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Generally speaking : exploring expressions of generality in secondary mathematics classrooms

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Generally Speaking:

Exploring Expressions of Generality in Secondary

Mathematics Classrooms

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ABSTRACT

It is widely recognised that generality is at the heart of the learning and teaching of mathematics. Motivated by a desire to understand what it is about generality which presents such an obstacle for so many students, this study examines the variety and complexity of ways in which generality is expressed in mathematics classrooms.

Systematic reflection on my own experience of teaching over a year revealed a wide range of types of generalisation taking place in mathematics classrooms. The main study then analyses transcripts of fifty-two lessons taught by six teachers teaching at least four hundred students, sampled over a period of two months. The focus is on 'ordinary' lessons where expression of generality is not the main objective. Informed by the literature, observation notes and student work, a framework is developed with five categories used to distinguish between types of generalisations, which emerge from the transcribed data . These categories are: the object of generalisation, its presumed longevity of relevance, its justification, its origin and the awareness being promoted.

Having established the ubiquitous richness and complexity of expression of generality in mathematics classrooms, the study looks in closer detail at the expression of generality pertinent to mathematical procedures and to mathematical concepts. The study uses the framework, and draws on second language education literature, to re-examine the fifty-two main study lessons. This analysis highlights the complexity of expressing generality through natural language, and suggests that natural language exhibits many of the pitfalls and ambiguities of algebraic expression. Further, it suggests that algebraic notation might offer a clearer means of expressing generality in many cases. The framework developed for considering characteristics of expressions of generality is then applied to the researcher's own classroom, demonstrating how awareness of ways in which generality is expressed can inform pedagogic choices as well as provide a structure for reflection on practice.

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CHAPTER 1: INTRODUCTION

This chapter introduces and explains the scope (1.1), research questions (1.2) and background (1.3) of the study, and discusses its significance as a piece of contemporary educational research (1.4). The final section provides an overview of the remainder of the thesis (1.5).

1.1 RESEARCH FOCUS

This study investigates the role of generalisation in secondary mathematics classrooms. It is concerned with the ways in which such generalisations are expressed by teachers and students in teacher-led discourse. The importance of students' appreciation of generality has long been recognised. When examples are used to illustrate a whole space of mathematical possibilities, it is important that students appreciate this general space, and are able to generalise. Krutetskii (1976) writing originally in 1968 observed that the more successful mathematics students in his study were those who could generalise on the basis of analysis of just one phenomenon. Following this argument, it would seem desirable for mathematics classrooms to be places where all students become better able to see the general through the particular (Whitehead, 1932). The discussion of generality is thus a central part of mathematics education, and the role of the teacher in promoting and guiding such discussion is worth serious consideration.

As Cobb *et al.* (1997: 258) observed, "The current reform movement in mathematics education places considerable emphasis on the role that classroom discourse can play in supporting students' conceptual development". When teaching, I find that leading

class discussions results in considerable personal and professional pressure. My philosophies and approaches feel particularly exposed, and the effects of the decisions I make appear magnified by my working with up to thirty-two students at a time. Having chosen an appropriate task, the teacher-led discourse phase influences the extent to which students' activity on the task will be mathematically profitable. During teacher-led discourse, whether implicitly or explicitly, misconceptions are addressed; the classroom culture is set out; the purpose of the activity is shared. Although all of these things are also taking place through choice of activity, and interactions with individuals and small groups, they are particularly prominent in teacher-led discourse phases.

1.2 RESEARCH QUESTIONS

This study sets out to explore the issues surrounding algebra as a language of the general, both through the literature and through reflection on my own practice. I listened for opportunities for algebraic expression of mathematical generality in my own and others' classrooms, and attempted to embrace such opportunities in my own classroom, to demonstrate potential links between research findings and implications for classroom practice.

The research questions that guided this study, however, did not set out a specific intention to work on emergent algebra (as defined by Ainley, 1999a and in section 2.5.4). The focus on emergent algebra emerged from immersion in the classroom observations, the data created from these observations, and reflection on my own practice. My intention in writing the research questions for the study was that they

should be sufficiently exploratory and open so as not to anticipate any particular findings.

Research questions serve a variety of functions. Five main functions are delineated by Punch (1998): organising the project and giving it direction and coherence; delimiting the project, showing its boundaries; keeping the researcher focused during the project; providing a framework for writing up the project; and pointing to the data that will be needed. Three research questions were posed in this study, in order to set the study in its broad context, refine its focus, and guide the analysis. The remainder of this section explores the origin, meaning and function of each research question in greater detail.

Research Question One	**What generalisations are being expressed in secondary mathematics classrooms?**
Research Question Two	**How are procedural generalisations expressed in mathematics classrooms?**
Research Question Three	**How are conceptual generalisations expressed in mathematics classrooms?**

The first research question – *What generalisations are being expressed in secondary mathematics classrooms?* – seeks to explore the shape and direction of class discourse, and describe how expressions of generality are arrived at in whole class discussions.

This question served an additional function at a later stage in the research when it provided a focus for synthesising the findings in response to the other research

questions and for identifying the main substantive contributions offered in this study. The response to this question is presented in chapter seven, and runs throughout the thesis.

The second research question asks – *How are procedural generalisations expressed in mathematics classrooms?* The distinction made between general mathematical procedures and general mathematical concepts, both in sections 2.2 and 2.3 and chapters 8-I and 8-II, enables greater insight into classroom expressions of generalisation. It is nevertheless the case that the same mathematical idea can be viewed as both a procedure and a concept. For example, in order to achieve the highest grades at GCSE, students are expected to be able to 'complete the square'. This involves three elements of concept appreciation: to know when it is an appropriate technique to employ (scope), to be able to refer to it as 'completing the square', and recognise the expression when used by others (naming), and to understand that the technique is a general one (abstraction). Similarly, learning about 'perimeter' involves understanding the name, scope and meaning of the concept 'perimeter', as well as learning the procedure for 'finding the perimeter' of various shapes.

The third research question – *How are conceptual generalisations expressed in mathematics classrooms?* – seeks to address the ways in which mathematical concepts are expressed in mathematics lessons. In addressing this question I employ Davydov's (1972/1990) description of what it means to understand a concept.

This work focuses on the classroom practices of six mathematics teachers teaching students across five academic years (seven to eleven). The three research questions are all explored both within the context of the researcher's own practice, and of fifty-two 'main study lessons' taught by six different teachers. All of these points will be elaborated on in subsequent chapters.

1.3 RESEARCHER'S BACKGROUND

This section describes the background to the present study in terms of the researcher's story of coming to undertake this piece of work. The study's focus on students' appreciation of generality and teachers' direction of whole class discussions relate strongly to my own background in teaching, and before that, studying mathematics. It is for this reason that a personal sketch of my path to commencing this study is seen as both necessary and important. This acknowledges that:

> The past experience of the researcher is of significance to any research study he or she conducts [for...] each new experience spreads a layer upon the researcher's existing fund of professional knowledge and influences the establishment of the reearcher's relationship with the focus of the investigation.
> Munby and Spafford, 1987: 507

It is always difficult to pin-point the beginning of an interest or a concern that, through time and subsequent experiences and interactions, grows into a motivation for research. Rather than attempt to do so here, I merely note that teacher-led discourse has, for some time, seemed to me to possess almost daunting potential for developing students' understanding of mathematics. As a beginning teacher, the experience of thirty pairs of eyes and thirty energetic students engaging to some extent with a topic that I was responsible for initiating, sustaining and concluding was both daunting and inspiring. When engaging with research literature, I was frequently frustrated that

teacher-led discourse was held up as the solution to a multitude of issues in mathematics education. Whilst the benefits of students sharing and discussing their ideas are frequently expounded, detailed descriptions of the direction and content of these discussions are rarely offered (see section 4.5.1). The intention when commencing this study was to explore teacher-led discourse in 'ordinary' classrooms, alongside reflecting on such discourse in my own classrooms, in order to offer detailed description of the decisions and tensions involved. My interest in generality and its role in mathematics education is explained in section 2.1.

1.4 THE SIGNIFICANCE OF THE STUDY

The significance of this study lies in the substantive insights and the methodological developments it has generated. These, it is argued, have potential relevance for mathematics education research and practice, and curriculum research and curriculum development more generally.

This work provides detailed, empirical, classroom-based data on the nature and dynamics of whole class discussion in a secondary school setting. Such data are relatively limited at present. Little is known from the perspective of empirical research about how teachers structure whole class discussions. This study, therefore, supports an important development in mathematics education, following researchers such as Davis (1997) who emphasise the role of 'listening' both as a teacher and as a researcher. In view of this, the substantive conclusions from this study would seem of particular importance both to the researcher and practitioner audience within mathematics education.

This study also has significance within the field of language in education more generally, through illustrating empirically how classroom interactions take place. Furthermore, it does this for a curriculum subject (mathematics) that has received comparatively little attention within recent research on language and discussion.

Much research sets out to find the best way to teach certain topics or skills. It considers the tensions, but with an implicit or explicit assumption that there will be a 'best' way, and that the research will be able to go some way towards finding that. I believe that teachers need to be fully aware of the tensions and choices they are making, and the potential advantages and drawbacks of each possible decision, so that they can make a considered and informed decision. Rather than focus on how teachers do teach, or how teachers 'should' teach, I ask what decisions and tensions are and could be experienced, and indicate where choices or decisions might be made.

In order to explore the mathematics classroom through the perspective of generalisation, this study has been required to innovate methodologically. It has applied and combined a number of new ideas to the context of language in mathematics education research. The insights that have emerged from this process will, it is hoped, contribute to the ongoing methodological dialogue within the field. I refer particularly to the ideas that have emanated concerning combining observing others and researching own practice and the challenges of trying to make research meaningful and beneficial to its practising teachers. As with the substantive insights, such methodological concerns, it is hoped, would also be of significance to the broader fields of mathematics education research more generally.

Both the investigation of a sample of teachers' teaching and action research into the researcher's own practice are commonly used and accepted methods of investigating a research question. What is innovative here is the combining of both of these methods in order gain greater insight into an enquiry. Combining approaches to explore a question more fully (triangulation) is common practice. This might involve, for example, interviewing teachers and/or students alongside observations of their lessons. This study sets out to use a notion of triangulation not to gain a more realistic conception of what did happen in a particular situation, but to gain a deeper understanding of what is possible.

1.5 THE STRUCTURE OF THE THESIS

The thesis is organised into ten chapters. This introductory chapter has outlined the scope, background, theoretical approach and significance of the study. In chapter two, attention turns to the research literature that has informed this work. This reviews studies from the fields of generalisation and algebra, language, teacher-led discourse and teaching decisions and locates the study at the intersection of these four areas. Its central theme is the expression of generality in teacher-led discourse.

Chapter three describes and explains the procedures by which the research was undertaken (methods) and the principles and intentions that underpinned their use (methodology). Emphasis is placed on providing detailed examples of how the nature of the data generation procedures evolved over the course of the study, in line with its 'emergent approach'. In chapter four the inter-related questions of how the data was analysed are addressed, along with what measures were taken to ensure validity in the claims made.

Reflection on my own practice played a significant role throughout this study. This is reflected in this thesis through the inclusion of chapters five and nine. In addressing the first research question - *what generalisations are being expressed in secondary mathematics classrooms?* – reflection on my own practice, as described in chapter five, was significant in initial categorisation of generalisations. Generalisations about algebra, about the activity in progress, and about behaviour in the mathematics classroom are considered. The insights gained through reflecting on my own practice through the perspective of 'types' of generalisation being expressed contributed to the decision to distinguish between generalisations in the main study lessons.

Before these distinctions are introduced, chapter six introduces the main study, through provision of a contextual sketch of the six main study teachers. The purpose of this chapter is to present specific details and more general information about the individual case settings to give the reader a context within which to understand the cross-case discussion of findings in the subsequent three analysis chapters.

The main study's findings are, then, provided in three separate chapters, each relating to one of the study's research questions. Chapter seven focuses on two teacher-led discourses, and develops a framework for distinguishing the characteristics of expressions of generality. This, therefore, responds to Research Question 1: *What generalisations are being expressed in secondary mathematics classrooms?* Chapter eight-I looks more closely at those generalisations identified as dealing with mathematical procedures. This, then, responds to Research Question 2 - *How are procedural generalisations expressed in mathematics classrooms?* Chapter eight-II

then explores the expression of mathematical concepts in greater detail. Its reference is Research Question 3 - *How are conceptual generalisations expressed in mathematics classrooms?*

Chapter nine returns to my own classroom, and addresses the emergent question of whether it might be effective to increase the use of algebraic notation in expressing generalisations, such as the procedures and concepts explored in chapters 8-I and 8-II, in secondary mathematics classrooms. The chapter also offers an illustration of how the findings of chapters 7, 8-I and 8-II can inform a teacher's practice.

Chapter ten summarises the findings, and considers their implications for research and practice in the area of whole class mathematical discussion. It ends with a critical evaluation of the design and undertaking of this study.

CHAPTER 2: LITERATURE REVIEW

Having outlined the scope, research questions, background, theoretical approach and curriculum context of the research in chapter one, this chapter considers the intellectual background against which this study was carried out. It situates the present research in relation to previous theoretical and empirical investigations in the fields of mathematics education research and research on classroom language, and demonstrates how it has been influenced by work within these areas. This chapter reviews research from four main areas: research on generalisation and algebra (2.1-2.3); research on language (2.3-2.6), research on class discussion (2.6-2.8) and research on teacher decision-making (2.8). These four areas overlap in sections 2.3, 2.6 and 2.8, reflecting the inter-connected nature of the issues. For each of these there will be a combination of brief overview of work in that area with more detailed consideration of the studies of most relevance to the present work. A final section will summarise the discussion by locating the present study in relation to the literature reviewed (2.9).

Various terms are examined throughout this thesis, including, fundamentally, *generalisation*, *algebra*, and *discourse*. Although it was not my intention that this chapter should focus on the definitions of these terms, it soon became clear that any serious exploration of the interplay between them required a detailed investigation of the scope of each. *Generality* and *generalisation* are consequently discussed in section 2.1, *algebra* in section 2.4, and *discourse* in section 2.7.

Literature informed this study from its initial conception to the writing-up of findings. It therefore seems inappropriate to confine my discussion of its influence to a single isolated chapter. Whilst the main background literature is discussed here in chapter two, some research that proved central to the study is introduced in later chapters where pertinent. Although my awareness of the studies introduced in later chapters also affected my research questions and design, they resonated most vividly once the initial phases of the study had been carried out. I try to show where the literature played an influential role in the initial design of the study, and where findings from the study prompted exploration of a particular area of the literature.

2.1 WHY GENERALITY?

This section focuses on the importance of generality as a topic for research in mathematics education. I begin by exploring the role that generalisation plays in the learning of mathematics, asking what distinguishes mathematics from other school subjects, and what links mathematics to generality. I then focus on four separate areas of the mathematics curriculum, and examine the role of generality in each.

I use the term *generalisation* to refer to the process of stressing some features and ignoring others (Gattegno, 1963) so as to express or appreciate the shared structure of a set of particulars. Generalisation has a particular role to play in mathematics. The abstract nature of the subject requires that generalisations be made about general relationships between general concepts. I take as assumed background the perspective advocated and adumbrated in Mason *et al.* (1985) and developed over many years, its most recent expression being in Mason *et al.* (2005). The authors argue the case for expressing generality as lying at the heart of school mathematics in general, with

algebra as its final expression, leading to rules for manipulating expressions so as to resolve problems.

Krutetskii (1976) carried out a major study of mathematical ability in students, which demonstrated the importance of students being able to appreciate that examples illustrate a whole space of possibilities. Through analysis of observations of, and conversations with, students, Kruteskii compiled a list of nine components of mathematical ability. Alongside spatial awareness, logical reasoning, flexibility, proof and memory, these include three components that particularly relate to appreciation and expression of generalisation:

1) An ability to extract the formal structure from the content of a mathematical problem and to operate with that formal structure.
2) An ability to generalize from mathematical results.
3) An ability to operate with symbols, including numbers.

Krutetskii observed that the more successful mathematics students in his study were those who could generalise "'on the spot', on the basis of an analysis of just one phenomenon":

> They recognise every specific problem at once as the representative of a class of problems of a single type and solve it in a general form – that is, they work out a general method (an algorithm) for solving problems of the given type.
>
> Krutetskii, 1976: 262

In their nine characteristics of good problem solvers, Suydam and Weaver (1977:42) list "ability to note irrelevant detail" and "ability to generalise on the basis of few examples", both of which relate strongly to generalisation. Hadamard (1945) used evidence from studies of famous mathematicians to argue that mathematical ability could not be distinguished from general ability. He did, however, separately consider the case of the 'prodigious calculator', arguing that many prodigious calculators did

not appear to be in any real sense mathematicians. This seems reasonable, as high facility in handling numbers often appears to stem from having committed numerous number facts to memory, and short-term memory enabling retention of more that the normal 7 ± 2 units (Miller, 1956). These are not deemed to be mathematical skills. Hope (1985) argues, however, that prodigious calculators also make use of mathematical relationships, such as $a^2 - b^2 = (a - b)(a + b)$ which assists in computations such as $63^2 - 37^2$. Although many students meet these general rules, few use them in calculation. This suggests that appreciation of generality might aid calculation, and skilled calculation might then be considered a mathematical skill.

Following these arguments, it would seem desirable for mathematics classrooms to be places where all students become increasingly able to see the general through the particular (Whitehead, 1911; Mason, 2002a). The discussion of generality is thus a central part of mathematics education, and the role of the teacher in promoting and guiding such discussion is worth serious consideration.

However, the finding that successful mathematics students find the process of generalisation easy, while students with low prior attainment find it hard, does not lead researchers unanimously to the conclusion that generality should occupy such a significant position in the mathematics classroom. Some believe that we should be *reducing* the emphasis on generalisation in the mathematics classroom, in order to give lower attaining students a greater chance of success. For example, Rees (1981) used factor analysis to conclude that there are two relatively distinct types of mathematical ability. One type (the 'g-factor') was dependent only on instrumental understanding, while the other was more dependent on relational understanding, and

was related to making valid inferences. Tasks involving inference were found to be more difficult for students than those requiring only the 'g-factor'. Rees suggested that very able students should be positively encouraged to develop inferential powers while average and less able students should concentrate on intellectual development via more instrumental approaches, with the possibility that relational understanding might develop in some domains.

Rees's suggestion seems ill-advised, on the grounds that all students have demonstrated the power to generalise. Lower attaining students should be offered more opportunities to express generality, in order that they may develop this natural power. Mason among others argues that everyone has powers of generalisation, all that we need to do is learn to apply them to mathematics.

> Children enter school already having displayed immense powers of imagining and expressing (describing what they see or imagine using language, displaying using their bodies, depicting), generalising and specialising (in picking up and using language), and conjecturing and reasoning (detecting patterns in language so as to be able to make up their own sentences to express themselves). Exercising, developing and extending one's powers is a source of pleasure and self-confidence. Failure to use those powers is at best throwing away an opportunity, and worst, turns students off mathematics and off school. So as a teacher I am faced with the question, 'Am I stimulating my students to use their powers, or am I trying to do the work for them?'.
>
> Mason, 2002: 107

Issues such as this seem to come down to uncertainty or disagreement about the underlying purpose of mathematics education in schools. In the following section I discuss one approach to delineating the contents of the mathematics curriculum, and show the various roles that generality plays in each of the categories identified.

In seeking to distinguish mathematics from other disciplines, much disparity is encountered regarding a definition of mathematics. Brown (1978) suggested that there were four types of mathematical learning, namely *simple recall*, *algorithmic learning*, *conceptual learning* and *problem solving*. The Cockcroft Report (1982: 71) suggested that there are three elements in mathematics teaching - *facts* and *skills*, *conceptual structures*, and *general strategies and appreciation*. In 1985, Her Majesty's Inspectorate listed five main categories of objectives for mathematics learning, the four cognitive categories of which (*facts*, *skills*, *conceptual* structures and general *strategies*) bear a close resemblance to those of Brown (they included a fifth category of 'personal qualities'). In sections 2.1.1 – 2.3, Brown's four types of mathematical learning are examined from the perspective of there being four types of mathematical *generality*. Sections 2.1.1 and 2.1.2 clarify what is intended by the terms 'general facts' and 'general approaches', while sections 2.2 and 2.3 contain more detailed discussion concerning 'general procedures' and 'general concepts', as literature related to these two categories informs research questions 2 and 3, and the analysis in chapters 8-I and 8-II.

2.1.1 General Facts

Although the memorisation of facts might bring to mind rote learning, rather than generalisation, there are numerous ways in which mathematical facts are linked with generality. Many facts can be derived from other facts, and students may not be remembering the facts themselves, but the general process of deducing the fact from others. Rather than learning that '7 x 9 = 63', for example, a student might use the fact that '$9n = 10n - n$', and quickly calculate 70 minus 7 to find their answer. The facts themselves may be general, such as 'all even numbers end in 0, 2, 4, 6 or 8'. General

strategies might be developed for learning mathematical facts, such as application, repetition or proof.

2.1.2 General Approaches

General approaches are defined here as procedures which guide the choice of which skills to use and which knowledge to draw on. Crucially they enable a problem to be approached with confidence and with the expectation that a solution will be possible. With these strategies is associated an awareness of the nature of mathematics and attitudes towards it.

It is often found that something that is first encountered in one part of our experience turns out to be useful in other areas as well. Having solved a problem in one context, we don't have to solve it again, because we solved it abstractly. For example, a common problem is to find the number of sides and diagonals of a polygon. It turns out that the same solution applies also to a question about the number of handshakes that occur if everyone in a room shakes hands with everyone else, and also to a problem about the number of different ways a student could choose two classes to take. They all look the same when you think of them abstractly. If I know the solution to one of these problems, I can transform a new problem into the known problem, and quickly find the answer. This thinking, this appreciation of general approaches, is central to mathematics.

Section 2.1 demonstrated the importance of generality as a topic for research, and outlined an approach to dividing the mathematics curriculum that informs the structure of the study. Having looked at ways in which *mathematical facts* and

mathematical approaches might be considered to involve generalisation, sections 2.2 and 2.3 look at the remaining two aspects of mathematics education, *procedures* and *concepts*, in connection with the process of generalisation.

2.2 GENERAL PROCEDURES

One of the objectives of mathematics teaching is for students to learn mathematical techniques and procedures. The extent to which students appreciate that these procedures are general is therefore of critical importance. For a procedure introduced in a lesson to be appropriately applied in future situations, students must understand the scope of its relevance. Various educational researchers have considered the question of how best to ensure students appreciate the scope of general procedures. In this section I consider some of the distinctions that have been made in this area in order to be able to discuss the issues more clearly.

It is important to emphasise that the term *procedure* is used throughout this study to refer to a mathematical method or technique. This could be as specific as 'how to multiply an integer by ten', or as widely applicable as 'how to approach word problems'. The term was selected in preference to *algorithm*, on the grounds that a student might develop a procedure for, say, multiplying two fractions, without the procedure having the formality one might associate with an algorithm. Although the term *procedure* can also be used to refer to a formal series of steps to solve a problem, in this study a looser interpretation is adopted. A student might 'see the general' in one problem, and apply that procedure to subsequent problems, without considering themselves to be following the rigorous steps of an algorithm, and perhaps even without consciously noticing the generality.

2.2.1 Inductive and deductive approaches

In this section I discuss the distinction between deductive and inductive approaches to teaching general rules. These two approaches might also be seen to apply to mathematical facts, concepts or approaches, but seem to apply most directly to the teaching of procedures. The examples used by advocates of both approaches tend to be drawn from the teaching and learning of procedures.

Spencer (1878) advocated the use of learners' powers, and reported on the beginnings of getting students to generalise for themselves:

> The particulars first, and then the generalizations, is the new method ... which, though 'the reverse of the method usually followed, which consists in giving the pupil the rule first' is yet proved by experience to be the right one. Rule-teaching is now condemned as imparting a merely empirical knowledge – as producing an appearance of understanding without the reality...While the rule-taught youth is at sea when beyond his rules, the youth instructed in principles solves a new case as readily as an old one.
>
> Spencer, 1878: 56-7

The term inductive refers to students detecting and expressing similarities and differences for themselves, and so reaching and expressing their own generalities. Halmos (1994) advocates a method where the teacher is encouraged to,

> concentrate attention on the definite, the concrete, the specific. Once a student understands, really and truly understands, why 3x5 is the same as 5x3, then he quickly gets the automatic but nevertheless exciting and obvious conviction that 'it goes the same way' for all other numbers.
>
> Halmos, 1994: 852

This contrasts with the deductive approach in which students are expected to commit definitions, rules and principles to memory. These are often illustrated with a few examples, and then applied to exercises. Teachers might teach deductively or inductively: teaching rules first and applications later or offering examples from

which learners use their own powers with guidance. A mixed approach might be thought most likely to benefit a variety of learners. Through viewing procedures as generalisations, this section has shown that mathematical procedures can be taught inductively (Spencer, 1878; Halmos, 1994) or deductively, and has briefly considered the advantages of each approach.

2.2.2 Empirical and structural generalisation

Another distinction I found to be useful in looking more closely at the process of generalisation is Bills and Rowland's (1999) distinction between 'empirical' and 'structural' generalisation. They define 'empirical' generalisations as those achieved by considering the form of the examples, while 'structural' generalisations are made by looking at underlying meanings, structures or procedures. One example from my own teaching experience that this brings to mind is that of 'collecting like terms', where an empirical generalisation can be relatively simply perceived through considering a set of examples such as:

$$3n + 4n = 7n$$
$$6x + 2x + 10x = 18x$$
$$12a + 5b + 3a = 15a + 5b$$

A general procedure can be appreciated from the above examples that might be expressed as 'add the numbers but keep the letters the same'. However, this general procedure can be appreciated on an empirical level without students appreciating, for example, that '$3n$' stands for the product of 3 and a number, rather than 'thirty-something'. The structural generalisation might be gleaned from working with examples of the form:

$$3 \times 9 + 4 \times 9 = 7 \times 9$$

This distinction between empirical and structural generalisation is exemplified by Mason through his task "Do Thou Likewise".

> Here is a worked example of a calculation on some abstruse number-like objects represented as pairs of numbers:
>
> (a; b) + (c; d) = (ac; bc + ad)
> so
> (4; 6) + (3; 5) = (4 x 3; 6 x 3 + 4 x 5) = (12; 18 + 20) = (12; 38)
> Now cover everything above, and do (1; 2) + (3; 4) yourself.
>
> Mason, 2005

For me, this is a powerful reminder that procedures can be learnt without being understood. Under Bills and Rowland's (1999) distinction between 'empirical' and 'structural' generalisation, "Do Thou Likewise" is 'empirical'. Due to the way this procedure is presented, it can only be appreciated empirically. The structural generalisation is actually that the procedure is an upside-down version of adding fractions. It is conceivable that many students who 'can add fractions' are applying an empirical generalisation similar to this. The question then arises of the value of such a procedure.

The distinction between empirical and structural generalisation (Bills and Rowland, 1999) introduced in this section raises the question for researchers and practitioners of how teachers can support structural generalisation.

2.2.3 Procedural and conceptual understanding

Students often encode in their memory any correlations between surface-level features of a problem and the method used for solving that problem and proceed to execute that method when detecting these surface features in other problems (Ben-Zeev, 1998; Ben-Zeev & Star, 2001; Chi & Bassok, 1989; Schoenfeld, 1988). Arguably, when such surface-level encoding happens during instruction, students do not acquire the

ability to discriminate between cases when a certain arithmetical operation is required and when it is not appropriate, but rather learn mindless, stereotyped copying behaviour. This 'when I see something like that, I do this' understanding is sometimes referred to as 'procedural'. Many researchers have drawn a distinction between these two types of 'knowing' or 'understanding', cautioning that where student understanding is limited to the procedural, answers cannot be checked, because the objects are without meaning.

Lampert (1986) draws a similar distinction, referring to procedural understanding as 'computational knowledge' as contrasted with 'principled conceptual knowledge' which represents the understanding of abstract principles and concepts that govern and define mathematical thinking and procedures. Following Mellin-Olsen (see Skemp, 1976), Skemp distinguishes between 'relational understanding', which he defines as "what I have always meant by understanding" (1971: 153), and 'instrumental understanding'. This latter he describes as 'rules without reasons'. A similar distinction is illustrated in this example from Davis:

> Some teachers want a student to begin solving the equation $2x + 3 = x + 8$ by thinking, "I'll move the 3 across the equals sign and change its sign." These teachers hope the student will then write $2x = x + 8 - 3$ but this approach is surely a case of regarding mathematics as a collection of small, meaningless rituals. Why "move the 3 across the equals sign?" And why on earth should such an act "change (the) sign of the 3?" Certainly, if we want the student to think of mathematics as consisting of reasonable responses to reasonable challenges, it will be far better if we encourage the student to think, "I can subtract 3 from each side of that equation, without changing its truth set." If the student has seen pictures of balance scales or has worked with actual balances, there can by very straightforward imagery underlying the idea of subtracting the same thing from each side of an equation.
>
> Davis, quoted in Wagner and Kieran, 1989: 270

Although fluency and understanding are often held up as if in conflict, they are both required for successful learning in mathematics. Time spent emphasising and developing the one may well contribute to increasing the other, but at a slower rate, perhaps, than if it had been the central focus of the activity. With respect to how these forms of knowing come to bear on the teaching of mathematics, Graeber (1999), in examining forms of knowing mathematics, finds that it is important for preservice teachers to understand that executing an algorithm, or getting the right answer, does not imply conceptual understanding. Prospective teachers, according to Graeber (1999), must understand that students who possess one form of knowledge do not necessarily possess other forms of knowledge. For instance, students may hold procedural knowledge of how to divide two fractions but have poor conceptual knowledge of either fractions or division. If preservice teachers enter the classroom without making the distinction between conceptual and procedural knowledge, they are apt to take existence of one type as evidence of existence of the other. It has been frequently asserted (e.g. Hart, 1981; Orton, 1992) that algorithms are taught too quickly, and are either forgotten, or remembered in a form different from that which was taught.

2.3 GENERAL CONCEPTS

The New Shorter Oxford English Dictionary (1993) defines a *concept* as "a product of the faculty of conception; an idea of a class of objects, a general notion; a theme, a design". A word in a language, such as *dog*, rarely applies to a single entity and nothing else. With the exception of proper names, words that refer to one particular case will also refer to other individuals that belong to the same category or kind. To learn a common noun, then, requires some understanding of the conditions of

category membership. This is usually described as a concept. Understanding of concepts necessitates generalisation as, in working with a concept such as *fraction*, or *quadrilateral*, we isolate one or a few features of a type of object for study, and see what we can learn about the behaviour of those features while ignoring everything else about them: features like number, shape, or direction. For example, when working with numbers we take the concept of counting away from all other details about the things we are counting, such as colour or name, and just think about how many there are. Once students have learnt to work with numbers as an abstract entity, they can add two numbers (such as fifty and thirty-two) without having to think of them as representing fifty apples and thirty-two apples. Once they have finished their calculations with numbers, they can come back to the material world and know the total number of apples.

Why is understanding of these general concepts necessary for language use? We could ask, following Locke, why such *general terms* exist in natural language. Why do we need common nouns? Locke's answer, having considered the impossibility of storing separate words for every individual thing, and the difficulty of sharing ideas with others who had experienced different individual things, is that:

> a distinct name for every particular thing would not be of any great use for the improvement of knowledge: which, though founded in particular things, enlarges itself by general views; to which things reduced into sorts, under general names, are properly subservient.
>
> Locke, 1690/1964, bk.2: 15

Each experience that we have with a particular object – a fire that is hot and burns us, a piece of chocolate that tastes nice – can only be learnt from if the particular object is seen as representative of a general category. The advantage of having a general

concept, is that once you know that something belongs to such a category, you know

further facts about it.

> The concept of an object or of a property is always abstract. Every
> concept is an abstraction, regardless of what it represents.
>
> Davydov, 1972/1990: 44

Links between language and generalisation are manifold. The abstract, general nature

of all words applies particularly to mathematical language, where words are often

used to describe the abstract concepts of a set of abstract concepts. 'Three', for

example, is both an abstract concept itself, and also part of the set of concepts we call

'odd numbers'. One of the arguments supporting children's capacity to generalise is

their understanding of the meanings of words.

Using a process involving testing people on the meanings of a sample of words from a

dictionary, including only those words whose meanings cannot be guessed through

morphology or analogy, Nagy & Herman (1987) estimated that American high school

graduates have a vocabulary of approximately 45,000 words. Given that this estimate

does not include words such as proper names or idioms, the figure is probably closer

to 60,000. This equates to about 10 new words a day between the age of 12 months

and the completion of high school (Pinker, 1994).

It seems convincing that if students can generalise word meaning, they can generalise

mathematically. Why, then, is mathematical generalisation deemed an advanced skill,

while the learning of a first language is considered easy? One distinction that can be

made is that the particular cases of mathematical generalisation tend to be more

abstract than those of everyday concepts.

Section 2.3 has sought to indicate the ways in which understanding a concept involves generalisation. It showed how every word is general, as each name (with the exception of proper nouns) belongs to a general concept. The argument was put forward that if students can generalise word meaning, they can generalise mathematically. Given the large number of new words that students show themselves to be capable of learning (Nagy & Herman, 1987; Pinker, 1994), it is suggested that students possess and demonstrate natural powers of generalisation.

Section 2.3.1 looks in greater detail at those concepts that relate specifically to mathematics, and at the distinctions that have been made between Ordinary English and Mathematical English. Having linked these two registers (Halliday, 1975) with Vygotsky's (1934/1987) distinction between everyday concepts and scientific concepts, his (1933/1975) definition of the *sublated* concept is introduced.

2.3.1 The vocabulary of mathematics

Limited understanding of mathematical concepts is a barrier for many students' mathematical development (Slavit, 1998). Ability to formulate, describe and compare mathematical ideas is central to mathematics learning (Ernest, 1999; Laborde, 1990). Kane (1967) makes a useful distinction between 'Ordinary English' and 'Mathematical English'. This distinction emphasises that mathematics teachers are not merely supporting students in extending their existing vocabulary to include technical mathematical words. Mathematics teachers also introduce new meanings for words that are already part of students' vocabulary. The meaning that these words are given 'in maths' may differ significantly from their meaning in everyday English.

> Mathematical English (ME) is a hybrid language. It is composed of ordinary English (OE) commingled with various brands of highly stylised formal symbol systems. The mix of these two kinds of language varies greatly from elementary school text books to books written for graduate students.
>
> Kane, 1967

One justification for not simply taking Mathematical English to be a sub-set of the vocabulary of Ordinary English is that, while some technical mathematical vocabulary has a meaning only in Mathematical English, some words have the same meaning in Mathematical and Ordinary English, and still others (arguably the most confusing for students) have a different meaning in Mathematical English than in Ordinary English. While some words have entirely different meanings in ME and OE, others have a similar meaning in the two, but the ME meaning is more specialised or refined. Some classic examples of these words, of which there are many, include *difference*, *product*, *similar*, *odd*, and *mean*.

Pimm (1995) uses Halliday's (1975) notion of a mathematics register as a starting point for exploring the ambiguities and potential confusions related to mathematical communication. Following Halliday, Pimm takes a register to be "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1975: 65). Pimm proposes that students must become aware that there are different registers and that the grammar, meanings and uses of certain terms and expressions vary within and between them.

In Earp and Tanner's (1980) investigation into the words used in an American sixth-grade textbook, only 197 out of 716 words were used in a technical mathematical way, such as average, millilitre, scale, thousandths. They interviewed fifty students to

assess their comprehension of the vocabulary in the book, and found 98% comprehension accuracy for the non-mathematical words and only 50% for the mathematical words. Mathematical texts, even those that use little technical vocabulary, are often written with fewer context clues than passages in ordinary English (Earp, 1971; Shuard & Rothery, 1984), which gives students fewer clues about unknown words or phases. The paucity of provision of context clues in mathematics means that questions often demand that students know the exact meaning of all words in order to know what is being asked of them. A study by Otterburn & Nicholson (1976), followed up by Nicholson (1977) found that offering context clues, by placing a word in a sentence or paragraph, improved students' understanding of the word. A word which appears in isolation is harder to read than if it is part of a sentence.

Williams (1981) showed that the choice of vocabulary used is not as significant a factor in students' understanding as the way in which the text helps the reader to understand its vocabulary. Although his focus was on written texts in OE, many of techniques apply to mathematics, and could be adapted for use by teachers. He offers the following checklist for effective lexical familiarisation.

(a) Is it clear to the reader when a lexical item is being familiarised? In other words, does the book have a typographic system for familiarisation?
(b) Is this system *consistent*?
(c) Does the book contain an index, and does that index distinguish (again by a typographical device) between lexical items that are familiarised in the main body of the book, and those that are not?
(d) Are the lexical familiarizations likely to do their job *for the target readership*?
(e) Are the familiarisation forms chosen the most appropriate? In particular, has the author made sufficient use of non-verbal devices?
(f) If illustrative familiarizations are included, what degree of interaction between illustration and text is there, and how successful is it?

(g) Are familiarisations followed up – by presentation in a different context, in an exercise, in a diagram, in an end-of-chapter summary etc.?

(h) If the book contains a glossary, is the language of the glosses suitable for the target readership?

(i) Are there instances of *non*-familiarization of key terminology?

<div align="right">Williams, 1981</div>

Vygotsky (1934/1987) distinguished between everyday concepts and scientific concepts, on the basis of whether or not they were based on a system. He argued that everyday concepts are based in rich daily contexts, rather than being based on a system, whilst scientific concepts are defined according to a system that has developed in human history, and therefore lacks concrete contexts. According to Kozulin, Vygotsky's scientific concepts are based on "formal, logical, and decontextualized structures" (1990:168). As *mathematical concepts* are within a system and are characterized by formal and decontextualized structures, in this thesis, I will take *mathematical concepts* to be an example of the *scientific concepts* discussed by Vygotsky. The following definitions are drawn from Vygotsky (1934/1987) and others who discussed this distinction (Panofsky, John-Steiner, and Blackwell, 1990; Kozulin, 1990; Davydov, 1972/1990; and Schmittau, 1993).

Everyday concepts are concepts that originate from children's daily lives through communication with their family, friends, or community and thus are closely connected to concrete personal contexts. Children express them through their own words and use them in their ways of thinking without conscious awareness. As a result, everyday concepts are not systems; rather they are based on subjectivity.

Mathematical concepts are scientific concepts that are connected to mathematics. They are based on a system, and therefore they have logic and objectivity. They are

expressed in a mathematical language and introduced to children in a formal, highly organized education. Mathematical concepts enable children to develop mathematical thinking and to require conscious awareness and voluntary behaviour for concepts.

As Zack (1999) discussed, the relation between spontaneous and scientific concepts might be regarded as either mutually dependent or dichotomous (mutually exclusive). On the one hand, the relation could be thought of as dichotomous on the surface because of their conflicting characteristics. Yet Vygotsky's words suggest he may have taken another position: "There is a mutual dependence in the relation between the processes of development of children's concepts in daily lives and in school. It enables such relation that the processes of these two concepts pass in the different ways" (Vygotsky, 1933/1975: 122). The subsequent sections describe in detail the relation between the two kinds of concepts and their development. However, care must be taken when inferring what Vygotsky said about the relation of scientific and everyday concepts, for while mathematical concepts are here equated with scientific concepts, Vygotsky himself did not deal specifically with *mathematical concepts.*

Vygotsky (1934/1987) thought of *concepts* as follows:

> We know that the concept is ... a "complex and true act of thinking" that cannot be mastered through simple memorization. ... At any stage of its development, the concept is an "act of generalization." The most important finding of all research in this field is that the concept – represented psychologically as word meaning – develops. The essence of the development of the concept lies in the transition from one structure of generalization to another. ... This process is completed with the formation of true concepts.
>
> Vygotsky, 1934/1987: 169-170

Accordingly, a concept is not static or unchangeable, but a complex and dynamic act of thinking and an act of generalisation. In addition, psychologically this concept develops from one structure of generalisation to another. It is apparent that this notion of concepts applies to everyday concepts but does not apply to mathematical concepts because mathematical concepts have been developed through history. Thus, the distinction between the two types of mathematical concepts which have developed in human history and which develop based on children's everyday concepts as their psychological development must be made clearer than it is in some of Vygotsky's writing.

Once clearly distinguished from one another, we now need to consider the relationship between these two kinds of concepts in mathematics education. In the cognitive development of children, everyday concepts arise from below to above in some sense when children learn formal and systematic mathematical concepts in school (Vygotsky, 1933/1975; 1934/1987). In other words, everyday concepts are reorganized and raised to a higher level by the appearance of mathematical concepts. However, the everyday concepts that are raised to higher level might not be called everyday concepts after this elevation because they now include elements of more systemic thinking. Likewise, mathematical concepts also change from their proper definition, for after this elevation they now include notions derived from experience in concrete contexts in addition to their systematic characteristics.

Though Vygotsky (1974) did not give detailed accounts of how these two mutually dependent concepts develop in children, he pointed out that the relation between higher and lower forms could be expressed well using the idea of a dialectic, where

from the interrelating of everyday and mathematical concepts a new form- the

sublated concept- gradually develops. Brushlinsky (1968) described this idea as

follows:

> In [Vygotsky's] words, any higher stage of developments does not replace [lower stages] but *subordinate* them as their parts. That is, the higher stages contradict the lower. However, they do not eliminate the lower; but the higher stages include the lower as their *sublated* parts. Categories become components within a system, contradicting one another in the system. (L. S. Vygotsky reminds us of the duality of Hegelian meaning of *sublate* – elimination [of original form] together with preservation [of crucial features of the category])
>
> Brushlinsky, 1968: 12

As Vygotsky (1933/1975) recognized, contradictions between everyday and

mathematical concepts are factors that promote the intellectual development and bring

new possibilities for their development of children. Therefore, it is useful to use the

idea *sublated* to explain how the concepts develop. These are defined as follows.

> (1) Mathematical concepts *contradict* a part of the children's everyday concepts;
> (2-A) The everyday concepts are *lifted* to a higher level, based on a system in mathematics;
> (2-B) The mathematical concepts are *lifted* to a higher level in which the daily contexts according to children's everyday concepts are accompanied with them;
> (3) They are *preserved* as a unified concept, that is, a *sublated concept*.

Consequently, sublated concepts are defined as follows: Concepts developed through

the sublated process, having both a system in mathematics and rich daily contexts,

wherein children are free to move back and forth between the everyday and the

mathematical world. Furthermore, children should be able to use the sublated

concepts with conscious awareness and voluntary behaviour.

Having briefly considered the complexity of word learning in mathematics, the next

section considers the process by which such concepts are thought to be learnt.

2.3.2 How do students learn words?

Whilst the importance of students learning and using subject-specific vocabulary is frequently emphasised, the process by which students learn these words, or words of any kind, remains relatively mysterious. Of all the language learning that children do, including grammar and sentence structure, it is word learning that seems to be central in terms of generalising. When a student learns that a particular word or name applies to a set of particular cases, they are generalising. This is because, as St. Augustine emphasised around 1600 years ago, it is not simply the word itself that is being learnt, but the general class of 'things' that the word signifies:

> From words we can learn only words. Indeed we can learn only their sound and their noise. We learn nothing new when we know the words already, and when we do not know them we cannot say we have learned anything unless we also learn their meaning. And their meaning we learn not from hearing their sound when they are uttered, but from getting to know the things they signify...
>
> St. Augustine, 4th C AD

Where consideration is given to what this process described by St. Augustine might consist of, it is often explained by the process of 'association'. The term 'association', is used to describe the way in which a learner hears a word being used, and comes to associate the word with what it refers to. It is assumed that this is the process by which dogs learn to obey the commands of their owner, as they associate the right behaviour with the right sound.

Although some studies (Collins, 1977; Harris, Jones & Grant, 1983) have found that for very young children there may be a spatial and temporal concurrence between word and meaning, this occurs only about 70 percent of the time. There is experimental evidence that children are able to learn words for objects and actions

that are not observable to them at the time the words are being used. The associative

principle therefore fails to capture certain facts about language development. The

complexity of word learning is emphasised by Quine (1960), who tells an anecdote in

which a linguist attempts to learn a language from its native speakers (Quine, 1960:

29). "A rabbit scurries by, the native says 'Gavagai,' and the linguist notes down the

sentence 'Rabbit' (or 'Lo, a rabbit'), as tentative translation, subject to testing in

further cases." The case that Quine puts forward is that the linguist can never be

certain of the accuracy of this translation. The word might be used to describe

mammals, animals, things that run, the act of running itself, the fur or colour of the

rabbit. Even once the linguist can be sure that *gavagai* is a name that refers to the

rabbit, a problem of generalisation arises, regarding how this word should be used on

future occasions.

Goodman (1983) similarly pointed out that for any act of induction, there is an infinite

number of equally logical generalisations that can be made, each one equally

consistent with the particular experience. Students need to be clear about what the

speaker intends to refer to when using the word.

Having formed a conjecture about a one-on-one pairing between a word and a general

concept, how do children check whether it is correct? Various studies have shown that

children do not appear to rely on adults to correct them, reinforcing correct naming

and correcting those that are incorrect. Not all cultures include the practice of parents

of correcting children's word use (Lievan, 1994). Although a child who cannot talk

obviously cannot get feedback on their speech, a study of a four-year-old who could

not speak showed that he possessed a vocabulary and grammatical understanding

appropriate to his age (Stromswold, 1994). This suggests that students need only to be exposed to the appropriate vocabulary to learn, and that testing and correcting students' word use does not necessarily significantly contribute to their language development. It follows from this that if students heard others using mathematical vocabulary frequently, they would learn the vocabulary even without the careful naming. This leads to the conjecture that teachers should use mathematical words frequently, and not worry overly about explaining them each time.

> No matter how good they are at understanding the minds of others, children cannot learn a word without the ability to grasp the associated concept. Suppose, for the sake of argument, that two-year-olds have the same theory of mind as adults. Still, two-year-olds will not be able to learn words such as modem and stockbroker (even though these refer to observable middle-sized objects) because they don't yet know what such categories are. What theory of mind does for children is enable them to establish the mapping between a word and a concept. But this presupposes the availability of the concept.
>
> Bloom, 2002: 86

The mathematics classroom exposes students to many signs, symbols and diagrams. After numerals and their combinations have been introduced as representing numbers, letters begin to be used to designate quantities and numbers.

> The principal functions of a symbol in mathematics are to designate with precision and clarity and to abbreviate. The demands of precision require that the meaning of each symbol or symbol string be razor sharp and unambiguous.
>
> Davis and Hersh, 1981: 123-124.

Skemp (1971) offered an explanation of how we learn concepts. He suggested that there is no way we can help an adult born blind but given sight by an operation understand the concept of "redness" by means of a definition. We could make the person abstract the idea by pointing to different objects that are red or to ones that are not red. This would explain what is meant by 'redness.' Skemp claimed that learning of mathematical concepts is similar. We should not expect students to learn concepts

through definitions. Teaching through examples (rather than general definitions) carries the implicit pedagogic intention of encouraging students to attend to aspects of the particular that will appear as important features of the general (Mason and Pimm, 1984).

Skemp argues that this examples-to-definitions route is cognitively necessary:

> Concepts of a higher order than those which people already have cannot be communicated to them by definition, but only by collecting together, for them to experience, suitable examples.
>
> Skemp, 1971:14

The approach of teaching general concepts through offering particular examples (rather than giving a general definition) has parallels with teaching general procedures inductively (see section 2.2.1). Generalisation entails the transition from a description of the properties of a particular object to locating and isolating a whole class of similar objects. It is the essential attributes of a concept, those that are common and remain stable, that delineate each concept from others.

There is a tendency in the language and psychology literature to suggest that general concepts are learnt through exposure to them. Bloom (2002) explains that, whilst it is a truism that children need to be exposed to words in contexts in which they can infer their meanings in order to learn them, the words do not need to be presented in a labelling context (they can be merely overheard), the children do not need to be able to see the word's referent, and they do not require encouragement or feedback on their word use.

'Lexical familiarisation' is used to describe the process of weaving new words into sentence contexts in order to render the meaning clear. It might therefore be expected that mathematical texts and teachers would use mathematical vocabulary at every available opportunity, to optimise lexical familiarisation. However, conceptions are often developed and discussed initially using a 'transient' concept. For instance, Shuard & Rothary (1984: 14) describe how in SMP Book C: 28, movement to the right on the number line is referred to as a 'blue shift' while movement to the left is termed a 'green shift'. Later in the chapter the student is told that the set of directed numbers will henceforth be used to describe such shifts. This notion of a distinction between a transient and a universal concept emerged as a useful distinction during the data analysis, and is discussed further in chapter 7.

Davydov (1972/1990) considers concept development (especially in young children) to proceed from perception to conception to concept. However the formation of theoretical concepts at the older age relies on the detection and delineation of certain 'invariables' abstracted from the formation of elementary concepts.

As the definition of 'meaning' is at the centre of this inquiry, it is unfortunate that no such definition appears to exist. Philosophers such as Quine propose that while sentences have meanings, such as their truth conditions or their method of verification, words do not. I will take Bloom's (2002) definition of what it is to know the meaning of a word, which is to have a certain mental representation or concept that is associated with a certain form.

> Under this view, two things are involved in knowing the meaning of a word – having the concept and mapping the concept onto the right form...Saying, for instance, that a two-year-old has mixed up the meanings of *cat* and *dog* implies that the child has the right concepts but

has mapped them onto the wrong forms. On the other hand, saying that the two-year-old does not know what mortgage means implies that the child lacks the relevant concept. People can also possess concepts that are not associated with forms. A child might have the concept of cat but not yet know the word, and even proficient adult users of a language can have concepts, such as of a dead plant or a broken computer, that they don't have words for.

<div align="right">Bloom 2002: 17-18</div>

This separates the knowing of meaning into two requirements: (1) having the concept and (2) mapping the concept onto the right form. For Davydov, however, in order for someone to have a concept of something, they must also know the word. Otherwise it remains a *conception*. A student who says "I couldn't do question 9, I don't know what *integer* means" may be deemed to possess the concept but not yet know the word, while saying that a student mixes up the meanings of *mean*, *mode* and *median* implies that the student has the right concepts but maps them onto the wrong forms (words). Saying of a year 7 student they don't know what *differentiation* means implies that the student lacks the relevant concept.

So what is this concept that is required for the first of these requirements? The concept need not contain every aspect of the knowledge someone connects with a word. If the meaning of a word were determined by all thoughts related to that word, then there would be no sense in which two people, or even a single person over time, could ever have the same meaning of a word. These 'meaning holistic' ideas were introduced into analytic philosophy in the early 1950's, in works by Hempel (1950) and Quine (1951), both concerned with the meaning of theoretical sentences in the formulation of a scientific theory. Semantic holism can be roughly characterized as the doctrine that the meaning of a person's words is a function of all of one's beliefs involving those words. A more detailed account of holism can be found in Block

(1998). Holistic theories of meaning are often criticized for entailing that *any* difference between beliefs will result in a difference in meaning. For instance, if what I mean by "gold" is a function of my "gold"-beliefs, and one of these beliefs changes, then what I mean by "gold" would seem to change as well. Holistic theories of meaning are thus typically accused of (among other things) entailing that no two people (or no person at two times) ever mean the same thing by any of their words.

There are two traditional ways in which a word's meaning can be context sensitive. The first is to be ambiguous. The word "bank," for instance, can be used to designate either a financial institution, or the edge of a river. The context-sensitivity of "bank" is thus explained in terms of the different lexical entries being accessed in different contexts. The second is for the word's meaning to incorporate an 'indexical' component, allowing the entry for the word in one's mental lexicon to make reference to various contextual features. The word "here," for instance, is context-sensitive because the entry for it in one's lexicon makes reference to its place of utterance. However, there are many cases where words seem to refer to different things in different contexts without being straightforwardly ambiguous or indexical. The phenomenon can also show up with some proper names. The classic illustration of this is Wittgenstein's discussion of "Moses".

> If one says "Moses did not exist", this may mean various things. It may mean: the Israelites did not have a single leader when they withdrew from Egypt -- or: their leader was not called Moses -- or: there cannot have been anyone who accomplished all that the Bible relates of Moses --or: etc. etc.
>
> Wittgenstein, 1953: §79

"Moses" seems as if it can be used to mean a number of things, but, as Wittgenstein

points out later, the suggestion that the term is ambiguous is not especially plausible.

> Learning a word is a social act. When children learn that rabbits eat
> carrots, they are learning something about the external world, but when
> they learn that rabbit refers to rabbits, they are learning an arbitrary
> convention shared by a community of speakers, an implicitly agreed-upon
> way of communicating. When children learn the meaning of a word, they
> are – whether they know it or not – learning something about the thoughts
> of other people.
>
> <div align="right">Bloom, 2002: 55</div>

Bloom emphasises that, "Just because the relationship between a word and its

meaning is a social fact doesn't entail that one needs social competence or knowledge

to learn this fact." (Bloom, 2002: 55)

> Word learning really is a hard problem, but children do not solve it
> through a dedicated mental mechanism. Instead, words are learned
> through abilities that exist for other purposes. These include an ability to
> infer the intentions of others, an ability to acquire concepts, an
> appreciation of syntactic structure, and certain general learning and
> memory abilities. These are both necessary and sufficient for word
> learning: children need them to learn the meanings of words, and they
> need nothing else.
>
> <div align="right">Bloom, 2002: 10</div>

Bloom suggests that the phenomena that such constraints have been posited to explain

(such as children's tendencies to treat words as object names, to avoid words with

overlapping references, and to generalise object names on the basis of shape) are

better explained in terms of other facts about how children think and learn.

