{\,2} = \left[\sum_{i=1}^{N} D_i^{\,2} M_{ii} + \sum_i \sum_{i \neq j} D_i D_j M_{ij} \right] \qquad (4)$$

Since the X's are fixed, the M_1 remains fixed even when the D's are permuted in all possible ways. Now expanding the R.H.S. of the equation (4) and taking the expectation we have

$$E_p(V_E^{\,2}) = E_p(D_i^{\,4}) \sum_i M_{ii}^{\,2} + E_p(D_i^{\,2} D_j^{\,2}) \left(2 \sum_i \sum_{j \neq i} M_{ij}^2 + \sum_i \sum_{j \neq i} M_{ii} M_{jj}\right)$$

$$+ E_p(D_i D_j^{\,2} D_K) \left(4 \sum_i \sum_{K \neq i} \sum_{j \neq i,K} M_{ij} M_{jk} + 2 \sum_i \sum_{K \neq i} \sum_{j \neq K,i} M_{ii} M_{jk}\right)$$

$$+ E_p(D_i^{\,3} D_j)\, 4 \sum_i \sum_{j \neq i} M_{ii} M_{ij} + E_p(D_i D_j D_K D_\ell)$$

$$\sum_i \sum_{j \neq i} \sum_{K \neq j,i} \sum_{\ell \neq K,j,i} M_{ij} M_{K\ell} \qquad (5)$$

Now, using David and Kendall's table (1949) we find

(i) $E_p(D_i^{\,4}) = \dfrac{V_4}{N}$

(ii) $E_p(D_i D_j\, D_K) = \dfrac{2V_4 - V_2^{\,2}}{N(N-1)(N-2)}$

(iii) $E_p(D_i D_j D_K D_\ell) = \dfrac{3V^2{}_2 - 6V_4}{N(N-1)(N-2)(N-3)}$ (6)

(iv) $E_p(D_i^{\,2} D_j^{\,2}) = \dfrac{V^2{}_2 - V_4}{N(N-1)}$

(v) $E_p(D_i\ D_j) = -\dfrac{V_4}{N(N-1)}$

Also, using the relation

$$M = M', \ M\, 1_n = 1_n, \ \ M^2 = M \text{ and trace } (M) = p$$

we find that the sums in (5) can be expressed in terms of m, N, p, where m is the sum of the squares of the diagonal elements of the matrix $M = \{M_{uv}\}$.

We derive (see Appendix D)

(vi) $\sum_i \sum_{j \neq i} M_{ij} M_{ji} = p-m$

(ix) $\sum_i \sum_{K \neq j,} \sum_{j \neq i,K} M_{ii} M_{jK} = Np-2p-p^2+2m$

(vii) $\sum_{i} \sum_{j\neq i} M_{ii}M_{jj} = p^2-m$

(x) $\sum_{i} \sum_{j\neq i} M_{ii}M_{ij} = p-m$

(viii) $\sum_{i} \sum_{K\neq i} \sum_{j\neq i,K} M_{ij}M_{jK} = N-3p+2m$

(xi) $\sum_{i} \sum_{j\neq i} \sum_{K\neq j,i} \sum_{\ell\neq K,j,i} M_{ij}M_{K\ell} = N^2-2Np-4N+10p+p^2-6m$

(7)

Now substituting (7) and (6) in (5) and writing V_2 and V_4 in terms of Fisher's K-statistics i.e.

$$V_2 = (N-1)\ K_2$$

$$V_4 = (N-1)\ (N-2)\ (N-3)\ K_4 + 3(N-1)^3\ K_2{}^2$$

we have

$$V_p(Z) = E_p\{\frac{V_E{}^2}{V_2{}^2}\} - \{E_p(Z)\}^2$$

$$= \frac{2(p-1)(N-p)}{(N-1)^2(N+1)} + \frac{K_4/K_2{}^2}{(N-1)^2}\left[m - \frac{p^2}{N} - \frac{2(p-1)(N-p)}{N(N+1)}\right]$$

