SPECTRAL EDGE PROPERTIES OF PERIODIC ELLIPTIC OPERATORS

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2017

Major Subject: Mathematics

ABSTRACT

In this dissertation, we study some spectral problems for periodic elliptic operators arising in solid state physics, material sciences, and differential geometry. More precisely, we are interested in dealing with various effects near and at spectral edges of such operators. We use the name "threshold effects" for the features that depend only on the infinitesimal structure (e.g., a finite number of Taylor coefficients) of the dispersion relation at a spectral edge.

We begin with an example of a threshold effect by describing explicitly the asymptotics of the Green's function near a spectral edge of an internal gap of the spectrum of a periodic elliptic operator of second-order on Euclidean spaces, as long as the dispersion relation of this operator has a non-degenerate parabolic extremum there. This result confirms the expectation that the asymptotics of such operators resemble the case of the Laplace operator.

Then we generalize these results by establishing Green's function asymptotics near and at gap edges of periodic elliptic operators on abelian coverings of compact Riemannian manifolds. The interesting feature we discover here is that the torsion-free rank of the deck transformation group plays a more important role than the dimension of the covering manifold.

Finally, we provide a combination of the Liouville and the Riemann-Roch theorems for periodic elliptic operators on abelian co-compact coverings. We obtain several results in this direction for a wide class of periodic elliptic operators. As a simple application of our Liouville-Riemann-Roch inequalities, we prove the existence of non-trivial solutions of polynomial growth of certain periodic elliptic operators on noncompact abelian coverings with prescribed zeros, provided that such solutions grow fast enough.

DEDICATION

To my father Xuong Kha, my mother Sang Lam, and my sister Loan Kha.

ACKNOWLEDGMENTS

I don't even know how to begin thanking my family. Thank you, Mom and Dad, for giving birth to me and your unconditional love, endless care, and spiritual support throughout my life. There is no way I could have accomplished this without you.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a thesis (or) dissertation committee consisting of Professor Peter Kuchment [advisor], Professor Gregory Berkolaiko, and Professor Kenneth Dykema of the Department of Mathematics and Professor Valery Pokrovsky of the Department of Physics.

The results in Chapter 3 were obtained from a collaboration of the student with Professor Peter Kuchment (Texas A&M University) and Professor Andrew Raich (University of Arkansas), and this will be published in an article listed in the Biographical Sketch (see [40]). The results in Chapter 5 were obtained together with Professor Peter Kuchment.

All other work conducted for the dissertation was completed by the student independently (see [41, 42]).

Funding Sources

Graduate study was supported by a Graduate Teaching Assistantship from the Department of Mathematics at Texas A&M University and also partially supported by the National Science Foundation.

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1. INTRODUCTION

Periodic media play a crucial role in solid state physics as well as in other areas (e.g., meta- and nano-materials). A remarkable example is the study of crystals: The atoms in a perfect crystal are placed in a periodic order and this order induces many interesting properties of the material. Mathematically speaking, to describe the one-electron model of solid state physics [6], one uses the stationary Schrödinger operator $-\Delta + V$ with a periodic potential V that represents the field created by the lattice of ions in the crystal. In general, one can study elliptic PDEs with periodic coefficients arising naturally from other contexts. For instance, periodic magnetic Schrödinger operators, overdetermined systems like Maxwell equations in a periodic medium, periodic elliptic operators on abelian coverings of compact manifolds or finite graphs are also of interest. It has been known for a long time that in mathematical physics, the Floquet-Bloch theory is a standard technique for studying such operators. Although periodic elliptic PDEs have been studied extensively, there are still many theoretical problems to address as well as new applications to explore such as metamaterials (e.g., graphene and photonic crystals), carbon nanostructures, topological insulators, waveguides and so on.

In Chapter 2, we briefly discuss periodic elliptic operators and then give a quick review of the Floquet transform and the Floquet-Bloch theory. Here we introduce definitions, notations, and preliminary results that will be needed in the subsequent chapters.

In Chapter 3, we describe the behavior at infinity of Green's functions near gap edges for periodic elliptic operators in $\mathbb{R}^d$ ($d \geq 2$), which is an example of a threshold effect. These asymptotics are relevant for many important applications in Anderson localization, impurity spectra, Martin boundaries and random walks. The original motivation comes from the well-known asymptotic behavior of the Green's function of the Laplacian in $\mathbb{R}^d$

outside its spectrum: along with an algebraic decay factor, the Green's function decays exponentially at infinity with the rate of decay controlled by the distance to the spectrum. One may ask how the Green's functions of other classes of elliptic operators behave at infinity: Would it resemble the Laplacian case? A natural candidate is the class of periodic self-adjoint elliptic operators of the second-order.

Exponential decay estimates of Schwartz kernels of resolvents, such as the Combes-Thomas estimates [18] are well-known. However, as most of them are obtained by operator-theoretic techniques, they do not capture precisely the anisotropy of the asymptotics for periodic operators. Moreover, an additional algebraically decaying factor besides the exponential decay rate is lost in this approach.

Since one can compute the asymptotics of the Green's function of the Laplacian by using Fourier transform, it is natural to use Floquet theory in the periodic case. Consider a periodic self-adjoint elliptic operator L on $\mathbb{R}^d$, where $d \geq 2$. Its spectrum has a band-gap structure, so it is reasonable to look at the Green's function around the spectral edges of a gap. A special case is of the bottom of the spectrum, for which results were established by Babilott [9] and then by Murata-Tsuchida [60].

Here we attack the case of an internal spectral edge, where the corresponding band function $\lambda_j(k)$ has an isolated nondegenerate extremum.[1] By using Floquet transform and a localization argument, we reduce the problem to studying the behavior of a scalar integral that involves a germ of the branch λ_j. Here only the local structure around the edge of the branch λ_j is solely responsible for the main term of the scalar integral, and this is where nondegeneracy of the extremum is used to compute the asymptotics. This also confirms the expectation that the asymptotics resemble the ones for the Laplacian. Indeed, the asymptotics of the Green's function $G_\lambda(x, y)$ of L at an energy level λ that is

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[1]Non-degeneracy means that the Hessian matrix of the corresponding band function $\lambda_j(k)$ at the gap edge is non-degenerate.

sufficiently close to a simple and nondegenerate spectral edge can be described roughly as follows:

$$G_\lambda(x,y) \sim \frac{e^{-\gamma_s|x-y|}}{|x-y|^{(d-1)/2}} \times C(\lambda,x,y), \quad \text{where} \quad s := \frac{(x-y)}{|x-y|}. \qquad (1.1)$$

Here $\gamma_s > 0$ depends on the direction s only and $C(\lambda,x,y)$ is a bounded term depending on the local structure of the dispersion relation of L around the spectral edge. Hence, the rate of exponential decay is usually non-isotropic, unlike in the Laplacian case. The additional algebraic decay is also captured.

In Chapter 4, we extend the above results to periodic elliptic operators on an abelian cover of a compact base. An abelian cover X is a normal Riemannian cover such that its deck transformation group G is abelian and the quotient space $M = X/G$ is a compact Riemannian manifold. Without loss of generality, one can assume $G = \mathbb{Z}^d$. Let $d_X(\cdot,\cdot)$ be the Riemannian distance on X. Note that the dimension n of the manifold X could be different from the rank d of the deck group G. Let A be a self-adjoint elliptic operator of second-order on X such that it is periodic, i.e., A commutes with actions of G. One can define a Floquet transform on X so that the machinery of Floquet theory still works well. The notions of dispersion relations, band-gaps, and spectral edges are defined similarly to the flat case and thus, the question of finding the asymptotics of Green's function $G_{A,\lambda}(x,y)$ of A at a level λ that is near to a non-degenerate gap edge still makes sense. At first glance, one may expect that the leading term of the Green's function of A can be represented in terms of the distance $d_X(x,y)$ like the formula (1.1) in the flat case. However, this is just 'almost' true as we show that the asymptotics should look like

$$G_{A,\lambda}(x,y) \sim \frac{e^{-\gamma_s|h(x)-h(y)|}}{|h(x)-h(y)|^{(d-1)/2}} \times \tilde{C}(\lambda,x,y), \quad \text{where} \quad s = \frac{h(x)-h(y)}{|h(x)-h(y)|}. \qquad (1.2)$$

Here, the mapping h from X to $\mathbb{R}^d$ is an analog on an abelian covering of the coordinate function x in the flat case. [2] It satisfies $|h(x) - h(y)| \sim d_X(x, y)$ when x and y are sufficiently far apart. The positive constant γ_s depends on s, and the term $\tilde{C}(\lambda, x, y)$ is again bounded and does not contribute much to the asymptotics at infinity. An interesting feature is that the rank d of G plays a more important role than the dimension n of X, since n does not enter explicitly into (1.2). This feature is in line with Gromov's idea that the large scale geometry of X is captured mostly by its deck group G. Furthermore, the result of [52] concerning the behavior at exactly a spectral edge is also extended to co-compact abelian coverings.

In Chapter 5, we combine the Riemann-Roch and the Liouville theorems for periodic elliptic operators on abelian coverings. The classical Riemann-Roch theorem for compact Riemann surfaces has been extended in various ways to higher dimensional settings. Instead of taking the viewpoint from algebraic geometry, Gromov and Shubin took a motivation from classical analysis of solutions of general elliptic equations with point singularities [32] and beyond [33]. Consider a compact manifold X of dimension n, a point divisor is an element in the free abelian group generated by the points of X. For an elliptic operator P of order m and a point divisor μ on X, they define the space $L(\mu, P)$ of "meromorphic" solutions of P associated to μ by taking all solutions that are allowed to have some poles at points that enter μ with a positive degree and are required to have zeros at the points that enter μ with a negative degree. Also, the multiplicities of these poles and zeros are controlled from above and below by quantities involving the corresponding degrees. The Riemann-Roch type formula appeared in [32] is a link between the dimensions of the space $L(\mu, P)$ and the 'dual' one $L(\mu^{-1}, P^*)$, where μ^{-1} is the dual divisor of μ and

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[2]A topological approach to defining such mappings on **any** Riemannian co-compact coverings can be found in Appendix A.

P^* is the adjoint operator of P:

$$\dim L(\mu, P) = \dim L(\mu^{-1}, P^*) + \operatorname{ind} P + \deg_P(\mu), \tag{1.3}$$

where $\operatorname{ind} P$ is the Fredholm index of P and $\deg_P(\mu)$ is the degree of μ that is written in terms of binomial coefficients involving m, n and μ. The classical Riemann-Roch formula is a special case of (1.3) if P is the $\bar\partial$ operator on a compact Riemann surface X. Later in [33], Gromov and Shubin proved a much more general version of (1.3) for solutions of general elliptic equations with singularities supported on arbitrary compact nowhere dense sets of a manifold (e.g., a Sierpinsky carpet). In this version, the analogs of point divisors are rigged divisors.[3] A version of (1.3) on non-compact manifolds is also given in [32, 33].

On the other hand, the classical Liouville theorem says that a harmonic function that grows polynomially is a harmonic polynomial and so, the space of all harmonic functions in $\mathbb{R}^n$ that are bounded by $C(1 + |x|)^N$ is of finite dimension:

$$h_{n,N} := \binom{n + N}{n} - \binom{n + N - 2}{n}. \tag{1.4}$$

This leads to a natural problem concerning the finite dimensionality of the spaces of solutions of an assigned polynomial growth, estimates of their dimensions, and descriptions of the structures of these solutions for more general elliptic operators on certain non-compact manifolds. In the flat case $\mathbb{R}^n$, M. Avellaneda-F.H. Lin [8] and J. Moser-M. Struwe [59] answered the question for any second order divergence form elliptic equation with periodic coefficients by using tools of homogenization theory: they proved that the dimensions of the spaces of all solutions of these equations of polynomial growth of order at most N are equal to $h_{n,N}$. However, homogenization techniques have significant limitations, e.g., they

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[3]A rigged divisor μ is a tuple $(D^+, L^+; D^-, L^-)$, where D^+, D^- are disjoint nowhere dense compact sets and L^+, L^- are finite-dimensional spaces of distributions supported on D^+, D^-, respectively.

work at the bottom of the spectrum only. Using Floquet theory and duality arguments, in [50], P. Kuchment and Y. Pinchover obtained Liouville type results for a wide range of elliptic periodic operators on abelian co-compact coverings. The main result of [50] says that the Liouville type result happens iff the corresponding Fermi surface is finite, which normally happens at a spectral edge. This is another example of threshold effects: the dimensions are calculated explicitly based on the lowest order term of a non-zero Taylor expansion term of the dispersion relations at a spectral edge.

A solution of polynomial growth of order N can be considered as a solution that is allowed to have a "pole at infinity" of order at most N, while a solution that is "zero at infinity" with multiplicity at least N can be regarded as a solution with a rate of decay of polynomial of order N. So one may think a Liouville type result as a Riemann-Roch type result for divisors located at infinity. It is thus natural to try to combine these theorems.

Consider $p \in [1, \infty]$, $N \in \mathbb{R}$ and an abelian cover X whose deck group is $G(= \mathbb{Z}^d)$. Suppose that μ is a rigged divisor (defined in Section 5.3) and A is a periodic elliptic operator on X. We denote by $L_p(\mu, A, N)$ the space of all solutions u of the equation $Au = 0$ with zeros enforced and poles allowed by the divisor μ, and of polynomial growth (in L^p-sense) of order N at infinity (see Section 5.4). Now we can give a rough statement (somewhat vague at this point) to outline some of our main results in Chapter 5.

Theorem 1.0.1. *Assume that $N \geq 0$, $p \in [1, \infty]$ and μ is a rigged divisor on X. We define* $p' := p/(p - 1)$.

 (i) *If 0 is in the resolvent set of A, then the following Liouville-Riemann-Roch equality holds:*

$$\dim L_p(\mu, A, N) = \dim L_{p'}(\mu^{-1}, A^*, -N) + \deg_A(\mu).$$

 (ii) *Suppose that 0 is in the spectrum of A and the operator A satisfies the Liouville property. Let us denote by $V_N^p(A)$ the space $L_p(\mu_0, A, N)$, where μ_0 is the trivial*

divisor $(\emptyset, 0; \emptyset, 0)$. *Assume either* $p = \infty$, $N \geq 0$ *or* $p \in (1, \infty)$, $N > d/p$. *Then under some local conditions on the dispersion relation of* A, *we have the following Liouville-Riemann-Roch inequality:*

$$\dim L_p(\mu, A, N) \geq \dim V_N^p(A) + \deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A, -N). \qquad (1.5)$$

The inequality (1.5) can be strict in some cases, e.g., when $A = -\Delta$ *on* $\mathbb{R}^3$, $N = 0$ *with some* μ. *However, when* μ *contains only poles, the equality in (1.5) occurs.*

Here $\deg_A(\mu)$ is the integer depending on μ and the differential operator A (see Section 5.3).

In particular, we are able to show that the Liouville-Riemann-Roch inequalities hold for 'generic' periodic Schrödinger operators at their gap edges and for two-dimensional periodic Schrödinger operators with honeycomb lattice potentials. An immediate application is the non-triviality of the space $L_\infty(\mu, A, N)$ in many situations, e.g., when 0 is at the bottom of the spectrum of $A = -\Delta + V$ and $h_{d,N} + \deg_A(\mu) > 0$. This also implies the fact that one can always find a non-zero solution u of polynomial-growth of A with prescribed zeros as long as N is large enough. The latter fact can be considered as an analog for noncompact abelian coverings of the following well-known property which could be deduced from the classical Riemann-Roch equality: the existence of a non-trivial meromorphic function on a compact Riemann surface with prescribed zeros, provided that poles of sufficiently high orders are allowed.

The results of this dissertation have been published in:

1. (Joint with Peter Kuchment, Andy Raich), *Green's function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral gap interior*, To appear in Journal of Spectral Theory (2016), arXiv:1508.06703.

2. *Green's function asymptotics of periodic elliptic operators on abelian coverings of compact manifolds*, Submitted, arXiv:1511.00276.

3. *A short note on additive functions on Riemannian co-compact coverings*, Preprint, arXiv:1511.00185.

4. (Joint with Peter Kuchment), *A Liouville-Riemann-Roch theorem on abelian coverings*, in preparation.

The results of this dissertation have been presented in the following talks:

1. *Green's function asymptotics of periodic elliptic operators on abelian coverings of compact manifolds*, Contributed talk, Ohio River Analysis Meeting, University of Kentucky, Kentucky, 03/12/2016.

2. *Green's function asymptotics of periodic elliptic operators on abelian coverings of compact manifolds*, Invited talk, Seminar in Applied Mathematics, Department of Mathematics and Statistics, Texas Tech University, TX, 03/02/2016.

3. *Green's function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral gap interior.*, Invited talk, Special Session on Analysis of Differential and Integral Equations, AMS Fall Southeastern Sectional Meeting, University of Memphis, TN, 10/18/2015.

4. *Green's function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral gap interior.*, Contributed talk, Prairie Analysis Seminar, Kansas State University, Manhattan, KS, 09/26/2015.

5. *Green's function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral gap interior.*, Mathematical Physics and Harmonic Analysis Seminar, Texas A&M University, 03/06/2015.

2. FLOQUET-BLOCH THEORY FOR PERIODIC ELLIPTIC OPERATORS

2.1 Periodic elliptic operators

Let a linear differential expression

$$Lu(x) = L(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x) \tag{2.1}$$

of order $m \geq 1$ be given in $\mathbb{R}^d$. Here if $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, the notation D^α stands for $D_1^{\alpha_1} \ldots D_d^{\alpha_d}$, where $D_k := -i\partial_k = \dfrac{1}{i}\dfrac{\partial}{\partial x_k}$. All coefficients a_α are smooth functions on $\mathbb{R}^d$ and they are periodic with respect to the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, i.e., $\forall x \in \mathbb{R}^d, n \in \mathbb{Z}^d$: $a_\alpha(x+n) = a_\alpha(x)$.

The formal adjoint expression (or transpose) is defined as

$$L^t u(x) = L(x, D)^t u = \sum_{|\alpha| \leq m} D^\alpha(\overline{a_\alpha(x)}u(x)),$$

i.e., $\langle Lu, v \rangle = \langle u, L^t v \rangle$ for any $u, v \in C_c^\infty(\mathbb{R}^d)$, where $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product in $\mathbb{R}^d$. If $L = L^t$, then L is called a formally self-adjoint differential expression and the operator L with the domain $C_c^\infty(\mathbb{R}^d)$ is symmetric.

The full symbol of L is the polynomial (in $\xi \in \mathbb{R}^d$) $L(x, \xi) = \sum\limits_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha$, while the principal symbol of L is the polynomial $L_0(x, \xi) = \sum\limits_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. Note that $(L^t)_0(x, \xi) = \overline{L_0(x, \xi)}$ and hence, the principal symbol of a formally self-adjoint expression is always real.

The scalar differential expression L is said to be (uniformly) elliptic if its principal

symbol satisfies the inequality

$$|L_0(x,\xi)| \geq c|\xi|^m, \quad c > 0.$$

Under these assumptions, the differential expression (2.1) is defined for $u \in L^2(\mathbb{R}^d)$ in the distribution sense. To consider L as a linear operator in $L^2(\mathbb{R}^d)$, we have to choose a domain [1] $\mathcal{D} \subset L^2(\mathbb{R}^d)$ of L such that $Lu \in L^2(\mathbb{R}^d)$ for any $u \in \mathcal{D}$. We denote by $L_{|\mathcal{D}}$ the corresponding operator (or a *realization*) of the differential expression L. There are two important realizations of L. The maximal operator $L_{\max}$ is the linear operator $L_{|D_{\max}}$, where its domain $D_{\max}$ is the largest possible one, i.e.,

$$D_{\max} := \{u \in L^2(\mathbb{R}^d) \mid Lu \in L^2(\mathbb{R}^d)\}.$$

The minimal operator $L_{\min}$ is defined to be the closure of the operator $L_{|C_c^\infty(\mathbb{R}^d)}$. Since L is uniformly elliptic and its coefficients are smooth, $D_{\min} = H^m(\mathbb{R}^d)$. When L is a formally self-adjoint expression, it follows that the maximal operator $L_{\max}$ is the adjoint $L_{\min}^*$ of the minimal operator $L_{\min}$. If $L_{\min} = L_{\max}$ or equivalently, $D_{\max} = D_{\min}$, then $L_{\min}$ is a self-adjoint operator and in this case, L is said to be essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$. In other words, L has a unique self-adjoint realization. It is known [71] that when $L = L^t$ and the coefficients of L are smooth and periodic, the differential expression L is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$. We also use the same notation L to denote the self-adjoint operator $L_{\min} = L_{\max} = L_{|H^m(\mathbb{R}^d)}$. For simplicity, we also say that L is self-adjoint in this case.

We now describe some important examples of periodic elliptic operators, which will come up in the next chapters.

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[1] I.e., a dense linear subspace of $L^2(\mathbb{R}^d)$.

(a) The one-electron model of solid state physics can be described by the *periodic Schrödinger operator* in $L^2(\mathbb{R}^d)$

$$L = -\Delta + V(x),$$

where the electric potential V is sufficiently smooth [2] and periodic with respect to the group $\mathbb{Z}^d$. The domain of this operator is the Sobolev space $H^2(\mathbb{R}^d)$. This Schrödinger operator is self-adjoint whenever the potential V is real. Note that many of the techniques and results do not require self-adjointness, i.e., they can also hold for complex-valued potentials V.

(b) One can also consider the *magnetic Schrödinger operators*

$$L = (-i\nabla + A(x))^2 + V(x),$$

where $A(x)$ and $V(x)$ are periodic magnetic and electric potentials. More generally, the Schrödinger operators with the presence of periodic metrics

$$L = -\nabla \cdot g(x)\nabla + iA(x) \cdot \nabla + V(x)$$

are also of interest. Usually, these operators are more complicated to study.

(c) *Periodic elliptic operators of higher than second-order* are also worthy of studying. A typical example is the polyharmonic Schrödinger operator

$$L = (-\Delta)^{m+1} + V(x),$$

where V is a real and periodic potential and m is a positive integer. We would like to remark that unlike the second-order case, one may encounter more difficulties with

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[2] L^∞ suffices (see e.g., [66]).

these operators since they may fail even the weakest version of unique continuation.

(d) In this chapter, we mainly discuss $\mathbb{Z}^d$-periodic elliptic operators on $\mathbb{R}^d$. Most of the techniques and some of the results in this thesis can be generalized to the case of periodic elliptic operators on abelian coverings of compact Riemannian manifolds (see Chapter 4 and Chapter 5).

2.2 Floquet-Bloch theory

Notation 2.2.1.

(a) Let $W = [0, 1]^d \subset \mathbb{R}^d$ be the unit cube, which is a **fundamental domain** of $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$ (**Wigner-Seitz cell**).

(b) The **dual** (or **reciprocal**) **lattice** is $2\pi\mathbb{Z}^d$ and the **Brillouin zone** is its fundamental domain $B = [-\pi, \pi]^d$.

(c) The d-dimensional tori with respect to the lattices $\mathbb{Z}^d$ and $2\pi\mathbb{Z}^d$ are denoted by $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ and $(\mathbb{T}^*)^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$, respectively.

Definition 2.2.2. For any $k \in \mathbb{C}^d$, the subspace $H_k^s(W) \subset H^s(W)$ consists of restrictions to W of functions $f \in H_{loc}^s(\mathbb{R}^d)$ that satisfy for any $\gamma \in \mathbb{Z}^d$ the **Floquet-Bloch condition** (also known as automorphicity condition or cyclic condition)

$$f(x + \gamma) = e^{ik\cdot\gamma} f(x) \quad \text{for a.e} \quad x \in W. \tag{2.2}$$

Here H^s denotes the standard Sobolev space of order s. Note that when $s = 0$, the above space coincides with $L^2(W)$ for any k. In this definition, the vector k is called the **quasimomentum** [3].

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[3]The name comes from solid state physics [6].

Due to periodicity, the operator $L(x, D)$ preserves condition (2.2) and thus it defines an operator $L(k)$ in $L^2(W)$ with the domain $H_k^m(W)$. In this model, $L(k)$ is realized as a k-independent differential expression $L(x, D)$ acting on functions in W with boundary conditions depending on k (which can be identified with sections of a linear bundle over the torus $\mathbb{T}^d$). An alternative definition of $L(k)$ is as the operator $L(x, D + k)$ in $L^2(\mathbb{T}^d)$ with the domain $H^m(\mathbb{T}^d)$. In the latter model, $L(k)$ acts on the k-independent domain of periodic functions on W as follows:

$$e^{-ik \cdot x} L(x, D) e^{ik \cdot x} = \sum_{|\alpha| \leq m} a_\alpha(x)(D + k)^\alpha. \tag{2.3}$$

Due to ellipticity and embedding theorems (see [47, Theorem 2.1]), the operators $L(k) = L(x, D + k) : H^m(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ are Fredholm for $k \in \mathbb{C}^d$. In addition, note that the condition (2.2) is invariant under translations of k by elements of the dual lattice $2\pi\mathbb{Z}^d$. Moreover, the operator $L(k)$ is unitarily equivalent to $L(k + 2\pi\gamma)$, for any $\gamma \in \mathbb{Z}^d$. In particular, when dealing with real values of k, it suffices to restrict k to the Brillouin zone $[-\pi, \pi]^d$ (or any its shifted copy).

Fourier transform is a major tool of studying linear constant coefficient PDEs, due to their invariance with respect to all shifts. The periodicity of the operator L suggests that it is natural to apply the Fourier transform with respect to the period group $\mathbb{Z}^d$ to block-diagonalize L. In fact, it is an analog of the Fourier transform on the group $\mathbb{Z}^d$ of periods. The group Fourier transform we have just mentioned is the so called **Floquet transform** $\mathcal{F}$ (see e.g., [26, 47, 48]). Now let us consider a sufficiently fast decaying function $f(x)$ (to begin with, compactly supported functions) on $\mathbb{R}^d$.

Definition 2.2.3. The **Floquet transform** $\mathcal{F}$

$$f(x) \to \widehat{f}(k, x)$$

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maps a function f on $\mathbb{R}^d$ into a function $\widehat{f}$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ in the following way:

$$\widehat{f}(k, x) := \sum_{\gamma \in \mathbb{Z}^d} f(x + \gamma) e^{-ik \cdot (x+\gamma)}.$$

From the above definition, one can see that $\widehat{f}$ is $\mathbb{Z}^d$-periodic in the x-variable and satisfies a cyclic condition with respect to k:

$$\begin{cases} \widehat{f}(k, x + \gamma) = \widehat{f}(k, x), & \forall \gamma \in \mathbb{Z}^d \\ \widehat{f}(k + 2\pi\gamma, x) = e^{-2\pi i \gamma \cdot x} \widehat{f}(k, x), & \forall \gamma \in \mathbb{Z}^d \end{cases}.$$

Thus, it suffices to consider the Floquet transform $\widehat{f}$ as a function defined on $[-\pi, \pi]^d \times \mathbb{T}^d$. Usually, we will regard $\widehat{f}$ as a function $\widehat{f}(k, \cdot)$ in k-variable in $[-\pi, \pi]^d$ with values in the function space $L^2(\mathbb{T}^d)$.

We list some well-known results of the Floquet transform (see e.g., [47, 48]):

Lemma 2.2.4.

I. The transform $\mathcal{F}$ is an isometry of $L^2(\mathbb{R}^d)$ onto

$$\int_{[-\pi,\pi]^d}^{\oplus} L^2(\mathbb{T}^d) = L^2([-\pi, \pi]^d, L^2(\mathbb{T}^d))$$

and of $H^2(\mathbb{R}^d)$ into

$$\int_{[-\pi,\pi]^d}^{\oplus} H^2(\mathbb{T}^d) = L^2([-\pi, \pi]^d, H^2(\mathbb{T}^d)).$$

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II. The inversion $\mathcal{F}^{-1}$ is given by the formula

$$f(x) = (2\pi)^{-d} \int\limits_{[-\pi,\pi]^d} e^{ik\cdot x}\widehat{f}(k,x)\,\mathrm{d}k, \quad x \in \mathbb{R}^d. \tag{2.4}$$

By using cyclic conditions of $\widehat{f}$, we obtain an alternative inversion formula

$$f(x) = (2\pi)^{-d} \int\limits_{[-\pi,\pi]^d} e^{ik\cdot x}\widehat{f}(k,x-\gamma)\,\mathrm{d}k, \quad x \in W + \gamma. \tag{2.5}$$

III. The action of any $\mathbb{Z}^d$-periodic elliptic operator L (not necessarily self-adjoint) in $L^2(\mathbb{R}^d)$ under the Floquet transform $\mathcal{F}$ is given by

$$\mathcal{F}L(x,D)\mathcal{F}^{-1} = \int\limits_{[-\pi,\pi]^d}^{\oplus} L(x,D+k)\,\mathrm{d}k = \int\limits_{[-\pi,\pi]^d}^{\oplus} L(k)\,\mathrm{d}k,$$

where $L(k)$ is defined in (2.3).

Equivalently,

$$\widehat{Lf}(k) = L(k)\widehat{f}(k), \quad \forall f \in H^2(\mathbb{R}^d).$$

*IV. (**A Paley-Wiener theorem for $\mathcal{F}$**.) Let $\phi(k,x)$ be a function defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that for each k, it belongs to the Sobolev space $H^s(\mathbb{T}^d)$ for $s \in \mathbb{R}^+$ and satisfies the cyclic condition in k-variable. Then*

(a) Suppose the mapping $k \to \phi(k,\cdot)$ is a C^∞-map from $\mathbb{R}^d$ into the Hilbert space $H^s(\mathbb{T}^d)$. Then $\phi(k,x)$ is the Floquet transform of a function $f \in H^s(\mathbb{R}^d)$ such that for any compact set K in $\mathbb{R}^d$ and any $N > 0$, the norm $\|f\|_{H^s(K+\gamma)} \leq C_N|\gamma|^{-N}$. In particular, by Sobolev's embedding theorem, if $s > d/2$, then the

pointwise estimation holds:

$$|f(x)| \le C_N(1+|x|)^{-N}, \quad \forall N > 0.$$

(b) *Suppose the mapping* $k \to \phi(k, \cdot)$ *is an analytic map from* $\mathbb{R}^d$ *into the Hilbert space* $H^s(\mathbb{T}^d)$*. Then* $\phi(k, x)$ *is the Floquet transform of a function* $f \in H^s(\mathbb{R}^d)$ *such that for any compact set* K *in* $\mathbb{R}^d$*, one has* $\|f\|_{H^s(K+\gamma)} \le Ce^{-C|\gamma|}$*. In particular, by Sobolev's embedding theorem, if* $s > d/2$*, then the pointwise estimation holds:*

$$|f(x)| \le Ce^{-C|x|}.$$

From Lemma 2.2.4, one can deduce the well-known result (see [20, 47, 66]) that the spectrum of L is the union of all the spectra of $L(k)$ when k runs over the Brillouin zone, i.e.

$$\sigma(L) = \bigcup_{k \in [-\pi, \pi]^d} \sigma(L(k)). \tag{2.6}$$

We now remind some notions that play a crucial role in studying periodic PDEs (see e.g., [47, 48]).

Definition 2.2.5.

(a) A **Bloch solution** of the equation $L(x, D)u = 0$ is a solution of the form

$$u(x) = e^{ik \cdot x} \phi(x),$$

where the function ϕ is 1-periodic in each variable x_j for $j = 1, \ldots, d$. The vector k is the **quasimomentum** and $z = e^{ik} = (e^{ik_1}, \ldots, e^{ik_d})$ is the **Floquet multiplier** of the solution. In our formulation, allowing quasimomenta k to be complex is essential.

(b) The (complex) **Bloch variety** B_L of the operator L consists of all pairs $(k, \lambda) \in \mathbb{C}^{d+1}$ such that λ is an eigenvalue of the operator $L(k)$:

$$B_L = \{(k, \lambda) \in \mathbb{C}^{d+1} : \lambda \in \sigma(L(k))\}.$$

In another word, the pair (k, λ) belongs to B_L if and only if the equation $Lu = \lambda u$ in $\mathbb{R}^d$ has a non-zero Bloch solution u with a quasimomentum k. Also, the Bloch variety B_L is also called the **dispersion relation/curve**, i.e., the graph of the multivalued function $\lambda(k)$.

(c) The (complex) **Fermi surface** $F_{L,\lambda}$ of the operator L at the energy level $\lambda \in \mathbb{C}$ consists of all quasimomenta $k \in \mathbb{C}^d$ such that the equation $L(k)u = \lambda u$ has a nonzero solution. Equivalently, $k \in F_{L,\lambda}$ means the existence of a nonzero periodic solution u of the equation $L(k)u = \lambda u$. In other words, Fermi surfaces are level sets of the dispersion relation. By definitions, $F_{L,\lambda}$ is $2\pi\mathbb{Z}^d$-periodic.

(d) We denote by $B_{L,\mathbb{R}}$ and $F_{L,\lambda,\mathbb{R}}$ the **real Bloch variety** $B_L \cap \mathbb{R}^{d+1}$ and the **real Fermi surface** $F_{L,\lambda} \cap \mathbb{R}^d$, respectively.

(e) Whenever $\lambda = 0$, we will write F_L and $F_{L,\mathbb{R}}$ instead of $F_{L,0}$ and $F_{L,0,\mathbb{R}}$, correspondingly. This is convenient, since being at the spectral level λ, we could consider the operator $L - \lambda I$ instead of L and thus, $F_{L,\lambda} = F_{L-\lambda}$ and $F_{L,\lambda,\mathbb{R}} = F_{L-\lambda,\mathbb{R}}$. In other words, we will be able to assume, w.l.o.g. that $\lambda = 0$.

Some important properties of Bloch variety and Fermi surface are stated in the next proposition (see e.g., [47, 48, 50]).

Proposition 2.2.6.

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(a) The Fermi surface and the Bloch variety are the zero level sets of some entire ($2\pi\mathbb{Z}^d$-periodic in k) functions of finite orders on $\mathbb{C}^d$ and $\mathbb{C}^{d+1}$ respectively.

(b) The Bloch variety is a $2\pi\mathbb{Z}^d$-periodic, complex analytic subvariety of $\mathbb{C}^{d+1}$ of codimension one.

(c) The real Fermi surface $F_{L,\lambda}$ either has zero measure in $\mathbb{R}^d$ or coincides with the whole $\mathbb{R}^d$.

(d) $(k, \lambda) \in B_L$ if and only if $(-k, \bar{\lambda}) \in B_{L^*}$. In other words, $F_{L,\lambda} = -F_{L^*,\bar{\lambda}}$ and $F_{L,\mathbb{R}} = -F_{L^*,\mathbb{R}}$.

The analytical and geometrical properties of dispersion relations encode significant information about spectral features of the operator. For example, the absolute continuity of the spectrum of a self-adjoint periodic elliptic operator (which is true for a large class of periodic Schrödinger operators) can be reformulated as the absence of flat components in its Bloch variety, which is also equivalent to a seemingly stronger fact that the Fermi surface at each energy level has zero measure (due to Proposition 2.2.6).

If L is self-adjoint, then by (2.3), $L(k)$ is self-adjoint in $L^2(\mathbb{T}^d)$ and has domain $H^m(\mathbb{T}^d)$ for each $k \in \mathbb{R}^d$. Due to the ellipticity of L, each $L(k)$ is bounded from below and has compact resolvent. This forces each of the operators $L(k)$, $k \in \mathbb{R}^d$ to have discrete spectrum in $\mathbb{R}$. Therefore, we can label its eigenvalues in non-decreasing order:

$$\lambda_1(k) \leq \lambda_2(k) \leq \ldots \quad . \tag{2.7}$$

Hence, we can single out continuous **band functions** $\lambda_j(k)$ for each $j \in \mathbb{N}$ [80]. The range of the band function λ_j constitutes exactly **the band** $[\alpha_j, \beta_j]$ of the spectrum of L in (2.8) (e.g., see Figure 2.1). Hence, when L is self-adjoint, the spectrum of the operator L in $L^2(\mathbb{R}^d)$ has a **band-gap structure** [20,47,66], i.e., it is the union of a sequence of closed

bounded intervals (**bands** or **stability zones** of the operator L) $[\alpha_j, \beta_j] \subset \mathbb{R}$ $(j = 1, 2, \dots)$:

$$\sigma(L) = \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j], \tag{2.8}$$

such that $\alpha_j \leq \alpha_{j+1}$, $\beta_j \leq \beta_{j+1}$ and $\lim_{j \to \infty} \alpha_j = \infty$. The bands can (and do) overlap when $d > 1$, but they may leave open intervals in between, called **spectral gaps**. Thus, a spectral gap is an interval of the form (β_j, α_{j+1}) for some $j \in \mathbb{N}$ for which $\alpha_{j+1} > \beta_j$. A **finite spectral gap** is of the form (β_j, α_{j+1}) for some $j \in \mathbb{N}$ such that $\alpha_{j+1} > \beta_j$, and the **semifinite spectral gap** is the open interval $(-\infty, \alpha_1)$, which contains all real numbers below the bottom of the spectrum of L.
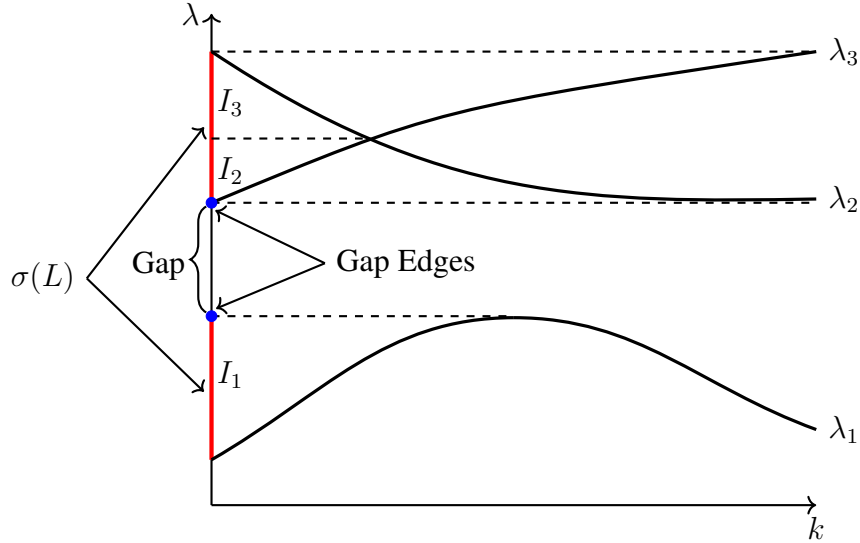
Figure 2.1: An example of $\sigma(L)$.

From Proposition 2.2.6 and the proof of [47, Lemma 4.5.1] (see also [80]), the band functions $\lambda(k)$ are piecewise analytic on $\mathbb{C}^d$.

Remark 2.2.7.

(a) It is worthwhile to mention that in the first three statements of Lemma 2.2.4, one can replace the Brillouin zone $[-\pi, \pi]^d$ by any other fundamental domain of $\mathbb{R}^d$ with respect to the dual lattice $2\pi\mathbb{Z}^d$ (due to $2\pi\mathbb{Z}^d$-periodicity in quasimomentum k). In the next chapters, we will use this lemma with some fundamental domain of the form $[-\pi, \pi]^d + k_0$, where k_0 is a fixed quasimomentum in $\mathbb{R}^d$.

(b) It is sometimes useful to employ an alternative version of Floquet transform for which the reader can find more details in Chapter 5. Analogs of the Plancherel and Paley-Wiener theorem for these Floquet transforms are also obtained like we just see in Lemma 2.2.4.