> Learning a word requires memorizing an arbitrary relationship between a
> form and a meaning, and the rote learning of paired associates is
> notoriously slow and difficult. Consider how hard it is to learn the capitals
> of different countries or the birthdays of particular people.
>
> <div align="right">Bloom, 2002: 25</div>

Although we think of word learning as something that 'children' do, "All we can say with certainty is that word learning typically reaches its peak not at 18 months but somewhere between 10 and 17 years." (Bloom, 2002: 44).

Carey and Bartlett (1978) carried out experiments with young children (see Bloom, 2002: 26) in which they were casually introduced to a new colour word whilst involved in another, unrelated activity. When the three- and four-year-old children were tested a week later on their comprehension of the word, over half remembered something about its meaning.

Bloom and Markson (1997) carried out an experiment (described in Bloom 2002: 28 onwards) that seemed to show that both children and adults learn properties of objects as effectively as they learn names for objects. So is an object's name just a special kind of fact about it?

In order for a conception to become a concept, Davydov (1972/1990) states that three requirements must be met. The class of objects that the concept contains must be unambiguously distinguishable from others through the possession of *essential attributes*. A term must be assigned to the concept. The meaning of this term need not rely on visual images, but can have an abstract character. This means that it might be difficult or impossible to have a mental picture of the concept, which may be defined by reference to previously understood concepts.

Davydov (1972/1990: 45-6) offers the following 3 criteria as the basis of developing a concept:

- Knowing the precise scope.

- Knowing the name.

- Understanding the general concept without being offered a particular example.

Conceptions, at least, are perpetually being formed in mathematics lessons. Some of these develop into concepts through discussion, naming, and clarification of scope. Many, however, do not. Students may have a conception of 'the sorts of questions you get where you have to factorise in two brackets' without the concept of 'quadratic'. They may have a conception of 'fractions that can be changed into each other' without a concept of 'equivalent fractions'.

The three criteria given by Davydov (1972/1990, 45-6) for forming a concept, described above, provide the basis of three important decisions in teaching:

Scope: when and how do you establish the precise scope of a concept?

Naming: when and how do you introduce a name?

Abstraction: when do you move from exemplifying the conception with the particular and assume that students can appreciate meaning from the general?

Davydov's (1972/1990) distinction between perception, conception and concept proved particularly valuable in the analysis (discussed in chapter eight-II) of classroom discourse related to general mathematical concepts. Davydov's (1972/1990) analysis led me to inquire whether the development of a conception was

a necessary prerequisite for a new concept. The movement through perception, conception and concept may not have to be a step-by-step progression. You could, for example, develop a conception without perception, by basing the new conception on a set of previous conceptions, rather than on a set of actual perceptions. Is it possible that the same can be said of concepts without conceptions? If the name, scope and meaning of a concept are introduced with reference to previously understood concepts, students may be able to get a sense of the abstract without recourse to either perception or conception. But their understanding of that concept might well be weaker as a consequence. Watson (2003) explores the role of learnt phrases such as "vertically opposite angles are equal" in the context of a South African township school. She explains that this chorused statement was not subsequently applied by many students in exercises following its chanting, and attributes this to poor understanding of the concept of angle and difficulty in recognising angles in unfamiliar orientations, as well as a disjunction between the memorised statement and its meaning.

Another example of introducing formal concepts before developing the relevant conceptions would be if students were told that 'equivalent fractions' are found when you "multiply the numerator and denominator by the same number". Alternatively, students could develop a conception of equivalent fractions through investigating, perhaps using diagrams to solve a problem. This conception could then be discussed, named, and so become a concept. Spending time developing conceptions before introducing formal concepts seems intuitively to be a valuable way to enhance understanding. Such a time investment, however, inevitably runs into issues of pace,

coverage and efficiency. The decision of how long to spend and what approach to take to help students build strong conceptions is a difficult and important one for teachers.

However powerful language may be, words alone do not a mathematician make. Definitions, however clear and precise, may not offer students the opportunity to develop a full concept.

> A person who hears or reads a detailed verbal formulation (the definition of a concept) can actually fail to have a definite visual image corresponding to the integral meaning of this formulation during this period, and yet "understand" it, know how to "explain" it. This reveals a characteristic feature of the concept as a particular form of reflection – the *nonvisuality* of its content. However, visual concepts have to lie beyond the particular attributes themselves, which are expressed in words."
>
> Davydov, 1972/1990: 53

This section set out to offer a brief overview of the state of understanding of how we learn words. It began by establishing that the learning of words actually requires much more than this: alongside the word itself, we must learn the general class of 'things' to which that word refers. The notion of 'association', by which each word is linked to its referent, was introduced. It is established that experimental data indicates that there is rarely a spatial and temporal concurrence between word and meaning, and that even where this is the case, the meaning of a word cannot be clearly deduced from the presence of the referent (Quine, 1960; Goodman, 1983). Once a one-on-one pairing between a word and its meaning has been adopted through exposure to the appropriate vocabulary, we do not necessarily rely on feedback and correction to ensure that we have correctly learnt the word (Lievan, 1994; Stromswold, 1994). Of course, before such a pairing can take place, we must possess the necessary concept. These can be developed through exposure to examples (Skemp, 1971) and encouragement to attend to aspects of the particular that also appear as features of the general (Mason and

Pimm, 1984). Davydov (1972/1990) described the process of coming to understand the meaning of a concept as a three stage movement from perception to conception to concept. The term 'lexical familiarisation' was introduced to describe the process through which new concepts are introduced. Whilst psychologists often separate word learning into two components: having the concept and mapping the concept onto the right form (Bloom, 2002), Davydov (1960) distinguishes three components: knowing the precise scope; knowing the name, and understanding the general concept without being offered a particular example.

2.4 ALGEBRA: THE LANGUAGE OF GENERALISATION

Whilst the separation of the mathematics curriculum into constituent parts, as in sections 2.1 – 2.3, does offer insights into the subject, it seems likely that many other subjects on the secondary curriculum could be reasonably broken down into 'general facts', 'general approaches', 'general procedures' and 'general concepts'. Yet my focus on the expression of generality seems particularly pertinent to mathematics education. Seeking a justification for my intuitive sense that the role played by generality in mathematics is somehow greater or more significant than its role in other secondary school curriculum subjects, I consistently arrived at the conclusion that *algebra* lay at the heart of the claim that generalisation is more important in mathematics than other subjects. Many of the problems discussed above might apply to other subjects. A new piece of vocabulary, or a new concept, might need to be learnt, or a new procedure might be taught. Learning in any subject, then, involves generalisation: students need to realise that this particular fact, concept, procedure or approach can be applied in other circumstances. The *expression* of this generality, however, does not usually form such a central part of a subject as it does in algebra.

One of algebra's most obvious purposes is to express general properties of numbers, operations on number, functions, and many other referents (Stacey and MacGregor, 2001: 141).

Algebra is not merely the use of particular symbols and letters, or the interpretation or transformation of expressions involving these symbols. The process of appreciating and expressing generality about a mathematical procedure or concept might be considered by many to be 'algebraic thinking'. Perhaps in response to the case that algebraic thinking could be viewed as seeing the general through the particular, Kieran (1989a) argued that *algebraic* thinking required not only that the general should be expressed, but that "one must also be able to express it algebraically" (Kieran, 1989a: 165). Whilst I acknowledge that other interpretations of algebraic thinking can be useful, including those that do not require use of algebraic notation to express the thinking, the definition of algebraic thinking that proved most valuable in this thesis runs along similar lines to Kieran's (1989a) idea of expressing the general algebraically. Throughout the remainder of this study, the term 'algebra' is used to mean use of 'algebraic notation', rather than adopting a looser sense in which consideration of generalised arithmetic, even when expressed in natural language, is included in the definition of 'algebra'. Section 2.4.1 offers illumination of the different interpretations of 'algebra' through history, as a background to its interpretation in this study.

2.4.1 Historical significance of algebraic notation

For an insight into why the word 'algebra' might have such a diversity of meanings I next consider the historical development of algebra. In 1842 G. H. F. Nesselman

categorized the historical development of algebraic symbolism into three stages, rhetorical, syncopated, and symbolic algebra (Bell, 1945). Rhetorical algebra writes the solution of a problem without any abbreviations or symbols. Syncopated algebra uses shorthand abbreviations for some of the more frequently used operations, quantities, and relations. Symbolic algebra writes the solutions to problems in a type of mathematical shorthand made up of symbols, some with less than obvious connections to the ideas and things they represent.

Expressions and equations have been a vital part of the history of mathematics. Starting with the ancient Egyptians and Babylonians about 3000 years ago, rhetorical algebra was used in the form of words to solve linear equations. Rhetorical algebra is also "the kind of algebra encountered by today's school children well before any formal notation is introduced" (Sfard & Linchevski, 1994: 197). Before the time of Diophantus of Alexandria (around A.D. 250), all algebra appears to have been rhetorical.

Diophantus (3rd century AD) introduced some symbolism in to Greek algebra, but rhetorical algebra endured in most of the world, except India, for many centuries. Diophantus and the Hindus were some of the first to use some type of shorthand or symbols in their algebra. In Diophantus's *Arithmetica* there are abbreviations for the unknown up through the sixth power, subtraction, equality, and reciprocals, a type of notation that came to be known as "syncopated algebra" (Sfard, 1995: 18). Not until the fifteenth century did some type of syncopation begin to appear in Western Europe, and in the sixteenth century symbolic algebra started to be used there. Its development did not become wide spread until around the middle of the seventeenth century.

Around 820AD the Persian mathematician Al-Khwarizmi introduced the term *Al-Jabr*, which gave algebra its name. The term, which means *to combine*, refers to the process of combining like terms when simplifying an equation in an attempt to solve it. Centuries later, and specifically in 1591, Francois Viete wrote an algebra book which is very similar to modern algebra texts, formally giving rise to symbolic algebra (Sfard, 1995). Then, in 1830, the British mathematician George Peacock proposed that in algebra letters replace numbers and that, in general, algebra was arithmetic using symbols (Sfard, 1995).

Having considered, in this section, the development of algebraic notation and its uses through time, the following section considers its place on the secondary mathematics curriculum, and the research findings concerning students' difficulties with learning to use and understand algebra.

2.4.2 Difficulties in using algebraic notation to express generality

> Every learner who starts school has already displayed the power to generalise and abstract from particular cases, and this is the root of algebra.
>
> Mason, 2005: 2

As described in section 2.3, students' capacity to learn words is often used, as in the Mason quote above, to argue that students are capable of thinking algebraically. However, research tends to focus on the difficulties students experience with algebra, rather than their natural ability to generalise. There is much more to learning algebra than merely using algebraic notation to express generalised arithmetic. But even the transition from arithmetic to algebra alone is fraught with issues.

Equals and equations

Different concepts associated with the equal sign were explored by Kieran (1981). As young students tend to use the sign while practising arithmetic, it can become viewed as a "a left-to-right directional signal" (Kieran, 1989b: 393). There is consequently a risk that students will view the equal sign as merely a signal to "do something" rather than as a symbol of equivalence and balance (Kieran, 1990; Stacey and MacGregor, 1997).

Kieran and Herscovics (1994: 59) make a case for the existence of a *cognitive gap* between arithmetic and algebra that can be characterised as "the student's inability to operate spontaneously with or on the unknown". The fundamental concept of equivalence was discussed by Pirie (1995), who focussed on low attaining pupils' use of the equals sign when working with equations. The students were encouraged to see the equation with unknowns on both sides as a single entity rather than as consisting of two separate sides. By seeing the equation as a mathematical *object*, with the equals sign as a 'fence', students were successful in solving the equations. The equals sign was seen as part of the whole and not taken as a signal for action (Kieran, 1992).

Objects and processes

Sfard (1991) and Gray and Tall (1994) called attention to the challenges faced by students in understanding that algebraic expressions are, at the same time, both objects and processes. Tall & Thomas (1991) describe this *process-product obstacle*. This obstacle refers to the inability of students to view algebraic expressions as having a dual nature; that of a *process* and of a *product*. An expression such as $2 + 5n$, for

example, indicates both the instructions to perform a calculation (process) and it is also the result of such a calculation when a value is not assigned to the variable (product). If an algebraic expression is only viewed as a process then "the powerful way in which it can be manipulated and linked to other expressions makes little sense and failure with algebra becomes inevitable" (French, 2002: 16). This relates to the *lack of closure obstacle* (Collis, 1975) which refers to students' view of 2 + 5*n* as an incomplete answer.

Sfard and Linchevski (1994) develop their *theory of reification* according to which there is an inherent process-object duality in the majority of mathematical concepts. They argue that the operational (process orientated) conception emerges first and that the mathematical objects (structural conceptions) develop afterwards through *reification* of the processes. They maintain that this is a difficult stage for many learners to achieve and that it is seldom accomplished quickly or without difficulty. They propose that the development of algebraic thinking is accomplished by means of a sequence of ever more advanced transitions from the operational to the structural. In particular they consider two especially crucial transitions: that from the purely operational algebra to the structural algebra 'of fixed value' i.e. an unknown, and then from there to the functional algebra of a variable.

According to Kieran (1989c; 1990), in relation to an algebraic or arithmetic expression, the *surface structure* of an algebraic expression refers to "the given form or arrangement of the terms and operations, subject ... to the constraints of the order of operations" (1989c: 34), while *systemic structure* refers to the properties of the operations, and relationships between them. One example of *systemic structure* is the

equality relationship between the left- and right-hand expressions of an equation. Kieran states that, for many students, "the equation is simply not seen as a balance between right and left sides nor as a structure that is operated on symmetrically" (1989c).

Lacking meaning

Because we read from left to right, many students tend to interpret expressions such as $2 + 5n$ as $7n$. Tall and Thomas (1991) refer to this as the *parsing obstacle*. Tall and Thomas (1991) also draw attention to what they call the *expected answer obstacle*. From prior experiences with arithmetic students expect to perform some calculation when they encounter an operation sign such as +. So, when faced with algebraic expressions such as $2 + 5n$ they expect to produce an answer.

Various studies have also been conducted that looked at student difficulties when dealing with the concept of equation. Equations are defined as open number sentences consisting of two expressions which are set equal to one another. This is what Kieran calls the "*surface structure* of an equation" and it is an aspect that students find challenging to recognize (1989c: 34). In addition, students have trouble recognizing the "*systemic structure* of an equation" which includes the equivalent forms of the two expressions given in the equation (1989c: 34). Kieran claims that students who "view the right-hand side of an equation as the answer and who prefer to solve equations by transposing," lack an understanding of the balance between the right and left hand sides of the equation (1989c: 52). Moreover, in one of her studies Kieran found that many algebra students "could not assign meaning to a in the expression $a+3$ because the expression lacked an equal sign and right-hand member" (1990, p. 104). Relating

to this, a 1984 study by Wagner, Rachlin, and Jensen found that students added "=0" to any expression they were asked to simplify.

Students also face difficulties when asked to work with equivalent equations. A study by Steinberg *et al.* (1991) examined the knowledge of eighth and ninth grade students related to equivalent equations. Pairs of equations were given to the participants who had to identify whether the equations in each pair were equivalent. The researchers found that students who gave incorrect solutions could not distinguish between $3x$ and $3+x$. In addition some thought that subtracting a number from both sides of an equation would alter the answer because "-4 on each side is subtracting 4 twice" (Steinberg *et al.*, 1991: 117).

Another study of secondary school students by Hall (2002), examined the errors that students make when attempting to solve simple linear equations. The results showed that many students "find the process of collecting "like" terms so difficult that they cannot confidently simplify an expression such as $3x+2x$" (2002: 46). Hall reports that some students have difficulties combining "like" terms in expressions such as "$3x+2y+4x$" which involve "unlike" terms within the expression (p. 46). The author concludes that students who have such difficulties with combining like terms in the case of expressions will have even more difficulties in finding the appropriate strategy to solve simple linear equations.

MacGregor & Stacey (1992, 1995) drew attention to the limits of X-Y numerical tables in the generalization of patterns. It was apparent that these tables, which list the inputs and outputs of an algebraic function, were emphasizing a formulaic aspect of

generality based on trial and error heuristics, hence confining algebraic notations to the status of place holders bearing very limited algebraic meaning.

The wide variety of difficulties outlined in section 2.4.2, including interpretation of equals and equations, distinguishing between objects and processes, and manipulating algebraic symbols without meaning, provide real challenges for teachers. The following section considers the research into teaching approaches that might help students to overcome these difficulties.

2.4.3 Issues in teaching algebra

Whilst the diagnosis of students' algebraic issues is interesting in its own right, the researchers' intention is usually that their study might inform teachers' practises. Having found and categorised nine errors carried out by students when solving linear equations, for instance, Hall (2002) offers a number of ideas concerning use of manipulatives to support student understanding. His main advice for practising teachers, however, relies on teacher-led discussion to 'prevent the formation of bad habits'.

> Familiarity with these nine errors should enable me, and perhaps other teachers, to be better equipped to forestall the most common mistakes. Indeed, such errors could be discussed at the appropriate point, both in lessons and in textbooks. This may be especially important at the introductory level because it prevents the formation of bad habits as well as the development of inaccurate constructions on the part of the learner.
>
> Hall, 2002: 62

As a practising teacher, faith in teacher-led discussion such as Hall's above can appear daunting, as researchers offer such interactions as the solution to all difficulties, but are often vague about the form such discussion might take. Another

example where teacher-led discourse is seen as effective, but where the reader is left

unsure of the form the discourse takes can be found in Radford's description of a class

working on using algebraic notation:

> When the students reached an impasse, the teacher intervened: "If the
> figure I have here is 'n', which one comes next?" Thinking of the letter in
> the alphabet that comes after n, Josh replied: "o". In the end they ended up
> with the following formula: "$(n+1) + n$".
>
> Radford 2006: 13

I found myself wondering *how* the teacher had intervened in order that they should

have 'ended up with' a correct formula.

An increasing number of studies are taking these errors and issues as their starting

point, and developing activities or teaching strategies designed to respond to the

identified issues. Nickson (2003) distinguishes two main clusters of research

concerned with the teaching and learning of algebra. There is a group of studies

relating to what teachers do with their students, and a group relating to the teaching of

particular topics within algebra. She points out that the focus of the first group of

studies is not the conceptual content of the lessons, while studies in the second group

address matters concerning particular content (Nickson, 2003: 33). In my study, with

its focus on teacher-led discourse, more can be discovered about 'what teachers do'

than about student understanding. I strive to address both areas in the analysis,

although theory regarding student understanding remains hypothetical or is based

closely on previous studies.

Whitney (1985) argues that students should grow in their natural powers of seeing the mathematical elements in a situation, reasoning with these elements to come to relevant conclusions, and carrying out the process with confidence and responsibility.

Hudson, Elliott and Johnson (1999) carried out a study exploring how a focus on language and meaning can assist students in reconstructing algebraic knowledge. The researchers' initial perspectives included:

- An interest in the use of vocabulary and terminology, such as *solve* and *simplify*
- David Pimm's (1995) discussion of the notion of a mathematical register
- An emphasis on the language competence underpinning the development of symbolic representation
- The Vygotskian (1962) emphasis on the social and communicative aspects of language and on speech as an instrument of thought itself, that is, as a psychological tool.

Hudson *et al.*, 1998: 2

These relate to my own starting points. Their study involved video recording a series of one-hour discussions with four groups of students, based around tasks designed to get students talking together about their understanding of algebra. Their aim was to, "relate our background reading to the development of a theoretical framework taking account of our emphasis on language and meaning in the interpretation of our data" (Hudson *et al.*, 1998: 4). After in-depth and detailed analysis of particular sections of the videotape transcripts, they found that their broad background understanding of teaching and learning algebra could not fully account for the phenomena observed. They therefore turned to social practice theory (e.g. Lave and Wenger, 1991) in search of a 'wide(r) angle lens' (Dengate and Lerman, 1995) through which to view their data.

These issues are explored further in section 2.5, through use of the metaphor of algebra as a second language. I discuss five main areas under consideration by maths educators concerned with the teaching of algebra: its aims, categorisation, place within the maths curriculum, use of rules, and choice of exercise or task. As the teaching of algebra is not the central focus of my research, I have not attempted to fully encompass the range and detail of literature on these five themes, but merely to give an overview that helps to situate the present study.

Is it the aim of teaching algebra that students should know the rules of algebra, or apply them in meaningful contexts? Should students devote time to manipulating expressions or creating them? Syntax of algebra involves manipulation of algebraic symbols. Semantic algebra is concerned with meaning. In the early stages, this would involve using algebraic notation to express arithmetic generality.

The general is present and involved in both these interpretations, as a lesson on algebraic manipulation (i.e. one that emphasised syntax) would expect students to understand and apply a rule such as 'multiply the term in front of the brackets by each of the terms within the bracket', or to follow several particular examples of this rule and generalise it for further particular cases. The general plays a huge role in manipulating algebraic syntax.

There is both syntactic and semantic generality involved in algebraic notation. It is possible to express a generality about the syntax, and also to use its semantics to express generality.

Malara and Navarra advocate "a didactical contract which tolerates initial 'promiscuous' syntactical moments" (2003: 230). They introduce the metaphor of 'algebraic babbling', emphasising the similarities between learning natural language and learning algebraic language. They promote early introduction of algebraic thinking, with ideas of generalised arithmetic expressed using some algebraic language. 'Algebraic babbling' allows for unconventional use of the syntax, as the emphasis is on semantics in the early stages.

Many researchers have questioned the extent to which algebraic notation is meaningful for students. "It is generally accepted that this meaning cannot only be a syntactical meaning, which would mean just knowing the rules of symbolic manipulation and conforming strictly to them – to some extent independently of any sense making" (Balacheff, 2001: 251).

> Be this as it may, I hardly believe that the didactic situations susceptible to leading our students to deeper layers of symbolic or other forms of generality can be reduced to the choice of fortuitously good mathematical problems. Powerful though it may be, the plane *subject-object* is not, epistemologically speaking, strong enough. The plane of *social interaction* must be included. The students have to learn to see the objects of knowledge from others' (teachers and students) perspectives. This is why, in the classroom, we often organized an exchange of ideas and solutions and the discussion of them between groups, followed by general class discussions (Radford & Demers, 2004). The idea, however, is not merely to 'share' solutions in order to *catalyze* the attainment of deeper layers of generality. It is rather that the objectification of knowledge presupposes the encounter with an object whose appearance in our consciousness is only possible through contrasts. Our awareness and understanding of an object of knowledge is only possible through the encounter with other individuals' understanding of it (Bakhtin, 1990; Hegel, 1977; Vygotsky, 1962). In this encounter, our understanding becomes entangled with the understandings of others and the historical intelligence embodied in cultural artifacts (e.g. language, writing) that we use to make our experience of the world possible in the first place.
>
> Radford, 2006: 17

I am interested in the teacher's role in this process and, more specifically, in the teacher's use of language as an influence on students.

Kieran (1996) expresses concern that of three kinds of activities that have been found to be beneficial in learning algebra, the early stages of the secondary school curriculum tends to focus on *transformational activities*, which are rule-based activities such as simplifying expressions and solving equations. Time is devoted to these *transformational activities* at the expense of *generational activities* where students generate expressions and equations to express general relationships, and *global, meta-level activities* such as problem solving, justifying and finding structure.

A point is reached in a student's mathematics education, often at key stage 5 (aged 16-18) where algebraic notation is deemed the most appropriate means for communication of generality. The literature reviewed in section 2.4.3 has pointed to a tension that can be summed up briefly as being between the following two conjectures:

1. For students to become more familiar with, and gain a better appreciation of, algebraic notation for expressing generality, teachers might beneficially use such language as frequently as possible.

2. Students find algebraic notation confusing, so teachers should avoid using it where a generality can be expressed using language with which the student is more comfortable.

2.5 ALGEBRA: A NON-NATIVE LANGUAGE?

The observation in section 2.4 that algebra could be viewed as a language through which general mathematical procedures and concepts can be expressed led me to investigate the research literature concerned with acquisition of non-native languages. The teaching of algebra as a purposeful communication tool is a recent idea in comparison with similar arguments in second language acquisition theory. As second language teaching is not the central focus of this study, this brief literature summary of methods and approaches is intended to indicate areas of relevance and interest to the field of mathematics teaching, rather than to provide an exhaustive review of the area.

In line with writings related to non-native language teaching, I use the term 'L2' to refer to the language being learnt. In this section I consider the extent to which algebra might gainfully be regarded as an L2. First I must make it clear that it is not my intention to suggest that algebra *is* an L2, but merely to draw attention to some interesting parallels. L2 studies offer a vocabulary, and theoretical and empirical richness from which to make greater leaps in the field of mathematics education. The vocabulary has been developed to enable concepts etc. to be discussed. In coming across much of this material, I found it resonated with ideas that I had developed or was developing, but was finding difficult to express through limited vocabulary. Rather than create and introduce a new set of terminology for mathematics education, it might make sense to use the terms adopted by the L2 research community. Whether or not these terms prove useful for my data analysis and sharing my conclusions, I have found them useful personally when grappling with the ideas involved.

The four main issues that repeatedly emerged from the L2 education literature were the extent to which the teacher communicates using the target language in the classroom, the extent to which classroom L2 use is purposeful and communicative, whether the focus is on correct manipulation of symbols, or on symbol meaning, and how grammatical rules should be taught. These can be seen as closely analogous with issues in the learning of algebra, as discussed in section 2.4.

2.5.1 Use of target language

Although Morgan and Neil (2001) insist that modern languages are the only subject in the curriculum where the language is both the content and the medium of instruction, it is possible to listen to some mathematics lessons, especially those focussing on algebra, and conclude that this 'unique' attribute could also be applied to mathematics.

One major element of controversy regarding the introduction of the UK National Curriculum guidelines was its insistence on use of the target language (TL) for 95-100 per cent of the time:

> Students are expected to use and respond to the TL, and to use English
> only when necessary (for example, when discussing a grammar point or
> when comparing English and the TL).
>
> DfEE/QCA, 1999: 16

The desire to use the target language for the majority of the time in language classrooms stems from the aspiration of creating a naturalistic environment for L2 learning. The National Curriculum requires that students are given opportunities for:

- Communicating in the target language in pairs and groups, and with their teacher
- Using the target language creatively and imaginatively

- Using the target language for real purposes

DfEE/QCA, 1999: 17

This led to my wondering whether the same criteria could be applied to students' use

of algebraic language in the mathematics classroom:

- Communicate algebraically in pairs and groups, and with their teacher
- Use algebraic notation creatively and imaginatively
- Use algebraic notation for real purposes

One textbook for beginning teachers in teaching L2 suggests:

> Use previously taught/learned structures for genuine communication ...
> You can create a bank of structures which students should be encouraged
> to use when dealing with the immediate classroom environment ... This
> bank of phrases can be added to in a systematized fashion during the
> course of their language learning classes.
>
> Morgan and Neil, 2001

This is challenging for teachers when placed in the context of algebraic notation. It

might roughly translate as: introduce notation and rules, remember you've introduced

them, and use them meaningfully in future lessons.

Atkinson (1993) gives some reasons why total target language use is not ideal.

Although when only the naturalisation variable is considered, total use of the target

language might seem desirable (as it most closely mimics total immersion), there are

many variables associated with maximising student learning. Some effective activities

can only be effectively carried out in students' native language. What sets out to be a

communicative ideology can inhibit the communication it was originally designed to

encourage.

Salters *et al.* (1995) carried out small-scale research that examined the teaching of science through the medium of French, taught by a native speaker. They found that the teacher's status as a native speaker can be a major factor in students' reacting positively to the lesson and the L2. If teachers try to pretend that they do not understand English, this can lead to an artificial situation that is far from the communicative ideal. As the mathematics teacher may be the student's only source of mathematical or algebraic vocabulary, it is tempting to conclude that such vocabulary should be used wherever relevant. The teacher has the task of providing a mathematically rich environment within the confines of the classroom.

In response to the insistence in government publications and inspectors' reports on use of the target language, some critics argue for a more balanced view of the place of the students' native language in learning the L2 (Buckby, 1985; Harbord, 1992; Atkinson, 1993; Collins, 1993). Common sense would suggest that it is possible to use more of the target language with students who have been learning the language for some time, than with beginners. It is surprising, then, that recent research does not find this to be the case (Dickson, 1996; Dobson, 1998).

Researchers engaged in the debate over appropriate use of the target language distinguish between two uses of the native language: codeswitching and decoding. The term 'codeswitching' refers to the practice of switching between the target language and the mother tongue for specific purposes. 'Decoding' refers to the practice of making an utterance in the target language followed by the translation in the native language. Wong-Fillmore (1985) observes that such decoding encourages students to wait for the translation and so deprives them of the process of deciphering

the target language used. Buckby (1985: 51) advocates use of the 'English sandwich',

in which a phrase is uttered in the target language, followed by the English

translation, and immediately followed again by the target language version.

Advice for teachers in encouraging student use of the target language:

- Accept incomplete utterances and allow them to supplement with English
- Encourage inaccurate language which does not impede communication
- Refuse to accept English where it is not necessary

<div align="right">Morgan and Neil, 2001: 148</div>

It is strikingly difficult to find such succinct recommendations to mathematics

teachers regarding students' use of algebra to express ideas, even as a starting point

for debate and discussion.

Section 2.5.1 was intended to point to the differences between the debate in L2

education literature concerning 'use of the target language' and use of algebra in

mathematics classrooms. Distinctions that have been developed in L2 teaching

research, such as between 'codeswitching' and 'decoding' might also be applied to

different ways teachers might use algebraic language to communicate with their

students. This literature resonates with main study findings (see chapters six to eight-

II) and reflection on my own practice (chapter nine), which suggest that there may be

advantages in mathematics education researchers and teachers engaging in a parallel

debate concerning use of algebra – a target language for mathematics.

2.5.2 Communicative language teaching

Communicative language teaching strives to give learners authentic, meaningful and purposeful tasks for language use. Communicative language courses do not focus on content learning, but use the content as a means for obtaining language learning. Communicative language teaching is at one extreme of a continuum of content-based approaches to language teaching, which ranges from language-driven to content-driven. As teachers and students are not held accountable for content outcomes, communicative language teaching can be described as the language-driven end of the spectrum of content-based approaches. At the opposite extreme, content-driven approaches are those in which student and teacher are not held accountable for language outcomes. For example, immersion programmes, in which the L2 is the medium of instruction for the curriculum. This resonates with the 'purposeful algebra' approach advocated by Ainley *et al.* (2004) and Coles and Brown (1998).

At the content-driven end, there is the introduction of mathematical vocabulary and algebraic notation in the midst of an investigation or exploration of a topic. The emphasis on content, rather than language, might be what I am seeing in the observed classrooms.

The problem here is that teachers could be 'sheltering' too much, for too long. This is perhaps why the National Curriculum places specific emphasis on acquisition of mathematical language and algebraic notation. The potential problem with this reemphasis on language is that the emphasis might shift to the other end of the spectrum, with a language-driven approach in which algebraic conventions are drilled and practised in unconvincing 'contexts'.

2.5.3 Meaningful study

From the 1980s onwards, task-based alternatives to conventional L2 syllabuses have been developed, as a response to both dissatisfaction with conventional linguistically-based syllabuses, and research findings into second language acquisition. Three distinct approaches can be identified (Doughty and Williams, 1998): courses with a structured approach to learning the grammar and structure of the L2 (*focus on forms*), courses where the L2 is used to explore interesting and diverse topics without focussing on vocabulary and grammar (*focus on meaning*), and courses that pay attention to formal elements of the L2 in the context of meaningful study (*focus on form*). The development of these three approaches is explored in this section.

Structurally graded materials tend to provide stilted language models, which reduces student motivation. Pedagogical devices such as translation, grammar rule exposition, pattern drills and error correction are used to teach small aspects of the language, such as a word or grammatical structure. The focus for learning, such as a word or grammatical structure, is presented by the teacher or textbook in chunks. This method is referred to as *focus on forms*. The learner is expected to try to learn each separate item when presented, and then to synthesise the parts in order to communicate when necessary. Wilkins (1976) calls this the *synthetic* syllabus. The model assumed by this teaching approach involves the accumulation of isolated linguistic entities, one-by-one.

One response to frustration with the *focus on forms* approach has been to abandon all attempts to teach specific units of vocabulary and grammar, and instead to develop an

approach to L2 learning which mimics that of students' learning of their native language. This approach, referred to as *focus on meaning*, is seen in immersion teaching and some content-based courses. This approach is limited by time restrictions, as L2 language exposure, even in an immersion situation, is rarely as extensive as students have experienced when learning their native language. It is also constrained by the less purposeful and necessary nature of L2 learning, in comparison with the purpose and need for the native language.

The late 1990s saw the development of a proposal for language teaching known as *focus on form* (Doughty and Williams, 1998; Long, 1998; Long and Robinson, 1998). This approach involves task- or content-based lessons, where grammar and vocabulary are introduced in context. The sequence and timing is determined by the students' internal syllabus, rather than an externally imposed one. Advocates of *focus on form* hold that this approach is more helpful for students than an externally imposed linguistic syllabus, with features such as explicit grammar rules, translation, structural pattern drills. Focus on form involves the use of a variety of pedagogic procedures designed to shift students' attention briefly to linguistic code features during an otherwise meaning-oriented lesson. These attention shifts are prompted by the learner's internal syllabus, as problems arise incidentally with comprehension or production while students are engaged in pedagogic tasks.

This approach should ensure that attention to linguistic code features occurs at the point when students have a perceived need for the new item. The meaning and function of the linguistic code are consequently more likely to be evident to the students. This is the time at which students are deemed to be psycholinguistically

ready to begin to learn the code. The *focus on form* approach does not require the use of any particular pedagogic procedure to achieve the focus on form.

Doughty and Williams (1998: 4) clarify the distinction between focus on form, on meaning and on forms, stating that focus on form *entails* attention to formal elements of a language, whereas focus on forms is *limited* to such a focus, and focus on meaning *excludes* it. Focus on form allows teachers and students to complete interesting, motivating courses dealing with content they recognise as relevant to their needs, while still successfully addressing language problems. Several attempts have been made (e.g. Long and Crookes, 1992; Skehan, 1998) to harness the benefits of a focus on meaning via adoption of an analytic syllabus, while simultaneously, through use of focus on form, to deal with its shortcomings, such as slow rate of development and poor grammatical accuracy. This can be seen as parallel to frustration with traditional approaches to mathematics teaching. There is movement towards the equivalent to a *focus on form* approach to algebra teaching, with research by Ainley *et al.* (2005) and Coles and Brown (2001), for example, advocating purposeful activity in mathematics classrooms, where algebraic notation is not the central focus, but attention is drawn to it where pertinent.

2.5.4 Teaching rules of manipulation (grammar)

Grammatical rules are often formed for learners, as pedagogic authors attempt to 'teach' grammatical rules that are only expressed in rule form for the purpose of teaching. Dirven (1990) found that the oversimplified, inaccurate rule formations in textbooks can and do mislead learners into incorrect generalisations. These arise because there is relatively little coherent theory underlying rule formation. The area of

rule formation is comparatively uncharted (Westney, 1994), although some grammarians have attempted to give a theoretical basis to their rules (e.g. Leech and Svartvik, 1975; Swan, 1994; Newby, 1989).

The methodology of teaching grammar is a highly contentious topic, with disagreement over whether the central aim of grammar teaching is that students should know about it, or use it, and whether they should be able to manipulate sentences or freely produce them. There is also discrepancy over the categorisation of grammar into units for the purposes of creating a syllabus or learning objective, and questions over the extent to which grammar should be dealt with separately from other aspects of language. Divergence of view can also be found regarding the extent to which a cognitive focus on explicit grammar rules assists acquisition, and the type of exercises and activities which will lead to grammatical fluency.

The traditional approach to teaching grammar is that learning is seen as a conscious process and grammar rules are used deductively (they are explained by teacher or textbook prior to being applied in exercises). According to this approach, grammar (seen as a set of forms and structures) is central to the textbook syllabus. The aim is for students to be able to form correct sentences, and this is to be achieved through presentation, explanation and practice. Exercises considered appropriate include gapped sentences, pattern drills and transformation of sentences. The emphasis is on form rather than context.

The communicative grammar approach, in contrast, not only sees language as a formal system, but as the process used to communicate messages between human beings in actual contexts. The central aim has shifted from focussing on formal correctness towards communicative effectiveness. In extreme cases, grammar was dispensed with altogether. Rather than concentrate on *analysis*, *use* became of greater importance, as a distinction was made between knowing *about* grammar ('declarative knowledge') and knowing how to *use* it ('procedural knowledge') (Johnson, 1994). This communicative approach, with a 'learning-by doing', inductive methodology, led to an apparent (but non-existent!) 'grammar versus communication' dichotomy, in which understanding was thought to emerge from use, rather than the other way round.

Just as L2 teaching suffered from a 'grammar versus communication' dichotomy, so there is a dichotomy in mathematics education between theory and manipulation.

> Much criticism has been levelled against the practice by which "symbol pushing" dominates early experiences in algebra. We call it "blind" manipulation when we criticise; "automatic" skills when we praise. Ultimately everyone desires that students have enough facility with algebraic symbols to deal with the appropriate skills abstractly. The key question is, What constitutes "enough facility"?
>
> Usiskin, Z. (1988: 16)

Krashen (1981) distinguished between learning – with a conscious focus on grammar and explicit rules and terminology – and automatic, unconscious acquisition. He argued that acquisition, rather than conscious learning, was the way to achieve communicative competence. He proposed a teaching method where 'comprehensible input' was automatically processed by students' innate acquisition mechanism. Various teaching methodologists took a similar approach during the 1980s, founded

on the belief that many of the processes that were so successful in first language acquisition could also be applied to the learning of L2s.

Some educational psychologists, focussing on the various cognitive processes linked to learning, and to learning a language in particular, argue in favour of learner autonomy. According to this method, teachers should guide learners towards focussing on aspects of language, and then be encouraged to use various cognitive strategies in order to explore for themselves how language works. Rather than 'imposing' their own grammatical knowledge on learners, teachers are seen as facilitators in the learning process. Grammar rules explained by the teacher gave way to consciousness-raising or discovery techniques and tasks (Rutherford, 1987; Rutherford and Sharwood Smith, 1988; Bolitho and Tomlinson, 1995).

The theories introduced in this section are illuminating when applied to the teaching and learning of algebra. Johnson's (1994) distinction between 'declarative knowledge' (knowing *about* grammar) and 'procedural knowledge' (knowing how to *use* it) can be revealingly applied to algebra, and have parallels with distinctions between conceptual and procedural understanding that were discussed in section 2.2. Likewise Krashen's (1981) distinction between learning and automatic, unconscious acquisition is one that has also been made in the context of mathematics education. These similarities indicate that there may be some merit in considering how L2 educationalists have resolved these tensions, as their approaches may offer something of use in supporting students' emergent algebra (Ainley, 1999a).

Algebraic notation as a language?

Whereas a learner of L2 would find most sentences easier to express in their natural language, some expressions are easier to express in algebra than they are in natural language. For instance, the rule that:

$$x^a \times x^b = x^{a+b}$$

In addressing the three research questions, the findings of which are revealed in chapters 7 and 8, I became increasingly aware of the unharnessed potential of algebra as a medium for mathematical thinking, and the expression of mathematical ideas. One aspect in which algebraic notation differs from an L2 is that (syntactic) manipulation of sentences in the language can give semantic insights. The syntax of the algebraic language, its grammar, tends to become the focus in lessons at the expense of its semantics. Algebra can be used as a communication medium, but can be reduced to 'grammar exercises' in the classroom, as its syntax is significantly different from those in natural language.

2.6 LANGUAGE AND LEARNING

In a study that focuses on use of language and teacher-led discourse, the role of language in learning is of central importance. It is therefore unsurprising that theories that explored the role of language in learning would play a significant role. The reader seeking a full exploration of alternative theories of learning is advised to look elsewhere, as word limit constrains this to a brief description of the Vygostkian approach to language and learning that frames this study.

Vygotsky explored the inter-relationship of language development and thought, and established an explicit and profound connection between speech (which may be silent inner speech or oral language) and the development of mental concepts and metacognition. Vygotsky's (1962) work is underpinned by a central assumption that socio-cultural factors are essential in the development of mind. Intellectual development is seen in terms of meaning making, memory, attention, thinking, perception and consciousness which evolves from the interpersonal to the intrapersonal. The individual dimension is considered to be derivative and secondary to the social dimension.

Vygotsky describes how self-talk develops, soon after the development of language as a tool for social interaction, as a tool to guide a child's activities, and this self-talk is used for self-directed and self-regulated behaviour. Self-talk "develops along a rising not a declining, curve; it goes through an evolution, not an involution. In the end, it becomes inner speech" (Vygotsky, 1987: 228). Around the age the child starts school, their self-talk is internalised. Language can thus be viewed as two separate systems. Firstly, a system of social communication, and secondly, inner speech. This is not to say that thinking cannot take place without language, but that it is mediated by it and thus develops to a higher degree of sophistication. Inner speech (involving signs based on those of communicative speech) provides much deeper meaning that the lower psychological functions would otherwise allow. While external speech is the process of turning thought into words, inner speech is the opposite; the conversion of speech into inward thought. For example, inner speech has no need for subjects, and contains predicates only. Word use is more economical, as one word in inner speech may be so

replete with sense to the individual that it would take many words to express it in external speech.

These ideas have significant implications for the teaching of mathematical concepts, suggesting as they do that students need language for concepts in external speech in order to manipulate those concepts in their inner speech.

Kozulin (1990) argues that language, rather than being correlative of thought, is correlative of consciousness, and that the mode of language correlative to consciousness is meanings. He maintains that to study human consciousness is to study a meaningful structure, of which verbal meaning is the methodological unit.

The London School of Linguistics, based on Malinowski's (1923, 1935) concept of context of situation, argues that utterances are comprehensible only in the context of the entire way of life of which they form part. They see language as essentially a social and cultural phenomenon which has evolved to fulfil our human needs.

Michael Halliday (1978) works within this tradition, which developed systemic-functional grammar. It terms all chunks of speech and writing 'text'. Systemic theory assumes that all languages have developed as a consequence of two general human needs: 'ideational meaning': the need to express ideas and 'interpersonal meaning': the need to express our social relationships with other people. There is a third component of meaning: 'textual': the way in which the other two types of meaning are brought together in speech or writing.

The authors referred to in this section helped to shape my understanding of the significance of language, and its relationship with thought. Vygotsky's concept of 'inner speech' supports the importance of developing mathematical concepts, suggesting that supporting students to become fluent in 'ME' (see section 2.3.1) may have more significant advantages for students than merely improved exam comprehension. Linguistic theories (Kozulin, 1990; Malinowski, 1923; Halliday, 1978) supported the emphasis that this study places on the spoken word, emphasising the importance of context in considering meanings, and the different uses of language.

2.7 TEACHER-LED DISCOURSE

Having established the importance of language and social interaction in mathematics classrooms, this section explores the issues related to teacher-led discussion in the mathematics classroom. The past decade has seen increased interest in the role played by language in mathematics education.

2.7.1 What is teacher-led discourse?

With the intention of placing my analysis of teacher-led discourse within the broader context of discourse analysis, I looked to the various strategies and spectra that have been developed to describe the variety of discourse that has been subjected to analysis within education research and beyond. One such spectrum concerns the extent to which students participate in the discourse. This extends from entirely one-way uninterrupted discourse with no student contribution, often termed 'lecture', to 'discussion' where students generate at least fifty percent of the talk. As shown in chapter six, the teacher-led discourse in the main study lessons lay in between these

two extremes. In discourse-analysis this type of discourse, where the teacher predominates, is often termed 'recitation'.

The lecture is the least interactive of the three main categories of teaching discourse. The term 'lecture' is used to describe one-way uninterrupted discourse, as when giving a speech (Hills, 1979). In a classroom context, this would involve the teacher 'delivering' a lesson without seeking student interaction. This category also includes other forms of unquestionable or unalterable content, such as books, radio or television. Much interest has been shown in how students learn through the lecture style of discourse. As the negotiation of meaning with the teacher and other students has been shown to play a central role in learning researchers have enquired as to how learning can take place effectively without interaction. It is argued that students participate in an "internal didactic conversation" (Holmberg, 1986) wherein they interact with course materials and "talk to themselves" about the new information and ideas it contains. Distance educationalists describe this as *learner-content* interaction. For Moore (1989) this is the "defining characteristic" of education: "Without it there cannot be education, since it is the process of intellectually interacting with content that results in changes in the learner's understanding, the learner's perspective, or the cognitive structures of the learner's mind" (1989: 2). Whilst this intrapersonal communication between a student and the instructional content is essential to learning (Dillon & Gunawardena, 1995; Hillman, Willis & Gunawardena, 1994; Holmberg, 1988; Moore, 1989; Wagner, 1994) interpersonal interaction must also be included. Holmberg (1988) insists that although pre-packaged materials for distance education can represent a kind of "simulated communication," it is the interaction between humans that "represents real communication" (1988: 116). Although the lecture

format is effective for disseminating information, it does so at the expense of validating this knowledge and making it meaningful to the student (Shale, 1988; Shale & Garrison, 1990). In their study of student nurses participating in distance learning, Gabriel and Davey (1995) found that the self-study packs were sufficient for students to learn elementary facts, but face-to-face interaction with other students was required for abstract or complex ideas. The Open University and other distance learning institutions have developed ways of encouraging self-reflection and engagement without discussion. Pirie and Schwarzenberger (1988) emphasised the role of mathematical discussion in developing understanding.

Recitation is the term used to describe the most common form of classroom interaction. In this mode of interaction, the teacher is the predominant speaker. The teacher guides the class through the use of questions, instructions and information (Edwards & Furlong, 1978; Hodge, 1993; Sinclair & Brazil, 1982; Sinclair & Coulthard, 1975). In 1985, Dillon found that on average the teacher speaks for 59% to 69% of the time (Dillon, 1985; 1994). Kramarae and Treichler (1990) report that it is typical for teachers in college classrooms in the United States to speak for 75% of the time. Bellack, Kliebard, Hyman & Smith (1966) found in their experimental social studies classes, taught to seventeen-year-olds, that teacher speech varied from 60% to 93% of all classroom discourse. These numbers are similar in the United Kingdom (Barnes, 1976).

This classroom domination is evidenced in the artificial interactions that take place in the classroom. Although classrooms frequently contain between twenty and thirty potential communicators, discourse within them is often characterised by a 'central

communication system (Adams and Biddle, 1970). This centralised communication is reinforced and maintained by means of rhetorical techniques such as responding to questions by asking another question, traditionally a technique used by teachers (Gere & Stevens, 1985). Flanders (1970) devised the "two-thirds" rule: two-thirds of every class is made up of talk, and two-thirds of the talk comes from the teacher.

The second characteristic of recitation is that the interaction between the teacher and students will follow a regular pattern. The teacher will initiate some form of action, usually a question, the student will respond, and the teacher will acknowledge the student's response (Atkinson, 1981; Dillon, 1985; 1994; Hodge, 1993; Mehan, 1978; Sinclair & Brazil, 1982; Sinclair & Coulthard, 1975; Stubbs, 1983). This mode of interaction is described by Sinclair and Coulthard (1975) as Initiation-Response-Feedback. They propose that it is the quintessential teaching exchange: (teacher's) *initiation*, (student's) *response* and (teacher's) *feedback* (Stubbs, 1983). The last stage is also known as *evaluation* (Mehan, 1978). "One aspect of this form of interaction is that the teacher retains control of the conversation, and another is that renewed initiation is not always required." (Pimm, 1987: 28). Pimm (1987:28) offers an extract in which the student appears to take the teacher's negative evaluation of a response to indicate that further suggestions are required.

These two characteristics of recitation are interrelated. Since the teacher is controlling the class by means of initiation and feedback, he or she will tend to do most of the talking (Atkinson, 1981). When the student asks a question, however, the structure is reduced to initiation-response since students do not "overtly evaluate teachers' answers" (Stubbs, 1983b).

Two main features distinguish discussion from the other two forms of classroom interaction. Firstly, unlike a lecture or recitation where the teacher will do all or two-thirds of the talking respectively, the students in a discussion will generate at least fifty per cent of the talk (Dillon, 1985; 1994). Secondly, students' participation in the discussion does not follow the initiation-response-feedback model of recitation. Both teachers and students offer statements and questions (Dillon, 1994). Shale & Garrison (1990: 29f) argue that discussion enables students to validate their "emerging knowledge through collaborative and sustained interaction with a teacher and other students".

Teachers whose predominant mode of classroom interaction is discussion reinforce the idea that the teacher is an active, communicative partner in learning (Jones & Mercer, 1993). Discussion also fosters co-operative learning between students and teacher (Fowler & Wheeler, 1995). The merits of this form of interaction are supported by Amidon and Giammatteo (1967) who found that the classrooms of those elementary school teachers that were found to be 'more effective' involved approximately 12% more student participation. These teachers were interrupted by more student questions, and tended to encourage and build on student ideas. Although teacher self-report indicates that teachers believe themselves to be using discussion, external observation indicates that this is not the case (Alvermann *et al.*, 1990; Connor & Chalmers-Neubauer, 1989).

Analysis of the main study transcript, and calculation of proportions of teacher and student talk (discussed in chapter 6) demonstrated the limitations of relying only on

comparisons of *who* is talking, or *how much* students contribute. It was therefore necessary to explore further into the qualitative differences between discourses.

2.7.2 Analysing teacher-led discourse

Analysis of the teacher-led discourses in the main study lessons, as will be discussed in chapter six, placed them in the category of 'recitation' as defined in section 2.7.1. However, deeper analysis was required in order to distinguish characteristics that might account for qualitative differences between the discourses. Researchers tend to develop their own systems of discourse analysis, or adapt others' for their particular purposes. This tailoring of the analysis method to the subject being studied offers the potential for greater insight into the specific research questions and data. It is nevertheless essential to benefit from the background of discourse analysis, in order to learn from both the methods and the findings of previous studies.

Flanders' categories of description for classroom verbal behaviour (1970) established the Interaction Analysis tradition, where classroom language is analysed with the intention of revealing something about the teaching and learning process. Since his initial development of ten Flanders' Interaction Analysis Categories (FIAC), many hundreds of classroom observation instruments have been developed along similar lines. His original categories were divided into *teacher talk* (seven categories including 'accepts or uses ideas of pupils', 'asks questions', and 'lecturing'), *pupil talk* (either 'response' or 'initiation'), and *silence*. FIAC has been developed into subject-specific techniques. For the language-teaching classroom, for example, Moskowitz (1976) adapted and added to FIAC in the development of his Foreign

Language Interaction (FLINT) categories. In his analysis tool, each of the FIAC categories is sub-divided to give a more detailed description of the interaction.

It is noteworthy that such descriptive categories include both pedagogic and social language. There is significant overlap between the two. Bowers (1980) sought to distinguish more clearly between language used directly for teaching and learning, and language used for social or organisational reasons, through developing seven categories of 'move'.

Brown's Interaction Analysis System (BIAS) (Brown, 1975) was developed in the Interaction Analysis tradition, and is a simplification and reduction of Flanders' original ten categories. Rather than coding acts or moves, these categories can be used with a time-line, and coded at regular time intervals (every three seconds, for example).

The aim of using tools such as those described above was to find out about the teaching and learning that goes on in a classroom. Classroom based language studies are also carried out with the intention of finding out more about how language works. Sinclair and Coulthard (1975), for example, carried out research into classroom language for linguistic purposes, to find out more about the structure of spoken discourse. They devised a classification system consisting of twenty-one acts, including *prompt* (such as 'Have a guess', 'Come on quickly'), *metastatement* (helping pupils see the purpose and structure of the lesson) and *conclusion* (summarising what has proceeded).

The potential disadvantage of systems of analysis such as these is that, by focusing on individual utterances, they neglect the bigger picture. Malamah-Thomas recommends that researchers should focus on whether or not the 'key message' is understood by students:

> ...these systems tend to concentrate on the various parts of the lesson. In order to analyse, they must fragment. And, in stressing the parts, they all overlook the whole; the whole lesson which is greater than the sum of its parts. For the crux of any classroom lesson lies in the learning that occurs in it. The crucial factor is whether the teacher gets his or her message across, whether the students learn what the teacher sets out to teach them. Any worthwhile analysis of classroom interaction must focus on this factor, and should also point up why the lesson succeeds, if it is successful, and why it fails, if it is unsuccessful.
>
> Malamah-Thomas, 1987: 31

The difficulty with this insistence is that there is no known method for accurately measuring whether a lesson 'succeeds' or not.