APPENDIX D

$M_1 = 1$ $\qquad$ $\Sigma M_{iK} = 1$ (1)

$M_2 = M$ $\qquad$ $\Sigma M_{ij} M_{jK} = M_{iK}$ (2)

M is symmetric $\quad M_{ij} = M_{ji}$

$$p = \sum_{i=1}^{N} M_{ii}, \quad m = \Sigma M^2_{ii} \qquad M_{ii} = \sum_j M_{ij}M_{ji}$$

(1) $$\sum_i M^2_{ii} = \sum_i \sum_{j \neq i} M_{ij}M_{ji} = \sum_i [\sum_j M_{ij}M_{ji} - M_{ii}M_{ii}]$$

$$= \sum_i [M_{ii} - M_{ii}M_{ii}]$$

$$= \sum_i M_{ii} - \sum_i M_{ii}M_{ii} = \underline{\underline{p - m}}$$

(2) $$\sum_i \sum_{j \neq i} M_{ii}M_{jj} = \sum_i [\sum_j M_{ii}M_{jj} - M_{ii}M_{ii}]$$

$$= \Sigma M_{ii} \, \Sigma M_{jj} - \Sigma M_{ii}M_{ii}$$

$$= \underline{\underline{p^2 - m}}$$

(3) $$\sum_i \sum_{K \neq i} \sum_{j \neq i,K} M_{ij}M_{jK} = \sum_i \sum_{K \neq i} [\sum_j M_{ij}M_{jK} - M_{ii}M_{iK} - M_{iK}M_{KK}]$$

$$= \sum_i \sum_{K \neq i} [M_{iK} - M_{ii}M_{iK}M_{KK}] \qquad \text{from (1)}$$

$$= \sum_i \{\sum_K (M_{iK} - M_{ii}M_{iK} - M_{iK}M_{KK}) - [M_{ii} - M_{ii}M_{ii} - M_{ii}M_{ii}]\}$$

$$= \sum_i (\sum_K M_{iK}) - \Sigma M_{ii}(\sum_K M_{iK}) - \sum_K M_{KK}(\sum_i M_{iK}) - \Sigma M_{ii} + 2\Sigma M_{ii}M_{ii}$$

$$= N - p - p - p + 2m = \underline{\underline{N - 3p + 2m}}$$

(4) $$\sum_j \sum_{K\neq j} \sum_{i\neq J,K} M_{ii}M_{jK} = \sum_j \sum_{K\neq j} [\sum_i M_{ii}M_{jK} - M_{jj}M_{jK} - M_{KK}M_{jK}]$$

$$= \sum_j \sum_{K\neq j} [p(M_{jK}) - M_{jj}M_{jK} - M_{KK}M_{jK}]$$

$$= \sum_j [\sum_K \{p(M_{jK}) - M_{jj}M_{jK} - M_{KK}M_{jK}\} - \{pM_{jj} - M_{jj}M_{jj} - M_{jj}M_{jj}\}]$$

$$= p\sum_j (\sum_K M_{jK}) - \sum_j M_{jj} (\sum_K M_{jK}) - \sum_K M_{KK} (\sum_j M_{jK}) - p\sum_j M_{jj} + 2\sum_j M_{jj}M_{jj}$$

$$= Np-p-p-(pXp) + 2m$$

$$= \underline{\underline{Np - 2p - p^2 + 2m}}$$

(5) $$\sum_i \sum_{K\neq i} M_{ii}M_{iK} = \sum_i [\sum_K M_{ii}M_{iK} - M_{ii}M_{ii}]$$

$$= \sum M_{ii} (\sum_K M_{iK}) - \sum_i M_{ii}M_{ii}$$

$$= \underline{\underline{p - m}}$$

(6) $$\sum_i \sum_{j\neq i} \sum_{K\neq j,i} \sum_{\ell\neq K,j,i} M_{ij}M_{K\ell}$$

$$= \sum_i \sum_{j\neq i} \sum_{K\neq j,i} [\sum_\ell M_{ij}M_{K\ell} - M_{ij}M_{KK} - M_{ij}M_{Kj} - M_{ij}M_{Ki}]$$