(c) Our above discussion about Floquet-Bloch theory (e.g., Floquet transform and its properties in Lemma 2.2.4) can be transferred without any major change to the case of periodic elliptic operators on co-compact abelian coverings.

3. GREEN'S FUNCTION ASYMPTOTICS NEAR THE INTERNAL EDGES OF SPECTRA OF PERIODIC ELLIPTIC OPERATORS. SPECTRAL GAP INTERIOR[*]

3.1 Introduction

The behavior at infinity of the Green function of the Laplacian in $\mathbb{R}^n$ outside and at the boundary of its spectrum is well known. Analogous results below and at the lower boundary of the spectrum have been established for bounded below periodic elliptic operators of the second order in [9, 60] (see also [81] for the discrete version). Due to the band-gap structure of the spectra of such periodic operators, the question arises whether similar results can be obtained at or near the edges of spectral gaps. The corresponding result at the internal edges of the spectrum was established in [52]. The main result of this chapter, Theorem 3.2.5, is the description of such asymptotics near the spectral edge for generic periodic elliptic operators of second-order with real coefficients in dimension $d \geq 2$, if the spectral edge is attained at a symmetry point of the Brillouin zone.

It is well known that outside of the spectrum the Green function decays exponentially at infinity, with the rate of decay controlled by the distance to the spectrum. See, e.g., Combes-Thomas estimates [10, 18]. However, comparison with the formulas for the case of the Laplacian shows that an additional algebraically decaying factor (depending on the dimension) is lost in this approach. Moreover, the exponential decay in general is expected to be anisotropic, while the operator theory approach can provide only isotropic estimates. The result of this chapter provides the exact principal term of asymptotics, thus resolving these issues.

3.2 Assumptions, notation and the main result

Consider a linear second order elliptic operator in $\mathbb{R}^d$ with periodic coefficients

$$L(x, D) = \sum_{k,l=1}^{d} D_k(a_{kl}(x)D_l) + V(x) = D^*A(x)D + V(x). \tag{3.1}$$

Here $A = (a_{kl})_{1 \leq k,l \leq d}$, $D = (D_1, \ldots, D_d)$, and $D_k := -i\partial_k = -i\dfrac{\partial}{\partial x_k}$. All coefficients a_{kl}, V are smooth real-valued functions on $\mathbb{R}^d$, periodic with respect to the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, i.e., $a_{kl}(x + n) = a_{kl}(x)$ and $V(x + n) = V(x)$, $\forall x \in \mathbb{R}^d, n \in \mathbb{Z}^d$. The operator L is assumed to be uniformly elliptic, i.e., the matrix A is symmetric and

$$\sum_{k,l=1}^{d} a_{kl}(x)\xi_k\xi_l \geq \theta|\xi|^2, \tag{3.2}$$

for some $\theta > 0$ and any $x \in \mathbb{R}^d$, $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. We recall from the previous chapter that the operator L, with the Sobolev space $H^2(\mathbb{R}^d)$ as the domain, is an unbounded, self-adjoint operator in $L^2(\mathbb{R}^d)$. Moreover, the spectrum $\sigma(L)$ of the operator L has the band-gap structure:

$$\sigma(L) = \bigcup_{j=1}^{\infty}[\alpha_j, \beta_j]. \tag{3.3}$$

We consider the open interval $(-\infty, \alpha_1)$, which contains all real numbers below the bottom of the spectrum of L, as an infinite spectral gap. However, we will be mostly interested in finite spectral gaps.

In this chapter, we study Green's function asymptotics for the operator L in a spectral gap, near a spectral gap edge. More precisely, consider a finite spectral gap (β_j, α_{j+1}) for some $j \in \mathbb{N}$ and a value $\lambda \in (\beta_j, \alpha_{j+1})$ which is close either to the spectral edge β_j or to the spectral edge α_{j+1}. We would like to study the asymptotic behavior when $|x - y| \to \infty$ of the Green's kernel $G_\lambda(x, y)$ of the resolvent operator $R_{\lambda,L} := (L - \lambda)^{-1}$. The case of

the spectral edges (i.e., $\lambda = \alpha_{j+1}$ or $\lambda = \beta_j$) was studied for the similar purpose in [52]. All asymptotics here and also in [52] are deduced from an assumed "generic" spectral edge behavior of the **dispersion relation** of the operator L, which we will briefly review below.

From now on, we fix L as a self-adjoint elliptic operator of the form (3.1), whose band-gap structure is as (2.8). By adding a constant to the operator L if necessary, we can assume that the spectral edge of interest is 0. It is also enough to suppose that the adjacent spectral band is of the form $[0, a]$ for some $a > 0$ since the case when the spectral edge 0 is the maximum of its adjacent spectral band is treated similarly.

Suppose there is no spectrum for small negative values of λ and hence there is a spectral gap below 0. Thus, there exists at least one band function $\lambda_j(k)$ for some $j \in \mathbb{N}$ such that 0 is the minimal value of this function on the Brillouin zone.

To establish our main result, we need to impose the following analytic assumption on the dispersion curve λ_j as in [52]:

Assumption A

There exists $k_0 \in [-\pi, \pi]^d$ and a band function $\lambda_j(k)$ such that:

A1 $\lambda_j(k_0) = 0.$

A2 $\min_{k \in \mathbb{R}^d, i \neq j} |\lambda_i(k)| > 0.$

A3 k_0 *is the only[1] (modulo $2\pi\mathbb{Z}^d$) minimum of λ_j.*

A4 $\lambda_j(k)$ *is a Morse function near k_0, i.e., its Hessian matrix $H := \mathrm{Hess}\,(\lambda_j)(k_0)$ at*

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[1]Finitely many such points can be also easily handled.

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k_0 is positive definite. In particular, the Taylor expansion of λ_j at k_0 is:

$$\lambda_j(k) = \frac{1}{2}(k - k_0)^t H(k - k_0) + O(|k - k_0|^3).$$

It is known [44] that the conditions **A1** and **A2** 'generically' hold (i.e., they can be achieved by small perturbation of coefficients of the operator) for Schrödinger operators. Although this has not been proven, the conditions **A3** and **A4** are widely believed (both in the mathematics and physics literature) to hold 'generically'. In other words, it is conjectured that for a 'generic' selfadjoint second-order elliptic operator with periodic coefficients on $\mathbb{R}^d$ each of the spectral gap's endpoints is a unique (modulo the dual lattice $2\pi\mathbb{Z}^d$), nondegenerate extremum of a single band function $\lambda_j(k)$ (see e.g., [50, Conjecture 5.1]). It is known that for a non-magnetic periodic Schrödinger operator, the bottom of the spectrum always corresponds to a non-degenerate minimum of λ_1 [43]. A similar statement is correct for a wider class of 'factorable' operators [13, 14]. The following condition on k_0 will also be needed:

A5 *The quasimomentum k_0 is a* **high symmetry point of the Brillouin zone**, *i.e., all components of k_0 must be either equal to 0 or to π.*

It is known [37] that the condition **A5** is not always satisfied and spectral edges could occur deeply inside the Brillouin zone. However, as it is discussed in [37], in many practical cases (e.g., in the media close to homogeneous) this condition holds.

We would like to introduce a suitable fundamental domain with respect to the dual lattice $2\pi\mathbb{Z}^d$ to work with.

Definition 3.2.1. Consider the quasimomentum k_0 in **Assumption A**. Due to **A5**, $k_0 = (\delta_1\pi, \delta_2\pi, \ldots, \delta_d\pi)$, where $\delta_j \in \{0, 1\}$ for $j \in \{1, \ldots, d\}$. We denote by $\mathcal{O}$ the fundamen-

tal domain so that k_0 is its center of symmetry, i.e.,

$$\mathcal{O} = \prod_{j=1}^{d} [(\delta_j - 1)\pi, (\delta_j + 1)\pi].$$

When $k_0 = 0$, $\mathcal{O}$ is just the Brillouin zone.

We now introduce notation that will be used throughout the chapter.

Notation 3.2.2. (a) Let $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}^{d-1}$, $z_3 \in \mathbb{C}^d$ and r_i be positive numbers for $i = 1, 2, 3$. Then we denote by $B(z_1, r_1)$, $D'(z_2, r_2)$ and $D(z_3, r_3)$ the open balls (or discs) centered at z_1, z_2 and z_3 whose radii are r_1, r_2 and r_3 in $\mathbb{C}$, $\mathbb{C}^{d-1}$ and $\mathbb{C}^d$ respectively.

(b) The real parts of a complex vector z, or of a complex matrix A are denoted by $\Re(z)$ and $\Re(A)$ respectively.

(c) The standard notation $O(|x - y|^{-n})$ for a function f defined on $\mathbb{R}^{2d}$ means there exist constants $C > 0$ and $R > 0$ such that $|f(x, y)| \leq C|x - y|^{-n}$ whenever $|x - y| > R$. Also, $f(x, y) = o(|x - y|^{-n})$ means that

$$\lim_{|x-y| \to \infty} |f(x, y)| / |x - y|^n = 0.$$

(d) We often use the notation $A \lesssim B$ to mean that the quantity A is less or equal than the quantity B up to some multiplicative constant factor, which does not affect the arguments.

Note that $L(z)$ is non-self-adjoint if $z \notin \mathbb{R}^d$. Note that $L(z) - L(0)$ is an operator of lower order for each $z \in \mathbb{C}^d$. Therefore, for each $z \in \mathbb{C}^d$, the operator $L(z)$ has discrete spectrum and is therefore a closed operator with non-empty resolvent set (see pp.188-190

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in [4]). These operators have the same domain $H^2(\mathbb{T}^d)$ and for each $\phi \in H^2(\mathbb{T}^d)$, $L(z)\phi$ is a $L^2(\mathbb{T}^d)$-valued analytic function of z, due to (4.2.14). Consequently, $\{L(z)\}_{z \in \mathbb{C}^d}$ is an analytic family of type $\mathcal{A}$ in the sense of Kato [39][2]. Due to **A1-A2**, $\lambda_j(k_0)$ is a simple eigenvalue of $L(k_0)$. By using analytic perturbation theory for the family $\{L(z)\}_{z \in \mathbb{C}^d}$ (see e.g., [66, Theorem XII.8]), there is an open neighborhood V of k_0 in $\mathbb{C}^d$ and some $\epsilon_0 > 0$ such that

(P1) λ_j is analytic in a neighborhood of the closure of V.

(P2) $\lambda_j(z)$ has algebraic multiplicity one, i.e., it is a simple eigenvalue of $L(z)$ for any $z \in \overline{V}$.

(P3) The only eigenvalue of $L(z)$ contained in the closed disc $\overline{B}(0, \epsilon_0)$ is $\lambda_j(z)$. Moreover, we may also assume that $|\lambda_j(z)| < \epsilon_0$ for each $z \in V$.

(P4) For each $z \in \overline{V}$, let $\phi(z, x)$ be a nonzero $\mathbb{Z}^d$-periodic function of x such that it is the unique (up to a constant factor) eigenfunction of $L(z)$ with the eigenvalue $\lambda_j(z)$, i.e., $L(z)\phi(z, \cdot) = \lambda_j(z)\phi(z, \cdot)$. We will also use sometimes the notation ϕ_z for the eigenfunction $\phi(z, \cdot)$.

By elliptic regularity, $\phi(z, x)$ is smooth in x. On a neighborhood of $\overline{V}$, $\phi(z, \cdot)$ is a $H^2(\mathbb{T}^d)$-valued holomorphic function.

(P5) By condition **A4** and the continuity of $\mathrm{Hess}\,(\lambda_j)$,[3] we can assume that for all $z \in V$,

$$2\Re(\mathrm{Hess}\,(\lambda_j)(z)) > \min \sigma(\mathrm{Hess}\,(\lambda_j)(k_0))I_{d \times d}.$$

(P6) V is invariant under complex conjugation. Furthermore, the smooth function

$$F(z) := (\phi(z, \cdot), \phi(\overline{z}, \cdot))_{L^2(\mathbb{T}^d)} \tag{3.4}$$

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[2]It is also an analytic family in the Banach space of bounded linear operators acting from $H^2(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$.

[3]The Hessian matrix of λ_j.

is non-zero on V, due to analyticity of the mapping $z \mapsto \phi(z, \cdot)$ and the inequality $F(k_0) = \|\phi(k_0)\|_{L^2(\mathbb{T}^d)}^2 > 0$.

The following lemma will be useful when dealing with operators having real and smooth coefficients:

Lemma 3.2.3. *(i) For k in $\mathbb{R}^d$ and $i \in \mathbb{N}$,*

$$\lambda_i(k) = \lambda_i(-k). \tag{3.5}$$

In other words, each band λ_i of L is an even function on $\mathbb{R}^d$.

(ii) If $k_0 \in X$, we have $\lambda_i(k + k_0) = \lambda_i(-k + k_0)$ for all k in $\mathbb{R}^d$ and $i \in \mathbb{N}$.

Proof. Let ϕ_k be an eigenfunction of $L(k)$ corresponding to $\lambda_j(k)$. This means that ϕ_k is a periodic solution to the equation

$$L(x, \partial + ik)\phi_k(x) = \lambda_j(k)\phi_k(x). \tag{3.6}$$

Taking the complex conjugate of (3.6), we get

$$L(x, \partial - ik)\overline{\phi_k}(x) = \lambda_j(k)\overline{\phi_k}(x).$$

Therefore, $\overline{\phi_k}$ is an eigenfunction of $L(-k)$ with eigenvalue $\lambda_j(k)$. This implies the identity (3.5).

(ii) By (i), $\lambda_i(k + k_0) = \lambda_i(-k - k_0) = \lambda_i(-k + k_0)$ since $2k_0 \in 2\pi\mathbb{Z}^d$. $\square$

Corollary 3.2.4. *If $\beta \in \mathbb{R}^d$ such that $k_0 + i\beta \in \overline{V}$ then $\lambda_j(k_0 + i\beta) \in \mathbb{R}$.*

Proof. Indeed, the statement (ii) of Lemma 3.2.3 implies that the Taylor series of $\lambda(k)$ at k_0 has only even degree terms and real coefficients. $\square$

Corollary 3.2.4 allows us to define near $\beta = 0$ the real analytic function $E(\beta) :=$ $\lambda_j(k_0 + i\beta)$ near 0. Since its Hessian at 0 is negative-definite (by **A4**), there exists a connected and bounded neighborhood V_0 of 0 in $\mathbb{R}^d$ such that $k_0 + iV_0 \subseteq V$ and $\mathrm{Hess}\,(E)(\beta)$ is negative-definite whenever β belongs to V_0. Thus, E is strictly concave on V_0 and $\sup_{\beta \in V_0} E(\beta) = E(0) = 0, \nabla E(\beta) = 0$ iff $\beta = 0$. Note that at the bottom of the spectrum (i.e., $j = 1$), we could take V_0 as the whole Euclidean space $\mathbb{R}^d$.

By the Morse lemma and the fact that 0 is a nondegenerate critical point of E, there is a smooth change of coordinates $\Phi : U_0 \to \mathbb{R}^d$ so that $0 \in U_0 \subset\subset V_0$, U_0 is connected, $\Phi(0) = 0$ and $E(\Phi^{-1}(a)) = -|a|^2, \forall a \in \Phi(U_0)$. Set $K_\lambda := \{\beta \in U_0 : E(\beta) \geq \lambda\}$ and $\Gamma_\lambda := \{\beta \in U_0 : E(\beta) = \lambda\}$ for each $\lambda \in \mathbb{R}$. Now, we consider λ to be in the set $\{-|a|^2 : a \in \Phi(U_0), a \neq 0\}$. Then K_λ is a strictly convex d-dimensional compact body in $\mathbb{R}^d$, and $\Gamma_\lambda = \partial K_\lambda$ is a compact hypersurface in $\mathbb{R}^d$. The compactness of K_λ follows from the equation $-|\Phi(\beta)|^2 = E(\beta) \geq \lambda$ which yields that $|\beta| = |\Phi^{-1}(\Phi(\beta))| \leq$ $\max\{|\Phi^{-1}(a)| : a \in \Phi(U_0), |a|^2 \leq -\lambda\}$. Additionally, $\lim_{\lambda \to 0^-} \max_{\beta \in K_\lambda} |\beta| = 0$.

Let $\mathcal{K}_\lambda$ be the Gauss-Kronecker curvature of Γ_λ. Since the Hessian of E is negative-definite on Γ_λ, $\mathcal{K}_\lambda$ is nowhere-zero. For the value of λ described in the previous paragraph and each $s \in \mathbb{S}^{d-1}$, there is a unique vector $\beta_s \in \Gamma_\lambda$ such that the value of the Gauss map of the hypersurface Γ_λ at this point coincides with s, i.e.

$$\nabla E(\beta_s) = -|\nabla E(\beta_s)|s. \tag{3.7}$$

This is due to the fact that the Gauss map of a compact, connected oriented hypersurface in $\mathbb{R}^d$, whose Gauss-Kronecker curvature is nowhere zero, is a diffeomorphism onto the sphere $\mathbb{S}^{d-1}$ (see e.g., [77, Theorem 5, p.104] or [27, Corollary 3.1]). Thus, β_s depends

smoothly on s. We also see that

$$\lim_{|\lambda|\to 0} \max_{s\in\mathbb{S}^{d-1}} |\beta_s| = 0.$$

Note that β_s could be defined equivalently by using the support functional h of the strictly convex set K_λ. Recall that for each direction $s \in \mathbb{S}^{d-1}$,

$$h(s) = \max_{\xi\in K_\lambda}\langle s, \xi\rangle.$$

Then β_s is the unique point in Γ_λ such that $\langle s, \beta_s\rangle = h(s)$.

By letting $|\lambda|$ close enough to 0, we can make sure that $(-\lambda)^{1/2} = |a|$ for some $a \in \Phi(U_0)$. Then

$$\{k_0 + it\beta_s, (t, s) \in [0, 1] \times \mathbb{S}^{d-1}\} \subset V. \tag{3.8}$$

We can now state the main result of the chapter.

Theorem 3.2.5. *Suppose conditions **A1-A5** are satisfied. For $\lambda < 0$ sufficiently close to 0 (depending on the dispersion branch λ_j and the operator L), the Green's function G_λ of L at λ admits the following asymptotics as $|x - y| \to \infty$:*

$$G_\lambda(x, y) = \frac{e^{(x-y)(ik_0-\beta_s)}}{(2\pi|x - y|)^{(d-1)/2}} \frac{|\nabla E(\beta_s)|^{(d-3)/2}}{\det\left(-\mathcal{P}_s \operatorname{Hess}(E)(\beta_s)\mathcal{P}_s\right)^{1/2}} \frac{\phi_{k_0+i\beta_s}(x)\overline{\phi_{k_0-i\beta_s}(y)}}{(\phi_{k_0+i\beta_s}, \phi_{k_0-i\beta_s})_{L^2(\mathbb{T}^d)}}$$

$$+ e^{(y-x)\cdot\beta_s}r(x, y).$$

$$\tag{3.9}$$

Here $s = (x - y)/|x - y|$, $\mathcal{P}_s$ is the projection from $\mathbb{R}^d$ onto the tangent space of the unit sphere $\mathbb{S}^{d-1}$ at the point s, and when $|x - y|$ is large enough, the remainder term r satisfies $|r(x, y)| \leq C|x - y|^{-d/2}$ for some constant $C > 0$ (independent of s).

This result achieves our stated goal of showing the precise (anisotropic) rates of the

exponential decay of the Green's function and capturing the additional algebraic decay factor.

3.3 Proof of the main theorem 3.2.5 and some remarks

Theorem 3.2.5 is a direct consequence of its local (with respect to the direction of $(x - y)$) version:

Theorem 3.3.1. *Under the hypotheses of Theorem 3.2.5 and when $\lambda \approx 0$, for each $\omega \in \mathbb{S}^{d-1}$, there are a neighborhood $\mathcal{V}_\omega$ in $\mathbb{S}^{d-1}$ containing ω and a smooth function $e(s) = (e_{s,2}, \ldots, e_{s,d}) : \mathcal{V}_\omega \to (T_s \mathbb{S}^{d-1})^{d-1}$, which $e(s)$ is an orthonormal basis of the tangent space $T_s \mathbb{S}^{d-1}$ for each unit vector $s \in \mathcal{V}_\omega$, such that following asymptotics*

$$
G_\lambda(x, y) = \frac{e^{(x-y)(ik_0 - \beta_s)}}{(2\pi|x-y|)^{(d-1)/2}} \left(\frac{|\nabla E(\beta_s)|^{(d-3)}}{\det\left(-e_{s,p} \cdot \operatorname{Hess}(E)(\beta_s) e_{s,q}\right)_{2 \leq p,q \leq d}} \right)^{1/2}
$$
$$
\times \frac{\phi_{k_0 + i\beta_s}(x)\overline{\phi_{k_0 - i\beta_s}(y)}}{(\phi_{k_0 + i\beta_s}, \phi_{k_0 - i\beta_s})_{L^2(\mathbb{T}^d)}} + e^{(y-x) \cdot \beta_s} r(x, y),
\tag{3.10}
$$

hold for all (x, y) such that $s = (x - y)/|x - y| \in \mathcal{V}_\omega$. Furthermore, there is a positive constant $C(\omega)$ depending on ω such that $|r(x, y)| \leq C(\omega)|x - y|^{-d/2}$.

Proof of Theorem 3.2.5.

Proof. Observe that for any orthonormal basis $\{e_{s,l}\}_{2 \leq l \leq d}$ of the tangent space $T_s \mathbb{S}^{d-1}$,

$$
\det\left(-\mathcal{P}_s \operatorname{Hess}(E)(\beta_s)\mathcal{P}_s\right) = \det\left(-e_{s,p} \cdot \operatorname{Hess}(E)(\beta_s) e_{s,q}\right)_{2 \leq p,q \leq d}.
$$

Now, using of a finite cover of the unit sphere by neighborhoods $\mathcal{V}_{\omega_j}$ in Theorem 3.3.1, one obtains Theorem 3.2.5. □

Remark 3.3.2.

- The asymptotics (3.9) (or (3.10)) resemble the formula (1.1) in [60, Theorem 1.1] when λ is below the bottom of the spectrum of the operator. Moreover, as in [61, Theorem 1.1], by using the Gauss-Kronecker curvature $\mathcal{K}_\lambda$, the main result (3.9) could be restated as follows:

$$G_\lambda(x,y) = \frac{e^{(x-y)(ik_0-\beta_s)}}{(2\pi|x-y|)^{(d-1)/2}} \frac{1}{|\nabla E(\beta_s)|\mathcal{K}_\lambda(\beta_s)^{1/2}} \frac{\phi_{k_0+i\beta_s}(x)\overline{\phi_{k_0-i\beta_s}(y)}}{(\phi_{k_0+i\beta_s},\phi_{k_0-i\beta_s})_{L^2(\mathbb{T}^d)}}$$
$$+ e^{(y-x)\cdot\beta_s}O(|x-y|^{-d/2}).$$

- Although (3.9) is an anisotropic formula, it is not hard to obtain from (3.9) an isotropic upper estimate for the Green's function G_λ based on the **distance from λ to the spectrum of the operator** L,[4] e.g., there are some positive constants C_1, C_2 (depending only on L and λ_j) and C_3 (which may depend on λ) such that whenever $|x-y| > C_3$, the following inequality holds:

$$|G_\lambda(x,y)| \leq C_1|\lambda|^{(d-3)/4}\frac{e^{-C_2|\lambda|^{1/2}|x-y|}}{|x-y|^{(d-1)/2}}.$$

- If the band edge occurs at finitely many points, rather than a single k_0, one just needs to combine the asymptotics coming from all these isolated minima.

Now we outline the proof of Theorem 3.3.1. In Section 3.5, we use the tools of Floquet-Bloch theory in Chapter 2 to reduce the problem to that of finding the asymptotics of a scalar integral. The purpose of Section 3.4 is to prepare for Section 3.5, by shifting an integral from the fundamental domain $\mathcal{O}$ along some purely imaginary directions in $\mathbb{C}^d$. This reduces finding the asymptotics of the Green's function G_λ to an auxiliary Green's function $G_{s,\lambda}$ via the formula (3.14). Next, we single out a principal term G_0 of the Green's function $G_{s,\lambda}$ and then represent this kernel G_0 as a scalar integral in (3.19). We also prove

---

[4]Recall that the spectral edge is assumed to be zero.

that the error kernel $G_{s,\lambda} - G_0$ decays rapidly (see Theorem 3.5.2). Then in (3.22), our reduced Green's function G_0 can be expressed in terms of the two integrals I and J. Here the integral I is mainly responsible for the asymptotics of G_0 and the integral J decays fast enough to be included in the remainder term $r(x, y)$ in the asymptotics (3.9). The first part of Section 3.6 is devoted to achieving the asymptotics of the main integral I (see Theorem 3.6.2) by adapting the method similar to the one used in the discrete case [81], while the second part of Section 3.6 provides an estimate of J (see Proposition 3.6.6). In order to not overload the main text with technicalities, the proofs of some auxiliary statements are postponed till Sections 3.7-3.9.

3.4 On local geometry of the resolvent set

The following proposition shows that for any $s \in \mathbb{S}^{d-1}$, $k_0 + i\beta_s$ is the only complex quasimomentum having the form of $k + it\beta_s$ where $k \in \mathcal{O}, t \in [0, 1]$ such that λ is in the spectrum of the corresponding fiber operator $L(k + it\beta_s)$. In other words, by moving from $k \in \mathcal{O}$ in the direction $i\beta_s$, the first time we **hit the Fermi surface** $F_{L,\lambda}$ (i.e., the spectrum of $L(k)$ meets λ) is at the value of the quasimomentum $k = k_0 + i\beta_s$. This step is crucial for setting up the scalar integral in the next section, which is solely responsible for the main term asymptotics of our Green's function.

Proposition 3.4.1. *If $|\lambda|$ is small enough (depending on the dispersion branch λ_j and L), then $\lambda \in \rho(L(k + it\beta_s))$ if and only if $(k, t) \neq (k_0, 1)$.*

The proof of this proposition is presented in Subsection 3.9.3.

3.5 A Floquet reduction of the problem

We will use the Floquet transform (see Chapter 2) to reduce our problem to finding asymptotics of a scalar integral expression, which is close to the one arising when dealing with the Green's function of the Laplacian at a small negative level λ. As in [52], the idea

is to show that only the branch of the dispersion relation λ_j appearing in the **Assumption A** dominates the asymptotics.

3.5.1 The Floquet reduction

The Green's function G_λ of L at λ is the Schwartz kernel of the resolvent operator $R_\lambda = (L - \lambda)^{-1}$. Fix a $\lambda < 0$ such that the statement of Proposition 3.4.1 holds. For any $s \in \mathbb{S}^{d-1}$ and $t \in [0,1]$, we consider the following operator with real coefficients on $\mathbb{R}^d$:

$$L_{t,s} := e^{t\beta_s \cdot x} L e^{-t\beta_s \cdot x}. \tag{3.11}$$

For simplicity, we write $L_s := L_{1,s}$ and note that $L_{0,s} = L$. Due to self-adjointness of L, the adjoint of $L_{t,s}$ is

$$L_{t,s}^* = L_{-t,s}. \tag{3.12}$$

By definition, $L_{t,s}(k) = L(k + it\beta_s)$ for any k in $\mathbb{C}^d$ and therefore, (2.6) yields

$$\sigma(L_{t,s}) = \bigcup_{k \in \mathcal{O}} \sigma(L(k + it\beta_s)) \supseteq \{\lambda_j(k + it\beta_s)\}_{k \in \mathcal{O}}. \tag{3.13}$$

The Schwartz kernel $G_{s,\lambda}$ of the resolvent operator $R_{s,\lambda} := (L_s - \lambda)^{-1}$ is

$$G_{s,\lambda}(x,y) = e^{\beta_s \cdot x} G_\lambda(x,y) e^{-\beta_s \cdot y} = e^{\beta_s \cdot (x-y)} G_\lambda(x,y). \tag{3.14}$$

Thus, instead of finding asymptotics of G_λ, we can focus on the asymptotics of $G_{s,\lambda}$.

By (3.13) and Proposition 3.4.1, λ is not in the spectrum of $L_{t,s}$ for any $s \in \mathbb{S}^{d-1}$ and $t \in [0,1)$. Let us consider

$$R_{t,s,\lambda}f := (L_{t,s} - \lambda)^{-1}f, \quad f \in L^2_{comp}(\mathbb{R}^d),$$

where L^2_{comp} stands for compactly supported functions in L^2.

Applying Lemma 2.2.4, we have

$$\widehat{R_{t,s,\lambda} f}(k) = (L_{t,s}(k) - \lambda)^{-1} \widehat{f}(k), \quad (t,k) \in [0,1) \times \mathcal{O}.$$

We consider the sesquilinear form

$$(R_{t,s,\lambda} f, \varphi) = (2\pi)^{-d} \int_{\mathcal{O}} \left((L_{t,s}(k) - \lambda)^{-1} \widehat{f}(k), \widehat{\varphi}(k) \right) dk,$$

where $\varphi \in L^2_{comp}(\mathbb{R}^d)$.

In the next lemma (see Subsection 3.9.3), we show the weak convergence of $R_{t,s,\lambda}$ in L^2_{comp} as $t \nearrow 1$ and introduce the limit operator $R_{s,\lambda} = \lim\limits_{t \to 1^-} R_{t,s,\lambda}$. The limit operator $R_{s,\lambda}$ is central in our study of the asymptotics of the Green's function.

Lemma 3.5.1. *Let $d \geq 2$. Under Assumption A, the following equality holds:*

$$\lim_{t \to 1^-} (R_{t,s,\lambda} f, \varphi) = (2\pi)^{-d} \int_{\mathcal{O}} \left(L_s(k) - \lambda)^{-1} \widehat{f}(k), \widehat{\varphi}(k) \right) dk. \tag{3.15}$$

The integral in the right hand side of (3.15) is absolutely convergent for f, φ in $L^2_{comp}(\mathbb{R}^d)$. Thus, the Green's function $G_{s,\lambda}$ is the integral kernel of the operator $R_{s,\lambda}$ defined as follows

$$\widehat{R_{s,\lambda} f}(k) = (L_s(k) - \lambda)^{-1} \widehat{f}(k). \tag{3.16}$$

3.5.2   Singling out the principal term in $R_{s,\lambda}$

By (3.16), the Green's function $G_{s,\lambda}$ is the integral kernel of the operator $R_{s,\lambda}$ with the domain $L^2_{comp}(\mathbb{R}^d)$. The inversion formula (2.4) gives

$$R_{s,\lambda} f(x) = (2\pi)^{-d} \int_{\mathcal{O}} e^{ik \cdot x} (L_s(k) - \lambda)^{-1} \widehat{f}(k,x) \, dk, \quad x \in \mathbb{R}^d.$$

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The purpose of this part is to single out the part of the above integral that is responsible for the leading term of the Green's function asymptotics.

To find the Schwartz kernel of $R_{s,\lambda}$, it suffices to consider functions $f \in C_c^\infty(\mathbb{R}^d)$. Our first step is to localize the integral around the point k_0. Let us consider a connected neighborhood V of k_0 on which there exist nonzero $\mathbb{Z}^d$-periodic (in x) functions $\phi_z(x)$, $z \in V$ satisfying 1) $L(z)\phi_z = \lambda_j(z)\phi_z$ and 2) each ϕ_z spans the eigenspace corresponding to the eigenvalue $\lambda_j(z)$ of the operator $L(z)$. According to **(P3)**, $\lambda_j(V) \subseteq B(0, \epsilon_0)$ and $\partial B(0, \epsilon_0) \subseteq \rho(L(z))$ when $z \in V$. For such z, let $P(z)$ be the Riesz projection of $L(z)$ that projects $L^2(\mathbb{T}^d)$ onto the eigenspace spanned by ϕ_z, i.e.,

$$P(z) = -\frac{1}{2\pi i} \oint_{|\mu|=\epsilon_0} (L(z) - \alpha)^{-1} \, d\mu.$$

Taking the adjoint, we get

$$P(z)^* = -\frac{1}{2\pi i} \oint_{|\mu|=\epsilon_0} (L(\bar{z}) - \mu)^{-1} \, d\mu = P(\bar{z}),$$

which is the Riesz projection from $L^2(\mathbb{T}^d)$ onto the eigenspace spanned by $\phi_{\bar{z}}$. Recall that due to (3.8), by choosing $|\lambda|$ small enough, there exists $r_0 > 0$ (independent of s) such that $k \pm i\beta_s \in V$ for $k \in \overline{D}(k_0, r_0) \cap \mathbb{R}^d$. We denote $P_s(k) := P(k+i\beta_s)$ for such real k. Then $P_s(k)$ is the projector onto the eigenspace spanned by $\phi(k+i\beta_s)$ and $P_s(k)^* = P(k-i\beta_s)$. Additionally, due to **(P6)**,

$$P_s(k)g = \frac{(g, \phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}}{(\phi(k+i\beta_s), \phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}}\phi(k+i\beta_s), \quad \forall g \in L^2(\mathbb{T}^d). \tag{3.17}$$

Let η be a cut-off smooth function on $\mathcal{O}$ such that $\text{supp}(\eta) \Subset D(k_0, r_0)$ and $\eta = 1$ around k_0.

We decompose $\widehat{f} = \eta\widehat{f} + (1-\eta)\widehat{f}$. When $k \neq k_0$, the operator $L_s(k) - \lambda$ is invertible by

Proposition 3.4.1. Hence, the following function is well-defined and smooth with respect to (k, x) on $\mathbb{R}^d \times \mathbb{R}^d$:

$$\widehat{u_g}(k, x) = (L_s(k) - \lambda)^{-1}(1 - \eta(k))\widehat{f}(k, x).$$

Using Lemma 2.2.4, smoothness of $\widehat{u_g}$ implies that u_g has rapid decay in x. Now we want to solve

$$(L_s(k) - \lambda)\widehat{u}(k) = \eta(k)\widehat{f}(k). \tag{3.18}$$

Let $Q_s(k) = I - P_s(k)$ and we denote the ranges of projectors $P_s(k)$, $Q_s(k)$ by $R(P_s(k))$, $R(Q_s(k))$ respectively. We are interested in decomposing the solution $\widehat{u}$ into a sum of the form $\widehat{u_1} + \widehat{u_2}$ where $\widehat{u_1} = P_s(k)\widehat{u_1}$ and $\widehat{u_2} = Q_s(k)\widehat{u_2}$. Let $\widehat{f_1} = P_s(k)\eta(k)\widehat{f}$ and $\widehat{f_2} = Q_s(k)\eta(k)\widehat{f}$. Observe that since the Riesz projection $P_s(k)$ commutes with the operator $L_s(k)$ and $R(P_s(k))$ is invariant under the action of $L_s(k)$, we have $Q_s(k)L_s(k)P_s(k) = P_s(k)L_s(k)Q_s(k) = 0$ and $Q_s(k)L_s(k)Q_s(k) = L_s(k)Q_s(k)$. Thus, the problem of solving (3.18) can be reduced to the following block-matrix structure form

$$\left(\begin{array}{c|c} (L_s(k) - \lambda)P_s(k) & 0 \\ \hline 0 & (L_s(k) - \lambda)Q_s(k) \end{array} \right) \left(\begin{array}{c} \widehat{u_1} \\ \widehat{u_2} \end{array} \right) = \left(\begin{array}{c} \widehat{f_1} \\ \widehat{f_2} \end{array} \right).$$

When k is close to k_0,

$$B(0, \epsilon_0) \cap \sigma(L_s(k)_{|R(Q_s(k))}) = B(0, \epsilon_0) \cap \sigma(L(k + i\beta_s)) \setminus \{\lambda_j(k + i\beta_s)\} = \emptyset.$$

Since $\lambda = \lambda_j(k_0 + i\beta_s) \in B(0, \epsilon_0)$, λ must belong to $\rho(L_s(k)|_{R(Q_s(k))})$. Hence, the operator function $\widehat{u_2}(k) = (L_s(k) - \lambda)^{-1}Q_s(k)\widehat{f_2}(k)$ is well-defined and smooth in k and hence by Lemma 2.2.4 again, u_2 has rapid decay when $|x| \to \infty$. More precisely, we have

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the following claim:

Theorem 3.5.2. *For each $s \in \mathbb{S}^{d-1}$, let $K_s(x, y)$ be the Schwartz kernel of the operator T_s acting on $L^2(\mathbb{R}^d)$ as follows:*

$$T_s = \mathcal{F}^{-1}\left(\int_{\mathcal{O}}^{\oplus} T_s(k)\, \mathrm{d}k\right)\mathcal{F},$$

where $\mathcal{F}$ is the Floquet transform (see Definition 2.2.3) and

$$T_s(k) = (1 - \eta(k))(L_s(k) - \lambda)^{-1} + \eta(k)((L_s(k) - \lambda)|_{R(Q_s(k))})^{-1}Q_s(k).$$

Then the kernel $K_s(x, y)$ is continuous away from the diagonal and furthermore, as $|x - y| \to \infty$, we have

$$\sup_{s \in \mathbb{S}^{d-1}} |K_s(x, y)| = O(|x - y|^{-N}), \quad \forall N > 0.$$

The proof of this claim shall be provided in Section 3.7.

The u_1 term contributes the leading asymptotics for the Schwartz kernel $G_{s,\lambda}$. Therefore, we only need to solve the equation $(L_s(k) - \lambda)P_s(k)\widehat{u_1} = \widehat{f_1}$ on the one-dimensional range of $P_s(k)$.

Applying (3.17), we can rewrite

$$\widehat{f_1}(k) = \frac{\eta(k)(\widehat{f}, \phi(k - i\beta_s))_{L^2(\mathbb{T}^d)}}{(\phi(k + i\beta_s), \phi(k - i\beta_s))_{L^2(\mathbb{T}^d)}}\phi(k + i\beta_s),$$

so that equation becomes

$$(L_s(k) - \lambda)\frac{(\widehat{u_1}, \phi(k - i\beta_s))_{L^2(\mathbb{T}^d)}\phi(k + i\beta_s)}{(\phi(k + i\beta_s), \phi(k - i\beta_s))_{L^2(\mathbb{T}^d)}} = \frac{\eta(k)(\widehat{f}, \phi(k - i\beta_s))_{L^2(\mathbb{T}^d)}\phi(k + i\beta_s)}{(\phi(k + i\beta_s), \phi(k - i\beta_s))_{L^2(\mathbb{T}^d)}}.$$

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So,

$$\frac{(\lambda_j(k+i\beta_s)-\lambda)(\widehat{u_1},\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}\phi(k+i\beta_s)}{(\phi(k+i\beta_s),\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}} = \frac{\eta(k)(\widehat{f},\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}\phi(k+i\beta_s)}{(\phi(k+i\beta_s),\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}}.$$

In addition to the equation $\widehat{u_1} = P_s(k)\widehat{u_1}$, $\widehat{u_1}$ must also satisfy

$$(\lambda_j(k+i\beta_s)-\lambda)(\widehat{u_1},\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)} = \eta(k)(\widehat{f},\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}.$$

Thus, we define

$$\widehat{u_1}(k,\cdot) := \frac{\eta(k)\phi(k+i\beta_s,\cdot)(\widehat{f},\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}}{(\phi(k+i\beta_s),\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}(\lambda_j(k+i\beta_s)-\lambda)}.$$

By the inverse Floquet transform (2.4),

$$u_1(x) = (2\pi)^{-d}\int_{\mathcal{O}} e^{ik\cdot x}\frac{\eta(k)\phi(k+i\beta_s,x)(\widehat{f},\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}}{(\phi(k+i\beta_s),\phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}(\lambda_j(k+i\beta_s)-\lambda)}\,\mathrm{d}k,$$

for any $x \in \mathbb{R}^d$.