Section 2.7.2 has given an overview of categories and systems of analysis, such as Flanders' Interaction Analysis Categories (1970), that have been used to offer insight into classroom verbal behaviour. In this study I strive to heed the warning of Malamah-Thomas (1987) that fragmentation, concentration on individual utterances, can lead researchers away from, rather than towards, appreciating what makes a lesson successful. In classifying expressions of generality, my intention is to retain a sense of the overall 'journey towards the general' (see section 6.5).

2.7.3 Characteristics of teacher-led discourse

Brendefur and Frykholm (2000) develop and use a framework of four constructs that can be used to analyse various forms of classroom communication. Each successive level necessarily assumes the characteristics of its predecessor. *Uni-directional*

communication is where teachers dominate discussion through lecturing, asking closed questions. Opportunities for students to communicate their ideas are limited. *Contributive* communication describes interactions to assist or share, between students and between teacher and students. *Reflective* communication is related to Cobb *et al.*'s (1997: 258) concept of "reflective discourse". The sharing of ideas described in contributive communication here become "springboards for deeper investigations and explorations" (Brendefur and Frykholm, 2000: 127). *Instructive* communication takes place where subsequent instruction is shaped by the student-teacher conversations. Brendefur and Frykholm question the extent to which teachers are, "consciously aware of conversations that are moving from reflective to instructive levels of discourse?" (2000: 150).

Researchers such as Barnes *et al.* (1967) demonstrated that detailed linguistic analysis of transcripts often results in insights that can easily be missed in the rapid exchanges of classroom interaction. Through use of such detailed analysis, they distinguished between questions that were genuinely open and those that were *pseudo-open*, a term introduced for those that appeared on the surface to be inviting a variety of responses, but actually sought the one answer intended by the teacher.

Some of the researchers into classroom language have come from a linguistic, rather than educational, research background. Sinclair and Coulthard (1975: 6), for example, studied classroom discourse as a topic for investigation in its own right, rather than to make inferences about significance, intention and meaning. They "wanted a situation where all participants were genuinely trying to communicate and where potentially ambiguous utterances were likely to have just one accepted meaning".

Teacher-led discourse helps students and teachers to construct and develop their interpretations of mathematical meaning. Effective classroom discussion can assist in extracting the concept's essential attributes and negotiating between every day and mathematical word use. If classroom practices and discourse focus predominantly on routine calculations and finding answers, then the meaning associated with some signs may be very narrow. One frequently stated example of this phenomenon is that students interpret 'equal' to mean 'answer'.

Several studies offer both a classroom context, and a linguistic framework of useful constructs in the consideration of class discussion. Chapman (1997) linguistically characterises 'more mathematical' language. She emphasises that using such language is integral to the learning of mathematics, and describes the actions of a teacher encouraging students to shift towards 'more mathematical' language. Ainley (1987) proposed three categories of questions: focussing, rehearsing and enquiring. The allocation of questions to categories cannot be an exact science, as they are delineated on the basis of teacher intentions, not merely the words spoken.

Gerofsky (1996) uses techniques from linguistic analysis to compare word problems and college lectures, and makes suggestions as to how each of these genres (Bakhtin, 1986) could be adapted in order to make them more realistic. Morgan (1996, 1998) uses Halliday's functional grammar (1985) to analyse students' assessed written work.

Ball (1987) addresses the importance of 'intellectual honesty' between teachers and students in the mathematics classroom. For Ball, this means giving students space and

freedom to make sense of mathematics, ask questions, connect new experiences with their own knowledge, and listen to others' reasoning. She emphasizes the need for teachers to learn to hear, and carefully listen to, the things children care and think about in the classroom.

Discussion amongst groups of students is considered to be an important vehicle for sorting out ideas. Gagné and Smith (1962) found that students who were encouraged to talk about what they were doing were more successful than when talk played little part. Wall (1965) argued that group work resulted in increased production of conjectures. He believed that this both increased the likelihood of a solution being reached, and the quality of that solution. This is because conjectures and suggested solutions receive a higher level of criticism in a group.

Barnes (1976) outlined several ways in which teachers can influence groups' activity and discussion while remaining at a distance. Students should fully comprehend the purposes of the activity, should be convinced that their contribution will be valued, and should not be constrained by formal language or by trying to guess what the teacher wants. Students will need help in organising materials and ideas, and in preparing to share findings with the class. For group work to be effective, it requires a significant time investment.

Hewitt (1997) introduces the metaphor of amplification and editing to describe a way in which a teacher can help students to shift their attention (Mason, 1989). By editing student contributions, asking them to repeat themselves missing some of the sentence out, teachers can amplify that which is left. "The role of teacher as amplifier/editor

can also help focus the attention of a whole class on certain aspects of what has already been said, written, or drawn by someone in the class" (Hewitt, 1997).

A conjecturing atmosphere is defined as one in which ideas that are still uncertain and unformed can be expressed and shared. By expressing these ideas, they can be worked on and modified. As the quote below shows, listening plays a central role in the creation of a conjecturing atmosphere.

> The essence of working in a conjecturing atmosphere is therefore listening to and accepting what others say as a conjecture which is intended to be modified. Consequently, it is well worth noticing how you go about:
> * Developing and using a vocabulary which fosters conjecturing, (e.g. use words such as 'I suggest that ...' or 'Perhaps ...' rather than 'No!' or 'That's right!').
> * Listening to others and being listened to.
>
> Mason, 1988: 6

Brown and Coles' early papers reflect on partnerships between teacher and researcher (Brown and Coles, 1997), leading to a case study of a change in teacher behaviour which focuses on 'listening' (Coles, 2001). This 'listening' focus resulted in the development of an increasingly supportive 'conjecturing' atmosphere in which students valued each other's contributions.

The Zone of Proximal Development (ZPD) is the collection of actions which can be caused and triggered by others, and which the students could soon initiate use of for themselves. Vygostky introduced the term to indicate the area where the most sensitive assistance should be given. The intention is that students develop higher mental functions through this assistance, and develop skills that they will then use independently. This assistance, enabling the student to achieve a task within their ZPD, is termed scaffolding.

Findings from analysis of the main study lessons contributed to the development and extension of Zones of Proximal Development into the concept of Zones of Proximal Generality (Mason *et al.*, 2007). As this emerged from the data and contributed to the framework developed in chapter seven, discussion of this theory can be found in section 7.3.2.

An explanation of scaffolding is given in Tharp & Gallimore (1997) and other authors. Levels of support may vary in form, substance and context. Support is given when the teacher models the targeted performance of a task, giving verbal explanations that identify the elements of the task and strategy. Limited support would be the provision of cues to some aspects of the task to complement what students have already mastered. In a similar vein, Beed, Hawkins and Roller (1991) have described levels of support that lie between these two extremes:

> Assisted modelling: Teachers provide some coaching and models that enable the completion of the task.
> Element identification: The teacher identifies the elements of the desired approach or strategies to help students complete the task.
> Strategy naming: The teacher articulates a relevant strategy and students employ it on their own.

Roehler and Cantlon (1997) focused on the types and characteristics of scaffolding in learning conversations and several different types were found:

- *Offering explanations*: Explicit statements are given by an expert to elaborate on the learners' emerging understandings.
- *Inviting students' participation*: Learners are given opportunities to assume control of the knowledge building process.
- *Verification and clarification of students' understandings*: If emerging understandings are reasonable, the teacher verifies the students' responses. If the understandings are erroneous, the teacher offers clarification.
- *Modelling of desired behaviours*, this includes:

Making thinking visible, as in *think aloud*, showing what someone
thinks about the process at a given moment.
Generating questions and comments as in *talk aloud*, for example when
a teacher shows how to perform by talking through the steps. Teachers
generate questions and comments but later students take over.
Inviting students to *contribute actively*. Learners are encouraged to
contribute clues in order to complete a task and to articulate their
understandings of task demands.

Meyer and Turner (2002) describe how scaffolded instruction during whole-class

mathematics lessons can provide the knowledge, skills, and supportive context for

developing students' self-regulatory processes. In examining classroom interactions

through discourse analysis, these qualitative methods reflect a theoretical change from

viewing self-regulation as an individual process to that of a social process. Their

article illustrates how studying instructional scaffolding through the analyses of

instructional discourse helps further the understanding of how self-regulated learning

develops and is realized in mathematics classrooms. Qualitative methods, such as

discourse analyses, and their underlying theoretical frameworks have great potential

to help "unlock" theories of learning, motivation, and self-regulation through

exploring the reciprocity of teaching and learning in classrooms.

Bauersfeld (1988) identified the phenomenon of *funnelling* in which the teacher

realises that a student has difficulties, and opens with a short question with the

intention of stimulating the student to correct themselves. On receiving an inadequate

response, the teacher goes further back, aiming to receive an 'adequate' response. The

quality of discussion deteriorates. The teacher's questions become increasingly

narrow, as the teacher reduces their presumption of the student's actual abilities.

Tension increases as the student realises that demands are becoming simplified and

more urgent. Over time, the discussion may be reduced to recitation or sentence

completion, until the student produced the expected answer, which is often a single word.

In his study exploring how students ascribe mathematical meanings to real-world situations, Voigt (1998) focuses on how the students and teacher negotiate mathematical meanings when they ascribe different meanings to phenomena. He describes a textbook picture of 6 people in a swimming pool and 3 people leaving the pool. He uses a fictitious discourse to illustrate how a teacher might direct a student towards the intended mathematical statement of 9 – 3 = 6.

- How many persons are in the picture? Students' answer: 9.
- How do we have to calculate if some go away? Answer: Minus.
- How many persons left the pool? Answer: 3.
- The result: How many persons remain in the pool? Answer: 6.

Through this procedure, the picture becomes a specific arithmetical task, and a "number sentence" represents the solution. Details of the picture become clearly related to mathematical signs. The sequence of questions occurs concurrently with writing the sentence. This procedure is an example of a pattern of interaction; it will be termed the pattern of 'direct mathematization'. Particular empirical phenomena are related directly to mathematical signs; the sequence of questions and answers establishes the close correspondence step by step. Neither empirical nor mathematical coherence is addressed in its own right.

Voigt, 1998: 209

Voigt points out that students might learn how to relate parts of pictures to particular numerals step-by-step, without thinking about the relationship among numerals. He argues that explicit negotiation of meaning is more helpful than direct mathematization (Voigt, 1998: 217). This could be viewed as a specific case of the tension between procedures and understanding that was described in section 2.2. Voigt's 'direct mathematization' might result in students being offered a 'procedure' for converting 'real world' situations into mathematical calculations, without the understanding required to evaluate the reasonableness of their interpretation. Voigt

argues that explicit discussion of alternative interpretations leads to improved understanding of mathematical signs.

2.8 DECISIONS AND TENSIONS

In each lesson, whenever something happens, it is at the expense of something else. Each decision that a teacher makes, while enabling one collection of happenings, restricts others. Berlak and Berlak studied teachers in the early 1980s, and found that one of the stresses of teaching comes both from the number of decisions teachers have to make, and from the nature of the decisions they have to take. Teaching decisions often require teachers to choose between mutually exclusive options, such as whether to treat all students the same, or allow for individual differences, or whether to intervene or ignore a situation. They pointed out that a single decision may involve the resolution of several such dilemmas. Through analysis of their observations, Berlak and Berlak (1981) delineate three sets of dilemmas, enabling teachers or researchers to assess a situation to discover if any dilemmas are present.

In the proceedings of a workshop that focussed on the mathematical knowledge required by teachers in order to teach effectively, and the ways in which such knowledge might be developed, Smith (2001) reports on a session investigating teacher management of whole class discussion. This paper contrasts with my own study, in that it explicitly focuses on teachers' mathematical knowledge in relation to class discussion, whereas I am seeking to describe the phenomena of expressing generality in class discourse, rather than seeking to explain it. Session participants looked in depth at a single classroom example where a problem is posed, worked on in small groups, and then the groups' solutions are shared. Participants were asked to

put themselves first in the position of the class teacher, deciding how to discuss the students' contributions, and then in the position of the reflective teacher, considering the mathematical understandings that they had drawn on in making the decisions.

> In the discussion, we were able to describe the general mathematical knowledge a teacher might draw upon in this situation ... what was more difficult was to describe what knowing would be supportive for a teacher "in the moment" and the ways that knowing would be connected to the mathematical worlds of the students ... That this group of experienced professionals did not seem to have the language to describe some of the essential kinds of knowledge and ways of knowing that might be important in a particular teaching situation is disquieting.
>
> Smith, 2001: 61-63

Smith emphasises the value of research that is "embedding the mathematics into classroom contexts, students' work on mathematics, and teacher interactions" (Smith, 2001: 63). He believes that a shared language will develop through the increased use of such materials, "for better describing the mathematical knowledge that supports teaching" (Smith, 2001: 63). I agree with the importance of developing a framework and language for discussing such issues, and seek to do so for issues extending beyond teachers' mathematical knowledge. One of the aims of this current study, consequently, is to embed theoretical findings within the classroom context, and so develop language for dealing with the central issues. In Smith's words, "as that happens, sessions such as the one described here may offer even stronger insights into the mathematics of teaching" (Smith, 2001: 63).

To teach requires a conscious process of reflection (Dewey, 1933; Russell & Munby, 1991; Schön, 1983; Van Manen, 1995, Grimmett and Erickson, 1988; Clift *et al.*, 1990; Henderson, 1992). There is much divergence surrounding the precise definition of reflection. Dewey defined reflective thought as the "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds

that support it and the further conclusions to which it tends" (Dewey, 1933: 9). Schön

(1983) considers the teaching profession to be characterised by "experience, trial and

error, ...[and] intuition in the face of complex problems". More recently, the term has

been used to describe how teachers make regular, in-the-moment decisions in non-

routine situations (Norlander-Case *et al.*, 1998; Reiman, 1999; Kelchtermans &

Ballet, 2002).

In the current study, I use the term 'teacher tensions' to describe the conscious or

unconscious choices that teachers make whilst teaching. I choose to use this term

inclusively, including all choices made, whether or not the teacher is aware of making

the choice.

Romano (2006) draws attention to those moments in the "complex teaching act"

where teachers are required to "engage in reflection to make critical decisions about

how to respond to particular problems in practice" (2006: 293). In seeking a stimulus

for capturing these reflections, Romano asks practising teachers to describe their

"bumpy moments" in teaching, with the intention that these will offer insights into the

teachers' thoughts, knowledge and beliefs. She argues that her findings have

implications for capturing reflection during teaching, ongoing practising teacher

professional development, and pre-service teacher education, as a tool for examining

the reflective process involved with the act of teaching.

Romano's work initially appears to have close parallels with the current study, in its

appreciation of teaching as a "highly complex activity" (2006: 973), and its intention

to gain insight into that complexity through asking teachers to write about moments in

lessons. "Bumpy moments" are defined as, "teaching incidents that require the teacher to engage in reflection to make an immediate decision about how to respond to a particular problem in practice" (2006: 974) "The problem is not easily solved for any number of reasons, has importance to the teacher, and is perceived to have future implications or an effect on the students in the classroom" (2006:974). This appears similar to my own idea of identifying 'teaching tensions' in lessons.

However, although the definition above and four examples were provided to teachers, they appeared to interpret the term with relation to classroom organisation and management, rather than explanation and discussion. One teacher in the study explicitly redefined the term "bumpy moment" to mean "teaching flubs" or errors, while another considered it to be a complete stop in learning. The author considered these reinterpretations to be consistent with her original intention that "bumpy moments" were incidents that require the teacher to engage in reflection to make an immediate decision about how to respond to a particular problem in practice (2006: 977).

It is possible that the choice of the four examples influenced the teachers' interpretation of the definition:

> The following examples were provided for the teachers: (1) an instructional dilemma which occurred when the range of student behaviours and abilities made it difficult for all students to complete what was considered to be a fairly simple assignment; (2) an instance in which a parent helper did not come in to class, forcing the teacher to reconsider how the day's activities might be affected; (3) a leaking roof which caused disruption in the classroom; and (4) a problem in teaching that directly resulted from the teacher not being prepared for the day's events. Consistent with the definition of "bumpy moment" set forth earlier, all examples described teaching incidents that required the teacher to engage in reflection to make in immediate decision about how to respond to a particular problem in practice.

Romano, 2006: 975

These difficulties influenced my decision to theorise about potential tensions and decisions, rather than attempt to access the in-the-moment thinking of the main study teachers.

Schifter and Lester (2005) examine several moments of discontinuity or "openings in the curriculum", and conclude that successful facilitation requires deep content knowledge, awareness of learning objectives, and appreciation of the beliefs and understandings of seminar participants. Although their analysis focuses on professional development seminars, rather than lessons, they believe that their conclusions apply to other mathematical discussion settings.

> Responding to openings for teacher learning, however, is not just a matter
> of having the right cognitive dispositions. It is just as important to
> understand that effective facilitation requires courage—courage to
> challenge the thinking of other adults, to redirect a discussion that is
> moving in an unproductive direction, and to face the agitation, sometimes
> even tears, that result when firmly held ideas begin to crack.
>   Schifter and Lester (2005: 118)

Borthwick (2001) focuses on the decision teachers make when they are surprised by a student's question or response, and must choose whether to explore the 'moment', or carry on as if the student had not contributed in this way.

2.9 CHAPTER SUMMARY

This chapter has located the present study at the intersection of three fields: research on students' appreciation of generalisation and algebra, research on language use, and research on teacher-led discourse. I have explored the central place that the

appreciation and expression of generality has in mathematics, and argued that the relationship of generality within mathematics is distinct from that which it has with other school subjects.

The plethora of generality in mathematics classrooms could be subjected to any number of frameworks and categories. Research literature and inspection and policy reports (Brown, 1978; Her Majesty's Inspectorate, 1985; Cockcroft Report, 1982) have framed my separation of mathematical generality into four types of learning: facts, concepts, algorithms and problem solving strategies. I have shown that each of these four aspects of mathematics has its own distinct link to the general, and particular issues linked with this generality. I have argued that these might be observable in the mathematics classroom, as generalities are expressed related to each of the four categories. These categories formed the basis for my first set of distinctions in chapter 7, although the category of 'mathematical facts' is subsumed by the others (see 7.3.1 for further explanation), and the term 'procedures' is used in preference to 'algorithms' (see section 2.2).

Two of these four categories, general procedures and general concepts, were examined in greater detail in sections 2.2 and 2.3. Section 2.2 made use of some of the available distinctions between different ways of teaching, and ways of understanding, mathematical procedures. These included distinctions between inductive and deductive approaches, empirical and structural generalisations, and procedural and conceptual understanding. These are taken as a theoretical starting point in chapter seven when considering different characteristics of expressions of generality.

Section 2.3 makes use of Davydov's (1972/1990, 45-6) three criteria for concept

formation to consider three important decisions for teachers in introducing students to

mathematical concepts:

1. when and how do you establish the precise *scope* of a concept?

2. when and how do you introduce a *name*?

3. when do you move from exemplifying the conception with the particular and

    assume that students can appreciate meaning from the general?

One difference between generality in a mathematical context, as compared with other

curriculum subjects, is that there is often the possibility of expressing these

generalities using algebraic notation. In section 2.4 the connections between

generality and algebra were explored. Having examined the historically developing

and currently various definitions of the term 'algebra', I explained that the focus in

this thesis is not what some have termed 'algebraic thinking', but rather 'algebra' is

used here to indicate formal algebraic notation. I take the question asked by

Sutherland (1991) "Can we develop a school algebra culture in which pupils find a

need for algebraic symbolism to express and explore their mathematical ideas?" and

ask whether it is the school *mathematics* culture, rather than merely the *algebra*

culture, that might benefit from change.

Section 2.5 looked to literature in the field of teaching non-native languages, and

considered whether any insight could be gained from regarding algebra as an L2 (the

term used in the literature for non-native languages). The literature discussed in this

section shaped my thinking and opened my awareness of alternative perspectives on

the expression of generality in mathematics classrooms. However, it did not

contribute directly into the framework developed in chapter 7, or the subsequent analyses in chapter 8. It was included in this chapter with the intention of offering an insight into the background literature that influenced this study.

Vygotskian approaches to the importance of language for thought were introduced in section 2.6. The impact of students using and understanding 'ME' (see section 2.3.1) is emphasised through consideration of Vygotsky's concept of 'inner speech', which supports the importance of developing mathematical concepts. Section 2.6 also considered how linguistic theorists (Kozulin, 1990; Malinowski, 1923; Halliday, 1978) can contribute to educational research, such as this, that focuses on the spoken word.

Section 2.7 examined previous research that analyses teacher-led discourse. The distinction was made between lecture, recitation and discussion, with the discourse that forms the main study expected to exhibit the characteristics of discourse towards the centre of this spectrum (recitation).

In section 2.8, literature regarding teacher decisions was reviewed. In this section it was noted that a teacher might have chosen a task designed to encourage meaningful use of algebra. This same teacher may be aware of and experienced in the importance of language. They may want to foster a conjecturing atmosphere in their classroom. Even if each of these three factors has been put in place, during each conversation they have with their students, they must continually make decisions that will impact on their students' learning, with concomitant stressing of some aspects and

consequently down-playing of others. The literature reviewed in this chapter calls attention to the complexity of these teaching decisions within the context of expressing generality.

CHAPTER 3: METHODOLOGY I – RESEARCH DESIGN AND DATA COLLECTION

Having considered the previous research that informs the study in chapter two, this chapter focuses on the processes by which this research was undertaken. The objective here is to describe the methodology and methods of this study, and to explain the basis on which these were decided and the manner in which they were carried out.

3.1 INTRODUCTION

Although the terms 'methodology' and 'method' are sometimes used synonomously, an important distinction must be made between the two. Methodology is the study of methods and deals with the philosophical assumptions underlying the research process, while a method is a specific technique for data collection under those philosophical assumptions. Inevitably, the methodology described in section 3.3 informs the development of the research methods employed, and so there is a necessary relationship between the methodological discussion in section 3.3, and the details of method that follows in section 3.4. The role of methodology can be held in very high regard:

> Methodology is a theory of how inquiry should proceed. It involves analysis of the assumptions, principles, and procedures in a particular approach to inquiry (that, in turn, governs the use of particular methods).
>
> Schwandt, 2001: 161

The imperative indicated by language such as the 'should' used by Schwandt in the previous quote can be somewhat daunting. The impression can be gleaned that

methodology is concerned with philosophising about the discovery of the 'one correct approach' to defining the problem, framing the hypotheses, generating and analysing data. My approach to the question of methodology more closely follows that of Blackburn (1996) who argues that, "The more modest task of methodology is to investigate the methods that are actually adopted at various historical stages of investigation into different areas, with the aim not so much of criticizing but more of systematizing the presuppositions..." (1996: 242).

3.2 LEARNING FROM THE PILOT STUDY

Before the three research questions of this present study were finalised, my research began with a more change-oriented approach. The pilot study sought to address the question: 'how can teachers use class discourse to support students in expressing generality?' With the intention of remaining open to a variety of successful approaches, I invited seven teachers to participate in the pilot study, and observed two different teaching groups with each teacher. As I wanted differences in the lessons to be comparable and definable, I constrained the participating teachers by offering them a particular task, whilst leaving them free to choose the way in which they adapted the task to design something that would result in appropriate activity for their students. I selected the task below (Mason, 2005: 56), on the basis that it provided rich opportunities for expressing generality, yet was open to different classroom applications. Word restrictions prevent detailed discussion of the pilot study here, but a forthcoming paper will be accessible at http://ermeweb.free.fr/proceedings.php.

Task 3.3.4a Multiple Expressions

Take a picture sequence and imagine building each picture from sticks, perhaps like the one shown here (all sticks are assumed to be the same length).
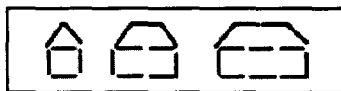
In how many different ways can you work out how many matchsticks will be needed to make the pth picture? How many sticks will be needed to make just the perimeter?

Mason, 2005: 56

This research method offered an opportunity to explore how different teachers approach the same task, and look at the different 'end-points' with different groups of students. The 'Algebra House' pilot study influenced the methodological principles of the main study, discussed in section 3.3 below. The pilot was particularly significant in establishing the fourth principle , to observe 'ordinary', rather than 'intervention' lessons. Findings from the pilot study also informed the central study, as I was particularly struck by the contrast between the teacher and students' use of algebra and terms such as 'general' and 'particular' in the pilot study, compared to their almost complete absence in central study lessons.

Further piloting activities were orientated more specifically to developing and trialling observation, transcription and analysis techniques, learning about appropriate researcher roles, and deciding which and how many lessons to incorporate into the full study. During July 2005 three days per week were spent in school, observing over 15 lessons (taught by six different teachers), talking to students, and trying out techniques. All pilot-data were analysed in order to judge the relative efficacy of different approaches. In the following sections reference is made to pilot work in the context of specific methodological decisions.

3.3 METHODOLOGICAL PRINCIPLES

This section is concerned with the ideas and intentions which underpinned the ways in which this study was carried out. Although some information is provided about the specific data generation procedures that were followed, the central purpose was to explain the spirit in which such methods were used and the aims their use was meant to achieve. Further detail about the research design can be found in section 3.4.

The study's methodological task was to construct a process for generating and analysing data that would follow ethical guidelines, function within practical constraints, and address the research questions being posed,.

The first criterion for the research design was that all practices should follow ethical guidelines set out by BERA (2004). The effects of following this principle are discussed in section 3.5. The second criterion involved practical considerations given the time-frame for completion and the desire to conduct ongoing analysis. The third criterion was that all aspects of the research design would help to address the research questions (Rossman and Rallis 1998):

1. What generalisations are being expressed in secondary mathematics classrooms?
2. How are procedural generalisations expressed in mathematics classrooms?
3. How are conceptual generalisations expressed in mathematics classrooms?

There was a need, therefore, for a methodology that would enable the generation of authentic accounts of whole class discussions in mathematics classrooms. The

descriptive and exploratory nature of the three research questions led to the choice of

a qualitative approach (see first principle), as this was considered more appropriate for

offering detailed insight and opportunities for careful analysis of discourse. For

practical reasons it was necessary to narrow the focus of questions such as "What

generalisations are being expressed in secondary mathematics classrooms?" to

considering particular secondary mathematics classrooms, which entailed adopting

some aspects of a case study approach (see second principle). The exploratory nature

of the research questions also made it appropriate to begin with the classroom

practice, rather than starting with a preconceived framework or a hypothesis about

findings (see principle three) and to collect data about what *is* happening, rather than

what could, should or might (see principle four). The last principle was also

specifically suited to my research questions. As the research questions are descriptive

of existing practices, I strove in the research design to minimise disruption to lessons.

This was a factor in my decision to maintain a teacher-persona during the data

collection phase. Although a digital recorder was used to enable subsequent analysis

of discourse, more intrusive techniques such as video recording, which might have

impacted on what teachers and students said more profoundly, were not employed.

The research questions' emphasis on discourse, and the *expression* of generality,

influenced my use of voice recorder and transcription as the main data collection

technique.

The three criteria described above (that the study would follow ethical guidelines,

function within practical constraints, and address the research questions being posed)

were complemented by critical engagement with the following sources of information

and experience:

1. Piloting experience (see section 3.2)
2. Relevant published research (see chapter 2)
3. Methodological literature (see below)

Arguments about the nature and strengths of particular strategies or techniques from methodological literature did not direct decisions but rather supported those made on the basis of pilot work and critical reviews of related studies.

The three criteria of the methodological task were considered in connection with the above sources of information and experience in order to develop the six methodological principles that underpinned the research design. The methodology of this study can be understood in terms of these six principles, which together comprise the theoretical and philosophical ideas underpinning the methods and their use. These methodological principles are:

- to adopt a qualitative approach;

- to work on a small scale, using case studies;

- to begin with the classroom practice;

- to observe 'ordinary' lessons;

- to engage in on-going reflection on my own teaching practice;

- to maintain a teacher-persona.

Each of these six principles will be explained in turn and a discussion of how they manifested themselves in the research design will be presented.

First principle: A qualitative approach

> By comparison with numbers, meanings may seem shifty and unreliable.
> But often they may also be more important, more illuminating and more fun.
>
> Dey, 1993: 11

The nature of the three research questions is descriptive and exploratory, and they are consequently most effectively approached qualitatively. Yates argues that research carried out with the intention of "constructing generalised laws" is mostly quantitative, while qualitative methods are more suited to achieving "detailed description of particular circumstances" (Yates, 2004: 135). Those working with qualitative methods,

> stress the importance of such things as the subjective experiences of the researcher and the participants, the central importance of meaning to social life, and the importance of social and cultural context in situating different meanings and interpretations.
>
> Yates, 2004: 137-8

Wolcott argues that, "the real mystique of qualitative inquiry lies in the processes of using data rather than in the process of gathering data" (Wolcott, 1994: 1). He distinguishes between description, analysis and interpretation. The aim of description is, "to stay close to the data as originally recorded. . . The underlying assumption, or hope, is that the data 'speak for themselves'." (Wolcott, 1994: 10).

Ethnography, or participatory action research, draws on participant observation and field notes as data sources.

> Instead of requiring myself to explain everything I see, I simply concentrate on recording every detail. I have learnt that rich descriptions yield the stuff of later explanations, whereas thin descriptions are worthless.
>
> Oran, 1998: 28

One of the major arguments made against quantitative methods, especially surveys and experiments, is that data is often collected in 'artificial' situations. As one of the intentions of the experimental method is to create as 'closed' a system as possible by controlling and excluding many aspects of the situation, they tend to create an

artificial setting where behaviour may differ significantly from when they teach 'real'

lessons.

By working qualitatively, I had access to an extensive range of material. This enabled

me to gain deeper and richer insights into the finer detail of the people, social contexts

and social practices involved. There is a very broad range of identifiable methods for

qualitative data collection. These include participant and non-participant observation,

unstructured interviewing, group interviews, and the collection of documentary

materials.

Montgomery-Whicher, who uses a phenomenological approach to inquire into the

lived meaning of drawing experience, draws an interesting analogy between

phenomenology and drawing:

> As a practice of inquiry – a way of questioning our experience of the
> world – a phenomenological approach to research shares three important
> characteristics with drawing from observation: One, it begins in the
> everyday world in which we live; two, it is directed towards a renewed
> contact with the world; and three, learning to do this kind of research, like
> learning to draw, it is largely a matter of relearning to see.
> <div align="right">Montgomery-Whicher, 1997</div>

Second principle: Small scale study

The research strategy adopted can best be described in terms of Hammersley *et al.*'s

(1998:2) definition of case study research; that is "research that involves collecting

detailed data relating to a relatively small number of cases, data that are

predominantly unstructured in character, and that are subjected to qualitative

analysis". While this definition does not necessarily correspond with others in the

literature (such as Stake, 1994; Yin, 1994) it is the most accurate description of the

undertakings of this study. Hammersley *et al.* (1998: 2) suggest that: "it is useful to distinguish between the focus of a piece of research and the case or cases studied which provide information that is hoped will illuminate that focus". Applying this distinction to the present study,

- the focus (or "the most general set of phenomena about which a study draws conclusions") is *the use of whole class discussion in improving students' appreciation of generality in mathematics.*
- the cases (or 'the particular objects, specifically located in place and time, about which data were collected") are *the teacher-led discourses of seven secondary school teachers with classes of students at Key Stage 3 and 4.*

As this study adopts a question-led approach to research design, presenting arguments about the strengths of case study research to justify the strategy adopted is inappropriate. Such a justification is also rendered problematic because the term 'case study' lacks specificity and clarity. While Stake (1995) suggests that case studies may be singular or multiple and of intrinsic or instrumental interest, Yin (2003) distinguishes explanatory or descriptive purposes and embedded or holistic approaches. He sets out arguments made by others that case studies can be characterised by their focus on decisions, processes, events, individuals or organisations. Robson (1993) also adopts a broad definition, describing case study as:

> A strategy for doing research which involves an empirical investigation of a particular contemporary phenomenon within its real life context using multiple sources of evidence.
>
> Robson, 1993: 146

He suggests that the contemporary phenomenon, or case, can be "virtually anything" (1993: 146). Views also differ about the methods used in case studies and their associated strengths. Stake (1995) characterises case study research as qualitative

while Yin (2003) highlights the potential for incorporating quantitative methods. Stake argues in favour of 'naturalistic generalisation', but Yin suggests that generalisations from case studies may be made theoretically or on the basis of established reliability and external reliability. Further diversity can be found in arguments relating to the role of the researcher, notions of objectivity and interpretation, units of analysis, and relationships between case studies and other designs.

A justification of case study as a strategy would thus be fraught with problems as to what exactly is being justified. Decisions made in this research are identified and justified with primary reference to the research questions and design principles before supporting arguments from methodological literature are presented. There remain issues of terminology and in light of the arguments made above this study is not labelled as a case study. However fit between the small-scale approach adopted and arguments that case study research is well-suited to addressing exploratory, open-ended questions may be noted.

Third principle: 'Ordinary' lessons

Lesson observations generated the bulk of data gathered in this study, so identifying which lessons would be best to observe was a central concern. The intention was not to seek lessons where generality was a central concern, but rather to observe 'ordinary' lessons and consider the extent to which generality played a role. I observed teachers on multiple occasions, teaching different topics to several year groups and attainment groupings, at a variety of times of day. This gave me an

opportunity to see a broad range of ways they both expressed generality themselves, and promoted students to so do.

This contrasts with the 'Algebra House' pilot study, in which, by keeping the task constant, I hoped to look more closely at the classroom culture, and how the ethos created by the students and teacher contributed to the aspects of generality on which I was focussing. The intention remained that the research would reveal what *did* happen in some mathematics classrooms, rather than what *could* happen as a consequence of significantly different circumstances. However, time spent discussing the lesson with the researcher and extra time spent preparing, for example, are both properties of 'intervention' lessons that arguably could not realistically be extended to all lessons.

In the 'Algebra House' pilot study, teachers commented that they had used different lesson structures and teaching techniques because they were 'doing something different'. If the teachers feel differently about the lesson, it is quite possible that the students do also. And yet it was my belief from the literature and my own teaching experience that there were opportunities to appreciate and express generality in every mathematics lesson. Surely, if this were truly the case, then a lesson such as 'Algebra houses' would fit right in to mathematics lessons, with teachers and students alike using teaching and learning skills that were considered standard. Why, then, did 'Algebra House' seem so out of the ordinary?

One answer to this is methodological. I had provided teachers with a specific activity. They had discussed their plans for the lesson during INSET. The 'Algebra House' observations may also have put more pressure on teachers to perform. Although the

teachers and students were accustomed to my presence in the classroom, there were some necessary differences. For example, a particular lesson had to be allocated to the activity, to ensure that I would be available to observe. BG's year 9 class, for example, had their 'Algebra House' lesson during a sequence of lessons on practice statistics coursework. For most of the teachers this was an unusual task type, which may have led to their questioning the appropriateness of their 'usual' teaching style.

I believe that the unusual nature of the 'Algebra House' lessons resulted from something beyond the research methods used, however. It was unusual for the teachers to focus on using algebra to express generality meaningfully. Several teachers expressed uncertainty about what to write as the 'objective' for the lesson. Based on pre- and post-lesson discussions with the project teachers, analysis of schemes of work and discussion with students, it appears that most of the teaching of algebra to express generality could be described as empirical, rather than structural generalisation (Bills and Rowland, 1999; this distinction is outlined in section 2.1.3.2). This observation emerged as a central theme of this study, and is discussed further in chapter nine.

The main study set out to explore what happens in ordinary lessons, to try and escape the methodological impact described above. Having stated this aspiration to observe representative lessons, it is important to emphasise that this representativeness is limited by sample size. As my sample is small, any research findings are indicative of what *can* happen, rather than what *does*; a representative description would require a much larger sample and consequently, particularly given my time constraints, less 'thick' data.

Fourth principle: Beginning with the classroom practice

The fourth methodological principle concerned the project's commitment to start with classroom practice. By this it is meant that the study deliberately took the speech-acts of actual lessons as the starting point for the investigation and sought to work from these events to generate understandings of the expression of generality. This principle is seen as important for a number of reasons.

Although interviews, both semi-structured and informal, were used during the pilot stage, teacher and students views were not systematically collected and analysed in this study. This was due to my increasing awareness of the inadequacies of asking people to give explanations, justifications or even descriptions of their practices. Keats (2000: 60-61) identifies eight potential interviewee behaviours that might affect interview responses. These include inconsistency, non-cooperation, evasion, inaccuracy in recall, lack of verbal skills, conceptual difficulty, emotional state and bias. Were I to do the study again, one possibility might be to collect and analyse this type of data, but to remain aware of the possibility of such effects. It would be difficult to not be influenced by what was said, so I decided on this occasion that it would be better not to ask. I did, however, record the conversations of teachers planning the task, which I used to triangulate with my lesson observation findings. I also had informal conversations with the teachers, some of which I recorded.

> Using research interviews (or focus groups) involves actively creating data which would not exist apart from the researcher's intervention (researcher-provoked data). By contrast, observation or the analysis of written texts, audiotapes or visual images deals with activities which seem to exist independently of the researcher. This is why we call such data

> naturally occurring: they derive from situations which exist independently
> of the researcher's intervention.
>
> Silverman, 2001: 159

A theoretical distinction made by Argyris and Schön (1974) also supports the suggestion that making sense of teachers' classroom practice requires the direct study of that practice. They distinguish between two types of theories of action, a person's 'espoused theory' defined as the theory they believe guides their actions, and a person's 'theory-in-use' defined as the theory that actually governs their actions. Argyris and Schön argue convincingly that an individual's 'theory in use', "may not be compatible with his espoused theory" and in light of this put forward the case for direct observation of an individual in order to learn about his/her 'theory-in-use'. They state that:

> We cannot learn what someone's theory-in-use is simply by asking him.
> We must construct his theory-in-use from observations of his behaviour.
>
> Argyris and Schön, 1974: 6-7

A further reason for this study's commitment to begin with classroom practice is motivated by arguments that research that is grounded in the events of real classrooms is more likely to generate findings of relevance to practitioners in the school context. This recognises a gap that is often seen to exist between research in education and actual practice and suggests that one way in which this gap can begin to be bridged is through research that focuses on exploring the practice that occurs in real classrooms.

Fifth principle: Continued reflection on own practice

The importance of educational research that can directly impact on the classroom was the initial impetus for this study. As a consequence of this, although the central

investigation is based on observation, transcription and analysis of six teachers' classroom practice, this thesis sets out to show both how the main study emerged from my own classroom practice, and how the study influenced my practice during and after data analysis.

The benefits of setting the theoretical and observation-based research findings of chapters six to eight in the context of the researcher's own practice are manifold. Chapter five shows how the questions emerged from the researcher's own practice, setting the main study in context. Chapter nine illustrates one way in which the central research findings might influence practising teachers.

Constraints of time and space inhibit a full discussion of the methodological concerns related to researching one's own practice. My approach in this instance is strongly influenced by Mason (2002).

Sixth principle: Continued teacher-persona

Doctoral theses often describe a transition that has taken place from teacher to researcher. It has been my aim to develop my researcher self without losing any of the teacher I was developing into being. I believe that there is a need for researchers who have remained in the teachers' world, of bells and timetables and pastoral responsibilities. I continued to teach at the main study school throughout my research. As described above (principle three), my focus was on non-intervention. I wanted to portray what actually does happen in secondary school mathematics classrooms. Ethnographic studies often involve the researcher taking on a role within the research

environment. The advantages of action research as a teacher are well-documented (see, for example, Carr and Kemmis, 1986; Ruddock and MacIntyre, 1998). The advantages of a researcher also being a teacher have also been considered, though perhaps to a lesser extent.

During all the phases of the research it was clear to me that I was in a good position to create and maintain trust with the teachers and students. The main study involved the observation and recording of lessons of teachers who had been, and continued to be, my colleagues. Having taught at the school for two years, I had worked closely with the teachers in the mathematics department. As the team has an 'open door' policy, I had observed their lessons both formally and informally. I had taken on the role of 'Induction Tutor' to two of the teachers, which involved regular formal observations. Although this had significant advantages in terms of gaining access (the ethical implications of which are discussed in section 3.6), it made the shift from 'teacher' to 'researcher' more challenging.

> In the study of education, the *action* in action research is located in and around the classroom where teachers teach students, or where better yet they educate each other. One can see that action research would bridge the great divide between research and its object, between research and researched. This is research which, in a democratic spirit, does not keep itself apart from the researched. After all, many of us who research were teachers in the schools once, and while we may figure that we are teachers still, we have found a considerable divide has grown up between our once and former colleagues. A form of research work that recognised our original work as teachers would seem attractive.
>
> Willinsky, 1997: 329-330

Brennan and Noffke describe a similar convergence of roles. In carrying out a piece of action research with student teachers, they point out that,

> There was no neat dividing line between our teaching and our research. Rather, we tried to improve our teaching as we reflected on our project,

often with input from the students as to how our approach to action research was contributing to their development...

Brennan and Noffke, 1997: 24

Ainley (1999b) discusses some of the tensions involved in taking on the dual role of teacher and researcher, principally between deciding when to support or advance the learning, and when to stay back and observe. In line with Ainley (1999b) and Wilson (1995), I do not view the situation as a conflict, but I am aware that there are tensions, choices that have to be made. As an adult in the room, especially one who was known to be a teacher, I was regularly looked to for advice and structure. A student considering inappropriate behaviour might look first to me to see whether I would prevent them. In one lesson, when the teacher went outside to speak to a student, I went to the front by the board and talked through a few examples with the students. They did not seem to find this surprising. Although outside researchers might hypothesise that students are better behaved in their presence, or that they answer their questions more honestly than they would answer those of a teacher, I did not feel that I missed out on these advantages.

Being known as a mathematics teacher by both students and teachers in some ways made the adoption of the role of a researcher a more challenging one. When deciding to carry out my research study in the school in which I taught, I was quickly aware of the many advantages this would have in terms of access to data and insights that might otherwise be difficult to gain. What I was less aware of, was the need for some distance between researcher and subject. I needed to be able to set aside time to just be a researcher. On a practical level, I found that the best times for analysis were

during the school holidays, where I could take some distance from the school,

teachers and students. As Ely *et al.* (1997: 204) comment:

> One of the many tensions or paradoxes of qualitative research is that when
> we as researchers begin to move into specifically interpretive modes, we
> must take a step back from the immediacy of the field and of our data and
> see them again as "the other".
>
> Ely *et al.*, 1997: 204

The time spent teaching in the school, attending departmental meetings etc. produced

case studies of the teaching approaches, drawing on some of the techniques of

ethnographic research. In terms of the ethnographic approach, I follow the broad

definition of ethnography offered by Hammersley and Atkinson:

> In its most characteristic form it involves the ethnographer participating,
> overtly or covertly, in people's daily lives for an extended period of time,
> watching what happens, listening to what is said, asking questions – in
> fact, collecting whatever data are available to throw light on the issues that
> are the focus of the research.
>
> Hammersley and Atkinson, 1995: 1

I took notes throughout the school day and also kept a journal of analytical notes

about each teacher, in order to identify salient features of each one's teaching

approach. The informal discussions held with teachers in the staff rooms also

generated data and ideas, which I noted.

I was given the opportunity to present the main features of the research project at one

departmental meeting. Thus, all teachers were aware of the purpose and procedure of

the research project, and were able to ask relevant questions from the start of the

research period. However, generally in maths team meetings I deliberately refrained

from making frequent references to research. I found this difficult to adhere to, as I

felt I could benefit the department through sharing the literature that I had the time to

access, but felt it was justified in terms of remaining 'one of the team'.

Following the methodology of Brown and McIntyre (1993), I tried to show positive regard for the main study teachers and their work; I refrained from negative judgements, took an interest in all that they said, and when a teacher appeared disappointed with a lesson, encouraged a balanced approach by asking what had gone well. I made it clear that I was not there to evaluate them, but to learn from them.

3.4 RESEARCH STRATEGY: DETAILS AND RATIONALE

In this section the research strategy is discussed in detail, identifying links to the principles and sources outlined in section 3.3. I observed fifty-two secondary mathematics lessons, from years 7-11 (10 to 16 years old), taught by six mathematics teachers in an Oxfordshire comprehensive school. Having previously taught full-time at the school, both students and teachers tended to relate towards me as they would a member of staff, rather than an outside researcher. Although I initially asked before each lesson whether I could observe and record, I established with all six teachers, after differing time periods, an agreement that I could come and go as I pleased. My research formed part of a department-wide policy of increased observations, in which all members of the department were encouraged to spend time in others' lessons. In this environment, my presence appeared to be unremarkable to both teachers and students. An important feature of the methodology was to make as little impact on the class as possible, and I wanted to avoid any influence on the teacher. I consequently felt confident that I was observing a realistic sample of the department's teaching.

I observed all of the recorded lessons, and four sources of data were collected:

- researcher's field notes, including observations of students' working
- audio-recordings from a digital recorder
- pre- and post-lesson teacher comments
- examples of students' written work

When main study lessons are referred to in the study, I refer to them in the form "[Lesson number] Teacher initials Year (set) Topic".

3.4.1 Sampling

Two factors influenced the number of teachers I invited to take part in the study: the possibility of comparison and the practicalities of the situation. I wanted to have more than one teacher in the study, in order than points of commonality and difference might emerge, but knew from my pilot study of lesson observations (see section 3.2) that the data would be rich and plentiful and thus require considerable time to analyse and digest. I also felt that to understand the strategies and styles of teaching used by the participants, I needed to observe them working with more than one year group, and teaching different types of lessons. What seemed feasible given these priorities was to join six teachers as they taught different year groups four or five times each over a two month period – a total of fifty-two lessons.

The department offered the possibility of observing teachers who differed along several potentially important dimensions. The six teachers had a spread of teaching experience, with two teachers in their fifth year of teaching or more, and three in their first three years of teaching. Two of those involved were female, and three men. More detail on the teachers as individuals is provided in chapter 6. Given this variety, two

months of full-time lesson observations was deemed sufficient to incorporate most of this variation. This seemed reasonable given ethical considerations (BERA, 2004) that no more demands than necessary be made of participants, and practical given the overall timeframe for the study.

3.4.2 Observing

Due to the extensive time scales over which patterns of classroom behaviour are established (Coles and Brown, 1998) the main study observations took place at the start of the academic year, when teacher expectations are made most explicit.

During whole class teaching I sat at the back of the classroom, noting teachers' and students' utterances and actions. During individual or group work a more interactive role was adopted, giving assistance to students if they asked, and initiating dialogue that helped me gain a fuller understanding of the task and the students' response to it. Aware that my presence as a participant may have changed teacher and student behaviour in lessons, discussions were held with teachers about this issue. None noted any changes in student behaviour, and field notes document several instances of students apparently ignoring my presence and engaging in off-task behaviour.

The audio recording of the whole class appeared much less intrusive than others have reported with recording of particular students. Students at the front of the classroom sometimes touched the digital recorder, and occasionally the recorder prompted comment, such as "don't say that..", but mostly its presence was not remarked upon.

Detailed field notes were made, including board work. At the end of each lesson pertinent student work was photocopied. Either before or immediately after each lesson, copies of handouts or worksheets were made. As with field notes, these data tend to complement the audio data, rather than form a major evidence base.

3.4.3 Recording

The teacher's interactions with the class were recorded at two levels. All whole class interactions were recorded and, as the recorder was placed on the teacher's desk, their interactions with individual students and small groups were also recorded on occasion.

This thesis focuses on teacher-led discourse, and consequently not all classroom discourse is subjected to analysis. This includes conversations between pairs or groups of students, discussions between the teacher and individual students, and those sections of the class discussion where several participants talk at once. These students may make a constructive contribution to the discussion, but as it is not available for all participants in the discussion to interpret, it is not included in the analysis of student or teacher talk. On occasion, a teacher chooses to amplify or edit this talk (Hewitt, 1997, see section 2.7.3) in which case they ask a student to repeat the contribution for all to hear. There were very few occurrences of general student chatter or suggestions amongst the sections of the lesson where mathematical ideas were being developed. Had it been practical to transcribe each individual one, and include them in the quantification of student talk, the effect on the proportions of teacher and student talk would have been minimal.

The majority of the lessons were successfully recorded. On one occasion a teacher (having given consent to the recording) mistook the digital recorder for a student's mobile phone. In the subsequent class telling-off, confusion and realisation, the record button was switched off.

I became particularly aware during this phase of the research that, "the idea that we 'collect' data is a bit misleading" (Dey, 1993: 16), as any data are essentially 'produced' by the researcher. It had initially seemed to me that my experience would be the data, that by observing and recording lessons I would be collecting data. But then 'the data' is my own experiences (unrecorded and, perhaps, unrecordable) and the digitally recorded speech. Both of these are difficult to analyse formally. For these experiences to become analysable data, I needed to process them in some way. I hesitated to convert my experiences to data in this way, however, as by processing or recording them as data, I am necessarily interpreting them. In the words of Brennan and Noffke (1997: 37), "Data do not exist, except under the social conditions of their making".

Of the two areas of experience, what I see and hear personally as researcher, and what is recorded by the digital recorder, it is tempting to conclude that the latter is the least susceptible to distortion through my interpretation in the data collection phase. To avoid subjectivity, it is tempting to remove myself completely from the picture, and "faithfully (alas, sometimes too faithfully) to preserve and report every word spoken" (Wolcott, 1994: 13).

> In the very act of constructing data out of experience, the qualitative
> researcher singles out some things as worthy of note and relegates others
> to the background. Because it takes a human observer to accomplish that,
> there goes any possibility of providing "pure" description, sometimes
> referred to lightheartedly as "immaculate perception".
>
> Wolcott, 1994: 13

The decision to use transcripts as the main unit of analysis, rather than 'thick'

descriptions of lessons, is described in section 4.5. Either method of 'describing' is

subject to Wolcott's warning, and I strived to take the following advice into account:

> Never forget that in your reporting, regardless of how faithful you attempt
> to be in describing what you observed, you are creating something that has
> never existed before. At best it can only be similar, never exactly the same
> as what you observed. And at worst...
>
> Wolcott, 1994: 15

3.4.4 Transcribing

It was initially intended that written accounts of observed lessons would form the

units of analysis. Through the pilot study (see section 3.2), it became apparent that the

drawbacks of this approach outweighed the advantages.

> Traditionally, data have been seen as a way to objectify an aspect of the
> relationship of subject and world, in order for the detached observer to
> subject it to critical scrutiny. However, if our epistemological stance is
> such that we do not accept either the detached, unitary subject, or this
> particular representation of the separation of subject and world, then this
> function of data in the research process is no longer viable.
>
> Brennan and Noffke, 1997: 37

I transcribed the lessons, as a strategy to familiarise myself with the data, and to allow

quotations that would be as accurate as possible. However, a transcription is a

necessarily incomplete record of an audio-tape, as it cannot include the physical

gestures, intonation, location and the full interaction between the participants. As

Kvale (1996) insists:

> The originally lived face-to-face conversations disappear in endless
> transcripts, only to reappear butchered into fragmented quotes.
>
> Kvale, 1996: 182

It is important to acknowledge that a transcription is not the raw data, but rather an

interpretation of that data (Powney and Watts, 1987). A certain level of editing is

deemed appropriate in the social sciences, as contrasted with transcripts written purely

for discourse analysis, if such editing improves readability and access to the "ideas,

logic, beliefs and understandings" (Arksey and Knight, 1999:146).

> There are some areas of research, notably linguistics, where it is vital that
> transcripts are literal records of the sounds on the tape, or as nearly as
> possible, and that pauses are exactly timed and recorded. Unfortunately,
> the tone of voice – enthusiastc, bored, confrontational, mocking – easily
> and routinely does not make it in to the transcript. So too with body
> language...The question is whether it is more suitable to try for richer
> descriptions of the interview, or whether it is acceptable to settle for an
> 'accurate' rendering of the spoken words.
>
> Arksey and Knight, 1999: 146-7

I found Mishler's (1991) parallel between a transcript and a photograph to be a

helpful one, emphasising, as it does, that a transcript is one frozen, contexted, printed

and edited version of reality. My aim then, should be to create one careful attempt to

represent some aspects of the teacher-led discourse. Accuracy is necessarily a relative

concept, for a number of reasons. One of these is that decisions about punctuation

mean that more than one 'accurate' transcription could be produced. It is not possible

to render an accurate account. Punctuation, for example, can never be ascertained with

certainty. I made many decisions about how 'accurate' the transcriptions should be. I

decided to include repetition, indicate hesitation (marked with dots for each second),

and laughter, and not to correct grammatically confused sentences. Some contractions

and colloquial language were removed or edited, where I felt it increased readability

and comprehension without significantly affecting meaning. For example, teachers

frequently pronounced 'want to' as 'wanna' (e.g. [21] 1:05), but this made the

transcripts more difficult to comprehend. I therefore decided to record 'wanna' as

'want to'. Abbreviation conventions such as 'isn't', 'aren't', and 'weren't' were kept

to, and verbal tics, such as 'er' and 'um' have been included. I decided to use this

relatively high level of accuracy of representation, to enable elementary discourse

analysis to be carried out should this prove helpful/enlightening. To capture as much

detail as possible, I transcribed with all pauses and hesitations included. In reporting

the data, however, I have left open the possibility of removing these, for ease of

reading.