$$= \sum_i \sum_{j\neq i} \sum_{K\neq j} [M_{ij} - M_{ij}M_{KK} - M_{ij}M_{Kj} - M_{ij}M_{Ki}]$$

$$= \sum_i \sum_{j\neq i} [\sum_K \{M_{ij} - M_{ij}M_{KK} - M_{ij}M_{Kj} - M_{ij}M_{Ki}\}$$

$$- \{-M_{ij}M_{jj} - M_{ij}M_{jj} - M_{ij}M_{ji}\} - \{-M_{ij}M_{ii} - M_{ij}M_{ij} - M_{ij}M_{ii}\}]$$

$$= \sum_i \sum_{j \neq i} [\{(N-2)M_{ij} - p(M_{ij}) - M_{ij} - M_{ij}\} + 3M_{ij}M_{jj} + 2M_{ij}M_{ji} + 2M_{ij}M_{ii}]$$

$$= \sum_i \sum_{j \neq i} [(N-p-4)M_{ij} + 2M_{ij}M_{jj} + 2M_{ij}M_{ji} + 2M_{ij}M_{ii}]$$

$$= \sum_i [\sum_{j \neq i} \{(N-p-4)M_{ij} + 2M_{ij}M_{jj} + 2M_{ij}M_{ji} + 2M_{ij}M_{ii}\}$$

$$- \{(N-p-4)M_{ii} - 6M_{ii}M_{ii}\}]$$

$$= \sum_i (N-p-4) + 2p\sum_i M_{ij} + 2\sum_i M_{ii} + 2\sum_i M_{ii}$$

$$- (N-p-4)\Sigma M_{ii} - 6\Sigma M_{ii}M_{ii}$$

$$= N(N-p-4) + 2p + 2p + 2p - p(N-p-4) - 6m$$

$$= N^2 - N - 4N + 6p - Np + p^2 + 4p - 6m$$

$$= \underline{\underline{N^2 - 2Np - 4N + 10p + p^2 - 6m}}$$

APPENDIX E

Construction of Tables IA and IB and II

The construction of Tables IA and IB for p_{11} corresponding to the 5% and 1% levels of significance was carried out in the following manner.

(1) $V^{-1} = NN'$ (where V is the error variance-covariance matrix)

(2) $M_1^* = V^{-1}x(x'V^{-1}x)^{-1}\,x'V^{-1}$

(3) $M_2^* = V^{-1}-V^{-1}x(x'V^{-1}x)^{-1}x'V^{-1}$

(4) Latent roots and latent vectors of $N'M_1^*N$. Let a_i $(i=1,2..n)$ be the latent roots of $N'M_1^*N$ and K be the orthogonal matrix of the latent vectors of $N'M_1^*N$.

(5) $b = K'N^{-1}\mu^*$ where $\mu^*(\beta-x\beta^*)$

(6) c_j

$$g_m = \sum_i \left(1 - \frac{1}{a_i}\right)^m + m\sum_i (b_i^2/a_i)\left(1 - \frac{1}{a_i}\right)^{m-1} \qquad m = 1,\ 2\ \ldots$$

$$c_o = e^{-\frac{1}{2}\Sigma b_i^2} \prod_{i=j} \left(\frac{1}{a_i}\right)^{\frac{1}{2}}$$

$$c_j = (2j)^{-1} \sum_{r=o}^{j-1} g_{j-r}\, c_r \qquad (j = 1,\ 2\ \ldots)$$

(7) Latent roots of $N'M_2^*N$ (let a_j $(j=1,2...n)$ be these latent roots)

(8) d_i

$$g_n = \sum_j \left(1 - \frac{1}{a_j}\right) \qquad n = 1,\ 2\ \ldots$$

$$d_o = \prod_j \left(\frac{1}{a_j}\right)^{\frac{1}{2}}$$

$$d_i = (2i)^{-1} \sum_{r=o}^{i-1} g_{i-r}\, d_r \qquad (i = 1,\ 2\ \ldots)$$

(9) $u_o = \frac{p}{n-p}F_\varepsilon$ where F_ε is the value of F at the chosen level of significance with p and n-p d.f's. The value of ε is taken as 0.05 and 0.01 respectively.