3.5.3 A reduced Green's function.

We are now ready for setting up **a reduced Green's function** G_0, whose asymptotic behavior reflects exactly the leading term of the asymptotics of the Green's function $G_{s,\lambda}$. We introduce $G_0(x,y)$ (roughly speaking) as the Schwartz kernel of the restriction of the operator $R_{s,\lambda}$ onto the one-dimensional range of P_s (which is the direct integral of idempotents $P_s(k)$) as follows:

$$u_1(x) = \int_{\mathbb{R}^d} G_0(x,y)f(y)dy, \quad x \in \mathbb{R}^d,$$

where f is in $L^2_{comp}(\mathbb{R}^d)$.

We recall from (3.4) that $F(k+i\beta_s)$ is the inner product $(\phi(k+i\beta_s), \phi(k-i\beta_s))_{L^2(\mathbb{T}^d)}$. As in [52], we notice that

$$
\begin{aligned}
u_1(x) &= (2\pi)^{-d} \int_{\mathcal{O}} \int_{\mathbb{T}^d} e^{ik\cdot x} \eta(k) \widehat{f}(k,y) \frac{\overline{\phi(k-i\beta_s,y)}\phi(k+i\beta_s,x)}{F(k+i\beta_s)(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}y \, \mathrm{d}k \\
&= (2\pi)^{-d} \int_{\mathcal{O}} \eta(k) \int_{[0,1]^d} \sum_{\gamma\in\mathbb{Z}^d} f(y-\gamma) e^{ik\cdot(x+\gamma-y)} \frac{\overline{\phi(k-i\beta_s,y)}\phi(k+i\beta_s,x)}{F(k+i\beta_s)(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}y \, \mathrm{d}k \\
&= (2\pi)^{-d} \int_{\mathcal{O}} \eta(k) \sum_{\gamma\in\mathbb{Z}^d} \int_{[0,1]^d-\gamma} f(y) e^{ik\cdot(x-y)} \frac{\overline{\phi(k-i\beta_s,y+\gamma)}\phi(k+i\beta_s,x)}{F(k+i\beta_s)(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}y \, \mathrm{d}k \\[2mm]
&= (2\pi)^{-d} \int_{\mathcal{O}} \eta(k) \sum_{\gamma\in\mathbb{Z}^d} \int_{[0,1]^d-\gamma} f(y) e^{ik\cdot(x-y)} \frac{\overline{\phi(k-i\beta_s,y)}\phi(k+i\beta_s,x)}{F(k+i\beta_s)(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}y \, \mathrm{d}k \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} f(y) \left(\int_{\mathcal{O}} \eta(k) e^{ik\cdot(x-y)} \frac{\overline{\phi(k-i\beta_s,y)}\phi(k+i\beta_s,x)}{F(k+i\beta_s)(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}k \right) \mathrm{d}y.
\end{aligned}
$$

Therefore, our **reduced Green's function is**

$$
G_0(x,y) = (2\pi)^{-d} \int_{\mathcal{O}} \eta(k) e^{ik\cdot(x-y)} \frac{\phi(k+i\beta_s,x)\overline{\phi(k-i\beta_s,y)}}{F(k+i\beta_s)(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}k. \tag{3.19}
$$

3.6 Asymptotics of the Green's function

Let $(e_1,\ldots,e_d)$ be the standard orthonormal basis in $\mathbb{R}^d$. Fixing $\omega \in \mathbb{S}^{d-1}$, we would like to show that the asymptotics (3.10) will hold for all (x,y) such that $x-y$ belongs to a conic neighborhood containing ω. Without loss of generality, suppose that $\omega \neq e_1$.

Now let $\mathcal{R}_s$ be the rotation in $\mathbb{R}^d$ such that $\mathcal{R}_s(s) = e_1$ and $\mathcal{R}_s$ leaves the orthogonal complement of the subspace spanned by $\{s, e_1\}$ invariant. We define $e_{s,j} := \mathcal{R}_s^{-1}(e_j)$, for all $j = 2,\ldots,d$. Then, $\langle s, e_{s,p}\rangle = \langle e_1, e_p\rangle = 0$ and $\langle e_{s,p}, e_{s,q}\rangle = \langle e_p, e_q\rangle = \delta_{p,q}$ for $p,q > 1$. In other words,

$$\{s, e_{s,2}, \ldots, e_{s,d}\} \text{ is an orthonormal basis of } \mathbb{R}^d.$$

Then around ω, we pick a compact coordinate patch $\mathcal{V}_\omega$, so that the $\mathbb{R}^{d(d-1)}$-valued

function $e(s) = (e_{s,l})_{2\le l\le d}$ is smooth in a neighborhood of $\mathcal{V}_\omega$.

We use the same notation for $\mathcal{R}_s$ and its $\mathbb{C}$-linear extension to $\mathbb{C}^d$.

3.6.1 The asymptotics of the leading term of the Green's function

We introduce the function $\rho(k, x, y)$ on $D(k_0, r_0) \times \mathbb{R}^d \times \mathbb{R}^d$ as follows:

$$\rho(k, x, y) = \frac{\phi(k + i\beta_s, x)\overline{\phi(k - i\beta_s, y)}}{F(k + i\beta_s)}.$$

where F is defined in (3.4) and $D(k_0, r_0)$ is described in Subsection 3.5.2.

Due to Proposition 3.9.6, the function ρ is in $C^\infty(\overline{D(k_0, r)} \times \mathbb{R}^d \times \mathbb{R}^d)$. For each (x, y), the Taylor expansion around k_0 of $\rho(k)$ gives

$$\rho(k, x, y) = \rho(k_0, x, y) + \rho'(k, x, y)(k - k_0), \tag{3.20}$$

where $\rho' \in C^\infty(\overline{D(k_0, r_0)} \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^d)$. Note that for $z \in V$, $\phi(z, x)$ is $\mathbb{Z}^d$-periodic in x and thus, ρ and ρ' are $\mathbb{Z}^d \times \mathbb{Z}^d$-periodic in (x, y). Since our integrals are taken with respect to k, it is safe to write $\rho(k_0)$ instead of $\rho(k_0, x, y)$. We often omit the variables x, y in ρ if no confusion can arise.

Let $\mu(k) := \eta(k + k_0)$ be a cut-off function supported near 0, where η is introduced in Subsection 3.5.2. We define

$$
\begin{aligned}
I &:= (2\pi)^{-d} \int_{\mathcal{O}} e^{i(k-k_0)\cdot(x-y)} \frac{\mu(k - k_0)}{\lambda_j(k + i\beta_s) - \lambda} \, dk, \\
J &:= (2\pi)^{-d} \int_{\mathcal{O}} e^{i(k-k_0)\cdot(x-y)} \frac{\mu(k - k_0)(k - k_0)\rho'(k, x, y)}{\lambda_j(k + i\beta_s) - \lambda} \, dk.
\end{aligned}
\tag{3.21}
$$

Hence, we can represent the reduced Green's function as

$$G_0(x, y) = e^{ik_0\cdot(x-y)}(\rho(k_0)I + J). \tag{3.22}$$

The rest of this subsection is devoted to computing the asymptotics of the main integral I, which gives the leading term in asymptotic expansion of the reduced Green's function $G_0(x, y)$ as $|x - y| \to \infty$.

By making the change of variables $\xi = (\xi_1, \xi') = \mathcal{R}_s(k - k_0)$, we have

$$I = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i|x-y|\xi_1} \frac{\mu(\xi_1, \xi')}{(\lambda_j \circ \mathcal{R}_s^{-1})(\xi + \mathcal{R}_s(k_0 + i\beta_s)) - \lambda} \, \mathrm{d}\xi. \qquad (3.23)$$

We introduce the following function defined on some neighborhood of 0 in $\mathbb{C}^d$:

$$W_s(z) := (\lambda_j \circ \mathcal{R}_s^{-1})(-iz + \mathcal{R}_s(k_0 + i\beta_s)) - \lambda.$$

It is holomorphic near 0 (on $i\mathcal{R}_s(V)$) and $W_s(0) = 0$. Then $W_s(iz)$ is the analytic continuation to the domain $\mathcal{R}_s(V)$ of the denominator of the integrand in (3.23). For a complex vector $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, we write $z = (z_1, z')$, where $z' = (z_2, \dots, z_d)$.

The following proposition provides a factorization of W_s that is crucial for computing the asymptotics of the integral I.

Proposition 3.6.1. *There exist $r > 0$ and $\epsilon > 0$ (independent of $s \in \mathcal{V}_\omega$), such that W_s has the decomposition*

$$W_s(z) = (z_1 - A_s(z'))B_s(z), \quad \forall z = (z_1, z') \in B(0, r) \times D'(0, \epsilon).^5 \qquad (3.24)$$

Here the functions A_s, B_s are holomorphic in $D'(0, \epsilon)$ and $B(0, r) \times D'(0, \epsilon)$ respectively such that $A_s(0) = 0$ and B_s is non-vanishing on $B(0, r) \times D'(0, \epsilon)$. Also, these functions and their derivatives depend continuously on s. Moreover for $z' \in D'(0, \epsilon)$,

$$A_s(z') = \frac{1}{2}z' \cdot Q_s z' + O(|z'|^3), \qquad (3.25)$$

---

[5]See Notation 3.2.2 in Section 3.2 for the definitions of $B(0, r)$ and $D'(0, \epsilon)$.

where $O(|z'|^3)$ is uniform in s when $z' \to 0$ and Q_s is the positive definite $(d-1) \times (d-1)$

matrix

$$Q_s = -\frac{1}{|\nabla E(\beta_s)|} \Big(e_{s,p} \cdot \mathrm{Hess}\,(E)(\beta_s) e_{s,q} \Big)_{2 \leq p,q \leq d}. \tag{3.26}$$

Proof. By Cauchy-Riemann equations for W_s and (3.7),

$$\frac{\partial W_s}{\partial z_1}(0) = \frac{\partial W_s}{\partial \xi_1}(0) = -i\nabla\lambda_j(k_0 + i\beta_s) \cdot \mathcal{R}_s^{-1} e_1 = -\nabla E(\beta_s) \cdot s = |\nabla E(\beta_s)| > 0. \tag{3.27}$$

Thus 0 is a simple zero of W_s. Due to smoothness in s of W_s and β_s, we have

$$c := \min_{s \in \mathcal{V}_\omega} \frac{\partial W_s}{\partial z_1}(0) \geq \min_{s \in \mathbb{S}^{d-1}} |\nabla E(\beta_s)| > 0. \tag{3.28}$$

Applying the Weierstrass preparation theorem (see Theorem 3.9.2), we obtain the decomposition (3.24) on a neighborhood of 0.

To show that this neighborhood can be chosen such that it does not depend on s, we have to chase down how the neighborhood is constructed in the proof of [38, Theorem 7.5.1] (only the first three lines of the proof there matter) and then show that all steps in this construction can be done independently of s.

In the first step of the construction, we need $r > 0$ such that $W_s(z_1, 0') \neq 0$ when $0 < |z_1| < 2r$. The mapping $(s, z) \mapsto \frac{\partial W_s}{\partial z_1}(z) = -i\nabla\lambda_j(-i\mathcal{R}_s^{-1} z + k_0 + i\beta_s) \cdot s$ is jointly continuous on $\mathcal{V}_\omega \times \mathcal{R}_s(V)$ and the value of this mapping at $z = 0$ is greater or equal than c due to (3.27) and (3.28). Therefore, $\left| \frac{\partial W_s}{\partial z_1}(z) \right| > c/2$ in some open neighborhood $X_s \times Y_s$ of $(s, 0)$ in $\mathcal{V}_\omega \times \mathbb{C}^d$. By compactness, $\mathcal{V}_\omega \subseteq \bigcup_{k=1}^{N} X_{s_k}$ for a finite collection of points $s_1, \ldots, s_N$ on the sphere. Let Y be the intersection of all Y_{s_k} and let $r > 0$ be such that $D(0, 2r) \subseteq Y$. Note that r is independent of s. We claim r has the desired property. Observe that for $|z| < 2r$, we have $\left| \frac{\partial W_s}{\partial z_1}(z) \right| > \frac{c}{2}$ for any s in $\mathcal{V}_\omega$. For a proof by

contradiction, suppose that there is some z_1 such that $0 < |z_1| < 2r$ and $W_s(z_1, 0') = 0 = W_s(0, 0')$ for some s. Applying Rolle's theorem to the function $t \in [0, 1] \mapsto W_s(tz_1, 0')$ yields $\dfrac{\partial W_s}{\partial z_1}(tz_1, 0') = 0$ for some $t \in (0, 1)$. Consequently, $(tz_1, 0') \notin D(0, 2r)$ while $|tz_1| < |z_1| < 2r$ (contradiction!).

For the second step of the construction, we want some $\delta > 0$ (independent of s) such that $W_s(z) \neq 0$ when $|z_1| = r, |z'| < \delta$. This can be done in a similar manner. Let $S(0, r) \subset \mathbb{C}$ be the circle with radius r. Now we consider the smooth mapping $W : (s, z_1, z') \mapsto W_s(z_1, z')$ where $z_1 \in S(0, r)$. Its value at each point $(s, z_1, 0')$ is equal to $W_s(z_1, 0')$, which is non-zero due to the choice of r in the first step of the construction. Thus, it is also non-zero in some open neighborhood $\tilde{X}_{s,z_1} \times \tilde{Y}_{s,z_1} \times \tilde{Z}_{s,z_1}$ of $(s, z_1, 0')$ in $\mathcal{V}_\omega \times S(0, r) \times \mathbb{C}^{d-1}$. We select points $s_1, \ldots, s_M \in \mathcal{V}_\omega$ and $\gamma_1, \ldots, \gamma_M \in S(0, r)$ so that the union of all $\tilde{X}_{s_k,\gamma_k} \times \tilde{Y}_{s_k,\gamma_k}, 1 \leq k \leq M$ covers the compact set $\mathcal{V}_\omega \times S(0, r)$. Next we choose $\delta > 0$ so that $D'(0, \delta)$ is contained in the intersection of these $\tilde{Z}_{s_k,z_k}$. Note that δ is independent of s and also z_1. Of course $W_s(z_1, z') \neq 0$ for all s and $z \in \{|z_1| = r, |z'| < \delta\}$. According to [38], the decomposition (3.24) holds in the polydisc $\{|z_1| < r, |z'| < \delta\}$.

Also, from the proof of [38, Theorem 7.5.1], the function A_s is defined via the following formula

$$z_1 - A_s(z') = \exp\left(\frac{1}{2\pi i} \int_{|\omega|=r} \left(\frac{\partial W_s(\omega, z')}{\partial \omega} \Big/ W_s(\omega, z')\right) \log(z_1 - \omega) \, \mathrm{d}\omega\right). \quad (3.29)$$

The mappings $(s, z') \mapsto A_s(z')$ and $(s, z) \mapsto B_s(z)$ are jointly continuous due to (3.24) and (3.29). There exists $0 < \epsilon \leq \delta$ such that $\max_{s \in \mathcal{V}_\omega} |A_s(z')| < r$ whenever $|z'| < \epsilon$. We have the identity (3.24) on $B(0, r) \times D'(0, \epsilon)$. Now, we show that this is indeed the neighborhood that has the desired properties. Since $|z'| < \epsilon$ implies that the points

$z = (A_s(z'), z') \in B(0, r) \times D'(0, \epsilon)$, we can evaluate (3.24) at these points to obtain

$$W_s(A_s(z'), z') = 0, \quad z' \in D'(0, \epsilon). \tag{3.30}$$

By differentiating (3.30), we have

$$\frac{\partial W_s}{\partial z_p}(A_s(z'), z') + \frac{\partial W_s}{\partial z_1}(A_s(z'), z')\frac{\partial A_s}{\partial z_p}(z') = 0, \quad \text{for } p = 2, \ldots, d. \tag{3.31}$$

Observe that from the above construction, the term $\dfrac{\partial W_s}{\partial z_1}(A_s(z'), z')$ is always non-zero whenever $|z'| < \epsilon$. Consequently, all first-order derivatives of A_s are jointly continuous in (s, z). Similarly, we deduce by induction on $n \in \mathbb{N}^d$ that all derivatives of the function A_s depend continuously on s since after taking differentiation of the equation (3.30) up to order n, the n-order derivative term always goes with the nonzero term $\dfrac{\partial W_s}{\partial z_1}(A_s(z'), z')$ and the remaining terms in the sum are just lower order derivatives. Hence the same conclusion holds for all derivatives of B_s by differentiating (3.24).

In particular, set $z' = 0$ in (3.31) to obtain

$$\frac{\partial W_s}{\partial z_p}(0) + \frac{\partial W_s}{\partial z_1}(0)\frac{\partial A_s}{\partial z_p}(0) = 0, \quad \text{for } p = 2, \ldots, d. \tag{3.32}$$

Note that for $p > 1$,

$$\frac{\partial W_s}{\partial z_p}(z) = -i\nabla\lambda_j(-i\mathcal{R}_s^{-1}z + k_0 + i\beta_s) \cdot \mathcal{R}_s^{-1}e_p. \tag{3.33}$$

By substituting $z = 0$,

$$\frac{\partial W_s}{\partial z_p}(0) = -i\nabla\lambda_j(k_0 + i\beta_s) \cdot \mathcal{R}_s^{-1}e_p$$

$$= -\nabla E(\beta_s) \cdot e_{s,p} = -|\nabla E(\beta_s)|s \cdot e_{s,p} = 0. \tag{3.34}$$

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(3.27), (3.32) and (3.34) imply

$$\frac{\partial A_s}{\partial z_p}(0) = 0, \quad \text{for } p = 2, \ldots, d. \tag{3.35}$$

Taking a partial derivative with respect to z_q $(q > 1)$ of (3.33) at $z = 0$, we see that

$$\begin{aligned}
\frac{\partial^2 W_s}{\partial z_p \partial z_q}(0) &= \sum_{m=1}^{d} -\nabla(\frac{\partial \lambda_j}{\partial z_m}(k_0 + i\beta_s)) \cdot \mathcal{R}_s^{-1} e_p (\mathcal{R}_s^{-1} e_q)_m \\
&= -\sum_{m,n=1}^{d} \frac{\partial^2 \lambda_j}{\partial z_m \partial z_n}(k_0 + i\beta_s)(e_{s,p})_m (e_{s,q})_n \\
&= e_{s,q} \cdot \text{Hess}\,(E)(\beta_s) e_{s,p}.
\end{aligned} \tag{3.36}$$

A second differentiation of (3.31) at $z = (A_s(z'), z')$ gives

$$\begin{aligned}
0 &= \left(\frac{\partial^2 W_s}{\partial z_p \partial z_q}(z) + \frac{\partial W_s}{\partial z_1}(z) \frac{\partial^2 A_s}{\partial z_p \partial z_q}(z') \right) \\
&\quad + \left(\frac{\partial^2 W_s}{\partial z_1 \partial z_q}(z) \frac{\partial A_s}{\partial z_p}(z') + \frac{\partial^2 W_s}{\partial z_p \partial z_1}(z) \frac{\partial A_s}{\partial z_q}(z') + \frac{\partial^2 W_s}{\partial z_1^2}(z) \frac{\partial A_s}{\partial z_p}(z') \frac{\partial A_s}{\partial z_q}(z') \right).
\end{aligned} \tag{3.37}$$

At $z = 0$, the sum in the second bracket of (3.37) is zero due to (3.35). Thus,

$$\frac{\partial^2 A_s}{\partial z_p \partial z_q}(0) = -\left(\frac{\partial W_s}{\partial z_1}(0) \right)^{-1} \frac{\partial^2 W_s}{\partial z_p \partial z_q}(0) \quad (2 \le p, q \le d). \tag{3.38}$$

Together with (3.27) and (3.36), the above equality becomes

$$\frac{\partial^2 A_s}{\partial z_p \partial z_q}(0) = -\frac{1}{|\nabla E(\beta_s)|} \left(e_{s,p} \cdot \text{Hess}\,(E)(\beta_s) e_{s,q} \right)_{2 \le p, q \le d} = Q_s. \tag{3.39}$$

Consequently, by (3.35) and (3.39), the Taylor expansion of A_s at 0 implies (3.25).

Finally, the remainder term $O(|z'|^3)$ in the Taylor expansion (3.25), denoted by $R_{s,3}(z')$,

can be estimated as follows:

$$
\begin{aligned}
|R_{s,3}(z')| &\lesssim |z'|^3 \max_{|\alpha|=3, 0 \le t \le 1} \left| \frac{\partial^\alpha A_s}{\partial z^\alpha}(tz') \right| \\
&\lesssim |z'|^3 \max_{|\alpha|=3, |y| \le |z'|} \left| \frac{\partial^\alpha A_s}{\partial z^\alpha}(y) \right|.
\end{aligned}
$$

Due to the continuity of third-order derivatives of A_s on $\mathcal{V}_\omega \times D'(0, \epsilon)$,

$$
\lim_{|z'| \to 0} \max_{s \in \mathcal{V}_\omega} \frac{|R_{s,3}(z')|}{|z'|^3} < \infty. \tag{3.40}
$$

This proves the last claim of this proposition. $\qquad\square$

We can now let the size of the support of η ($\Subset \mathcal{O}$) be small enough such that the decomposition (3.24) in Proposition 3.6.1 holds on the support of μ, i.e., $\operatorname{supp}(\mu) \Subset B(0, r) \times D'(0, \epsilon)$. Therefore, from (3.23), we can represent the integral I as follows:

$$
I = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i|x-y|\xi_1} \frac{\mu(\xi_1, \xi')}{W_s(i\xi)} \, d\xi_1 \, d\xi' = (2\pi)^{-d} \int_{|\xi'|<\epsilon} \int_{\mathbb{R}} \frac{e^{i|x-y|\xi_1} \tilde{\mu}_s(\xi_1, \xi')}{i\xi_1 - A_s(i\xi')} \, d\xi_1 \, d\xi', \tag{3.41}
$$

where $\tilde{\mu}_s(\xi) = \mu(\xi)(B_s(i\xi))^{-1}$. We extend $\tilde{\mu}_s$ to a smooth compactly supported function on $\mathbb{R}^d$ by setting $\tilde{\mu}_s = 0$ outside its support. Since all derivatives of $\tilde{\mu}_s$ depend continuously on s, they are uniformly bounded in s. Let $\nu_s(t, \xi')$ be the Fourier transform in the variable ξ_1 of the function $\tilde{\mu}_s(-\xi_1, \xi')$ for each $\xi' \in \mathbb{R}^{d-1}$, i.e.,

$$
\nu_s(t, \xi') = \int_{-\infty}^{+\infty} e^{it\xi_1} \tilde{\mu}_s(\xi_1, \xi') \, d\xi_1.
$$

By applying the Lebesgue Dominated Convergence Theorem, the function ν_s is continuous in (s, t, ξ') on $\mathcal{V}_\omega \times \mathbb{R}^d$. For such ξ', $\nu_s(\cdot, \xi')$ is a Schwartz function in t on $\mathbb{R}$. Due to Lemma 3.9.1, for any $N > 0$, $\nu_s(t, \xi') = O(|t|^{-N})$ uniformly in s and ξ' as $t \to \infty$. We also choose

ϵ small enough such that whenever $|\xi'| < \epsilon$, the absolute value of the remainder term $O(|\xi'|^3)$ in (3.25) is bounded from above by $\frac{1}{4}\xi' \cdot Q_s\xi'$. Note that ϵ is still independent of s, because the term $O(|\xi'|^3)/|\xi'|^3$ is uniformly bounded by the quantity in (3.40). Meanwhile, each positive definite matrix Q_s dominates the positive matrix $\gamma_\omega I_{(d-1)\times(d-1)}$, where $\gamma_\omega > 0$ is the smallest among all the eigenvalues of all matrices Q_s ($s \in \mathcal{V}_\omega$). This implies that if $0 < |\xi'| < \epsilon$, then

$$\Re(i\xi_1 - A_s(i\xi')) = -\Re(A_s(i\xi')) = -\Re(-\frac{1}{2}\xi' \cdot Q_s\xi' + O(|\xi'|^3))$$
$$= \frac{1}{2}\xi' \cdot Q_s\xi' - \Re(O(|\xi'|^3)) > \frac{1}{4}\gamma_\omega|\xi'|^2 > 0.$$

We thus can obtain the following integral representation for a factor in the integrand of I (see (3.41)):

$$\frac{1}{i\xi_1 - A_s(i\xi')} = \int_{-\infty}^0 e^{(i\xi_1 - A_s(i\xi'))w}\,\mathrm{d}w, \quad (\xi_1, \xi') \in \mathbb{R} \times (D'(0,\epsilon) \setminus \{0\}). \qquad (3.42)$$

Therefore,

$$I = \frac{1}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^0 e^{-wA_s(i\xi')} \int_{-r}^r e^{i(w+|x-y|)\xi_1}\tilde{\mu}_s(\xi_1, \xi')\,\mathrm{d}\xi_1\,\mathrm{d}w\,\mathrm{d}\xi'$$
$$= \frac{1}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^{|x-y|} e^{(-t+|x-y|)A_s(i\xi')}\nu_s(t, \xi')\,\mathrm{d}t\,\mathrm{d}\xi'. \qquad (3.43)$$

Now our remaining task is to prove the following asymptotics of the integral I:

Theorem 3.6.2. *We have*

$$I = \frac{|\nabla E(\beta_s)|^{(d-3)/2}|x - y|^{-(d-1)/2}}{(2\pi)^{(d-1)/2} \det\left(-e_{s,p} \cdot \text{Hess}\,(E)(\beta_s)e_{s,q}\right)_{2\leq p,q\leq d}^{1/2}} + O(|x - y|^{-d/2}). \qquad (3.44)$$

Here the term $O(|x - y|^{-d/2})$ is uniform in $s \in \mathcal{V}_\omega$ as $|x - y| \to \infty$.

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The next lemma reduces the leading term of the right hand side of (3.44) to a scalar integral as follows

Lemma 3.6.3.

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \nu_s(t,0) \, \mathrm{d}t \, \mathrm{d}x' = \frac{(2\pi)^{(d+1)/2} |\nabla E(\beta_s)|^{(d-3)/2}}{\det\left(-e_{s,p} \cdot \mathrm{Hess}\,(E)(\beta_s) e_{s,q}\right)_{2 \le p,q \le d}^{1/2}}.$$

Proof. By applying the Fourier inversion formula to ν_s, we get

$$\frac{1}{2\pi} \int_{\mathbb{R}} \nu_s(t,0) \, \mathrm{d}t = \tilde{\mu}_s(0) = (B_s(0))^{-1} = \left(\frac{\partial W_s}{\partial z_1}(0)\right)^{-1} = \frac{1}{|\nabla E(\beta_s)|}. \tag{3.45}$$

Here (3.27) is used in the last equality. Thus,

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \nu_s(t,0) \, \mathrm{d}t \, \mathrm{d}x'$$

$$= \frac{1}{(\det Q_s)^{1/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \exp\left(-\frac{1}{2} |u'|^2\right) \nu_s(t,0) \, \mathrm{d}u' \, \mathrm{d}t = \frac{(2\pi)^{(d-1)/2}}{(\det Q_s)^{1/2}} \int_{\mathbb{R}} \nu_s(t,0) \, \mathrm{d}t$$

$$= \frac{(2\pi)^{(d+1)/2}}{(\det Q_s)^{1/2} |\nabla E(\beta_s)|} = \frac{(2\pi)^{(d+1)/2} |\nabla E(\beta_s)|^{(d-3)/2}}{\det\left(-e_{s,p} \cdot \mathrm{Hess}\,(E)(\beta_s) e_{s,q}\right)_{2 \le p,q \le d}^{1/2}}.$$

Note that we use the change of variables $u' := Q_s^{1/2} x'$, (3.45), (3.26) in the first, the third and the last equality respectively. $\square$

For clarity, we introduce the notation $x_0 := |x - y|$. The purpose of the following two lemmas is to truncate some unnecessary (rapidly decreasing) parts of the main integrals we are interested in.

Lemma 3.6.4. *i) For any $\alpha \in (0,1)$ and $n > 0$, one has*

$$\sup_{s \in \mathcal{V}_\omega} \int_{|\xi'| < \epsilon} \int_{(-\infty, -x_0^\alpha) \cup (x_0^\alpha, x_0)} \exp\left((x_0 - t) A_s(i\xi')\right) \nu_s(t, \xi') \, \mathrm{d}t \, \mathrm{d}\xi' = O(x_0^{-n})$$

and

$$\sup_{s \in \mathcal{V}_\omega} \int_{\mathbb{R}^{d-1}} \int_{|t| > x_0^\alpha} \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \nu_s(t, 0) \, \mathrm{d}t \, \mathrm{d}x' = O(x_0^{-n}).$$

ii) For any $\beta < 1/2$, $n > 0$ and each fixed $t \in [-x_0/2, x_0/2]$, one obtains

$$\sup_{s \in \mathcal{V}_\omega} \int_{\epsilon\sqrt{x_0-t} > |x'| \geq x_0^\beta} \left| \exp\left((x_0 - t) A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right) \right| \mathrm{d}x' = O(x_0^{-n})$$

and

$$\sup_{s \in \mathcal{V}_\omega} \int_{|x'| \geq x_0^\beta} \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \mathrm{d}x' = O(x_0^{-n}).$$

Proof.   i) We recall that $\sup_{s, \xi'} |\nu_s(t, \xi')| = O(|t|^{-n})$ for any $n > 0$. Observe that when $t \leq x_0$, $|e^{(x_0-t)A_s(i\xi')}| \leq 1$. Thus, we have

$$\sup_{s \in \mathcal{V}_\omega} \left| \int_{|\xi'| < \epsilon} \int_{(-\infty, -x_0^\alpha) \cup (x_0^\alpha, x_0)} \exp\left((x_0 - t) A_s(i\xi') \right) \nu_s(t, \xi') \, \mathrm{d}t \, \mathrm{d}\xi' \right|$$

$$\lesssim \int_{|\xi'| < \epsilon} \int_{(-\infty, -x_0^\alpha) \cup (x_0^\alpha, x_0)} |t|^{-n/\alpha - 1} \, \mathrm{d}t \, \mathrm{d}\xi' \lesssim \int_{|t| > x_0^\alpha} |t|^{-n/\alpha - 1} \, \mathrm{d}t = O(x_0^{-n}).$$

$$(3.46)$$

Since $\left| \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \right| \leq 1$, the second integral in this part also decays rapidly by the same argument.

ii) When $t < x_0$, we can substitute $\xi' = x'(x_0 - t)^{-1/2}$ into (3.25) to obtain

$$(x_0 - t) \cdot A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) = -\frac{1}{2} x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}} \right). \qquad (3.47)$$

Due to our choice of ϵ and the definition of γ_ω, we get the following estimate when

$|x'| < \epsilon\sqrt{x_0 - t}$:

$$\sup_{s \in \mathcal{V}_\omega} \left| \exp\left((x_0 - t)A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right) \right| = \sup_{s \in \mathcal{V}_\omega} \exp\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}} \right) \right)$$

$$\leq \exp\left(-\frac{1}{4}\gamma_\omega |x'|^2 \right).$$

(3.48)

Hence, the two integrals in the statement can be estimated from above by:

$$\int_{|x'| \geq x_0^\beta} \exp\left(-\frac{1}{4}\gamma_\omega |x'|^2 \right) dx' \lesssim \int_{x_0^\beta}^\infty \exp\left(-\frac{1}{4}\gamma_\omega r^2 \right) r^{d-2} \, dr$$

$$\lesssim \int_{x_0^\beta}^\infty r^{-n/\beta - 1 - (d-2)} r^{d-2} \, dr = O(x_0^{-n}).$$

$\square$

Lemma 3.6.5. *If $\alpha \in (0, 1)$, we have*

$$\sup_{s \in \mathcal{V}_\omega} \int_{|t| \leq x_0^\alpha} \int_{|x'| < \epsilon\sqrt{x_0 - t}} \left((1 - \frac{t}{x_0})^{-(d-1)/2} - 1 \right) \exp\left((x_0 - t)A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right)$$

$$\times \nu_s\left(t, \frac{x'}{\sqrt{x_0 - t}} \right) dx' \, dt = O(x_0^{2\alpha - 1}).$$

Proof. As we argued in the proof of Lemma 3.6.4 (ii), this integral is majorized by

$$\int_{-x_0^\alpha}^{x_0^\alpha} \int_{|x'| < \epsilon\sqrt{x_0 - t}} \exp\left(-\frac{1}{4}x' \cdot Q_s x' \right) \cdot \left| \nu_s\left(t, \frac{x'}{\sqrt{x_0 - t}} \right) \right| \cdot \left| \left(1 - \frac{t}{x_0} \right)^{-(d-1)/2} - 1 \right| dx' \, dt.$$

It suffices to estimate the factor $\left(1 - \frac{t}{x_0} \right)^{-(d-1)/2} - 1$ since ν_s is uniformly bounded on $\mathbb{R} \times D'(0, \epsilon)$. But this is straightforward, since

$$\int_{-x_0^\alpha}^{x_0^\alpha} \left| \left(1 - \frac{t}{x_0} \right)^{-(d-1)/2} - 1 \right| dt \leq 2x_0^\alpha \left(\left(1 - x_0^{\alpha-1} \right)^{-(d-1)/2} - 1 \right) = O(x_0^{2\alpha - 1}).$$

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This finishes the proof of this lemma. □

Proof of Theorem 3.6.2.

Proof. Thanks to Lemma 3.6.3, it is enough to prove the relation

$$I = (2\pi)^{-d} x_0^{-(d-1)/2} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \nu_s(t, 0) \, \mathrm{d}t \, \mathrm{d}x' + O(x_0^{-d/2}).$$

Due to Lemma 3.6.4 (i) with $\alpha = 1/4$, we only need to show that

$$\widetilde{I} = x_0^{-(d-1)/2} \int_{\mathbb{R}^{d-1}} \int_{|t| \leq x_0^{1/4}} \exp\left(-\frac{1}{2} x' \cdot Q_s x'\right) \nu_s(t, 0) \, \mathrm{d}t \, \mathrm{d}x' + O(x_0^{-d/2})$$

where

$$\widetilde{I} = \int_{|\xi'| < \epsilon} \int_{|t| \leq x_0^{1/4}} \exp\left((x_0 - t) A_s(i\xi')\right) \nu_s(t, \xi') \, \mathrm{d}t \, \mathrm{d}\xi'.$$

Then we substitute $x' = \xi'\sqrt{x_0 - t}$ to the integral $\widetilde{I}$ to get

$$\widetilde{I} = x_0^{-(d-1)/2} \int_{|t| \leq x_0^{1/4}} \int_{|x'| < \epsilon\sqrt{x_0 - t}} \left(1 - \frac{t}{x_0}\right)^{-(d-1)/2} \exp\left((x_0 - t) A_s\left(\frac{ix'}{\sqrt{x_0 - t}}\right)\right)$$
$$\times \nu_s\left(t, \frac{x'}{\sqrt{x_0 - t}}\right) \mathrm{d}x' \, \mathrm{d}t.$$

By Lemma 3.6.5 with $\alpha = 1/4$, we have

$$\int_{|t| \leq x_0^{1/4}} \int_{|x'| < \epsilon\sqrt{x_0 - t}} \exp\left((x_0 - t) A_s\left(\frac{ix'}{\sqrt{x_0 - t}}\right)\right) \nu_s\left(t, \frac{x'}{\sqrt{x_0 - t}}\right) \mathrm{d}x' \, \mathrm{d}t$$
$$= x_0^{(d-1)/2} \widetilde{I} + O(x_0^{-1/2}). \tag{3.49}$$

Next, it is clear that for $|t| \leq x_0^{1/4}$, one has

$$\left| \nu_s\left(t, \frac{x'}{\sqrt{x_0 - t}}\right) - \nu_s(t, 0) \right| \leq \frac{|x'|}{\sqrt{x_0 - t}} \sup_{s, \xi'} |\nabla_{\xi'} \nu_s(t, \xi')| \lesssim \frac{|x'|}{\sqrt{x_0}} \sup_{s, \xi'} |\nabla_{\xi'} \nu_s(t, \xi')|.$$

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Also, from the definition of the function ν_s, it follows that $\sup_{s,\xi'} |\nabla_{\xi'} \nu_s(t, \xi')| = O(|t|)^{-n}$ for any $n > 0$. Consequently,

$$\int_{|t| \le x_0^{1/4}} \int_{|x'| < \epsilon \sqrt{x_0 - t}} \left| \exp\left((x_0 - t) A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right) \left(\nu_s \left(t, \frac{x'}{\sqrt{x_0 - t}} \right) - \nu_s(t, 0) \right) \right| dx' \, dt$$
$$\lesssim \frac{1}{\sqrt{x_0}} \int_{\mathbb{R}^{d-1}} \exp\left(-\frac{1}{4} \gamma_\omega |x'|^2 \right) |x'| \, dx' \cdot \int_{\mathbb{R}} \sup_{s,\xi'} |\nabla_{\xi'} \nu_s(t, \xi')| \, dt = O(x_0^{-1/2}).$$

(3.50)

Using (3.49), (3.50) and Lemma 3.6.4 (i), it remains to derive the relation

$$\int_{|t| \le x_0^{1/4}} \int_{|x'| < \epsilon \sqrt{x_0 - t}} \exp\left((x_0 - t) A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right) \nu_s(t, 0) \, dx' \, dt$$
$$= \int_{|t| \le x_0^{1/4}} \int_{\mathbb{R}^{d-1}} \exp\left(-\frac{1}{2} x' \cdot Q_s x' \right) \nu_s(t, 0) \, dx' \, dt + O(x_0^{-1/2}).$$

(3.51)

Due to Lemma (3.6.4) (ii) with $\beta = 1/6$, we obtain

$$\sup_{s \in \mathcal{V}_\omega} \int_{\epsilon \sqrt{x_0 - t} > |x'| \ge x_0^{1/6}} \left| \exp\left((x_0 - t) A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right) \right| dx' = O(x_0^{-n}),$$

$$\sup_{s \in \mathcal{V}_\omega} \int_{|x'| \ge x_0^{1/6}} \exp\left(-\frac{1}{2} x' \cdot Q_s x' \right) dx' = O(x_0^{-n}).$$

On the other hand,

$$\sup_{s \in \mathcal{V}_\omega} \int_{|x'| < x_0^{1/6}} \left| \exp\left((x_0 - t) A_s \left(\frac{ix'}{\sqrt{x_0 - t}} \right) \right) - \exp\left(-\frac{1}{2} x' \cdot Q_s x' \right) \right| dx'$$
$$= \sup_{s \in \mathcal{V}_\omega} \int_{|x'| < x_0^{1/6}} \exp\left(-\frac{1}{2} x' \cdot Q_s x' \right) \left| \exp\left(O\left(\frac{|x'|^3}{\sqrt{x_0}} \right) \right) - 1 \right| dx'$$
$$\lesssim \int_{|x'| < x_0^{1/6}} \exp\left(-\frac{1}{2} \gamma_\omega |x'|^2 \right) \frac{|x'|^3}{\sqrt{x_0}} \, dx' = O(x_0^{-1/2}).$$

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Hence, we deduce

$$\int_{|x'|<\epsilon\sqrt{x_0-t}} \exp\left((x_0-t)A_s\left(\frac{ix'}{\sqrt{x_0-t}}\right)\right)\,\mathrm{d}x' - \int_{\mathbb{R}^{d-1}} \exp\left(-\frac{1}{2}x'\cdot Q_s x'\right)\,\mathrm{d}x' = O(x_0^{-1/2})$$

for each $t \in [-x_0^{1/4}, x_0^{1/4}]$. Finally, we multiply the above relation with $\nu_s(t,0)$ and then integrate over the interval $[-x_0^{1/4}, x_0^{1/4}]$. Since $\sup_s |\nu_s(t,0)|$ is integrable over $\mathbb{R}$, the right hand side is still $O(x_0^{-1/2})$. Thus, we derive (3.51) as we wish. $\qquad\square$

3.6.2 Estimates of the integral J

In this part, we want to show that the expression J decays as $O\left(|x-y|^{-d/2}\right)$. Thus, taking into account (3.44), we conclude that J does not contribute to the leading term of the reduced Green's function.