> ...committing verbal exchanges to paper seems to result in their
> immediate deterioration: context, empathy, and other emotional dynamics
> are often lost or diminished, and the language seems impoverished,
> incoherent, and ultimately embarrassing for those who have cause to read
> back over their contributions (including the interviewer/researcher!).
>
> <div align="right">Poland, 1995: 299</div>

Although Powney and Watts (1987: 145) argue that, "transcription is very slow and

expensive of resources", I decided to transcribe the lessons in full. My original

intention was to fully transcribe only a small number of the recorded lessons. I

expected that this process would sensitise me to the information required to address

the research questions, and that I would then transcribe selectively. However, to

preserve the coherence of the lessons, I ultimately created full lesson transcripts of

fifteen lessons, including at least two different classes with each of the six central

study teachers. The lessons selected were as follows:

[05] CB 11(3) Expanding brackets
[07] SJ 10(3) Approximation
[10] SJ 11(1) Quadratic formula
[12] SJ 10(3) Squares and cubes
[14] CB 10(1) Percentages
[15] LR 7(4) Sheep Pens
[21] BG 10(2) Order of Operations

[25] BG 10(2) Data
[26] KT 8(1) Angles
[27] PF 9(1) Fractions
[31] CB 8(2) Sequences
[34] LR 10(1) Rational numbers
[35] PF 7(mixed) Subtraction
[36] KT 10(4) Angles
[38] KT 8(1) Sequences

Five of these lessons are discussed in greater detail in chapter 8-I ([14], [15], [21], [26] and [12]), and a further four in chapter 8-II ([05], [07], [27] and [35]). In addition, an excerpt from lesson [10] was transcribed for use in chapter 7 to illustrate the dimensions of the framework developed in the central study (see section 7.2). The framework is then applied to lesson [35] in section 7.4. The selection of these particular lessons and episodes to illustrate research findings is explained in the context of each particular case, and so can be found in section 7.1, at the start of 8-I.2, and throughout 8-II.3.

3.5 ETHICS

The distinctive ethnographic methodology adopted has two clear consequences for the ethics of the investigation. Firstly, no activity is taking place that would not normally take place in a mathematics classroom. Secondly, participants (both teachers and students) may not be aware of their part in the research. I wanted to overcome the potential ethical issues associated with this second point, whilst retaining the advantages of being 'just Miss Drury'. I did this by explaining clearly at the start of my research what I was intending to do, but then deliberately avoiding discussing my research with teachers or students from then onwards.

> If researchers do not want their potential hosts and/or subjects to know too
> much about specific hypotheses and objectives, then a simple way out is

to present an explicit statement at a fairly general level with one or two examples of items that are not crucial to the study as a whole.

                                                    Cohen and Manion, 1994: 357

I adopted this 'vague yet explicit' approach to sharing my research hypotheses and objectives. It was central, as Frankfort-Nachmias and Nachmias emphasise below, that informed consent should be acquired from participants in the study.

> The practice of ensuring informed consent is the most general solution to the problem of how to promote social science research without encroaching on individual rights and welfare.
>   Frankfort-Nachmias and Nachmias, 1996: 85

Diener and Crandall (1978) define *informed consent* as "the procedures in which individuals choose whether to participate in the investigation after being informed of facts that would be likely to influence their decisions". I have endeavoured to observe all four elements of this definition in my research method: competence, voluntarism, full information, and comprehension.

Competence requires that the individuals involved in the research are sufficiently responsible and mature to make correct decisions given relevant information. *Volunteerism* implies that individuals freely choose whether or not to participate in the research, so that any exposure to risks is undertaken knowingly and voluntarily. *Full information* requires that participants' consent is fully informed. However, researchers themselves do not always know everything about the investigation. As Reynolds (1979: 95) points out, if "there were full information, there would be no reason to conduct the research – research is only of value when there is ambiguity about a phenomenon". The strategy of *reasonably* informed consent is therefore often applied. *Comprehension* entails that participants fully understand the nature of the

research project. This may be achieved through a time lag between the request for

participation and the decision to take part in the study.

Cohen and Manion (1980) note that there are two stages to the process of seeking

informed consent with regard to minors. First, the permission of the adults responsible

for the prospective subjects must be sought and obtained, secondly the young people

themselves should be consulted. In this case, I first gained permission from the

school's Headteacher and Head of Mathematics, before seeking consent from the

main study teachers to observe their lessons.

Fine and Sandstrom (1988) argue that researchers must provide a credible and

meaningful explanation of their research intentions, and that children must be given a

real and legitimate opportunity to say that they do not want to take part. Should a

child refuse, the authors recommend that they should not be questioned, their actions

should not be recorded, and they should not be included in any book or article (even

under a pseudonym). They might be included anonymously as part of a group. Fine

and Sandstrom (1988) advise that while it is desirable to lessen the asymmetry of the

power differential between children and adult researchers, the difference will remain

and its elimination may be ethically inadvisable. Individual characteristics of

particular students or their personal learning were not the focus of this study. In

chapters seven, eight-I and eight-II, students' names are replaced by '[student name]'

in the transcript to protect their anonymity. Where particular students are quoted or

paraphrased in chapters five and nine, they are referred to using gender-preserving

pseudonyms. These are used on those occasions when the thread of a discourse can be

followed more clearly if individual contributing students are identified. The initials given to the six central study teachers are also pseudonyms.

3.6 CHAPTER SUMMARY

Chapter three describes the methodological principles that drive this study. It describes the justification for, and implications of, adopting a qualitative approach, with a small-scale study of 'ordinary' lessons. Analysis began with the classroom practice, rather than the study being designed to test a preconceived hypothesis. Throughout the study, I carried out continued reflection on my own practice, and maintained a teacher-persona. The ethical challenge was to observe lessons with participants behaving as 'ordinarily' as possible, whilst ensuring that participants were fully informed about their part in the research. This was overcome through ensuring informed consent at the start of the study, then maintaining a low research profile during the observation period.

CHAPTER 4: METHODOLOGY II - DATA ANALYSIS

AND INTERPRETATION

In chapter four, I describe how the data collection discussed in chapter three was analysed and interpreted. Analysis operated at three levels. Level 1 analysis was conducted during the fieldwork period, documenting immediate substantive and methodological responses to the data (see section 4.2). Level 2 analysis (section 4.3) addressed research question one and resulted in an analytical framework describing generalisations expressed in the observed lessons (presented in chapter seven). Level 3 analysis (section 4.4) looked further into the types of generality defined at level two and addressed the second and third research questions.

4.1 INTRODUCTION

I needed to create a system of breaking down the complex social realities of classrooms in order to describe, explain and theorise about them in the form of written propositions, but without losing their subtleties. The approach to analysis that has been adopted for this study is similar to that described by Coffey and Atkinson:

> Analysis is a cyclical process and a reflexive activity; the analytical process should be comprehensive and systematic but not rigid; data are segmented and divided into meaningful units, but connection to the whole is maintained; and the data are organised according to a system derived from the data themselves... Analysis is not about adhering to any one correct approach or set of right techniques; it is imaginative, artful, flexible, and reflexive. It should also be methodical, scholarly, and intellectually rigorous.
>
> Coffey and Atkinson, 1996: 10

Analysis in the main study focused on transcribed class discourse, supplemented by observation notes and copies of students' written work. Analysis involved what Miles

and Huberman (1994) affirm are common features of qualitative analytic methods, including annotating and coding data, identifying commonalities and differences, and gradually elaborating claims. Whilst the claims made in chapters six to eight have grounded qualities and are based on evidence, they also reflect the researcher's insight, imagination, judgements, interpretations, orientation towards particular research questions, and the desire to respond to those questions in a particular way. Guba and Lincoln (1989) argue that the "findings or outcomes of an inquiry are themselves a literal creation or construction of the inquiry process" (1989: 143) and Dey (1993) emphasises that outcomes are "those of the analyst, and related to the overall direction and purpose of the research" (1993: 98). *An* rather than *the* interpretation of the data was sought: different researchers might construct different categories in analysing data from the present study, or use the same data to address different research questions.

However the aim was not to develop *any* response to the research questions, but a response based on analysis that attends to the links between claims and evidence, and is oriented towards the research questions. Through following these principles, the production of trustworthy and relevant outcomes was facilitated. The similarity between these principles and those guiding methods of data collection (section 3.1) demonstrates the coherent approach adopted across the study's methodology.

A framework based on that developed by Srivastava (2005) ensured an appropriate balance between more grounded and more researcher-directed analysis. The following three questions were asked throughout analysis: 1) What are the data telling me?; 2) What do I want to know?; 3) What is the relationship between 1 and 2? The first of

these questions reflected the first analytic principle requiring close relationships between claims and evidence, and guided more grounded aspects of analysis. This was not an exercise in grounded theory (Glaser and Strauss, 1967), but analysis did have grounded qualities in the emergence (through purposive interaction between the researcher and the data) of categories or concepts that had not previously been considered. Data were not read, interpreted and analysed free of prior ideas, expectations, values or aims. Question 2 ensured that these were made explicit and that the second analytic principle was adhered to. Question 3 assessed the dynamic interplay between more grounded and more researcher-driven aspects of analysis. While both questions influenced all analysis at all times, analysis became an increasingly focussed activity.

Dey notes that distinctions made by researchers during analysis may "not [be] recognised explicitly or even implicitly by the subjects themselves" (1993: 98). One can question why participants, in this study the students and teachers, should be expected to confirm research findings presented to them in a form that reflects analytical processes which involve the researcher's creative insight, imagination and judgements, and lead to outcomes at a remove from the participants' spoken and written words. For these reasons participants were not asked to comment on findings.

Several criteria were used to evaluate outcomes of analyses, ensuring that they were not accepted prematurely, and that analysis was deemed complete at an appropriate juncture. These related closely to the analytic principles.

- The analysis helps address the research questions.
- All claims can be related to specific evidence.

- All potential difficulties in interpreting the meaning of data have been identified and I have only used interpretations in which I was sufficiently confident as the basis of claims.

- The analysis accounts for most of the relevant data.

If the analyses met all four criteria, then they were judged trustworthy and accepted as a basis for making claims.

It is important to emphasise that the three levels of analysis were inter-related in that they represented three sequential periods of investigation of a *common and evolving focus*. In essence, the research was an ongoing process of becoming progressively clearer about expression of generality in whole class discourse, and the three levels of analysis were all part of this broader process. In this sense they need to be understood as inter-linked, rather than separate, entities.

4.2 LEVEL 1 ANALYSIS (FIELDWORK)

Level 1 refers to ongoing analysis throughout the fieldwork phase. It did not lead directly to a response to any of the research questions, but it was crucial to methods of data description and other forms of analysis. Level 1 analysis played a key role in section 6.2, where the different teaching styles of teachers are depicted, and enabled me to take account of the situation of the lesson, as well as of the discussion recorded. Hull refers to what he calls, "a 'black market' record of events and on-the-spot interpretations" (Hull, 1985: 27). This 'black-market' record was also available in my memory of the lessons. Such understandings are inevitable when the same researcher both gathers and analyses data. I have attempted to make use of their value whilst being aware of their dangers.

Analysis was an on-going process throughout the study. Although the relevant

literature on qualitative research approaches often deals with data analysis as if it were

a separate process, Taylor and Bogdan observe that this is "perhaps misleading"

(1998: 141). Grounded theory is an approach that is concerned with 'the discovery of

theory from data' (Glaser and Strauss, 1967: 1; see also Strauss and Corbin, 1990).

Although this was not an exercise in grounded theory, one aspect that I took from

grounded theory was to consider analysis as an ongoing process, and allow theories to

emerge from the data:

> The analysis of the data gathered in a naturalistic inquiry begins the first
> day the researcher arrives at the setting. The collection and analysis of the
> data obtained go hand-in-hand as theories and themes emerge during the
> study.
>
>   Erlandson *et al.* 1993: 109

I was very aware that the process of selecting the data was also part of the analysis.

This was initially debilitating, as I wanted as full and accurate a picture as possible,

and was reluctant to select and reduce my data. It soon became clear, however, that I

was required to do so, in order to reduce my data into a 'handleable form' (Powney

and Watts, 1987: 161). The shift from description to analysis and interpretation was

challenging. The nuances between these three activities are often very fine (see

Wolcott, 1994, chapter two). I also acknowledge that description is never 'mere

description', but already some form of analysis (see Wolcott, 1994: 15).

It was important that all sources of data be recognised. In some cases, structure was

imposed, in others, I endeavoured to recognise the influence of informal

observations/discussions on my findings. Lesson observation notes were written up

the day they took place, and this was immediately followed by annotation of the data

with memos, creating evidential narratives. These were narrative accounts of

immediate reactions to and reflections on the data. Both substantive and

methodological issues were noted (Burgess 1984) while Dey's (1993) comments

describe the nature of these annotations succinctly:

> We may put down a jumble of confused ideas. We may ask confused
> questions. Memoing should be a creative activity, relatively
> unencumbered by the rigours of logic and the requirements of
> corroborating evidence. Memos should be suggestive, they needn't be
> conclusive.
>
> Dey, 1993: 89

Level 1 analyses ensured the principles for data collection and analyses were adhered

to, through the creation of an audit trail of initial and evolving understandings of data.

In recording evolving ideas and interpretations, the danger that later developments

would automatically be accepted as 'better' was avoided. Instead later analyses could

be compared and contrasted with earlier attempts and informed judgements made.

4.3 LEVEL 2 ANALYSIS (TRANSCRIPT CODING)

This section deals with what Ely *et al.* (1997: 160) describe as 'overtly analytical

acts'. Level 2 analysis began only once I had collected the entirety of the data, and

completed the basic data management (sorting field notes, transcribing recordings).

The main source of data consisted of transcripts of interactions between teachers and

groups of students. With the intention of avoiding imposing preconceptions onto the

data, I used the data itself, informed by the theory, to shape categories for coding. The

process of analysis began with consideration of all fifty-two observed and recorded

lessons attending to the manifestation and expression of different types of generality. Initial analysis was based on familiarisation with the data and repeated listening to the lesson recordings and reading and re-reading observation notes, interview notes and transcripts.

The transcribed lessons were coded using Nvivo software to identify examples of different kinds of expression of generality. I coded examples of teachers and students expressing generalisations, then sub-coded these into distinguishing categories that emerged from the data. One advantage of the software was that expressions of generality could be coded and compared without removing them from their contexts. Although considering the utterances in isolation diminished their richness, it was possible to access the preceding and subsequent discourse, up to the full lesson transcript. I also had my field notes from the lessons, the full audio-recording, and notes made during level 1 analysis, to refer back to. Throughout my analysis, I aimed to remain aware of the context and background of each expression of generality. Although this richness of context cannot be fully shared with the reader, as I cannot expect the reader to read the full transcript of every lesson so the observation experience can be accessed only through the written word, every effort has been made to ensure descriptions are as full as possible (see section 4.5).

Having coded the expressions of generality, the data were examined to see what themes and ideas emerged. The decision to use these techniques from grounded theory (Strauss and Corbin, 1990) was based on the aim of the research to portray what *is happening* in 'ordinary' classrooms. I then grouped the codes into categories, and checked across the discourses that there was a fit for all of the lesson transcripts. Most

of the categories emerged from all the discussions, although some lessons had relatively more or less of certain categories. The categories began to emerge from my own practice, as indicated in chapter five, but were developed and clarified through analysis of the full transcribed data set. The sub-themes that emerged and developed during the coding process are introduced and explained in chapters seven and eight.

The conclusions reached through this analysis should be directly attributable to, and justifiable by recourse to, the data, so that someone else could understand the conclusions. That person may not have reached the same conclusions themselves. It is the intention of my research design that my study benefits from my total submersion in the data, my presence while the lessons took place, and transcribing the lessons myself.

4.4 LEVEL 3 ANALYSIS (DEEPER ANALYSIS)

The findings from the level 3 analysis are primarily reported in chapters 8-I and 8-II. Analysis at level 3 took as a starting point the framework developed with respect to generalisations through level 2 analysis. At all times the evidential narratives from level 1 analysis acted as a reference point and record of previous ideas and insights. The aim was not to re-code the data but to enhance the discussion of findings from level 2 (see section 1.2). In this way all three levels of analysis served complementary and cumulative purposes. The analytic principles (see 4.1) continued to guide analysis at level 3, and the questions 'what is the data telling me?' and 'what do I want to know?' remained important in ensuring that analysis remained respectful of evidence and focused on the research questions. According to Bogdan & Biklen (1992):

> Analysis involves working with data, organising them, breaking them into
> manageable units, and synthesising them, searching for patterns,
> discovering what is important and deciding what you will tell others.
>   Bogdan & Biklen, 1992: 153

The main motivation for data collection, description, interpretation and analysis was to aim to explore the generalities being expressed in secondary mathematics classrooms. It would have run counter to this objective to place a pre-existing theoretical framework on the data. Instead, the analysis is a mixture of 'direct interpretation of the individual instances' and some 'aggregation of instances until something can be said about them as a class' (Stake, 1995: 74). However, since it is neither the main nor explicit aim of this research to generalise to other cases, the main modes of analysis were narrative description and direct interpretation, rather than categorical aggregation.

Throughout all three levels of data analysis, and particularly at level two, the fifty-two recordings of teacher-led discourse were the main source of data for analysis. The unit of analysis at level two was the transcribed discourse, coded into episodes where the discourse appeared to be 'journeying towards' a generality. However, classroom observation notes, samples of students' work and pre- and post-lesson unstructured interviews with teachers were used to support or counter the conclusions of analysis, enabling deeper insights to be gained, and hypotheses to be verified. For example, conjectures that are made about how teachers viewed the procedures and concepts involved in their lessons, which emerged from analysis of the transcripts, were supported by comments made by teachers before and after lessons.

4.5 CREATING ACCOUNTS OF CLASSROOMS

I found a number of texts useful when I was starting to write the so-called 'final' version of the study. In terms of writing strategies, I gained support from texts by Bassey (1999), Becker (1986), Ely *et al.* (1997), Walford (1998) and Wolcott (1990). Reading about writing was a necessary part of the writing-up process. I was repeatedly reminded that it was not acceptable to assume that the data could speak for itself, and that I should resist the temptation to let the quotes make my points. Taylor and Bogdon (1998: 175) point out that analysis is essential, and that it is not enough simply to quote from an interesting interaction, and hope that the points make themselves. They also argue against indulging in colourful quotations or examples, and against either supplying insufficient or overly lengthy quotations. They also advise letting the reader know where the argument is going, using direct and concise writing, grounding the writing in specific examples and editing early drafts carefully. I have striven to honour this advice in this study.

I focus in this section on methodological issues involved in describing lesson observations. Mason says of such experiences that they,

> "...become data only when they are constructed as such by someone taking a research stance, with the intention to analyse it in relation to other data. Thus data occupy a space between observations and analysis."
>
> Mason, 2002b: 158

Having explained why there is a need to focus on describing lessons, I consider three examples of lesson descriptions taken from mathematics education research. A lesson from the pilot study is then described, with reflection on the interpretive tensions experienced during its writing.

As described in section 3.4.4, the original conception that accounts of classrooms would provide the main unit of analysis for the main study yielded to the appeal of the relative objectivity of lesson transcripts. This was partly a result of the breadth and exploratory nature of the research questions, and the large quantity of data collected. These two factors combined to render creation of detailed and comparable accounts of the main study classrooms overwhelming. Although creating classroom accounts did not play as large a part in the 'writing-down' of data as originally intended, it retained an important role in the 'writing-up'. Whilst both of these are equally matters of textual construction, "the second phase of 'writing up' carries stronger connotations of a constructive side to the writing" (Atkinson, 1990: 61).

This section consequently plays a dual role. Firstly, it serves to explain why the decision was made to use lesson transcripts, rather than lesson descriptions, as the main unit of analysis. The justification for this lies in the complexity and tensions related to the creation of such descriptions, and the difficulties of writing full and comparable accounts of lessons without concurrently analysing and interpreting (and so 'accounting for') what took place. Secondly, given that such written accounts of lessons play an important part in the sharing of analysis findings, the complexities and tensions needed to be resolved in order that 'accurate' accounts can be created. I argue that all description is interpretative to some degree, and that responsibility falls on the researcher to be aware and honest about the extent to which they have interpreted and analysed.

4.5.1 Desiring description

The process of analysis of the transcripts of classroom discussion is influenced by my experiences in the classroom as an observer. The principle of contextualisation is based on Gadamer's insight that there is an inevitable difference in understanding between the reader and the writer of a text (Gadamer, 1976: 133). If the subject matter be set in context the intended audience can see how the situation under investigation emerged.

> If you want to be in a position to analyse some event, some situation, then we must first be clear on what that event or situation consists of, as impartially as possible.
>
> Mason, 2002b: 40

It is the natural inclination of readers (perhaps especially those who are practising teachers) to compare the data with their own experiences and to consider aspects not intended by the researcher when collecting or describing the data. Experiences recorded are available to be shared, and readers can reach their own conclusions based on the data. My own practice is significantly influenced by reading about others' lessons. Although much research does not contain detailed lesson descriptions, I find that it is to these descriptions that I relate most strongly.

In this section I consider the work of three mathematics education researchers who use classroom descriptions in different ways. Mason (2002b) draws a distinction between writing an 'account-of' an experience and 'accounting-for' it. The intention with accounts-of experiences is that others can recognise the experience.

> An account-of describes as objectively as possible by minimising emotive terms, evaluation, judgements and explanation. It attempts to draw attention to or to resonate with experience of some phenomenon. . . A 'phenomenon' is a pattern discerned or distinguished by an observer de-

sensitised in certain ways, and so from an account we learn about the sensitivities of the observer as well as about the incident.

Mason, 2002b: 40

I have been influenced by maths researchers who have included accounts as data. One such description appears in Houssart (2004), where selected sections of particular lessons are included at the start of each chapter. For example, in a chapter titled 'Easy tasks, hard tasks, elastic tasks', she begins with the following extract.

> The teacher was introducing multiplication of two-digit numbers by single-digit numbers using vertical format. The first example on the blackboard was 14 x 1. The teacher went through this, multiplying first the 4 by 1 and the 1 by 1. I was sitting next to Matthew, who had a whispered conversation with me about 'One times anything' and arrived at the answer 14 in a single calculation. Matthew was asked to go to the board and do the second calculation, 14 x 2, which he did correctly. As he came back to his seat, he said, 'Pips, mate.'
>
> Houssart, 2004:127

This is an 'account of'. Choices have necessarily been made about what to include and what to leave out. We do not hear the words of the teacher, whether she praised the student for his answer. Nor are we offered a description of what the other students are doing. The choice has also been made to describe this particular section of this particular lesson, possibly partly because it relates to the chapter theme. Houssart recorded unofficial talk, or 'whispers', in all but two lessons observed throughout her study (2004: 32). She chooses to give us a sense of this even when illustrating other findings.

Fernandez and Yoshida (2004) give very detailed accounts of two lessons observed as part of a 'Lesson Study' in a Japanese elementary school. Ms. Nishi's lesson is described in eighteen pages, in the following style:

One of the students described the expression as 12 minus 2, and another student provided the answer 10. The student replied: "I subtracted 2 from 12." Then she asked him again how he got the answer 10. The student answered again "I subtracted 2 from 12." Ms. Nishi insisted, "Can you explain it in more detail?," but she got no further response. It seemed as if it was not clear to the student what else the teacher wanted to hear. Ms. Nishi looked a little bit frustrated with the difficulty she was having getting the student to answer her question.

Fernandez and Yoshida, 2004: 95

No analysis is included of this lesson. The following chapter describes the discussion of the other teachers, but does not explicitly include the thoughts of the researchers. Yet the eighteen page description is not merely an 'account of'. The researcher makes decisions about when the exact words used are important, and must be quoted, and when the speaker can be paraphrased. This makes the lesson easier for readers to access than if it had been written as an annotated transcript. Describing the lesson in such detail enables the readers to make sense of the data for themselves. It may also persuade the reader of the veracity of their explicit and implicit conclusions.

French takes a different approach, offering a 'typical' classroom discourse:

T: Let the number we start with be x.
We added 7. How can we write that?
A: $x + 7$
T: We then doubled, so what shall we write next?
B: $2x + 7$
T: C. What number did you start with?

C: 3
T: So, what is $x + 7$ if $x = 3$?
C: 10
T: D, what is $2x + 7$ with $x = 3$?
D: 13, but that's not double.
T: So, what should double $x + 7$ be?
D: $2x + 14$

French 2002: 43

The point he emphasises here is that "establishing a correct expression at each stage requires discussion about appropriate algebraic procedures and conventions together with verification through numerical checks". (French, 2002: 43). However, his

'typical' classroom discussions leave me wondering, as a teacher, how closely they can or could be emulated.

4.5.2 'Thick' Descriptions

The term 'thick description' was introduced by a philosopher, Ryle (1971), and made popular by an anthropologist, Geertz (1973), in the discussion of anthropological methods. In general terms, a thick description of human behaviour is one that retains and is faithful to the meanings which that behaviour has for the people involved. The method stems from the belief that brief numerical descriptions of behaviour often distort meaning. The definition of behaviour here is different from that used by a behaviourist. A behaviourist describes behaviour exclusively in terms of overt physical events, removing inventions, purposes, beliefs, conjectures and thoughts. An advocate of thick description would tend to avoid describing thoughts and behaviour in terms derived from a general model of behaviour. Instead, they aim to use terms and concepts that the individuals being described would recognise and judge appropriate. This is not to say, however, that an interaction between people must have a single meaning or interpretation. Where there are several participants, they may have different, perhaps unstable, purposes or intentions.

An advocate of thick descriptions would favour a rich narrative account, perhaps accompanied by photographs, film, taped conversation, although the ethics of including such sources in a study involving children are complex. The intention here is to make the reader feel acquainted with the individuals described. An advocate of thick description would be sceptical of a claim that a quantitative data file is always a

complete and accurate record of an experience. Available data may omit important

measurements. Through thick description, hidden biases can be revealed.

Ryle's examples of the need for thick description are drawn from every day life, such

as this one:

> [consider] ... the notion of waiting - waiting for a train perhaps. ... The 'thick' description of what I am doing on the platform requires mention of my should-be train-catching. Here there is [nothing] in particular that I must be positively doing in order to qualify as waiting. I may sit or stand or stroll, smoke or tackle a crossword puzzle, chat or hum or keep quiet. All that is required is that I do not do anything or go anywhere or remain anywhere that will prevent me catching the train. Waiting is abstaining from doing things that conflict with the objective.
>
> Ryle, 1971: 479

'Waiting' is readily characterised in terms of purposes, and poorly in terms of specific

behaviours.

My dual aims of wanting my analysis to benefit from my immersion in the field,

whilst being able to share my findings as fully as possible, led to my desire to write

'thick' descriptions of the lessons I observed. I wanted to give others a sense of the

classrooms derived from more than the transcripts, including observation notes,

discussions with teachers, and students' work. Events are not reducible to simplistic

interpretation, hence 'thick descriptions' are essential. Morrison (1993: 88) argues

that by "being immersed in a particular context over time not only will the salient

features of the situation emerge and present themselves but a more holistic view is

gathered of the interrelationships of factors". Such immersion facilitates the

generation of descriptions which lend themselves to accurate explanation and

interpretation of events rather than relying on the researcher's own inferences.

The relevance of thick description to empirical research is emphasised by Becker

(1996). Geertz states that the aim of ethnography is "to render obscure matters

intelligible by providing them with an informing context" (1983: 152).

> I actually think the well-publicised tension between quantitative and
> qualitative approach has a greater ring of truth when formulated as a
> problem in ontology rather than as a problem in method (or epistemology)
> ... quantitative research (with its methodological emphasis on pointing,
> sampling, counting, measuring, calculating, and abstracting) is premised
> on the notion that the subjective involves illusions that should be rejected.
> The basic idea is that it is only when all subjectivity has been subtracted
> from the world that the really real world remains. And what remains that
> is really real is the world of quanta ... In contrast, qualitative research
> (with its procedural emphasis on empathy, interpretation,
> thematization/enplotment, narration, contextualisation, and
> exemplification/concreteness/substance), is premised on the notion that
> the objective conception of the real world is partial or incomplete. The
> basic idea is that one of the very important things left out of the real world
> by the objective conception are *qualia*. Think of *qualia* as things that can
> only be understood by reference to what they mean, signify, or imply…
>
> Shweder 1996: 177

In considering how to share my observed classroom experiences with readers, I also

made use of the notion of hypotyposis. Rather than merely present the conclusions

attained through data analysis, it is essential that the reader be able to access and

experience the data for themselves. In this way, they can get a sense of its authenticity

and reliability. The construction of versions of social reality offers an opportunity to

persuade the reader of the authenticity, plausibility and significance of my

representations (Atkinson, 1990: 57).

> One of the important devices whereby the narrative contract is invited in
> the text is via the rhetorical device known as hypotyposis: that is, the use
> of a highly graphic passage of descriptive writing, which portrays a scene
> or action in a vivid and arresting manner. It is used to conjure up the
> setting and its actors, and to 'place' the implied reader as a first-hand
> witness.
>
> Atkinson, 1990: 71

Crapanzano (1986) has discussed the role of ethnographic hypotyposis in relation to

the writing of George Catlin's early descriptions of North American Indian scenes:

'His aim is to impress his experience of what he has seen so strongly, so vividly, on

his readers that they cannot doubt its veracity. It is the visual that gives authority'.

(Crapanzano, 1986: 57). Such hypotyposis is "used to establish and reaffirm the

relationship of co-presence of reader and author 'at the scene'. (Atkinson, 1990: 70).

> There is a close relationship between the 'authenticity' of these vivid
> accounts and the authority of the account – and hence of the author.
> Authenticity is warranted by virtue of the ethnographer's own first-hand
> attendance and participation. It is therefore mirrored in the 'presence' of
> the reader in the action that is reproduced through the text. The
> ethnograher is a virtuoso – a witness of character and credibility. It is
> therefore important that 'eye-witness' evidence be presented which
> recapitulates that experience.
>
> Atkinson, 1990: 73

This study, with its main study data set of fifty-two observed, recorded lessons, with

student work and observation notes, can benefit from ideas related to 'thick'

description and hypotyposis.

I will give an 'account of' (Mason, 1994, and defined in section 4.5.1) the lessons

based on my lesson observation notes, full transcription from audio recordings, and

students' work. My aim in writing about this intervention is to recreate these lessons

for the reader. By sharing my experiences of the lessons, I hope to offer opportunities

for readers to reach their own conclusions about what they can tell us about language

and generalisation in mathematics classrooms.

4.5.3 Interpretive Tensions when creating 'thick' descriptions

It is important to acknowledge that, when creating 'thick' descriptions of classrooms, researchers must go some way towards analysis. My interpretation of the pilot study lesson discussed later in this section may have begun even before the start of the lesson. My reading of relevant research literature, my teaching experience and my research design, contributed to the way in which I read the teacher's lesson plan, and doubtless affected the way in which I expected the group work to be carried out. These unformed expectations provided implicit assumptions which resulted in certain student actions being particularly noteworthy (because they illustrated or conflicted with my expectations). Although as an ethnographer, I aimed to be open to all possibilities, I acknowledge the impossibility of eliminating all conjecture and expectation. Better, I believe, to work hard at being open minded, whilst being aware of possible structures that I may be imposing on my experiences. As Gadamer (1976) argues, prejudice is the necessary starting point of our understanding. The critical task lies in distinguishing between "true prejudices, by which we understand, from the false ones by which we misunderstand" (Gadamer, 1976: 124).

The data is also subjected to the interpretation taking place during the lesson. This interpretation may be as simple as writing down something that a student has said in my observation notes. A choice has been made to record this comment, and to ignore others. I therefore have an idea, though not necessarily a conscious one, about the story I plan to tell, and the data that will be useful in telling that story.

Even if I were to record my experiences fully, in order to share my data with others I would need to condense it. Interpretation would come into play. Before the experiences can be analysed they must be rendered *analysable* – tapes transcribed, experiences noted in journals and written up in detail. At this stage, the data must be structured, to make analysis possible. Interpretation is involved again; the way I choose to structure my data affects subsequent analysis. I try to be as explicit as possible about the choices I make in interpreting my data in order to present it.

While writing the lesson description that follows, which offers an account of a lesson from the pilot study, I was aware of various interpretive tensions. I was required to make numerous choices about what to include, and how to include it. My writing about my writing is in italics, in the same style as this paragraph.

As students entered the room at the start of the lesson, the tables were arranged for groups of 4 students so that most chairs were sideways to the board. The students were told that they could choose who to sit with, although the teacher intervened in deciding that a 'fifth' person on one table should move to join a table with only three students. There was an A3 sheet of questions on each table.

The register was taken, with students giving special answers if their name's position in the register was an odd number, or if it appeared in the sequence $3n + 2$. They then discussed how frequently a student would be both of these things. Reference was made by students to multiples of six.

The teacher then introduced the main theme of the lesson.

Teacher:	We're going to be looking at sequences today, ok, and in particular we're going to be looking at picture sequences. It's easy enough for most of us, especially if you've had somebody tell you how to do it, to work out n^{th} terms for things and work out what sequences should look like. What's a much harder skill, that we love to develop in you, is to be able to look at pictures, and try to ascertain sequences from the pictures.

Students were invited to work in their groups, and given around ten minutes, subject to what the teacher could see and hear happening in the groups. The teacher told them that he was very interested in what ideas they would come up with:

Teacher:	I'd love you to feed back your ideas to me as I'm coming round as well, so, maybe, you know, as I'm approaching your table you can speak up that much louder or say one of your very good ideas, as a group, ok.

Most groups quickly began lively discussion, which generally appeared to be related to the task. One student in each group seemed to take on a 'chairing' role, reading out the questions and writing down the answers, and the layout of the tables appeared to make it difficult for some of the other group members to see.

With the permission of one group, I left a recorder on their table. When listening to their discussion later, while they were deciding which matchsticks were 'added on each time', I was struck by how limited their descriptions were.

Student 1	It's these
Student 2	Just these, yes.
Student 3	I think it's that one, that one and that one. Those three.
Student 4	I think its,
Student 1	No, I want that one.
Student 3	It's that one, that one and that one.

The students seem to be relying on pointing to matchsticks, rather than describing them verbally. Given that some group members were finding it difficult to see the sheet, this seems restrictive and prone to misunderstanding. There was a contrast here with the teachers' discussion during the INSET session, where they used expressions such as 'the walls', 'the roof' to make clear what they meant.

This paragraph is clearly interpretive. Other observers might not have made this observation, or might have noticed other things that I have not included. I have therefore ensured that there is sufficient evidence that, had their attention been drawn to it, a different observer could have seen the same thing. If something in the description strikes me, should I include my accounting for, or my noticing, in the description? Although this is intended as description, rather than analysis, interpretation has a role to play.

After fifteen minutes of group work, several students came up to the board to describe their 'way of seeing'. These explanations have been transcribed, and are being analysed using discourse analysis techniques. In the last five minutes, the teacher invited the class to consider four expressions that could be used to describe the number of matches in the nth pattern. In their groups, they were asked to choose an expression and think about what 'way of seeing' it related to. Some groups built up the 'n^{th}' house using longer sticks as groups of n matches, and shorter sticks as individual matches. One group considered $2(2 + n) + (n - 1)$. Their work is shown in figure 1 below.
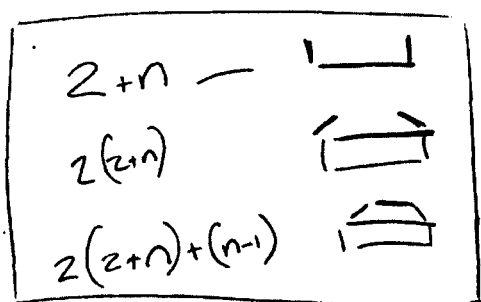
Figure 1

The first diagram, $2 + n$, shows a line indicating n matches along the base of the house, with a single match on each side. In the second diagram, the expression is doubled, and written as $2(2 + n)$, and the diagram shows this doubling, with the addition of another line of n matches along the base of the roof, with a single match on each side. The third diagram shows the addition of $(n - 1)$ to the expression, with the roof drawn on.

The students' diagram does not seem to provide a convincing explanation of why the total length of the roof is $n - 1$. The line at the top of the roof (which has a length of $n - 1$) is of ambiguous length in the diagram, only slightly longer than the single match and about half the length of the line used to denote n matches. This may be because this had already been discussed as a class, and was taken as a given.

An interpretive tension: Should I be using language to describe the algebraic expressions that the students would themselves have used?

The second way of directing attention towards the general rule, or locating the n, proved even more popular: looping the sections corresponding to the variable. Asked to justify others' rules, many groups worked on $3(p+1)$. One group's work on this expression is shown in Figure 2. Thus, the first diagram has 3 groups of 2 looped, the second has 3 groups of 3, the third 3 groups of 4 and the fourth 3 groups of 5.
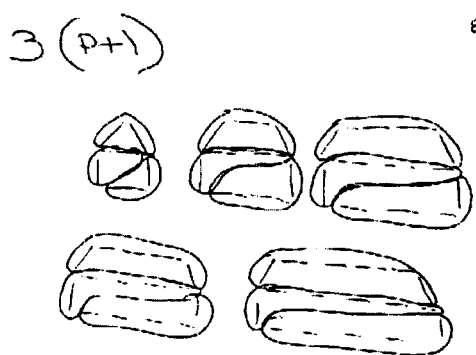
Figure 2

The difficulty here is giving the impression that all communication was successful. There is tension in trying to give an accurate description of all that happened without offering every student's work in full. Arguably, if the description is intended to show what is possible, it is acceptable to focus on the positive. This may, however, lead to readers developing infeasible expectations that may lead to feelings of disillusionment.

The 'looping' technique became confusing when students chose to use the same set of diagrams for all their general rules as shown on the right, below.

But now the reader is offered 50% successful and 50% unsuccessful examples. Does this accurately represent the students' work? A judgement needs to be made at this stage about students' understanding, so that the description allows the reader to have some idea of how representative is each piece of work offered. A tension arises here between offering a wholly qualitative account, and using quantitative measures to convey how representative each example might be.
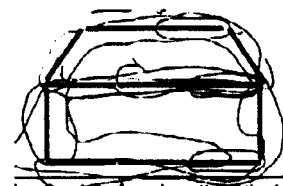
The looping technique enabled many students to express and share their general understandings without the algebra, as shown by the work below.
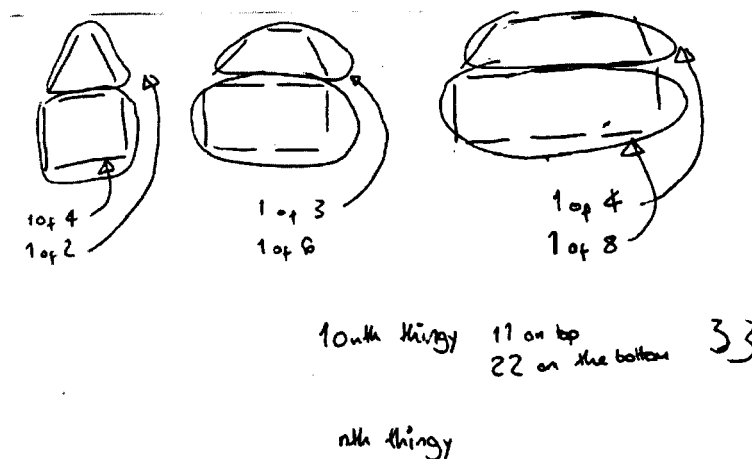
Figure 3

The following two paragraphs offer alternative descriptions of what is shown by the students' work in figure 3. There seems to be value in describing the work, and drawing the reader's attention to key features. In the first paragraph, I set out to describe only what could have been observed by the students. In the second paragraph, I intended to describe the work as clearly as possible to a reader with some background in algebra, and to indicate the generalities observable in the work.

The students' work shows the first three houses with the roof and the main body of the house separately circled. They have recorded the number of matchsticks in each looped section. Underneath this it is written that the '10th thingy' would have '11 on top' and '22 on the bottom'. There is a large '33' that appears to be their calculation of the total number of matchsticks used in the 10th diagram.

In the students' work, three diagrams have been drawn, resembling the 1st, 2nd and 3rd diagrams (i.e. where n = 1, 2 and 3). The students here have indicated that the roof of the house consists of $n + 1$ matches, and the body of the house consists of twice this amount. They have circled the roof section, and written the corresponding value of $n + 1$, and circled the body of the house, writing the corresponding value of $2(n + 1)$.

Again, in describing the students' work, my attempt to create an 'account of' suffers from interpretative tensions. My desire to offer a 'true' description of the lesson leaves me tempted merely to offer the students' work with no explanation. As a reader, however, I know I gain more from the data by reading a description, or even analysis, of it, as it provides a structure around which I can build my own 'story'. As long as I remain aware of the distinction between the two, there is a role for both 'account of' and 'accounting for' here. It may be useful for a reader to be told that the generality that the students seem to be working on is some approximation of (n + 1) + 2(n + 1). I cannot be sure how to express this algebraically, as the students did not complete the expression for n.

These students are being algebraic (Hewitt, 1998) in that they are structuring their counting. They seem to be seeing the roof as $n+1$ and the building as $2(n+1)$ but they have not recorded this as an algebraic statement. They have, however, applied their

structure to an un-drawn example – *the 10ᵗʰ thingy* – and correctly predicted that it will require 33 matchsticks.

Not all groups were successful in explaining the 'ways of seeing' that had resulted in the four expressions. A student in one group wrote the comment on the right.

> We couldn't do any "i"

The choice to include this in my description of the lesson reflects my interpretation of the lesson as a whole. It illustrates that the tools offered for understanding the generality had not been successful for all students. It also enables me to point out that this was unusual – this was the lowest level and all other students achieved more. In that sense, I am using a negative example for a positive purpose. I felt that the students were confident and honest. There was a conjecturing atmosphere. The apparently willing acknowledgement of not being able to 'do any' suggests that the students did not feel constrained by pressure to succeed (Boaler, 1997).

In summary, there are numerous interpretative tensions when writing 'thick' descriptions of lesson observations. I have shown how research might benefit from clarity about what is interpretation, and what description. I note the following interpretive tensions:

- If something in the description particularly strikes me, should I include my accounting for, or my noticing, in the description?
- Is it appropriate or justifiable to use language to describe student work that the students might not themselves have used?
- If the intention is to show what is possible, is it acceptable to focus on the positive?
- How to give a realistic overview of the lesson without being quantitative?
- How can I make students' comments or workings clearer for the reader, without putting words into students' mouths?

- Can I describe the lesson vividly and convincingly without conveying a value judgment about the effectiveness of the teaching and learning?

There are no simple answers to such questions. What is important is that they are considered by researchers as they 'create' their data. I have illustrated the value of this approach using a pilot study lesson. By being open about the tensions I have experienced in describing observed lessons, I found myself making more considered and informed choices. This resulted in a more considered lesson description. Wolcott (1990) argues that 'writing is thinking', and points out the value of writing during the process of qualitative research. I would go further than this, and argue that writing about decisions made while writing is also of significant value.

The tensions and decisions explored in this section served to illustrate both the complexity of using 'thick descriptions', and the insights they can offer. In carrying out level 2 analyses on the central study lessons, focusing on expressions of generality, sections of transcript emerged as a more effective unit of analysis than thick descriptions, as they enabled comparison of the discourse on a sentence- and word- level that would not otherwise have been possible. 'Thick descriptions' are used in chapters five to nine (and particular in the first and last of these chapters) in order to share research findings with the reader.

4.6 EVALUATION OF METHODOLOGY

Throughout chapters three and four I have discussed the various tensions and conflicts encountered when designing the research methods. Choices have necessarily been made between alternative imperfect methods of data collection and analysis. In this

section I consider the extent to which the research methods adopted were appropriate

to the methodological task. That is to say, did the methods generate and analyse data

that would answer the research questions being posed, follow ethical guidelines, and

function within practical constraints (see section 3.3).

The ethical question is addressed in section 3.5, whilst the practical constraints must

be taken as a given; inevitably they were either operated within or overcome, and

either of these is an acceptable outcome. The remaining question, therefore, is

whether the methods adopted were appropriate to answer the research questions

posed:

Research Question One	**What generalisations are being expressed in secondary mathematics classrooms?**
Research Question Two	**How are procedural generalisations expressed in mathematics classrooms?**
Research Question Three	**How are conceptual generalisations expressed in mathematics classrooms?**

The 'received view' of science (Agar, 1986: 11) based on the systematic test of

explicit hypotheses is inappropriate to research problems concerned with 'What is

going on here?'(1986: 12), which involve learning about the world firsthand. In a

sense, my analysis in this study involved my looking for things that resonated within

me. As my reading and thinking focuses on language and generalisation, the issues

and questions prompted within me by the lessons were mostly related to these areas.

Although I had key questions for consideration, I wanted to remain open to new

directions of study, new possibilities for the findings. As a consequence, Mason's

observation that we often learn as much from research about the researcher as about

the researched (2002b) is pertinent to this study.

Dey (1993) argues that there is no single set of categories waiting to be discovered in

qualitative data analysis. Instead of subjecting outcomes to external replication he

suggests the procedure of data generation and analysis should be described in

sufficient detail so that judgements about them can be made (this is the purpose of this

chapter), and that evidence should be presented to support the claims made (as is done

in chapters five to nine).

> In qualitative research, little is ever usually written about the process of
> analysis at all ... little is said about who the analysts are,... which
> particular   perspective   they   adopt...how   disagreements   are
> resolved...whether full transcripts are used, how much is reported, what
> level of uncodable or unsortable data is tolerable, what basis is used for
> filtering data...
>
> <div align="right">Powney and Watts, 1987: 174</div>

Since Powney and Watts lamented the paucity of insight offered into qualitative data

analysis, some progress appears to have been made. Through rendering the process of

data collection and analysis as clear as possible, the findings of the analysis are open

to the critical review of the reader, which in itself renders these analyses more

trustworthy.

I have striven to avoid the anecdotal approach described by Bryman through

systematic selection of data, with lengthier transcripts.

> There is a tendency towards an anecdotal approach to the use of data in
> relation to conclusions or explanations in qualitative research. Brief
> conversations, snippets from unstructured interviews . . . are used to
> provide evidence of a particular contention. There are grounds for disquiet
> in that the representativeness or generality of these fragments is rarely
> addressed.

<div align="center">156</div>

Bryman, 1988: 77

At the start of my recording and analysis I aimed to overcome this worry and potential bias of selecting snippets and anecdotes by considering everything as important. However, this quickly rendered any description or analysis impossible. My intention is not to set these worries aside, as they are both real and present. My awareness of them makes me more cautious about the judgments I make, but I cannot take it to its logical conclusion that, as no analysis is 'true' or 'accurate', there is no point in doing any analysis at all.

As Mason (2002b) has emphasised (see section 4.5.1), it is important that the researcher is able to distinguish between describing what happened ('accounts of') and interpreting its meaning ('accounting for'). By striving to make a distinction, wherever possible, between these two types of accounting, my intention is that the findings of chapters five to nine are described in sufficient detail so that judgements about them can be made (Dey, 1993).

Section 10.3 contains further evaluation of the study's research methods, including exploration of how three main tensions (depth versus breadth, researching others versus reflecting on own practice and understanding versus changing practice) were resolved in the study.

4.7 CHAPTER SUMMARY

This chapter has set out the ways in which the data of this study was analysed, by giving an account of the principles that underpinned the analytical procedures and of

the analytical procedures themselves. Furthermore, it has described and discussed how the final account was written and the measures that were employed to support the plausibility, credibility and trustworthiness of this final account.

Analysis was carried out at three levels. Firstly, during the fieldwork phase. Secondly, transcript coding using Nvivo. Thirdly, deeper analysis was carried out focussing on the expression of general procedures and concepts, in order to address research questions two and three.

The 'Algebra House' pilot study demonstrated the use of 'thick' descriptions of lessons. It was shown that these can be an effective tool for reporting research findings, although the number of lessons observed in the central study, and their complexity, rendered 'thick descriptions' less appropriate as the unit of analysis at level 2. Such descriptions were used to report the findings of the central study, and were made use of particularly in chapters five and nine, when recounting episodes in my own classroom. Although lesson transcripts were therefore adopted as the main unit of analysis, use was made of 'thick' descriptions and hypotyposis when disseminating the findings of analysis with the intention of increasing the study's plausibility, credibility and trustworthiness.

CHAPTER 5: IN THE RESEARCHER'S CLASSROOM

During the initial stages of this study, while the research questions were being refined and an initial review of the literature was being carried out, I focussed concurrently on my own classroom teaching. Through considering the role of the teacher in focussing students' attention on generality, my own experiences shaped the direction of the study. In this chapter an excerpt from one of my own lessons is used to show how the decision to focus on the categorisation of types of generality emerged from my own classroom experience.

5.1 INTRODUCTION

The literature discussed in section 2.1 emphasised the desirability of mathematics classes as places where all students become better able to see the general through the particular. The observations of Krutetskii (1976) and others demonstrate that discussion of generality should thus form a central part of mathematics education, and the role of the teacher in promoting and guiding classroom discussion of generality is worth serious consideration.

Much of my focus as a teacher, whether considering language use, task design, or lesson structure, centres on structuring awareness. I regard it as my responsibility as a teacher to help students to focus their attention on those things that I think are particularly worth attending to. In developing students' sense of generality, for example, their attention can be drawn to which numbers in an example are structural, and which are particular, or whether a statement is sometimes, always or never true.

Analysis of the relevant literature heightened my sensitivity to movement between levels or types of attention when working with students on practice activities such as exercises and games. Various attempts have been made to categorise different 'levels' of attention. For example, the van Hiele 'levels' in geometry have been interpreted by some (e.g. Burger and Shaughnessy, 1986) as a way to differentiate between different students, labelling some as 'higher level thinkers' than others. The van Hiele 'levels' can also be used to describe "thinking in the moment" (Mason and Johnston-Wilder, 2004: 59) without classifying learners. The levels can also be viewed as describing neither learner nor thinking, but rather the activity, or questioning, on offer for the learner to think about. Different tasks or questions might encourage thinking on different levels. At the same time, a single question or activity may require a number of different 'levels' of thinking, as specific examples are sought to check a general conjecture, for example.

Through reflecting on my own classroom practice, I began to see a spectrum of levels that spanned between the extremes of *particular* and *general*. While students work on a particular example or set of examples, the teacher can be considered to be directing their attention either towards the more particular or more general ends of this spectrum in order to enhance student appreciation of an area of mathematics.

Activities where skills are practised offer potential to direct attention away from the particular toward general patterns and findings. The discussion surrounding such practice activities may appear more relevant and interesting, or less important and note-worthy, as a consequence of being triggered by such an activity.

5.2 LESSON SEGMENT: ALGEBRA BINGO

The discourse that follows took place during a game of 'algebra

7	6	9
11	15	1
2	12	3

bingo'. In this game, each student draws a 3-by-3 grid and fills it with

their choice of integers from 1-15 inclusive, similar to the one on the

right. I ask a student to choose a letter, then another student to decide what number

the chosen letter will stand for, with the intention of emphasising that any letter could

stand for any number. Another letter and number are then chosen. An algebraic

expression is written on the board, and students calculate the value and cross off the

answer in their grids, if they have it. The winner is the first student who crosses out all

nine of their numbers. On this occasion the students chose $a = 13$ and $g = 4$. As

explained in section 3.5, student names given here are gender-preserving pseudonyms.

1	Me:	All the questions are having brackets today. I'll start, let me know if you want to suggest a question.

I write an expression on the board, saying it aloud as I do so, and students work out

what number the expression represents. I then write the 'answer' up, forming an

equation.

My first three bingo questions were:

$2(a-8)$		10
$3(7-g)$	with corresponding answers of	9
$2(a-9)$		8

2	Grace:	You're doing a pattern Miss. We'll stop thinking!
3	Me:	Oh yes. But if I carry on like that I'll have to do 7 next, and that seems tricky.
4	Lizzie:	You could put 7 in front.
5	Sam:	You'll have to put 7 in front.
6	Me:	Will I?
7	Grace:	Or you could put 1, but//
8	Sam:	//That would be silly, you don't need to multiply it by 1.
9	Natalie:	Multiplying by 1 doesn't do anything.
10	Me:	[Writing $7(a-12)$]. Does anyone want to make up a question?
11	Louise:	I want a. Can I have a?

12	Me:	No. I want brackets today.
13	Louise:	Ok. Put it in brackets! Can you do that?
14	Me:	[Writing (a)]. I suppose it would mean the same as a, but that doesn't count.
15	Chris:	[To Louise] Put a 1 in front of the brackets.
16	Rosie:	[To Louise] Do $(a - 0)$ in the bracket.
17	Students:	[Talk amongst themselves. Some of the talk is relevant, but not all.]
18	Me:	Ok. [Pause for quiet]. Let's think about this together. How many different ways can we make 13?
19	Chris:	1, a in brackets.
20	Me:	[Writing $1(a)$]. What do you think?
21	Rosie:	Aren't the brackets supposed to be for doing something? Like $a - 0$?
22	Chris:	Or a times 0.
23	Me:	[Writing $1(a - 0)$ and $1(a \times 0)$ on the board]. Are they both the same?
24	Louise:	a times 0 is 0.
25	Chris:	Oh yeah.
26	Me:	Are they both the same?
27	Chris:	No, my one doesn't work. $a + 0$ would though.