(10) $$I_{\frac{u_o}{1+u_o}}\left(\frac{p+2j}{2},\ \frac{n-p+2i}{2}\right) \qquad j = 0,\ 1,\ 2,\ldots 15 \quad i = 0,\ 1,\ 2,\ldots 15$$

where $I_x\ (p,q)$ is the incomplete Beta function.

(11) $p_{11} = \sum_{i=o}^{15} \sum_{j=o}^{15} C_j d_i \cdot I_{\frac{u_o}{1+u_o}} \left(\frac{p+2j}{2}, \frac{n-p+2i}{2}\right)$

In calculation of p_{11} the summations of i and j are considered up to the value of 15 because the value of p_{11} is not changed in the four places of decimal by summing over any more extra terms for i and j greater than 15.

To enter the table IA, the unequal diagonal elements in the diagonal variance covariance matrix $\underset{\sim}{V}$ are given under the heading 'error variances'. The noncentrality parameter is given by $\lambda = (\frac{1}{2}\, \underset{\sim}{b}'\underset{\sim}{b})^{\frac{1}{2}}$. ν_1 and ν_2 are the d.f. and ε is the level of significance. Hence $\nu_1 = p$ and $\nu_2 = n-p$ respectively.

In table IB, ρ represents the first order serial correlation of the errors. We note that the matrix $\underset{\sim}{V} = \underset{\sim}{\delta}$ where

$$\underset{\sim}{\delta} = \begin{pmatrix} 1 & \rho & 0 & \cdots\cdots & 0 & 0 \\ \rho & 1 & \rho & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & \rho & 1 \end{pmatrix}$$

ν_1 and ν_2 are the degrees of freedom, and the non-centrality parameter is given by $\lambda = (\frac{1}{2}\, \underset{\sim}{b}'\underset{\sim}{b})^{\frac{1}{2}}$. In the first row of table IA a particular case of our general formula is obtained by substituting unity for the diagonal elements of the variance covariance matrix $\underset{\sim}{V}$. Similarly in table IB we have substituted $\rho = 0$ for the particular case. Since $\underset{\sim}{\delta}$ is by definition a positive definite matrix it is necessary that the value of ρ lies between $-\frac{1}{2} \leq \rho \leq \frac{1}{2}$ (see SCHEFFÉ (1959) pp. 334).

TABLE 1A: Table of ε = 0.01, 0.05 and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2, \nu_2 = 2\right)$

Error Variances	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.0
1:1:1:1	.938	.987	.903	.980	.848	.968	.777	.851	.694	.929	.596	.886
1:2:3:4	.825	.871	.660	.706	.419	.457	.195	.216	.062	.070	.013	.014
1:8:4:2	.830	.911	.597	.670	.319	.366	.118	.139	.029	.034	.004	.005
1:1:1:9	.039	.877	.618	.651	.342	.364	.132	.142	.034	.036	.005	.006

TABLE 1A: Table of ε = 0.01, 0.05 and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2 \ \nu_2 = 3\right)$

Error Variances	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.
1:1:1:1:1	.933	.986	.883	.974	.797	.952	.682	.917	.548	.868	.408	.79
1:2:3:4:5	.786	.850	.693	.791	.525	.643	.310	.407	.130	.180	.036	.05
1:8:9:3:2	.802	.909	.568	.669	.298	.367	.109	.140	.026	.035	.004	.00
1:1:1:1:9	.780	.813	.564	.594	.304	.324	.114	.123	.028	.030	.004	.00