In (3.20), we set the coordinate functions of ρ' as $(\rho_1, \ldots, \rho_d)$. Let us introduce the smooth function $\mu^{(l)}(k, x, y) = \rho_l(k + k_0, x, y)\mu(k)$ for any $k \in \mathbb{R}^d$. The support of $\mu^{(l)}$ (as a function of k for each pair (x, y)) is contained in the support of μ and $\mu^{(l)}(k, \cdot, \cdot)$ is $\mathbb{Z}^d \times \mathbb{Z}^d$-periodic. We denote the components of a vector k in $\mathbb{R}^d$ as $(k_1, \ldots, k_d)$. Observe that J is the sum of integrals J_l $(1 \le l \le d)$ if we define

$$J_l := (2\pi)^{-d} \int_{\mathcal{O}} e^{i(k-k_0)\cdot(x-y)} \frac{\mu^{(l)}(k-k_0, x, y)(k-k_0)_l}{\lambda_j(k+i\beta_s) - \lambda}\,\mathrm{d}k. \tag{3.52}$$

Proposition 3.6.6. *As $|x-y| \to \infty$, we have $J_1 = O\left(|x-y|^{-(d+1)/2}\right)$ and $J_l = O\left(|x-y|^{-d/2}\right)$ if $l > 1$. In particular, $J = O\left(|x-y|^{-d/2}\right)$.*

Proof. Indeed, to treat these integrals, we need to re-examine the calculation in the previous subsection done for the integral I. After applying the orthogonal transformation $\mathcal{R}_s$

on each integral J_l, we rewrite them under the form of (3.41) as

$$J_l = (2\pi)^{-d} \int_{|\xi'|<\epsilon} \int_{\mathbb{R}} e^{i|x-y|\xi_1} \frac{\tilde{\mu}_s^{(l)}(\xi_1, \xi', x, y)\xi_l}{i\xi_1 - A_s(i\xi')} \, d\xi_1 \, d\xi', \tag{3.53}$$

where $\tilde{\mu}_s^{(l)}(\xi, x, y)$ is $\mu^{(l)}(\xi, x, y)(B_s(i\xi))^{-1}$ on the support of $\mu^{(l)}$ and vanishes elsewhere. Let $\nu_s^{(l)}(t, \xi', x, y)$ be the Fourier transform in ξ_1 of $\tilde{\mu}_s^{(l)}(\xi_1, \xi', x, y)$. If the parameter s is viewed as another argument of our functions here, then $\nu_s^{(l)}(\cdot, \xi', x, y)$ is a Schwartz function for each quadruple (s, ξ', x, y). It is elementary to check that the Fourier transform $\nu_s^{(l)}(t, \xi', x, y)$ is jointly continuous on $\mathcal{V}_\omega \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^d \times \mathbb{R}^d$ due to the corresponding property of $\tilde{\mu}_s^{(l)}(\xi, x, y)$. Periodicity in (x, y) of $\nu_s^{(l)}$ and Lemma 3.9.1 imply the following decay:

$$\lim_{t \to \infty} |t|^N \sup_{(s,\xi',x,y) \in \mathcal{V}_\omega \times \overline{D'(0,\epsilon)} \times \mathbb{R}^d \times \mathbb{R}^d} |\nu_s^{(l)}(t, \xi', x, y)| = 0, \quad N \geq 0. \tag{3.54}$$

In particular,

$$\max_{1 \leq l \leq d} \sup_{(s,t,\xi',x,y) \in \mathcal{V}_\omega \times \mathbb{R} \times \overline{D'(0,\epsilon)} \times \mathbb{R}^d \times \mathbb{R}^d} |\nu_s^{(l)}(t, \xi', x, y)| < \infty \tag{3.55}$$

and

$$S := \max_{1 \leq l \leq d} \int_{\mathbb{R}} \sup_{(s,\xi',x,y) \in \mathcal{V}_\omega \times \overline{D'(0,\epsilon)} \times \mathbb{R}^d \times \mathbb{R}^d} |\nu_s^{(l)}(t, \xi', x, y)| \, dt < \infty. \tag{3.56}$$

Recall that when $0 < |\xi'| < \epsilon$, $\Re(A_s(i\xi')) < 0$ and thus from (3.55),

$$\lim_{t \to -\infty} e^{(-t+|x-y|)A_s(i\xi')} \nu_s^{(1)}(t, \xi', x, y) = 0. \tag{3.57}$$

Case 1: $l = 1$.

Using (3.42), (3.57) and integration by parts, we obtain

$$
\begin{aligned}
J_1 &= \frac{1}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^{0} e^{-wA_s(i\xi')} \int_{-r}^{r} \xi_1 e^{i(w+|x-y|)\xi_1} \tilde{\mu}_s^{(1)}(\xi_1, \xi', x, y)\, d\xi_1\, dw\, d\xi \\
&= -\frac{i}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^{|x-y|} e^{(-t+|x-y|)A_s(i\xi')} \frac{d}{dt} \nu_s^{(1)}(t, \xi', x, y)\, dt\, d\xi' \\
&= -\frac{i}{(2\pi)^d} \int_{|\xi'|<\epsilon} \Bigg(\nu_s^{(1)}(|x-y|, \xi', x, y) + \int_{-\infty}^{|x-y|} A_s(i\xi') e^{(-t+|x-y|)A_s(i\xi')} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \nu_s^{(1)}(t, \xi', x, y)\, dt \Bigg)\, d\xi'.
\end{aligned}
\tag{3.58}
$$

Recall the notation $x_0 = |x - y|$. The term

$$
\int_{|\xi'|<\epsilon} \nu_s^{(1)}(x_0, \xi', x, y)\, d\xi'
$$

decays rapidly in x_0, due to (3.54). We decompose the other term

$$
\int_{-\infty}^{x_0} A_s(i\xi') e^{(x_0-t)A_s(i\xi')} \nu_s^{(1)}(t, \xi', x, y)\, dt
$$

into two parts, where the first integral is taking over $(x_0/2, x_0]$ and the second one over $(-\infty, x_0/2]$. The first part decays rapidly, as in Lemma 3.6.4 (i). Now we need to prove that the second part decays as $O(x_0^{(d+1)/2})$. To do this, we use the change of variables $x' = \xi'\sqrt{x_0 - t}$ to rewrite the remaining integral as

$$
\begin{aligned}
&x_0^{(d+1)/2} \int_{|\xi'|<\epsilon} \int_{-\infty}^{x_0/2} A_s(i\xi') e^{(-t+x_0)A_s(i\xi')} \nu_s^{(1)}(t, \xi', x, y)\, dt\, d\xi' \\
&= \int_{-\infty}^{x_0/2} \left(1 - \frac{t}{x_0}\right)^{-(d+1)/2} \int_{|x'|<\epsilon\sqrt{x_0-t}} \left(-\frac{1}{2} x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0-t}}\right)\right) \\
&\quad \times \exp\left(-\frac{1}{2} x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0-t}}\right)\right) \nu_s^{(1)}\left(t, \left(\frac{x'}{\sqrt{x_0-t}}\right), x, y\right) dx'\, dt.
\end{aligned}
\tag{3.59}
$$

From (3.56), we derive

$$\int_{-\infty}^{x_0/2} \left(1 - \frac{t}{x_0}\right)^{-(d+1)/2} \sup_{(s,\xi',x,y)\in\mathcal{V}_\omega\times\overline{D'(0,\epsilon)}\times\mathbb{R}^d\times\mathbb{R}^d} |\nu_s^{(1)}(t,\xi',x,y)|\, dt \leq 2^{\frac{(d+1)}{2}} S. \quad (3.60)$$

On the other hand, we recall that

$$\Re\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}}\right)\right) \leq -\frac{1}{4}\gamma_\omega |x'|^2.$$

The exponential term is majorized as follows:

$$\left|\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}}\right)\right) \exp\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}}\right)\right)\right|$$
$$\leq \left(\frac{1}{2}x' \cdot Q_s x' + O(\epsilon |x'|^2)\right) \exp\left(-\frac{1}{4}\gamma_\omega |x'|^2\right).$$

Consequently,

$$\int_{|x'|<\epsilon\sqrt{x_0-t}} \left|\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}}\right)\right) \exp\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0 - t}}\right)\right)\right| dx'$$
$$\lesssim \int_{\mathbb{R}^{d-1}} |x'|^2 \exp\left(-\frac{1}{4}\gamma_\omega |x'|^2\right) dx' < \infty.$$
$$(3.61)$$

Combining (3.58) through (3.61), we deduce $J_1 = O(x_0^{-(d+1)/2})$.

Case 2: $l > 1$.

Using (3.42) and decomposing J_l into two parts as in **Case 1**, we get

$$
\begin{aligned}
J_l &= \frac{1}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^{0} \xi_l e^{-wA_s(i\xi')} \int_{-r}^{r} e^{i(w+|x-y|)\xi_1} \tilde{\mu}_s^{(l)}(\xi_1, \xi', x, y) \, d\xi_1 \, dw \, d\xi \\
&= \frac{1}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^{|x-y|} \xi_l e^{(-t+|x-y|)A_s(i\xi')} \nu_s^{(l)}(t, \xi', x, y) \, dt \, d\xi' \\
&= \frac{1}{(2\pi)^d} \int_{|\xi'|<\epsilon} \int_{-\infty}^{|x-y|/2} \xi_l e^{(-t+|x-y|)A_s(i\xi')} \nu_s^{(l)}(t, \xi', x, y) \, dt \, d\xi' + o(|x-y|^{-d/2}).
\end{aligned}
$$

$$(3.62)$$

By changing the variables as before,

$$
\begin{aligned}
x_0^{d/2} &\int_{|\xi'|<\epsilon} \int_{-\infty}^{x_0/2} \xi_l e^{(-t+x_0)A_s(i\xi')} \nu_s^{(l)}(t, \xi', x, y) \, dt \, d\xi' \\
&= \int_{-\infty}^{x_0/2} \left(1 - \frac{t}{x_0}\right)^{-d/2} \int_{|x'|<\epsilon\sqrt{x_0-t}} x_l' \exp\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0-t}}\right)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \nu_s^{(l)}\left(t, \left(\frac{x'}{\sqrt{x_0-t}}\right), x, y\right) \, dx' \, dt.
\end{aligned}
$$

$$(3.63)$$

In a similar manner, we obtain

$$
\begin{aligned}
\int_{-\infty}^{x_0/2} &\left(1 - \frac{t}{x_0}\right)^{-d/2} \int_{|x'|<\epsilon\sqrt{x_0-t}} \left| x_l' \exp\left(-\frac{1}{2}x' \cdot Q_s x' + O\left(\frac{|x'|^3}{\sqrt{x_0-t}}\right)\right) \right| \\
&\qquad\qquad\qquad\qquad \times \left| \nu_s^{(l)}\left(t, \left(\frac{x'}{\sqrt{x_0-t}}\right), x, y\right) \right| \, dx' \, dt \\
&\qquad\qquad\qquad \leq 2^{d/2} S \int_{\mathbb{R}^{d-1}} |x'| \exp\left(-\frac{1}{4}\gamma_\omega |x'|^2\right) dx' < \infty.
\end{aligned}
$$

This final estimate and (3.62)-(3.63) imply $J_l = O(x_0^{-d/2})$. $\qquad\square$

3.7 The full Green's function asymptotics

The main purpose of this section is to give a detailed proof of Theorem 3.5.2. Essentially, this theorem is needed for showing that full Green's function $G_{s,\lambda}$ has the same asymptotics as the reduced Green's function G_0 as $|x - y| \to \infty$.

First, we recall that for each unit vector s, $T_s(k) = (1 - \eta(k))(L_s(k) - \lambda)^{-1} + \eta(k)((L_s(k) - \lambda)|_{R(Q_s(k))})^{-1}Q_s(k)$ and the operator T_s is unitarily equivalent (via the Floquet transform) to the direct integral of the operators $T_s(k)$ over $\mathcal{O}$. Now we observe that the kernel of each projector $P_s(k)$ (see Subsection 3.5.2) is the smooth function:

$$\frac{\phi(k + i\beta_s, x)\overline{\phi(k - i\beta_s, y)}}{F(k + i\beta_s)},$$

for each k in the support of η. Thus, $(1 - \eta(k))P_s(k)$ is a finite rank smoothing operator on $\mathbb{T}^d$. Moreover, we also have $(L_s(k) - \lambda)T_s(k) = T_s(k)(L_s(k) - \lambda) = I - \eta(k)P_s(k)$. Each $T_s(k)$ is a parametrix (i.e., an inverse modulo a smoothing operator) of the elliptic operator $L_s(k) - \lambda$ when $(s, k) \in \mathbb{S}^{d-1} \times \mathcal{O}$. This suggests to study parametrices of the family of elliptic operators $L_s(k) - \lambda$ simultaneously.

3.7.1 Parameter-dependent periodic pseudodifferential operators

First, we briefly recall some basic definitions of **periodic** (or **toroidal**) pseudodifferential operators (i.e., ΨDO on the torus $\mathbb{T}^d$). We also introduce some useful classes of symbols with parameters and describe some of their properties that we will use.

There are several approaches to defining pseudodifferential operators on the torus. The standard approach based on Hörmander's symbol classes (see e.g., [72]) uses local smooth structure on the torus $\mathbb{T}^d$ and thus ignores the group structure on $\mathbb{T}^d$. An alternative approach uses Fourier series with the difference calculus and avoids using local coordinate charts on $\mathbb{T}^d$ (the details in [67, Chapter 4])[6]. To make a distinction, Ruzhansky and Turunen in [67] refer to the symbols in the first approach as **Euclidean symbols** and the symbols in the latter one as **toroidal symbols** (see [67, Section 4.5]). We recall their definitions for only the Kohn-Nirenberg symbol classes, which we need here:

---

[6]A different approach to periodic ΨDOs is introduced by A. Sobolev [73].

Definition 3.7.1. Let m be a real number.

(a) The class $S^m(\mathbb{T}^d \times \mathbb{R}^d)$ consists of all smooth functions $\sigma(x, \xi)$ on $\mathbb{T}^d \times \mathbb{R}^d$ such that for any multi-indices α, β,

$$|D_\xi^\alpha D_x^\beta \sigma(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m - |\alpha|},$$

for some constant $C_{\alpha,\beta}$ that depends only on α, β. Symbols in $S^m(\mathbb{T}^d \times \mathbb{R}^d)$ are called **Euclidean symbols** of order m on $\mathbb{T}^d$.

(b) The class $S^m(\mathbb{T}^d \times \mathbb{Z}^d)$ consists of all functions $\sigma(x, \xi)$ on $\mathbb{T}^d \times \mathbb{Z}^d$ such that for each $\xi \in \mathbb{Z}^d$, $\sigma(., \xi) \in C^\infty(\mathbb{T}^d)$ and for any multi-indices α, β,

$$|\Delta_\xi^\alpha D_x^\beta \sigma(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m - |\alpha|},$$

for some constant $C_{\alpha,\beta}$ that depends only on α, β. Here we recall the definition of the forward difference operator Δ_ξ^α with respect to the variable ξ [67]. Let f be a complex-valued function defined on $\mathbb{Z}^d$ and $1 \leq j \leq d$. Then we define

$$\Delta_j f(\xi) := f(\xi_1, \ldots, \xi_{j-1}, \xi_j + 1, \xi_{j+1}, \ldots, \xi_d) - f(\xi),$$

and for any multi-index α,

$$\Delta_\xi^\alpha := \Delta_1^{\alpha_1} \ldots \Delta_d^{\alpha_d}.$$

Symbols in $S^m(\mathbb{T}^d \times \mathbb{Z}^d)$ are called **toroidal symbols** of order m on $\mathbb{T}^d$.

(c) The intersection of all the classes $S^m(\mathbb{T}^d \times \mathbb{R}^d)$ $(S^m(\mathbb{T}^d \times \mathbb{Z}^d))$ is denoted by $S^{-\infty}(\mathbb{T}^d \times \mathbb{R}^d)$ $(S^{-\infty}(\mathbb{T}^d \times \mathbb{Z}^d))$, which are also called **smoothing symbols**.

Due to [67, Theorem 4.5.3], a symbol is toroidal of order m if and only if it could be extended in ξ to an Euclidean symbol of the same order m. Such an extension is unique

modulo a smoothing symbol. Consequently, we will use the notation $S^m(\mathbb{T}^d)$ for both classes $S^m(\mathbb{T}^d \times \mathbb{R}^d)$ and $S^m(\mathbb{T}^d \times \mathbb{Z}^d)$. The two approaches are essentially equivalent in defining pseudodifferential operators on $\mathbb{T}^d$ whenever the symbol is in the class $S^m(\mathbb{T}^d)$. Following [36], this motivates us to define periodic pseudodifferential operators as follows:

Definition 3.7.2. Given a symbol $\sigma(x, \xi) \in S^m(\mathbb{T}^d)$, we denote by $Op(\sigma)$ the corresponding periodic pseudodifferential operator defined by

$$(Op(\sigma)f)(x) := \sum_{\xi \in \mathbb{Z}^d} \sigma(x, \xi)\tilde{f}(\xi)e^{2\pi i \xi \cdot x}, \tag{3.64}$$

where $\tilde{f}(\xi)$ is the Fourier coefficient of f at ξ. The right hand side of (3.64) converges absolutely if, for instance, $f \in C^\infty(\mathbb{T}^d)$.

We also use the notation $Op(S^m(\mathbb{T}^d))$ for the set of all periodic pseudodifferential operators $Op(\sigma)$ with $\sigma \in S^m(\mathbb{T}^d)$.

Since we must deal with parameters s and k, we introduce a suitable class of symbols depending on parameters $(s, k) \in \mathbb{S}^{d-1} \times \mathcal{O}$.

Definition 3.7.3. The parameter-dependent class $\tilde{S}^m(\mathbb{T}^d)$ consists of symbols $\sigma(s, k; x, \xi)$ satisfying the following conditions:

- For each $(s, k) \in \mathbb{S}^{d-1} \times \mathcal{O}$, the function $\sigma(s, k; \cdot, \cdot)$ is a symbol in the class $S^m(\mathbb{T}^d)$.

- Consider any multi-indices α, β, γ. Then for each $s \in \mathbb{S}^{d-1}$, the function $\sigma(s, \cdot; \cdot, \cdot)$ is smooth on $\mathcal{O} \times \mathbb{T}^d \times \mathbb{R}^d$, and furthermore,

$$\sup_{s \in \mathbb{S}^{d-1}} |D_k^\alpha D_\xi^\beta D_x^\gamma \sigma(s, k; x, \xi)| \leq C_{\alpha\beta\gamma}(1 + |\xi|)^{m - |\alpha| - |\beta|},$$

for some constant $C_{\alpha\beta\gamma} > 0$ that is independent of s, k, x, and ξ.

Thus, taking derivatives of a symbol in $\tilde{S}^m(\mathbb{T}^d)$ with respect to k improves decay in ξ. We also denote

$$\tilde{S}^{-\infty}(\mathbb{T}^d) := \bigcap_{m \in \mathbb{R}} \tilde{S}^m(\mathbb{T}^d).$$

Definition 3.7.4. For each $m \in \mathbb{R} \cup \{-\infty\}$, we denote by $Op(\tilde{S}^m(\mathbb{T}^d))$ the set of all families of periodic pseudodifferential operators $\{Op(\sigma(s, k; \cdot, \cdot))\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$, where σ runs over the class $\tilde{S}^m(\mathbb{T}^d)$.

Example 3.7.5.

- Suppose that $|\lambda|$ is small enough so that $\max_{s \in \mathbb{S}^{d-1}} |\beta_s| < 1$. Then the family of symbols $\{(1 + (\xi + k + i\beta_s)^2)^{m/2}\}_{(s,k)}$ belongs to the class $\tilde{S}^m(\mathbb{T}^d)$ for any $m \in \mathbb{R}$.

- If $a_\alpha(x) \in C^\infty(\mathbb{T}^d)$ and $m \geq 0$, then

$$\{\sum_{|\alpha| \leq m} a_\alpha(x)(\xi + k + i\beta_s)^\alpha\}_{(s,k)} \in \tilde{S}^m(\mathbb{T}^d).$$

- The family of elliptic operators $\{(L_s(k) - \lambda)\}_{(s,k)}$ is in $Op(\tilde{S}^2(\mathbb{T}^d))$.

- If $a = \{a(s, k; x, \xi)\}_{(s,k)} \in \tilde{S}^l(\mathbb{T}^d)$ and $b = \{b(s, k; x, \xi)\}_{(s,k)} \in \tilde{S}^m(\mathbb{T}^d)$ then $ab = \{ab(s, k; x, \xi)\}_{(s,k)} \in \tilde{S}^{l+m}(\mathbb{T}^d)$.

- $a(s, k; x, \xi) \in \tilde{S}^l(\mathbb{T}^d)$ implies $D_k^\alpha D_\xi^\beta D_x^\gamma a(s, k; x, \xi) \in \tilde{S}^{l-|\alpha|-|\beta|}(\mathbb{T}^d)$.

The following result will be needed in the next subsection:

Theorem 3.7.6. *There exists a family of parametrices $\{A_s(k)\}_{(s,k)}$ in the class $Op(\tilde{S}^{-2}(\mathbb{T}^d))$ for the family of elliptic operators $\{(L_s(k) - \lambda)\}_{(s,k)}$.*

The reader can refer to Section 3.8 for the proof of this result as well as some other basic properties of parameter-dependent toroidal ΨDOs.

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3.7.2 Decay of the Schwartz kernel of T_s

Lemma 3.7.7. *For all k on a sufficiently small neighborhood of the support of η, λ (< 0) is in the resolvent of the operator $L_s(k)Q_s(k)$ acting on $L^2(\mathbb{T}^d)$. Furthermore, for such k, we have the following identity:*

$$((L_s(k) - \lambda)|_{R(Q_s(k))})^{-1}Q_s(k) = \lambda^{-1}P_s(k) + (L_s(k)Q_s(k) - \lambda)^{-1}. \tag{3.65}$$

Proof. In the block-matrix form, $(L_s(k)Q_s(k) - \lambda)$ is

$$\left(\begin{array}{c|c} -\lambda P_s(k) & 0 \\ \hline 0 & (L_s(k) - \lambda)|_{R(Q_s(k))} \end{array} \right). \tag{3.66}$$

This gives the first claim of this lemma. The inverse of (3.66) is

$$\left(\begin{array}{c|c} -\lambda^{-1}P_s(k) & 0 \\ \hline 0 & ((L_s(k) - \lambda)|_{R(Q_s(k))})^{-1} \end{array} \right),$$

which proves the identity (3.65). $\qquad\square$

The identity (3.65) implies that for each (s, k), the operator

$$\eta(k)((L_s(k) - \lambda)|_{R(Q_s(k))})^{-1}Q_s(k)$$

is a periodic pseudodifferential operator in $S^{-2}(\mathbb{T}^d)$. Thus, each of the operators $T_s(k)$ is also in $S^{-2}(\mathbb{T}^d)$ and its symbol is smooth in (s, k) since $P_s(k)$ and $Q_s(k)$ are smooth in (s, k). Actually, more information about the family of operators $\{T_s(k)\}_{(s,k)}$ and their Schwartz kernels can be obtained.

At first, we want to introduce a class of family of operators whose kernels behave

nicely.

Definition 3.7.8. We denote by $\mathcal{S}$ the set consisting of families of smoothing operators $\{U_s(k)\}_{(s,k)}$ acting on $\mathbb{T}^d$ so that the following properties hold:

- For any $m_1, m_2 \in \mathbb{R}$, the operator $U_s(k)$ is smooth in k as a $B(H^{m_1}(\mathbb{T}^d), H^{m_2}(\mathbb{T}^d))$-valued function[7].

- The following uniform condition holds for any multi-index α:

$$\sup_{s,k} \|D_k^\alpha U_s(k)\|_{B(H^{m_1}(\mathbb{T}^d), H^{m_2}(\mathbb{T}^d))} < \infty.$$

We remark that if the family of smoothing operators $\{U_s(k)\}_{(s,k)}$ is in $Op(\tilde{S}^{-\infty}(\mathbb{T}^d))$, then this family also belongs to $\mathcal{S}$.

In order to obtain information on Schwartz kernels of a family of operators in $\mathcal{S}$, we need to use the following standard lemma on Schwartz kernels of integral operators acting on $\mathbb{T}^d$.

Lemma 3.7.9. *Let A be a bounded operator in $L^2(\mathbb{T}^d)$. Suppose that the range of A is contained in $H^m(\mathbb{T}^d)$, where $m > d/2$ and in addition,*

$$\|Af\|_{H^m(\mathbb{T}^d)} \leq C\|f\|_{H^{-m}(\mathbb{T}^d)}$$

for all $f \in L^2(\mathbb{T}^d)$.

Then A is an integral operator whose kernel $K_A(x,y)$ is bounded and uniformly continuous on $\mathbb{T}^d \times \mathbb{T}$ and the following estimate holds:

$$|K_A(x,y)| \leq \gamma_0 C, \tag{3.67}$$

---

[7]We remind the reader that $B(E, F)$ denotes the space of all bounded linear operators from the Banach space E to F.

where γ_0 is a constant depending only on d and m.

The fact can be found in [1, Lemma 2.2].

Now we can state a useful property of Schwartz kernels of a family of operators in $\mathcal{S}$.

Corollary 3.7.10. *If $\{U_s(k)\}_{(s,k)}$ is a family of smoothing operators in $\mathcal{S}$, then the Schwartz kernel $K_{U_s}(k, x, y)$ of the operator $U_s(k)$ satisfies*

$$\sup_{s,k,x,y} |D_k^\alpha K_{U_s}(k, x, y)| < \infty,$$

for any multi-index α.

Proof. We pick any $m > d/2$. Then by Definition 3.7.8, we have

$$\sup_{s,k} \|D_k^\alpha U_s(k)f\|_{H^m(\mathbb{T}^d)} \le C_\alpha \|f\|_{H^{-m}(\mathbb{T}^d)}.$$

Applying Lemma 3.7.9, the estimates (3.67) hold for kernels $D_k^\alpha K_{U_s}(k, x, y)$ of the operators $D_k^\alpha U_s(k)$ uniformly in (s, k). $\qquad\square$

We now go back to the family of operators $T_s(k)$.

Proposition 3.7.11. *There is a family of periodic pseudodifferential operators $\{A_s(k)\}_{(s,k)}$ in $Op(\tilde{S}^{-2}(\mathbb{T}^d))$ such that the family of operators $\{T_s(k) - A_s(k)\}_{(s,k)}$ belongs to $\mathcal{S}$.*

Proof. Due to Theorem 3.7.6, there is a family of operators $\{A_s(k)\}_{(s,k)}$ in $Op(\tilde{S}^{-2}(\mathbb{T}^d))$ and a family of operators $\{R_s(k)\}_{(s,k)}$ in $Op(\tilde{S}^{-\infty}(\mathbb{T}^d))$ such that

$$(L_s(k) - \lambda)A_s(k) = I - R_s(k).$$

Since $T_s(k)(L_s(k) - \lambda) = I - \eta(k)P_s(k)$, we deduce that

$$T_s(k) = A_s(k) - \eta(k)P_s(k)A_s(k) + T_s(k)R_s(k). \tag{3.68}$$

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Now it remains to show that the two families of smoothing operators $\{T_s(k)R_s(k)\}_{(s,k)}$ and $\{\eta(k)P_s(k)A_s(k)\}_{(s,k)}$ are in $\mathcal{S}$. Let us fix any two real numbers m_1, m_2 and a multi-index α. Notice that $(L_s(k) - \lambda)$ is analytic in k as a $B(H^{m_2}(\mathbb{T}^d), H^{m_2-2}(\mathbb{T}^d))$-valued function and also,

$$\sup_{s,k} \|D_k^\alpha(L_s(k) - \lambda)\|_{B(H^{m_2}(\mathbb{T}^d), H^{m_2-2}(\mathbb{T}^d))} < \infty.$$

Due to Lemma 3.7.7,

$$T_s(k) = (1 - \eta(k))(L_s(k) - \lambda)^{-1} + \eta(k)\lambda^{-1}P_s(k) + \eta(k)(L_s(k)Q_s(k) - \lambda)^{-1}.$$

Thus, $T_s(k)$ is smooth in k as a $B(H^{m_2-2}(\mathbb{T}^d), H^{m_2}(\mathbb{T}^d))$-valued function and moreover,

$$\sup_{s,k} \|D_k^\alpha T_s(k)\|_{B(H^{m_2-2}(\mathbb{T}^d), H^{m_2}(\mathbb{T}^d))} < \infty. \tag{3.69}$$

Since $\{R_s(k)\}$ is in $Op(\tilde{S}^{-\infty}(\mathbb{T}^d))$, $R_s(k)$ is smooth in k as a $B(H^{m_1}(\mathbb{T}^d), H^{m_2-2}(\mathbb{T}^d))$-valued function and furthermore,

$$\sup_{s,k} \|D_k^\alpha R_s(k)\|_{B(H^{m_1}(\mathbb{T}^d), H^{m_2-2}(\mathbb{T}^d))} < \infty. \tag{3.70}$$

By (3.69), (3.70) and Leibnitz's rule, we deduce that $T_s(k)R_s(k)$ is smooth in k as a $B(H^{m_1}(\mathbb{T}^d), H^{m_2}(\mathbb{T}^d))$-valued function and the corresponding uniform estimate also holds. Hence, we conclude that the family $\{T_s(k)R_s(k)\}_{(s,k)}$ belongs to $\mathcal{S}$. Meanwhile, since $\{\eta(k)P_s(k)\}_{(s,k)}$ is in $\mathcal{S}$ and $\{D_k^\alpha A_s(k)\}_{(s,k)}$ is a toroidal pseudodifferential operator of order $2 - |\alpha|$ for any multi-index α, we could repeat the above argument to show that the family $\{\eta(k)P_s(k)A_s(k)\}_{(s,k)}$ is also in $\mathcal{S}$. $\qquad\square$

We need the following important estimate of Schwartz kernels of operators $T_s(k)$:

Corollary 3.7.12. *Let $K_s(k, x, y)$ be the Schwartz kernel of the operator $T_s(k)$. Let $N > d - 2$. If α is a multi-index such that $|\alpha| = N$, then each $D_k^\alpha K_s(k, x, y)$ is a continuous function on $\mathbb{T}^d \times \mathbb{T}^d$ and the following estimate also holds uniformly with respect to (x, y):*

$$\sup_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}} |D_k^\alpha K_s(k, x, y)| < \infty.$$

Proof. Due to Proposition 3.7.11, the operator $T_s(k)$ is a sum of operators $A_s(k)$ and $U_s(k)$ such that $\{A_s(k)\}_{(s,k)} \in Op(\tilde{S}^{-2}(\mathbb{T}^d))$ and $\{U_s(k)\}_{(s,k)} \in \mathcal{S}$. In particular,

$$K_s(k, x, y) = K_{A_s}(k, x, y) + K_{U_s}(k, x, y).$$

Recall that in the distributional sense, the Schwartz kernel $K_{A_s}(k, x, y)$ of the periodic pseudodifferential operator $A_s(k)$ is given by

$$\sum_{\xi \in \mathbb{Z}^d} \sigma(s, k; x, \xi) e^{2\pi i \xi \cdot (x-y)},$$

where $\sigma(s, k; x, \xi)$ is the symbol of the operator $A_s(k)$.

Since $\{\sigma(s, k; x, \xi)\}_{(s,k)}$ is in $\tilde{S}^{-2}(\mathbb{T}^d)$,

$$|e^{2\pi i \xi \cdot (x-y)} D_k^\alpha \sigma(s, k; x, \xi)| \lesssim (1 + |\xi|)^{-2-N}.$$

Since $-(2 + N) < -d$, the sum

$$\sum_{\xi \in \mathbb{Z}^d} D_k^\alpha \sigma(s, k; x, \xi) e^{2\pi i \xi \cdot (x-y)}$$

converges absolutely and moreover,

$$\sup_{(s,k,x,y)\in\mathbb{S}^{d-1}\times\mathcal{O}\times\mathbb{T}^d\times\mathbb{T}^d} |D_k^\alpha K_{A_s}(k,x,y)| \lesssim \sum_{\xi\in\mathbb{Z}^d} (1+|\xi|)^{-(d+1)} < \infty.$$

Combining this with Corollary 3.7.10, we complete the proof. □

Notation 3.7.13. Let ψ be a function on $\mathbb{R}^d$ and γ be a vector in $\mathbb{R}^d$, then $\tau_\gamma\psi$ is the γ-shifted version of ψ. Namely, it is defined as follows:

$$\tau_\gamma\psi(\cdot) = \psi(\cdot + \gamma).$$

We denote by $\mathcal{P}$ the subset of $C_0^\infty(\mathbb{R}^d)$ consisting of all functions ψ such that its support is connected, and if γ is a **non-zero vector in** $\mathbb{Z}^d$, then the support of $\tau_\gamma\psi$ does not intersect with the support of ψ.

Definition 3.7.14. Since $\mathbb{R}^d$ is the universal covering space of $\mathbb{T}^d$, we can consider the covering map

$$\pi : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d.$$

In particular, $\pi(x+\gamma) = \pi(x)$ for any $x \in \mathbb{R}^d$ and $\gamma \in \mathbb{Z}^d$.

A **standard fundamental domain** (with respect to the covering map π) is of the form $[0,1]^d + \gamma$ for some vector γ in $\mathbb{R}^d$. Thus, a standard fundamental domain is a fundamental domain of $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$.

Using Definition 2.2.3 of the Floquet transform $\mathcal{F}$, we can obtain the following formula:

Lemma 3.7.15. *Let ϕ and θ be any two smooth functions in $\mathcal{P}$. Then the Schwartz kernel*

$K_{s,\phi,\theta}$ *of the operator* $\phi T_s \theta$ *satisfies the following identity for any* $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$K_{s,\phi,\theta}(x, y) = \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik \cdot (x-y)} \phi(x) K_s(k, \pi(x), \pi(y)) \theta(y) \, dk.$$

Proof. Since both $\phi, \theta \in \mathcal{P}$, there are standard fundamental domains W_ϕ and $W_\theta \subset \mathbb{R}^d$ so that

$$\mathrm{supp}(\phi) \subset \mathring{W}_\phi, \quad \mathrm{supp}(\theta) \subset \mathring{W}_\theta.$$

Then, it suffices to show that $\langle \phi T_s \theta f, g \rangle$ equals

$$\frac{1}{(2\pi)^d} \int_{W_\phi} \int_{W_\theta} \int_{\mathcal{O}} e^{ik \cdot (x-y)} (\phi \overline{g})(x) K_s(k, \pi(x), \pi(y))(\theta f)(y) \, dk \, dy \, dx,$$

for any f, g in $C^\infty(\mathbb{R}^d)$.

We observe that

$$\begin{aligned}
\langle \phi T_s \theta f, g \rangle &= \langle \mathcal{F} \phi T_s \theta f, \mathcal{F} g \rangle \\
&= \frac{1}{(2\pi)^d} \left\langle (\mathcal{F} \phi \mathcal{F}^{-1}) \left(\int_{\mathcal{O}}^{\oplus} T_s(k) \, dk \right) \mathcal{F}(\theta f), \mathcal{F} g \right\rangle \\
&= \frac{1}{(2\pi)^d} \left\langle \left(\int_{\mathcal{O}}^{\oplus} T_s(k) \, dk \right) \mathcal{F}(\theta f), \mathcal{F}(\overline{\phi} g) \right\rangle.
\end{aligned}$$

Since $\theta \in \mathcal{P}$, for any y in W_θ, we have

$$\mathcal{F}(\theta f)(k, \pi(y)) = (\theta f)(y) e^{-ik \cdot y}.$$

Similarly,

$$\mathcal{F}(\overline{\phi} g)(k, \pi(x)) = (\overline{\phi} g)(x) e^{-ik \cdot x}, \quad \forall x \in W_\phi.$$

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We also have

$$\left(\int_{\mathcal{O}}^{\oplus} T_s(k)\, \mathrm{d}k \right) (\mathcal{F}(\theta f))(k, \pi(x)) = T_s(k)(\mathcal{F}(\theta f)(k, \cdot))(\pi(x)).$$

Consequently,

$$
\left\langle \left(\int_{\mathcal{O}}^{\oplus} T_s(k)\, \mathrm{d}k \right) \mathcal{F}(\theta f), \mathcal{F}(\overline{\phi} g) \right\rangle
$$
$$
= \int_{\mathcal{O}} \int_{W_\phi} T_s(k)(\mathcal{F}(\theta f)(k, \cdot))(\pi(x)) \overline{(\overline{\phi} g)(x) e^{-ik \cdot x}}\, \mathrm{d}x\, \mathrm{d}k
$$
$$
= \int_{\mathcal{O}} \int_{W_\phi} \int_{W_\theta} K_s(k, \pi(x), \pi(y)) \mathcal{F}(\theta f)(k, \pi(y))(\phi \overline{g})(x) e^{ik \cdot x}\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}k
$$
$$
= \int_{\mathcal{O}} \int_{W_\phi} \int_{W_\theta} e^{ik \cdot (x-y)} K_s(k, \pi(x), \pi(y))(\theta f)(y)(\phi \overline{g})(x)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}k.
$$

Using Fubini's theorem to rewrite the above integral, we have the desired identity. $\qquad \square$

Proposition 3.7.16. *Consider any two smooth compactly supported functions ϕ and θ on $\mathbb{R}^d$ such that their supports are disjoint. Then the kernel $K_{s,\phi,\theta}(x, y)$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and moreover, it satisfies the following decay:*

$$\sup_s |K_{s,\phi,\theta}(x, y)| \le C_N |\phi(x)\theta(y)| \cdot |x - y|^{-N},$$

for any $N > d - 2$. Here, the constant C_N is independent of ϕ and θ.

Proof. By using partitions of unity, any smooth compactly supported function can be written as a finite sum of smooth functions in the set $\mathcal{P}$. Thus, we can assume without loss of generality that both ϕ and θ belong to $\mathcal{P}$.

First, observe that for any $(k, n) \in \mathcal{O} \times \mathbb{Z}^d$,

$$T_s(k + 2\pi n) = \mathcal{M}_n^{-1} T_s(k) \mathcal{M}_n,$$

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where $\mathcal{M}_n$ is the multiplication operator on $L^2(\mathbb{T}^d)$ by the exponential function $e^{2\pi i n \cdot x}$.