Several students then put their hands up to suggest other expressions, and the game continued.

5.3 ANALYSIS AND INTERPRETATION

Analysis of this transcript led to an increasing appreciation of a variety of categories and levels of generality contained within it. The subsequent discussion is divided according to the type of generalisation that it involves. I have spoken thus far of generality being contained within or involved in the discussion. A slightly different claim would be to say that generality is potentially present. The generalities examined below are those that I can perceive in the discussion, but another reader may discern further examples, or dispute those that I have perceived. The question, then, is one of perception of generality. An important question for consideration, and one that recurs throughout this study, is what generality the students are perceiving during the lesson. In analysing a lesson, the value of finding a generality seems questionable unless the

students could also perceive it. Whilst a teacher striving to encourage students to think mathematically and to form generalisations may be pleased to note an increase in the 'presence' of such occurrences in their lessons, such generality needs to be present in some sense for the students as well as for the teacher.

There is something within me that would be hesitant to notice something in this extract that I didn't think could be noticed on other occasions with this group, but no claims to generality are being made here. This short extract is not chosen for being representative, but nor is it unusual. There was nothing deliberately extraordinary about this task, the class, or my approach. That is not to say, however, that mathematical discussion is equally likely or equally rich with all groups of students. Just as while all tasks offer an opportunity for generalisation and mathematical thought, some lend themselves to it more than others, so groups of students respond differently to any given task. As a teacher, there is a need to adapt a given task or teaching approach to suit each group of students. The risk here is that the adaptations teachers make for less mathematically confident groups of students give them fewer opportunities for mathematical thinking (see Dweck, 1999; Boaler, 1997). This 'dumbing down' has been highlighted as problematic by Stein *et al.* (1996), amongst others.

My reason for choosing this excerpt is that during whole class conversations such as this I am particularly aware of my role in structuring, or seemingly choosing not to structure, students' attention. The game provides opportunities for the students' attention to shift from the repetition and practice of algebraic substitution to considering more general properties. Having considered the various generalities

(explicit and implicit, mathematical and behavioural) I examine the extent to which students in the class might be aware of each generality, and the possible direction of their attention.

5.3.1 Generalisations about... algebra

Algebra is both the language of generality, and an area of mathematics about which generalisations can be made. Partially as a consequence of this, there is some ambiguity in the extract about whether students are using a general rule in a particular case, or moving from the particular to express a general rule. Because the students are using algebra, some of their applications of generality may be interpreted as expressions. For example, when Chris and Rosie suggest ways that Louise can achieve the answer a, they offer the suggestions $1(a)$ and $(a - 0)$. They could be interpreted as focussing on the particular case of $a = 13$ and stating that multiplying 13 by 1 gives 13, or that $13 - 0$ is 13. An alternative interpretation might consider them to be claiming that $1(a)$ and $(a - 0)$ would equal a for any value of a. We cannot tell whether they, or the other students, are aware of this. A listener could interpret their suggestions as general statements, or as being true only when $a = 13$. Their assertions, in lines 15 and 16, seem to be applications of general rules. They do appear to be applying the general rules that multiplying by 1 or subtracting 0 leave the original number unchanged, and may also be expressing this generality.

Students also appear to be using general rules about algebra. Rather than ask for the number she wanted (13, in line 11), Louise said "I want a. Can I have a?". This seems to be an application of the rule that a letter can represent a number. For her, it seems,

for this game, *a* and 13 may have become interchangeable, but she may retain '*a* is currently 13'. The distinction between variable and unknown, and the understanding that letter represents number are considered to be an essential foundation of algebraic understanding (see Rosnick & Clement, 1980; MacGregor & Stacey, 1997).

In line 5, Sam suggests that an answer of 7 can only be achieved if a 7 is put in front of the brackets:

5   Sam: You'll have to put 7 in front.

With this distinction between seven as a possible answer, and the only way to get seven as an answer, Sam appears to demonstrate some general sense of seven as a prime number. His expression could perhaps be interpreted as 'seven is the only (significant) factor of seven', which seems to be a more general statement (though incorrect, as one is also a factor of seven) than 'seven is a factor of seven'.

Natalie appears to be justifying Sam's particular statement about multiplying a bracketed expression by one to obtain the answer seven with the expression of a general rule:

8   Sam: That would be silly, you don't need to multiply it by 1.
9   Natalie: Multiplying by 1 doesn't do anything.

With her use of the present tense here, Natalie seems to imply the general 'multiplying any number by one doesn't do anything' rather than the more particular 'multiplying this number by one wouldn't do anything'. Tense-use by both students

and teachers can be seen as an indication of the level of generality of a statement, with

past-tense for the particular and present for the more general.

Uncertainty about the level of generality experienced while analysing the transcript

might also be felt by students trying to make sense of the discussion for themselves.

Not only might they be unsure about whether a statement represents an exception or

an example of a generality, they might also be uncertain about whether the claim is

true. With the intention of encouraging students to think about each others'

conjectures, and to realise that the truth or falsity of mathematics can be determined

without an external authority (the teacher), I tend not to correct students' imperfect

conjectures immediately. I believe that my classroom stance is generally founded on

the belief that "If I'm having to remember..., then I'm not working on mathematics."

(Hewitt, 1999: 9).

> A teacher taking a stance of deliberately not informing students of
> anything which is necessary is aware that developing as a mathematician
> is about educating awareness rather than collecting and retaining
> memories. Furthermore, this stance clarifies for the students the way of
> working which is appropriate for any particular aspect of the curriculum –
> the arbitrary has to be memorised, but what is necessary is about
> educating their awareness.
>
> Hewitt, 1999: 9

There is a huge responsibility here on the teacher to "provide a task which will make

properties accessible through awareness" (Hewitt, 1999: 9). Students may become

aware of the general effect of multiplying by one through their own experience. A

student referring to this rule might provide a prompt for students to test the conjecture

and develop their understanding. Multiplication by one deserves serious consideration

and thought, there is much discussion to have, and plenty of particular cases, with or

without context, to consider. It would be debilitating, however, if such deep

consideration were required on every occasion a generality was implied or inferred. Although the student may well generalise the rule, and even verbalise the rule, for themselves, the teacher might encourage the process by emphasising its importance. Some generalities will be more useful than others, and the teacher may be able to indicate to the student which these might be.

5.3.2 Generalisations about... the game

I write *game* here, but few of the students appear to see competition as the main purpose of the activity. I find myself frustrated when the activity is interrupted by students telling us "I've got a full house!". Other students seem to feel the same. When someone won and we stopped, for example, Louise quietly said "I got full house ages ago". She apparently didn't think it was interesting enough to tell us about at the time. It is tempting to conclude that the game is an irrelevance, but I suspect that it acts to focus students' attention in the first place, offering them an opportunity to engage with the algebra. There is an issue of vocabulary here, for while *give* and *make* seem too strong, *offer* is often too weak. Although the opportunity to think mathematically is *offered* by a wide variety of tasks, there is something about some of these tasks that makes it more likely that the opportunity will be taken up. Perhaps it could be claimed that the game prompts, stimulates or provokes engagement with algebra.

It is unclear how students distinguish between the rules of mathematics and the rules of the game. Grace's comment in line 2 suggested to me that she knew that the 'game' was just a disguise. When Louise asks, in line 13, whether she can put *a* in brackets, it

is unclear whether she is clarifying the rules of the game or of the mathematical world. Likewise, when Rosie questioned whether you would or could have brackets that weren't for "doing something" (line 21) she may have been asking about the rules of conventional algebraic notation, or the rules of the game. As I had insisted that all expressions must have brackets, Rosie may have been interpreting my game rule as 'all expressions must have brackets that do something'. Her contribution can be seen either as a reminder of the rules of the game, or a general statement about the meaning and role of brackets in mathematics. In either case, her observation is a non-trivial one, especially given that in written language brackets are often used to designate the inessential, provide clarification or direct attention.

With statements such as "all the questions are having brackets today" (line 1), and similar in line 12, I seem to need to have the arbitrary control of judging what is and isn't allowed. It is possible to distinguish levels of convention or arbitrariness in mathematics, of which games, tasks and exercises are perhaps the most arbitrary and transient. For example, while the 'fact' that there are 360° around a point is a convention, the related rule that the sum of angles in a planar triangle is half the sum of angles around a point is a necessary mathematical truth. Hewitt (1999) explores this distinction between the arbitrary and the necessary, arguing that effectiveness of teaching practice may be improved by viewing the curriculum in terms of things that can be worked out by someone (necessary) and those things that all students *need* to be informed about (arbitrary). The National Strategy for Key Stage 3 mathematics (DfEE, 2001: 178) states that students should be able to "distinguish between conventions, facts, definitions and derived properties". It is possible to conceive of a continuum from apparently arbitrary rules for activities, investigations or games,

through the conventions of the mathematics community, to mathematical truths. Students' awareness of whether a generality is a mathematical necessity, a mathematical convention, or an arbitrary rule for a game or task seems crucial to their understanding. This distinction became increasingly pertinent through analysis of the main study data, and is discussed more fully in section 7.3.4.

The arbitrary nature of general rules in classroom activities often acts to limit the range of permissible change associated with an aspect of mathematics. "Pick any three numbers", for example, may always mean (for a given group, with a given teacher) "pick any three 2-digit integers". If many activities require this sort of number, then time is saved by establishing 2-digit integers as the range of permissible values for these questions. If students understand that this is a classroom convention, and are still aware of the huge range of numbers that is actually available, then this practice is unproblematic. Unfortunately, this is unlikely to be the case.

I experienced the effect of falsely reducing the dimensions-of-possible-variation for myself during the lesson, when I expressed the difficulty of 'making 7' within the restriction with brackets. I was restricting my interpretation of 'with brackets' to only those expressions with the form $a(x + y)$. In retrospect, I could have introduced $2(g + 1) + 3$, or an equivalent, thereby allowing for many more possibilities. None of the students suggested an expression of the form $a(x + y) + b$, which suggests that, for this activity at least, the dimensions of possible variation of an expression with brackets did not include such a form.

Although my awareness of dimensions-of-possible-variation seemed to influence the students' apparent awareness, many of my contributions in this extract indicate that I am following, rather than leading the students. My response to Grace's indicating the pattern (in line 3) was to demonstrate surprise and interest (you noticed something that I hadn't noticed), to ensure that other students knew what she meant (by indicating that 7 would be needed next and pointing to the pattern on the board), and then to wait for students to take control. If a student had suggested that we break the pattern at that stage, I would have done so, as I believe there is value in following a student lead, and tend to do so almost automatically except in situations where I feel there is a mathematical or pedagogical reason to continue with my original intention. The group attention, however, seemed to focus on how 7 could be achieved. It has often been suggested that mathematical activity is at its most fruitful when it is initiated by the students themselves (e.g. Spencer, 1878; Banwell *et al.*, 1972; Ainley, 1982). In a whole class situation all students would generally be expected to consider the same problem at a given time, so a question immediately arises about whether a given student benefits from a different student's initiation of a mathematical task.

Sam's contribution in line 5 indicates that his focus is on the mathematics, rather than the game. This is no longer 'find a way to make 7', but a consideration of what makes 7 different from the previous three answers. Line 10 might have been a good opportunity to ask "Could I put anything else, or just 7 and 1?", and discuss prime numbers and factors. I could have let the students think of alternative ways to make 7, but I felt that the pace of the activity would suffer. Many classroom activities have an extra purpose, alongside that of learning a particular topic. With tests the focus can move from learning mathematics to 'getting a good mark', while with games it might

move to 'winning'. Group work carries a wide variety of objectives including social as well as mathematical or psychological, some made explicit by the teacher (such as 'make sure everyone contributes', 'give everyone a role') and others that may operate sub-consciously. While these objectives are generally seen as less important than the mathematics, to remove them completely would often render the activity effectively meaningless.

5.3.3 Generalisations about... behaviour and purpose

Many of the students' suggestions and assertions give insight into more than their knowledge of mathematics. Just as rules about mathematics can be formed on the basis of several particular examples (or even just one), rules about behaviour are being formed and revised based on particular instances in lessons. The general understandings of a group at a time combine to constitute a community of practice (Lave and Wenger, 1991). These generalities can be discerned from particular instances.

One such generality concerns the role of errors and misconceptions in maths lessons. Chris's readiness to accept that "my one doesn't work" (line 27) and to suggest an alternative is not exceptional in this group. In another lesson with the same students, for example, Louise built positively on Tom's misconception to help him understand. We were converting fractions to percentages using equivalent fractions with a denominator of a hundred. When we were discussing 1/25, Tom insisted that it must be equivalent to 25% , and a quarter, "because 25 is a quarter of a hundred". Louise responded, "I can see why you think that, Tom, but 1/25 is much smaller". Grace

added, "Like if you had a cake cut into 25, the pieces would be really small, they wouldn't be quarters".

In the discussion transcribed above, Chris's comment can be seen as a particular example of it being 'safe' to admit when you are wrong, which may be used by other students to create a general rule. His comment can also be viewed as an application of his own general views about what maths lessons are for.

Exploiting learner's errors is a commonly accepted strength in teaching. What is more difficult to agree upon, however, is how this 'exploitation' is to be effected. Whilst discussions such as that between Tom, Louise and Grace may contribute to students' understanding of fractions, they are unlikely to resolve the misunderstanding universally. It has been suggested that taking time to explore misconceptions more fully may provide opportunities to develop understanding (Mason *et al.*, 2005). One such suggestion is to invite learners to find instances in which the incorrect interpretation would give the correct answer (1/10 comes to mind in Chris's case).

> Getting the learners to articulate the conjecture and then to test it for themselves turns the initiative over to them, and reinforces the notion that in mathematics the authority lies within mathematics, not with individual people.
>
> Mason *et al.*, 2005: 279.

Such an exploration would require some consideration as to an accessible approach to the conjecture, as well as reworking of its exact phrasing.

A second generality that I can perceive in the discussion is the rule that 'when other students are talking amongst themselves, it's ok for us to do that too'. Students were

attentive during most of the game, with a large proportion offering questions, answers, or other contributions. This is perhaps the nearest I can come, as an observer, to saying that they were 'listening'. The social contract that ensures that students listen during such discussions seems to break down in line 17. This particular occurrence is an example of a general tendency for students to talk amongst themselves if I appear to have taken a step back from the discussion.

Inter-student conversations such as that started by Louise's declaration in line 11 that she 'wanted *a*' are something that I aim for in my teaching. In theory a problem suggested by a student and contributed to by other students should seem more interesting and relevant to the other students. Lines 11 to 16 lead to a period of time with all students talking. There is an assumption, often explicit, that if a student is talking to me, the whole class should be attending to it. This is somehow more difficult to encourage with student-student discussions, perhaps because the students who are speaking are less aware than I am of the need to keep everyone's attention. The students perhaps trust me not to use too many words they don't understand, and to watch all their faces closely to see if they are following the line of argument, but do not trust their peers to do the same.

As their teacher, I might be able to share with them explicitly the objective that they listen carefully to discussions in which I am not participating. This kind of 'communication about how we communicate' seems to have had an impact in other areas. In line 2, for example, Grace appears to be discouraging me from making it too easy so that they 'stop thinking'. In other lessons also, these students appear to see thinking as the central purpose of mathematics lessons. I believe that this is partly due

to the emphasis that I placed, especially at the start of the year, on the value of thinking. As a teacher, I can influence students' sense of satisfaction, and partly determine what they aim to achieve in maths lessons (Coles, 2004). Coles decided that, in order to alter students' views of mathematics, he would need to engage in *meta-communication*. This involved him promoting the idea of each student 'becoming a mathematician'. He argues that: "It is through meta-communication about what it means to be 'becoming a mathematician' that I believe I can help create the context in my classroom in which learning 2 [changing the process of learning] can occur." (Coles, 2004: 23). The concept of *meta-communication* proved useful later in the study in attempting to account for the practice of main study teachers, and is discussed further in section 8-II.3 in relation to use of mathematical language.

5.4 REFLECTIONS: UNFOUNDED GENERALISATIONS AND LEVELS OF AWARENESS

The bingo game offers an illustration of how tasks with the principal objective of practising skills can provide an opportunity for discussion and development of ideas. The extent to which students take up this opportunity is greatly shaped by the role played by the teacher, and the stance they choose to take. In just a few minutes of classroom discussion, generalities seemed present concerning algebra, the rules of the game, and behaviour both in mathematics lessons and in mathematics more widely. I am interested in the 'presence' of these ideas. While they may be present for the students expressing and applying them, arguably they are not present for students struggling with the mechanics of substituting in the values of a and g. If there is a value for students struggling with the ideas in being in the presence of higher level

ideas, then it is one that is difficult to measure. Such a benefit might be increased by emphasising the generalities, and drawing students' attention to what is happening in the discussion. One way to focus attention on important contributions would be to push for more reasoning and justifying. Unless I question and probe for this justification, the possibility remains that surface-level observations will have the same status amongst listening students as those that are much more profound.

Ellis (2005) found that teacher requests for justification led to "more productive" generalisation, and that repeated use of "why" before a generalisation had been expressed formally did indeed lead to improved, higher-level restatements of the generality. Much is often expected of teachers at the 'discussion' stage of a task. For example:

> Students can be asked to complete a table like this, simplifying the numerical fractions where possible. Discussion can then focus on how the results indicate which of the algebraic forms simplify and how that is achieved by seeking genuine common factors.
>
> French, 2002: 61

In this, the 'findings' of the activity, the generalisation, is supposed to be remembered and referred back to. The next task French describes, however, results in a generalisation that does not need to be remembered and relied upon. How do students know the difference? In the first activity they benefit from attending to the general need to seek genuine common factors of algebraic fractions. In the second activity their attention is expected to focus on the technique of adding algebraic fractions. In the first the particular provides illustration and explanation for the required generalisation. In the second the general provides a meaningful context for practising the required particular skill. What are students attending to in each? What are they

learning? As a teacher, I feel that the words I choose to accompany each task, and the discussion that I guide, will have a major impact on students' learning in each case.

5.5 CHAPTER SUMMARY

This chapter shared a lesson segment from my own teaching practice to illustrate how reflection on my practice influenced the direction and structure of the study. Consideration of the overarching research question 'how is generality expressed in the secondary mathematics classroom?' whilst teaching lead me to develop the first research question, as discussed in section 1.2: what types of generality are expressed in secondary mathematics classrooms?

I was interested in what it means for students to 'be generalising'. Although Hendrix (1961) argued that a student's understanding of a generality may be greatly temporally removed from their expression of it as a rule (with appreciation and expression occurring in either order), I believe that listening to learners offers an opportunity to gain insight into the process of generalising. In respect of the particular lesson segment, I asked how students' attention was directed towards the general, whether students were generalising for themselves, and what types of generality were being discussed. In order to address the first research question, I resolved to develop a framework for considering 'types' of generality. This, for example, distinguishes between generalities that hold in a particular lesson or for a certain activity, and those that are mathematically necessary. It also distinguishes between generalities that have been 'told' or surface-level 'spotted' and those that can be justified and explained. This latter is not a characteristic of the generality itself, but of the perceptions of the

students and teacher considering it. This distinction became increasingly important as

the study progressed.

CHAPTER 6: THE MAIN STUDY LESSONS

This chapter offers an overview of the lessons at the centre of this study. Having introduced the chapter in section 6.1, each of the teachers in the main study is introduced in turn in section 6.2. To begin to discover who is expressing generality, I began by exploring who was expressing themselves verbally on any topic. Section 6.3 reports findings concerning the proportion of teacher and student talk in the main study lessons. Sections 6.4 and 6.5 explore some of the issues and questions arising from the previous sections. The chapter findings are summarised in section 6.6.

6.1 INTRODUCTION

Before examining the discourse observed, recorded, transcribed and analysed in the main study classrooms, this chapter offers the reader some insight into the pervading classroom culture in the mathematics department as a whole, and in the classrooms of the six individual study teachers. Different classrooms place different expectations on students regarding their participation and the verbal expression of their ideas. As discussed in chapter three, it is intended that the greatest insight into the main study classrooms will be gained by 'thick' descriptions when reporting the findings of analysis (chapters seven and eight). It is hoped that the introduction offered in this chapter will contribute to the depth of understanding of the classroom episodes offered in subsequent chapters.

6.2 THE MAIN STUDY TEACHERS

The six mathematics teachers whose work informs the main study have varying professional experience, having taught for between one and seven years at the time of data collection. They teach 'good' or 'outstanding' lessons as 'measured' by institutional guidelines and criteria. Classrooms were generally well-ordered, with clear expectations shared implicitly or explicitly with students. They are not offered as typical, but examination of their work reveals the complexity of generality within mathematics classrooms.

A concern is that teachers establish relationships with classes over the course of a year or more, so a single lesson might not be 'representative'. An extremely 'grade and exam' focussed teacher might not make explicit reference to exams in every lesson. A teacher who emphasises the importance of pair or group work might have rendered this so apparent in previous lessons that it is now an established way of working, and no reference is made to it.

I have ensured that my analysis of teacher intentions in this section is consistent with all the data collected in the main study. This gives the advantage that I am not looking at a single lesson in isolation. Since the observations were carried out at the start of an academic year, where classroom culture and expectations are more likely to be made explicit, I have a sense (from observation notes and lesson transcripts) of the way teachers work with their classes.

Whilst emphasising that teachers' beliefs and practices change and develop over time, I found myself searching for frameworks I could use to describe the apparent differences in the main study teachers' orientations. Although constraints prohibit a full exploration of the plentiful theory (Thompson, 1992) of teacher beliefs and practices, I make some use of the relevant literature in this section to enhance teacher descriptions. Fang (1996) suggests that, rather than attempting to access teacher's beliefs through interview questions involving educational 'jargon', beliefs and practices can more effectively be approached by examining how often different teaching behaviours occur. As Ernest (1989) described it, there is sometimes a significant difference between espoused and enacted practice.

I used Swan's (2006:199) *statements used to assess teachers' practices* as a guide for distinguishing between different teacher's beliefs and practices during the main study lessons. Swan generated twenty-eight statements categorised as 'teacher-centred' or 'student-centred'. The teacher-centred statements describe practices arising from a transmission-oriented belief system, with teaching seen as the transmission of definitions and methods to be practised. 'Student-centred' describes practices that one would expect to arise from a constructivist position, with mathematics seen as a subject open for discussion.

Teacher-centred statements (transmission oriented)	Student-centred statements (constructivist)
1. Students learn through doing exercises.	15. Students learn through discussing their ideas.
2. Students work on their own, consulting a neighbour from time to time.	16. Students work collaboratively in pairs or small groups.
3. Students use only the methods I teach them.	17. Students invent their own methods.
4. Students start with easy questions and	18. Students work on substantial tasks

work up to harder questions.	that can be worked on at different levels.
19. I tell students which questions to tackle.	5. Students choose which questions they tackle.
20. I find myself encouraging students to work more quickly.	6. I encourage students to work more slowly.
21. I only go through one method for doing each question.	7. Students compare different methods for doing questions.
8. I teach each topic from the beginning, assuming they know nothing.	22. I find out which parts students already understand and don't teach those parts.
9. I teach the whole class at once.	23. I teach each student differently according to individual needs.
10. I try to cover everything in a topic.	24. I only cover important ideas in a topic.
25. I try to teach each topic separately.	11. I draw links between topics and move back and forth between topics.
26. I know exactly what maths the lesson will contain.	12. I am surprised by the ideas that come up in a lesson.
13. I avoid students making mistakes by explaining things carefully first.	27. I encourage students to make and discuss mistakes.
14. I tend to follow the textbook or worksheets closely.	28. I jump between topics as the need arises.

From Swan, 2006:199

SJ

SJ appears to believe that the most effective way to 'cover' the syllabus is to 'deliver' it. At key stage 4 she uses the textbook as a structure for lessons, relying on it as an indicator as to what content should be covered, and setting students questions from the exercises to work on individually. Each lesson begins with five quick questions, which test students' recall of topics previously 'covered'. In the transcribed lessons, exam success was referred to frequently (at least twice in each lesson) although this may be partly as a consequence of all observed SJ lessons being at key stage 4.

In departmental meetings, SJ is enthusiastic about ideas contributed by colleagues involving students developing their own methods or working collaboratively in groups. However, these ideas are rarely used in her own classroom. Although

apparently interested in constructivist approaches, her observed style is more transmission-oriented. Unstructured questioning of students suggested that they regard SJ as a 'good' teacher, because she "explains things clearly" and "knows exactly what we need to learn".

BG

BG is a confident and experienced teacher. Her lessons tend to involve a starter, which may or may not be connected to the objective for the lesson. The lesson objective is displayed on the whiteboard throughout. There is then an explanation or discussion phase, followed by students working individually or with others sat near them on practice questions. BG works as a marker for SATs and GCSE exam papers, and has a particularly good knowledge of the expectations and demands of external exams. She uses this, and her awareness of common misconceptions, to support her clear, careful explanations, and to inform the order in which she teaches topics. In many aspects of her practice, BG could be thus described as taking a transmission-orientated approach. However, she has also been influenced by continuing professional development, and is enthusiastic about research findings and pedagogic ideas along more constructivist lines. BG strongly believes that different students learn in different ways and at a different pace. She speaks with affection and impressive detail about her impressions of individual students in her classes, including their approach to learning mathematics, their participation, and their understanding of various topics, and spends a substantial proportion of each lesson working with individuals and small groups of students. BG also values students expressing their mathematical ideas, and rarely asks her classes to work without discussion.

LR

LR was in her second year of teaching when the main study data was collected, and her practice was still evolving and developing. She was keen to hear about new innovations and to try them in her classroom, although she was sometimes disappointed that the students did not respond as well as she would have liked.

LR often introduces relatively open-ended problems for the class to work on, with the intention that students will take control of their learning, but then attempts to scaffold their work, apparently wary of allowing students to 'struggle' for themselves. In lesson [15], discussed in section 8-I.2, LR introduces a problem where a rectangular pen is to be made using 20m of fencing. The students seem reluctant to attempt to draw diagrams of possible pens. After two minutes, some students have not yet written in their exercise books, while others have drawn one or two rectangles that did not have perimeters of 20 units, and are beginning to express frustration. LR responds by asking the students to stop and listen, and asking them if anyone can explain an easier way of approaching the problem.

> 10:52 Teacher: Has anyone worked out an easier way, or a way you can work out what different sizes of rectangles we can use? Has anyone thought of a way? Erm, [student name], have you?

Her intention seems to be to simplify the problem into a step-by-step process, to avoid students making mistakes. She appears to hold constructivist beliefs such as students learning through discussing their ideas, inventing their own methods and working on substantial tasks. However, her classroom practice is often transmission oriented, as

she generally teaches the whole class at once, and seems to feel responsible for explaining things carefully to avoid student mistakes.

PF

PF places strong emphasis on the role of discussion in students' developing ideas. He is overtly excited by mathematics, and by students' ideas for methods and approaches. Perhaps due to departmental setting policy, these more student-centred beliefs are not always manifested in his teaching. Lessons tend to combine whole class discussion, in which PF introduces an idea or method using regular student questioning, and students working individually on worksheets or textbook exercises. Although he held beliefs that were related to a constructivist view of learning, when students were working individually he operated in a transmissive mode. PF recognises that his beliefs are sometimes in conflict with his actions. During the pilot study, he decided to arrange the students in groups to work on the *matchstick houses* task, and stated in the post-lesson interview that he wanted to "do more group-work". As a teacher in his NQT year, it is likely that PF's practice was still developing rapidly, and that differences between his apparent beliefs and his practice might partially be explained by his relative inexperience.

PF begins lesson [27], which is discussed in more detail in section 8-II.3, by showing his year 9 students two diagrams of rectangles, and asking them to think about how they could represent a quarter and a third of each one. Each rectangle was divided into twelve equal parts, and he asked the students to think about what the diagrams could be used for.

> 05:01 PF: Ok. Erm, have a look at what I've got on the board so far. We've got two grids almost, ok? {?} Next to the first one it says "how can we represent a quarter?", and next to the second one it says "how can we represent a third?". Um, and don't just think "Ok, what boxes can I shade? And I can clearly shade in that many boxes to give us a third, that many boxes gives us a quarter" – what I need you thinking about is why I've chosen those particular grids, why I've asked you for those particular fractions, ok, and what possibly we could do with these things. Just have ten fifteen seconds, talk to the person next to you if you need to, have a think about what we could do.

This excerpt demonstrates PF's belief that students learn through discussing their ideas and having time to ponder. He then responds to student suggestions with enthusiastic statements such as "a key observation", "I really liked that", showing that he values their contributions.

However, when asking the students to consider what they could do with the two fraction diagrams, he seems interested in getting the *right* answer that he has in mind (in this case, add the fractions). He is quite dismissive of a student's suggestion that they might multiply them:

> (10:33)PF: We've had an idea from Charlie that we could probably multiply them, that's not a bad idea. Erm, wouldn't be talking about common factors, lowest common factors if we were talking about multiplication.

Rather than offer students the time to consider whether and how the diagrams might support multiplication of a third by a quarter, PF explains (possibly erroneously) that they would not be appropriate for supporting multiplication, and asks for further suggestions.

CB

CB is a confident, experienced teacher, and qualified as an 'Advanced Skills Teacher' during the academic year following the main study lesson observations. A large

proportion of CB's lessons are spent on discourse of some kind. Predominantly, this takes the form of CB thinking aloud on some aspect of the curriculum. Students then work independently on questions related to the topic that has been discussed. Rather than demonstrating a method or technique, CB often begins lessons with a challenging question or issue that he then works through methodically and reflectively. He asks students for contributions throughout these discourses, and encourages them to ask 'what if...' style questions, or to challenge the claims he makes.

CB appears to take student contributions seriously. In lesson [14], which is discussed more fully in section 8-I.2, he responds to student explanations of the result of increasing a number by 10% then decreasing it by 10% with comments such as "That's nice, yeah I like that" (04:40) and "Absolutely" (05:02). He then sets students a set of questions including '0.9 x 0.9' (10:02). Having received the correct answer of 0.81 (10:21), he agrees that it is correct, then asks "Let's just see what kind of, can anyone tell me a wrong answer that they wrote down. I'm just interested, but there's nothing wrong with getting it wrong, it just means you had a misdirection of thought" (10:22).

10:39 Student:	Times by eight point one.
10:40 Teacher:	Times by eight point one. Really, really common, I think other people in here would have done that. Any other wrong answers you wrote down? Be proud of these wrong answers.
10:48 Student:	Nought point nought eight one.
10:50 Teacher:	Nought point nought eight one. Oh an extra divided by ten, rather than, I... A new type of weird {?}. Another wrong answer?
11:04 Student:	I put nought point one nine.
11:07 Teacher:	Nought point one nine? [unclear comments]
11:11 Student:	I thought nought point eight one was too easy so I changed it round.
11:16 Teacher:	Too easy [chuckles]. Why did you change it to nought point one nine? Eight one, and then...
11:23 Student:	One nine {?}

11:25 Teacher:	Yeah, explain why the one nine {?}
11:29 Student:	Oh, eight-one plus nineteen is a hundred.
11:32 Teacher:	We're definitely coming on to that, you've got one stage further... [unclear] Erm, I don't know if you're technically wrong or not but decrease by ten percent and then ten percent of your new number.
11:44 Student:	Ok.

[14] CB 10(1) Percentage change

CB is listening sufficiently to spot that the student who offers 0.19 as the answer has "got one stage further" (11:32). By this it seems that CB believes the student to be calculating what percentage would be left after a 10% decrease followed by a 10% increase.

As this extract from lesson [14] shows, there are many aspects of CB's practice that could be considered constructivist. However, his lessons remain predominantly teacher-led. He generally teaches the whole class at once, and his questions progress from easy through to more challenging. When working on exercises, students mostly work on their own, consulting a neighbour where appropriate. In contrast with PF and LR, who appear to aspire to more constructivist practices, CB had spent seven years, at the time of the observations, reflecting on and developing his teaching practice, and appears satisfied that this approach is effective.

KT

KT's teaching combines aspects of the transmission and constructivist approaches. He emphasises the importance to students of expressing their ideas clearly, and values alternative student methods and approaches. Alongside these student-centred practices, he places great emphasis on full 'coverage' of the curriculum, with charts tracking topics that have been 'covered' filling the display boards on classroom walls.

Most lessons begin with students writing the answers to the previous lesson's homework on the board, and explaining their answers.

> 08:27 Teacher: Now today, what we're gonna do is we're gonna go through, erm, some of the facts that we were talking about. And we're gonna try and prove them, ok? It's gonna be quite a, erm, it's gonna be quite a talky lesson. I'm gonna do lots of talking today. What I want, what I want you to do is to listen very carefully and try and follow things as I write them on the board, ok, cos what we're gonna do is we're go through and we're gonna try and prove some of the things that you already know about angles. Things like that. Yes, [student name]?
>
> 09:25 Student: Do you want us to copy everything off the board into our books?
>
> 09:38 Teacher: Yes, I'd like you to – but what I don't want you to do, I don't want you to just copy it. I want you to think about it as you're writing it, try and understand it as you're writing it. So, you might need to add some notes about the things that I say as I go along, ok? Right, erm, once we've done that, then we're gonna go and do an exercise on how to do things, and then we're also gonna go back and add something to our splurge diagram picture thing that we did at the beginning.

[26] KT 8(1) Angles

KT's apparent increased hesitancy in comparison with other teachers is more noticeable in the transcripts than when observing, when the pauses feel like comfortable punctuations in his speech. The splurge diagrams are used to make links between different aspects of a topic, and between topics. These are added to throughout the topic, and frequently reviewed.

6.3 WHO IS TALKING?

What had seemed during the observation and level one analysis to be a reasonably balanced, interactive process, once transcribed, appeared more teacher-dominated than expected. Whilst it was apparent during level one analysis that the teacher dominated class discourse, the extent to which this was true only emerged through

further analysis. A node for teacher talk and one for student talk were created and all

the sources (lessons) were coded for teacher talk and student talk (so all *discourse* was

coded, ignoring comments, description and initial analysis). The percentage of the

discourse that was contributed by the teacher and by the students was then calculated.

Through immersion in the data, including observations of lessons, transcription and

early analysis, a sense had emerged that the culture in many of the lessons was that of

developing general understandings of concepts and procedures. Although I was aware

from classroom observations that the teachers studied did dominate the talk time, the

extent to which this was the case, as revealed through data analysis, was still

surprising. Many of the lessons observed had felt like lessons in which the whole class

was working together to develop a conceptual or procedural generality, and this

feeling of mutual discovery and expression was at odds with the finding that, on

average, almost *eighty-nine percent* of the 'class discourse' was teacher talk.

The key to further developing this analysis was the observation that the classroom

with the highest proportion of student talk, which was lesson [26] with 14.87%, did

not seem to be the one where ideas developed the most collaboratively. In fact, there

did not seem to be a strong correlation between my personal sense of a collaborative

development of generality, and the proportion of teacher and student talk. In order to

explore this aspect of the research question more fully, it therefore became necessary

to examine more closely the source of this sense of collaborative development of

ideas, and how it could be identified through transcription analysis.

This is the number of characters, rather than the number of words, or length of time

spent speaking.

Lesson	Teacher	Year	Set	Prior attainment	Topic		Teacher talk	Student talk
[10]	SJ	11	1	higher	Quadratic formula	Algebra	83.42%	16.58%
[14]	CB	10	1	higher	Percentages	Number	89.30%	10.70%
[26]	KT	8	1	higher	Angles	SSM	82.95%	17.05%
[27]	PF	9	1	higher	Add & subtract fractions	Number	89.91%	10.09%
[34]	LR	10	1	higher	Rational numbers	Number	86.65%	13.35%
[38]	KT	8	1	higher	Sequences	Algebra	79.36%	20.64%
[7]	SJ	10	2	upper middle	Approximation	Number	89.88%	10.12%
[25]	BG	10	2	upper middle	Data	Data	95.26%	4.74%
[31]	CB	8	2	upper middle	Sequences	Algebra	92.20%	7.80%
[35]	PF	7	2	middle	Subtraction	Number	89.88%	10.12%
[5]	CB	11	3	lower middle	Expanding brackets	Algebra	94.61%	5.39%
[12]	SJ	10	3	lower middle	Squares and cubes	Number	86.46%	13.54%
[21]	BG	10	3	lower middle	Order of operations	Number	87.24%	12.76%
[15]	LR	7	4	lower	Area and perimeter	SSM	94.59%	5.41%
[36]	KT	10	4	lower	Angles	SSM	85.94%	14.06%
							88.51%	11.49%

The average teacher talk percentage was 88.51%, with 11.49% student talk. Given

Flanders' (1970) "two-thirds rule", discussed in section 2.6, it is perhaps unsurprising

to find teachers dominating classroom talk time, but the extent of their domination is

remarkable. Although teachers' perceptions of their practice were not a focus of this

study, there is some indication that they were unaware of the extent of their majority

of talk time. For example, in lesson [31], CB suggests to the class that this is a lesson

in which student talk will dominate (line 10:09), though in fact only 7.8% of talk is

student contributions.

10:03 Student:	Three.
10:03 Teacher:	Three, good.
10:06 Student:	Oh, I know. Five.
10:09 Teacher:	Five, brilliant. Saves my voice, this lesson, period five, it's lovely.
10:17 Student:	Seven?

10:17 Teacher:	Yeah!
10:20 Student:	Nine.
10:21 Teacher:	Nine.
10:22 Student:	Eleven?
10:25 Teacher:	Eleven, absolutely.

Lesson [31] CB 8(2) Sequences

The percentage of teacher talk varied between lessons, from a minimum of seventy nine percent to a maximum of ninety five percent. The high percentage in lesson [25] might be explained by the small amount of total talk. Although ninety five percent of the talk was by the teacher (BG), class discourse of any sort takes up a relatively small proportion of the lesson. The first fifteen minutes are spent with students working on the 'starter' questions on the whiteboard, after which students are asked to read a page in the textbook that reminds them how to calculate the mean, mode and median, and explains why one of three averages might be most appropriate for use in different situations.

The lesson with the smallest proportion of teacher talk was lesson [38], in which the teacher (KT) asks his year 8 set 1 to share their understanding of *sequence* and related concepts. Seventy-nine percent of the class discourse in this lesson was teacher talk. The class discourse alternated throughout the lesson between teacher and students, with a student speech act being followed by a speech act from another student only when alternative numerical answers were being offered. Student speech acts in lesson [38] are longer than in many other lessons, with apparent correlation between the length of the teacher prompt and the student response.

19:12 Teacher:	Erm. Can anybody think of the formula for that sequence? What's the formula for that sequence? [student name].
19:38 Student:	Add three to the prev-, the previous term.
19:40 Teacher:	So that's add three to the previous term, that's the term to term rule. Does anybody know what the formula is? [silence] No?

19:54 Student:	*n* equals the term times three.
19:59 Teacher:	Yeah?
20:01 Student:	Is *n* equal to the term times three?
20:03 Teacher:	The term number times...?
20:05 Student:	Three.
20:06 Teacher:	Three, yes.
20:06 Student:	Cos the first term's three, second term's six, third term's nine.
20:10 Teacher:	The formula, which you might have seen written down before, is three *n*. Three times *n*. If *n* is one, if *n*'s two, then three times that number. No? No? Ok, I'm gonna, I'll take that off. Go through that a bit more. Ok? So we add three to the previous term.

The average proportion of teacher talk with top set groups was eighty-five percent, while the average for all other sets was ninety-one percent. Students in sets two to four contributed an average of nine percent to the class discourse, while the average for students in the top set was fifteen percent.

This tendency holds for each individual teacher, with the exception of PF, whose set one and set four lessons both had class discourse with around ninety percent teacher and ten percent student talk. The other five teachers in the study all talked for a smaller proportion of the amount of talking with their top set classes than with their other groups, as shown by the table below.

Lesson	Teacher	Year	Set	Prior attainment	Teacher talk	Student talk
[34]	LR	10	1	higher	86.65%	13.35%
[15]	LR	7	4	lower	94.59%	5.41%
[14]	CB	10	1	higher	89.30%	10.70%
[31]	CB	8	2	upper middle	92.20%	7.80%
[5]	CB	11	3	lower middle	94.61%	5.39%
[10]	SJ	11	1	higher	83.42%	16.58%
[7]	SJ	10	2	upper middle	89.88%	10.12%
[12]	SJ	10	3	lower middle	86.46%	13.54%
[25]	BG	10	2	upper middle	95.26%	4.74%
[21]	BG	10	3	lower middle	87.24%	12.76%
[26]	KT	8	1	higher	82.95%	17.05%
[38]	KT	8	1	higher	79.36%	20.64%
[36]	KT	10	4	lower	85.94%	14.06%
[27]	PF	9	1	higher	89.91%	10.09%
[35]	PF	7	2	middle	89.88%	10.12%

This is a tendency that might merit further investigation. Analysis of the data collected in this small-scale study is not sufficient to show that this is a general tendency of teachers.

Teacher	Average % student talk
CB	7.96
BG	8.75
LR	9.38
PF	10.1
SJ	13.42
KT	17.25

Closer examination of the lessons taught by the two teachers with the highest percentage of student talk (SJ and KT) suggested that there might be less similarity between the characteristics and expectations of student talk in these classrooms than there was between, for example, that of KT and CB, whose lessons exhibited the greatest difference in the ratio of student : teacher talk time. Whilst there are many studies that advocate the use of student discussion as an effective way to improve student understanding and develop mathematical reasoning, there is no reason to expect that more talk of any kind is desirable (Pirie & Schwarzenberger, 1988). The intention of this phase of the analysis was to explore the quality of student speech acts, and to analyse whether any general rules are discernible regarding students' contributions. Quantitative analysis of speech acts reveals rather little of what might be happening for students.

It became apparent that the question "who is expressing generality" is more complex than can be answered quantitatively. The question had originally been conceived in order to distinguish between lessons in which generalities were expressed *at* students, and those in which students expressed the generalities for themselves. What emerges from this phase of the data analysis is the possibility that a further alternative is available, in which the teacher expresses the generalities *with* or *for* the students.

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6.4 WHO IS GENERALISING?

Language provides partial insight into the generality available to students in the mathematics classroom. Just as the researcher's insight is limited through accessing only the words and gestures, but not the thoughts of the speakers, so is the students'. There is no direct link between what is being expressed and what is being thought. In chapters seven and eight, the form of expression of these generalities is more closely examined.

I often feel the urge to write about there being "a sense of ..." or "an idea that ..." in the classroom. A sense for who, and how explicit is this sense? Is the teacher aware that their language might "suggest the idea that ...", and do they consciously choose it for this purpose? If students "pick up" the required conception, to what extent do they do so consciously? Are there occasions when it might be desirable to be more explicit about what students are supposed to be "developing an awareness of"?

Expression lies somewhere between or is a mixture of, the enactive and cognitive modes or worlds, and often precipitates a movement between the two. Through talking, we can move from knowing how to do (enactive) to knowing why it works (cognitive). These three modes of (re)presentation are discussed in more detail in section 7.3. Expression plays various roles in the affective mode also, as overly formal expression might result in 'maths anxiety'.

6.5 JOURNEYING TOWARDS THE GENERAL

Throughout this chapter there has been much discussion of the 'direction' of class discussions. The analogy of travel seems a useful one, and worth further exploration. In analysing the data, I felt I had an appreciation of the 'direction' of the discussion. This was not always flagged up explicitly to students, and I wondered whether they were aware of the purpose of the discussion.

Having discussed the observed teaching traits of main study teachers in section 6.2, and analysed the proportion of teacher and student talk in section 6.3, section 6.4 considered the theoretical possibility of students being cognitively involved without participating in the discussion, or conversely of participating without significant cognitive awareness. The style of this section is different again, and contains my own musings on questions related to the metaphor of teacher-led discussion as a journey towards the general.

How direct is the route?

There may be detours along the route that are designed to point out key mathematical features that will not be explored on today's route. While some students might be inspired by the possibilities for future exciting journeys, it is possible that others, already struggling with the length and challenge of today's journey, will be put off by knowing that there is so much else to explore, and consequently, may 'switch off'. Others yet might not realise that the detour is a detour, confuse it for part of the main journey, wander off down it and get lost.

Who is leading the way?

If students are merely following along the path, head down, attending only to each next step, they may not engage with the decisions being made along the way. Although they appear to reach the destination, they have merely been 'following', 'assenting' rather than 'asserting' and would be unlikely to be able to find their way there again unaided.

What pace is being travelled at?

Do you wait for all to be sure of each step before moving on, or enable some students to explore ahead while others lag behind? How will we know when we've got there? Who will announce the successful completion of the journey? Is it for the teacher to decide what has been achieved, or are students deciding for themselves what is of importance? If students do not feel a sense of achievement at the journey they have made, they may be reluctant to engage in another journey tomorrow.

6.6 CHAPTER SUMMARY

This chapter provides an introduction to the teachers involved in the study, including generalities relating to the classroom culture they create. It also demonstrates the surprising predominance of teacher talk in the main study lessons. It is observed that the answer to the question 'who is generalising?' is not as strongly correlated with who is speaking as might be expected. The class discourse can be regarded as a journey that the whole class participates in to some extent, whether or not they contribute verbally. The question then arises as to the extent to which each student appreciates the generality that forms the 'destination' of this journey, and the factors

that might affect this appreciation. Chapters seven and eight, in addressing the three

research questions, set out to cast light on the complexity of this issue.

CHAPTER 7: TYPES OF GENERALITY

The chapter is intended to offer an insight into the complexity of generality that is present in secondary mathematics classrooms. Having introduced this section of the research findings (7.1), a lesson is described, with six generalisations identified and examined in detail (7.2). In section 7.3 a framework is elaborated that offers descriptive insight into the generalisations encountered. The same lesson is used throughout sections 7.1 – 7.3, in order that the reader may familiarise themselves with the data. This lesson is used to introduce and illustrate the five dimensions of the framework. Section 7.4 shows how the framework can be applied to a different lesson transcript. The chapter findings are summarised in section 7.5. In chapters 8-I and 8-II, two of the identified types of generality, namely those relating to mathematical procedures and to mathematical concepts, are examined in more detail, responding to the second and third research questions.

7.1 INTRODUCTION

This chapter addresses the first of the three research questions that have driven this study: *what generalities are being expressed in secondary mathematics classrooms?* The chapter is intended to offer an insight into the complexity of generality that is present in secondary mathematics classrooms. Alongside continued reflection on my own teaching practice, as described in chapter five, I spent six months observing and recording the lessons of six mathematics teachers in one secondary school. This enabled the creation of a data set of fifty-two recorded lessons in Olympus DSS. These were listened to with the intention of identifying thick descriptions of excerpts from the lessons where generalities were being expressed. However, I very quickly

became aware of two findings that led to an adaptation of the research methods. First, every lesson observed contained numerous sections of transcript that could be considered to be expressions of generality. Second, many of the interesting and more striking distinctions between expressions of generality related to the teachers' or students' use of language, and consequently could be both analysed and shared with a reader most effectively through careful and accurate transcription.

Due to practicalities of time, and with the intention of preserving the breadth and variety of the fifty-two central study lessons, fifteen lessons were transcribed and analysed using Nvivo software. These fifteen were selected from the fifty-two with the intention of retaining the breadth of the central study. To fulfil this aim, lessons were transcribed that were taught by all six of the central study teachers. For each of the six teachers, I selected a lesson from each of key stage 3 and key stage 4, and from classes with higher, lower and average prior attainment where available. The analysis of these lessons enabled me to develop my ideas, and to clarify concepts and theories both through personal reflection, and reference to the literature.

In carrying out this analysis, I began to distinguish between the episodes of discourse in the transcript where generalities were being expressed or 'journeyed towards'. These distinctions were along several dimensions, which proved illuminating in addressing research question one as they highlighted the range of types of generality that were being expressed. In addressing research question 1, I wanted to articulate these distinctions, which had emerged from analysing the data by coding expressions of generality, and from consideration of research literature, in the context of ongoing reflection on my own teaching experience. In order to formalise these dimensions,

with the intention of using them for further analyses, and of sharing them with the reader, a framework for considering 'types' of generality was developed. This distinguishes, for example, between generalities that hold in a particular lesson or for a certain activity, and those that are mathematically necessary. It also distinguishes between generalities that have been 'told' or surface-level 'spotted' and those that can be justified and explained. This latter is not a characteristic of the generality itself, but of the perceptions of the students and teacher considering it. The five dimensions of the framework fall into two groups, as the framework is concerned with both the journey being made towards a particular generalisation, and the nature of the generalisation itself. Three of the dimensions are used to distinguish between aspects of the generality that appears to be underlying the discourse; that which seems to be being generalised about. By its very 'underlying' nature, this is necessarily subject to interpretation. Within the classroom, there may well have been divergence over what the teacher, and the different students, thought they were discussing. The framework is intended to offer insight into possible ambiguities, differences, and the variety of generality that might be said to be 'being discussed' in the observed secondary mathematics classrooms; the coding is not intended to be absolute, nor can it be wholly objective. Two dimensions were developed in order to describe aspects of the journey towards the underlying generality. These relate to the way in which the generality is derived in the particular instance, and the awareness that appears to be being promoted through the discourse.

It is not the separation into categories itself that offers insight into classroom generalisation. Rather, their separation allows for closer examination of the generalities and their expression, and the research questions can thereby be addressed.

There are potentially as many ways of categorising expressions of generality as there are groups of people to undertake their analysis. The possibilities outlined here, which emerged from both theoretical literature and lesson observations, are those that offer the most insight for this study. They might offer a framework for teachers to consider when resolving, or preparing to resolve, classroom tensions.

For reasons of economy of words and clarity, the framework is introduced through the particular example of lesson [10]. This segment from lesson [10] was not the basis from which the framework was developed (emerging as it did from a desire to distinguish between perceived differences between expressions of generality throughout all fifty-two central study lessons), but was chosen as it offers episodes which illustrate the detail of the framework.

7.2 LESSON ANALYSIS

It is intended that the generalities illustrated in the lesson analysis that follows will bring to the reader's mind similar interactions in their own classroom or wider experience. This should enable the development of a shared sense of how the general can feature in mathematics teaching. The aim, then, is not limited to demonstrating the role of generality in one particular mathematics classroom. The intention is also to develop categories of generality that aid future thought and discussions.

Every maths lesson provided opportunities for generalisation. With the intention of illustrating the different types of generality, and different teacher methods of making students aware of them, I focus in this section on a twenty minute lesson segment

from lesson [10]. Six generalities are focussed on that are present in the discourse, which are labelled as *[10]A* through *[10]F* in order to facilitate future reference.

Lesson [10] CB 10(1) Quadratics

This analysis focuses on the first part of the lesson, which involves a twenty minute discussion of the general shape of a quadratic graph. The actual term 'quadratic' is notably absent throughout this sequence of lessons. This may be because the teacher wanted the students to be exploring and thinking for themselves, rather than merely accepting predetermined language and rules. He appears to be encouraging students to think about why certain generalities are true, rather than merely accepting them. For example, after about five minutes of discussion, one student offers, "isn't it, as long as it's got the x squared it'll always be a smiley graph?". The teacher responds with "ah, now you're quoting rules at us, which is fair enough, but what about the people in here who don't want to learn rules and they want to say, well wait a minute...[and think about why]". The issue of when it is more or less beneficial to use the name 'quadratic' whilst encouraging students to develop their own understanding of the related maths is explored further in chapter seven. It is possible that the use of an official term, imposed from the outside, would give students the impression that the rules and methods are also to be imposed in this way.

The lesson began with reference to the homework task, which had involved students choosing any four quadratic equations (although this mathematical term had not been used) from the ten on the whiteboard, and drawing them on a graph. The teacher walked round looking at them, and observed that he could tell which of them were

right. He emphasised that, even though he didn't know which students had chosen which graph, and he wasn't looking carefully at the precise values, he could tell from their shape which of them were right and which were wrong "and we should see why in a second".

Generality [10]A:

> *There is something that all of these equations have in common, and they consequently share a common shape.*

This might be intended to assist students' developing conception of *quadratic*, prior to introduction of the terminology on a future occasion. The teacher directed attention towards this through increasing the example-space, using 'distributed examples', then making statements about *all* the examples looked at. This implies a greater scope for the generality than might be apparent from only looking at four examples.