TABLE 1A: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values of P_{11}

degrees of freedom $(\nu_1 = 2, \nu_2 = 4)$

Error Variances	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.[illegible]
1:1:1:1:1:1	.930	.985	.867	.969	.757	.936	.608	.880	.442	.800	.287	.68[illegible]
1:2:3:4:5:6	.803	.870	.635	.714	.401	.472	.189	.232	.062[illegible]	.079	.013	.01[illegible]
1:8:9:2:5:3	.784	.891	.555	.662	.292	.369	.107	.143	.026	.036	.004	.00[illegible]
1:1:1:1:1:9	.698	.730	.499	.529	.265	.285	.098	.106	.023	.026	.003	.00[illegible]

TABLE 1A: Table of ε = 0.01, 0.05 and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2\ \nu_2 = 5\right)$

Error Variances	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ϵ =0.05	ϵ =0.01	ϵ=0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ=0.05	ϵ =0.01	ϵ=0.05	ϵ =0.01	ϵ =0.05	ϵ =0
1:1:1:1:1:1:1	.927	.984	.855	.964	.727	.920	.554	.845	.371	.736	.214	.59
1:2:3:4:5:6:7	.793	.872	.621	.716	.388	.476	.181	.236	.059	.081	.013	.01
1:8:9:5:7:2:1	.778	.906	.544	.661	.281	.361	.102	.138	.024	.034	.003	.00
1:1:1:1:1:1:9	.606	.640	.430	.461	.225	.247	.082	.092	.019	.022	.003	.00

TABLE 1A: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2, \nu_2 = 6\right)$

Error Variances	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0.01	ϵ =0.05	ϵ =0
1:1:1:1:1:1:1:1	.925	.983	.845	.960	.702	.097	.513	.814	.321	.679	.167	.5
1:2:3:4:5:6:7:8	.783	.874	.605	.717	.372	.476	.171	.237	.055	.082	.012	.0
1:4:8:9:7:5:3:2	.747	.879	.521	.657	.270	.370	.098	.146	.024	.038	.004	.0
1:1:1:1:1:1;1:9	.517	.554	.362	.398	.187	.212	.067	.078	.015	.019	.002	.0

TABLE 1B: Table of $\epsilon = 0.01, 0.05$ and the corresponding values of P_{11}

degrees of freedom $(\nu_1 = 2, \nu_2 = 2)$

Serial Correlation of Errors	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.$
$\rho = 0.0$	.938	.987	.903	.980	.848	.968	.777	.951	.694	.929	.596	.88
$\rho = 0.2$	.945	.980	.918	.974	.876	.965	.819	.953	.752	.937	.674	.91
$\rho = 0.4$	.890	.917	.867	.913	.831	.905	.783	.894	.722	.877	.628	.81
$\rho = -0.2$	.919	.991	.874	.981	.804	.964	.716	.941	.613	.904	.482	.80

TABLE 1B: Table of ε = 0.01, 0.05 and the corresponding values of P_{11}

degrees of freedom $(\nu_1 = 2 \ \nu_2 = 3)$

Serial Correlation of Errors	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.
ρ = 0.0	.933	.986	.883	.974	.797	.952	.682	.917	.548	.868	.408	.79
ρ = 0.2	.953	.986	.917	.978	.856	.964	.767	.940	.658	.907	.536	.86
ρ = 0.4	.907	.928	.884	.923	.843	.914	.781	.899	.702	.878	.609	.84
ρ = -0.2	.883	.970	.811	.950	.695	.915	.550	.860	.396	.772	.248	.60

TABLE 1B: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2, \nu_2 = 4\right)$

Serial Correlation of Errors	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.$[illegible]
$\rho = 0.0$	.930	.985	.867	.969	.757	.936	.608	.880	.442	.800	.287	.68[illegible]
$\rho = 0.2$	.957	.989	.916	.980	.839	.960	.725	.926	.584	.873	.434	.80[illegible]
$\rho = 0.4$	.919	.938	.892	.932	.840	.920	.759	.898	.652	.864	.530	.81[illegible]
$\rho = -0.2$	.872	.970	.777	.939	.626	.880	.447	.787	.276	.649	.142	.45[illegible]

TABLE 1B: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2 \quad \nu_2 = 5\right)$