Hence,

$$\nabla_k^\alpha K_s(k + 2\pi n, \pi(x), \pi(y)) = e^{-2\pi i n \cdot \pi(x)} \nabla_k^\alpha K_s(k, \pi(x), \pi(y)) e^{2\pi i n \cdot \pi(y)},$$

for any multi-index α. Since $e^{2\pi i n \cdot x} = e^{2\pi i n \cdot \pi(x)}$ for any $x \in \mathbb{R}^d$, we obtain

$$e^{i(k+2\pi n)\cdot(x-y)}\nabla_k^\alpha K_s(k + 2\pi n, \pi(x), \pi(y)) = e^{ik\cdot(x-y)}\nabla_k^\alpha K_s(k, \pi(x), \pi(y)). \qquad (3.71)$$

Applying Lemma 3.7.15, we then use integration by parts (all boundary terms vanish when applying integration by parts due to (3.71)) to derive that for any $|\alpha| = N$,

$$(2\pi)^d (i(x - y))^\alpha K_{s,\phi,\theta}(x, y) = \phi(x)\theta(y) \int_{\mathcal{O}} e^{ik\cdot(x-y)} \nabla_k^\alpha K_s(k, \pi(x), \pi(y))\, \mathrm{d}k.$$

Suppose $N > d - 2$. Then by applying Corollary 3.7.12, the above integral is absolutely convergent and it is also uniformly bounded in (s, x, y). Consequently, the kernel $K_{s,\phi,\theta}(x, y)$ is continuous. Furthermore,

$$\sup_s |K_{s,\phi,\theta}(x, y)| \lesssim |\phi(x)\theta(y)| \cdot \min_{|\alpha|=N} |(x - y)^\alpha|^{-1} \lesssim |\phi(x)\theta(y)| \cdot |x - y|^{-N}.$$

$\square$

We now have enough tools to approach our goal:

Proof of Theorem 3.5.2.

Proof. Let us fix a point (s, x) in $\mathbb{S}^{d-1} \times \mathbb{R}^d$. Now we consider a point $y = x + st$, where t is a real number. When $|t| > 0$, we can choose two cut-off functions ϕ and θ such that ϕ and θ equal 1 on some neighborhoods of x and y, respectively, and also, the supports of

these two functions are disjoint. Then, Proposition 3.7.16 implies that the kernel $K_s(x, y)$ is continuous at (x, y) since it coincides with $K_{s,\phi,\theta}$ on a neighborhood of (x, y). This yields the first statement about the continuity off diagonal of K_s. Again, by Proposition 3.7.16, we obtain

$$\sup_s |K_s(x, y)| = \sup_s |K_{s,\phi,\theta}(x, y)| \leq C_N |x - y|^{-N},$$

which proves the last statement. $\qquad\square$

3.8   Some results on parameter-dependent toroidal ΨDOs

The aim in this section is to provide some results needed to complete the proof of Theorem 3.7.6. We adopt the approach of [36] to periodic elliptic differential operators.

The next two theorems are straightforward modifications of the proofs for non-parameter toroidal ΨDOs:

Theorem 3.8.1. *(The asymptotic summation theorem) Given families of symbols $b_l \in \tilde{S}^{m-l}(\mathbb{T}^d)$, where each family $b_l = \{b_l(s, k)\}_{(s,k)}$ for $l = 0, 1, \ldots$, there exists a family of symbols b in $\tilde{S}^m(\mathbb{T}^d)$ such that*

$$\{b(s, k) - \sum_{i<l} b_i(s, k)\}_{(s,k)} \in \tilde{S}^{m-l}(\mathbb{T}^d). \tag{3.72}$$

We will write $b \sim \sum_l b_l$ if b satisfies (3.72).

Proof. Step 1. Let $n = m + \epsilon$ for some $\epsilon > 0$. Then

$$|b_l(s, k; x, \xi)| \leq C_l(1 + |\xi|)^{m-l} = \frac{C_l(1 + |\xi|)^{n-l}}{(1 + |\xi|)^\epsilon}.$$

Thus, there is a sequence $\{\eta_l\}_{l \geq 1}$ such that $\eta_l \to +\infty$ and

$$|b_l(s, k; x, \xi)| < \frac{1}{2^l}(1 + |\xi|)^{n-l}$$

for $|\xi| > \eta_l$. Let $\rho \in C^\infty(\mathbb{R})$ satisfy that $0 \leq \rho \leq 1$, $\rho(t) = 0$ whenever $|t| < 1$ and $\rho(t) = 1$ whenever $|t| > 2$. We define:

$$b(s, k; x, \xi) = \sum_l \rho\left(\frac{|\xi|}{\eta_l}\right) b_l(s, k; x, \xi).$$

Since only a finite number of summands are non-zero on any compact subset of $\mathbb{T}^d \times \mathbb{R}^d$, $b(s, \cdot; \cdot, \cdot) \in C^\infty(\mathcal{O} \times \mathbb{T}^d \times \mathbb{R}^d)$. Moreover, $b(s, k) - \sum_{r<l} b_r(s, k)$ is equal to:

$$\sum_{r<l} \left(\rho\left(\frac{|\xi|}{\eta_r}\right) - 1\right) b_r(s, k) + \rho\left(\frac{|\xi|}{\eta_l}\right) b_l(s, k) + \sum_{r>l} \rho\left(\frac{|\xi|}{\eta_r}\right) b_r(s, k).$$

The first summand is compactly supported while the second summand is in $S^{m-l}(\mathbb{T}^d)$. Now let $\epsilon < 1$. Then, the third summand is bounded from above by

$$\sum_{r>l} \frac{1}{2^r}(1 + |\xi|)^{n-r} \leq (1 + |\xi|)^{n-l-1} \leq (1 + |\xi|)^{m-l}.$$

Consequently,
$$\sup_{s \in \mathbb{S}^{d-1}} \left| b(s, k) - \sum_{r<l} b_r(s, k) \right| \leq C(1 + |\xi|)^{m-l}.$$

Step 2. For $|\alpha| + |\beta| + |\gamma| \leq N$, one can choose η_l such that

$$\sup_{s \in \mathbb{S}^{d-1}} \left| D_k^\alpha D_\xi^\beta D_x^\gamma b_l(s, k; x, \xi) \right| \leq \frac{1}{2^l}(1 + |\xi|)^{n-l-|\alpha|-|\beta|}$$

for $\eta_l < |\xi|$. The same argument as in Step 1 implies that

$$\sup_{s \in \mathbb{S}^{d-1}} \left| D_k^\alpha D_\xi^\beta D_x^\gamma (b(s,k) - \sum_{r<l} b_r(s,k)) \right| \leq C_N (1 + |\xi|)^{m-l-|\alpha|-|\beta|}. \qquad (3.73)$$

Step 3. The sequence of η_l's in Step 2 depends on N. We denote this sequence by $\eta_{l,N}$ to indicate this dependence on N. By induction, we can assume that for all l, $\eta_{l,N} \leq \eta_{l,N+1}$. Applying the Cantor diagonal process to this family of sequences, i.e., let $\eta_l = \eta_{l,l}$ then b has the property (3.73) for every N. $\qquad \square$

Theorem 3.8.2. *(The composition formula) Let $a = \{a(s,k)\}$ be a family of symbols in $\tilde{S}^l(\mathbb{T}^d)$ and $Q(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a differential operators of order $m \geq 0$ with smooth periodic coefficients $a_\alpha(x)$. Then the family of periodic pseudodifferential operators $\{Q(x, D + k + i\beta_s) Op(a(s,k))\}_{(s,k)} \in Op(\tilde{S}^{l+m}(\mathbb{T}^d))$. Indeed, we have:*

$$Q(x, D + k + i\beta_s) Op(a(s,k)) = Op((Q \circ a)(s,k)),$$

where

$$(Q \circ a)(s,k;x,\xi) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D_\xi^\alpha Q(x, \xi + k + i\beta_s) D_x^\alpha a(s,k;x,\xi) \qquad (3.74)$$

Proof. The composition formula (3.74) is obtained for each (s,k) is standard in pseudodifferential operator theory (see e.g., [36, 67, 72]). We only need to check that the family of symbols $\{(Q \circ a)(s,k;x,\xi)\}_{(s,k)}$ is in $\tilde{S}^{l+m}(\mathbb{T}^d)$. But this fact follows easily from (3.74) and Leibnitz's formula. $\qquad \square$

We now finish the proof of Theorem 3.7.6.

Theorem 3.8.3. *(The inversion formula) There exists a family of symbols $a = \{a(s,k)\}_{(s,k)}$*

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in $\tilde{S}^{-2}(\mathbb{T}^d)$ *and a family of symbols* $r = \{r(s,k)\}_{(s,k)}$ *in* $\tilde{S}^{-\infty}(\mathbb{T}^d)$ *such that*

$$(L_s(k) - \lambda)Op(a(s,k)) = I - Op(r(s,k)).$$

Proof. Let

$$L_0(s,k;x,\xi) := \sum_{|\alpha|=2} a_\alpha(x)(\xi + k + i\beta_s)^\alpha,$$

$$\|a\|_\infty := \sum_{|\alpha|=2} \|a_\alpha(\cdot)\|_{L^\infty(\mathbb{T}^d)},$$

and

$$M := \max_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}} \left(|k|^2 + \theta^{-1}\|a\|_\infty|\beta_s|^2 + \theta^{-1} \right),$$

where θ is the ellipticity constant in (3.2). Whenever $|\xi| > (2M)^{1/2}$,

$$|L_0(s,k;x,\xi)| \geq \Re(L_0(s,k;x,\xi)) \geq \theta|\xi + k|^2 - \sum_{|\alpha|=2} a_\alpha(x)(\beta_s)^\alpha$$

$$\geq \theta\left(\frac{|\xi|^2}{2} - |k|^2 \right) - \|a\|_\infty|\beta_s|^2 > 1.$$

Let $\rho \in C^\infty(\mathbb{R})$ be a function satisfying $\rho(t) = 0$ when $|t| < (2M)^{1/2}$ and $\rho(t) = 1$ when $|t| > 2M^{1/2}$. We define the function

$$a_0(s,k)(x,\xi) = \rho(|\xi|)\frac{1}{L_0(s,k;x,\xi)}. \tag{3.75}$$

Then $a_0 := \{a_0(s,k)\}_{(s,k)}$ is well-defined and belongs to $\tilde{S}^{-2}(\mathbb{T}^d)$. The next lemma is the final piece we need to complete the proof of the theorem.

Lemma 3.8.4. *(i) If* $b = \{b(s,k)\}_{(s,k)} \in \tilde{S}^l(\mathbb{T}^d)$ *then* $b - (L - \lambda) \circ (a_0 b) \in \tilde{S}^{l-1}(\mathbb{T}^d)$.

(ii) There exists a sequence of families of symbols $a_l = \{a_l(s,k)\}_{(s,k)}$ *in* $\tilde{S}^{-2-l}(\mathbb{T}^d)$, $l = 0, 1, \ldots$ *and a sequence of families of symbols* $r_l = \{r_l(s,k)\}_{(s,k)}$ *in* $\tilde{S}^{-l}(\mathbb{T}^d)$, $l = 0, 1, \ldots$

such that a_0 is the family of symbols in (3.75), $r_0(s, k) = 1$ for every (s, k) and for all l,

$$(L - \lambda) \circ a_l = r_l - r_{l+1}.$$

Proof. (i) Let $p(s, k) = (L(s, k) - \lambda)(x, \xi) - L_0(s, k; x, \xi)$ so that $p = \{p(s, k)\}_{(s,k)} \in \tilde{S}^1(\mathbb{T}^d)$ and hence, $p \circ (a_0 b)$ is in $\tilde{S}^{l-1}(\mathbb{T}^d)$ due to Theorem 3.8.2. Moreover, $b - L_0 a_0 b = (1 - \rho(|\xi|))b$ is a family of symbols whose ξ-supports are compact and thus it is in $\tilde{S}^{-\infty}(\mathbb{T}^d)$. We can now derive again from the composition formula (3.74) when $P := L_0$ that

$$(L - \lambda) \circ (a_0 b) = L_0 \circ (a_0 b) + p \circ (a_0 b) = L_0 a_0 b + \cdots = b + \dots,$$

where the dots are the terms in $\tilde{S}^{l-1}(\mathbb{T}^d)$.

(ii) Recursively, let $a_l = a_0 r_l$ and $r_{l+1} = r_l - (L - \lambda) \circ a_l$. By part (i), $r_{l+1} \in \tilde{S}^{-(l+1)}(\mathbb{T}^d)$. □

Now let a be the asymptotic sum of the families of symbols a_l, i.e., $a \sim \sum_l a_l$. Then

$$(L - \lambda) \circ a \sim \sum_l (L - \lambda) \circ a_l = \sum_l r_l - r_{l+1} = r_0 = 1,$$

which implies that $1 - (L - \lambda) \circ a \sim 0$. In other words, this means that $r := 1 - (L - \lambda) \circ a \in \tilde{S}^{-\infty}(\mathbb{T}^d)$. Hence, there exists a family of symbols a in $\tilde{S}^{-2}(\mathbb{T}^d)$ and a family of symbols r in $\tilde{S}^{-\infty}(\mathbb{T}^d)$ satisfying $(L - \lambda) \circ a = 1 - r$. Finally, an application of Theorem 3.8.2 completes the proof of Theorem 3.7.6. □

3.9 Some auxiliary statements

3.9.1 A lemma on the principle of non-stationary phase

Lemma 3.9.1. *Let M be a compact manifold (with or without boundary) and $a : \mathbb{R} \times M \to \mathbb{C}$ be a smooth function with compact support. Then for any $N > 0$, there exists a constant*

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$C_N > 0$ *so that the following estimate holds for any non-zero* $t \in \mathbb{R}$:

$$\sup_{x \in M} \left| \int_{-\infty}^{\infty} e^{ity} a(y, x) \, \mathrm{d}y \right| \leq C_N |t|^{-N}. \tag{3.76}$$

Here C_N *depends only on* N, *the diameter* R *of the* y-*support of* a *and* $\sup_{x,y} |\partial_y^N a|$.

Proof. Let $t \neq 0$. Applying integration by parts repeatedly (N-times), it follows that

$$\left| \int_{-\infty}^{\infty} e^{ity} a(y, x) \, \mathrm{d}y \right| = |t|^{-N} \left| \int_{-\infty}^{\infty} e^{ity} \partial_y^N a(y, x) \, \mathrm{d}y \right| \leq R \sup_{x,y} |\partial_y^N a| \cdot |t|^{-N}.$$

$\square$

3.9.2 The Weierstrass preparation theorem

Theorem 3.9.2. *Let* $f(t, z)$ *be an analytic function of* $(t, z) \in \mathbb{C}^{1+n}$ *in a neighborhood of* $(0, 0)$ *such that* $(0, 0)$ *is a simple zero of* f, *i.e.:*

$$f(0, 0) = 0, \quad \frac{\partial f}{\partial t}(0, 0) \neq 0.$$

Then there is a unique factorization

$$f(t, z) = (t - A(z))B(t, z),$$

where A, B *are analytic in a neighborhood of* 0 *and* $(0, 0)$ *respectively. Moreover,* $B(0, 0) \neq 0$ *and* $A(0) = 0$.

The proof of a more general version of this theorem could be found in [38, Theorem 7.5.1].

3.9.3   Proofs of Proposition 3.4.1 and Lemma 3.5.1

Remark 3.9.3. Consider a domain $\mathcal{D}$ of $\mathbb{C}^d$ and let $f : \mathcal{D} \to \mathbb{C}$ be a holomorphic function. For $z \in \mathbb{C}^d$, write $z = x + iy$ where $x, y \in \mathbb{R}^d$. Now we fix a vector β in $\mathbb{R}^d$ and denote $\mathcal{D}_\beta = (\mathcal{D} - i\beta) \cap \mathbb{R}^d$. If this intersection is non-empty, we may consider the restriction $k \to f(k + i\beta)$ as a real analytic function defined on a subdomain $\mathcal{D}_\beta$ of $\mathbb{R}^d$. Thanks to Cauchy-Riemann equations of f, we do not need to make any distinction between derivatives of f with respect to x (when f is viewed as a real analytic one) or z (when f is considered as a complex analytic one) at every point in $\mathcal{D}_\beta$ since

$$\frac{\partial f}{\partial x_l}(k + i\beta) = \frac{\partial f}{\partial z_l}(k + i\beta) = -i\frac{\partial f}{\partial y_l}(k + i\beta), \quad 1 \le l \le d.$$

For higher order derivatives, we use induction and the above identity to obtain

$$\partial_x^\alpha f(k + i\beta) = \partial_z^\alpha f(k + i\beta) = (-i)^{|\alpha|}\partial_y^\alpha f(k + i\beta),$$

for any multi-index α. We use these facts implicitly for the function λ_j. When dealing with the analytic function $f = \lambda_j$ in this part, denote $\partial^\alpha \lambda_j$ to indicate either its x or z-derivatives.

We also want to mention this simple relation between derivatives of λ_j and E:

$$\partial^\alpha E(\beta) = \partial_y^\alpha \lambda_j(k_0 + i\beta) = i^{|\alpha|}\partial^\alpha \lambda_j(k_0 + i\beta).$$

Proof of Proposition 3.4.1.

Proof. We recall from Section 2 that V is an open neighborhood of k_0 in $\mathbb{C}^d$ such that the properties **(P1)-(P6)** are satisfied. Note that V depends only on the local structure at k_0 of the dispersion branch λ_j of L. Denote $\mathcal{O}_s = \{k + it\beta_s : k \in \mathcal{O}, t \in [0, 1]\}$ for each

$s \in \mathbb{S}^{d-1}$. For $C > 0$ (which is defined later), set $M_{s,C} = \mathcal{O}_s \cap \{z \in \mathbb{C}^d : |\Re(z) - k_0| < C\}$ and $N_{s,C} = \mathcal{O}_s \setminus M_{s,C}$ (e.g., see Figure 3.1). For C and $|\lambda|$ small enough, we can suppose $M_{s,C} \Subset V$ since β_s is small too. We also assume that $|\lambda| \leq \epsilon_0$.[8]
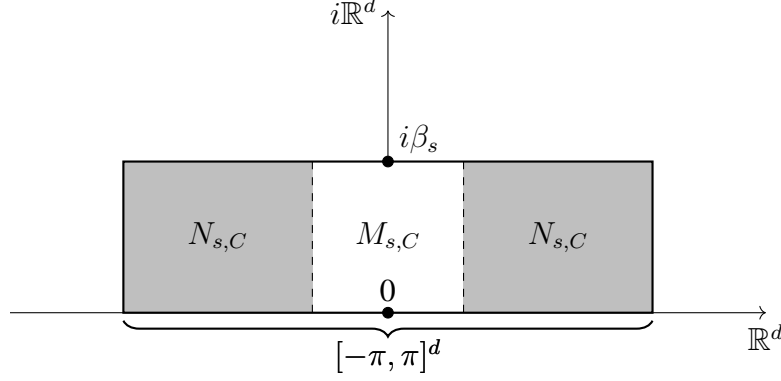
Figure 3.1: An illustration of the regions $M_{s,C}$ and $N_{s,C}$ when $k_0 \equiv 0$.

For any point $z = k + it\beta_s \in M_{s,C}$, we want to show if $\lambda \in \sigma(L(z))$, it forces $z = k_0 + i\beta_s$. By **(P3)**, this is the same as showing the equation $\lambda_j(z) = \lambda$ has no solution z in $M_{s,C}$ except for the trivial solution $z = k_0 + i\beta_s$. Suppose for contradiction $\lambda_j(k + it\beta_s) = \lambda = \lambda(\beta_s)$ for some $t \in [0, 1]$ and k in $\{k \in \mathcal{O} \mid 0 < |k - k_0| < C\}$. By Taylor expanding around $k_0 + it\beta_s$, there is some $\gamma \in (0, 1)$ such that

$$\lambda - \lambda_j(k_0 + it\beta_s) = \left((k - k_0) \cdot \nabla\lambda_j(k_0 + it\beta_s) + \sum_{|\alpha|=3} \frac{(k - k_0)^\alpha}{\alpha!} \partial^\alpha \lambda_j(k_0 + it\beta_s) \right) +$$
$$\frac{1}{2}(k - k_0) \cdot \mathrm{Hess}\,(\lambda_j)(k_0 + it\beta_s)(k - k_0) + \sum_{|\alpha|=4} \frac{(k - k_0)^\alpha}{\alpha!} \partial^\alpha \lambda_j(\gamma(k - k_0) + k_0 + it\beta_s).$$

$$(3.77)$$

---

[8]Recall the definition of ϵ_0 from **(P3)**.

If $|\alpha|$ is odd, then by Remark 3.9.3 and the fact that E is real, we have

$$\partial^\alpha \lambda_j(k_0 + it\beta_s) = \frac{1}{i^{|\alpha|}}\partial^\alpha E(t\beta_s) \in i\mathbb{R}.$$

Taking the real part of the equation (3.77) to get

$$E(\beta_s) - E(t\beta_s) = -\frac{1}{2}(k - k_0) \cdot \text{Hess}(E)(t\beta_s)(k - k_0) + \sum_{|\alpha|=4} \frac{(k - k_0)^\alpha}{\alpha!} \times$$

$$\times \Re(\partial^\alpha \lambda_j(\gamma(k - k_0) + k_0 + it\beta_s)).$$

The left-hand side is bounded above by $(1 - t)\lambda \leq 0$ because of the concavity of E ($E(t\beta_s) \geq tE(\beta_s) = t\lambda$). On the other hand, by **(P5)**,

$$-\frac{1}{2}(k - k_0) \cdot \text{Hess}(E)(t\beta_s)(k - k_0) \geq \frac{1}{4}|k - k_0|^2 \min \sigma(\text{Hess}(\lambda_j)(k_0))$$

$$\left| \sum_{|\alpha|=4} \frac{(k - k_0)^\alpha}{\alpha!}\Re(\partial^\alpha \lambda_j(\gamma(k - k_0) + k_0 + it\beta_s)) \right| \leq C(d)|k - k_0|^4 \max_{z \in \overline{V}, |\alpha|=4} |\partial^\alpha \lambda_j(z)|.$$

We simply choose $C^2 < \dfrac{\min \sigma(\text{Hess}(\lambda_j)(k_0))}{C(d) \max_{z \in \overline{V}, |\alpha|=4} |\partial^\alpha \lambda_j(z)|}$ to get a contradiction if $k \neq k_0$.

For the remaining part, we just need to treat points $k + it\beta_s$ in $N_{s,C}$. We have $\lambda \in \rho(L(k)), \forall k \in \mathbb{R}^d$. The idea is to adapt the upper-semicontinuity of the spectrum of an analytic family of type $\mathcal{A}$ on $\mathbb{C}^d$, following [39]. For any $k \in \mathcal{O}$ and $z \in \mathbb{C}^d$, the composed operators $(L(k + z) - L(k))(L(k) - \lambda)^{-1}$ are closed and defined on $L^2(\mathbb{T}^d)$ and by closed graph theorem, these are bounded operators. Clearly,

$$L(k + z) - \lambda = (1 + (L(k + z) - L(k))(L(k) - \lambda)^{-1})(L(k) - \lambda).$$

Thus, λ is in the resolvent of $L(k+z)$ if the operator $1 + (L(k+z) - L(k))(L(k) - \lambda)^{-1}$ is invertible. Hence, it is enough to show that there is some positive constant τ such that for any $k \in \mathcal{O}$ and $|z| < \tau$,

$$\|(L(k+z) - L(k))(L(k) - \lambda)^{-1}\|_{op} < 1/2, \quad |k - k_0| \geq C, \tag{3.78}$$

where the operator norm on $L^2(\mathbb{T}^d)$ is denoted by $\| \cdot \|_{op}$. Indeed, if $|\lambda|$ is small enough so that we have $\max_{s \in \mathbb{S}^{d-1}} |\beta_s| < \tau$ and then (3.78) implies that $\lambda \in \rho(L(k + it\beta_s))$ for any $t \in [0, 1]$.

Finally, we will use some energy estimates of linear elliptic equations and spectral theory to obtain (3.78). Observe that,

$$L(k+z) - L(k) = z \cdot A(x)(D + k) + (D + k) \cdot A(x)z + z \cdot A(x)z.$$

For $v \in H^1(\mathbb{T}^d)$ and $|z| < 1$, there is some constant $C_1 > 0$ (independent of z) such that

$$\|(z \cdot A(x)(D + k) + (D + k) \cdot A(x)z + z \cdot A(x)z)v\|_{L^2(\mathbb{T}^d)} \leq C_1 |z| \cdot \|v\|_{H^1(\mathbb{T}^d)}. \tag{3.79}$$

Set $v := (L(k) - \lambda)^{-1}u$ for $u \in L^2(\mathbb{T}^d)$. Ellipticity of $L(k)$ yields $v \in H^2(\mathbb{T}^d)$ and in particular, we obtain (3.79) for such v. Testing the equation $(L(k) - \lambda)v = u$ with the function v, we derive the standard energy estimate

$$\|Dv\|_{L^2(\mathbb{T}^d)} \leq C_2(\|v\|_{L^2(\mathbb{T}^d)} + \|u\|_{L^2(\mathbb{T}^d)}). \tag{3.80}$$

Note that both C_1 and C_2 in (3.79) and (3.80) are independent of k and λ since we take k in the bounded set $\mathcal{O}$ and consider $|\lambda|$ to be small enough.

Suppose that $|\lambda|$ is less than one-half of the length of the gap between the dispersion

branches λ_j and λ_{j-1}. Due to functional calculus of the self-adjoint operator $L(k)$, we get

$$\|(L(k) - \lambda)^{-1}\|_{op} = \text{dist}(\lambda, \sigma(L(k)))^{-1} = \min\{(\lambda_j(k) - \lambda), (\lambda - \lambda_{j-1}(k))\}^{-1}.$$

Now let $\delta_1 = -\frac{1}{2} \max \lambda_{j-1}(k) > 0$ and $\delta_2 = \min_{k \in \mathcal{O}, |k-k_0| \geq C} \lambda_j(k)$. Then due to **A3**, $\delta_2 > 0$. Moreover,

$$\lambda - \lambda_{j-1}(k) \geq \lambda - \max_{k \in \mathcal{O}} \lambda_{j-1}(k) > \delta_1,$$

and

$$\lambda_j(k) - \lambda \geq \min_{k \in \mathcal{O}, |k-k_0| \geq C} \lambda_j(k) - \lambda > \delta_2.$$

Hence,

$$\|(L(k) - \lambda)^{-1}\|_{op} < \delta := \min\{\delta_1, \delta_2\}^{-1}. \tag{3.81}$$

In other words, $\|v\|_{L^2(\mathbb{T}^d)} \leq \delta \|u\|_{L^2(\mathbb{T}^d)}$. Applying this fact together with (3.79) and (3.80), we have

$$\begin{aligned}
\|(L(k+z) - L(k))(L(k) - \lambda)^{-1} u\|_{L^2(\mathbb{T}^d)} &\leq |z| C_1 \|v\|_{H^1(\mathbb{T}^d)} \\
&\leq |z| C_1 C_2 (\|v\|_{L^2(\mathbb{T}^d)} + \|u\|_{L^2(\mathbb{T}^d)}) \\
&\leq |z| C_1 C_2 (1 + \delta) \|u\|_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Now (3.78) is a consequence of the above estimate if we let

$$\tau \leq \min\left(\frac{1}{2C_1 C_2(1 + \delta)}, 1\right).$$

□

Proof of Lemma 3.5.1.

Proof. From Proposition 2.2.6, the complex Bloch variety $\Sigma := B_L$ of the operator L is

an analytic subset of codimension one in $\mathbb{C}^{d+1}$. By [47, 80], there exist an entire scalar function $h(k, \mu)$ and an entire operator-valued function $I(k, \mu)$ on $\mathbb{C}^{d+1}$ such that

(1) h vanishes only on Σ and has simple zeros on Σ, i.e., its normal derivative is not zero at all smooth parts of Σ.

(2) In $\mathbb{C}^{d+1} \setminus \Sigma$, $(L(k) - \mu)^{-1} = h(k, \mu)^{-1} I(k, \mu)$.

In particular, $(L_{t,s}(k) - \lambda)^{-1} = h(k + it\beta_s, \lambda)^{-1} I(k + it\beta_s, \lambda)$ for $k \in \mathbb{R}^d$ and $t \in [0, 1)$ by Proposition 3.4.1. Due to **Assumption A** and **(P2)**, if $k_0 + it\beta_s \in V$, the k-variable function $h(k, \lambda)^{-1}$ is equal (up to a non-vanishing analytic factor) to $(\lambda_j(k + it\beta_s) - \lambda)^{-1}$ on an open disc $D(k_0, 2\varepsilon) \subseteq V$ in $\mathbb{C}^d$ for some $\varepsilon > 0$. Hence, we can write the sesquilinear form for such values of k as

$$(R_{t,s,\lambda} f, \varphi) = R_1 + R_2,$$

where

$$R_1 = (2\pi)^{-d} \int_{\mathcal{O} \cap D(k_0, \varepsilon)} \frac{\left(M(k, \lambda) \widehat{f}(k), \widehat{\varphi}(k) \right)}{\lambda_j(k + it\beta_s) - \lambda} \, \mathrm{d}k$$

and

$$R_2 = (2\pi)^{-d} \int_{\mathcal{O} \setminus D(k_0, \varepsilon)} \left(L(k + it\beta_s) - \lambda)^{-1} \widehat{f}(k), \widehat{\varphi}(k) \right) \mathrm{d}k.$$

Here $M(k, \lambda)$ is a $L^2(\mathbb{T}^d)$-valued analytic function on $D(k_0, \varepsilon)$ when $|\lambda|$ is small. Since f and φ have compact supports, their Floquet transforms $\widehat{f}(k), \widehat{\varphi}(k)$ are analytic with respect to k. To prove the equality (3.15), we apply the Lebesgue Dominated Convergence Theorem. For R_1, it suffices to show that the denominator in the integrand when $t \to 1^-$ is integrable over $D(k_0, \varepsilon)$ for $d \geq 2$. Indeed,

$$|\lambda_j(k + i\beta_s) - \lambda| \geq \delta \left| i\nabla E(\beta_s) \cdot (k - k_0) - \frac{1}{2}(k - k_0) \cdot \mathrm{Hess}\,(E)(\beta_s)(k - k_0) \right|,$$

for some $\delta > 0$ if ε is chosen small enough so that in the Taylor expansion of λ_j at $k_0 + i\beta_s$, the remainder term $O(|k - k_0|^3)$ is dominated by the quadratic term $|k - k_0|^2$. Furthermore,

$$\left| i\nabla E(\beta_s) \cdot (k - k_0) - \frac{1}{2}(k - k_0) \cdot \mathrm{Hess}\,(E)(\beta_s)(k - k_0) \right|^2 \geq C(|\langle k - k_0, s \rangle|^2 + |k - k_0|^4),$$

for some constant $C > 0$ (independent of k). Now let $v := (k - k_0)$ and so the right hand side of the above estimate is just $|\langle v, s \rangle|^2 + |v|^4$ (up to a constant factor). One can apply Hölder's inequality to obtain

$$|\langle v, s \rangle|^2 + |v|^4 \geq |\langle v, s \rangle|^2 + |v'|^4 \geq C|\langle v, s \rangle|^{3/2}|v'|,$$

where $v = (v_1, v') \in \mathbb{R} \times \mathbb{R}^{d-1}$. Thus, we deduce

$$|\lambda_j(k + i\beta_s) - \lambda|^{-1} \leq C|\langle v, s \rangle|^{-3/4}|v'|^{-1/2}. \tag{3.82}$$

Since the function $|x|^{-n}$ is integrable near 0 in $\mathbb{R}^d$ if and only if $n < d$, $|v'|^{-1/2}$ and $|\langle v, s \rangle|^{-3/4}$ are integrable near 0 in $\mathbb{R}^{d-1}$ and $\mathbb{R}$ respectively. Therefore, the function in the right hand side of (3.82) is integrable near 0.

The integrability of R_2 as $t \to 1^-$ follows from the estimation (3.78) in the proof of Proposition 3.4.1. Indeed,

$$\|(L(k + it\beta_s) - \lambda)^{-1}\|_{op} = \|(1 - (L(k + it\beta_s) - L(k))(\lambda - L(k))^{-1})^{-1}(\lambda - L(k))^{-1}\|_{op}$$

$$\leq \frac{\|(L(k) - \lambda)^{-1}\|_{op}}{1 - \|(L(k + it\beta_s) - L(k))(\lambda - L(k))^{-1}\|_{op}}. \tag{3.83}$$

By decreasing $|\lambda|$, if necessary, and repeating the arguments when showing (3.78) and

(3.81) we derive:

$$1 - \|(L(k + it\beta_s) - L(k))(\lambda - L(k))^{-1}\|_{op} \geq 1/2, \quad \forall k \in \mathcal{O} \setminus D(k_0, \varepsilon) \qquad (3.84)$$

and

$$\sup_{k \in \mathcal{O} \setminus D(k_0, \varepsilon)} \|(L(k) - \lambda)^{-1}\|_{op} < \infty. \qquad (3.85)$$

Thanks to (3.83), (3.84), (3.85), Cauchy-Schwarz inequality and Lemma 2.2.4, we have:

$$\sup_{t \in [0,1]} \left| \left(L(k + it\beta_s) - \lambda)^{-1} \widehat{f}(k), \widehat{\varphi}(k) \right) \right| \leq 2\|(L(k) - \lambda)^{-1}\|_{op} \cdot \|\widehat{f}(k)\|_{L^2(\mathbb{T}^d)} \|\widehat{\varphi}(k)\|_{L^2(\mathbb{T}^d)}$$

$$\lesssim \|\widehat{f}(k)\|_{L^2(\mathbb{T}^d)} \|\widehat{\varphi}(k)\|_{L^2(\mathbb{T}^d)} \quad , \forall k \in \mathcal{O} \setminus D(k_0, \varepsilon)$$

and

$$\int_{\mathcal{O} \setminus D(k_0, \varepsilon)} \|\widehat{f}(k)\|_{L^2(\mathbb{T}^d)} \|\widehat{\varphi}(k)\|_{L^2(\mathbb{T}^d)} \, \mathrm{d}k \leq \|f\|_{L^2(\mathbb{R}^d)} \|\varphi\|_{L^2(\mathbb{R}^d)} < \infty.$$

This completes the proof of our lemma. $\qquad \square$

3.9.4 Regularity of eigenfunctions $\phi(z, x)$

In this subsection, we study the regularity of the eigenfunctions $\phi(z, x)$ of the operator $L(z)$ with corresponding eigenvalue $\lambda_j(z)$ (see **(P4)**). It is known that for each $z \in V$, the eigenfunction $\phi(z, x)$ is smooth in x. We will claim that these eigenfunctions are smooth in (z, x) when z is near to k_0. The idea is that initially, $\phi(z, \cdot)$ is an analytic section of the Hilbert bundle $V \times H^2(\mathbb{T}^d)$ and then by ellipticity, it is also an analytic section of the bundle $V \times H^m(\mathbb{T}^d)$ for any $m > 0$ (for statements related to Fredholm morphisms between analytic Banach bundles, see e.g., [82]) and hence smoothness will follow.

For the sake of completeness, we provide the proof of the above claim by applying standard bootstrap arguments in the theory of elliptic differential equations.

Lemma 3.9.4. *The function $\partial_x^\alpha \phi(z, x)$ is jointly continuous on $V \times \mathbb{R}^d$ for any multi-index α.*

Proof. By periodicity, it suffices to restrict x to $\mathbb{T}^d$. Let $K := \overline{V}$. Due to **(P4)**, the function $z \mapsto \phi(z, \cdot)$ is a $H^2(\mathbb{T}^d)$-valued analytic on some neighborhood of K. Thus,

$$\sup_{z \in K} \|\phi(z, \cdot)\|_{H^2(\mathbb{T}^d)} < \infty.$$

Then, we can apply bootstrap arguments for the equation

$$L(z)\phi(z, \cdot) = \lambda_j(z)\phi(z, \cdot)$$

to see that $M_m := \sup_{z \in K} \|\phi(z, \cdot)\|_{H^m(\mathbb{T}^d)}$ is finite for any nonnegative integer m.

Now we consider z and z' in K. Let $\phi_{z,z'}(x) := \phi(z, x) - \phi(z', x)$. Then, $\phi_{z,z'}$ is a (classical) solution of the equation

$$L(z)\phi_{z,z'} = f_{z,z'},$$

where $f_{z,z'} := (\lambda_j(z)\phi(z, \cdot) - \lambda_j(z')\phi(z', \cdot)) + (L(z') - L(z))\phi(z', \cdot)$.

By induction, we will show that for any $m \geq 0$,

$$\|\phi_{z,z'}\|_{H^m(\mathbb{T}^d)} \lesssim |z - z'|. \tag{3.86}$$

The case $m = 0$ is clear because **(P4)** implies that $z \mapsto \|\phi(z, \cdot)\|_{L^2(\mathbb{T}^d)}$ is Lipschitz continuous.

Next, we assume that the estimate (3.86) holds for m. As in (3.79),

$$\|(L(z) - L(z'))\phi(z', \cdot)\|_{H^m(\mathbb{T}^d)} \lesssim |z - z'| \cdot \|\phi(z', \cdot)\|_{H^{m+1}(\mathbb{T}^d)} \lesssim M_{m+1}|z - z'|. \tag{3.87}$$

Using triangle inequalities, the estimates (3.86), (3.87) and analyticity of λ_j, we get

$$
\begin{aligned}
\|f_{z,z'}\|_{H^m(\mathbb{T}^d)} &\lesssim \|\lambda_j(z)\phi(z,\cdot) - \lambda_j(z')\phi(z',\cdot)\|_{H^m(\mathbb{T}^d)} + \|(L(z) - L(z'))\phi(z',\cdot)\|_{H^m(\mathbb{T}^d)} \\
&\lesssim |\lambda_j(z)| \cdot \|\phi_{z,z'}\|_{H^m(\mathbb{T}^d)} + M_m|\lambda_j(z) - \lambda_j(z')| + M_{m+1}|z - z'| \\
&\lesssim |z - z'|.
\end{aligned}
$$

$$(3.88)$$

Notice that for any $m \geq 0$, the following standard energy estimate holds (see e.g., [22, 28, 53]):

$$
\|\phi_{z,z'}\|_{H^{m+2}(\mathbb{T}^d)} \lesssim \|f_{z,z'}\|_{H^m(\mathbb{T}^d)} + \|\phi_{z,z'}\|_{L^2(\mathbb{T}^d)}. \tag{3.89}
$$

Combining (3.88) and (3.89), we deduce that $\|\phi_{z,z'}\|_{H^{m+2}(\mathbb{T}^d)} \lesssim |z - z'|$. Hence, (3.86) holds for $m + 2$. This finishes our induction.

Applying the Sobolev embedding theorem, $\|\phi_{z,z'}\|_{C^m(\mathbb{T}^d)} \lesssim |z - z'|$ for any $m \geq 0$. In other words, $\phi \in C(K, C^m(\mathbb{T}^d))$ for any m. Since $C(K \times \mathbb{T}^d) = C(K, C(\mathbb{T}^d))$, this completes the proof. $\qquad\square$

Notation 3.9.5. Consider a z-parameter family of linear partial differential operators $\{L(z)\}$ where $z \in \mathbb{R}^d$. Suppose $L(x, \xi, z)$ is the symbol of $L(z)$. Whenever it makes sense, the differential operator $\dfrac{\partial L(z)}{\partial z_l}$ is the one whose symbol is $\dfrac{\partial L}{\partial z_l}(x, \xi, z)$ for any $l \in \{1, 2, \ldots, d\}$.

Proposition 3.9.6. *Assume D is an open disc centered at k_0 in $\mathbb{R}^d$ such that $D \pm i\beta_s \Subset V$ for any $s \in \mathbb{S}^{d-1}$. Then all eigenfunctions $\phi(k \pm i\beta_s, x)$ are smooth on a neighborhood of $\overline{D} \times \mathbb{R}^d$. Furthermore, all derivatives of $\phi(k \pm i\beta_s, x)$ are bounded on $\overline{D} \times \mathbb{R}^d$ uniformly in s, i.e., for any multi-indices α, β:*

$$
\sup_{(s,k,x)\in\mathbb{S}^{d-1}\times\overline{D}\times\mathbb{R}^d} |\partial_k^\alpha \partial_x^\beta \phi(k \pm i\beta_s, x)| < \infty.
$$

Proof. Pick any open disc D' in $\mathbb{R}^d$ so that $\overline{D} \pm i\beta_s \subset D' \pm i\beta_s \Subset V$. We will prove that

all eigenfunctions are smooth on the domain $D' \times \mathbb{R}^d$. Also, it is enough to consider the function $\phi(k + i\beta_s)$ since the other one is treated similarly.