They then look at the specific quadratic '$x^2 + 2x - 5$', using a table with which they had worked in the previous lesson:

x	-3	-2	-1	0	-1	-2	-3
(1)							
(2)							
(3)							
$x^2 + 2x - 5$							

Teacher	Can anyone tell me what goes in this little column here [indicating cell (1)]. This is if you decided to be one of those people who does all the working bit by bit. Linden, what goes in that bit there? [indicating cell (1)].
Student	Um. x squared.
Teacher	Then what? I'm gonna do it all in one go.
Student	2x, minus 5.
Teacher	Absolutely. You get the two x and the minus five. [Writes these in cells (2) and (3))

Lesson [10] CB 10(1) Quadratics

Generality [10]B:

> *When we're drawing the graph for a formula 'like this', this is a general method that can be used.*

Attention may have been directed towards this through the teacher's referring to specific numbers or expressions with the definite article. It is noteworthy that while the student says only "$2x$, minus 5", the teacher uses the definite article. This may act to emphasise the general and to locate what is being discerned as an object rather than as 'what you write'. Although there isn't always a $2x$ or a minus 5, use of 'the' might imply that we were looking for 'bx' and 'c'. Although $2x$ and -5 are specific examples of this, use of 'the' seems to indicate that they are acting as examples of a generality.

Teacher   The minus 5 bit which, from what I remember was Luke's bit.

Lesson [10] CB 10(1) Quadratics

Generality [10]C:

> *Formulae 'like this' can be broken into chunks, which themselves have general properties.*

All of the chunks that are 'like minus five' have certain general properties. The teacher appears to be directing attention towards this distinction through referring to specific numbers or expressions with reference to a particular student. Going through lots of examples in a previous lesson, Luke had always supplied the 'c' answer. Rather than call this bit 'c', it was being referred to as 'Luke's bit'. This also has the

benefit of student 'ownership' of the maths, which can have positive affective effects

(Maher, 2002; Nesher & Winograd, 1992).

| Teacher | The minus 5 bit will just say minus 5 all the time, and will look like that (pressing reveal on screen to reveal line of −5s). |

Lesson [10] CB 10(1) Quadratics

Is this a particular statement about a specific row in the current problem, or a general

statement along the lines of "the *c* bit will just say *c* all the time". "All the time"

seems to say more here than just 'in all these boxes'. It hints at the generality that this

part is *always* constant. That whatever number they use, it will be constant. The scope

of a generality comes to mind here.

| Teacher | The plus $2x$, won't have very big numbers on it. The plus $2x$ will just have, er 8, 6, 4, 2, 0, -2, -4, -6 and -8 on it. Can anyone tell me what's *special* about these numbers here? (indicating x^2 line). |

Lesson [10] CB 10(1) Quadratics

Generality [10]D:

The number in the x^2 row is always positive.

The teacher suggests with a question that there is a generality about all the numbers.

"Can anyone tell me what's special about these numbers here?".

The teacher is offered the answer that "they're all positive numbers", which he

confirms before emphasising the comparatively large size of the x^2 row of the table.

X		-4	-3	-2	-1	0	1	2	3	4
x^2		16	9	4	1	0	1	4	9	16

$2x$	-8	-6	-4	-2	0	2	4	6	8
-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
$x^2 + 2x - 5$	3	-2	-5	-6	-5	-2	3	10	19

Teacher:	Can anyone describe to me what happens to the numbers as I work along from right to left? The 4 squared is big. [student name].
Student:	Um. There's sort of a symmetrical pattern. As you go along. So, it will go down and then go up.
Teacher:	Absolutely. That's what I meant. It will go down and go up again.
	Because I've said the minus 5 doesn't do much at all, it just takes away 5 from them all, it doesn't really change that fact that [student name]'s said. That it goes go down, then they go up again. And I've also said, this 2x, it's not really that *impressive* a number. 2x just turns out to be 8 for this, 6 for this, 4 for this, 2 for this, it's not all that *big* a number. (2 secs). With me saying that, what must the graph look like? (2 secs) If I'm saying that it's *only* what [student name] said which is important, *only* that top row which is important, what must the graph look like? [Student name]
Student:	Um, like a horseshoe.
Teacher:	Like a horseshoe. And it is actually in a lot of textbooks called a *smiley* graph. And you can get away with calling it a smiley graph.
	x^2 plus 2x minus 5 is a smiley graph.

Lesson [10] CB 10(1) Quadratics

The reference to being able to "get away with" calling it a smiley graph can be taken to be a reference to the GCSE exam. The "it" here must refer to more than just $x^2 + 2x - 5$, and we can assume that it refers to all quadratic graphs with a positive coefficient of x^2. Whilst this is all comparatively clear to post key stage four mathematicians, a student might be justified in wondering why they would want to call "it" anything at all.

Teacher:	I reckon that x^2 plus 3x minus 5 will also be a smiley graph. I also reckon, oh, I didn't go any higher, x^2 plus 5x minus 5 will also be a smiley graph. What about x^2 plus a million x minus 5?

Lesson [10] CB 10(1) Quadratics

Generality [10]E:

Quadratic graphs (always) look like smiley faces.

Attention is drawn to this through looking at extreme particular cases. What would a

quadratic look like with large values of *b* or *c*? Why is it always a smiley face?

Teacher:	Because I was saying this middle row was, you know, not very significant numbers, they're all tiny. Ten, er no 20, er what's that, 15, 10, 5, 0, -5, -10, -15, -20. What if I change that to a million? x squared plus a million x minus 5? [student name]
Student:	It'll probably just be a smiley graph.
Teacher:	It'll probably just be a smiley graph. [student name]? That'll be 4 million, that'll be 3 million, and that'll be 2 million, that'll be 1 million. Maybe it won't be a smiley graph.
Student:	No it wouldn't, would it, 'cause the amount of *x*s is gonna be more than the squared number, so it's gonna make more than itself.
Teacher:	So the squared number will be tiny won't it? It'll be 16, 9, 4, 1, nought.
Student:	Compared to the middle.
Teacher:	The question I'm asking is will x^2 plus a million *x* minus 5 be a smiley graph?
Student:	I think it would be but it wouldn't be as It wouldn't be as kind of like squashed.
Teacher:	Wouldn't be as squashed. So it'll be kind of, like that? So it'll be kind of not like that but like that?
Student:	Um, it will be a smiley graph 'cause it's got the *x* squared. Isn't it, as long as its got the *x* squared it'll always be a smiley graph?
Teacher:	Ah, now you're quoting rules at us, which is fair enough, but what about the people in here who don't want to learn rules and they want to say, well wait a minute, if that was a million, then it doesn't do a nice uppy downy thing. You're actually absolutely right. I can't argue with what you said, it's true.

Lesson [10] CB 10(1) Quadratics

Generality [10]F:

There is (always) value in thinking about why generalities apply, rather than

merely expressing them.

Student:	It is going to be a smiley graph, because it's like (.), it's like different points. You've got the 4, 3, 2, 1, zero, and then back round. See what I mean. If you're timesing it by that certain number then they're all going to be (done?) like that and there's still going to be a gap in between so there's still going to be a smiley.

The student says that the graph is "still going to be a smiley", as if the examples are changing over time. A comparison is being made between the extreme examples and the 'more standard' examples before. How often do students experience mathematical objects being changed over time?

The teacher talks through the example, and emphasises the impact of the term in x^2.

Student:	If it's a smiley face does that mean it's got to start and end at the same point, or can it start lower and then go bigger?
Teacher:	It could, perhaps be like the one I showed you where all you see is that bit.
Student:	No but could it, could it be, like that, where it starts high, goes down, and then doesn't up as high?
Teacher:	Yeah, because you're not seeing the whole, the smiley face doesn't have to be centred on that, that line. So it might be, that it's just this is a bad window and you can't see the whole of it. You've seen this bit, and then you can't see the bit on the side. But that's not a problem. So if it looks like (.) like that [lop-sided smiley face] if you move along a bit it looks like this [smiley face] probably.
Student:	Is it possible to have one like, one that goes like...

Lesson [10] CB 10(1) Quadratics

In this extract, the students and teacher talk about the singular "It could...", whilst seeming to refer to the whole set of quadratic graphs. Having asked , "If it's a smiley face does that mean...", the subsequent discourse is expressed in the singular, although it apparently refers to some sub-set of, or to all, graphs of quadratic functions.

The six generalities offered in this section are not intended to be illustrative or representative of the types of generality, and means of expressing them, that might be identified in all maths lessons. The emphasis in this lesson was on students thinking

for themselves, and developing their own generalisations. They were encouraged to go beyond appreciating generality, and even expressing generality, to having a sense of *why* the generality held. Although this topic may lend itself particularly well to this approach, it seems that many other areas of mathematics could be explored in a similar way. Mason (2002) emphasised this when he said that,

> A lesson without the opportunity for learners to generalise is not a mathematics lesson.
>
> Mason and Johnston-Wilder, 2002: 137

7.3 THE EMERGENT FRAMEWORK FOR GENERALITY

As explained in section 7.1, the six expressions of generality from lesson [10] that were identified in section 7.2 are being offered in this chapter with the intention of explaining the five dimensions of the framework. For the purposes of illustration and explanation, for each of the five framework dimensions that are introduced below, I show which category of the dimension might be applied to each of the expressions of generality discussed in section 7.2 above. In analysing the data, applying these codings to other lessons in the central study, it was clear that not all five dimensions of the framework would be as illuminative as each other for a particular lesson episode. It may be that only one or a few offer insights into any given episode, although which ones will vary according to the particular episode. For the purpose of establishing the nature of the framework, however, in this chapter, all five aspects are addressed for each identified lesson episode.

7.3.1 Subject of generalisation

The subjects of generality in section 7.2 vary considerably, and following the distinctions made in sections 2.1.2-2.1.4 can be categorised as generalisations about

concepts, procedures and behaviours. The distinction between these subjects of generalisation emerged from analysis of the data informed by knowledge of the literature. In the analysis it was helpful to distinguish between mathematical concepts, procedures, and a variety of behaviours. There were a number of generalities which concerned behaviours belonging to a spectrum extending form the purely social to the predominantly mathematical. These I have termed 'behaviour on a social-mathematical spectrum', or where the context is clear, just 'behaviour'. The "socio-mathematical norms" developed by Cobb (1996) and his colleagues, which include mathematical activity with a social dimension (classroom social norms, sociomathematical norms and classroom mathematical practices) and a psychological dimension (beliefs about roles and mathematical activity in school, mathematical beliefs and values, and mathematical conceptions) lie towards the middle of this category.

The literature review had led to the distinguishing of four categories, including mathematical facts (discussed in section 2.1.1) along with the three illustrated here. In analysing the data, however, these factual generalities could be seen as a subset of conceptual or procedural generality. In the majority of the observed lessons mathematical facts appeared to be approached from the viewpoint either of contributing to students' conception of a concept, or as part of a procedure.

It is important to emphasise that these categories are not mutually exclusive. As the example below illustrates, an individual expression of generality can be classified, for instance, as both a concept (lowest common multiple) and as a procedure (how to find the lowest common multiple of two numbers).

> Asked what 'the lowest common multiple of eight and ten' meant, the student answered:
>
> Student: Um, I went up in eights and then went up in tens, and then forty was the first one.
>
> Teacher: Went up in eights. Twenty-four, thirty-two, forty. Then work in tens, ten, twenty, thirty, forty. Forty's in both, well it's the *lowest* one in *both* the lists.

[20] SJ 10(3) Fractions

Despite this lack of exclusivity, it proved beneficial to distinguish between these groups in level 2 analysis, as shown in chapter 8. The same expression of generality might be analysed both as a general procedure and as a general concept, with different insights each time.

Generality [10]A: *There is something that (all of) these equations have in common, and they consequently share a common shape.*	General *concept*: quadratic Quadratics share a common shape.
Generality [10]B: *When we're drawing the graph for a formula 'like this', this is a general method that can (always) be used.*	General *procedure* for drawing graphs of quadratic functions.
Generality [10]C: *Formulae 'like this' can (all) be broken into chunks, which themselves have general properties.*	General *concept*: the constant term '*c*'.
Generality [10]D: x^2 *is (always) positive.*	General *concept*: square numbers
Generality [10]E: *Quadratic graphs (always) look like smiley faces.*	General *concept*: quadratics
Generality [10]F: *There is (always) value in thinking about why generalities apply, rather than merely expressing them.*	General *behaviour*: asking why?

Some generalities, such as *generality C*, which generalises about the concept of a constant term in an algebraic expression, seem reasonably straightforward to classify as conceptual generalities. Others, such as *generality A*, are more difficult to classify. The generality asserts that the graphs the students have been drawing have something in common to do with their shape, could be described as a mathematical fact, but could also be seen as contributing to the conception of quadratic equation/expression.

The generality that 'quadratic graphs [with positive leading coefficient] look like smiley faces' could be considered to be part of the concept of 'quadratic', and thus classified as conceptual generality. In this case, as the term 'quadratic' is not being employed, it seems that the generality is being explored for its own merit. Likewise, this generality could be viewed as a check or tip for drawing quadratic graphs; checking that the finished graph looked like a smiley face could be part of the algorithm for drawing quadratics. It could then be classed as *procedural*. However, the teacher does not emphasise this point. Another utterance that might have been considered to be a mathematical fact is **generality D** which states that whether x is positive or negative, x^2 is always positive. However, the utterance can also be classed as developing the general concept of a square number and of the action of squaring. The decision as to whether the generalisation can most appropriately be classified as a general fact or a general concept seems to depend on the form of the generality. The third generality that might have been considered factual is **generality E**, which generalises that graphs of quadratic functions always look like smiley faces. This I also coded as conceptual. While numerous mathematical concepts are invoked during this lesson, few are referred to by name. One concept that the teacher does appear to be attempting to direct students' attention towards is that of the three distinct terms within a quadratic function, in **generality C**. This lesson illustrates the main finding from the lesson observations regarding mathematical concepts; they are often used or introduced very differently from what might be expected from the literature. Just as this lesson on quadratics does not include use of the word 'quadratic', many central concepts were worked on and apparently developed without being named. The use of the student or teacher's own terminology, rather than that used by the established mathematics community, is also a common feature. In this lesson, a student describes

a parabolic graph as a horseshoe, and the teacher introduces his own preferred expression: smiley face. This may be because it enables them to distinguish between positive and negative coefficients of x^2 more easily (smiley face and sad face), or because they are used to using the term themselves over many years, or for various other reasons. Either way, it is interesting that the teacher introduces the affective metonymy of 'smiley face' in preference to the technical metonymy of 'quadratic', especially as quadratics with negative leading coefficients require an alternative 'scowling' metonymy. Given Davydov's insistence that to understand a concept fully one must know its name (Davydov, 1972/1990), the use and misuse of mathematical terminology is a focus for Chapter 8-II.

Generality B is a procedure for calculating the values for plotting quadratic graphs, and thus can be classified as a procedural generality.

Generality F is a mixture of attitudinal and behavioural, with CB indicating the value of a disposition to think about why mathematical rules work, rather than merely accepting, memorising and applying them. CB asserts that, in general, there is more to be learned and appreciated than just a general rule, there is value in considering *why* that rule always applies.

7.3.2 Awareness of generalisation

Analysis of the data revealed that the classification of generality into that concerned with behaviours, concepts or procedures was insufficient to encompass its full complexity. Each procedure, concept or behaviour that is available for students to

attend to can be brought to students' attention by teachers in various different ways. Some generalities were expressed in a way that encouraged cognitive consideration of their underlying meaning, justification and proof, while others could more fittingly be described as encouraging 'learning through doing'. When discussing the nature of a generality, a distinction is made here between enactive, cognitive and affective generalisation. If enactive generalisation involves 'being able to', while cognitive generalisation is associated with 'knowing', then affective generalisation includes 'wanting to', 'feeling able to', 'being disposed to' (see Kilpatrick *et al.,* 2001).

The term *awareness* is used in many ways. While Mason uses awareness in relation to cognition, I use the term to encompass the three strands or behaviours: cognitive, enactive and affective. I use the word awareness to refer to what informs the response of the person to its environment. These may be manifested physically, emotionally and intellectually, which is more commonly summarised as enactive, affective and cognitive. These three categories have been used in various contexts to describe mathematical activity.

One use of the three ideas is as the pairs of strands in the *structure-of-a-topic* framework, which was originally described in a series of booklets under the title of *Preparing To Teach a Topic* (Griffin and Gates, 1989; see also Mason, 2002c). The pairs of strands can be summarised by the expressions 'harnessing emotion', 'training behaviour', and 'educating awareness', which relate to the ideas of affective, enactive and cognitive awarenesses respectively.

Teachers can *enactively* train students' behaviour, by leading them through a sequence of tasks where the response becomes interiorised, or even a habit or ritual. The student's body may then perceive a generality, through copying or repeated exercises, before they become cognitively aware of it. Mason *et al.* (2005: 282) offer a task that illustrates how our attention can shift from thinking about what we are doing to letting our body almost unconsciously follow a pattern. However, this limited training of student awareness, whilst training students' behaviours, does not fully embrace the possibilities of their full awareness. When one or more of cognitive, enactive and affective dimensions is pushed to the background, student experience is impoverished, and learning made more difficult.

Analysis at level two led to growing realisation that these three aspects of awareness could also be used to consider how generalisations themselves could be cognitive, enactive or affective. Through discussion with John Mason, we therefore extended the thoroughly documented discussion of enactive and cognitive generalisations to include affective generalisation (Mason *et al.* 2007). Where cognitive generalisation involves student's reasoning about the generality and enactive generalisation requires repeated application of a technique, affective generalisation alters students' dispositions to engage in future generalisation. The process of thinking mathematically can be fostered and sustained by the teacher not only demonstrating this way of thinking, but also using techniques to promote it in students. Affective generalisation involves students generalising from particular *ways of working* (Tahta & Brookes, 1966) to enjoying or gaining satisfaction from choosing to use those approaches on future occasions. In Mason *et al.* (2007) we showed how this fits with Vygotsky's language of *zone of proximal development* which refers to actions which

currently usually have to be triggered by the teacher, but which are beginning to be available to the student to initiate use of for themselves (see section 2.7).

All three generalisations can be experienced in different mixtures differently by different students at the same point in the same discussion. While a particular generality may be intended by the teacher to direct students' attention cognitively, some students might generalise enactively and others have a more affective awareness of the generality. Each of these three types of generality can be negative or erroneous as well as positive and appropriate. For example, after 'practising' pages and pages of solving quadratic equations, a student could very easily form an enactive generalisation of an incorrect method or 'bugged procedure' (Brown & van Lehn, 1980; van Lehn, 1989). Affective generalisations can be formed such as 'maths is boring' or 'I can't do this'.

Generality [10]A: *There is something that (all of) these equations have in common, and they consequently share a common shape.*	**Affective**: it is *easy* to check whether quadratics are correct or incorrect because they share a general shape.
Generality [10]B: *When we're drawing the graph for a formula 'like this', this is a general method that can (always) be used.*	**Enactive**: splitting any given quadratic into a table with rows like these can become almost automatic – if you see this, do this.
Generality [10]C: *Formulae 'like this' can (all) be broken into chunks, which themselves have general properties.*	**Affective:** use of name encourages ownership of ideas. **Enactive**: method involves breaking down.
Generality [10]D: x^2 *is (always) positive.*	**Cognitive**: thinking about whether and why this is true.
Generality [10]E: *Quadratic graphs (always) look like smiley faces.*	**Affective**: the affective metonymy of 'smiley faces' is used.
Generality [10]F: *There is (always) value in thinking about why generalities apply, rather than merely expressing them.*	**Affective**: a disposition is encouraged wherein there is a value in considering why things work the way they do.

Generality B might be considered enactive, in that the procedure does not necessarily demand much cognitive attention. Given an equation of the form $y = ...$, students are encouraged by CB to break the equation down into its component parts.

Generality D could be taken to be enactive, with students remembering that any number squared is positive, or even some version of 'a minus times a minus is a plus', without considering cognitively why this might the case.

Generality F is an affective generality that encourages the process of cognitive generalisation. The teacher emphasises that different students work in different ways, and encourages those students who feel it is sufficient to have 'spotted' the rule to think about why it works.

Teacher	Ah, now you're quoting rules at us, which is fair enough, but what about the people in here who don't want to learn rules and they want to say, well wait a minute, if that was a *million*, then it doesn't do a nice uppy downy thing.

In the six identified lesson [10] generalities, there was a misleading tendency for the procedures to be enactive, the concepts cognitive, and the behaviours affective. For example, the *generality B* was classified as a *procedure* in 7.3.1, and as *enactive* in the current section. The *concept* of *generality D* was classified as *cognitive*, and the *behavioural generality F* was classified *affective*.

There are many reasons why these connections could predominate more generally. The link between procedural generality and enactive awareness, for instance, can derive from students being tested on the use of procedural techniques on routine questions. Teachers can consequently be tempted to follow textbooks in

demonstrating techniques using worked examples, and then invite learners to follow the template, or as Gillings (1972) reports from Egyptian papyry of 2000 BCE, to 'do thou likewise'.

It is not necessarily the case, however, that general processes must be appreciated enactively, or that concept awareness involves cognitive awareness. By way of illustration of this point, the table below refers the reader to lesson segments described in chapter eight where concepts and procedures are generalised enactively, cognitively and affectively.

	Concepts	*Processes*
Enactive	Understanding and using the concept of the lowest common multiple of the denominators when adding fractions, although the correct terminology was not used. [27] PF 9(1) Fractions (8-II.2)	Filling in the table of interior and exterior angles of regular polygons. [21] BG 10(2) Interior and exterior angles (8-I.2)
Cognitive	Defining rational and irrational numbers. [34] LR 10(1) Rational and irrational numbers (8-II.3)	Explaining whether you prefer a 10% increase then a 10% decrease or vice versa and why. [14] CB 10(1) Percentage change (8-I.2)
Affective	Praising students for writing the wrong name for the correct angle theorem. [31] CB 8(2) Angles (8-II.2)	Students using and naming their own methods for approximation. [07] SJ 10(2) Approximations (8-I.2)

7.3.3 Derivation of generalisation

The teacher can offer students an opportunity to use reasoning to derive a generality. Alternatively, they could offer students particular cases and invite them to spot a [the] general pattern. Some generalities discussed in the analysed lessons were recalled from previous lessons, while others the teacher simply told the students. CB uses some techniques to move away from telling towards students generalising for themselves.

Generality [10]A: *There is something that (all of) these equations have in common, and they consequently share a common shape.*	***Telling***: the teacher tells the students that there is a common general shape to their graphs.
Generality [10]B: *When we're drawing the graph for a formula 'like this', this is a general method that can (always) be used.*	***Telling***: the method comes from the teacher.
Generality [10]C: *Formulae 'like this' can (all) be broken into chunks, which themselves have general properties.*	***Telling***: this is CB's method for drawing graphs.
Generality [10]D: x^2 *is (always) positive.*	***Pattern-spotting*** or ***recall***: although there are numbers in the table from which students could 'pattern-spot', there are many other things to 'notice' about the numbers.
Generality [10]E: *Quadratic graphs (always) look like smiley faces.*	***Reasoning***: considering extreme examples
Generality [10]F: *There is (always) value in thinking about why generalities apply, rather than merely expressing them.*	***Telling***: CB values justifying the rules, rather than only applying them.

Generality A is told by the teacher, although in such a way that students who notice the generality through pattern-spotting later in the lesson may legitimately believe that they spotted it for themselves. Rather than all students drawing the same four graphs, the teacher has asked them to choose four from ten, to create 'distributed examples' (Watson and Mason, 2005). As well as increasing the example-space, this gives students some ownership over the task, especially as they have some awareness about which would be easier and harder to draw. Increasing the example-space, then making statements about *all* the examples looked at, suggests greater generality, as the scope of the sample-space is a better representation of the scope of the generality in question.

Generality D is to be recalled from a previous lesson, although it could also be pattern-spotting.

Teacher: I reckon that x^2 plus $3x$ minus 5 will also be a smiley graph.

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> I also reckon, oh, I didn't go any higher, x^2 plus 5x minus 5 will also be a smiley graph. What about x^2 plus a million x minus 5?

Lesson [10] CB 10(1) Quadratics

Having established that the particular example of $x^2 + 2x - 5$ makes a smiley face, CB moves very quickly towards the general through other examples of the form $x^2 + bx - 5$. This might have been an interesting opportunity to ask students to suggest graphs 'like this one' that would make a smiley face.

> Teacher: The plus 2x, won't have very big numbers on it. The plus 2x will just have, er 8, 6, 4, 2, 0, -2, -4, -6 and -8 on it. Can anyone tell me what's *special* about these numbers here? (indicating x^2 line)

Lesson [10] CB 10(1) Quadratics

CB directs their attention towards the relative size and significance of the term in x^2 through asking "can anyone tell me what's special about these numbers here?" This question suggests that there is only one 'special' thing about the numbers; the one CB is thinking of himself.

Graphing software might have supported student understanding of **generality E**, where the effect of changing the coefficients a, b and c is considered. The extreme example of $x^2 + 1000000x - 5$ is used to move towards conjecturing about $x^2 + bx - 5$. The intention here might be to direct students towards considering what a quadratic graph would look like with very large values of b or c, so as to offer them an opportunity to appreciate the relative significance of the term in x^2, which accounts for the quadratic graph's 'smiley face' shape.

7.3.4 Justification of generalisation

A further dimension to the generalities in the main study lessons that emerged in level two analysis was whether the generality was mathematically necessary, mathematically conventional, or social. Not all mathematical generalities expressed in the lessons were mathematically necessary. For example, the rule that there are 360° in a full turn is a mathematical convention.

Generality [10]A: *There is something that (all of) these equations have in common, and they consequently share a common shape.*	*Necessary*
Generality [10]B: *When we're drawing the graph for a formula 'like this', this is a general method that can (always) be used.*	*Conventional Behavioural/social*
Generality [10]C: *Formulae 'like this' can (all) be broken into chunks, which themselves have general properties.*	*Necessary*
Generality [10]D: x^2 *is (always) positive.*	*Necessary*
Generality [10]E: *Quadratic graphs (always) look like smiley faces.*	*Necessary*
Generality [10]F: *There is (always) value in thinking about why generalities apply, rather than merely expressing them.*	*Behavioural/social*

Generality A is mathematically necessary: quadratic graphs are in the shape of a 'smiley' (or 'sad') face. In fact, this mathematically necessary generality is the central focus of the entire twenty minute class discourse. From the way CB mentions it, however, students might view it as a special maths teacher skill. Their attention might focus on the 'I' in 'I can tell which are right and which are wrong', thereby supporting a social rather than a mathematical generality.

It is not mathematically necessary to use the procedure in **generality B** to find the values to plot in a quadratic graph, although it is mathematical necessity that ensures that such a procedure works (for the restricted set of equations provided in external examinations). The generality is a convention, one encouraged by the writers of the GCSE exams through the provision of extendable tables, and here encouraged by CB.

It is a mathematical necessity that algebraic expressions can be 'broken' into their constituent terms, and this section of the discourse could be seen to be referring to this. It is, however, conventional to refer to the term that is the subject of **generality C** as the *y-intercept*, or as *c*. CB, however, creates his own classroom convention by referring to it as 'Luke's bit'.

7.3.5 Longevity of generalisation

The generalities [10]A – [10]F selected in this lesson analysis can all be considered as universal generalities. By this it is meant that they can be applied in all situations. However, this is rarely made explicit in the discourse, and it therefore seems reasonable to question the extent to which the students are aware that these generalities apply beyond the particular example, task or lesson.

Generality [10]A: *There is something that (all of) these equations have in common, and they consequently share a common shape.*	*Universal*: shape of quadratic graphs
Generality [10]B: *When we're drawing the graph for a formula 'like this', this is a general method that can (always) be used.*	*Universally applicable method that may only be temporarily employed:* this a method that can be used to plot any graph, but students at A-level and beyond would be encouraged to plot graphs by considering values of the

	coefficients and constant, rather than using a table.
Generality [10]C: *Formulae 'like this' can (all) be broken into chunks, which themselves have general properties.*	**Transient naming of universal concept**: the y-intercept is a universal concept, but the name 'Luke's bit' is unlikely to be used in future lessons.
Generality [10]D: x^2 *is (always) positive.*	**Universal**: property of multiplication of positive and negative numbers.
Generality [10]E: *Quadratic graphs (always) look like smiley faces.*	**Universal**: as with generality [10]A.
Generality [10]F: *There is (always) value in thinking about why generalities apply, rather than merely expressing them.*	**Universal**: there is value in considering the underlying reason behind mathematical rules.

Generality C, where CB refers to the c term of the quadratic as Luke's bit, is a universal general concept, being given a transient name. By referring to this term separately, CB may be offering students an opportunity to develop their conception of the constant. However, he would possibly be surprised if a student referred to the *y-intercept* as *Luke's bit* in another lesson with a different context.

Generality D is an universal generality, as x^2 is positive for any real value of x, always. Given the focus in this lesson on equations of the form $ax^2 + bx + c$, there is the possibility that students may not appreciate the large scope of this generalisation. They might think that the rule only applies in the context of finding values to plot for quadratic graphs or when filling in tables of values.

Distinguishing between transient and universal generalities leads to the question of the extent to which students are accurately able to detect the scope of a given generality. Teachers in the analysed transcripts often made statements that students were intended to interpret as general statements, that applied in more than one particular case, but

which were transient rather than universal. For example, in lesson [21], which is discussed in more detail in section 8-I.2, BG asks the students to write their answers to two decimal places. The students appear to interpret correctly that her intention is for this general rule to apply only in this one lesson, but there seem to be limited linguistic indicators that this is the case.

36:38 Teacher:	Write the sum in your books first and then write the answers, yeah? So that when I'm checking...
37:18 Teacher:	Two decimal places is fine, yeah? Write your answers to two decimal places.
44:31 Teacher:	Erm, just check {?} of your answers so far. Check your answers – two point one nine, four point six eight, three point eight seven five, nought point five, erm, nought point nine two. Tick that so far if you've got them.
49:08 Teacher:	Right, how many of you have not done up to five? Ok, in that case, stop {?} and do these six instead. [general cries of dismay] Erm, just write the answers for those ones, yeah? You don't have to copy it down, just write the answers.

Lesson [21] BG 10(2) Interior and exterior angles

The procedure for drawing the graph of a quadratic is made to appear universally general in two main ways. Firstly, the teacher uses the same table to calculate values for all quadratic graphs, with x values always from -4 to 4, as below:

x	-4	-3	-2	-1	0	1	2	3	4
x^2									
$2x$									
-5									
$x^2 + 2x - 5$									

Secondly, although he talks in particulars, referring to the graph of $x^2 + 2x - 5$ rather than $ax^2 + bx + c$, he uses the definite article in front of the terms, which makes them appear more general. The student says "$2x$, minus 5", but the teacher echoes it as "the $2x$, the minus 5". This use of the definite article may act to emphasise the general. Although there isn't always a $2x$ or a minus 5, use of 'the' might imply that we were

looking for a '*bx*' and a '*c*'. Although 2*x* and –5 are specific examples of this, use of '*the*' seems to indicate that the terms are acting as examples of a generality.

He also emphasises the existence of three generally distinguishable parts to the quadratic (ax^2, *bx* and c) through calling the third of these parts "Luke's bit". Going through lots of examples in a previous lesson, Luke had always supplied the '*c*' answer.

| Teacher | The minus 5 bit will just say minus 5 all the time, and will look like that (pressing *reveal* on screen to reveal line of –5s). |

Lesson [10] CB 10(1) Quadratics

Is this a particular statement about a specific row in the current problem, or a general statement along the lines of "the *c* bit will just say *c* all the time"? "All the time" seems to say more here than just 'in all these boxes'. It hints at the generality that this part is *always* constant. That whatever number they use, it will be constant.

Perhaps by exposing students to the general algebraic form $ax^2 + bx + c$, a distinction could be made less ambiguously between when the particular was representing only the particular, and when it represented the general. This is explored more fully in chapter nine.

7.4 APPLYING THE FRAMEWORK

In this section I demonstrate how the emergent categories of generality described in section 7.3 can be used to provide a framework for analysing generalisations in a

mathematics lesson. Use of the framework offers a structure through which insights

into the nature of the generality can be attained.

Underlying generality	Subject Longevity Justification	procedure, concept or behaviour transient or universal mathematical necessity, conventional or behavioural/social
This given utterance	Derivation Awareness	telling, pattern-spotting or reasoning affective, cognitive or enactive

Lesson [35] PF 7(mixed) Subtraction

PF begins the lesson by clarifying some classroom expectations:

00:29 Teacher: Ok, can we get ready nice and quietly please? There'll be no talking. Books out in front of you, ready to start please. And of course, never anyone swinging on their chairs.

Lesson [35] PF 7(mixed) Subtraction

Generality [35]A:

There will never be anybody swinging on their chairs.

Underlying generality	Subject Longevity Justification	behaviour universal social
This given utterance	Derivation Awareness	telling enactive

The subject of this last general rule is classroom *behaviour*. The teachers' intended

awareness could reasonably be expected to be *enactive*. We may find that this is often

the case with generalities with a behavioural subject. A more cognitive awareness

might be encouraged with a question such as 'why do I ask you not to swing on your

chair?', or 'what might happen if...?'. The utterance can be categorised as *telling*,

although reasons may have been previously offered for why it is a good idea. The

generality is *behavioural/social*, as it forms part of the structure of a functional

classroom. It seems reasonable to assume that the teacher intends the generalisation to

apply *universally*, indeed the use of the term 'never' might be seen as signalling this longevity of relevance. That PF feels the need to reiterate the rule, however, suggests that at least one student is not applying the rule as if it is universal, but rather expects to be reminded in each lesson. What is more, the prompt for PF's utterance was that a student *was* indeed swinging on their chair. Such a statement is actually therefore self-contradicting.

the lesson continues...

PF then asks the class what they had been doing in the previous lesson.

03:23 Student:	We were rounding numbers to make sums easier.
03:30 Teacher:	Fantastic, that's a great way to put it. Erm, we were rounding numbers in order to make calculations a lot easier for us. Erm the kinds of examples we had on the board were things like, we started off with something like four point nine six minus three point eight seven divided by two point something. And we said "Well if you round all those numbers, instead of having hideous calculation, we've got all these decimals and we're trying to divide, you know, four point eight four over six point two one" – we can round all the things together and it makes our life a lot easier. Ok, when we're dealing with just whole numbers, it makes our life a lot easier, it makes the calculating easier. And that's what we want isn't it, really, we wanna make everything a lot easier. That's fantastic. I'm glad we all remember about that.

Lesson [35] PF 7(mixed) Subtraction

Generality [35]B:

PF offers a definition of 'rounding'.

Underlying generality	Subject Longevity Justification	concept, procedure universal mathematical necessity, conventional
This given utterance	Derivation Awareness	telling affective

There is a *mathematically necessary* background to rounding, but it could also be considered to be a *mathematical convention*. Although PF gets a 'reminder' of the

concept that students had been thinking about in the previous lesson, he does not ask

them for a full definition or description of what rounding involves, but instead he *tells*

them what rounding is about. One reason for PF asking the students what they had

studied in the previous lesson, especially given that it is not subsequently directly

linked with the topic of 'subtraction', which is the focus of the current lesson, might

be to emphasise the *universal* nature of the concept. The general concept is described

repeatedly in terms of 'making things easier', and so may be intended as an *affective*

generality that rounding is straightforward. Although two examples are offered of

"hideous calculations" ("something like four point nine six minus three point eight

seven divided by two point something" and "four point eight four over six point two

one"), PF does not work through the process of rounding the numbers and estimating

the answers. In fact, the description offered might more accurately be considered to be

a definition of the concept of estimation or approximation, with rounding used as part

of the approximating *procedure*.

the lesson continues...

PF then explains that the lesson will begin with a magic trick, which will be linked to

addition and subtraction of whole numbers.

04:17 Teacher: And it's gonna be – oohhh, all very excited – erm, and it's gonna
                   be, it's gonna be linked in to the adding and subtracting whole
                   numbers. Ok? Erm, and today I wanna think about adding and
                   subtracting whole numbers, I wanna do it in a nice formal way,
                   using nice columns.

        Lesson [35] PF 7(mixed) Subtraction

Generality [35]C:

Whole numbers should be added and subtracted in a formal way, using columns.

Underlying generality	Subject	**Procedure**: A formal method for adding and subtracting.
		Behaviour: The layout of the workings is being emphasised over the underlying method.
	Longevity	
		Transient: there are many other ways to carry out subtraction.
	Justification	**Conventional** and **behavioural/social**: The accepted traditional method for presenting these calculations.
This given utterance	Derivation	**Telling**: PF wants formal column subtraction.
	Awareness	**Affective**: PF requests use of this procedure, describing it as 'nice'.

Again the students are being *told* the generality, rather than spotting it or deducing it for themselves. On some occasions students will be encouraged to add and subtract integers mentally, on other occasions they will do so using informal pencil and paper methods. PF's stated preference for the formal method is not a universal rule, but this may not be clear from his choice of words. 'Today I wanna think about adding and subtracting whole numbers, I wanna do it in a nice formal way, using nice columns'. Although as a colleague of PF's I interpret his statement is meaning that *today* he wants to do it in a nice formal way using nice columns, this is not what he says literally. The *transient* nature of the preference may consequently not be apparent to the students.

the lesson continues...

05:04 Teacher:	...Erm, what I'd like to do in, in this trick is, the first thing I'd like you to do is choose a number, erm three digit number, ok, all non-zero, erm where –
05:56 Student:	I know...
05:57 Teacher:	Ok, erm, and I'd like you to make sure that the first digit is greater than the last digit. Ok, so you choose a three digit number, say an example could be five hundred and thirty-two. Three digits, the first digit has got to be greater than the last digit, ok. That's the only, only rules we've got for this, this three digit number we've got. So we can't have anything like a hundred and ninety-eight or something like that, that would have been...

...

What I'd like you to do, underneath that number, can you reverse that first number, so write the reverse of that number. Yeah, just write that in the back of your books.

And of course now you should have your first digits smaller than your last digits, hopefully in that second number. If you've done it all right.

Erm, can you then put a line under those two numbers? Ok, erm, put a little subtraction sign to the left hand side of your second number, and do your subtraction please. Could you do the subtraction of those numbers.

Lesson [35] PF 7(mixed) Subtraction

Generality [35]D:

Every three digit number must have non-zero digits, and the first digit must be greater than the last digit.

Underlying generality	Subject Longevity Justification	*Transient*: just for this 'magic trick'. *Social*: rules for this particular trick. *Mathematical necessity*: to ensure that the subtraction results in a positive answer, and that the method involves 'carrying'.
This given utterance	Derivation Awareness	*Telling*: PF tells students the rule. *Cognitive*: Students must think about whether their selected number fits PF's rule.

Generality [35]E:

Write the second number underneath the first, put a line under the two numbers, put a little subtraction sign to the left hand side of the second number.

Underlying generality	Subject Longevity	***Procedure***: how to subtract. ***Universal***: a method that should always be used to subtract?
	Justification	***Conventional***: although there are mathematical (place value) reasons behind the layout.
This given utterance	Derivation	***Telling***: PF describes the traditional procedure.
	Awareness	***Enactive***: the description is of 'doing', rather than 'thinking'.

Generality [D] and [E] are both expressed in terms of what PF would 'like', but further consideration reveals that the two generalities have very different natures. Generality [35]E is an *universal convention* for laying out the standard written method

The value of the framework lies in the process, rather than the product, of its application. Consideration of the five identified aspects of the generality leads to deeper consideration of its complexity.

7.5 CHAPTER SUMMARY

This section provides an overview of the findings relating to the first research question. It summarises the claims made, and indicates the need for new directions. Analysis of the data at level two revealed that a variety of perspectives could be used to describe the nature of a generalisation. These included considering the subject of the underlying generality (procedure, concept, or behaviour), its longevity of

relevance (whether it applies just in the current task or lesson, or is more universal) and its justification (mathematically necessary, conventional or behavioural/social). Generalisations could also be analysed through consideration of their derivation-origin in the given instance (telling, pattern-spotting or reasoning) and the awareness that was being promoted (affective, cognitive or enactive).

Underlying generality	Subject Longevity Justification	procedure, concept or behaviour transient or universal mathematical necessity, conventional or behavioural/social
This given utterance	Derivation Awareness	telling, pattern-spotting or reasoning affective, cognitive or enactive

Application of the framework to expressions of generality in the transcribed data revealed the value of a framework like this in bringing to attention areas of possible mismatch between teacher intention and student experience.

CHAPTER 8-I: PROCEDURAL GENERALITY

This chapter addresses the second research question: *how are procedural generalisations expressed in mathematics classrooms?* As was emphasised and illustrated in chapter seven, in describing the first dimension of the framework, a single utterance could be considered as involving behavioural, conceptual and procedural generalities. The intention is not to separate each category from the others, but to consider procedural and conceptual generalities separately in order to gain greater insight. For the purpose of analysis it became desirable to consider the types of generality separately, focussing in this chapter on procedural generality. This chapter develops from the analytical structures of chapter seven. The meaning and function of this question is considered in section 1.2, while the processes of analysis used to address this question were described in chapter 4.

8-I.1 INTRODUCTION

A substantial proportion of the secondary mathematics curriculum requires students to 'be able to' or 'know how to' carry out various procedures, and it was therefore to be expected that the exposition and practice of these procedures would form a significant part of the lessons observed in phase two of this study. This chapter focuses on the ways in which students and teachers communicate that a procedure used to solve one problem, or part of a problem, can be applied to a general class of similar problems.

Through addressing the research question, my attention shifted from considering how procedural generalisations were being expressed in the main study lessons, to questioning the research finding that teachers did not appear to consider the teaching

of mathematical procedures to involve generalisation. The intention of this chapter is to give the reader sufficient insight into the main study lessons that they appreciate the plausibility, credibility and trustworthiness of the overarching research findings related to research question two:

i)   every observed lesson contained mathematical procedures, each of which was an opportunity to generalise;

ii)   these procedures were rarely explicitly described as general, and the move between the particular and the general was implicit in those cases where a general expression was used at all;

iii)   viewing the teacher-led discourse regarding mathematical procedures as *procedural generalisation* offers insight into the numerous complexities and tensions involved in their introduction and appreciation.

8-I.2 LESSON ANALYSIS

Level three analysis of the lesson transcripts resulted in the identification of numerous episodes focussing on mathematical procedures. In this section, five lessons are used to illustrate how analysis resulted in greater insight into the process of generalising about mathematical procedures. Analysis of main study lessons revealed significant variety and complexity in the expression of general procedures. The purpose of this section is to offer the reader an opportunity to experience that complexity through descriptions of the observed lessons. The five episodes offered here were selected with the intention of reflecting the range of topics (including generalities that focus on number, algebra and geometry), of teachers, and of styles of journey in the fifty-two central study lessons. I explain the similarities and differences between the five lessons, and the central study lessons as a whole, at the end of section 8-I.2. The five

episodes were also selected because, in comparison with other observed lessons, the journeys were comparatively concise, and so could be shared with the reader within word constraints.

In lesson [14] we journey towards the generality of percentage change, focussing on the consequences of following an $n\%$ increase with an $n\%$ decrease, beginning with the particular case where $n = 10$. In lesson [15] we journey towards the general procedure for drawing a rectangle given a fixed perimeter. In lesson [21] our journey concerns interior and exterior angles, and we journey towards the general procedure for finding exterior angles in regular polygons. In lesson [26] we journey towards a general procedure for proving angle theorems such as those concerned with alternate and corresponding angles. In lesson [12] we journey towards the general procedure for finding the square or the cube of a given number. Whilst following these journeys, the framework developed in chapter seven is used as a tool to consider the possibilities and problems with each journey.

[14] CB 10(1) Percentage change

The students had been set a homework task of writing about what would happen to an amount if it was increased by ten percent and then decreased by ten percent.

00:30	Teacher:	Erm, to start off, before I {?} what your answer to... {?} Your answer to a ten percent increase followed by a ten percent decrease {?} ...Erm can anyone read what they've put, they may have started {?} I don't mind at all. Ten percent increase followed by ten percent decrease, you've {?}.
04:13	Student:	Erm you're decreasing ten percent for a hundred tenths so {?}
04:19	Teacher:	Yeah, that makes sense to me.
04:23	Student:	Or, erm, one percent is different in both cases so you can never...
04:31	Teacher:	That makes sense, yes, because you're talking about one percent {?}. Absolutely true.
04:35	Student:	{?}

04:40 Teacher:	That's nice, yeah I like that. After you've increased by ten percent you have a new hundred percent. Can you read me what you put {?}
04:45 Student:	If we increase them by ten percent and then decrease ten percent, we will not end up with the same number cos you, the ten percent that you increase is a {?} smaller number than after you've added ten percent {?} and you end up with less than what you started with.
05:02 Teacher:	Absolutely. That thing about ending up with less than what you started with, I always find really, really hard to work out. I can never work out, I always know "Oh yeah, they're different", I always know that's not the same number you end up with". And then someone says "Is that more or less?" and I think "One of those two. Yeah definitely, definitely either more or less, one of those".

[14] CB 10(1) Percentage change

Underlying generality	Subject Longevity Justification	*procedure* *universal* *mathematical necessity*
This given utterance	Derivation Awareness	*reasoning* *cognitive*

Giving students time to think about a particular case, and to express their thoughts in writing for homework, provides a considered start to this lesson. It is unfortunate that the recording of this discourse is indistinct, so the students' contributions cannot be fully transcribed. Rather than CB simply telling the students the consequences of following an *n*% increase with an *n*% decrease, he gives them the structured opportunity to use *reasoning* to discover this for themselves, thereby encouraging more *cognitive* awareness. As the generality is *mathematically necessary*, it lends itself well to student discovery, in a way that conventional or arbitrary procedures do not (Hewitt, 1999).

[26] KT 8(1) Angles

This lesson involved general procedures for finding missing angles on diagrams, and a general procedure for proving angle theorems. The lesson began with six students

explaining the answers to six homework questions. Their explanations referred to the general procedures for finding angles on a straight line, and in a triangle.

There was considerable consistency between the six students' styles of explaining their methods. They all combined stating the general 'angle fact' used with stating the sums they did in the particular case.

04:16 Student:	**Well there's one hundred and eighty degrees in a triangle,** so ninety add forty is one hundred and thirty, so you need another fifty to make a like proper triangle, so B must be fifty.
04:32 Teacher:	Excellent.
04:32 Student:	**And then cos that's a straight line, that has to have one hundred and eighty degrees in it as well,** so there's forty already there so you need another one hundred and forty to make it a proper triangle.

04:50 Student:	**Um, I did that one first and I needed, you need a hundred and eighty to make the straight line** so I took a hundred and ten away from a hundred and eighty, which is seventy.
05:07 Teacher:	Excellent.
05:08 Student:	And then erm, so I did {?} on seventy **and then a triangle has to be a hundred and eighty.**
05:19 Teacher:	Yep.
05:19 Student:	So I did seventy plus seventy is a hundred and forty and {?}
05:27 Teacher:	Good. So D is forty.

05:42 Student:	**Um I started with, um, knowing that a triangle is a hundred and eighty degrees in it,** so I took the forty to make a hundred and forty, so I know these two, and **I know these two are the same cos it's an isosceles,** so I divided hundred and forty by two to get seventy.
06:00 Teacher:	Excellent. What is it that tells you it's an isosceles triangle?
06:02 Student:	Um because those two are the same.
06:04 Teacher:	Those two marks are the same.
06:05 Student:	Yeah.
06:06 Teacher:	**Two sides are the same, so two angles are the same. Yeah?**

06:21 Student:	Erm, **with F, the opposite side's always the same** so F has to be eighty. And for G, erm, because F and fifty plus eighty is hundred and thirty, um, G has to be fifty **because there are a hundred and eighty degrees in a triangle.**

07:01 Student:	Um, ninety add sixty is a hundred and fifty, **and there have to be a hundred and eighty degrees in a triangle**, so that is... thirty.
07:15 Teacher:	Good.
07:17 Student:	Er, thirty add a hundred and fifty is **a hundred and eighty, cos that's a straight line.**
07:23 Teacher:	Yep. So I is a hundred and fifty.
08:15 Student:	Eighty add sixty is a hundred and forty, so it means J must be forty, erm, **opposite angles are always the same** so that's K is forty, forty add forty is eighty, and L must be a hundred.

[26] KT 8(1) Angles

Underlying generality	Subject Longevity Justification	*procedure* *universal* *mathematical necessity*
This given utterance	Derivation Awareness	*reasoning* *cognitive*

I was struck by the similarity in the discourse between expressions such as "F has to be eighty" (06:21) and "there have to be a hundred and eighty degrees in a triangle" (07:01). Whilst the "have to be" in the second might be taken to indicate a generality, in the first quote this is surely not the case. Without linguistic tools such as this to guide them, students wishing to distinguish between particular and general statements must use non-linguistic indicators.

The fact that the angle sum for a planar triangle is constant, and the convention that it is 180° is used and referred to in five of the explanations, in all of which it is referred to as 'a triangle', making the rule more clearly general, although the potential remains for 'a' to be interpreted as either singular or 'any' (Pimm, 1987). The straight line references appear more specific, with two of the three references being about 'that' straight line, rather than 'a' straight line. Both references to opposite angles use the word 'always', in an unambiguous signal of generality. The table below shows the expressions used by the students when referring to the general rules.

Triangle	(04:16) Well there's one hundred and eighteen degrees in a triangle, (05:08) and then a triangle has to be a hundred and eighty. (05:42) Um I started with, um, knowing that a triangle is a hundred and eighty degrees in it, (06:21) because there are a hundred and eighty degrees in a triangle. (07:01) there have to be a hundred and eighty degrees in a triangle
Straight line	(04:32) And then cos that's a straight line, that has to have one hundred and eighty degrees in it as well, (04:50) Um, I did that one first and I needed, you need a hundred and eighty to make the straight line (07:17) a hundred and eighty, cos that's a straight line.
Opposite angles	(08:15) opposite angles are always the same
Isosceles triangle	(06:21) the opposite side's always the same (05:42) I know these two are the same cos it's an isosceles (06:06) Two sides are the same, so two angles are the same

KT then leads students through a proof that opposite angles are equal, and that alternate angles are equal, before asking the students to write their own proof that corresponding angles are equal.

[15] LR 7(4) Perimeter and area

This lesson began with students asked to draw rectangles with a perimeter of 20cm. LR then tells the class that a farmer has 20m of fencing and wants to make the biggest possible pen for his four sheep. A student conjectures that if the perimeter has to stay constant at 20 metres, then changing the dimensions will not alter the amount of space the sheep have. LR asks the class to raise their hands to show whether or not they agree with this conjecture. They then count squares to find the areas of the 6 by 4 and 7 by 3 rectangles, and find that the conjecture is false.

A general procedure for finding a rectangle with a perimeter is offered by one student.

This is not expanded to a general procedure for drawing rectangles with any given perimeter.

10:52 Teacher:	Has anyone worked out an easier way, or a way you can work out what different sizes of rectangles we can use? Has anyone thought of a way? Erm, [student name], have you?
10:56 Student:	Erm, well we {?} you know you can get two numbers that add up to ten...
11:01 Teacher:	Two numbers that add up to ten.
11:02 Student:	Like seven and three, you could do like, do that and then just like double it. Because like ten is {?}
11:11 Teacher:	What a good idea.
11:14 Student:	Because then six add four makes ten, and then you know you just double it {?} makes twenty.
11:19 Teacher:	Six add four makes ten, excellent, and then another six add four makes ten, so we've just got, we've got two the same again, so six add four is ten so another six add four to make twenty. So those are like complements to ten aren't they? Complements to ten, six and four. [student name], what does add to seven to make ten?
11:38 Student:	Three.
11:39 Teacher:	Three. Good. So let's have a rectangle which is seven and three. Draw that one, see if that one gets a perimeter of twenty. Seven and three, let's try seven and three. Tried seven and three? Try seven and three, seven wide, three high. Ok, we're gonna see if we can draw all the rectangles, all the rectangles we can with a perimeter of twenty.

[15] LR 7(4) Perimeter and area

Underlying generality	Subject Longevity Justification	*procedure* *universal* *mathematical necessity*
This given utterance	Derivation Awareness	*reasoning* *cognitive*

The student's general procedure, begun at 10:56, is very broken up, and consequently difficult for the other students to follow. The tension here is between valuing students' thinking, valuing the clear articulation of their thinking, and valuing their answer. It is important to make clear and distinct time for both of these skills in the mathematics lesson. Mocking students who express their ideas unclearly, or who appear not to understand a procedure because they struggle to discuss it, diminishes the likelihood of creating a conjecturing atmosphere (Mason *et al.*, 1982). But students do need to

learn how to express mathematical ideas, so this must also be seen as valuable. This is supported by giving students time, in pairs, to construct an explanation before sharing it with the rest of the class, modelling explanations (as KT does in lesson [26]), or building time for students to explain to the class (as in lesson [38]).