Serial Correlation of Errors	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.0$
$\rho = 0.0$	.927	.984	.855	.964	.727	.920	.554	.845	.371	.736	.214	.594
$\rho = 0.2$	.960	.991	.915	.980	.827	.956	.694	.912	.531	.840	.365	.743
$\rho = 0.4$	.929	.945	.904	.940	.852	.927	.766	.903	.650	.863	.514	.804
$\rho = -0.2$	.856	.961	.742	.918	.566	.835	.369	.706	.200	.533	.087	.330

TABLE 1B: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values of P_{11}

degrees of freedom $\left(\nu_1 = 2, \nu_2 = 6\right)$

Serial Correlation of Errors	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.05$	$\epsilon=0.$
$\rho = 0.0$	.925	.984	.845	.964	.702	.920	.513	.845	.321	.736	.167	.59
$\rho = 0.2$	.962	.991	.913	.980	.817	.956	.669	.912	.490	.840	.315	.74
$\rho = 0.4$	.936	.945	.911	.940	.856	.927	.762	.903	.633	.863	.484	.80
$\rho = -0.2$	.846	.961	.717	.918	.524	.835	.318	.706	.155	.533	.059	.33

TABLE 1C: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values of P_{II} by T-method and K-method

Non Centrality Parameter λ	d.f. 2,2				d.f. 2,4			
	T		K		T		K	
	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.C$
0.5	.9382	.9875	.9382	.9875	.9300	.9853	.9300	.9853
1.0	.9032	.9801	.9037	.9801	.8670	.9691	.8670	.9691
1.5	.8489	.9680	.8489	.9680	.7571	.9360	.7576	.9360
2.0	.7778	.9511	.7778	.9511	.6079	.8808	.6086	.8808
2.5	.6950	.9300	.6947	.9300	.4421	.8008	.4428	.8005

T = Tang's Method K = Kanji's Method

TABLE IIA: Table value of ε = 0.01, 0.05 and the corresponding values of P_{II}

One-way Layout Analysis of Variance

Degrees of freedom ($\nu_1 = 2, \nu_2 = 12$)

Group sizes (n_1 5, $n_2 = 5$, $n_3 = 5$)

Group Variances			Non centrality parameter λ											
			0.05		1.0		1.5		2.0		2.5		3.0	
σ_1^2	σ_2^2	σ_3^2	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.
1	1	1	0.919	0.981	0.815	0.944	0.631	0.855	0.401	0.696	0.200	0.486	0.075	0.27
1	2	3	0.628	0.723	0.528	0.672	0.377	0.571	0.271	0.424	0.098	0.263	0.034	0.12
1	6	3	0.563	0.676	0.442	0.590	0.283	0.448	0.142	0.280	0.053	0.134	0.014	0.04
1	1	3	0.722	0.808	0.641	0.775	0.503	0.702	0.331	0.579	0.177	0.419	0.075	0.25

TABLE IIA: Table of ε = 0.01, 0.05 and the corresponding values of P_{II}

One-way Layout Analysis of Variance

Degrees of freedom ($\nu_1 = 2$, $\nu_2 = 12$)

Group sizes ($n_1 = 7$, $n_2 = 5$, $n_3 = 3$)

Group variances			Non centrality parameter λ											
			0.5		1.0		1.5		2.0		2.5		3.0	
σ_1^2	σ_2^2	σ_3^2	ε 0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0
1	1	1	0.919	0.981	0.815	0.944	0.631	0.855	0.401	0.696	0.200	0.486	0.075	0.2
1	2	3	0.615	0.716	0.511	0.660	0.356	0.552	0.200	0.400	0.087	0.240	0.028	0.1
1	6	3	0.544	0.663	0.410	0.561	0.246	0.402	0.113	0.229	0.037	0.096	0.008	0.0
1	1	3	0.709	0.802	0.628	0.767	0.491	0.693	0.323	0.576	0.172	0.410	0.071	0.2

TABLE IIB: Table of $\varepsilon = 0.01, 0.05$ and the corresponding values P_{II}