First, we show that $\frac{\partial \phi}{\partial k_l}(k + i\beta_s, x)$ is continuous for any $1 \leq l \leq d$. By Lemma 3.9.4, the function $(k, x) \mapsto \phi(k + i\beta_s, x)$ is continuous on $D' \times \mathbb{T}^d$. We consider any two complex-valued test functions $\varphi \in C_c^\infty(D')$ and $\psi \in C^\infty(\mathbb{T}^d)$. Testing the equation of the eigenfunction $\phi(k + i\beta_s, x)$ with ψ and $\frac{\partial \varphi}{\partial k_l}$, we derive

$$\int_{D'} \int_{\mathbb{T}^d} (L(k + i\beta_s) - \lambda_j(k + i\beta_s))\phi(k + i\beta_s, x)\overline{\psi(x)}\frac{\partial \varphi}{\partial k_l}(k)\, dx\, dk = 0.$$

Observe that $L(k + i\beta_s)^* = L(k - i\beta_s)$ and $\left(\frac{\partial L(k - i\beta_s)}{\partial k_l}\right)^* = \frac{\partial L(k + i\beta_s)}{\partial k_l}$. We integrate by parts to derive

$$\begin{aligned}
0 &= \int_{D'} \left((L(k + i\beta_s) - \lambda_j(k + i\beta_s))\,\phi(k + i\beta_s, x), \psi(x)\right)_{L^2(\mathbb{T}^d)} \frac{\partial \varphi}{\partial k_l}(k)\, dk \\
&= \int_{D'} \left(\phi(k + i\beta_s, x), \left(L(k - i\beta_s) - \overline{\lambda_j}(k + i\beta_s)\right)\psi(x)\right)_{L^2(\mathbb{T}^d)} \frac{\partial \varphi}{\partial k_l}(k)\, dk \\
&= \int_{D'} \left(-\frac{\partial \phi}{\partial k_l}(k + i\beta_s, x), \left(L(k - i\beta_s) - \overline{\lambda_j}(k + i\beta_s)\right)\psi(x)\right)_{L^2(\mathbb{T}^d)} \varphi(k)\, dk \\
&\quad - \int_{D'} \left(\phi(k + i\beta_s, x), \frac{\partial L(k - i\beta_s)}{\partial k_l}\psi(x) - \frac{\partial \overline{\lambda_j}}{\partial k_l}(k + i\beta_s)\psi(x)\right)_{L^2(\mathbb{T}^d)} \varphi(k)\, dk \\
&= \int_{D'} \left((-L(k + i\beta_s) + \lambda_j(k + i\beta_s))\frac{\partial \phi}{\partial k_l}(k + i\beta_s, x), \psi(x)\right)_{L^2(\mathbb{T}^d)} \varphi(k)\, dk \\
&\quad - \int_{D'} \left(\left(\frac{\partial L(k + i\beta_s)}{\partial k_l} - \frac{\partial \lambda_j}{\partial k_l}(k + i\beta_s)\right)\phi(k + i\beta_s, x), \psi(x)\right)_{L^2(\mathbb{T}^d)} \varphi(k)\, dk.
\end{aligned} \tag{3.90}$$

We introduce

$$\phi_l(k, x) := \frac{\partial \phi}{\partial k_l}(k + i\beta_s, x),$$

$$G(k) := -L(k + i\beta_s) + \lambda_j(k + i\beta_s),$$

$$H(k, x) := \left(\frac{\partial L(k + i\beta_s)}{\partial k_l} - \frac{\partial \lambda_j}{\partial k_l}(k + i\beta_s)\right)\phi(k + i\beta_s, x).$$

By invoking the previous lemma, the Lipschitz continuity of the $C^{m+2}(\mathbb{T}^d)$-valued function $\phi(k + i\beta_s, \cdot)$ implies that the mapping $k \mapsto H(k, \cdot)$ must be Lipschitz as a $C^m(\mathbb{T}^d)$-valued function on $\overline{D'}$ for any $m \geq 0$. On the other hand, the $H^2(\mathbb{T}^d)$-valued function $\phi_l(k, \cdot)$ is also Lipschitz on D' due to **(P4)**. Hence, both $(G(k)\phi_l(k, \cdot), \psi)_{L^2(\mathbb{T}^d)}$ and $(H(k, \cdot), \psi)_{L^2(\mathbb{T}^d)}$ are continuous on D' for any test function ψ. The continuity let us conclude from (3.90) that for every $k \in D'$, $\phi_l(k, \cdot)$ is a weak solution of the equation

$$G(k)\phi_l(k, x) = H(k, x). \tag{3.91}$$

We interpret (3.91) in the classical sense since all the coefficients of this equation are smooth. Consider any k_1, k_2 in D' and subtract the equation corresponding to k_1 from the one corresponding to k_2 to obtain the equation for the oscillation function $\phi_l(k_1, \cdot) - \phi_l(k_2, \cdot)$:

$$G(k_1)(\phi_l(k_1, x) - \phi_l(k_2, x)) = (G(k_2) - G(k_1))\phi_l(k_2, x) + (H(k_1, x) - H(k_2, x)).$$

Note that due to regularities of λ_j, H and the fact that the differential operator $G(k)$ depends analytically on k, we get

$$\|H(k_1, \cdot) - H(k_2, \cdot)\|_{H^m(\mathbb{T}^d)} + \|(G(k_1) - G(k_2))\phi_l(k_2, \cdot)\|_{H^m(\mathbb{T}^d)} = O(|k_1 - k_2|), \quad \forall m \in \mathbb{N}.$$

Combining this with the uniform boundedness in k of the supremum norms of all coefficients of the differential operator $G(k_1)$, we obtain

$$\|\phi_l(k_1, \cdot) - \phi_l(k_2, \cdot)\|_{H^m(\mathbb{T}^d)} = O(|k_1 - k_2|),$$

by using energy estimates as in the proof of Lemma 3.9.4. An application of the Sobolev

embedding theorem shows that $\partial_x^\beta \phi_l(k, x)$ is continuous on $D' \times \mathbb{T}^d$ for any multi-index β.

To deduce continuity of higher derivatives $\partial_x^\beta \partial_k^\alpha \phi(k + i\beta_s)$ ($|\alpha| > 1$, $|\beta| \geq 0$), we induct on $|\alpha|$ and repeat the arguments of the $|\alpha| = 1$ case.

Finally, the last statement of this proposition also follows since all of our estimates hold uniformly in s. $\qquad\square$

Observation 3.9.7. 1. The property **(P4)** is crucial in order to bootstrap regularities of eigenfunctions $\phi(k \pm i\beta_s)$.

2. If one just requires $\phi(k \pm i\beta_s) \in C^m(\overline{D} \times \mathbb{R}^d)$ for certain $m > 0$ then the smoothness on coefficients of L could be relaxed significantly (see [28, 53]).

3.10   Concluding remarks

1. The condition that the potentials A, V are infinitely differentiable is an overkill. The Fredholm property of the corresponding Floquet operators is essential, which can be obtained under much weaker assumptions.

2. The main result of this chapter assumes the central symmetry (evenness) of the relevant branch of the dispersion curve $\lambda_j(k)$, which does not hold for instance for operators with periodic magnetic potentials [24, 69]. Note that the result of [52] at the spectral edge does not require such a symmetry. It seems that in the inside-the-gap situation one also should not need such a symmetry. However, we have not been able to do so here, and thus were limited to the case of high symmetry points of the Brillouin zone.

3. In the case when λ is below the whole spectrum, the result of this chapter implies [60, Theorem 1.1] for self-adjoint operators.

4. GREEN'S FUNCTION ASYMPTOTICS OF PERIODIC ELLIPTIC OPERATORS ON ABELIAN COVERINGS OF COMPACT MANIFOLDS.

4.1 Introduction

Many classical properties of solutions of periodic Schrödinger operators on Euclidean spaces were generalized successfully to solutions of periodic Schrödinger operators on coverings of compact manifolds (see e.g., [3, 15, 16, 45, 50, 56, 74, 75]). Hence, a question arises of whether one can obtain analogs of the results of Chapter 3 and the paper [52] as well. The main theorems 4.3.1 and 4.3.4 of this chapter provide such results for operators on abelian coverings of compact Riemannian manifolds. The results are in line with Gromov's idea that the large scale geometry of a co-compact normal covering is captured mostly by its deck transformation group (see e.g., [17, 31, 68]). For instance, the dimension of the covering manifold does not enter explicitly to the asymptotics. Rather, the torsion-free rank d of the abelian deck transformation group influences these asymptotics significantly. One can find a similar effect in various results involving analysis on Riemannian co-compact normal coverings such as the long time asymptotic behaviors of the heat kernel on a noncompact abelian Riemannian covering [46], or the analogs of Liouville's theorem [50] (see also [68] for an excellent survey on analysis on co-compact coverings).

We discuss now the main thrust of this chapter.

Let X be a noncompact Riemannian manifold that is a normal abelian covering of a compact Riemannian manifold M with the deck transformation group G. For any function u on X and any $g \in G$, we denote by u^g the "shifted" function

$$u^g(x) = u(g \cdot x),$$

for any $x \in X$. Consider a bounded below second-order elliptic operator L on the manifold X with real and smooth coefficients. We assume that L is a **periodic** operator on X, i.e., the following invariance condition holds:

$$Lu^g = (Lu)^g,$$

for any $g \in G$ and $u \in C_c^\infty(X)$. The operator L, with the Sobolev space $H^2(X)$ as its domain, is an unbounded self-adjoint operator in $L^2(X)$.

As one could expect from Remark 2.2.7, the following fact for such operators on co-compact abelian coverings is well-known (see e.g., [16, 20, 47, 48, 66, 74, 75]):

Theorem 4.1.1. *The spectrum of the above operator L in $L^2(X)$ has a **band-gap struc-ture**:*

$$\sigma(L) = \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j],$$

such that $\alpha_j \leq \alpha_{j+1}$, $\beta_j \leq \beta_{j+1}$ and $\lim_{j \to \infty} \alpha_j = \infty$.

The bands can overlap when the dimension of the covering X is greater than 1.

In this chapter, we study Green's function asymptotics for the operator L at an energy level $\lambda \in \mathbb{R}$, such that λ belongs to the union of all closures of finite spectral gaps[1]. We divide this into two cases:

- **Case I:** (*Spectral gap interior*) The level λ is in a finite spectral gap (β_j, α_{j+1}) such that λ is close either to the spectral edge β_j or to the spectral edge α_{j+1}.

- **Case II:** (*Spectral edge case*) The level λ coincides with one of the spectral edges of some finite spectral gap, i.e., $\lambda = \alpha_{j+1}$ (lower edge) or $\lambda = \beta_j$ (upper edge) for some $j \in \mathbb{N}$.

---

[1]All of the results still hold for the case when λ does not exceed the bottom of the spectrum, i.e. for the semi-infinite gap.

In Case I, the Green's function $G_\lambda(x, y)$ is the Schwartz kernel of the resolvent operator $R_{\lambda,L} := (L - \lambda)^{-1}$, while in Case II, it is the Schwartz kernel of the weak limit of resolvent operators $R_{\lambda,L} := (L - \lambda \pm \varepsilon)^{-1}$ as $\varepsilon \to 0$ (the sign $\pm$ depends on whether λ is an upper or a lower spectral edge). Note that in the flat case $X = \mathbb{R}^d$, Green's function asymptotics of periodic elliptic operators were obtained in Chapter 3 for Case I, and in [52] for Case II. As in the previous chapter (see also [52]), we will deduce all asymptotics from an assumed "generic" spectral edge behavior of the dispersion relation of the operator L.

The organization of the chapter is as follows. In Subsection 4.2.1, we will review some general notions and results about group actions on abelian coverings, and then in Subsection 4.2.2, we introduce additive and multiplicative functions defined on an abelian covering, which will be needed for writing down the main formulae of Green's function asymptotics. Subsection 4.2.3 contains not only a brief introduction to periodic elliptic operators on abelian coverings, but also the necessary notations and assumptions for formulating the asymptotics. The main results of this chapter are stated in Section 4.3. In Section 4.4, the Floquet-Bloch theory is recalled and the problem is reduced to studying a scalar integral. Some auxiliary statements that appeared in Chapter 3 and [52] are collected in Section 4.5 for reader's convenience, and the final proofs of the main results are provided in Section 4.6. Section 4.7 provides the proofs of some technical claims that were postponed from previous sections.

4.2 Notions and preliminary results

4.2.1 Group actions and abelian coverings

Let X be a noncompact smooth Riemannian manifold of dimension n equipped with an **isometric, properly discontinuous, free**, and **co-compact** action of an **finitely generated abelian discrete group** G. The action of an element $g \in G$ on $x \in X$ is denoted by $g \cdot x$. Due to our conditions, the orbit space $M = X/G$ is a **compact** smooth Riemannian

manifold of dimension n when equipped with the metric pushed down from X. We assume that X and M are connected. Thus, we are dealing with a normal abelian covering of a compact manifold

$$X \xrightarrow{\pi} M(= X/G),$$

where G is the deck group of the covering π.

Let $d_X(\cdot, \cdot)$ be the distance metric on the Riemannian manifold X. It is known that X is a complete Riemannian manifold since it is a Riemannian covering of a compact Riemannian manifold M (see e.g., [17]). Thus, for any two points p and q in X, $d_X(p, q)$ is the length of a length minimizing geodesic connecting these two points.

Let S be any finite generating set of the deck group G. We define the **word length** $|g|_S$ of $g \in G$ to be the number of generators in the shortest word representing g as a product of elements in S:

$$|g|_S = \min\{n \in \mathbb{N} \mid g = s_1 \ldots s_n, s_i \in S \cup S^{-1}\}.$$

The **word metric d_S on G with respect to S** is the metric on G defined by the formula

$$d_S(g, h) = |g^{-1}h|_S$$

for any $g, h \in G$.

We introduce a notion in geometric group theory due to Gromov that we will need here (see e.g., [57, 62]).

Definition 4.2.1. Let Y, Z be metric spaces. A map $f : Y \to Z$ is called a **quasi-isometry**, if the following conditions are satisfied:

- There are constants $C_1, C_2 > 0$ such that

$$C_1^{-1} d_Y(x, y) - C_2 \leq d_Z(f(x), f(y)) \leq C_1 d_Y(x, y) + C_2$$

  for all $x, y \in Y$.

- The image $f(Y)$ is a net in Z, i.e., there is some constant $C > 0$ so that if $z \in Z$, then there exists $y \in Y$ such that $d_Z(f(y), z) < C$.

We remark that given any two finite generating sets S_1 and S_2 of G, the two word metrics d_{S_1} and d_{S_2} on G are equivalent (see e.g., [62, Theorem 1.3.12]).

The next result, which directly follows from the Švarc-Milnor lemma (see e.g., [57, Lemma 2.8], [62, Proposition 1.3.13]), establishes a quasi-isometry between the word metric $d_S(\cdot, \cdot)$ of the deck group G and the distance metric $d_X(\cdot, \cdot)$ of the Riemannian covering X of a closed connected Riemannian manifold M.

Proposition 4.2.2. *For any $x \in X$, the map*

$$(G, d_S) \to (X, d_X)$$

$$g \mapsto g \cdot x$$

given by the action of the deck transformation group G on X is a quasi-isometry.

Since G is a finitely generated abelian group, its torsion free subgroup is a free abelian subgroup $\mathbb{Z}^d$ of finite index. Hence, we obtain a normal $\mathbb{Z}^d$-covering

$$X \to M'(= X/\mathbb{Z}^d),$$

and a normal covering of M with a finite number of sheets

$$M' \to M.$$

Then M' is still a compact Riemannian manifold. By switching to the normal subcovering $X \xrightarrow{\mathbb{Z}^d} M'$, we **assume from now on that the deck group** G **is** $\mathbb{Z}^d$ and substitute M' for M. This will not reduce generality of our results [2].

Notation 4.2.3. (a) Hereafter, we choose the symmetric set $\{-1, 1\}^d$ to be the generating set S of $\mathbb{Z}^d$. Then the function $z = (z_1, \ldots, z_d) \mapsto \sum_{j=1}^{d} |z_j|$ is the word length function $|\cdot|_S$ on $\mathbb{Z}^d$ associated with S.

(b) For a general Riemannian manifold Y, we denote by μ_Y the Riemannian measure of Y. We use the notation $L^2(Y)$ for the Lebesgue function space $L^2(Y, \mu_Y)$. Also, the notation $L^2_{comp}(Y)$ stands for the subspace of $L^2(Y)$ consisting of compactly supported functions. It is worth mentioning that in our case, the Riemannian measure μ_X is the lifting of the Riemannian measure μ_M to X. Thus, μ_X is a G-invariant Riemannian measure on X.

(c) We recall that a fundamental domain $F(M)$ for M in X (with respect to the action of G) is an open subset of X such that for any $g \neq e$, $F(M) \cap g \cdot F(M) = \emptyset$ and the subset

$$X \setminus \bigcup_{g \in G} g \cdot F(M)$$

has measure zero. One can refer to [7] for constructions of such fundamental domains. Henceforth, we use the notation $F(M)$ to stand for a fixed fundamental domain for M in X.

---

[2]The same reduction holds for any **finitely generated virtually abelian** deck group G.

Remark 4.2.4. The closure of $F(M)$ contains at least one point in X from every orbit of G, i.e.,

$$X = \bigcup_{g \in G} g \cdot \overline{F(M)}. \tag{4.1}$$

Thus, if $F : X \to \mathbb{R}$ is the lifting of an integrable function $f : M \to \mathbb{R}$ to X, then

$$\int_M f(x) d\mu_M(x) = \int_{\overline{F(M)}} F(x) d\mu_X(x). \tag{4.2}$$

4.2.2 Additive and multiplicative functions on abelian coverings

To formulate our main results in Section 4.3, we need to introduce an analog of exponential type functions on the noncompact covering X.

We begin with a notion of additive and multiplicative functions on X (see [56]) [3].

Definition 4.2.5.   • A real smooth function u on X is said to be **additive** if there is a homomorphism $\alpha : G \to \mathbb{R}$ such that

$$u(g \cdot x) = u(x) + \alpha(g), \quad \text{for all} \quad (g, x) \in G \times X.$$

• A real smooth function v on X is said to be **multiplicative** if there is a homomorphism β from G to the multiplicative group $\mathbb{R} \setminus \{0\}$ such that

$$v(g \cdot x) = \beta(g)v(x), \quad \text{for all} \quad (g, x) \in G \times X.$$

• Let $m \in \mathbb{N}$. A function α (resp. β) that maps X to $\mathbb{R}^m$ is called a vector-valued additive (resp. multiplicative) function on X if every component of α (resp. β) is also additive (resp. multiplicative) on X.

---

[3]The definition can apply to any covering manifold with a discrete deck group.

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Following [50, 56], we can define explicitly some additive and multiplicative functions for which the group homomorphisms α, β appearing in Definition 4.2.5 are trivial.

Definition 4.2.6. Let f be a nonnegative function in $C_c^\infty(X)$ such that f is strictly positive on $\overline{F(M)}$. For any $j = 1, \ldots, d$, we define the following function

$$E_j(x) = \sum_{g \in \mathbb{Z}^d} \exp(-g_j) f(g \cdot x).$$

We also put $E(x) := (E_1(x), \ldots, E_d(x))$.

Then E_j is a positive function satisfying the multiplicative property $E_j(g \cdot x) = \exp(g_j) E_j(x)$, for any $g = (g_1, \ldots, g_d) \in \mathbb{Z}^d$. The multiplicative function E plays a similar role to the one played by the exponential function e^x on the Euclidean space $\mathbb{R}^d$.

By taking logarithms, we obtain an additive function on X, which leads to the next definition.

Definition 4.2.7. We denote by h the smooth $\mathbb{R}^d$-valued function on X

$$h(x) := (\log E_1(x), \cdots, \log E_n(x)).$$

Then $h = (h_1, \ldots, h_d)$ with $h_j(x) = \log E_j(x)$. Thus, h satisfies the following additivity:

$$h(g \cdot x) = h(x) + g, \quad \text{for all} \quad (g, x) \in G \times X. \tag{4.3}$$

Here we use the natural embedding $G = \mathbb{Z}^d \subset \mathbb{R}^d$.

Clearly, the definitions of functions E and h depend on the choice of the function f and the fundamental domain $F(M)$. So, there is no canonical choice for constructing additive and multiplicative functions. Nevertheless, a more invariant approach to defining additive functions on co-compact coverings can be found in Appendix A, for instance.

The following important comparison between the Riemannian metric and the distance from the additive function h in Definition 4.2.7 will be needed later.

Proposition 4.2.8. *There are some positive constants R_h (depending on h) and $C > 1$ such that whenever $d_X(x, y) \geq R_h$, we have*

$$C^{-1} \cdot d_X(x, y) \leq |h(x) - h(y)| \leq C \cdot d_X(x, y).$$

Here $|\cdot|$ is the Euclidean distance on $\mathbb{R}^d$, and the constant C is independent of the choice of h.

As a consequence, the pseudo-distance $d_h(x, y) := |h(x) - h(y)| \to \infty$ if and only if $d_X(x, y) \to \infty$.

The proof of this statement is given in Section 4.7.

Definition 4.2.9. For any additive function h, $\mathcal{A}_h$ is the set consisting of unit vectors $s \in \mathbb{S}^{d-1}$ such that there exist two points x and y satisfying $d_X(x, y) > R_h$ and

$$s = (h(x) - h(y))/|h(x) - h(y)|.$$

The set $\mathcal{A}_h$ is called the **admissible set of the additive function** h, and its elements are **admissible directions** of h.

For the proof of the following proposition, one can see in Section 4.7.

Proposition 4.2.10. *For any additive function h on X, one has*

$$\mathbb{Q}^d \cap \mathbb{S}^{d-1} = \{g/|g| \mid g \in \mathbb{Z}^d \setminus \{0\}\} \subset \mathcal{A}_h. \tag{4.4}$$

Hence, the admissible set $\mathcal{A}_h$ of h is dense in the sphere $\mathbb{S}^{d-1}$. In particular, when $d = 2$, $\mathcal{A}_h$ is the whole unit circle $\mathbb{S}^1$.

Remark 4.2.11. When the dimension n of X is less than $(d-1)/2$ (e.g., if $d > 5$ and X is the standard two dimensional jungle gym JG^2 in $\mathbb{R}^d$, see [63]), the $(d-1)$-dimensional Lebesgue measure of the admissible set $\mathcal{A}_h$ of any additive function h on X is zero. To see this, we first denote by X_h the $2n$-dimensional smooth manifold $\{(x,y) \in X \times X \mid d_X(x,y) > R_h\}$, and then consider the smooth mapping:

$$\Psi : X_h \to \mathbb{S}^{d-1}$$
$$(x,y) \mapsto \frac{h(x) - h(y)}{|h(x) - h(y)|}.$$

Then $\mathcal{A}_h$ is the range of Ψ. Since $\dim X_h < \dim \mathbb{S}^{d-1}$, every point in the range of Ψ is critical and thus, $\mathcal{A}_h$ has measure zero by Sard's theorem.

Example 4.2.12.

- Here is a family of non-trivial examples of additive functions in the flat case, i.e., when the covering space X is $\mathbb{R}^d$ and the base is the d-dimensional torus $\mathbb{T}^d$. Let $d \geq 1$ and φ be a real smooth function in $\mathbb{R}^d$ such that φ is $\mathbb{Z}^d$-periodic. It is shown in [5] that there exists a unique map $F_\varphi = ((F_\varphi)_1, \ldots, (F_\varphi)_d) : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $F_\varphi(0) = 0$, the additive condition (4.3), i.e., $F_\varphi(x+n) = F_\varphi(x) + n$ for any $(x,n) \in \mathbb{R}^d \times \mathbb{Z}^d$, and the equation

$$\Delta(F_\varphi)_i = \nabla\varphi \cdot \nabla(F_\varphi)_i,$$

for any $1 \leq i \leq d$. Note that F_φ is just the identity mapping in the trivial case when $\varphi = 0$. Moreover, it is also known [5] that when $d = 2$, F_φ is a diffeomorphism of $\mathbb{R}^d$ onto itself. In particular, for any $\mathbb{Z}^2$-periodic function φ, $|F_\varphi(x) - F_\varphi(y)| \geq C_\varphi|x-y|$ for any $x, y \in \mathbb{R}^2$ for some $C_\varphi > 0$. However, when $d \geq 3$, F_φ may admit a critical point for some $\mathbb{Z}^d$-periodic function φ.

- Let $X \xrightarrow{p} M$, $Y \xrightarrow{q} N$ be normal $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ coverings of compact Riemannian manifolds M and N respectively. Then $X \times Y \xrightarrow{p \times q} M \times N$ is also a normal $\mathbb{Z}^{d_1+d_2}$ covering of $M \times N$. Consider any $\mathbb{R}^{d_1}$-valued function h_1 (resp. $\mathbb{R}^{d_2}$-valued function h_2) defined on X (resp. Y). Let us denote by $h_1 \oplus h_2$ the following $\mathbb{R}^{d_1+d_2}$-valued function on $X \times Y$:

$$(h_1 \oplus h_2)(x, y) = (h_1(x), h_2(y)), \quad (x, y) \in X \times Y.$$

Then it is clear that $h_1 \oplus h_2$ is additive (resp. multiplicative) on $X \times Y$ if and only if both functions h_1 and h_2 are additive (resp. multiplicative). Moreover, $\mathcal{A}_{h_1 \oplus h_2} \subseteq \{(a_1 \cdot \mathcal{A}_{h_1}, a_2 \cdot \mathcal{A}_{h_2}) \mid 0 < a_1, a_2 < 1 \text{ and } a_1^2 + a_2^2 = 1\}$.

4.2.3 Some notions and assumptions

Let L be a **bounded from below, real** and **symmetric second-order elliptic**[4] operator on X with **smooth**[5] coefficients such that the operator **commutes with the action of** G. An operator that commutes with the action of G is called a G-**periodic** (or sometimes **periodic**) operator for brevity.

Notice that on a Riemannian co-compact covering, any G-periodic elliptic operator with smooth coefficients is **uniformly elliptic** in the sense that

$$|L_0^{-1}(x, \xi)| \leq C|\xi|^{-2}, \quad (x, \xi) \in T^*X, \xi \neq 0.$$

Here $|\xi|$ is the Riemannian length of (x, ξ) and $L_0(x, \xi)$ is the principal symbol of L.

The periodic operator L can be pushed down to an elliptic operator L_M on M and thus, L is the lifting of an elliptic operator L_M to X. By a slight abuse of notation, we will use

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[4]The ellipticity is understood in the sense of the nonvanishing of the principal symbol of the operator L on the cotangent bundle of the underlying manifold (with the zero section removed).

[5]The smoothness condition is assumed for avoiding lengthy technicalities and it can be relaxed.

the same notation L for both elliptic operators acting on X and M.

Under these assumptions on L, the symmetric operator L with the domain $C_c^\infty(X)$ is essentially self-adjoint in $L^2(X)$, i.e., the minimal operator L_{min} coincides with the maximal operator L_{max} (see Chapter 2 or [71] for notation L_{min} and L_{max}). This fact can be found in [7, Proposition 3.1], for instance [6]. Hence, there exists a unique self-adjoint extension in the Hilbert space $L^2(X)$ of L, which we denote also by L. Since L is a uniformly elliptic operator on the manifold X of bounded geometry, its domain is the Sobolev space $H^2(X)$ [71, Proposition 4.1], and henceforward, we always work with this self-adjoint operator L.

From now on, we fix an additive function h (see Definition 4.2.7). The following lemma is a preparation for the next definition.

Lemma 4.2.13. *For any $k \in \mathbb{C}^d$, we have*

$$e^{-ik \cdot h(x)} L(x, D) e^{ik \cdot h(x)} = L(x, D) + B(k),$$

where $B(k)$ is a smooth differential operator of order 1 on X that commutes with the action of the deck group G. Thus by pushing down, the differential operators $e^{-ik \cdot h(x)} L(x, D) e^{ik \cdot h(x)}$ and $B(k)$ can be considered also as differential operators on M. Moreover, given any $m \in \mathbb{R}$, the mapping

$$k \mapsto e^{-ik \cdot h(x)} L(x, D) e^{ik \cdot h(x)}$$

is analytic in k as a $B(H^{m+2}(M), H^m(M))$-valued function.

Proof. It is standard that the commutator $[L, e^{ik \cdot h(x)}]$ is a differential operator of order 1

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[6]In [7], Atiyah proves for symmetric elliptic operators acting on Hermitian vector bundles over any general co-compact covering manifold (not necessary to be a Riemannian covering). Later, in [16], Brüning and Sunada extend Atiyah's arguments to the case including compact quotient space X/G with singularities.

on X. Now one can write

$$B(k) = e^{-ik\cdot h(x)}Le^{ik\cdot h(x)} - L = e^{-ik\cdot h(x)}[L, e^{ik\cdot h(x)}]$$

to see that $B(k)$ is also a smooth differential operator of order 1. Also, one can check that $B(k)$ commutes with the action of G by using G-periodicity of the operator L and additivity of h. This proves the first claim of the lemma. From a standard fact (see e.g., [36, Theorem 2.2]), the operator $e^{-ik\cdot h(x)}Le^{ik\cdot h(x)}$ defined on X can be written as a sum $\sum_{|\alpha|\leq 2} k^\alpha L_\alpha$, where L_α is a G-periodic differential operator on X of order $2 - |\alpha|$ which is independent of k. By pushing the above sum down to a sum of operators on M, the claim about analyticity in k is then obvious. $\qquad\square$

Definition 4.2.14. For any $k \in \mathbb{C}^d$, we denote by $L(k)$ the elliptic operator

$$e^{-ik\cdot h(x)}L(x, D)e^{ik\cdot h(x)}$$

in $L^2(M)$ with the domain the Sobolev space $H^2(M)$.

The operator $L(k)$ is self-adjoint in $L^2(M)$ for each $k \in \mathbb{R}^d$, with the domain $H^2(M)$. Due to ellipticity of L, each of the operators $L(k)$ ($k \in \mathbb{R}^d$) has discrete real spectrum and thus, we can single out continuous and piecewise-analytic band functions $\lambda_j(k)$ for each $j \in \mathbb{N}$ as before. By Lemma 4.2.13, the operators $L(k)$ are perturbations of the self-adjoint operator $L(0)$ by lower order operators $B(k)$ for each $k \in \mathbb{C}^d$. Consequently, the spectra of the operators $L(k)$ on M are all discrete (see [4, pp.180-190]). In a similar manner to the flat case, we have:

Theorem 4.2.15. [16, 20, 45, 47, 66, 74, 75] *The spectrum of L is the union of all the spectra of $L(k)$ when k runs over the Brillouin zone (or any its shifted copy), i.e., for any*

quasimomentum $k_0 \in \mathbb{R}^d$:

$$\sigma(L) = \bigcup_{k \in k_0 + [-\pi, \pi]^d} \sigma(L(k)) = \bigcup_{k \in k_0 + [-\pi, \pi]^d} \{\lambda_j(k) \mid j \in \mathbb{N}\}. \tag{4.5}$$

We recall the notions of Bloch variety and Fermi surface from Chapter 2.

Definition 4.2.16. A **Bloch solution with quasimomentum** k of the equation $L(x, D)u = 0$ is a solution of the form

$$u(x) = e^{ik \cdot h(x)} \phi(x),$$

where h is any fixed additive function on X and the function ϕ is invariant under the action of the deck transformation group G.[7]

Using the above definition of Bloch solution with quasimomentum k, one can define the Bloch variety and the Fermi surfaces of the operator L as in Definition 2.2.5.

Without loss of generality, it is enough to assume henceforth that 0 is the spectral edge of interest (by adding a constant into the operator L if necessary) and there is a spectral gap below this spectral edge 0. Therefore, 0 is the lower spectral edge of some spectral band [8], i.e., 0 is the minimal value of some band function $\lambda_j(k)$ for some $j \in \mathbb{N}$ over the Brillouin zone.

From now on, we will impose **Assumption A** in Chapter 3 on the band function λ_j.

Remark 4.2.17.

(a) For the flat case, the main theorem in [44] shows that the conditions **A1** and **A2** are 'generically' satisfied, i.e., they can be achieved by small perturbation of the potential of a periodic Schrödinger operator. The same proof in [44] still works for periodic Schrödinger operators on a general abelian covering.

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[7]It is easy to see that this definition is independent of the choice of h.
[8]The upper spectral edge case is treated similarly.

(b) We shall only use this condition **A5** for the *spectral gap interior* case.

(c) Due to results of [43] (in the flat case $X = \mathbb{R}^d$) and of [45] (in the general case), all these assumptions **A1-A5** hold at the bottom of the spectrum for non-magnetic Schrödinger operators.

Here are some notations that will be used throughout this chapter.

Notation 4.2.18. For any two functions f and g defined on $X \times X$, if there exist constants $C > 0$ and $R > 0$ such that $|f(x, y)| \leq C|g(x, y)|$ whenever $d_X(x, y) > R$, we write $f(x, y) = O(g(x, y))$.

We say that a set W in $\mathbb{C}^d$ is **symmetric** if for any $z \in W$, we have $\overline{z} \in W$.

The following proposition will play a crucial role in establishing Theorem 4.3.1. We omit the proof since it requires no change from our discussion in Chapter 3.

Proposition 4.2.19. *There exists an $\epsilon_0 > 0$ and a symmetric open subset $V \subset \mathbb{C}^d$ containing the quasimomentum k_0 from the Assumption A such that the band function λ_j in Assumption A has an analytic continuation into a neighborhood of $\overline{V}$, and the following properties hold for any z in a symmetric neighborhood of $\overline{V}$:*

(P1) $\lambda_j(z)$ is a simple eigenvalue of $L(z)$.

(P2) $|\lambda_j(z)| < \epsilon_0$ and $\overline{B}(0, \epsilon_0) \cap \sigma(L(z)) = \{\lambda_j(z)\}$.

(P3) There is a nonzero G-periodic function ϕ_z defined on X such that

$$L(z)\phi_z = \lambda_j(z)\phi_z.$$

Moreover, $z \mapsto \phi_z$ can be chosen analytic as a $H^2(M)$-valued function.

(P4) $2\Re(\mathrm{Hess}\,(\lambda_j)(z)) > \min\sigma(\mathrm{Hess}\,(\lambda_j)(k_0)) \cdot I_{d \times d}$.

(P5) $F(z) := (\phi(z, \cdot), \phi(\overline{z}, \cdot))_{L^2(M)} \neq 0$.

Let $\mathcal{V} = \{\beta \in \mathbb{R}^d \mid k_0 + i\beta \in \overline{V}\}$. Now we introduce the function $E(\beta) := \lambda_j(k_0 + i\beta)$, which is defined on $\mathcal{V}$. The next lemma (see also Chapter 3) is the only place in this chapter where the condition **A5** is used.

Lemma 4.2.20. *Assume the condition **A5**. Then E is a **real-valued** function. By reducing the neighborhood V in Proposition 4.2.19 if necessary, the function E can be assumed real analytic and strictly concave function from $\mathcal{V}$ to $\mathbb{R}$ such that its Hessian at any point β in $\mathcal{V}$ is negative-definite.*

For $\lambda \in \mathbb{R}$, we put $\Gamma_\lambda := \{\beta \in \mathcal{V} : E(\beta) = \lambda\}$.

Due to Lemma 4.2.20, there exists a diffeomorphism β from $\mathbb{S}^{d-1}$ onto Γ_λ such that

$$\nabla E(\beta_s) = -|\nabla E(\beta_s)|s.$$

In addition, $\lim_{|\lambda|\to 0} \max_{s\in\mathbb{S}^{d-1}} |\beta_s| = 0$. By letting $|\lambda|$ be sufficiently small, we will suppose that there is an $r_0 > 0$ (independent of s) such that

$$\{k + it\beta_s \mid (t,s) \in [0,1] \times \mathbb{S}^{d-1}, \ |k - k_0| \leq r_0\} \subset V. \tag{4.6}$$

4.3 The main results

We recall that h is a fixed additive function (see Definition 4.2.7).

First, we consider the case when λ is inside a gap and is near to one of the edges of the gap. The following result is an analog for abelian coverings of compact Riemannian manifolds of Theorem 3.2.5.

Theorem 4.3.1. *(Spectral gap interior)*

*Suppose that $d \geq 2$ and the conditions **A1-A5** are satisfied. For $\lambda < 0$ sufficiently close to 0 (depending on the dispersion branch λ_j and the operator L), the Green's function G_λ*

of L at λ admits the following asymptotics as $d_X(x,y) \to \infty$:

$$G_\lambda(x,y) = \frac{e^{(h(x)-h(y))(ik_0-\beta_s)}}{(2\pi|h(x)-h(y)|)^{(d-1)/2}} \cdot \frac{|\nabla E(\beta_s)|^{(d-3)/2}}{\det\left(-\mathcal{P}_s \operatorname{Hess}(E)(\beta_s)\mathcal{P}_s\right)^{1/2}}$$
$$\times \frac{\phi_{k_0+i\beta_s}(x)\overline{\phi_{k_0-i\beta_s}(y)}}{(\phi_{k_0+i\beta_s}, \phi_{k_0-i\beta_s})_{L^2(M)}} + e^{(h(y)-h(x))\cdot\beta_s}r(x,y). \tag{4.7}$$

Here

$$s = (h(x)-h(y))/|h(x)-h(y)| \in \mathcal{A}_h,$$

and $\mathcal{P}_s$ is the projection from $\mathbb{R}^d$ onto the tangent space of the unit sphere $\mathbb{S}^{d-1}$ at the point s. Also, there is a constant $C > 0$ (independent of s and of the choice of h) such that the remainder term r satisfies

$$|r(x,y)| \leq d_X(x,y)^{-d/2},$$

when $d_X(x,y)$ is large enough.

By using rational admissible directions (see (4.4)) in the formula (4.7), the large scale behaviors of the Green's function along orbits of the G-action admit the following nice form in which the additive function h is absent.