[21] BG 10(2) Interior and exterior angles, and order of operations

The starter activity for lesson 21 followed on from a previous lesson in which students had found the size of the interior angles of regular polygons. At (00:02), BG defines 'exterior angle' by drawing a diagram of a pentagon and labelling the interior and exterior angles. She then begins to fill in a large table that asks students to calculate (or copy from earlier in their exercise book) the interior angle of regular polygons with from three to 11 sides, and to use this to calculate the exterior angles, and thus the sum of exterior angles. To fill in the row concerning an equilateral triangle, BG draws a triangle on the board, marks one of the interior angles and its corresponding exterior angle, and works through the gaps in the table row as follows:

01:24 Teacher:	... so three, a triangle has three sides, we know it's a hundred and eighty degrees inside a triangle, one exterior angle, one interior angle is 60 because you're dividing it by three, ok? So how big is the exterior angle going to be?
01:47 Student:	I just don't know, erm...
01:49 Teacher:	The interior and exterior add up to?
01:51 Student:	Ah, a hundred and twenty!
01:53 Teacher:	A hundred and?
01:54 Student:	Twenty.
01:54 Student:	Eighty.
01:55 Teacher:	Eighty. A hundred and eighty. Right so that's going to be a hundred and twenty. That's where it was coming from. Ok?

[21] BG 10(2) Interior and exterior angles

Number of sides	Sum of interior angles	Size of each interior angle	Size of each exterior angle	Sum of exterior angles
3	180°	60°	120°	360°
4				
5				

And so on...				

Until the prompt at 01:49, the definition of exterior angle that had been offered did not include reference to the fact that the sum of the interior and exterior angles is 180°. Students were given a brief opportunity to notice this for themselves. BG then directs their attention towards it first by asking them to find the exterior angle of an equilateral triangle, hoping that they will become aware of the sum being 180°, and then by explicitly asking, at 01:49; "The interior and exterior add up to?". At this point, at least one student seems to notice that they sum to 180°, and use this to calculate that the exterior angle is 120°. Perhaps concerned that this will not make sense for those students listening to the thread of discourse, BG does not allow "The interior and exterior add up to?"(01:49) to be followed only by "Ah, a hundred and twenty!" (01:51), but waits until 180° is offered. Instances such as these when a student answers the original question asked when an extra question is inserted were identified as a phenomenon by Davis (1984).

BG then finishes the first row of the table by asking "Then, if one of them is a hundred and twenty, how much will be all three of them?" (01:55). Having received the answer 360°, she states "Ok? This is basically what you've got to do for all these other polygons".

Underlying generality	Subject Longevity Justification	*procedure* *universal* *mathematical necessity*
This given utterance	Derivation Awareness	*pattern-spotting* *enactive*

The general procedures here could be viewed in many different ways. Students are unlikely to have to fill in a table just like this one again, but might still perceive much

of the procedure to be about the filling in of tables, rather than calculation of angles. As well as potentially limiting student awareness to the *enactive*, this might result in students not appreciating the *universal* nature of the generality. Students may generalise from the particular line of the table that deals with triangles to completing other lines of the table through *pattern-spotting* without an awareness that the procedures can be generalised beyond the table. Students might also believe that the patterns are due to the social conventions of the way the table has been set up by BG, rather than appreciating that they are *mathematically necessary*.

[12] SJ 10(3) Squares and cubes

Lesson [12] is the first lesson SJ had with this year ten set three group at the start of their GCSE course. The first ten minutes are therefore taken up with explaining the course structure and other administrative details. The remaining fifty minutes are a 'discussion' of how to find square numbers and cube numbers, with some time spent for students to record the square and cube numbers on matching cards.

Thirteen minutes of discourse is offered in full in this section, as it seemed the most appropriate way to convey the seeming lack of *journeying towards the general* in this lesson. Although there is the potential for students to appreciate numerous procedural generalities, including how to square a number, cube a number, multiply by one, find multiples of nine, and multiply by four through doubling twice, the extent to which these are in fact appreciated is uncertain.

Underlying generality	Subject Longevity Justification	*procedure* *universal* *conventional*
This given utterance	Derivation Awareness	*pattern spotting* It is difficult to identify the

		promotion of any particular aspect of awareness.

The central general procedures in lesson [12] are how to find the square of a number and how to find the cube of a number. Each of these is introduced by SJ asking about particular cases, beginning with one (squared and cubed respectively), then asking for the general method after a few examples. As these students are in year ten, they have been exposed to the procedure for finding squares and cubes in previous years' mathematics lessons. The students offering the correct answers could be seen as working from the general to the particular, in applying their general procedure to SJ's particular numbers. The students who did not seem to recall the general procedure have the opportunity to work from the particular to the general, through spotting what the students offering answers are doing to each number.

Having invited students to create matching cards with all the square numbers between 1^2 and 15^2, and given them time to so do, SJ explains that the six remaining pairs of cards on their sheets are for the cube numbers. She does not give a general definition of this concept. SJ then writes 1^3, 2^3, 3^3, 4^3, 5^3, and 10^3 on the board, and asks the class what should go on the card that matches 1^3.

30:59 Teacher:	Right, what goes in one? [students offer answers of three, two, or one]
31:21 Teacher:	Hands up for one. Hands up for one. Hands up for three. [student name], why did you put your hand up for one? [student name], why did you put your hand up for one?
31:33 Student:	Because one times one times one equals three. One. [laughter]
31:38 Teacher:	Right. [student name] your mistake, as [student name] made earlier on, he said one times one was two, remember, he added it instead of multiplying it. So many of you will get this wrong, and it's –
31:48 Student:	Except me.
31:50 Teacher:	Except [student name] at the back there. Cos you've got one times one is one, one times one again is one. So what's one to the power of anything then? What's that, effectively, what's one to the power of a million?

32:01 Students:	One.
32:02 Student:	A million.
32:04 Students:	One!
32:06 Teacher:	What's one to the power of seven?
32:08 Student:	A million.
32:09 Student:	One.
32:09 Student:	Seven.
32:11 Students:	One!

[12] SJ 10(3) Squares and cubes

The student at 31:33, when asked to explain why 1^3 is one, says that she has calculated one times one times one. This might indicate to other students that the general procedure for finding n^3 is to calculate n times n times n. However, as the general procedure is explained with a particular example, other interpretations are available, including, in this case, multiplying $(n + 2)$ lots of n, or n^2 times n^2 times n^2. At 31:38 SJ confirms this general procedure by directing students' attention towards the possible error of multiplying the number by three instead of multiplying it by itself three times. Again, the procedure is discussed in the very special particular case of one, obscuring important generalisation.

At 31:50 SJ draws students attention to the generality $1^n = 1$, although it seems some students have not been following the discussion, as the answers of seven and a million either suggest that some students are continuing to add rather than multiply, or that they are being deliberately obtuse.

32:13 Teacher:	Are you winding me up now? I hope you are, I don't know if you are, else I'm going to be (?). Right, what about two cubed?
32:21 Student:	Ooh, ooh.
32:21 Student:	Six.
32:23 Student:	Eight.
32:26 Student:	Twelve.
	[general disagreement]
32:38 Teacher:	Right... Think before you shout a number. What sum do I have to do to work out what the answer to two cubed is? What am I working out, in my head, what am I making my brain do, if I say to you "two cubed"? What am I...? I know I've let you shout out

	but some people have put their hand up nicely...{?} I'll have a go at remembering your name – [student name].
33:08 Student:	Alright, would you do two times two times two?
33:10 Teacher:	You do two times two times two. Alright, because it's two to the power of three, so I want three lots of two. So, now then, I know we can all multiply by two. [student name].
33:24 Student:	It's eight.
33:25 Teacher:	It's eight isn't it. We're not adding. Who shouted six? Right, we just need to think about it – we're not adding, and we're not doing two add two add two, and it's – cos I can appreciate why you said six cos... [all talk at once]. So let's think about it, so multiplied by two would be...?
33:51 Student:	Four.
33:52 Teacher:	Four, four multiplied by two would be...?
33:54 Student:	Eight.
33:55 Teacher:	Eight. We have to think when we're doing our cubes or we're gonna get them wrong. Alright, so it takes us a few seconds to think about it to stop us from getting it wrong. [several students talk at once]

[12] SJ 10(3) Squares and cubes

SJ's use of the term 'sum' rather than 'calculation' at 32:38 is ambiguous, as in Mathematical English (see 2.3.1) 'sum' means add. Despite the student having explained that cubing one involved multiplying one by one then by one again, the common misconception that 2^3 is 6 arises in this section, along with the less easily explained suggestion that 2^3 might be 12. At 32:38, SJ directs students' attention towards the general procedure, asking "what am I working out, in my head, what am I making my brain do, if I say to you "two cubed"?". Having received an explanation from a student, SJ amplifies their response with "You do two times two times two. Alright, because it's two to the power of three, so I want three lots of two" (33:10). Her amplification contains the concerning ambiguity, which she explicitly attempts to clarify later in the discourse, as 'three lots of two' signals multiplication of three by two. The general procedure is still being expressed through particular examples, although the 'because' clause of the sentence links the form of the expression "two to the power of three" with the procedure required "three lots of two". Given SJ's

repeated emphasis that 2^3 means 2x2x2 rather than 2+2+2, her use of the expression "three lots of two" is potentially misleading.

34:23 Teacher:	Three cubed.
34:24 Student:	Oh, you're having a laugh.
34:30 Teacher:	Think about it right, think –
34:33 Student:	Nine.
34:34 Teacher:	Three cubed, what am I trying to do? Three times three times three. [all talk at once, volume increasing]
35:11 Teacher:	Guys! Guys! Guys! Hang on a minute! Manners, now, the lot of you. Alright, I appreciate that I have let you call out at some points in my lesson. But I do not appreciate the way some of you have just been acting then. And I do not want to spend the rest of my Maths lessons with you lot telling you how to behave. Am I making myself quite clear now?
36:01 Student:	Yes Miss.
36:03 Teacher:	And has anyone got a problem with it? You know the answer to that?
36:09 Student:	Twenty-seven.
36:10 Teacher:	Twenty-seven, how do you know it's twenty-seven?
36:12 Student:	Cos three times three is nine, times three is twenty-seven.
36:15 Teacher:	Alright. It just takes that few minutes to think about it, doesn't it. three to the power of three means there's three of them; three multiplied by three multiplied by three.
36:27 Student:	Nine.
36:28 Teacher:	Three multiplied – you're at it again. Three multiplied by three.
36:32 Student:	Ohhh! That's what I was doing.
36:34 Student:	Nine.
36:35 Student:	It's twenty-seven.
36:36 Student:	Eighteen.
36:37 Teacher:	Right [student name] – {?} four legs on your chair, four legs on your chair, or you'll be kneeling in my lessons from now on. Right, three multiplied by three, let's count the threes. Three, six, nine, yeah? [student name]?
36:52 Student:	Miss?
36:53 Teacher:	What's three multiplied by three?
36:66 Student:	Uh?
36:56 Teacher:	Three multiplied by three.
36:58 Student:	Nine.
36:59 Teacher:	Nine. I'm at nine now, I've got to multiply it by three again. Nine multiplied by three?
37:06 Student:	Twenty-seven.

[12] SJ 10(3) Squares and cubes

Due to the particular numbers involved in this example, it is difficult to know whether the students offering 'nine' are part way through the calculation 3x3x3, or have completed the inappropriate calculation 3+3+3. Different calculation speeds lead to

answers being shouted out at different times, with some students becoming inattentive

whilst waiting for others to follow the procedure.

37:07	Teacher:	Twenty-seven. You know the trick with your nine times table and your fingers?
37:11	Student:	Yep.
37:11	Student:	No, I don't
37:13	Student:	Yes I do.
37:14	Teacher:	Right, I'll show you then. What you have to do, ok, and I have to go like this because I find it easier that way – if I had to do nine multiplied by three, I would count along my fingers and I would go that's one, two, three, so I would put my third finger down. Which leaves me with... You put your fourth finger down. I'm doing times three.
37:37	Student:	Oh I'm doing times four!
37:38	Teacher:	Well, {?} we're doing three times nine.
37:40	Student:	Oh.
37:41	Teacher:	Right, third finger down, are you watching?
37:45	Student:	Oh yeah!
37:47	Teacher:	Twenty on there, and how many on that side? One, two, three, four, five, six, seven. Twenty-seven.
37:51	Student:	That is so smart!
37:53	Student:	But what if you've got fingers missing?
37:55	Teacher:	Ok, that would be a problem. Perhaps you could just draw some fingers and pretend you had five on one hand. [general chatter]
38:28	Teacher:	So it'll carry on working for any of your multiples of nine, your fingers, ok? So if I've got five times nine, I count to my fifth finger and put it down, that'd give me four and five on that side to give me forty-five. Right, it's just something that's worth knowing. So we were wanting to do nine times three which I think [student name] told us a bit ago would give us the answer of twenty-seven.
38:42	Teacher:	Right. What do you have to know next then? You have to know what four cubed is. And these, I expect for you to know them quite quickly. And that's our aim, to be able to do them quite quickly at the end, ok? Four cubed. How many – right let's think about it. Think about it, I want brains thinking. I don't want silly answers, I want brains in gear, thinking about what it means. If you can't do sixteen multiplied by four, you might want to double it and double it again. Double it and double it again. [general discussion]
40:09	Teacher:	Four times four is sixteen, we know that from our four squared, but we've got to multiply that by four. I appreciate that multiplying by four isn't always – [indistinct aside to a student] – isn't always the easiest thing to do.
40:25	Student:	Times four?
40:26	Teacher:	Times four, so that would be sixteen times four, I actually find it easier if I double it then double it again.
40:32	Student:	{?}
40:33	Teacher:	Do you need to go outside?
40:34	Student:	No.

40:35 Teacher:	Right.
40:39 Student:	It's forty-eight. No, forty-six. Forty-six, it is.
40:43 Teacher:	What's sixteen doubled? Double sixteen for me.
40:45 Student:	Thirty-eight.
40:46 Teacher:	Thirty-two, so double thirty-two for me.
40:46 Student:	It's sixty-four! Sixty-four.
	[discussion]
41:02 Teacher:	Alright, go! [student name], work that out. All of you, go. I've got to pick something up from {?}. I don't want to hear any answers, it's not fifteen, you're adding. I want you working it out now, in your heads.
41:16 Student:	Eighty-six.
41:17 Teacher:	No, in your heads. Thank you.
	[students chat]
41:43 Teacher:	Five times five is twenty-five, twenty-five times five – I find it easier to count in twenty-fives cos I can use my fingers – twenty-five, fifty, seventy-five, a hundred, a hundred and twenty-five is what I get to.

[12] SJ 10(3) Squares and cubes

Having found the value of five cubed, they do not discuss the cube of six, seven, eight

or nine as they are not required by the exam syllabus.

42:13 Teacher:	You're lucky in the fact you don't have to know six, seven, eight and nine. They don't reckon that you have to know them off the top of your head, so one, two, three, four and five you do and you actually do need to know –
42:22 Student:	Are we going to write these down in the front of our books?
42:23 Teacher:	You're going to do this in a second, alright? I just wanna go through them now and we'll, I'll tell you what I want to do with them in a minute. Ten cubed.
42:32 Student:	Three hundred.
42:34 Teacher:	No, and I don't want any calling out, remember. Ok? Ten cubed. What you are doing {?}
	Right what I'm doing is ten times ten times ten. Ten multiplied by ten multiplied by ten.
42:51 Student:	Four hundred.
42:54 Teacher:	Ten multiplied by ten multiplied by ten.
42:56 Student:	Ten multiplied by ten multiplied by ten.
42:59 Student:	Ten times ten is a hundred, times ten is a thousand.
43:02 Student:	Thirty.
43:06 Teacher:	[student name]. [student name]?
43:10 Student:	What?
43:11 Teacher:	Don't add.
43:11 Student:	Oh yeah.
43:16 Teacher:	{?}
43:17 Student:	It's a thousand.
43:19 Teacher:	Ten times ten times ten.
43:21 Student:	I think it is.
43:22 Teacher:	[student name]?
43:22 Student:	One thousand five hundred?

43:28 Teacher:	What's ten times ten? Sarah?
43:30 Student:	Hundred.
43:31 Teacher:	Hundred. Times ten?
43:32 Student:	Thousand.
43:44 Teacher:	A thousand {?} Alright? A thousand.

[12] SJ 10(3) Squares and cubes

Students do not seem to be taking notice of each others' answers. Although, at 42:59, a student offers the explanation and answer "ten times ten is a hundred, times ten is a thousand", students continue to offer suggested answers of 30 and 1500.

Whilst research question two focuses on procedural generality, and although the generalities in lesson [12] can be viewed as procedural, a greater insight can perhaps be gained into this lesson through consideration of those generalisations being made by students regarding classroom culture.

Summary of lesson analysis: similarities and differences

As might be expected, the full data set of fifty recorded lessons contains an enormous variety of styles of journey towards appreciation of general procedures. Level three analysis of the fifteen transcribed lessons developed my sense that while some of the journeys involved the majority of students progressing in their appreciation of the general procedure, others were leaving students behind, or some participants were on very different journeys to others. The five lessons offered in section 8-I.2 were selected to give the reader a sense of this distinction. My impression throughout the stages of analysis was that the discourse in lessons [14] and [26] offered greater opportunity for appreciation of the general than lessons [15], [21] and [12].

The clarity of student explanation in lesson [26], where they explain the reasoning behind their homework answers using angle facts, seems to contrast with the more 'choppy' style of much of the classroom discourse analysed. In lesson [15] the short student responses were considered as one reason for the student's general procedure not seeming to be adopted by other students in the class. The quality of student explanation arose as an issue in lessons [15] and [12], in each of which the teacher seems to be encouraging students to contribute ideas and share ideas, but the students' contributions do not seem conducive to supporting each others' learning. Lesson [15] is marked by a lack of student contributions, in which one student is prompted to explain her approach to an apparently unreceptive class. LR seems nervous of the students not understanding what they have been asked to do. One reason for asking a student to explain, rather than explaining herself, may have been to reassure herself, and the other students, that there is a student present who has followed LR's instructions, so the task is possible. Lesson [21] appears to reduce to a table filling exercise in which students are working with the grain, and missing opportunities to work across the grain (Watson, 2000).

8-I.3 FINDINGS

The review of the literature offered a variety of distinctions between approaches to teaching general procedures. Level three analysis involved applying these well-established distinctions to the transcribed lessons in pursuit of an explanation for the differences between the journeys. This section explains how these distinctions seemed insufficient to account for the significant differences in character of the five journeys towards generality explored in section 8-I.2.

In section 2.1.3 teaching was separated into deductive or inductive, while student understanding was categorised as either computational or conceptual. Both teaching and learning were seen to be focusing on either a structural or an empirical appreciation. These theoretical categories and definitions may prove useful for teachers in planning lessons, or to researchers examining certain aspects of teaching practice, but their contribution to accounting for the success or otherwise of a journey towards the general seems limited.

In accounting for differences between the lesson episodes, it is difficult to assess whether student understanding is computational or conceptual, structural or empirical. As discussed in section 2.2, it is possible for students to have procedural knowledge of how to divide two fractions but have poor conceptual knowledge of either fractions or division (Graeber, 1999). One way in which teachers encourage students to attain conceptual rather than merely computational knowledge is to promote structural generalisation, rather than empirical (this distinction was first introduced in section 2.2). This takes place through teachers directing students' attention to underlying meanings, structures or procedures involved in the particular cases, rather than towards form. In the five lessons explored in section 8-I.2, one might argue that lessons [14] and [26] seem more effective in developing students' conceptual knowledge than lessons [15], [21] or [12]. However, this distinction does not seem to take us any nearer to understanding why this might be the case.

One possibility, again outlined in chapter two, is that one way of promoting conceptual understanding is to offer students the opportunity to discover or at least to express the generality for themselves. The distinction between inductive and

deductive teaching methods for general procedures was described in section 2.2. Any given general procedure can be approached from either of two directions: beginning with the general or beginning with the particular. As discussed in section 2.2, teachers might teach deductively or inductively: teaching rules first and applications later or offering examples from which learners use their own powers with guidance.

Two of the lessons examined in section 8-I.2 might be defined as having an inductive approach to teaching general rules. In lesson [21], BG offers students a table to complete that gives them the opportunity to 'discover' that the sum of exterior angles is 360°. In lesson [14], CB uses an inductive method to generalise about repeated percentage change. Although these could both be classified as inductive approaches, they are qualitatively different.

Many of the general procedures identified in 7.3 were introduced in a way that does not appear to fit into either of these categories. The teacher does *tell* the students the generality (a characteristic of the deductive approach), but they do so having worked through various particular examples. In a sense, the teacher could be described as *modelling* the inductive approach. In accounting for the difference, in terms of apparent student understanding and involvement, between lesson segments, the distinction between deductive and inductive approaches does not seem critical.

Structural understanding does seem to be being encouraged in lessons [14] and [26] more effectively than in the other three episodes described. This might be partially attributable to the teachers' emphasis on proof and justification. Students' cognitive

awareness was being promoted through challenging them to explain why the rules worked the way they did.

Another possible contributory factor is that the students in lesson [14] had been asked to think about the topic for discussion for homework, and had written explanations of their thinking that may have supported them in participating in the discussion. Lesson [26] also began with student contributions based on homework, which led into the topic of the lesson. This student reflection need not have taken place at home, but offering students time to think independently before discussion might account for the qualitative differences in the discourses.

8-I.3.1 A role for algebra?

In this chapter, with the emphasis of analysis on the expression of procedural generality, it seems relevant to consider whether algebra might have been a useful tool for describing the general procedure in this case.

Each of the five lessons described in this chapter were concerned with generalisations that could be expressed using elementary algebra. In lesson [14] the teacher directs the students to offer particular cases where an $n\%$ increase is followed with an $n\%$ decrease, beginning with the particular case where $n = 10$. In lesson [15] the different 'ways of seeing' the perimeter of a rectangle could have been expressed in general, for example, using l to stand for the length and w for the width. These might include $2l + 2w$, $l + w + l + w$, or $2(l + w)$. In lesson [21] the findings from the table of interior and exterior angles could have been expressed, using e to stand for the exterior, and i for the interior angle. In an n-sided shape, $i + e = 180$, $e = 360/n$ and so on. In lesson

[26] algebra could be used to express the alternate and corresponding angle theorems, facilitating proof. In lesson [12] the terms 'square' and 'cube' might be defined less ambiguously through stating that '$n^2 = n \times n$' and $n^3 = n \times n \times n$.

Through consideration of the open and exploratory research question: *'How are procedural generalisations expressed in mathematics classrooms?'*, questions have emerged concerning the possible advantages of being more explicit about the *general* nature of the procedures being taught and used in mathematics classrooms. In particular, an awareness that many of the procedures on the secondary mathematics curriculum can be expressed using secondary level algebra appears to offer potential both for enhancing awareness of the general procedures themselves, and for using algebra purposefully (Ainley *et al.*, 2005).

8-I.4 CHAPTER SUMMARY

A lesson where mathematical procedures are being taught and learnt is necessarily one in which generality is being expressed. In this chapter, the complexities associated with expressing general procedures have been exposed. In order to highlight the complexity and variety of general procedures, I have offered detailed episodes from five of the fifty-two main study lessons.

Mathematically necessary procedures can be deduced by students themselves using their reasoning powers. This promotes cognitive awareness. Teachers can offer several examples, and ask students to express what is the same and what is different, or simply ask them what they notice about the examples offered. Where the procedure is revealed to students, rather than students deriving it for themselves, an alternative

possibility for supporting cognitive awareness of the generality might be to invite students to express the procedure algebraically.

CHAPTER 8-II: CONCEPTUAL GENERALITY

This chapter addresses the third research question: *how are conceptual generalisations expressed in mathematics classrooms?* Having introduced the themes of the chapter (8-II.1), Section 8-II.2 reports on analysis concerning the way in which new mathematical terminology is introduced. Section 8-II.3 explores the extent to which new concepts introduced by teachers are used subsequently by teachers and students to communicate ideas. In section 8-II.4 the development of the meaning and scope of general concepts is examined. The chapter findings are summarised in section 8-II.5.

8-II.1 INTRODUCTION

Reference to any mathematical concept requires generalisation, as a particular mathematical object is being seen as a member of a set of similar objects. Mathematics has developed an elaborate technical vocabulary, the use of which has been subject to much attention (e.g. Durkin and Shire 1991, Pimm 1987). The British government provides schools with a comprehensive list that year 7 (10-11 year old) students are required to "use, read and write, spelling correctly" (DfEE, 2001). The United Kingdom National Strategy for Key Stage 3 (DfEE, 2001) expects students to 'use', 'understand', 'distinguish between' and 'define' a large number of mathematical concepts.

These mathematical concepts are sometimes, but not always, formally named. For example, teachers talking about fractions often refer to the 'number on top', or even 'the top' perhaps because it almost as quick as saying 'numerator'. Both expressions are intended to indicate a generality. The teacher cannot be sure, however, that

students view the word or phrase as general rather than particular. The example of

numerator is unusual in that an approximate definition of the mathematical term (top

number) is hardly longer than the term itself. If the definition feels too cumbersome to

use, as might be the case with terms such as 'quadratic', or 'rational number', teachers

must choose between the general mathematical word, and a specific numeric example.

However often, as in the case of denominator (which 'denominates' or signals the

sizes of portions) and numerator (which enumerates the denominated portions, saying

how many there are), these mathematical terms offer some insight into meaning and

use. This chapter focuses on the comparative advantages of using ordinary English

and mathematical English, the factors that might influence the decision to use one or

the other, and the influence that they have on learners.

Alongside use of the concept's name, a multitude of options are available for the

expression of conceptual generality, of which some are exemplified in the following

list:

	for example...
Mathematical name	decimal fraction
Particular example	0.76
Colloquial name	pointy number
Placeholder name	thingy
Definition	proper fraction whose denominator is a power of 10

Often a combination of these terms will be used. "Is the answer a pointy number

thingy like 0.76?". Arguably the student who asks this question has an emerging

concept of 'decimal', although in this instance he or she is unaware of the option to,

or chooses not to, use the mathematical name.

A student whose concept of *decimal fraction* is not fully developed might be unclear about the specific numerical examples that would or would not be encompassed under the term. As discussed in chapter 2, I use Watson & Mason's (2005) expression *dimensions of possible variation*, adapted from Marton & Booth (1997).

There are several definitions available for decimal fraction. We could define it as:

[A] A decimal fraction is a fraction where the denominator (the bottom number) is a power of ten (such as 10, 100, 1000, etc).
www.mathsisfun.com

[B] A proper fraction whose denominator is a power of 10.
www.thefreedictionary.com

[C] A fraction in which tenths, hundredths, thousandths, etc are written in figures after a decimal point which follows the figure or figures expressing whole numbers, eg 0.5 = five tenths; or ½.
www.allwords.com

[D] Any number written in the form: an integer followed by a decimal point followed by a (possibly infinite) string of digits.
www.answers.com

There are some interesting differences between these definitions. Definition [D] does not include the requirement that the fraction be a proper fraction (and so includes fractions of magnitude greater than 1). The definitions focus on the way the fraction is written, but whilst [A] and [B] are written as fractions, [C] and [D] focus on the decimal form.

Analysis of the main study lessons and levels 1 and 2 highlighted three issues pertinent to the expression of conceptual generality in mathematics classrooms:

- When is the mathematical name used (and by whom)?

- How is the general definition expressed?

- How do students come to appreciate the dimensions of possible variation of a concept?

These relate to Davydov's (1972/1990) three criteria for understanding a concept, which will be used as a framework for the analysis addressed in this chapter.

Segments from four different lessons are discussed in this chapter. These are not intended to be representative of the fifty-two central study lessons. Rather, each excerpt is one that particularly struck me during the analysis, and illustrates the findings of level 3 analysis. In some cases the episode offered a particularly concise example of a phenomenon that was noted repeatedly in the fifty-two central study lessons. In others cases the episode was striking for being unusual in comparison with the other observed lessons. The rationale for including each of the four episodes is explained as each particular extract is introduced.

8-II.2 INTRODUCING CONCEPTS: NEW NAMING

The most obvious way of referring to a mathematical concept, one might think, would be to name it. When analysing the fifty-two classroom transcripts, coding for expressions of conceptual generality, it was somewhat surprising to find that names were not used as much as might be expected when expressing conceptual generalities. Of Davydov's three criteria for understanding a concept, introduced in sections 2.1.2 and 2.5, the criterion that the concept should be associated with a particular term, referred to as 'naming', consequently became a particular focus. I searched for instances of either teacher or students using specifically mathematical terminology, such as 'irrational number' or 'sequence' and considered possible reasons for the use of that particular word. I also noticed occasions when such a word *could* have been used, as the concept was being referred to, but the terminology was not mentioned.

The commonly accepted use of a word is to refer to a concept whose meaning is taken-as-shared. The term 'taken-as-shared' is attributed to Schutz (1962) who coined the term taken-to-be-shared, later adjusted to taken-as-shared by Streeck (1979). It refers to the meaning that is thought to be shared by others. This is not to deny that individuals hold their own unique meanings, merely to insist that communication is made possible only if a meaning is negotiated that is thought to be shared by others. Cobb, Yackel & Wood (1990) describe how classroom discussion is an opportunity for taken-as-shared meanings to emerge in the classroom, and that these meanings can become the shared mathematical meanings of society. Following Davydov, it would therefore be expected that names of mathematical concepts would be used, the scope and meaning of which were appreciated by all participants in the discussion.

Based on the main study data, it is argued that a word can 'make sense' even when unfamiliar, if it is used to point to a specific and indicate that it represents a more general case. This happened particularly in those classrooms where mathematical meaning was being negotiated and developed by students and teachers. There was a sense that precise definitions were not always valued. Words were used as tools; a means to the end of understanding.

The analysis focussed on those parts of lessons where a new concept was being introduced. Several examples were witnessed, as expected, where a new word was introduced with a specific example, or several, or a definition. On other occasions, a concept was discussed for some time before a new piece of vocabulary was attached to it. Cases were also observed where teachers appeared to have decided that a conception need not be named at all. A teacher emphasising that 5 + 7 is the same as 7

+ 5, and so $b + a$ is the same as $a + b$, appeared to be developing students' conceptions of the commutative law, although the term was not mentioned. The UK National Strategy (DfEE, 2001) explicitly discourages teachers from using this terminology, apparently due to its perceived complexity.

In observed lessons, there was a relatively high frequency of interactions where words were being used in the classroom apparently with the purpose of communication, but where context or subsequent discussion made it clear that only the speaker (and maybe a very few of the listeners) could understand the term being used. The word, then, was being given for a concept that either students had no conception of, or if they did have conception, it was not yet related to the word being used. I found examples of various ways in which these initially 'meaningless' words were discussed in order to link them to previous conceptions, or build new conceptions. These are discussed below.

The statement, "we're going to study quadratics, let's start with $3x^2 + 2x$" is different from "we're going to study $3x^2 + 2x$." If the word 'quadratic' does not yet have a reference for students, then the meaning of the two sentences is arguably identical. The sense of the first, however, emphasises that the particular case of '$3x^2 + 2x$' represents some kind of general concept.

Analysis revealed that in the lessons observed, teachers often seemed to use words with no shared reference, words that students did not yet 'have a concept' of. The development of a sense of 'how general' is crucial to students' appreciation of concepts. A full appreciation of the general requires more than realising that a concept

extends beyond the particular. One is also required to know the range of permissible change of the concept. Students must know the precise scope of the generality. As Davydov (1972/1990) argued, for a concept to be understood, the class of objects that the concept contains must be unambiguously distinguishable from any others. The provision of a definition is related to this later stage. If a definition of the general is provided before there is any sense that a general case exists, it is effectively meaningless. Arguing along these lines, Gattegno insists that meaning is antecedent to speech, and that, "Speech can come only after we have grasped the existence of meanings" (1970: 18).

This suggests that mathematical vocabulary is of use only once students have fully developed a conception. In the classrooms I observed, however, teachers often use such technical language in the very early stages of students beginning to develop a concept.

Aware of the view that words "makes sense to us only when they are familiar, when we recognise them from experience" (Mason 2002a: 47) I wondered whether it was really the case that all these new words were making no sense to students. This led me to distinguish between the sense that a word has alone, without context (Frege would have referred to this as the 'referent'), and the sense that a word can make at a given time, in a different place. Prior experience *is* required for us to recognise a concept through its associated word alone. A recently introduced word, however, can make sense if it is accompanied by a particular example. Understanding the word without any such particular example is Davydov's (1972/1990) third condition for 'having' a concept.

In the lessons observed, words were often used, either by teachers or students, which, according to Davydov's (1972/1990) definition, not everyone present 'had a concept' of, but this does not make them meaningless. Technical language may be used in the classroom, with sense-giving examples, to limit the possibility of students not knowing what the teacher is talking about. There is an assumption that if the word is used again in another lesson, it will be again be defined or exemplified. Although this repeated, exemplified use might enable students to discern the scope, and hence understand the concept, that the word relates to, there is a danger that students merely rely on the teacher to exemplify or explain mathematical words, and so there is less motivation to try to engage with its meaning.

This leads me to consider the introduction of mathematical concepts within the framework of scaffolding (Wood, Bruner and Ross, 1976). Although their definition of the term emphasised the teaching of methods and problem solving, the idea also seems applicable to the understanding of concepts. When a word is initially introduced it might be accompanied by explanation and example, but over time the scaffolding is reduced, with the intention that students can make sense of the word alone. In the same way as a mother working with a child might not want to "stifle his performance by doing too much herself" (Wood and Middleton 1975: 182), a teacher must avoid students becoming over-reliant on their explanations and examples of concepts.

The analysis of teacher-led discourse in the main study revealed that mathematical

vocabulary can fulfil an essential role very early on in the development of a general

concept, even when its scope, and hence definition, is incomplete.

> At an elementary level, 'multiplication makes bigger' expresses a valid
> generalisation about the operation of multiplication when applied to whole
> numbers. When the notion is extended, and the same words and symbols
> are applied metaphorically to the new situation (either to fractions or to
> negative numbers, for example), this observation makes sense, but it is no
> longer true. The failure of a valid generalisation, one which accords with
> the everyday connotations of the word multiply, can result from not
> perceiving the novel use to which the words are being put.
>
> Pimm 1987: 9

This seems to me to come down to *levels* of generality. A student's understanding of

number is not just a definition, in fact it may not include a definition at all, as with

Tall and Vinner's concept image (1981). While the concept definition is "a form of

words used to specify that concept" (ibid: 152), the concept image includes all mental

images and associations with the concept. Seemingly conflicting concept images may

be evoked at different times. The word *number* would be related to a collection of

general statements and images of the nature 'the integers 1, 2, 3...are all numbers', or

Pimm's example, 'all numbers are either odd or even'.

Vygotsky (1965) provides experimental evidence that meanings of words undergo

evolution during childhood, and defines the basic steps in that evolution.

> At any age, a concept embodied in a word represents an act of
> generalization. But word meanings evolve. When a new word has been
> learned by the child, its development is barely starting; the word at first is
> a generalization of the most primitive type; as the child's intellect
> develops, it is replaced by generalizations of a higher and higher type – a
> process that leads in the end to the formation of true concepts.
>
> Vygotsky 1965, 83

8-II.3 USING CONCEPTS: NEEDLESS AND NONSENSICAL NAMING

The linguistic literature reviewed in section 2.3 suggested that the closer the introduction of language in the classroom to that in the real world, the more effective it is likely to be. This leads me to believe that one of the practices observed could be improved upon. This is the introduction of a new concept which, though related to the topic under discussion, is not used to communicate the ideas in question.

[05] CB 11(3) Factorising

Lesson [05] is chosen as it illustrates effectively and concisely a phenomenon that occurred frequently across the fifty-two central study lessons; teachers repeatedly using mathematical vocabulary immediately accompanied by its definition.

In lesson [5] CB introduces the students to the term *factorise*:

21:16 Teacher:	But really, the ultimate skill to get you the grade you're aiming for, that's the people who are doing foundation paper, if you wanna get a D, you're gonna have to learn to do something called <u>factorising</u>. People who are doing the intermediate paper, if you wanna get a C, there is no way you're gonna get that C unless you can do something called <u>factorising</u>. And <u>factorise</u> means when you put the brackets back in. If I give you those seven problems, which you've got written on your page, can you work out what they came from? Can you work out what number went out the front of them, or for one of them, which one had a letter out the front of it?

[5] CB 11(3) Factorising

This is a year eleven class preparing for their GCSEs, and so this is almost certainly not the first time they have used the term *factorise*. CB's use of language, however, suggests that the term is completely new: "something called factorising".

Throughout the remainder of the lesson, however, every use of the word is accompanied by a definition.

22:01 Teacher: And so without me teaching you at all, can you write down, can you <u>factorise</u> those seven expressions? Can you get them back to having brackets in them?

[5] CB 11(3) Factorising

Perhaps CB is not yet confident that the term has been adequately defined. He defines it again once students have had time, perhaps, to begin to develop a conception of the term.

26:20 Teacher: Guys if you've finished, <u>factorise</u> means put brackets back in, multiply out means get rid of brackets.

[5] CB 11(3) Factorising

At the end of the lesson, CB uses the term *factorise* twice *before* defining it again.

54:33 Teacher: Um, next lesson we're gonna do something similar, I'm amazed actually. I am absolutely flabbergasted by the fact that I've come round and watched you <u>factorise</u> expressions which are C-grade expressions. I've come round and just watched you do it and think 'Oh God, I'm gonna have to help her on that one', and then I've watched you <u>factorise</u> stuff which I did not think you'd be able to do. You've learnt what factorise means, <u>factorise means put back in brackets</u>, but you can do it, that's really, really impressive.

[5] CB 11(3) Factorising

This occurs where one of the teacher's objectives for the lesson, stated or otherwise, is that students should 'use', 'understand', 'distinguish between' or 'define' (DfEE, 2001) certain concepts. As a result, the concept is named, explained, exemplified, and then perhaps discarded.

Naming allows you to talk about things. If you don't have a need to talk about something, there is no need to name it. Arguably, this argument could be expanded to include thinking, rather than merely talking, but the idea of 'needing' a name is nevertheless a central one. Pimm warns that despite its advantages for classification

and discussion, if the naming loses its purpose, "the process degenerates into a sort of feeble natural history" (Pimm, 1995: xiii).

> Just as with communicative language teaching, for communicative mathematics teaching, the pupils doing the communicating must have something they wish to express.
>
> <div align="right">Pimm, 1987: 202-3</div>

Analysis of the full set of main study lessons suggested that an awareness of Gattegno's (1970) recommendation concerning the need to develop meaning before terminology (discussed in section 8-II.2) is worth heeding. Perhaps because the school's schemes of work frequently require that students 'understand' a particular term, the main study teachers frequently introduced a definition and examples of a concept in a lesson, even when the concept name was not to be referred to subsequently. There were several examples of such arbitrary naming in the lessons observed. In lesson [20], for instance, the particular fraction 24/5 is on the whiteboard and the following conversation takes place:

13:18	Teacher:	Posh word for the top number?
13:20	Student A:	Nominator.
13:21	Student B:	Noooom
13:22	Student C:	Numerator.
13:24	Teacher:	Numerator. And denominator. So we need to remember those.

[20] SJ 10(3) Fractions

In the subsequent discussion of a method for converting from top-heavy fractions to mixed numbers, the technical vocabulary of numerator and denominator was not used at all. Instead, all discussion concerned the example of 24/5, with 'twenty-four', 'four' and 'five' functioning as specific labels for general concepts. In some ways these names are more effective. Confusion does arise though, as the equivalent mixed number ($4\frac{4}{5}$) involves two different fours, leading to ambiguity.

14:31 Student D:	Can you see how many fives go into twenty-four, and then the remainder goes on top?
14:35 Teacher:	Right.
14:36 Student E:	Oh yeah.
14:37 Teacher:	Ok? How many whole fives go into twenty-four?
14:39 Students:	Four.
14:40 Teacher:	With remainder what?
14:42 Students:	Four.
14:43 Teacher:	Remainder four. So I've done twenty four, divided by five, which gives me //
14:45 Student F:	// It would be five, the answer, wouldn't it?
14:46 Teacher:	Four remainder four. So what's my answer got to be?
14:47 Student F:	Five.
14:48 Student G:	Twenty.
14:49 Teacher:	How many whole ones have I got?
14:51 Students:	Four.
14:52 Student F:	Four, so four goes into four once. So it'd be five. Or is that not //
14:54 Teacher:	// I think you've gone //
14:55 Student H:	// Was I right?
14:57 Student I:	Miss, it's four and four fifths.
14:59 Teacher:	Right. Yeah?
15:01 Student F:	Oh yeah.
15:02 Teacher:	Happy with that?

[20] SJ 10(3) Fractions

Having introduced the terms 'numerator' and 'denominator', they are completely ignored. Perhaps this was because there was some hesitation from students when asked for the word, or because it is easier to discuss with specifics. If the latter were the case, however, the question arises as to why the technical terms were introduced at all. There seems to be a degree of confusion amongst the students over this explanation, which may or may not be reduced by use of the general concepts. If you read this dialogue through, replacing the specific numbers with your own choice of general term (eg. numerator, whole number), does the explanation appear more or less confusing?

Why introduce general terms if you are then going to talk in particulars? The benefit of the terminology to empower students in communicating mathematically is lost if they do not use them. It is important for teachers to think carefully about when to

introduce or define terms, to ensure that they are seen as a tool to expression and understanding, rather than merely 'extra things to learn'.

[07] SJ 10(2) Approximations

As with the excerpt from lesson [05] above, this episode from lesson [07] is chosen as a succinct example of a phenomenon repeated across the central study lessons: the use of 'made up' terms to refer to mathematical concepts or procedures. General mathematical procedures that are alluded to in lesson [07] involve using approximation both to find exact answers, and to check answers found on a calculator. The general procedures are not expressed generally, but are demonstrated using particular examples. The lesson begins with seven questions that require mental strategies including doubling, halving and approximating. After about five minutes, SJ asks the students for their answers.

14:12 Student:	I did boxes.
14:13 Teacher:	So what did you do?
14:14 Student:	Car boxes.
14:16 Student:	Ok, what I did...
14:18 Teacher:	Cardboard box.
14:20 Student:	Can I write it up on the board?
14:21 Teacher:	No, just tell me –
14:22 Student:	Ohh.
14:22 Teacher:	– what you did.
14:23 Student:	Ok. What I did –

[07] SJ 10(2) Approximations

It was usual practice in other observed SJ lessons for students to give fairly full explanations of how they found their answers. For example, in this lesson one student explained how he had found 50% of 498, by noticing that 498 is close to 500, then recording the two that had been added on by recording it in a box. He then halved the 500, and halved the two before subtracting the resulting 1 from the 250 to get the answer 249.

14:29 Student:	Ok. I got the four nine eight, and obviously that's close to five hundred.
14:34 Teacher:	Five hundred, ok.
14:35 Student:	So I put, so that means I put a little two in my little box, so I remembered it, cos that's how I do it.
14:41 Teacher:	Yep, ok.
14:43 Student:	And what I did is I halved it
14:45 Teacher:	Yep.
14:45 Student:	I halved it, oh ok, you've already got that on the board, two fifty. So this means I have to halve what is in the box, so I halved what was in the box, which equals one.
14:55 Teacher:	Ok.
14:56 Student:	So I take away one from that, it would equal two four nine.
15:00 Teacher:	Ok.

[07] SJ 10(2) Approximations

The student's term 'boxes' is not a generally accepted piece of mathematical terminology, but is used by both student and teacher to refer to this method.

Perhaps the most surprising finding in the data collected came from those lessons with a particular emphasis on discussion and students 'thinking for themselves'. Conceptions under discussion in these classrooms were sometimes given completely different names, or even an inaccurate name, and yet students' appreciation of the conceptions did not appear to suffer. It would seem that a sense of generality can be communicated by allocating a word to a particular number or example. The word means 'things like this'. The linguistic term for this is a 'placeholder'. A placeholder name occupies a syntactic space between noun and pronoun. Whilst functioning grammatically as nouns, their referents must be supplied by context, like pronouns. They serve as placeholders for names of objects that are otherwise unknown or unspecified. In everyday discourse, words such as 'thingummy' or 'whatsit' are used in this way. The meaning of these words changes depending on context. In lesson [07] a student ascribes the word 'boxy thing' to his method for finding 50% of 498. Such

use of placeholders has much in common with the use of letters for as-yet-unspecified

numbers, or place holders for later insertion of numbers in expressions and formulae.

In lesson [31], talking to class 8(2) about the work on angles that he had just marked,

CB explained that he has corrected their language but they shouldn't worry about

getting it wrong. "Trying to use the vocab" appears to be seen as a different skill from

understanding the maths. Whilst valued, it is somehow separate from understanding

and method.

3:50 Teacher:	I started writing on some people, um, brilliant explanation but that's not corresponding angles that's alternate angles. Because I just thought well I might, while you're trying to use the vocab, I might as well show you exactly how to use it. I think on one of them I wrote 'it's not alternative it's alternate'. That's not I'm telling you off, that's me just saying, they're doing so fantastically, why not have, make another little comment.

[31] CB 8(2) Sequences

To use Davydov's framework, the teacher's emphasis for this piece of homework

seems to be on the *meaning* of alternate and corresponding angles. Each correctly

found missing angle also indicates some appreciation of the concept's *scope*, although

the examples included in the worksheet did not differ significantly from those they

had discussed in the previous lesson, with Z-shaped and F-shaped diagrams that

involved few other lines. The concepts' *names* are apparently considered to be of

relatively little significance.

[27] PF 9(1) Fractions

In lesson [27], 'lowest common multiple' is confused with 'lowest common factor'.

This confusion goes unnoticed by the teacher, who also adopts the incorrect

terminology. Whilst in a sense this illustration could be considered unrepresentative

of the fifty-two central study lessons, as no other lesson contained such a lengthy teacher confusion of mathematical vocabulary, one of the most striking things about the episode was the lack of concern or surprise demonstrated by students, which left me as an observer wondering how much attention is being given to the use of particular mathematical vocabulary by students, who are perhaps more focused on the underlying meaning than the particular word used.

The students are asked to think about why the fractions 1/3 and 1/4 might have been chosen by the teacher for shading on a rectangle containing 12 squares.

07:43 Student:	Um, I was thinking along the lines of why. [Is it because]
07:45 Teacher:	[Good, good]
07:46 Student:	you've given three and four and the lowest common factor of three and four is twelve, so you've divided into twelve. And I was thinking if you'd given us three and six you'd have to divide the box into six because the lowest common factor of three and six is six.
07:59 Teacher:	Fantastic. And, um, great use of language there as well, I really liked that. Um, lowest common factor, really key idea. And as the lowest common factor of 3 and 4 is 12 it's no coincidence that I've got twelve boxes.

[27] PF 9(1) Fractions

The [placeholder] of 3 and 4 is 12 can be understood by the students as suggesting a link between the denominators of the fractions. Factor and multiple being commonly confused words linked to multiplication and to the idea of 'going in to', it almost doesn't seem to matter *in this instant* that the term is incorrect. Some may interpret 'lowest common factor' as 'product', others as 'lowest common multiple'. Perhaps many interpret it as 'slightly tricky thing to do with multiples'. When completing a worksheet of adding and subtracting fractions in the second half of the lessons, all of the students seemed to be accurately selecting the lowest common multiple of the two denominators. Understanding in the moment does not appear to have suffered from

inaccurate language. The confusion was corrected by the teacher, once the mistake had been realised, and the correct term was then used.

It seems that, when responding to the student, the teacher's attention was on the conception, not the concept. The student was being praised for "great use of language" because they were expressing a complex concept. It's the "really key idea" that was being praised. Coles' (2004) idea of meta-communication, first referred to in section 5.3.3 resonates again here, as PF's meta-message in praise of the use of mathematical language might be considered of greater, or at least equal, importance, to correcting the terminology used. In striving to encourage students to 'talk like mathematicians', some degree of 'babble' (Malara and Navarra, 2003; see 2.4.3) is perhaps inevitable. If each incorrect use of mathematical names was immediately corrected by the teacher, students might become reluctant to attempt its use.

In the same lesson [27], students were asked to think about how they would find the answer to $\frac{7}{12} + \frac{8}{9}$. A student suggested multiplying 9 by 12, and conversation followed between several students and the teacher, with the teacher explaining that it is best to use the lowest common multiple of the two denominators, rather than always using their product. Once the example has been worked through, and a denominator of 36 has been chosen, the teacher summed up the importance of using LCM rather than product.

PF: Ok, so that's the important thing, [name of student who suggested 108], there's nothing wrong with what you said, it's perfectly reasonable, and when we get on to harder and harder things, it's sometimes easier just to do it that way. Um, but it is good for us to look at a lowest common multiple, rather than just any old multiple, ok, and the lowest common multiple of these two things, is (?) is of course, 36, ok. 12, times 3, 36, and 9 times

	4 is 36. So it's definitely a common multiple and it's the lowest common multiple.
Student:	Do you expect me to remember that?
PF:	I would love you to remember that.

To what does 'that' refer in this instance? It might refer to the technical term of 'lowest common multiple', the specific example $\frac{7}{12} + \frac{8}{9}$, or perhaps the idea that it might not always be best to use the product of the denominators. This student's question raised an interesting question for me about what exactly students are expected to take away from whole class discussions. Do they need to know the precise scope, name and definition of each idea discussed? Often it seems it is the conceptions rather than the concepts that they are expected to have developed.

There appears to be a conflict between immediate and longer-term understanding in teacher's use of, and response to, students' language. The development of an accurate mathematical language will aid students in future discussions, but in the short-term the emphasis may be on communication, rather than accuracy. An overemphasis on short-term communication may result in missed opportunities for the development of the mathematical and algebraic registers.

8-II.4 DEVELOPING CONCEPTS: DEFINITION AND SCOPE

As section 8-II.2 describes, concept names are introduced by both teachers and students, sometimes without an associated meaning, and sometimes incorrectly. The meaning and scope of concepts are invoked by teachers and students without use of the appropriate name. Having explored the issue of naming in depth in section 8-II.2, this section focuses on the other two aspects of concept appreciation described by

Davydov (1972/1990). The focus here is on how class discussion provides an opportunity for students' appreciation of the concept's scope and meaning to develop.

[34] LR 10(1) Rational and irrational numbers

The excerpt from lesson [34] struck me, as it contained the longest amount of time spent explicitly developing the scope of a concept, and demonstrated how challenging this process can be. The episode is as an example of a process that I have observed happening informally in a number of mathematics classrooms, as a necessary part of students' developing concepts. I offer it here because it illustrates the high cognitive demand that this places on students, thus potentially making a case for more consideration and time to be spent on the development of mathematical concepts.

The lesson began as follows.

> 03:29 LR: The first thing we're going to do today is we're going to think
>   about rational and irrational numbers.

[34] LR 10(1) Rational and Irrational numbers

In the above example, the use of the specific terms 'rational' and 'irrational' indicate that a generality is present. The teacher seems to be suggesting to the students that there is something to look for. The board was then divided into two, and students suggested numbers that they thought were rational, or were irrational. De Morgan (1898) argues that individuals should participate in abstraction in this way. They should develop their own conceptions and concepts.

> It is by collecting facts and principles, one by one, and thus only, that we
> arrive at what are called general notions; and we afterwards make
> comparisons of the facts which we have acquired and discover analogies
> and resemblances which, while they bind together the fabric of our
> knowledge, point out methods of increasing its extent and beauty.
>
> de Morgan 1898, 33-4

When defining rational and irrational numbers, how many examples might you feel you needed to show a student for them to 'understand' the concepts? This lesson demonstrated that numerous examples were required to enable students to form a definition for themselves of the two concepts. The board was almost full of fractions, pi, and other symbols, before the suggested definitions were clarified to:

> 19:02 Student: Um, a rational is a number that has, it can be converted into a fraction that has a whole number as the numerator and the denominator.

[34] LR 10(1) Rational and Irrational numbers

Although the year 10 students involved would be expected to understand the concepts 'fraction', 'whole number', 'numerator' and 'denominator', merely offering them a definition such as this at the start of the lesson would have resulted in a 'concept based on concepts', as discussed above, which would arguably have resulted in less complete conceptual understanding. A general concept requires generalisation over a number of other general concepts. So in order to express a generalisation students may have to refer to previous generalisations, with their associated terminology.

This was a formalisation of a process that from my observations, seems to happen frequently in mathematics classrooms. This is where a name is used to indicate that there is a general concept, and various examples either are or aren't allocated the name. Over time, the scope becomes clearer to students.

In the post-lesson interview, the teacher in this lesson said they had been particularly pleased when students introduced π, as it demonstrated that their scope or range-of-permissible-change is not limited to surds. As the teacher's self-appointed role was to

choose students to offer examples, and to respond to their suggestions, she had the time to reflect on the appropriateness of each one. How often do teachers give themselves the chance to think this carefully about the examples that students are offered (or offer themselves) in developing a conception? What is it that gives us the sense that students are 'ready' for the scope to be extended?

The scope of a concept, then, is something that students must learn and develop over time, just as they do when learning vocabulary in their first language. As Pimm (1987) observes, the mathematics teacher acts as a role model of a native speaker of mathematics. Looking at it in this way, the ratio of 'native' to 'foreigner' in a classroom seems less than ideal. Arguably the best way to learn a language is to surround yourself with its speakers. You then meet lots of vocabulary in use, and as the words become more familiar, you develop a sense of their meaning, and discern their scope.