One-way Layout Analysis of Variance

Degrees of freedom ($\nu_1 = 2, \nu_2 = 12$)

Group Variances ($\sigma_1^2 = 1$, $\sigma_2^2 = 6$, $\sigma_3^2 = 3$)

Group Sizes			Non centrality parameter λ											
			0.5		1.0		1.5		2.0		2.5		3.0	
n_1	n_2	n_3	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.
7	5	3	0.544	0.663	0.410	0.561	0.246	0.402	0.113	0.229	0.037	0.096	0.008	0.02
5	5	5	0.563	0.676	0.442	0.590	0.283	0.448	0.142	0.280	0.053	0.134	0.014	0.04

TABLE IIIA: Table of ε = 0.01, 0.05 and corresponding values of P_{II}

Two-way Layout Analysis of Variance

Degrees of freedom ($\nu_1 = 2, \nu_2 = 4$)

Rows = n = 3, Columns = k = 3

Column Variances			Non centrality parameter λ											
			0.05		1.0		1.5		2.0		2.5		3.0	
σ_1^2	σ_2^2	σ_3^2	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0
1	1	1	0.930	0.985	0.867	0.969	0.757	0.936	0.608	0.880	0.442	0.800	0.287	0.5
1	1	3	0.790	0.860	0.743	0.847	0.660	0.822	0.547	0.780	0.418	0.718	0.289	0.6
1	2	3	0.726	0.804	0.665	0.785	0.564	0.748	0.435	0.682	0.295	0.563	0.164	0.3
2	3	5	0.649	0.736	0.579	0.702	0.465	0.623	0.318	0.473	0.168	0.274	0.062	0.1

TABLE IIIB : Table of ε = 0.01, 0.05 and the corresponding values of P_{II}

Two-way Layout Analysis of Variance

Degrees of freedom ($\nu_1 = 2$, $\nu_2 = 4$)

Rows = n = 3, Columns = k = 3

Serial Correlation within Rows ρ	Non centrality parameter λ											
	0.5		1.0		1.5		2.0		2.5		3.0	
	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0
0.0	0.930	0.985	0.867	0.969	0.757	0.936	0.608	0.880	0.442	0.800	0.287	0.6
0.2	0.898	0.947	0.847	0.934	0.755	0.908	0.626	0.846	0.475	0.799	0.327	0.7
-0.2	0.904	0.965	0.833	0.946	0.713	0.907	0.555	0.843	0.385	0.747	0.232	0.5
0.4	0.867	0.933	0.791	0.911	0.664	0.868	0.503	0.795	0.334	0.678	0.187	0.4

TABLE IV A One-way layout Analysis of Variance (Fixed Model)

The values of $\beta(\lambda)$ at 5% and 1% level of significance

Degrees of freedom ($\nu_1 = 2$, $\nu_2 = 6$)

Non-Centrality	Normal (Tang)		Normal (Simulation)		Erlangian Distribution				Contaminated Normal			
					K=4		K=1		A=.05 h=5		A=.01 h=10	
Parameter	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05	ϵ=0.01	ϵ=0.05
0.0	.010	.050	.010	.047	.015	.054	.013	.049	.007	.047	.008	.053
0.5	.016	.075	.018	.078	.019	.074	.025	.086	.015	.083	.020	.090
1.0	.039	.154	.032	.141	.047	.175	.048	.192	.077	.222	.092	.247
1.5	.092	.296	.084	.282	.101	.331	.163	.390	.193	.470	.205	.518
2.0	.184	.485	.191	.504	.211	.505	.292	.575	.355	.683	.388	.732
2.5	.319	.678	.317	.660	.354	.715	.450	.737	.549	.807	.605	.876
3.0	.480	.832	.496	.829	.511	.823	.582	.840	.710	.881	.787	.947
3.5	.641	.927	.646	.929	.650	.908	.702	.906	.794	.910	.876	.953
4.0	.779	.974	.776	.969	.785	.956	.783	.938	.836	.927	.920	.963
4.5	.879	.993	.882	.992	.875	.984	.867	.968	.881	.947	.946	.966