Corollary 4.3.2. *Under the same notations and hypotheses of Theorem 4.3.1 and suppose that $\lambda < 0$ is close enough to 0, as $|g| \to \infty$ ($g \in \mathbb{Z}^d$), we have*

$$G_\lambda(x, g\cdot x) = \frac{e^{g\cdot(ik_0-\beta_{g/|g|})}}{(2\pi|g|)^{(d-1)/2}} \cdot \frac{|\nabla E(\beta_{g/|g|})|^{(d-3)/2}}{\det\left(-\mathcal{P}_{g/|g|} \operatorname{Hess}(E)(\beta_{g/|g|})\mathcal{P}_{g/|g|}\right)^{1/2}}$$
$$\times \frac{\phi_{k_0+i\beta_{g/|g|}}(x)\overline{\phi_{k_0-i\beta_{g/|g|}}(g\cdot x)}}{(\phi_{k_0+i\beta_{g/|g|}}, \phi_{k_0-i\beta_{g/|g|}})_{L^2(M)}} + e^{g\cdot\beta_s}O(|g|^{-d/2}). \tag{4.8}$$

We also give another interpretation of Theorem 4.3.1 in the special case $X = \mathbb{R}^2$ as follows:

Corollary 4.3.3. *Let φ be any real, $\mathbb{Z}^2$-periodic and smooth function on $\mathbb{R}^2$, and we recall*

the notation F_φ from Example 4.2.12. Let s be any unit vector in $\mathbb{R}^2$ and $y \in \mathbb{R}^2$. Then as $|t| \to \infty$ $(t \in \mathbb{R})$, the Green's function G_λ of L at λ (≈ 0) has the following asymptotics

$$
G_\lambda(F_\varphi^{-1}(ts + F_\varphi(y)), y) = \frac{e^{ts \cdot (ik_0 - \beta_s)}}{(2\pi|\nabla E(\beta_s)| \cdot \det(-\mathcal{P}_s \operatorname{Hess}(E)(\beta_s)\mathcal{P}_s) \cdot |t|)^{1/2}}
$$
$$
\times \frac{\phi_{k_0 + i\beta_s}(F_\varphi^{-1}(ts + F_\varphi(y)))\overline{\phi_{k_0 - i\beta_s}(y)}}{(\phi_{k_0 + i\beta_s}, \phi_{k_0 - i\beta_s})_{L^2(\mathbb{T}^2)}} + e^{ts \cdot \beta_s} O(|t|^{-1}).
$$

We now switch to the case when λ is on the boundary of the spectrum. The following result is a generalization of [52, Theorem 2].

Theorem 4.3.4. *(Spectral edge case)*

*Let $d \geq 3$, the operator L satisfy the assumptions **A1-A4**, and $R_{-\varepsilon} = (L + \varepsilon)^{-1}$ for a small $\varepsilon > 0$ denote the resolvent of L near the spectral edge $\lambda = 0$ (which exists, due to Assumption A). Then:*

i) *For any $\phi, \varphi \in L^2_{comp}(X)$, as $\varepsilon \to 0$, we have:*

$$
\langle R_{-\varepsilon}\phi, \varphi \rangle \to \langle R\phi, \varphi \rangle.
$$

for an operator $R : L^2_{comp}(X) \to L^2_{loc}(X)$.

ii) *The Schwartz kernel $G(x, y)$ of the operator R, which we call **the Green's function of** L (at the spectral edge 0), has the following asymptotics when $d_X(x, y) \to \infty$:*

$$
G(x, y) = \frac{\Gamma(\frac{d-2}{2})e^{i(h(x) - h(y)) \cdot k_0}}{2\pi^{d/2}\sqrt{\det H}|H^{-1/2}(h(x) - h(y))|^{d-2}} \cdot \frac{\phi_{k_0}(x)\overline{\phi_{k_0}(y)}}{\|\phi_{k_0}\|^2_{L^2(M)}}
$$
$$
\times \left(1 + O\left(d_X(x, y)^{-1}\right)\right) + O\left(d_X(x, y)^{1-d}\right),
$$

(4.9)

where H is the Hessian matrix of λ_j at k_0.

Remark 4.3.5.

(a) An interesting feature in the main results is that the dimension n of the covering manifold X does not explicitly enter into the asymptotics (4.7) and (4.9) (especially, see also (4.8)). Anyway, it certainly influences the geometry of the dispersion curves and therefore the asymptotics too. However, as the Riemannian distance of x and y becomes larger, one can see that in the asymptotics, the role of the dimension n is rather limited, while the influence of the rank d of the torsion-free subgroup of the deck group G is stronger.

(b) Note that for a periodic elliptic operator of second order on $\mathbb{R}^d$, at the bottom of its spectrum, the operator is known to be critical when the dimension $d \leq 2$ (see [56, 60, 64]). So, the assumption $d \geq 3$ is needed in Theorem 4.3.4.

Proving Theorem 4.3.4 by generalizing [52, Theorem 2] is similar to showing Theorem 4.3.1 by generalizing Theorem 3.2.5. Thus, after finishing the proof of Theorem 4.3.1, we will sketch briefly the proof of Theorem 4.3.4 in Section 4.6.

We outline the general strategy of both the proofs of Theorem 4.3.1 and Theorem 4.3.4. As in Chapter 3 and [52], the idea is to show that only one branch of the dispersion relation λ_j appearing in the Assumption A will control the asymptotics.

- **Step 1:** We use the Floquet transform to reduce the problems of finding asymptotics of Green's functions to the problems of obtaining asymptotics of some integral expressions with respect to the quasimomentum k.

- **Step 2:** We localize these expressions around the quasimomentum k_0 and then we cut an "infinite-dimensional" part of the operator to deal only with the multiplication operator by the dispersion branch λ_j.

- **Step 3:** The dispersion curve around this part is almost a paraboloid according to the assumption **A4**, thus, we can reduce this piece of operator to the normal form in the free case. In this step, we obtain some scalar integral expressions which are close to the ones arising when dealing with the Green's function of the Laplacian operator at the level λ. Our remaining task is devoted to computing the asymptotics of these scalar integrals.

4.4 A Floquet-Bloch reduction of the problem

In this section, we will consider the Green's function $G_\lambda(x, y)$ at the level λ in Case I (i.e., *Spectral gap interior*).

4.4.1 The Floquet transforms on abelian coverings and a Floquet reduction of the problem

Notation 4.4.1. We introduce the following fundamental domain $\mathcal{O}$ (with respect to the dual lattice $2\pi\mathbb{Z}^d$):

$$\mathcal{O} = k_0 + [-\pi, \pi]^d.$$

If k_0 is a high symmetry point of the Brillouin zone (i.e., k_0 satisfies the assumption **A5**), then $\mathcal{O}$ is the fundamental domain we defined in Definition 3.2.1.

The following transform will play the role of the Fourier transform for the periodic case. Indeed, it is a version of the Fourier transform on the group $\mathbb{Z}^d$ of periods.

Definition 4.4.2. The **Floquet transform** $\mathcal{F}$ (which depends on the choice of h)

$$f(x) \rightarrow \widehat{f}(k, x)$$

maps a compactly supported function f on X into a function $\widehat{f}$ defined on $\mathbb{R}^d \times X$ in the following way:

$$\widehat{f}(k, x) := \sum_{\gamma \in \mathbb{Z}^d} f(\gamma \cdot x) e^{-ih(\gamma \cdot x) \cdot k}.$$

From the above definition, one can see that $\widehat{f}$ is $\mathbb{Z}^d$-periodic in the x-variable and satisfies a cyclic condition with respect to k:

$$\begin{cases} \widehat{f}(k, \gamma \cdot x) = \widehat{f}(k, x), & \forall \gamma \in \mathbb{Z}^d \\ \widehat{f}(k + 2\pi\gamma, x) = e^{-2\pi i \gamma \cdot h(x)} \widehat{f}(k, x), & \forall \gamma \in \mathbb{Z}^d \end{cases}.$$

Thus, it suffices to consider the Floquet transform $\widehat{f}$ as a function defined on $\mathcal{O} \times M$. Usually, we will regard $\widehat{f}$ as a function $\widehat{f}(k, \cdot)$ in k-variable in $\mathcal{O}$ with values in the function space $L^2(M)$.

We recall some properties of the Floquet transform on abelian coverings as in Lemma 2.2.4. Note that the proof for the abelian covering case does not require any change from the proof for the flat case.

I. The transform $\mathcal{F}$ is an isometry of $L^2(X)$ onto

$$\int_{\mathcal{O}}^{\oplus} L^2(M) = L^2(\mathcal{O}, L^2(M))$$

and of $H^2(X)$ onto

$$\int_{\mathcal{O}}^{\oplus} H^2(M) = L^2(\mathcal{O}, H^2(M)).$$

II. The following two equivalent inversion formulae $\mathcal{F}^{-1}$ are given by

$$f(x) = (2\pi)^{-d} \int_{\mathcal{O}} e^{ik \cdot h(x)} \widehat{f}(k, x) \, \mathrm{d}k, \quad x \in X. \tag{4.10}$$

and

$$f(x) = (2\pi)^{-d} \int_{\mathcal{O}} e^{ik \cdot h(x)} \widehat{f}(k, \gamma^{-1} \cdot x) \, dk, \quad x \in \gamma \cdot \overline{F(M)}. \tag{4.11}$$

III. (*Block-diagonalization*) The action of any periodic elliptic operator P in $L^2(X)$ under the Floquet transform $\mathcal{F}$ is given by

$$\mathcal{F}P(x, D)\mathcal{F}^{-1} = \int_{\mathcal{O}}^{\oplus} P(k) \, dk,$$

where $P(k)(x, D) = e^{-ik \cdot h(x)} P(x, D) e^{ik \cdot h(x)}$. In other words, $\widehat{Pf}(k) = P(k)\widehat{f}(k)$, for any $f \in H^2(X)$.

The next statement is proven in Proposition 3.4.1 for the flat case. We omit the proof.

Proposition 4.4.3. *If $|\lambda|$ is small enough (depending on the dispersion branch λ_j and L), then for any $(t, s) \in [0, 1] \times \mathbb{S}^{d-1}$, we have $\lambda \in \sigma(L(k+it\beta_s))$ if and only if $(k, t) = (k_0, 1)$.*

As in the previous chapter, the main ingredients in the proof of this statement are the upper-semicontinuity of the spectra of the analytic family $\{L(k)\}_{k \in \mathbb{C}^d}$ and the fact that E is a **real** function, whose Hessian is negative definite (Proposition 4.2.20).

We consider the following real, smooth linear elliptic operators on X:

$$L_{t,s} = e^{t\beta_s \cdot h(x)} L e^{-t\beta_s \cdot h(x)}, \quad (t, s) \in [0, 1] \times \mathbb{S}^{d-1}.$$

Notice that these operators are G-periodic, and when pushing $L_{t,s}$ down to M, we get the operator $L(-i\beta_s)$. We also use the notation L_s for $L_{1,s}$. By Floquet-Bloch theory for the operator $L_{t,s}$, we obtain

$$\sigma(L_{t,s}) = \bigcup_{k \in \mathcal{O}} \sigma(L_{t,s}(k)) = \bigcup_{k \in \mathcal{O}} \sigma(L(k + it\beta_s)) \supseteq \{\lambda_j(k + it\beta_s)\}_{k \in \mathcal{O}}. \tag{4.12}$$

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We now fix a real number λ such that the statement of Proposition 4.4.3 holds. By (4.12) and Proposition 4.4.3, λ is in the resolvent set of $L_{t,s}$ for any $(t,s) \in [0,1) \times \mathbb{S}^{d-1}$. Let $R_{t,s,\lambda}$ be the resolvent operator $(L_{t,s} - \lambda)^{-1}$. From the block-diagonalize property of the Floquet transform, for any $f \in L^2_{comp}(X)$, we have

$$\widehat{R_{t,s,\lambda}f}(k) = (L_{t,s}(k) - \lambda)^{-1}\widehat{f}(k), \quad (t,k) \in [0,1) \times \mathcal{O}.$$

Hence, the sesquilinear form $(R_{t,s,\lambda}f, \varphi)$ is equal to

$$(2\pi)^{-d} \int_{\mathcal{O}} \left((L_{t,s}(k) - \lambda)^{-1}\widehat{f}(k), \widehat{\varphi}(k) \right) \mathrm{d}k,$$

where $\varphi \in L^2_{comp}(X)$.

In the next lemma, the weak convergence as $t \nearrow 1$ of the operator $R_{t,s,\lambda}$ in $L^2_{comp}(X)$ is proved and thus, we can introduce the limit operator $R_{s,\lambda} := \lim\limits_{t \to 1^-} R_{t,s,\lambda}$.

Lemma 4.4.4. *Let $d \geq 2$. Under **Assumption A**, for f, φ in $L^2_{comp}(X)$, the following equality holds:*

$$\lim_{t \to 1^-} (R_{t,s,\lambda}f, \varphi) = (2\pi)^{-d} \int_{\mathcal{O}} \left(L_s(k) - \lambda)^{-1}\widehat{f}(k), \widehat{\varphi}(k) \right) \mathrm{d}k. \tag{4.13}$$

The integral in the right hand side of (4.13) is absolutely convergent.

This lemma is a direct corollary of the analyticity of the Bloch variety (see Proposition 2.2.6), Proposition 4.4.3 and the Lebesgue Dominated Convergence Theorem as being shown in Lemma 3.5.1. We skip the proof.

For any $(t,s) \in [0,1) \times \mathbb{S}^{d-1}$, let $G_{t,s,\lambda}$ be the Green's function of $L_{t,s}$ at λ, which is

the kernel of $R_{t,s,\lambda}$. Thus,

$$G_{t,s,\lambda}(x,y) = e^{t\beta_s \cdot (h(x) - h(y))} G_\lambda(x,y).$$

Taking the limit and applying Lemma 4.4.4, we conclude that the function

$$G_{s,\lambda}(x,y) := e^{\beta_s \cdot (h(y) - h(x))} G_\lambda(x,y)$$

is the integral kernel of the operator $R_{s,\lambda}$ defined as follows:

$$\widehat{R_{s,\lambda} f}(k) = (L_s(k) - \lambda)^{-1} \widehat{f}(k). \tag{4.14}$$

Hence, the problem of finding asymptotics of G_λ is now equivalent to obtaining asymptotics of any function $G_{s,\lambda}$, where s is an admissible direction in $\mathcal{A}_h$.

In addition, by (4.10) and (4.14), the function $G_{s,\lambda}$, which is also the Green's function of the operator L_s at λ, is the integral kernel of the operator $R_{s,\lambda}$ that acts on $L^2_{comp}(X)$ in the following way:

$$R_{s,\lambda} f(x) = (2\pi)^{-d} \int_{\mathcal{O}} e^{ik \cdot h(x)} (L_s(k) - \lambda)^{-1} \widehat{f}(k, x) \, dk, \quad x \in X. \tag{4.15}$$

This accomplishes Step 1 in our strategy of the proof.

4.4.2 Isolating the leading term in $R_{s,\lambda}$ and a reduced Green's function

The purpose of this part is to complete Step 2, i.e., to localize the part of the integral in (4.15), that is responsible for the leading term of the Green's function asymptotics.

Definition 4.4.5. For any $z \in V$, we denote by $P(z)$ the spectral projector $\chi_{B(0,\varepsilon_0)}(L(z))$,

i.e.,

$$P(z) = -\frac{1}{2\pi i} \oint_{|\alpha|=\epsilon_0} (L(z) - \alpha)^{-1} \, d\alpha.$$

By **(P2)**, $P(z)$ projects $L^2(M)$ onto the eigenspace spanned by ϕ_z. We also put $Q(z) :=$ $I - P(z)$ and denote by $R(P(z))$, $R(Q(z))$ the ranges of the projectors $P(z)$, $Q(z)$ correspondingly.

Using **(P6)** and the fact that $P(k+i\beta_s)^* = P(k-i\beta_s)$, we can deduce that if $|k-k_0| \leq$ r_0 (see (3.8)), the following equality holds

$$P(k + i\beta_s)u = \frac{(u, \phi_{k-i\beta_s})_{L^2(M)}}{(\phi_{k+i\beta_s}, \phi_{k-i\beta_s})_{L^2(M)}} \phi_{k+i\beta_s}, \quad \forall u \in L^2(M). \tag{4.16}$$

Let η be a cut-off smooth function on $\mathcal{O}$ supported on $\{k \in \mathcal{O} \mid |k - k_0| < r_0\}$ and equal to 1 around k_0.

According to (4.15), for any $f \in C_c^\infty(X)$, we want to find u such that

$$(L_s(k) - \lambda)\widehat{u}(k) = \widehat{f}(k).$$

Then the Green's function $G_{s,\lambda}$ satisfies

$$\int_X G_{s,\lambda}(x, y) f(y) \, d\mu_X(y) = \mathcal{F}^{-1}\widehat{u}(k, x) = u(x),$$

where $\mathcal{F}$ is the Floquet transform introduced in Definition 4.4.2.

By Proposition 4.4.3, the operator $L_s(k) - \lambda$ is invertible for any k such that $k \neq k_0$. Hence, we can decompose $\widehat{u}(k) = \widehat{u}_0(k) + (L_s(k) - \lambda)^{-1}(1 - \eta(k))\widehat{f}(k)$, where $\widehat{u}_0$ satisfies the equation

$$(L_s(k) - \lambda)\widehat{u}_0(k) = \eta(k)\widehat{f}(k).$$

Observe that $R(P(z))$ and $R(Q(z))$ are invariant subspaces for the operator $L(z)$ for

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any $z \in V$. Thus, if u_1, u_2 are functions such that $\widehat{u_1}(k) = P(k + i\beta_s)\widehat{u_0}(k)$ and $\widehat{u_2}(k) = Q(k + i\beta_s)\widehat{u_0}(k)$, we must have

$$(L_s(k) - \lambda)P(k + i\beta_s)\widehat{u_1}(k) = \eta(k)P(k + i\beta_s)\widehat{f}(k) \tag{4.17}$$

and

$$(L_s(k) - \lambda)Q(k + i\beta_s)\widehat{u_2}(k) = \eta(k)Q(k + i\beta_s)\widehat{f}(k).$$

Due to **(P2)**, when k is close to k_0, $\lambda = \lambda_j(k_0 + i\beta_s)$ must belong to the resolvent of the operator $L_s(k)|_{R(Q(k+i\beta_s))}$. Hence, we can write $\widehat{u_2}(k) = \eta(k)(L_s(k) - \lambda)^{-1}Q(k + i\beta_s)\widehat{f}(k)$. Therefore, $\widehat{u}(k)$ equals

$$\widehat{u_1}(k) + \left((1 - \eta(k))(L_s(k) - \lambda)^{-1} + \eta(k)((L_s(k) - \lambda)|_{R(Q(k+i\beta_s))})^{-1}Q(k + i\beta_s)\right)\widehat{f}(k).$$

The next theorem shows that we can ignore the infinite-dimensional part of the operator $R_{s,\lambda}$, i.e., the second term in the above sum of two operators.

Theorem 4.4.6. *Define*

$$T_s(k) := (1 - \eta(k))(L_s(k) - \lambda)^{-1} + \eta(k)((L_s(k) - \lambda)|_{R(Q(k+i\beta_s))})^{-1}Q(k + i\beta_s).$$

Let T_s be the operator acting on $L^2(X)$ as follows:

$$T_s = \mathcal{F}^{-1}\left((2\pi)^{-d}\int_{\mathcal{O}}^{\oplus} T_s(k)\,\mathrm{d}k\right)\mathcal{F}.$$

Then the Schwartz kernel $K_s(x, y)$ of the operator T_s is continuous away from the diagonal of X, and moreover, it is also rapidly decaying in a uniform way with respect to

$s \in \mathbb{S}^{d-1}$, *i.e., for any $N > 0$,*

$$\sup_{s \in \mathbb{S}^{d-1}} |K_s(x, y)| = O(d_X(x, y)^{-N}).$$

A proof using microlocal analysis is provided in Section 4.7.

Now let $V_s := R_{s,\lambda} - T_s$. Then the Schwartz kernel $G_0(x, y)$ of the operator V_s satisfies the following relation:

$$\int_X G_0(x, y) f(y) \, \mathrm{d}\mu_X(y) = \mathcal{F}^{-1} \widehat{u_1}(k, x) = u_1(x). \tag{4.18}$$

In what follows, we will find an integral representation of the kernel G_0. We will see that G_0 provides the leading term of the asymptotics of the kernel $G_{s,\lambda}$. For this reason, G_0 is called **a reduced Green's function**.

To find u_1, we use the equation (4.17) and apply (4.16) to deduce

$$(\lambda_j(k + i\beta_s) - \lambda)(\widehat{u_1}(k), \phi_{k-i\beta_s})_{L^2(M)} = \eta(k)(\widehat{f}(k), \phi_{k-i\beta_s})_{L^2(M)}.$$

Using $\widehat{u_1}(k) = P(k + i\beta_s)\widehat{u_1}(k)$ and (4.16) again, the above identity becomes

$$\widehat{u_1}(k, x) := \frac{\eta(k)\phi_{k+i\beta_s}(x)(\widehat{f}(k), \phi_{k-i\beta_s})_{L^2(M)}}{(\phi_{k+i\beta_s}, \phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k + i\beta_s) - \lambda)}, \quad k \neq k_0.$$

By the inverse Floquet transform (4.10), for any $x \in X$,

$$u_1(x) = (2\pi)^{-d} \int_{\mathcal{O}} e^{ik \cdot h(x)} \frac{\eta(k)\phi_{k+i\beta_s}(x)(\widehat{f}(k), \phi_{k-i\beta_s})_{L^2(M)}}{(\phi_{k+i\beta_s}, \phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k + i\beta_s) - \lambda)} \, \mathrm{d}k.$$

Now we repeat some calculations in Chapter 3 to have

$$
\begin{aligned}
u_1(x) &= \frac{1}{(2\pi)^d} \int_{\mathcal{O}} \int_M \frac{e^{ik\cdot h(x)}\eta(k)\widehat{f}(k,y)\overline{\phi_{k-i\beta_s}(y)}\phi_{k+i\beta_s}(x)}{(\phi_{k+i\beta_s},\phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}\mu_M(y) \, \mathrm{d}k \\
&= \frac{1}{(2\pi)^d} \int_{\mathcal{O}} \int_{F(M)} \sum_{\gamma\in G} \frac{e^{ik\cdot(h(x)-h(\gamma^{-1}\cdot y))}\eta(k)\overline{\phi_{k-i\beta_s}(y)}\phi_{k+i\beta_s}(x)}{(\phi_{k+i\beta_s},\phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}\mu_X(y) \, \mathrm{d}k \\
&= \frac{1}{(2\pi)^d} \int_{\mathcal{O}} \sum_{\gamma\in G} \int_{\gamma\cdot\overline{F(M)}} f(y) \frac{e^{ik\cdot(h(x)-h(y))}\eta(k)\overline{\phi_{k-i\beta_s}(\gamma^{-1}\cdot y)}\phi_{k+i\beta_s}(x)}{(\phi_{k+i\beta_s},\phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}\mu_X(y) \, \mathrm{d}k \\
&= \frac{1}{(2\pi)^d} \int_X f(y) \left(\int_{\mathcal{O}} \frac{e^{ik\cdot(h(x)-h(y))}\eta(k)\overline{\phi_{k-i\beta_s}(y)}\phi_{k+i\beta_s}(x)}{(\phi_{k+i\beta_s},\phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}k \right) \mathrm{d}\mu_X(y).
\end{aligned}
$$

In the second equality above, we use the identity (4.2).

Consequently, from (4.18), we conclude that our reduced Green's function is

$$
G_0(x,y) = \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik\cdot(h(x)-h(y))}\eta(k) \frac{\phi_{k+i\beta_s}(x)\overline{\phi_{k-i\beta_s}(y)}}{(\phi_{k+i\beta_s},\phi_{k-i\beta_s})_{L^2(M)}(\lambda_j(k+i\beta_s)-\lambda)} \, \mathrm{d}k.
\tag{4.19}
$$

4.5   Some auxiliary statements

In this part, we provide the analogs of some results from Chapter 3 and [52], which do not require any significant change in the proofs when dealing with the case of abelian coverings. Instead of repeating the details, we will make brief comments.

The first result studies the local smoothness in (z,x) of the eigenfunctions $\phi_z(x)$ of the operator $L(z)$ with the eigenvalue $\lambda_j(z)$.

Lemma 4.5.1. *Suppose that $B \subset \mathbb{R}^d$ is the open ball centered at k_0 with radius r_0 (see (4.6)). Then for each $s \in \mathbb{S}^{d-1}$, the functions $\phi_{k\pm i\beta_s}(x)$ are smooth on a neighborhood of $\overline{B} \times M$ in $\mathbb{R}^d \times M$. In addition, for any multi-index α, the functions $D_k^\alpha \phi_{k\pm i\beta_s}(x)$ are also*

jointly continuous in (s, k, x). In particular, we have

$$\sup_{(s,k,x)\in\mathbb{S}^{d-1}\times\overline{B}\times M} |D_k^\alpha \phi_{k\pm i\beta_s}(x)| < \infty.$$

To obtain Lemma 4.5.1, one can modify the proof of Proposition 3.9.6 without any significant change. Indeed, the three main ingredients in the proof are the smoothness in z of the family of operators $\{L(z)\}_{z\in V}$ acting between Sobolev spaces (Lemma 4.2.13), the property **(P3)** for bootstrapping regularity of eigenfunctions in k, and the standard coercive estimates of elliptic operators $L(z)$ on the compact manifold M (see e.g., [76, estimate (11.29)]) for bootstrapping regularity in x.

The next result is the asymptotics of the scalar integral expression obtained from the integral representation (4.19) of the reduced Green's function G_0.

Proposition 4.5.2. *Suppose that $d \geq 2$ and B is the open ball defined in Proposition 4.5.1. Let $\eta(k)$ be a smooth cut off function around the point k_0, and $\{\mu_s(k, x, y)\}_{s\in\mathbb{S}^{d-1}}$ be a family of smooth $\mathbb{C}^d$-valued functions defined on $\overline{B} \times M \times M$. We also use the same notation $\mu_s(k, x, y)$ for its lift to $\overline{B} \times X \times X$. For each quadruple $(s, a, x, y) \in \mathbb{S}^{d-1} \times \mathbb{R}^d \times X \times X$, we define*

$$I(s, a) := \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik\cdot a} \frac{\eta(k)}{\lambda_j(k + i\beta_s) - \lambda} \, \mathrm{d}k$$

and

$$J(s, a, x, y) := \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik\cdot a} \frac{\eta(k)(k - k_0) \cdot \mu_s(k, x, y)}{\lambda_j(k + i\beta_s) - \lambda} \, \mathrm{d}k.$$

Assume that the size of the support of η is small enough. Fix a direction $s \in \mathbb{S}^{d-1}$ and

consider all vectors a such that $s = \dfrac{a}{|a|}$. Then when $|a|$ is large enough, we have

$$I(s, a) = \frac{e^{ik_0 \cdot a}|\nabla E(\beta_s)|^{(d-3)/2}}{(2\pi|a|)^{(d-1)/2} \det\left(-\mathcal{P}_s \operatorname{Hess}(E)(\beta_s)\mathcal{P}_s\right)^{1/2}} + O(|a|^{-d/2}) \tag{4.20}$$

and

$$\sup_{(x,y)\in X\times X} |J(s, a, x, y)| = O(|a|^{-d/2}). \tag{4.21}$$

Moreover, if all derivatives of $\mu_s(k, x, y)$ with respect to k are uniformly bounded in $s \in \mathbb{S}^{d-1}$, then all the terms $O(\cdot)$ in (4.20) and (4.21) are also uniform in $s \in \mathbb{S}^{d-1}$ when $|a| \to \infty$.

The proof of Proposition 4.5.2 can be extracted from Section 3.6. The main ingredient (see Proposition 3.6.1) is an application of the Weierstrass Preparation Lemma in several complex variables to have a factorization of the denominator $\lambda_j(k + i\beta_s) - \lambda$ of the integrands of I, J into a form that is close to the normal form in the free case. This trick was used in [81] in the discrete setting.

The next result [52, Theorem 3.3] will be needed in the proof of Theorem 4.3.4.

Proposition 4.5.3. *Assume $d \geq 3$. Let $a \in \mathbb{R}^d$. Let η be a smooth function satisfying the assumptions of Proposition 4.5.2, and let $\mu(k, x, y)$ be a smooth G-periodic function from a neighborhood of $\overline{B} \times X \times X$ to $\mathbb{C}^d$. Then the following asymptotics hold when $|a| \to \infty$:*

$$\frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik \cdot a} \frac{\eta(k)}{\lambda_j(k)} \, \mathrm{d}k = \frac{\Gamma(\frac{d}{2} - 1)e^{ik_0 \cdot a}}{2\pi^{d/2}(\det H)^{1/2}|H^{-1/2}(a)|^{d-2}}(1 + O(|a|^{-1})),$$

and

$$\sup_{x,y\in X} \left| \int_{\mathcal{O}} e^{ik \cdot a} \frac{\eta(k)(k - k_0) \cdot \mu(k, x, y)}{\lambda_j(k)} \, \mathrm{d}k \right| = O(|a|^{-d+1}).$$

Here the notation $\Gamma(z)$ means the Gamma function $\Gamma(z) = \displaystyle\int_0^{\infty} x^{z-1}e^{-x} \, \mathrm{d}x$.

4.6 Proofs of the main results

Proof of Theorem 4.3.1.

Proof. We fix an admissible direction s of the additive function h and consider any $x, y \in X$ such that

$$\frac{h(x) - h(y)}{|h(x) - h(y)|} = s \in \mathcal{A}_h.$$

As we discussed in Section 4.4, the Green's function G_λ satisfies

$$G_\lambda(x, y) = e^{\beta_s \cdot (h(y) - h(x))} G_{s,\lambda}(x, y), \tag{4.22}$$

where $G_{s,\lambda}$ is the Schwartz kernel of the resolvent operator R_s. Also, $R_{s,\lambda} = V_s + T_s$. Due to Theorem 4.4.6, the Schwartz kernel of T_s decays rapidly (uniformly in s) when $d_X(x, y)$ is large enough. Hence, to find the asymptotics of the kernel of $R_{s,\lambda}$, it suffices to consider the kernel G_0 of the operator V_s. Define

$$a := h(x) - h(y) \tag{4.23}$$

and

$$\tilde{\mu}_\omega(k, p, q) := \frac{\phi_{k+i\beta_\omega}(p)\overline{\phi_{k-i\beta_\omega}(q)}}{(\phi_{k+i\beta_\omega}, \phi_{k-i\beta_\omega})_{L^2(M)}}, \quad (\omega, p, q) \in \mathbb{S}^{d-1} \times M \times M.$$

By Lemma 4.5.1, $\tilde{\mu}_\omega$ is a smooth function on $\overline{B} \times M \times M$. By Taylor expanding around k_0, $\tilde{\mu}_\omega(k, p, q) = \tilde{\mu}_\omega(k_0, p, q) + (k - k_0) \cdot \mu_\omega(k, p, q)$ for some smooth $\mathbb{C}^d$-valued function $\mu_\omega(k, p, q)$ defined on $\overline{B} \times M \times M$. From Lemma 4.5.1 and the definition of $\tilde{\mu}_\omega$,

$$\sup_{(\omega, k, x) \in \mathbb{S}^{d-1} \times \overline{B} \times M} |D_k^\alpha \tilde{\mu}_\omega(k, x, y)| < \infty,$$

for any multi-index α. Thus, all derivatives of μ_ω with respect to k are also uniformly

bounded in $\omega \in \mathbb{S}^{d-1}$. We now can rewrite (4.19) as follows:

$$
\begin{aligned}
G_0(x, y) &= \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik \cdot a} \frac{\eta(k)}{\lambda_j(k + i\beta_s) - \lambda} \left(\tilde{\mu}_s(k_0, x, y) + (k - k_0) \cdot \mu_s(k, x, y) \right) dk \\
&= I(s, a) \frac{\phi_{k_0 + i\beta_s}(x) \overline{\phi_{k_0 - i\beta_s}(y)}}{(\phi_{k_0 + i\beta_s}, \phi_{k_0 - i\beta_s})_{L^2(M)}} + J(s, a, x, y).
\end{aligned}
$$

Here the integrals $I(s, a)$ and $J(s, a, x, y)$ are defined in Proposition 4.5.2. Applying Proposition 4.5.2, we obtain the following asymptotics whenever $|a|$ is large enough:

$$
\begin{aligned}
G_0(x, y) &= \left(\frac{e^{ik_0 \cdot a} |\nabla E(\beta_s)|^{(d-3)/2}}{(2\pi|a|)^{(d-1)/2} \det\left(-\mathcal{P}_s \operatorname{Hess}(E)(\beta_s)\mathcal{P}_s\right)^{1/2}} + O(|a|^{-d/2}) \right) \\
&\quad \times \frac{\phi_{k_0 + i\beta_s}(x) \overline{\phi_{k_0 - i\beta_s}(y)}}{(\phi_{k_0 + i\beta_s}, \phi_{k_0 - i\beta_s})_{L^2(M)}} + O(|a|^{-d/2}),
\end{aligned}
\tag{4.24}
$$

where all the terms $O(\cdot)$ are uniform in s. Due to (4.23) and Proposition 4.2.8, $O(|a|^\ell) = O(d_X(x, y)^\ell)$ for any $\ell \in \mathbb{Z}$, provided that $d_X(x, y) > R_h$. Hence, by choosing the constant R_h larger if necessary, we can assume that when $d_X(x, y) > R_h$, the asymptotics (4.24) would follow. Finally, we substitute (4.23) to the asymptotics (4.24) and then use (4.22) to deduce Theorem 4.3.1. $\qquad\square$

Proof of Theorem 4.3.4.

Proof. We recall that $\lambda = \lambda_j(k_0) = 0$ and $R_{-\varepsilon}$ is the resolvent operator $(L + \varepsilon)^{-1}$ when $\varepsilon > 0$ is small enough. We will repeat the Floquet reduction approach in Section 4.4. Given any $f, \varphi \in L^2_{comp}(X)$, the sesquilinear form $\langle R_{-\varepsilon} f, \varphi \rangle$ is

$$
(2\pi)^{-d} \int_{\mathcal{O}} \left((L(k) + \varepsilon)^{-1} \widehat{f}(k), \widehat{\varphi}(k) \right) dk.
$$

The first conclusion of this theorem is achieved by a similar argument in [52, Lemma 2.3]. Hence the operator $R = \lim_{\varepsilon \to 0+} R_{-\varepsilon}$ is defined by the identity $\widehat{Rf}(k) = R(k)\widehat{f}(k)$ and

the Green's function G is the Schwartz kernel of the operator R. To single out the principal term in R, we first choose a neighborhood $V \subset \mathcal{O}$ of k_0 such that when $k \in V$, there is a non-zero G-periodic eigenfunction $\phi_k(x)$ of the operator $L(k)$ with the corresponding eigenvalue $\lambda_j(k)$ and moreover, the mapping $k \mapsto \phi_k(\cdot)$ is analytic in k as a $H^2(M)$-valued function. For such $k \in V$, let us denote by $P(k)$ the spectral projector of $L(k)$ that projects $L^2(M)$ onto the eigenspace spanned by ϕ_k. The notation $R(I - P(k))$ stands for the range of the projector $I - P(k)$. Then we pick η as a smooth cut off function around k_0 such that $\mathrm{supp}(\eta) \Subset V$. Define the operator

$$T := \frac{1}{(2\pi)^d} \int_{\mathcal{O}}^{\oplus} T(k) \, dk,$$

where

$$T(k) := (1 - \eta(k))L(k)^{-1} + \eta(k)(L(k)|_{R(I-P(k))})^{-1}(I - P(k)).$$

As in Theorem 4.4.6, the Schwartz kernel $K(x, y)$ of T is rapidly decaying as $d_X(x, y) \to \infty$. Thus, the asymptotics of the Green's function G are the same as the asymptotics of the Schwartz kernel G_0 of the operator $R - T$. To find G_0, we repeat the arguments in Section 4.4 to derive the formula

$$G_0(x, y) = \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik \cdot (h(x) - h(y))} \frac{\eta(k)}{\lambda_j(k)} \frac{\phi_k(x)\overline{\phi_k(y)}}{\|\phi_k\|^2_{L^2(M)}} dk, \quad x, y \in X.$$

As in the proof of Theorem 4.3.1, we set $a := h(x) - h(y)$ and rewrite the smooth function

$$\frac{\phi_k(x)\overline{\phi_k(y)}}{\|\phi_k\|^2_{L^2(M)}} = \frac{\phi_{k_0}(x)\overline{\phi_{k_0}(y)}}{\|\phi_{k_0}\|^2_{L^2(M)}} + (k - k_0) \cdot \mu(k, x, y),$$

for some smooth G-periodic function $\mu : \overline{B} \times X \times X \to \mathbb{C}^d$. Now by applying Proposition 4.5.3, the proof is completed. $\qquad \square$

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4.7 Proofs of technical statements

4.7.1 Proof of Proposition 4.2.8

Fixing a point $x_0 \in X$, we let

$$K := \overline{F(M)},$$

$$R := \max_{x \in K} d_X(x_0, x),$$

and

$$\widetilde{R_h} := \max_{(x,y) \in K \times K} |h(x) - h(y)|.$$

Due to Proposition 4.2.2 and the fact that $|\cdot|_S$ is equivalent to $|\cdot|$ on $\mathbb{Z}^d$, there exist $C_1 > 1$ and $C_2 > 0$ such that

$$C_1^{-1} \cdot d_X(g_1 \cdot x_0, g_2 \cdot x_0) - C_2 \leq |g_1 - g_2| \leq C_1 \cdot d_X(g_1 \cdot x_0, g_2 \cdot x_0) + C_2,$$

for any $g_i \in \mathbb{Z}^d$, $i = 1, 2$.

Now we consider any two points x, y in X. By (4.1), we can select $\tilde{x}, \tilde{y}$ in K such that $x = g_1 \cdot \tilde{x}$ and $y = g_2 \cdot \tilde{y}$ for some $g_1, g_2 \in \mathbb{Z}^d$. Since $\mathbb{Z}^d$ acts by isometries, we get

$$d_X(g_1 \cdot x_0, g_1 \cdot \tilde{x}) = d_X(x_0, \tilde{x}) \quad \text{and} \quad d_X(g_2 \cdot x_0, g_2 \cdot \tilde{y}) = d_X(x_0, \tilde{y}). \tag{4.25}$$

By (A.3), we have

$$h(x) - h(y) = h(\tilde{x}) - h(\tilde{y}) + g_1 - g_2.$$

Using triangle inequalities and (4.25), we obtain

$$|h(x) - h(y)| \leq \widetilde{R_h} + |g_1 - g_2| \leq C_1 \cdot d_X(g_1 \cdot x_0, g_2 \cdot x_0) + \widetilde{R_h} + C_2$$

$$\leq C_1 \cdot d_X(x, y) + C_1 \cdot (d_X(x_0, \tilde{x}) + d_X(x_0, \tilde{y})) + \widetilde{R_h} + C_2$$

$$\leq C_1 \cdot d_X(x, y) + (2C_1 R + \widetilde{R_h} + C_2).$$

Likewise,

$$|h(x) - h(y)| \geq |g_1 - g_2| - \widetilde{R_h} \geq C_1^{-1} \cdot d_X(g_1 \cdot x_0, g_2 \cdot x_0) - (\widetilde{R_h} + C_2)$$

$$\geq C_1^{-1} \cdot d_X(x, y) - (C_1^{-1} \cdot (d_X(x_0, \tilde{x}) + d_X(x_0, \tilde{y})) + \widetilde{R_h} + C_2)$$

$$\geq C_1^{-1} \cdot d_X(x, y) - (2C_1 R + \widetilde{R_h} + C_2).$$

The statement follows if we put $C := 2C_1$ and $R_h := 2C_1(2C_1 R + \widetilde{R_h} + C_2)$.

4.7.2 Proof of Proposition 4.2.10

By Definition 4.2.7, any rational point in the unit sphere $\mathbb{S}^{d-1}$ is an admissible direction of the additive function h and thus we have (4.4). By using the stereographic projection, one can see that the subset $\mathbb{Q}^d \cap \mathbb{S}^{d-1}$ is dense in $\mathbb{S}^{d-1}$. Hence, the density of $\mathcal{A}_h$ follows. Now we consider the case $d = 2$. For any point $x_0 \in X$, we denote by $\mathcal{A}_h(x_0)$ the subset of $\mathcal{A}_h$ consisting of unit vectors s such that there exists a point x in $\{x \in X \mid d_X(x, x_0) > R_h\}$ satisfying either $h(x) - h(x_0) = |h(x) - h(x_0)|s$ or $h(x_0) - h(x) = |h(x) - h(x_0)|s$. It is enough to prove that for any x_0, $\mathcal{A}_h(x_0) = \mathbb{S}^1$. Without loss of generality, we suppose that $h(x_0) = 0$. Let Y be the range of the continuous function $x \mapsto \dfrac{h(x)}{|h(x)|}$, which is defined on the connected set $\{x \in X \mid d_X(x, x_0) > R_h\}$. Then Y is a connected subset that contains $\mathbb{Q}^2 \cap \mathbb{S}^1$ since $h(n \cdot x_0) = n$ for any $n \in \mathbb{Z}^d$. Suppose for contradiction, there is a unit vector s such that $s \notin \mathcal{A}_h(x_0)$ and hence, $Y \subseteq \mathbb{S}^1 \setminus \{\pm s\}$. Thus, Y cannot be connected, which is a contradiction.