In this sense, every curriculum subject might be seen to have a lot to learn from the language classroom. There is a major difference, however, in that words in a foreign language are often learnt by attaching them to the corresponding concept in English. This is only possible when the word is already known, the concept already grasped. The language of mathematics is not a different way of describing previously used concepts, but a vocabulary associated with a whole new set of ideas and ways of perceiving and conceiving. The 'language of mathematics' seems to be more usefully thought of as a set of concepts than as a set of words. The question of 'scope' then, is particularly pertinent.
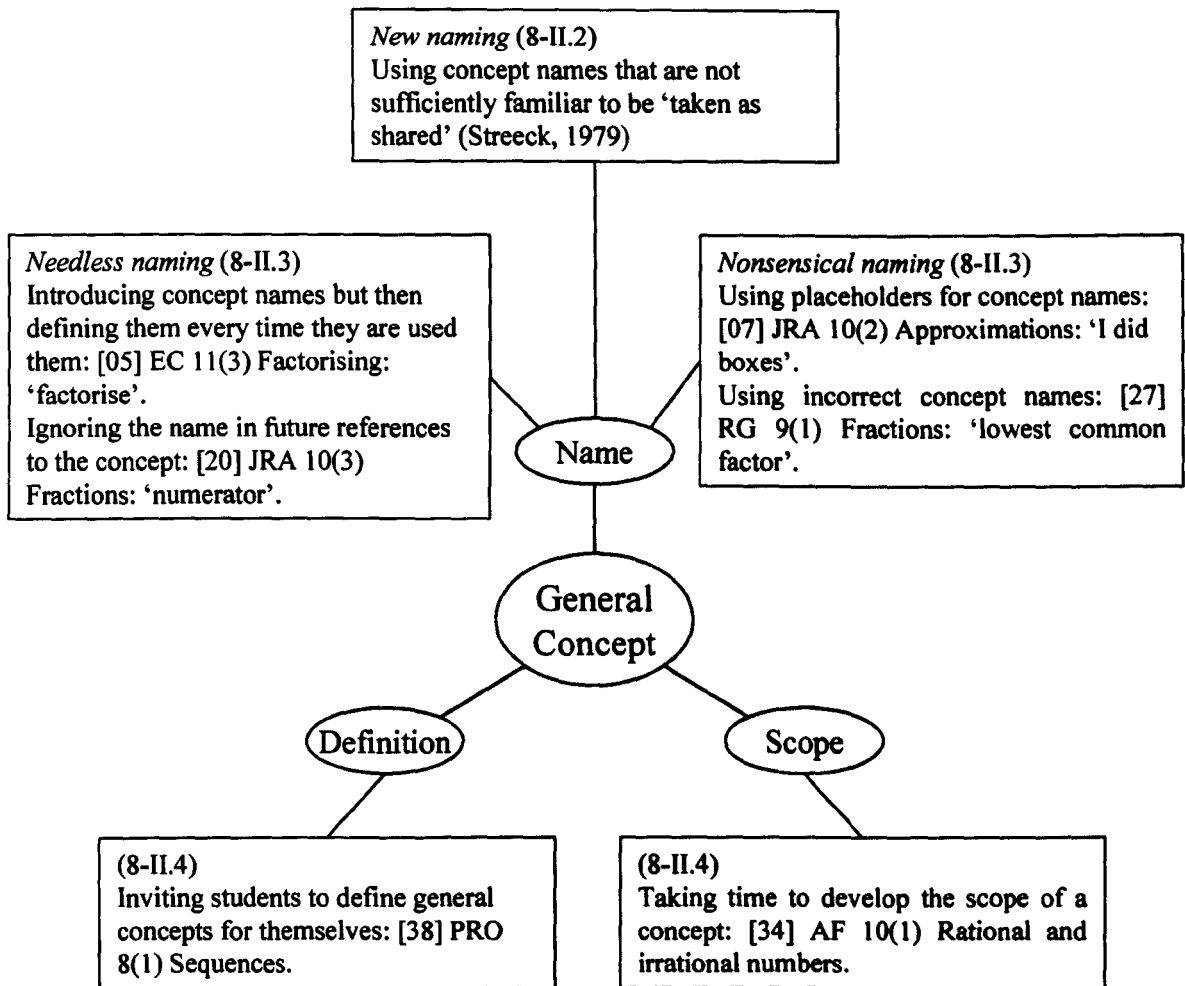
In a year 11 class where all students 'understand' the term 'rectangle', a classic misconception of scope (French 2004: 31) arose. The teacher was asking how many faces a cuboid drawn on the board had. The drawing showed six rectangular faces of which one opposite pair were square. When students were hesitant to answer, she rephrased the question to "how many rectangles does the net have?", to which a student replied "four". The scope of 'rectangle' for this student apparently did not include squares:

Student:	I was meaning the end two bits is a square, so they're not rectangles.
Teacher:	Right, ok. Right. What's the size of my first rectangle?

'Rectangle' was being used with the reference 'faces of the cuboid'. The student's misunderstanding was ignored because they worked out for themselves that the teacher meant all the faces of the cuboid. But their understanding of the scope of 'rectangle' remained incorrect. The focus was on nets and 3D shapes, so thinking a square isn't a rectangle doesn't impede understanding of the question. When is it helpful for the teacher to explore this misconception, and why? How can teachers help students to correctly identify the scope of a general concept?

8-II.5 CHAPTER SUMMARY

The focus of this chapter is teachers' use of language when introducing concepts and developing students' understanding of them. A conflict was detected between immediate and longer-term understanding in teacher's use of, and response to, students' language. The development of an accurate mathematical language will aid students in future discussions, but in the short-term the emphasis may be on communication, rather than accuracy.

New naming (8-II.2)
Using concept names that are not
sufficiently familiar to be 'taken as
shared' (Streeck, 1979)

Needless naming (8-II.3)
Introducing concept names but then
defining them every time they are used
them: [05] EC 11(3) Factorising:
'factorise'.
Ignoring the name in future references
to the concept: [20] JRA 10(3)
Fractions: 'numerator'.

Nonsensical naming (8-II.3)
Using placeholders for concept names:
[07] JRA 10(2) Approximations: 'I did
boxes'.
Using incorrect concept names: [27]
RG 9(1) Fractions: 'lowest common
factor'.

(Name)

General
Concept

(Definition)

(Scope)

(8-II.4)
Inviting students to define general
concepts for themselves: [38] PRO
8(1) Sequences.

(8-II.4)
Taking time to develop the scope of a
concept: [34] AF 10(1) Rational and
irrational numbers.

As shown in the summary diagram above, Davydov's definition of the three essential
aspects of concept appreciation provides a useful framework for examining how class
discussion offers students opportunities to develop general concepts. The episodes
illustrated above show how words and their scopes and meanings can become
separated in mathematics lessons. Section 8-II.2 showed how a name is often used
before the concept, or even the conception, has begun to develop. Section 8-II.3
showed how a concept can be introduced and defined but then no use subsequently
found for its meaning. An emerging conception might also be given a transient, even
an incorrect, name in the short-term, so that it its referent can be focused on, referred
to, discussed and developed without concern for linguistic precision. There seem to be
inherent concerns with each of these separations, and yet instances were found of

most of them in most of the classrooms observed, all of them effective lessons with competent teachers. Section 8-II.4 showed how conceptions are often discussed and developed in these mathematics classrooms without some or even any of Davydov's (1972/1990) criteria for their becoming concepts. Students may be unsure of the exact scope of a generality, it may be given no name, or an incorrect name, and it may be perpetually supported by particular examples as the speaker believes its meaning would not otherwise be shared. The decision as to how and when to develop students' conceptions into concepts is a complex and fascinating one.

An accurate appreciation of mathematical general terms (rectangle, numerator, lowest common multiple) is required for success in exams, and use of textbooks and worksheets. Students' use of appropriate terminology makes them part of a community of mathematicians, spanning both time and space. Orton (1994) emphasises that "mathematics is a unique universal language which transcends social, cultural and linguistic barriers, having symbols and syntax that are accepted the world over". (Orton 1994, 17). In a particular classroom at a particular time, however, it may be possible to communicate effectively using a different term for a conception, or without a name at all.

Teachers with the intention of *really listening* to students may choose to focus on what those students apparently mean, rather than what they say. There is perhaps a balance to be achieved between developing students' long-term mathematical language proficiency, and communicating effectively in the short term. The study findings suggest that it is beneficial for teachers to be aware that they are making that compromise.

There are numerous tensions related to the teaching of general mathematical concepts. A major finding of the analysis reported in this chapter concerns the dearth of discussions where students' attention is linguistically directed towards all three aspects of a concept: its name, scope and meaning. The separation of these three aspects of mathematical concepts could have implications for student understanding.

Through consideration of how general concepts are expressed in mathematics classrooms, the implications for using algebra to express generality begin to emerge. What is particularly striking is that the issues and tensions described in relation to use of mathematical concepts also apply to the use of algebraic notation, yet these issues tend to be treated as separate. In chapter nine, some implications of the chapter eight findings for use of algebraic notation are considered.

CHAPTER 9: BACK IN THE RESEARCHER'S

CLASSROOM

Throughout the analysis of chapters 7, 8-I and 8-II, the complexities of expressing

mathematical procedures and concepts have been examined in detail. Through using

the framework to examine my own and others' practices I became aware of the

considerable and perhaps over-looked complexity of expressing generality through

natural language, and of the potential benefits of expressing generality through

algebra. This is not a necessary conclusion of application of the framework, and it is

to be hoped and expected that others will apply the framework to their own and

others' practice in order to gain different insights into possibilities for expressing

generality. Similarly, it is intended that the theoretical considerations and practical

investigations of this chapter might be fruitfully explored by a teacher without their

necessarily having applied the framework of chapters 7, 8-I and 8-II to their own or

others' practice. Although the framework acted as a stimulus to further and deeper

analysis, causal links between the process of applying the framework (the outcome of

which is shown in table form as in previous chapters), and the subsequent insights are

not always transparent. By attempting to replicate the process of applying the

framework to the described episodes, the reader may be provoked to similar or

different insights into the choices that were and could have been made.

9.1 INTRODUCTION

This study considered the challenge of communicating general mathematical rules

(chapter 8-I) and concepts (chapter 8-II), and the challenge of teaching algebra with

understanding (pilot study). Analysis suggested that each of these can be done more effectively if they are intertwined. Whilst the use of new and imaginative 'purposeful algebra' tasks can be both effective and inspiring, the search for the innovative and creative can draw attention away from possibilities that are present even in very traditional-seeming tasks and lessons. Tasks where algebraic expression of generality is a central focus, at the forefront of both the task-designer and teacher's mind, have a significant part to play in students' developing appreciation of generality, and algebraic facility. This study has shown, however, that there is an apparent gulf between lessons 'about' algebra, and lessons where the focus is on other aspects of mathematics. This was particularly emphasised to me through the differences in the teaching styles of the project teachers between the pilot lessons with the matchstick houses and the main study lessons.

As is recounted in chapter 8-I, the analysis of the fifty-two main study lessons led me to an increased awareness both of the ambiguities and imperfections of using mathematical terminology and everyday language to express general mathematical procedures, and of the negligible use of algebra to express these in 'ordinary' lessons. This resulted in my adopting the question asked by Sutherland (1991), "Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas?". Analysis of the main study lessons revealed a tendency for algebra to be used to express patterns in shapes and in numbers, but to be ignored when working on other areas of mathematics.

Tall and Thomas (1991: 4) assert that "there is a stage in the curriculum when the introduction of algebra may make simple things hard, but not teaching algebra will

soon render it impossible to make hard things simple". There are a number of points

that I think are worthy of mention in connection with this observation. Firstly, it was

clear from the transcripts in the main study that students do not necessarily think that

'hard maths is bad, easy maths is good'. Where such an attitude does develop, it can

arise as the consequence of numerous socio-cultural factors from home, school, and

the wider community. Many of these can be overcome through the creation of a

classroom culture in which difficulty is embraced and relished. Making things more

complicated, or even making things *sound* more complicated, can be a satisfying

activity. Watson and Mason (2005: 144) give the example of a teacher asking students

for different questions where the answer is 16. The teacher introduced the idea to

students of his having "my favourite question", which was interpreted as meaning

"hard or complicated". Activities such as this one make achieving complexity into

both a game, and a worthwhile objective. In my experience, one of the pleasures for

students in learning trigonometry or Pythagoras' Theorem is that they can tell others

that they have so done with considerable pride. Mathematics can be made accessible

and enjoyable without needing to seem 'easy'. Secondly, it is possible for students

with very little experience of symbolic algebra to be offered tasks in which use of

algebraic notation makes hard things easier, for example problems involving

mathematical proof.

Many tasks have been offered to mathematics teachers that are designed to offer

students an opportunity to express generality. Rather than develop original and

innovative 'purposeful algebra' tasks, the research findings suggested that there was

scope to develop 'ordinary' practice in 'ordinary' lessons. This in itself is not a new

message. Mason (2002a) emphasises how language can be used to express generality,

and Ainley *et al.* (2004) write about purposeful algebraic tasks. Brown and Coles (1997) also suggest that algebra can be used meaningfully to express generalisations students have made during 'meaningful' activity. However research often uses exemplar tasks that involve the expression of 'extra-curricular' generalities. The analysis of the main study lessons demonstrated that there is still great scope for teachers to invite students to express 'curricular' generalities using algebraic language. This requires of teachers both that they recognise moments in classroom interaction where algebraic language is available as an effective alternative to natural or mathematical language, and that they make the decision in such moments to adopt algebraic language as a means of expressing the generality. Having completed level 2 and 3 analysis of the main study lessons (chapters 7 – 8-II), I endeavoured to notice, and make informed choices in response to, such moments in my own practice.

9.2 WHY USE ALGEBRA TO EXPRESS GENERALITY?

The use of tasks such as those discussed in the previous session can focus students' attention on the expression of generality, and establish it as a worthwhile activity in the mathematics classroom. Findings from the pilot and main study, however, suggested that the impact of such tasks was reduced by a separation between these 'expressing generality' tasks and normal classroom tasks.

The use of algebra to express general procedures and concepts that form part of the secondary mathematics curriculum potentially offers the following opportunities:

- show what algebra can do (9.3.1)

- increase relevance (9.3.2)

- increase facility (9.3.3)

- reduce ambiguity (9.3.4)

- direct attention towards the general (9.3.5)

9.2.1 To demonstrate the power of algebra

Main study lessons that focused on algebra tended to concentrate on the manipulation of expressions and solving of equations, generally in the absence of context. This corresponds with the literature, as discussed in section 2.4, which proposes that the teaching of algebra as defined by the national curriculum can be restricted to manipulating and transforming expressions and solving equations with no apparent purpose.

> To most lay people the defining characteristic of algebra is its use of symbols, but beyond that they would find it difficult to describe either what algebra is or what purpose it has. Indeed, many mathematics teachers find it difficult to answer those recurrent questions that students ask: 'Why are we learning algebra?' and 'What use is it?' Unfortunately, many people's experience of school algebra fails to give them a clear view of what algebra is. The questions arise because they acquire a very narrow and restricted image of the subject.
>
> French, 2002: 1

By using algebra as mathematicians use it, this problem may be overcome. Through endeavouring to use algebra to express generality across the whole curriculum, students may experience its power as a tool for communicating mathematical ideas.

9.2.2 To increase relevance

Through considering algebraic language as a second language like French or German, a new perspective can be gained on its teaching. Section 2.3 considered how the research and practice of second language teaching might have something to offer mathematics education. Mathematics teachers are often students' only ambassadors of

mathematical language, the only representative of the mathematics community with whom students come into contact. If students perceive that the only use teachers have for algebraic language is in the teaching of algebraic language, then the purpose of learning such a language becomes lost. It seems to be essential that students see how algebraic language can be used across all mathematical topics.

9.2.3 To increase facility

Some parallels can be drawn between use of algebraic language, and use of mathematical language. Lessons learnt in chapter 8-II regarding use of the mathematics register can be applied to use of algebraic notation. Findings in chapter 8-II suggested that teachers often appear concerned about using mathematical language, in case students do not understand, and 'switch off'. If teachers explain a task using complex mathematical vocabulary, students might be intimidated, and feel excluded by the mathematics community (here represented by the mathematics teacher). Research has shown that the language in which mathematics is expressed can act as an obstruction to students' mathematical understanding (Pimm, 1987; Laborde, 1990). The answer to this problem lies in frequent use, with explanations, definitions and 'translations' into regular English to ensure that the meaning of the technical terminology is clear to students. This scaffolding and fading approach is discussed in section 2.7.2. Lee's (2006: 20) argument that, "the teacher's role is to mediate between the discourse of mathematics and the discourse that pupils routinely use, to make bridges between the discourses so that the pupils become able to use mathematical language to conjure ideas and to explore and communicate those ideas" applies equally well to algebraic language.

9.2.4 To reduce ambiguity

Chapter 8-I showed how, in the main study lessons, examples often played an essential part in the introduction and explanation of mathematical concepts and procedures. The concept of 'fraction' is much easier to exemplify than to define in general, and I have heard experienced teachers debate whether or not such a definition should or should not include $\frac{2\pi}{3\pi}$. Likewise, use of the mathematical vocabulary associated with fractions, such as numerator and denominator, to explain the general rule for dividing fractions, can result in confusing and contorted sentences that fail to illuminate. Perhaps as a consequence, main study teachers often used examples to show general concepts and procedures.

The example that $2^2 = 2 \times 2$, for instance, can lead to the inappropriate generalisation that $3^2 = 2 \times 3$ and $4^2 = 2 \times 4$. Especially when working with more complex procedures, the same number often plays different roles in an example. Students might not always understand what is invariant and what can change, what is structural and what is particular. For a student who is familiar with algebraic notation, however, the rule that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ could clarify the procedure of dividing rational numbers. This algebraic expression might also lead to justification and proof.

Of course, exploration of examples can be an essential part of developing students' conceptual understanding. It is certainly not my intention that general algebraic expressions should replace exploration of particular examples. Rather, there are occasions when students benefit from hearing or developing the general rule and that

algebra is often a clearer medium for this than either natural or mathematical language.

9.2.5 To direct attention towards the general

As found in chapters 8-I and 8-II, examples and general rules are often offered with the intention of providing students access to a general concept or rule. The extent to which the students are aware of the generality present in such discussions is often unclear. The discussion required to develop an algebraic definition of a concept or description of a process offers the opportunity to emphasise the general nature of the concept or procedure.

One approach I have adopted is, having worked on several examples of a method in some area of mathematics, to ask students to write down a general method. They are then asked to read out their methods. I might then encourage students to offer a method description that is entirely general, with no particular examples. This description can be tested on more and more complex examples to make sure it deals with all possible cases. Through discussion, we then attempt to construct such a general method as a class, and see whether algebraic notation becomes useful. An episode involving this approach is recounted in section 9.5.

Analysis in chapters five to eight, particularly that of 8-I and 8-II, reveal potential for the plethora of general procedures and concepts that were found to be expressed in mathematics classrooms to act as an opportunity for emergent algebra (Ainley, 1999a). However, given the difficulties students have been found to have with understanding and using algebraic notation (as outlined in section 2.4.2), practitioners

might reasonably be concerned that their students would be daunted by such an activity, and that placing such an emphasis on emergent algebra would lead to a classroom culture marked by frustration and failure. In researching my own practice with regard to use of algebra to express procedural and conceptual generality, it was therefore important to consider ways in which correct algebraic notation can be encouraged, thus enabling algebraic expression of generality to become a central part of classroom practice.

In sections 9.3, 9.4 and 9.5 I offer accounts of my own teaching experience, with the hope that they will provide an opportunity for consideration of whether the reader might have made the same or different decisions. I aim to give a thick description, so that the reader can more clearly envisage the choices that were available to me, and consider the merits of the various options. These accounts are *not* intended to model best, or even effective, practice, but rather to act as stimulus to reflection. A distinction is made in these three sections between 'accounts of' the lesson (Mason, 2002b) and my reflections on the decisions made in the findings of this study, with 'accounts of' set in standard style, and the reflections as indented text.

9.3 EXPRESSING GENERALITY ALGEBRAICALLY

At the start of the lesson, the students unpacked their exercise books and copied the objective from the whiteboard: "describing a number you don't know, or a pattern that always works".

> As I have numerous on-going objectives, choosing a single one to share with students (in order to comply with whole school policy) can be difficult. Objectives can be 'shared' with students on other occasions, through what and how I choose to praise contributions, and where I direct attention, and writing them up at the start of the lesson does not always seem appropriate. Even where a 'learning objective' can be identified, the

impact of a lesson, for example, where students measure angles in triangles to deduce that the sum of the angles is 180°, would be significantly diminished by sharing the objective "to know that the sum of the angles in a triangle is 180°". In this case my intention was to direct attention away from what was being expressed, towards the method of its expression. The framework developed in chapter 7 distinguished between transient and universal generalities, and questioned whether students are always aware of which of the generalisations expressed in a lesson are universal, and which apply only for a particular activity. The learning objective in this case might direct students' attention towards the universal generalities associated with algebraic notation, rather than the transient rules of the particular activities engaged with.

I then began an activity that the group was familiar with. I read down the register, and

as I called each student's name, I threw an imaginary dice at them.

> The use of an imaginary dice emerged from throwing a die at students and asking them to square the number. Frustrated that we didn't have a die that went up to twelve in the department, I asked students to imagine one that did. The resultant humour from 'missed catches', 'losing the dice' and 'ow, it hit me' may not add much mathematically, but such student creativity feels like an important part of the culture I aim to create in my classroom.

The students were asked to catch the dice, look at the number, do something to it, and

tell the rest of the class what they had done, and what the answer was. Mason *et al.*

(2005: 22) offer several versions of this activity, wherein all students in a class are

invited to think of a number, and then to carry out a sequence of operations on that

number. In this lesson, I gave the example that I might catch the die, and tell the class

that if I multiplied the number by 7 and added 5, the answer would be 40.

> One of the generalities being expressed here is: *when it is your turn to 'catch the die', say 'something like this'*. The particular example of "if I multiply my number by 7 and add 5, the answer is 40" is intended to tell students the accepted general form of their statements.

Underlying generality	Subject Longevity Justification	procedure, behaviour transient behavioural/social
This given utterance	Derivation Awareness	telling difficult to classify

My attention is drawn by the framework to the potential ambiguity between my offering a general structure of statements that are acceptable in this activity (transient, behavioural/social), and my indicating a general structure of statements that are mathematically acceptable (universal, conventional).

The examples below are selected to give the reader an insight into the scope of examples offered by students in the subsequent discourse, rather than an exhaustive list of questions.

Student description	My board workings
Add 5, get 9.	$n + 5 = 9$
	$n = 4$
I square it and I subtract 1 and the answer's 99.	$n^2 - 1 = 99$
	$n^2 = 100$
	$n = 10$
n timesed by 7 plus 2 is 44	$7n + 2 = 44$
	$7n = 42$
	$n = 6$

I noticed that students were moving between natural and algebraic language, and considered attempting to direct students' attention towards the variety of language available. I felt that by drawing attention to this I would reduce its impact. Somehow the value of the students' language use being 'natural' would be lost if I, as the authority figure, indicated distinctions between the two. If there are students who are not noticing the distinction between natural language and the algebraic register, perhaps this is of benefit.

Again, I was sensitised by the framework to notice that some of the general rules about what was or was not acceptable to say or write on the board were universal conventions of the mathematics community (a number multiplied by itself is written as 'n^2', for example), while other rules were behavioural/social rules for this particular task, imposed by me (the 'problems' must have one or two steps, and must be sufficiently straightforward that other students in the class can solve them). Where students appeared to be complying with other general rules (n, the number on the imaginary die, must be an integer, for example), it is unclear whether they might consider this to belong to the former (universal, conventional) category, or to be part of the constraints of the particular task.

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Section 9.3 was intended both to give some insight into the ways of working I adopted to support the year 7 students to use algebraic notation, and to indicate ways in which the findings of the main study influenced my reflection on my own teaching. Sections 9.4 and 9.5 offer further classroom episodes from my own practice that illustrate the use of algebraic notation to express generality in 'ordinary' lessons, and show how the analysis of chapters 7, 8-I and 8-II shaped my thinking.

9.4 EXPRESSING GENERAL CONCEPTS ALGEBRAICALLY

In another lesson with the same year 7 group, the focus for the lesson was the mathematical topics that would be useful for their upcoming maths test. I reintroduced each topic, students offered their own definitions and examples, and then the students wrote their own booklet, textbook, poster or cartoon to explain the key concepts. I was aware of making numerous decisions when leading discussion of the terms *multiple* and *factor*. General definitions of these terms are often example-ridden, and I found it difficult to phrase a clear definition without offering particular examples.

When the 'teacher-led discourse' phase of the lesson had ended, Ben, who was writing a revision booklet on the six terms at which I had been directing students' attention, asked me, "How would you define multiple, Miss?". I paused, realising that the way Ben had phrased his question afforded me an opportunity to act upon my noticing that the general definitions of multiple and factor were awkward in both natural and mathematical language. "How would I define it?" I asked. "It would depend who I was defining it for and why" (here deliberately ignoring the manifest fact that I was defining it for Ben, in order that he could complete the task I had set

him) "I might define it by saying *m* is a multiple of *n* if, when you divide *m* by *n*, the answer's an integer".

Nick, sitting next to Ben, was listening by now. He asked me to repeat my definition, I raised my eyebrows at the pair of them, with the intention of indicating that there was little value in just writing down my words verbatim, and they began to reconstruct a general definition of multiple using *x* and *y*. The two students also, through their subsequent conversation, developed a definition of factor which ran something like: "*a* is a factor of *b* if *b* divided by *a* is a whole number", although the definition that Ben wrote down was "*a* is a factor of *b* if *b* is in the *a* times table".

This episode raises a substantial number of questions, which I offer as considerations for further research:

- Do Ben and Nick's expressions of the definitions using some algebraic notation indicate an understanding of the two terms?
- Does the students' attempt to verbalise the generality enhance their understanding of the terms?
- Why did I choose not to use a more formal definition, such as "*m* is a multiple of *n* if, $bn = m$, and *b*, *m* and *n* are integers"? What determines an appropriate level of formality?

The framework developed in chapter 7 offers a structure for further analysis of this interaction.

Underlying generality	Subject Longevity Justification	concept: 'factor' and 'multiple'. universal convention, mathematically necessary.
This given utterance	Derivation	telling, pattern-spotting: students were asked to give examples, then offer a general definition.
	Awareness	enactive: several examples.

Although all mathematical concepts are, to some extent, conventional, as their names tend not to be mathematically necessary, the concepts multiple and factor could be seen as necessary to some extent. Various components of the concept image (Tall and Vinner, 1981) of the two concepts are mathematically necessary attributes of number. Prompted by the framework, reflection on my teacher-led discourse related to the two concepts sensitised me to the extent to which I had been treating factor and multiple as conventional mathematical terms, rather than general concepts that can be explored and investigated. In association with concepts of divisibility and prime, these are areas of mathematics towards which students' cognitive awareness could be developed. Rather than focussing on the enactive, by asking students to find the factors and multiples of different numbers, there is an opportunity to direct attention towards the necessary, rather than the arbitrary (Hewitt, 1999). Use of the framework developed in chapter 7 to structure my reflection on the teacher-led discourse related to the general concepts heightened my awareness that, by focussing on examples and definitions in everyday language, opportunities to think mathematically about the concepts may have been missed.

Analysis in chapter 8-II resulted in increased sensitivity to the tensions and decisions involved in expressing general concepts. Analysis of the fifty-two main study lessons demonstrated that in lessons such as lesson [34] (8-II.4) where attention is directed to the scope of a concept by students suggesting particular cases that are or are not contained within the scope of the concept, many examples are required before students volunteer a general definition. This leads me to question whether students would have benefitted from offering examples such as "9 is a multiple of 3" and "40

is a factor of 1,600" and explaining whether and why they are true or false. An activity such as this might have given students such as Ben and Nick an opportunity to express their algebraic definitions and explain them to other students in the class.

The identification of needless naming (8-II.3) also resonated with me on reflecting on this lesson, as I realised that I rarely use the terms factor and multiple other than in those lessons where they form part of the main objective. I find myself choosing not to use these terms with the self-justification that they are sometimes confused by students, and might detract from the mathematics on which I intend students to focus. However, this study's findings suggest that it is often the concept itself, rather than the name, that students are still developing, and ordinary language can be at least as ambiguous as can mathematical language. As discussed in section 8-II.2, as teachers are the 'native speaker' of mathematical language in the classroom, we can support students' lexical familiarisation by using mathematical terminology frequently.

Section 9.3 has shown how the framework developed in chapter 7, and the findings of chapter 8-II, supported my reflection on my own practice in an interaction where the focus was on the general concepts *multiple* and *factor*. The following section offers a further account of a teacher-led discourse with the same year 7 'mixed-ability' group, where the generality being expressed was the procedure for converting between mixed numbers and improper fractions.

9.5 EXPRESSING GENERAL PROCEDURES ALGEBRAICALLY

During a series of lessons about fractions, one lesson was spent considering the meaning of mixed numbers (e.g. $7\frac{2}{5}$) and the different ways they might be represented. Alongside various diagrams and 'real world examples', students suggested that they could be expressed as improper fractions (in this case $\frac{37}{5}$), using diagrams to support the equivalence between the mixed number and improper fraction forms of notation.

At the start of the following lesson, I told the class that I was thinking of a mixed number, and I wanted them to explain to me how to convert it into an improper fraction. A few hands went up. One student said, "if you had $3\frac{1}{3}$, then you would times the three by the three, add the one...". I interrupted, and said I wanted to know how I could do it for *any* fraction, not just $3\frac{1}{3}$. Another student said "you times the number at the front by the num...". As she hesitated to find the term, a student sat near her supplied "numerator". I asked the first student whether she meant the number at the top or the bottom. She said it was the bottom and corrected herself to "denominator". I said that it was starting to sound complicated, and wondered aloud how a mathematician might describe my fraction. "If we're being mathematicians, then rather than say '3 1/3 or 2 1/9 or 6 2/3 or a fraction like that', how might we describe *any* mixed number?".

> In retrospect, and in consideration of the fairly small proportion of students who claimed to be happy with the algebraic expression of the general procedure, it might have been beneficial to have spent more time exploring how particular examples and the mathematics register can be used to express generality. This might have been useful for the six

students who did not apply the rules in the two questions asked in the following lesson. Regular exploration of these three approaches to expressing generality (through the particular, the mathematics register, and algebraically) might promote the creation of links between the three, as well as enhancing appreciation of their various merits and complexities. Whilst these decisions might seem particularly pertinent when working with a 'mixed-ability' class, findings from the fifty-two lessons in the main study suggest that the 'speed of travel' when engaging in teacher-led discourse with thirty students is a significant decision to make.

Perhaps due to the increased use of algebra with this class when expressing generality,

my prompt "If we were being mathematicians..." stimulated one student to offer

"algebra". Another student then suggested "n". "But how would we know that it was a

mixed number?" I asked them. "n could be any number, and could be written any

way, like a decimal, or a percentage, or an improper fraction. How can we use algebra

to show that it's a mixed number?" A boy near the front, increasingly loudly, was

muttering "n, n, n", so I asked him to come and write his idea on the board, and he

wrote $n\dfrac{n}{n}$ on the board.

> From this student's muttered contribution I was reasonably sure that this student would suggest an expression of this form, and so as I suggested that he share it with the whole class, I was anticipating that other students would challenge his suggestion. It is part of the creation of a 'conjecturing atmosphere' that students are confident to offer ideas that might be incorrect, but there is still an important decision to be made about when it is appropriate to 'expose' students to the corrections of their peers.

I paused and indicated that I wanted the students to be silent and think about what had

been suggested. A couple of hands went up, and one student said "all the numbers

would have to be the same, like two and two halves, or five and five fifths". I wrote

these mixed numbers on the board, and added a hundred and a hundred hundredths.

Other students nodded as if in understanding.

> Another decision was to draw attention to the meaning of 'n and n n[th]s',
> but not so much that it detracted attention from the conversion of mixed
> numbers to improper fractions, and the algebraic expression of this

process. I was really leading the discussion, and had a clear idea what I wanted the outcome to be. Although we wandered a little way into the patterns created by $n\frac{n}{n}$, and I was intrigued by the students' choice of letters to represent the three constituent parts, essentially I was directing the discussion to a predetermined end. It is important that I do not believe that I am following the students' line of enquiry when I am actually following my own, but as long as there is awareness that this is the case, it can be an effective strategy on occasion.

At the end of this discussion I asked the students to raise their hands if they had followed all or most of the discussion – six students put their hands up, and two put their hands half up. There are obvious limitations in this form of student assessment, but this did seem to match with the students who had been contributing. This at least can tell me that none of the students who had remained silent considered themselves to have been following (or wanted me to consider them to have been following), and all of the students who had contributed did consider themselves, or did want me to consider them, to have been following. Of course, definitions of *following* are open to interpretation here.

The framework developed in chapter 7 offers a structure for considering the approach I took to expressing the general procedure for converting between improper fractions and mixed numbers.

Underlying generality	Subject Longevity Justification	procedure universal mathematical necessity
This given utterance	Derivation Awareness	reasoning, pattern-spotting, telling cognitive or enactive

The first contribution of the framework is to focus attention on the underlying generality that was being expressed: the general procedure for converting between improper fractions and mixed numbers. This leads me to question the extent to which the students were attending to this general procedure, and whether this could have been emphasised more effectively.

The procedure is a mathematical necessity (although one that is closely related to the conventions of fraction notation), and in the previous lesson I had intended that the procedure for converting between improper

fractions and mixed numbers would be derived through reasoning. However, working with the framework heightened my awareness that, through starting this lesson by asking students to 'remember' the procedure, and spending very limited time on particular examples, a more 'pattern-spotting' approach might be being taken by students. Students who did not recall the procedure from the previous lesson were essentially being 'told' how to convert between the two different forms. In the process, we may have lost sense of the meaning of the procedure, including the idea that the improper and mixed forms of the fractions represent an equivalent quantity.

In the light of this reflection, I was led to consider what students' might have been attending to during the discourse, and what they might be able to draw on in future lessons.

9.5.1 Students' appreciation of generality

In the next lesson (which was the following day) I asked the students to write down:

1. an example of an improper fraction.
2. an example of a mixed number.
3. why, when we were discussing 'general fractions' at the end of yesterday's lesson, did we write n/d? What did the n and d stand for?
4. $3\frac{1}{4}$ as an improper fraction.
5. $\frac{16}{7}$ as a mixed number.
6. a general description of how to convert a mixed number to an improper fraction, using words, examples, algebra, as desired.
7. a general description of how to convert an improper fraction to a mixed number, using words, examples, or algebra, as desired.

I was interested to see whether any of the students who were able to apply the general procedures in a particular instance (as in questions four and five) did not write a general definition in questions six and seven. Six students did not complete question four and question five correctly, so it is perhaps unsurprising that they did not write a general method for question six and question seven. These six students' responses might be considered concerning, as one could argue that their apparent confusion over how to convert between mixed numbers and improper fractions manifested a failing

of the approach of using algebra to express general procedures. Without a control group, however, it remains possible that these six students would not have answered questions four and five correctly after a more traditional teaching experience. In section 9.4.2 I consider how the emphasis on using algebra to express generality could have been retained and improved upon so that these students might correctly apply the general procedure.

All of the students who applied the general procedure to the particular questions four and five also attempted a general description of the procedure (questions six and seven). One student correctly answered for question 5 that 16/7 = 2 2/7, and wrote a general description of how: "Get improper fraction eg. 16/7, see how many 7s go into 16 = 2 but two left so it is 2 2/7!" (Q7), but for question 4 and question 6 wrote "?????????????????????". It is interesting that this student made the link between the particular and the general here, but felt unable to carry out the inverse of the procedure shown in questions five and seven.

In question three, I had directed students' attention towards the vocabulary 'numerator' and 'denominator', through using the first letter of each term, so it is perhaps unsurprising that many of them used it in their descriptions. What might be considered of interest is that *none* of the students referred to 'top number' and 'bottom number', and all the students who used the terms numerator and denominator (which was all of those who attempted an answer to questions six and seven) used them correctly. Of course, offering them '*n/d*' in question three provided potential scaffolding for this, but their behaviour does suggest that links were being made.

In response to question six, students offered a variety of explanations of how to

convert from a mixed number to an improper fraction, from which the five examples

below have been selected to illustrate the range of responses given.

> You would see how many the denominator said and times it by the whole number then
> you would add the numerator and keep the denominator the same.

> Multiply the integer and the denominator and add on a extra numerator and put the
> answer over the denominator [sic].

> You times the denominator and the hole number and add the numerator [sic].

> Times the whole number by the denominator. Add the answer to the numerator, the
> answer will become the numerator and you will keep the original denominator.

> To convert a mixed number into an improper fraction you would take the mixed
> number (e.g. 2 2/7) and see that the denominator is a 7, and the whole number is a 2, so
> you must do 2 multiplied by 7 which is 14. The numerator of 2 2/7 is 2, so you must
> look at this as an extra two sevenths. So 14 sevenths plus 2 sevenths is 16 sevenths, and
> you must put 16 over the original denominator which was 7 so you get the answer 16/7.

Only one student chose to use algebra. This student wrote:

> To get a mixed number into an improper fraction you times the denominator by the
> whole number and then add the numerator

> m = mixed number

> $$\frac{md + n}{d}$$

This general rule appears partially confused, as it is not the mixed number (m) that is

multiplied by the denominator, but its integer component. However, it is perhaps

unrealistic to expect perfect accuracy from a student who is so new to expressing

themselves algebraically. This may be because their attention was on both the correct

algebraic notation required, and the general procedure they were trying to express.

The student's expression of the general procedure might be taken as *algebraic*

babbling, as defined by Malara and Navarra (2003, see section 2.4.3), and thus taken

as a natural and unconcerning consequence of a focus on semantics rather than syntax in early algebra.

A similar range of responses was offered in response to question seven, explaining how to convert from an improper fraction to a mixed number:

> See how many times the denominator goes into the numerator and how many times would be the whole number then what was left behind would be the numerator and the denominator is the same.

> You see how many times the denominator goes into the numerator and write that to the side. You see how many is remained from this and write that above the denominator.

> You see how many times the denominator goes into the numerator and make the remainder as a fraction.

> Divide the numerator by the denominator, the answer will be the whole number. The remainder will be the numerator. The denominator will stay the same.

> You must divide the numerator by the denominator and as many times it goes in with no remainder is the whole number on the left, and the left overs makes the fraction on the right. Eg:
> 11/5
> $11 \div 5 = 2$ r. 1 The 2 goes on the left and the remainder is the fraction 1/5 So the answer is 2 1/5.

As the examples above illustrate, the students used a wide variety of means to communicate their methods, including particular examples, technical mathematical vocabulary (numerator and denominator) and, in one case, algebra. In a 'mixed-ability', year 7 classroom, the number of students who offered a clear general description might be considered encouraging, but at the time I was disappointed that some students had felt unable to do so.

One student, with high prior attainment, wrote "Don't know myself, would ask [names of three students on his table]". He generally acts as if eager both to learn and to meet teacher expectations, so it is interesting that he chose to write this, and to read it out when I asked students to share their explanations with the class. He seemed to have humorous intent, as he was smiling himself, and did not seem surprised when I laughed, but I wonder at the extent to which it contains elements of the truth. Another student (sitting nearby) wrote a description of how to convert an improper fraction into a mixed number, but in response to question six wrote "sorry don't know ask brainy people" then provided a list of eleven names of students in the class, as well as my own name. I was intrigued to see that he had included me amongst this list, roughly in the middle. The culture I aspire to creating with this group is one in which the students' contributions are considered as valuable as my own, and that it is acceptable to ask other students in the class for help with understanding a concept or process. Perhaps because of my own focus on this, I am reading too much into this student's writing, which may not be representative or indicative of his genuine views of the mathematics classroom. Despite this, I feel it is a possible positive indication that he views asking others for help as an acceptable mathematical behaviour, and that he does not distinguish strongly the value of students' and teachers' contributions.

Sections 9.3, 9.4 and 9.5 gave an insight into how the findings of the main study, described in chapters 7, 8-I and 8-II, influenced reflection on my own teaching practice, and how algebra was used to express general mathematical procedures and concepts in teacher-led discourse with my year 7 class.

9.6 CHAPTER SUMMARY

This chapter serves both to indicate classroom implications of the main study findings and to suggest avenues for further research. The framework has been used to inform reflection on my own practice. The framework was used to trigger and structure reflections and analyses that might otherwise have been overlooked, and acts as a prompt to elicit deeper insight rather than as an ongoing underlying framework. As suggested by the findings of chapter 8-I, use of algebra to express general procedures and concepts could be more comprehensively included in schemes of work. Although my initial work with the year 7 group was not thoroughly and instantaneously effective, the (sometimes spontaneous) take-up of algebraic notation in these and other lessons with the group is encouraging. Further study would be required to monitor the effects of this change both on students' use of algebra to express generality, and their understanding and application of the procedures and concepts.

CHAPTER 10: CONCLUSION

This final chapter draws together the insights that have emerged from this study of the use of algebraic language to express generality. The discussion is in three parts. The first section summarises the findings reported in the preceding chapters, relating them to the study's three research questions (10.1). An overview of the study's findings is then offered in the next section (10.2). The chapter ends with a critical reflection on the design and undertaking of this research study (10.3).

10.1 SUMMARY OF FINDINGS

This section re-presents the findings of the study in relation to the foci of the three research questions:

Research Question One	**What generalisations are being expressed in secondary mathematics classrooms?**
Research Question Two	**How are procedural generalisations expressed in mathematics classrooms?**
Research Question Three	**How are conceptual generalisations expressed in mathematics classrooms?**

Research Question 1

What generalities are being expressed in secondary mathematics classrooms?

The study revealed the complexity of generality in secondary mathematics classrooms. The review of the literature revealed the importance of generality in

mathematics education. However, the significant role that generality plays in mathematics was not found to be emphasised in the main study lessons.

The teacher-led discourse from my own teaching practice in chapter five illustrated the wide variety of 'types' of generalisation. Distinctions were made between generalities concerning algebra, the activity in which the students are being invited to participate, and behaviour in mathematics classrooms. It was found that consideration of the lesson from the perspective of generality and generalisation offered insight into potential ambiguities and possible changes in teacher practice that might support student understanding.

In the main study the approach of identifying generalisations in teacher-led discourse was expanded to observations of fifty-two lessons taught by five teachers in a secondary comprehensive in Oxfordshire. Shaped by the literature, five categories for distinguishing between generalisations emerged from the transcribed data, informed by observation notes and student work. The categories that emerged included consideration of the object of the underlying generality (procedure, concept, or behaviour), its longevity of relevance (whether it applies just in the current task or lesson, or is more universal) and its justification (mathematically necessary, conventional or behavioural/social). Generalisations were also analysed through consideration of their derivation-origin in the given instance (telling, pattern-spotting or reasoning) and the awareness that was being promoted (affective, cognitive or enactive). These categories are introduced in section 7.3, and summarised in the table overleaf.

Underlying generality	Subject Longevity Justification	procedure, concept or behaviour transient or universal mathematical necessity, conventional or behavioural/social
This given utterance	Derivation Awareness	telling, pattern-spotting or reasoning affective, cognitive or enactive

This framework was shown to be useful in bringing to attention areas of potential mismatch between teacher intention and student experience.

Research Question 2

How are procedural generalisations expressed in mathematics classrooms?

This study has shown that regarding the teaching of mathematical procedures as a process of supporting students in their appreciation of procedural generalisations offers insight into teaching decisions that may improve practice. In the main study, as described in chapter 8-I, every lesson observed involved procedural generalisation, each of which was an opportunity to generalise. This generalisation was rarely made explicit in the discourse, with shifts between the particular and the general being made implicitly, if at all. Whilst the main study teachers, knowing that this study was focusing on *generalisation*, frequently apologised that there 'wasn't any generalising' in the lessons being observed, I identified numerous generalisations that were offered as methods to be remembered and practised, rather than general procedures whose scope and derivation can be tested and proven. The research findings suggest that teachers being explicit (in self-talk as well as with students) about use of particular examples, and expression of the general, when teaching students mathematical procedures, might increase student awareness of those procedures. Reflection on my own practice, discussed in chapter nine, explored how algebraic notation could be used in every mathematics lesson to focus attention on the general nature of the

procedures being taught and learnt, as well as to develop students' purposeful (Ainley *et al.*, 2004) use of algebraic notation.

Analysis of main study lessons, discussed in chapter 8-I, revealed that whilst numerous procedural generalisations were expressed, students' attention appeared to be directed towards them enactively rather than cognitively. Hewitt's (1999) distinction between the arbitrary and necessary was employed to suggest that mathematically necessary procedures could be deduced by students themselves using their powers of reason, thereby encouraging cognitive, rather than merely enactive, engagement with the procedure. Where procedures are arbitrary, alternative strategies might be more appropriate, such as offering several examples and asking students to express what is the same and what is different. One significant contribution of this study has been the proposal that cognitive awareness of procedural generalisations might also be promoted through inviting students to express the procedure algebraically.

Research Question 3

How are conceptual generalisations expressed in mathematics classrooms?

Chapter 8-II reported the findings of analysis at all three levels related to the expression of general concepts. Research findings suggested a tension for teachers between developing students' mathematical language, which will benefit future communication, and emphasising current communication. A teacher's correction of students' unconventional use of mathematical language could create the appearance of also criticising the mathematical conjecture the student is making, and might hamper the development of a conjecturing atmosphere.

Davydov's separation of concept appreciation into *scope, name* and *definition* was offered as an effective framework for researching the opportunities offered to students for developing general concepts during teacher-led discourse. The study has shown that names and their scopes and meanings can become separated during teacher-led discourse. In the main study lessons, concept names were often used before students had been offered an opportunity to develop the concept, or even the conception (8-II.2). It was found that concepts were introduced and defined in lessons where no use was subsequently found for the concepts' meaning (8-II.3).

The research findings reported in section 8-II.4 demonstrated that conceptions were often discussed and developed in main study classrooms without some or even any of Davydov's (1972/1990) criteria for their becoming concepts. Teacher-led discourse that focused on a particular conception often demonstrated lack of clarity about the exact scope of a concept. Concepts were often discussed without the correct name, occasionally even with an incorrect name. The majority of uses of concepts were supported by particular examples. Analysis suggested that the teaching decisions surrounding how and when to support students in developing conceptions into concepts is both complex and fascinating.

10.2 OVERVIEW OF FINDINGS

This study has probed the complexity of the manipulative and expressive roles of algebra, and brought to the surface some of the choices that teachers make, with the intention that they might become more aware of their impact. I have shown, through observation, recording and analysis of fifty-two lessons, that there is a multitude of

generalisation in mathematics classrooms. The effect of this research on me as the researcher was to sensitise me to the possibility that the potential of algebra as a medium for purposeful expression of mathematical generality has been overlooked. Whilst working towards addressing the three research questions discussed in section 10.1, throughout analysis of the fifty-two main study lessons, I became aware of an apparent mismatch between the huge potential of algebra as a tool to express generality, and of mathematical generality as a 'real context' for developing use of algebraic notation.

The study has shown that the language used to express general concepts and general procedures is often obtuse or appears to go unnoticed. Whilst awareness of the manifold complexities of algebraic symbols and expressions might lead teachers to neglect algebraic notation in those lessons where algebraic manipulation is not the main objective, the complexities of everyday language and mathematical terminology illustrated in this study revealed that algebra might offer a less ambiguous tool for the expression of general mathematical concepts and procedures.

Whilst students studying mathematics at 'A' level and beyond are expected to understand and use procedures and concepts expressed in algebraic notation, analysis of main study lessons found that this is not the case in younger classes. This study's research findings suggest that the advantages (conciseness, preciseness) of algebra to express generality in higher level mathematics courses could be used throughout secondary mathematics education. This study has demonstrated that there are many challenges and ambiguities involved in the use of natural and of mathematical language to express general concepts and procedures. It follows from this that

although using algebraic notation would bring its own challenges, it is far from obvious that students would find it was a barrier to understanding.

This study has offered rich data, with detailed descriptions of classroom discourse. The study's findings are made more relevant through detailed descriptions of 'ordinary' lessons. Through engaging in research as a teacher practitioner, I have also shown that engagement in enquiry is an important part of teaching. Implications for practice are illustrated in chapter nine, where the impact of increased sensitivity to expressions of generality on my own teaching is shown. Through studying the relevant literature, analysing others' practice, and reflecting on my own teaching, I have heightened my awareness of the decisions being made in teacher-led discourse, and the possible learning benefits of increased use of algebra in 'ordinary' lessons.

10.3 CRITICAL EVALUATION OF THE STUDY

Several limitations of this study were considered at the design stage and reflect the fact that decision-making in research involves compromise. Aspects that have strengths in one respect often have weaknesses in others. Some limitations relate to the generalisability of the study (10.3.1), others to the combination of researching my own practice and that of others (10.3.2). Further limitations are associated with impact on teaching practice (10.3.3). The concluding section considers these limitations with respect to the principles used to guide research design and analysis that were introduced in section 3.3.

Undertaking research involves continually trying to find a balance between different sets of objectives that seem to be pulling in conflicting directions. The challenge,

though, is not so much in making the trade-offs, but in trying to remain aware of
when, how and why the trade-offs are made. This research has involved making trade-
offs between:

1. depth of detail and breadth of generalisability;
2. studying other teachers and reflecting on my own practice
3. understanding practice and changing practice.

This section will critically reflect on the course that this study has steered between
these three sets of alternatives, and in so doing consider how they might better be
approached in the future.

10.3.1 Depth of detail and breadth of generalisibility

A perennial dilemma in all research is that of depth versus breadth. The former is
often championed in terms of understanding and insight while the latter is proposed as
the route towards generalisation and external validity. The nature of the focus and
questions that this study sought to investigate meant that a small-scale approach was
both sensible and necessary. The onus throughout the work has been on depth of
detail, rather than breadth of generalisibility. This is seen not only in the small
numbers of case study settings, teachers and students, but also in the analytical focus
on specific lessons. The small scale strategy afforded rich data relating to individual
class discourses, but this came at the expense of generalisability to other classrooms.

The advantage of this trade-off lies in the richness of the data, the detail of the
descriptions of classroom discourse and the relevance of the findings and implications
for practice. The disadvantage, though, is an inability to know with confidence the
extent to which the findings and implications of this study might be applicable to

settings beyond the five classrooms that were investigated. At this stage, it would seem important that the conclusions of this study are seen as ideas that are valid in relation to the contexts in which they were generated, but that will require further investigation in different settings to explore their potential generalisibility. I would add that I feel strongly that the question of the generalisibility of findings from a research study such as this is not simply a question of methodological theory, but should also be gauged through dialogue with practitioners about the applicability of the ideas that have emerged. In addition to this, the ideas could be considered as potential frameworks used to explore further settings. Consequently, any generalisations arising from this study should be constructed in a 'naturalistic' form. These are generalisations, not of a statistical nature, but instead, generalisations "that form the basis of hypotheses to be carried from one case to the next" (Brown and McIntyre, 1993:50). Further investigation in different settings, therefore, would enable the potential generalisability of the findings of this study to be explored.

10.3.2 Studying other teachers and reflecting on own practice

The process of reflecting on my own practice, informed by relevant literature and observation of others, is one that I regard as an essential part of my teaching. Whilst the research questions raised in this study could have been explored through structuring and formalising this existing informal practice, it was also important to make use of other research methods to gain deeper insight.

One alternative study design would have placed my own lessons amongst the main study lessons, and carried out all three levels of analysis on my own practice alongside that of my colleagues. Quite apart from the difficulties of fairly comparing

my own actions (about which I have some level of appreciation of motive and intention) with those of colleagues, I found it impossible to reflect on my own practice and possible alterations that might be more effective for student learning without implementing those changes in my future practice. I was unable to simultaneously systematically reflect on potential improvements to my practice and teach as I 'ordinarily' would. Whilst it was an important part of my research design that the lessons analysed were 'ordinary ' lessons, I could not carry out level 1 analysis of my practice without wanting to change it. In the following section I discuss the tension between understanding and changing practice, which relate to this point.

10.3.3 Understanding practice and changing practice

This study has been very much about understanding, rather than changing, mathematics whole class discussions. While I would argue that this was justified in light of the desirability of more exploratory work on maths classroom discussions, I feel it is also important to recognise the potential limitations of such a strategy. I refer particularly to the problem of academic research that does little to actually inform the development of teaching and learning within actual schools. In one sense this is a dissemination issue, in that there is a very crucial need to explore creative and constructive ways of communicating research findings to practitioners. It is also, though, a question of methodology, and I feel that great benefit could be gained, in terms of relevance for practice, by undertaking classroom curriculum research that is more collaborative in relationship, and change-orientated in its intentions. The potential for such approaches to build upon, use and refine, the insights generated from studies such as this one is a particularly exciting prospect. To this end, the

considerable literature on collaborative research with, and action research by, teachers, provides a wealth of ideas and experience on which to draw.

The limitations discussed above have not prevented the making of claims which, with their basis in both theory and evidence, constitute a response to the research questions posed at the outset of the study, and offer a novel contribution to knowledge.

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