TABLE IV B Two-way layout Analysis of Variance (Fixed Model)

The values of $\beta(\lambda)$ at 5% and 1% level of significance

Degrees of freedom ($\nu_1 = 2$, $\nu_2 = 4$)

Non-Centrality	Normal (Tang)		Normal (Simulation)		Erlangian Distribution				Contaminated Normal			
					K=4		K=1		A=.05 h=5		A=.01 h=10	
Parameter	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05	ε=0.01	ε=0.05
0.0	.010	.050	.012	.050	.008	.052	.009	.047	.007	.040	.010	.048
0.5	.014	.070	.017	.071	.016	.069	.015	.073	.014	.064	.017	.078
1.0	.031	.133	.027	.123	.036	.147	.033	.151	.053	.177	.056	.196
1.5	.064	.242	.065	.233	.069	.256	.099	.310	.122	.377	.131	.411
2.0	.119	.392	.132	.396	.120	.417	.195	.478	.225	.564	.242	.613
2.5	.199	.557	.191	.536	.222	.597	.304	.641	.355	.718	.401	.785
3.0	.301	.711	.317	.721	.329	.712	.409	.756	.511	.828	.579	.900
3.5	.417	.832	.405	.829	.433	.814	.516	.849	.625	.880	.708	.932
4.0	.538	.913	.542	.908	.559	.903	.607	.886	.696	.895	.790	.952
4.5	.652	.960	.640	.957	.689	.945	.703	.935	.800	.927	.868	.959

TABLE V A One-way layout analysis of variance (random model)

The values of $\beta(\lambda)$ at 5 percent and 1 percent level of significance
Degrees of freedom ($\nu_1 = 2$, $\nu_2 = 6$)

λ	Standard normal distribution		Erlangian distributions				Contaminated normal distribution			
			K = 4		K = 1		A = 0.05 h = 5		A = 0.01 h = 10	
	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$
0.0	.014	.058	.010	.052	.014	.042	.012	.045	.013	.052
1.0	.046	.156	.055	.165	.065	.201	.069	.216	.080	.243
1.5	.118	.2965	.119	.294	.159	.335	.175	.383	.198	.418
2.0	.175	.400	.201	.429	.259	.465	.310	.522	.352	.561
2.5	.292	.516	.308	.539	.357	.577	.414	.616	.456	.661
3.0	.408	.627	.420	.626	.434	.639	.514	.709	.556	.753
3.5	.476	.694	.494	.696	.537	.724	.574	.744	.627	.783
4.0	.577	.769	.557	.742	.618	.780	.647	.802	.681	.816
4.5	.622	.801	.638	.815	.677	.818	.700	.820	.744	.860
5.0	.675	.821	.690	.836	.705	.834	.731	.853	.764	.861

TABLE V B Two-way layout analysis of variance (random model)

The values of $\beta(\lambda)$ at 5 percent and 1 percent level of significance
Degrees of freedom ($\nu_1 = 2$, $\nu_2 = 4$)

λ	Standard normal distribution		Erlangian distributions				Contaminated normal distribution			
			K = 4		K = 1		A = 0.05 h = 5		A = 0.01 h = 10	
	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.05$
0.0	.012	.051	.011	.051	.009	.040	.010	.039	.010	.049
1.0	.032	.132	.035	.143	.045	.166	.044	.169	.054	.196
1.5	.071	.242	.081	.244	.096	.287	.107	.313	.121	.341
2.0	.123	.338	.137	.367	.171	.390	.215	.462	.237	.495
2.5	.193	.457	.209	.471	.262	.504	.298	.543	.344	.591
3.0	.282	.557	.306	.562	.320	.579	.399	.652	.437	.685
3.5	.348	.620	.370	.636	.424	.670	.470	.682	.508	.739
4.0	.437	.707	.439	.687	.498	.724	.535	.723	.577	.764
4.5	.498	.746	.512	.744	.540	.774	.603	.788	.634	.821
5.0	.546	.788	.554	.774	.588	.795	.620	.802	.669	.825