4.7.3 Proof of Theorem 4.4.6

It suffices to prove the following claim:

Theorem 4.7.1. *Let ϕ and θ be two functions in $C_c^\infty(X)$ such that the metric distance on X between the supports of these two functions is bigger than R_h. Let $K_{s,\phi,\theta}$ be the Schwartz kernel of the operator $\phi T_s \theta$. Then $K_{s,\phi,\theta}$ is continuous and rapidly decaying (uniformly in s) on $X \times X$, i.e., for any $N > 0$, we have*

$$\sup_{s \in \mathbb{S}^{d-1}} |K_{s,\phi,\theta}(x,y)| \leq C(1 + d_X(x,y))^{-N},$$

for some positive constant $C = C(N, \|\phi\|_\infty, \|\theta\|_\infty)$.

Let $K_s(k, x, y)$ be the Schwartz kernel of the operator $T_s(k)$. The next lemma is an analog for abelian coverings of Lemma 3.7.15.

Lemma 4.7.2. *Let ϕ and θ be any two compactly supported functions on X such that $\operatorname{supp}(\phi) \cap \operatorname{supp}(\theta) = \emptyset$. Then the following identity holds for any $(x, y) \in X \times X$:*

$$K_{s,\phi,\theta}(x,y) = \frac{1}{(2\pi)^d} \int_{\mathcal{O}} e^{ik \cdot (h(x) - h(y))} \phi(x) K_s(k, \pi(x), \pi(y)) \theta(y) dk,$$

where π is the covering map $X \to M$.

Proof. Let $\mathcal{P}$ be the subset of $C_c^\infty(X)$ consisting of all functions ψ whose support is connected, and if $\gamma \in G$ such that $\operatorname{supp} \psi^\gamma \cap \operatorname{supp} \psi \neq \emptyset$ then γ is the identity element of the deck group G. Since any compactly supported function on X can be decomposed as a finite sum of functions in $\mathcal{P}$, we can assume that both ϕ and θ belong to $\mathcal{P}$. Then the rest is similar to the proof of Lemma 3.7.15. $\square$

Another key ingredient in proving Theorem 4.7.1 is the following result:

Proposition 4.7.3. *Let* $\dim M = n$. *Then for any multi-index* α *such that* $|\alpha| \geq n$, $D_k^\alpha K_s(k, x, y)$ *is a continuous function on* $M \times M$. *Furthermore, we have*

$$\sup_{(s,k,x,y)\in\mathbb{S}^{d-1}\times\mathcal{O}\times M\times M} |D_k^\alpha K_s(k, x, y)| < \infty.$$

Before providing the proof of Proposition 4.7.3, let us use it to prove Theorem 4.7.1.

Proof of Theorem 4.7.1.

Proof. The exponential function $e^{2\pi i\gamma\cdot h(x)}$ is G-periodic for any $\gamma \in G$, and hence, it is also defined on M. We use the same notation $e^{2\pi i\gamma\cdot h(x)}$ for the corresponding multiplication operator on $L^2(M)$. Then we can write

$$T_s(k + 2\pi\gamma) = e^{-2\pi i\gamma\cdot h(x)} T_s(k) e^{2\pi i\gamma\cdot h(x)}, \quad (k, \gamma) \in \mathcal{O} \times G$$

It follows that for any multi-index α,

$$e^{i(k+2\pi\gamma)\cdot(h(x)-h(y))}\nabla_k^\alpha K_s(k + 2\pi\gamma, \pi(x), \pi(y)) = e^{ik\cdot(h(x)-h(y))}\nabla_k^\alpha K_s(k, \pi(x), \pi(y)).$$

$$(4.26)$$

Now we apply integration by parts to the identity in Lemma 4.7.2 to obtain

$$i^N (h(x) - h(y))^\alpha K_{s,\phi,\theta}(x, y) = \frac{\phi(x)\theta(y)}{(2\pi)^d} \int_{\mathcal{O}} e^{ik\cdot(h(x)-h(y))}\nabla_k^\alpha K_s(k, \pi(x), \pi(y)) \, dk.$$

$$(4.27)$$

Note that due to (4.26), when using integration by parts, we do not have any boundary term. If $|\alpha| \geq n$, then the above integral is uniformly bounded in (s, x, y) by Proposition 4.7.3. When $\phi(x)\theta(y) \neq 0$, we have $d_X(x, y) > R_h$ and so, $h(x) \neq h(y)$ by Proposition 4.2.8. Therefore, the kernel $K_{s,\phi,\theta}(x, y)$ is continuous on $X \times X$. Now fix (x, y) such that $\phi(x)\theta(y) \neq 0$. Next we choose $\ell_0 \in \{1, \ldots, d\}$ such that $|h_{\ell_0}(x) -$

$h_{\ell_0}(y)| = \max_{1 \le \ell \le d} |h_\ell(x) - h_\ell(y)| > 0$. Fix any $N \ge n$. Let $\alpha = (\alpha_1, \ldots, \alpha_d) = N(\delta_{1,\ell_0}, \ldots, \delta_{d,\ell_0})$, where $\delta_{\cdot,\cdot}$ is the Kronecker delta. Then $|(h(x) - h(y))^\alpha|^{-1} = |h_{\ell_0}(x) - h_{\ell_0}(y)|^{-N} \le d^{N/2}|h(x) - h(y)|^{-N}$. Consequently, from (4.27), we derive a positive constant C (independent of x, y) such that

$$\sup_{s \in \mathbb{S}^{d-1}} |K_{s,\phi,\theta}(x,y)| \le C|\phi(x)\theta(y)||(h(x) - h(y))^\alpha|^{-1} \le C d^{N/2}\|\phi\|_\infty \|\theta\|_\infty |h(x) - h(y)|^{-N}.$$

Using Proposition 4.2.8, the above estimate becomes

$$\sup_{(s,x,y) \in \mathbb{S}^{d-1} \times X \times X} (1 + d_X(x,y))^N |K_{s,\phi,\theta}(x,y)| < \infty,$$

which yields the conclusion. $\qquad\Box$

Back to Proposition 4.7.3, we first introduce several notions. Let $\mathcal{S}(M)$ be the space of Schwartz functions on M. The first notion is about the order of an operator on the Sobolev scale (see e.g. [67, Definition 5.1.1]).

Definition 4.7.4. A linear operator $A : \mathcal{S}(M) \to \mathcal{S}(M)$ is said to be **of order** $\ell \in \mathbb{R}$ **on the Sobolev scale** $(H^m(M))_{m \in \mathbb{R}}$ if for every $m \in \mathbb{R}$ it can be extended to a bounded linear operator $A_{m,m-\ell} \in B(H^m(M), H^{m-\ell}(M))$. In this situation, we denote by the same notation A any of the operators $A_{m,m-\ell}$.

A typical example of an operator of order ℓ on the Sobolev scale is any pseudodifferential operator of order ℓ acting on M.

Definition 4.7.5. Given $\ell \in \mathbb{R}$. We denote by $\mathcal{S}_\ell(M)$ the set consisting of families of operators $\{B_s(k)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$ acting on M so that the following properties hold:

- For any $(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}$, $B_s(k)$ is of order ℓ on the Sobolev scale $(H^p(M))_{p \in \mathbb{R}}$.

- For any $p \in \mathbb{R}$, the operator $B_s(k)$ is smooth in k as a $B(H^p(M), H^{p-\ell}(M))$-valued function.

- For any multi-index α, $D_k^\alpha B_s(k)$ is of order $\ell - |\alpha|$ on the Sobolev scale $(H^p(M))_{p \in \mathbb{R}}$ and moreover, for any $p \in \mathbb{R}$, the following uniform condition holds

$$\sup_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}} \|D_k^\alpha B_s(k)\|_{B(H^p(M), H^{p-\ell+|\alpha|}(M))} < \infty.$$

It is worth giving a separate definition for the class $\mathcal{S}_{-\infty}(M) = \bigcap_{\ell \in \mathbb{R}} \mathcal{S}_\ell(M)$ as follows:

Definition 4.7.6. We denote by $\mathcal{S}_{-\infty}(M)$ the set consisting of families of smoothing operators $\{U_s(k)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$ acting on M so that the following properties hold:

- For any $m_1, m_2 \in \mathbb{R}$, the operator $U_s(k)$ is smooth in k as a $B(H^{m_1}(M), H^{m_2}(M))$-valued function.

- The following uniform condition holds for any multi-index α:

$$\sup_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}} \|D_k^\alpha U_s(k)\|_{B(H^{m_1}(M), H^{m_2}(M))} < \infty.$$

We now introduce the class $\tilde{S}^\ell(\mathbb{T}^n)$ of parameter-dependent toroidal symbols on the n-dimensional torus [9].

Definition 4.7.7. The parameter-dependent class $\tilde{S}^\ell(\mathbb{T}^n)$ consists of symbols $\sigma(s, k; x, \xi)$ satisfying the following conditions:

- For each $(s, k) \in \mathbb{S}^{d-1} \times \mathcal{O}$, the function $\sigma(s, k; \cdot, \cdot)$ is a symbol of order ℓ on the torus $\mathbb{T}^n$ (see e.g., Definition 3.7.3 in Chapter 3).

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[9]Note that for the case $n = d$, the class of parameter-dependent toroidal symbols was introduced in Definition 3.7.3. Nevertheless, the techniques and results on parameter-dependent toroidal pseudodifferential operators obtained in Section 3.8 of Chapter 3 still apply for the general case $n \geq 1$.

- Consider any multi-indices α, β, γ and any $s \in \mathbb{S}^{d-1}$. Then the function $\sigma(s, \cdot; \cdot, \cdot)$ is smooth on $\mathcal{O} \times \mathbb{T}^n \times \mathbb{R}^n$. Furthermore, for some positive constant $C_{\alpha\beta\gamma}$ (independent of s,k,x,ξ), we have

$$\sup_{s \in \mathbb{S}^{d-1}} |D_k^\alpha D_\xi^\beta D_x^\gamma \sigma(s, k; x, \xi)| \leq C_{\alpha\beta\gamma}(1 + |\xi|)^{m-|\alpha|-|\beta|}.$$

We also define

$$\tilde{S}^{-\infty}(\mathbb{T}^n) := \bigcap_{\ell \in \mathbb{R}} \tilde{S}^\ell(\mathbb{T}^n).$$

The class of pseudodifferential operators on the torus $\mathbb{T}^n$ is also provided in the next definition.

Definition 4.7.8.

- Given a symbol $\sigma(x, \xi)$ of order ℓ on the torus $\mathbb{T}^n$, the corresponding periodic pseudodifferential operator $Op(\sigma)$ is defined by

$$(Op(\sigma)f)(x) := \sum_{\xi \in \mathbb{Z}^n} \sigma(x, \xi)\tilde{f}(\xi)e^{2\pi i \xi \cdot x},$$

where $\tilde{f}(\xi)$ is the Fourier coefficient of f at ξ.

- For any $\ell \in \mathbb{R} \cup \{-\infty\}$, the set of all families of periodic pseudodifferential operators $\{Op(\sigma(s, k; \cdot, \cdot))\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$, where σ runs over the class $\tilde{S}^\ell(\mathbb{T}^n)$, is denoted by $Op(\tilde{S}^\ell(\mathbb{T}^n))$.

Remarks 4.7.9.

(a) It is straightforward to check from definitions and the Leibnitz rule that for any $\ell_1, \ell_2 \in \mathbb{R} \cup \{-\infty\}$, if $\{A_s(k)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$, $\{B_s(k)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$ are two families of operators

in the class $\mathcal{S}_{\ell_1}(M)$ and $\mathcal{S}_{\ell_2}(M)$, respectively, then the family $\{A_s(k)B_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ belongs to $\mathcal{S}_{\ell_1+\ell_2}(M)$.

(b) If the family of operators $\{B_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ belongs to the class $\mathcal{S}_\ell(M)$ then by definition, the family of operators $\{D_k^\alpha B_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ is in the class $\mathcal{S}_{\ell-|\alpha|}(M)$ for any multi-index α.

(c) $\mathcal{S}_{-\infty}(\mathbb{T}^n)$ is the class $\mathcal{S}$ introduced in Definition 3.7.8.

(d) Given a family of symbols $\{\sigma(s,k;\cdot,\cdot)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}} \in \tilde{S}^\ell(\mathbb{T}^n)$, it follows from definitions here and boundedness on Sobolev spaces of periodic pseudodifferential operators (see e.g., [67, Corollary 4.8.3]) that the corresponding family of periodic pseudodifferential operators $\{Op(\sigma(s,k;\cdot,\cdot))\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ is in the class $\mathcal{S}_\ell(\mathbb{T}^n)$. In other words, $Op(\tilde{S}^\ell(\mathbb{T}^n)) \subseteq \mathcal{S}_\ell(\mathbb{T}^n)$ for any $\ell \in \mathbb{R} \cup \{-\infty\}$.

Roughly speaking, the next lemma says that we can deduce regularity of the Schwartz kernel of an operator provided that it acts "nicely" on Sobolev spaces.

Lemma 4.7.10. *Let A be a bounded operator in $L^2(M)$, where M is a compact n-dimensional manifold. Suppose that the range of A is contained in $H^m(M)$, where $m > n/2$ and in addition,*

$$\|Af\|_{H^m(M)} \leq C\|f\|_{H^{-m}(M)} \tag{4.28}$$

for all $f \in L^2(M)$.

Then A is an integral operator whose kernel $K_A(x,y)$ is bounded and uniformly continuous on $M \times M$ and the following estimate holds:

$$|K_A(x,y)| \leq \gamma_0 C, \tag{4.29}$$

where γ_0 is a constant depending only on n and m.

Proof. For the Euclidean case, this fact is shown in [1, Lemma 2.2]. To prove this on a general compact manifold, we simply choose a finite cover $\mathcal{U} = \{U_p\}$ of M with charts $U_p \cong \mathbb{R}^n$. Then fix a smooth partition of unity $\{\varphi_p\}$ with respect to the cover $\mathcal{U}$, i.e., $\operatorname{supp} \varphi_p \Subset U_p$. We decompose $A = \sum_{p,q} \varphi_p A \varphi_q$. Given any $f \in L^2(M)$, the estimate (4.28) will imply the estimate $\|\varphi_p A \varphi_q f\|_{H^m(U_p)} \leq C \|f\|_{H^{-m}(U_q)}$ for any p, q. Hence, we obtain the conclusion of the lemma for the kernel of each operator $\varphi_p A \varphi_q$, and thus for kernel of A too. $\qquad\square$

In what follows, we will show a nice behavior of kernels of families of operators in the class $\mathcal{S}_\ell(M)$ following from an application of the previous lemma.

Corollary 4.7.11. *Let $\ell \in \mathbb{R} \cup \{-\infty\}$. If $\{B_s(k)\}_{(s,k)}$ is a family of operators in $\mathcal{S}_\ell(M)$, then the Schwartz kernel $K_{B_s}(k, x, y)$ of the operator $B_s(k)$ satisfies*

$$\sup_{s,k,x,y} |D_k^\alpha K_{B_s}(k, x, y)| < \infty,$$

for any multi-index α satisfying $|\alpha| \geq n + \ell + 2$.

Proof. For such $|\alpha| \geq n + \ell + 2$, we pick some integer $m \in (n/2, (-\ell + |\alpha|)/2]$. Then by Definition 4.7.5, we have

$$\sup_{s,k} \|D_k^\alpha B_s(k) f\|_{H^m(M)} \leq C_\alpha \|f\|_{H^{-m}(M)}.$$

Applying Lemma 4.7.10, the estimates (4.29) hold for kernels $D_k^\alpha K_{B_s}(k, x, y)$ of the operators $D_k^\alpha B_s(k)$ uniformly in (s, k). $\qquad\square$

The next theorem shows the inversion formula (i.e., the existence of a family of parametrices) in the case of $\mathbb{T}^n$. The proof of this theorem just comes straight from the proof of Theorem 3.7.6. We omit the details.

Theorem 4.7.12. *Let $r \in \mathbb{N}$. Let us consider a family of $2r^{th}$ order elliptic operators $\{(\mathcal{Q}_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ on the torus $\mathbb{T}^n$. Assume that this family is in $Op(\tilde{S}^{2r}(\mathbb{T}^n))$ and moreover, for each $(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}$, the symbol $\sigma(s,k;x,\xi)$ of the operator $\mathcal{Q}_s(k)$ is of the form*

$$\sigma(s,k;x,\xi) = L_0(s,k;x,\xi) + \tilde{\sigma}(s,k;x,\xi),$$

where the families of parameter-dependent symbols $\{L_0(s,k;x,\xi)\}_{(s,k)}$, $\{\tilde{\sigma}(s,k;x,\xi)\}_{(s,k)}$ are in the class $\tilde{S}^{2r}(\mathbb{T}^n)$ and $\tilde{S}^{2r-1}(\mathbb{T}^n)$, respectively. Moreover, suppose that there is some constant $A > 0$ such that whenever $|\xi| > A$, we have

$$|L_0(s,k;x,\xi)| \geq 1, \quad (s,k,x) \in \mathbb{S}^{d-1} \times \mathcal{O} \times \mathbb{T}^n.$$

We call $L_0(s,k;x,\xi)$ the "leading part" of the symbol $\sigma(s,k;x,\xi)$.

Then there exists a family of parametrices $\{\mathcal{A}_s(k)\}_{(s,k)}$ in $Op(\tilde{S}^{-2r}(\mathbb{T}^n))$ such that

$$\mathcal{Q}_s(k)\mathcal{A}_s(k) = I - \mathcal{R}_s(k),$$

where $\mathcal{R}_s(k)$ is some family of smoothing operators in the class $\mathcal{S}_{-\infty}(\mathbb{T}^n)$.

To build a family of parametrices on a compact manifold, we will follow closely the strategy in [36] by working on open subsets of the torus first and then gluing together to get the final global result.

Theorem 4.7.13. *There exists a family of operators $\{A_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ in $\mathcal{S}_{-2}(M)$ and a family of operators $\{R_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}}$ in $\mathcal{S}_{-\infty}(M)$ such that*

$$(L_s(k) - \lambda)A_s(k) = I - R_s(k).$$

Proof. Let V_p ($p = 1, \ldots, N$) be a finite covering of the compact manifold M by evenly

covered coordinate charts. We also choose an open covering U_p $(p = 1, \ldots, N)$ that refine the covering $\{V_p\}$ such that $\overline{U_p} \subset V_p$ for any p. We can assume that each V_p is an open subset of $(0, 2\pi)^n$ in $\mathbb{R}^n$ and hence, we can view each V_p as an open subset of the torus $\mathbb{T}^n$.

To simplify the notation, we will suppress the index $p = 1, \ldots, N$ which specifies the open sets V_p, U_p until the final steps of the proof. Let us denote by i_U, r_U the inclusion mapping from $i_U : U \to \mathbb{T}^n$ and the restriction mapping $r_U : C^\infty(\mathbb{T}^n) \to C^\infty(U)$, correspondingly. We also use the same notation $L_s(k) - \lambda$ for its restrictions to the coordinate charts V, U if no confusion arises. Then $(L_s(k) - \lambda) r_U$ can be considered as an operator on $\mathbb{T}^n$.

Let us first establish the following localized version of the inversion formula

Lemma 4.7.14. *There are two families of symbols* $\{a(s, k; x, \xi)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$ *in* $\tilde{S}^{-2}(\mathbb{T}^n)$ *and* $\{r(s, k; x, \xi)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$ *in* $\tilde{S}^{-\infty}(\mathbb{T}^n)$ *so that*

$$(L_s(k) - \lambda) r_U \mathcal{A}_s(k) = r_U (I - \mathcal{R}_s(k)),$$

where $\mathcal{A}_s(k) = Op(a(s, k; \cdot, \cdot))$, $\mathcal{R}_s(k) = Op(r(s, k; \cdot, \cdot))$.

Proof. We denote by $(L_s(k) - \lambda)^T$ the transpose operator of $(L_s(k) - \lambda)$ on V. Now let ν be a function in $C_c^\infty(V)$ such that $\nu = 1$ in a neighborhood of $\overline{U}$ and $0 \leq \nu \leq 1$. Define

$$\mathcal{Q}_s(k) = (L_s(k) - \lambda)(L_s(k) - \lambda)^T \nu + (1 - \nu)\Delta^2.$$

Observe that each operator $\mathcal{Q}_s(k)$ is a globally defined 4th order differential operator on $\mathbb{T}^n$ with the following principal symbol

$$\nu(x)|\sigma_0(s, k; x, \xi)|^2 + (1 - \nu(x))|\xi|^4.$$

Here $\sigma_0(s,k;x,\xi)$ is the non-vanishing symbol of the elliptic operator $L_s(k) - \lambda$. Thus, each operator $\mathcal{Q}_s(k)$ is an elliptic differential operator on $\mathbb{T}^n$. In order to apply Theorem 4.7.12 to the family $\{\mathcal{Q}_s(k)\}_{(s,k)}$, we need to study its family of symbols $\{\sigma(s,k;x,\xi)\}_{(s,k)}$.

On the evenly covered chart V, we can assume that the operator $L_s(k) - \lambda$ is of the form

$$\sum_{|\alpha|\leq 2} a_\alpha(x)(D + (k + i\beta_s)^T \cdot \nabla\tilde{h})^\alpha,$$

for some functions $a_\alpha \in C^\infty(V)$ and $\tilde{h}$ is a smooth function obtained from the additive function h through some coordinate transformation on the chart V. Similarly, since $(L_s(k) - \lambda)^T = L(k - i\beta_s) - \lambda$, one can write the operator $(L_s(k) - \lambda)^T$ on V as follows:

$$\sum_{|\alpha|\leq 2} \tilde{a}_\alpha(x)(D + (k - i\beta_s)^T \cdot \nabla\tilde{h})^\alpha,$$

for some functions $\tilde{a}_\alpha \in C^\infty(V)$. Then, on $\mathbb{T}^n$, the operator $\mathcal{Q}_s(k)$ has the following form:

$$\sum_{|\alpha|,|\beta|\leq 2} a_\alpha(x)\tilde{a}_\beta(x)(D + (k + i\beta_s)^T \cdot \nabla\tilde{h})^\alpha(D + (k - i\beta_s)^T \cdot \nabla\tilde{h})^\beta \nu(x) + (1 - \nu(x))\Delta^2.$$

Put

$$L_0^{(1)}(s,k;x,\xi) := \sum_{|\alpha|=2} a_\alpha(x)(\xi + (k + i\beta_s)^T \cdot \nabla\tilde{h})^\alpha,$$

$$L_0^{(2)}(s,k;x,\xi) := \sum_{|\beta|=2} \tilde{a}_\beta(x)(\xi + (k - i\beta_s)^T \cdot \nabla\tilde{h})^\beta$$

and

$$L_0(s,k;x,\xi) = \nu(x)L_0^{(1)}(s,k;x,\xi)L_0^{(2)}(s,k;x,\xi) + (1 - \nu(x))|\xi|^4. \tag{4.30}$$

Then the symbol $\sigma(s, k; x, \xi)$ of the operator $\mathcal{Q}_s(k)$ can be written as

$$L_0(s, k; x, \xi) + \tilde{\sigma}(s, k; x, \xi),$$

where the family of symbols $\{\tilde{\sigma}(s, k; x, \xi)\}_{(s,k)}$ is in the class $\tilde{S}^3(\mathbb{T}^n)$. Using the boundedness of $\nabla\tilde{h}$ and coefficients a_α on the support of ν, we deduce that the family of the symbols of $\{\mathcal{Q}_s(k)\}_{(s,k)}$ is in $\tilde{S}^4(\mathbb{T}^n)$. Thus, our remaining task is to find a constant $A > 0$ such that whenever $|\xi| > A$, we obtain $|L_0(s, k; x, \xi)| > 1$. Note that by ellipticity, there are positive constants θ_1, θ_2 such that

$$\sum_{|\alpha|=2} a_\alpha(x) \geq \theta_1 |\xi|^2$$

and

$$\sum_{|\alpha|=2} \tilde{a}_\alpha(x) \geq \theta_2 |\xi|^2.$$

We define

$$\|a\|_\infty := \sum_{|\alpha|=|\beta|=2} \|a_\alpha(\cdot)\|_{L^\infty(\mathrm{supp}(\nu))} + \|\tilde{a}_\beta(\cdot)\|_{L^\infty(\mathrm{supp}(\nu))}$$

and

$$A_p := \max_{(s,k,x) \in \mathbb{S}^{d-1} \times \mathcal{O} \times \mathrm{supp}(\nu)} \left(|k^T \cdot \nabla\tilde{h}|^2 + \theta_p^{-1} \|a\|_\infty |\beta_s^T \cdot \nabla\tilde{h}|^2 + \theta_p^{-1} \right), \quad p = 1, 2.$$

Suppose that $|\xi|^2 > 2 \max_{p=1,2} A_p$, then for any $p = 1, 2$, we have

$$\sqrt{\nu(x)}|L_0^{(p)}(s, k; x, \xi)| \geq \Re \left(\sqrt{\nu(x)} L_0^{(p)}(s, k; x, \xi) \right)$$

$$\geq \sqrt{\nu(x)} \left(\theta_p |\xi + k^T \cdot \nabla \tilde{h}|^2 - \sum_{|\alpha|=2} a_\alpha(x)(\beta_s^T \cdot \nabla \tilde{h})^\alpha \right)$$

$$\geq \sqrt{\nu(x)} \left(\theta_p \left(\frac{|\xi|^2}{2} - |k^T \cdot \nabla \tilde{h}|^2 \right) - \|a\|_\infty |\beta_s^T \cdot \nabla \tilde{h}|^2 \right)$$

$$\geq \sqrt{\nu(x)}.$$

Thus, due to (4.30), if $|\xi|^2 > 2\max_{p=1,2} A_p + 1$ then $|L_0(s, k; x, \xi)| \geq (\sqrt{\nu(x)})^2 + (1 - \nu(x))|\xi|^4 \geq 1$ as we wish. Now we are able to apply Theorem 4.7.12 to the family of operators $\{\mathcal{Q}_s(k)\}_{(s,k)}$, i.e., there are families of operators $\{\mathcal{B}_s(k)\}_{(s,k)} \in Op(\tilde{S}^{-4}(\mathbb{T}^n))$ and $\{\mathcal{R}_s(k)\}_{(s,k)} \in \mathcal{S}_{-\infty}(\mathbb{T}^n)$ such that $\mathcal{Q}_s(k)\mathcal{B}_s(k) = I - \mathcal{R}_s(k)$.

Let $\mathcal{A}_s(k) := (L_s(k) - \lambda)^T \nu \mathcal{B}_s(k)$. Since $\nu = 1$ on a neighborhood of $\overline{U}$, we obtain

$$r_U(I - \mathcal{R}_s(k)) = r_U \mathcal{Q}_s(k)\mathcal{B}_s(k)$$

$$= r_U \left((L_s(k) - \lambda)(L_s(k) - \lambda)^T \nu \mathcal{B}_s(k) + (1 - \nu)\Delta^2 \mathcal{B}_s(k) \right)$$

$$= r_U (L_s(k) - \lambda)(L_s(k) - \lambda)^T \nu \mathcal{B}_s(k)$$

$$= (L_s(k) - \lambda)r_U (L_s(k) - \lambda)^T \nu \mathcal{B}_s(k) = (L_s(k) - \lambda)r_U \mathcal{A}_s(k).$$

In addition, $\{\mathcal{A}_s(k)\}_{(s,k)} \in Op(\tilde{S}^{-2}(\mathbb{T}^n))$ according to the composition formula in Theorem 3.8.2. Hence, the lemma is proved. $\qquad\square$

Let $\mu_p \in C_c^\infty(U_p)$ $(p = 1, \ldots, N)$ be a smooth partition of unity with respect to the cover $\{U_p\}_{p=1,\ldots,N}$ and for any $p = 1, \ldots, N$, let ν_p be a smooth function in $C_c^\infty(U_p)$ such that it equals one on a neighborhood of $\mathrm{supp}(\mu_p)$. By Lemma 4.7.14, there are families of

operators $\{\mathcal{A}_s^{(p)}(k)\}_{(s,k)} \in Op(\tilde{S}^{-2}(\mathbb{T}^n))$ and $\{\mathcal{R}_s^{(p)}(k)\}_{(s,k)} \in \mathcal{S}_{-\infty}(\mathbb{T}^n)$ such that

$$(L_s(k) - \lambda)r_{U_p}\mathcal{A}_s^{(p)}(k) = r_{U_p}(I - \mathcal{R}_s^{(p)}(k)). \tag{4.31}$$

Due to pseudolocality, $(1 - \nu_p)\mathcal{A}_s^{(p)}(k)\mu_p \in \mathcal{S}_{-\infty}(\mathbb{T}^n)$. This implies that $r_{U_p}\mathcal{A}_s^{(p)}(k)\mu_p - \nu_p\mathcal{A}_s^{(p)}(k)\mu_p \in \mathcal{S}_{-\infty}(\mathbb{T}^n)$, and thus,

$$(L_s(k) - \lambda)r_{U_p}\mathcal{A}_s^{(p)}(k)\mu_p - (L_s(k) - \lambda)\nu_p\mathcal{A}_s^{(p)}(k)\mu_p \in \mathcal{S}_{-\infty}(\mathbb{T}^n).$$

By (4.31), $\mu_p - (L_s(k) - \lambda)r_{U_p}\mathcal{A}_s^{(p)}(k)\mu_p \in \mathcal{S}_{-\infty}(\mathbb{T}^n)$. Hence,

$$\mu_p I - (L_s(k) - \lambda)\nu_p\mathcal{A}_s^{(p)}(k)\mu_p \in \mathcal{S}_{-\infty}(\mathbb{T}^n).$$

Since both operators $\mu_p I$ and $(L_s(k) - \lambda)\nu_p\mathcal{A}_s^{(p)}(k)\mu_p$ are globally defined on the manifold M, it follows that

$$\sum_p \left(\mu_p I - (L_s(k) - \lambda)\nu_p\mathcal{A}_s^{(p)}(k)\mu_p\right) \in \mathcal{S}_{-\infty}(M). \tag{4.32}$$

Because $Op(\tilde{S}^{-2}(\mathbb{T}^n)) \subset \mathcal{S}_{-2}(\mathbb{T}^n)$ (see Remark 4.7.9), each family $\{\mathcal{A}_s^{(p)}(k)\}_{(s,k)}$ is in the class $\mathcal{S}_{-2}(\mathbb{T}^n)$ for every p. Since $\{\nu_p\mathcal{A}_s^{(p)}(k)\mu_p\}_{(s,k)}$ is globally defined on M, we also have $\{\nu_p\mathcal{A}_s^{(p)}(k)\mu_p\}_{(s,k)} \in \mathcal{S}_{-2}(M)$ for any p. Now define $A_s(k) := \sum_p \nu_p\mathcal{A}_s^{(p)}(k)\mu_p$ and $R_s(k) := I - (L_s(k) - \lambda)A_s(k)$. Then $\{A_s(k)\}_{(s,k)} \in \mathcal{S}_{-2}(M)$ and moreover, due to (4.32), the family of operators $\{R_s(k)\}_{(s,k)}$ is in $\mathcal{S}_{-\infty}(M)$. $\qquad\square$

The statement of the following lemma is standard.

Lemma 4.7.15. *Let $\mathcal{M}$ be a compact metric space, $\mathcal{D}$ be a domain in $\mathbb{R}^m$ ($m \in \mathbb{N}$) and H_1, H_2 be two infinite-dimensional separable Hilbert spaces. Let $\{T_s\}_{s\in\mathcal{M}}$ be a*

family of smooth maps from $\mathcal{D}$ to $B(H_1, H_2)$ such that for any multi-index α, the map $(s, d) \mapsto D_d^\alpha T_s(d)$ is continuous from $\mathcal{M} \times \mathcal{D}$ to $B(H_1, H_2)$. Suppose that there is a family of maps $\{V_s\}_{s \in \mathcal{M}}$ from $\mathcal{D}$ to $B(H_2, H_1)$ such that $V_s(d)T_s(d) = 1_{H_1}$ and $T_s(d)V_s(d) = 1_{H_2}$ for any $(s, d) \in \mathcal{M} \times \mathcal{D}$. Then for each $s \in \mathcal{M}$, the map $d \in \mathcal{D} \mapsto V_s(d)$ is smooth as a $B(H_2, H_1)$-valued function. Furthermore for any multi-index α, the map $(s, k) \mapsto D_d^\alpha V_s(d)$ is continuous on $\mathcal{M} \times \mathcal{D}$ as a $B(H_2, H_1)$-valued function.

We now go back to the family of operators $\{T_s(k)\}_{(s,k) \in \mathbb{S}^{d-1} \times \mathcal{O}}$. The next statement is the main ingredient in establishing Proposition 4.7.3.

Proposition 4.7.16. *There is a family of operators $\{B_s(k)\}_{(s,k)}$ in $\mathcal{S}_{-2}(M)$ such that the family of operators $\{T_s(k) - B_s(k)\}_{(s,k)}$ belongs to $\mathcal{S}_{-\infty}(M)$.*

Proof. Using Theorem 4.7.13, we can find a family $\{A_s(k)\}_{(s,k)} \in \mathcal{S}_{-2}(M)$ and a family $\{R_s(k)\}_{(s,k)} \in \mathcal{S}_{-\infty}(M)$ such that

$$(L_s(k) - \lambda)A_s(k) = I - R_s(k).$$

From the definition of $T_s(k)$, we obtain $T_s(k)(L_s(k) - \lambda) = I - \eta(k)P(k + i\beta_s)$.

Using these equalities, we deduce

$$T_s(k) = A_s(k) - \eta(k)P(k + i\beta_s)A_s(k) + T_s(k)R_s(k).$$

We recall from Section 4.4 that $P(k+i\beta_s)$ projects $L^2(M)$ onto the eigenspace spanned by the eigenfunction $\phi_{k+i\beta_s}$. Hence, its kernel is the following function

$$\frac{\phi(k + i\beta_s)(x)\overline{\phi(k - i\beta_s)(y)}}{(\phi(k + i\beta_s), \phi(k - i\beta_s))_{L^2(M)}},$$

which is smooth due to Lemma 4.5.1. Thus, the family of operators $\{\eta(k)P(k + i\beta_s)\}_{(s,k)}$

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is in $\mathcal{S}_{-\infty}(M)$. Also, the family of operators $\{\eta(k)Q(k+i\beta_s)\}_{(s,k)}$ belongs to $\mathcal{S}_0(M)$.

We put $B_s(k) := A_s(k) - \eta(k)P(k+i\beta_s)A_s(k)$, then $\{B_s(k)\}_{(s,k)} \in \mathcal{S}_{-2}(M)$. Since $T_s(k) - B_s(k) = T_s(k)R_s(k)$, the remaining task is to check that the family of operators $\{T_s(k)R_s(k)\}_{(s,k)}$ belongs to the class $\mathcal{S}_{-\infty}(M)$.

Let us consider any two real numbers m_1 and m_2. By Lemma 4.2.13, the operators $L_s(k) - \lambda$ and $L_s(k)Q(k+i\beta_s) - \lambda$ are smooth in k as $B(H^{m_2}(M), H^{m_2-2}(M))$-valued functions such that their derivatives with respect to k are jointly continuous in (s,k). On the other hand, we can rewrite (see Lemma 3.65):

$$T_s(k) = (1-\eta(k))(L_s(k)-\lambda)^{-1} + \eta(k)\lambda^{-1}P(k+i\beta_s) + \eta(k)(L_s(k)Q(k+i\beta_s)-\lambda)^{-1}.$$

Hence, by Lemma 4.7.15, $T_s(k)$ is smooth in k as a $B(H^{m_2-2}(M), H^{m_2}(M))$-valued function and its derivatives with respect to k are jointly continuous in (s,k). Therefore, for any multi-index α, we have

$$\sup_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}} \|D_k^\alpha T_s(k)\|_{B(H^{m_2-2}(M),H^{m_2}(M))} < \infty.$$

Moreover since $\{R_s(k)\}_{(s,k)} \in \mathcal{S}_{-\infty}(M)$, $R_s(k)$ is smooth as a $B(H^{m_1}(M), H^{m_2-2}(M))$-valued function and for any multi-index α,

$$\sup_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}} \|D_k^\alpha R_s(k)\|_{B(H^{m_1}(M),H^{m_2-2}(M))} < \infty.$$

From the Leibniz rule, the composition $T_s(k)R_s(k)$ is smooth as a $B(H^{m_1}(M), H^{m_2}(M))$-valued function and for any multi-index α, the following uniform condition also holds

$$\sup_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}} \|D_k^\alpha(T_s(k)R_s(k))\|_{B(H^{m_1}(M),H^{m_2}(M))} < \infty.$$

Consequently, $\{T_s(k)R_s(k)\}_{(s,k)\in\mathbb{S}^{d-1}\times\mathcal{O}} \in \mathcal{S}_{-\infty}(M)$ as we wish. $\qquad\square$

We finish this subsection.

Proof of Proposition 4.7.3.

Proof. Proposition 4.7.16 provides us with the decomposition $T_s(k) = B_s(k) + C_s(k)$, where $\{B_s(k)\}_{(s,k)} \in \mathcal{S}_{-2}(M)$ and $\{C_s(k)\}_{(s,k)} \in \mathcal{S}_{-\infty}(M)$. Let $K_{B_s(k)}$, $K_{C_s(k)}$ be the Schwartz kernels of $B_s(k)$ and $C_s(k)$, correspondingly. It follows from applying Corollary 4.7.11 that for any multi-index α satisfying $|\alpha| \geq n$, the kernel $D_k^\alpha B_s(k)$ is continuous on $M \times M$ and

$$\sup_{(s,k,x,y)} |D_k^\alpha K_{B_s(k)}(k,x,y)| < \infty.$$

A similar conclusion also holds for the kernels $D_k^\alpha C_s(k)$ and thus, for $D_k^\alpha K_s(k,x,y)$ too.

$\qquad\square$

4.8 Concluding remarks

- The asymptotics (4.7) and (4.9) can be described in terms of the Albanese map and the Albanese pseudo-distance (see [46, Section 2]), provided that the additive function h is chosen to be harmonic (see also Appendix A).

- The main results in this chapter can be easily carried over to the case when the band edge occurs at finitely many quasimomenta k_0 in the Brillouin zone (instead of assuming the condition **A3**) by summing the asymptotics coming from all these non-degenerate isolated extrema.

  It was shown recently in [25] that for a wide class of two dimensional periodic elliptic second-order operators (including the class of operators we consider in this paper and periodic magnetic Schrödinger operators in $2D$), the extrema of any spectral band function (not necessarily spectral edges) are attained on a finite set of values of the quasimomentum in the Brillouin zone.

- The proofs of our results go through verbatim for periodic elliptic second-order operators acting on vector bundles over the abelian covering X.

5.   A LIOUVILLE-RIEMANN-ROCH THEOREM ON ABELIAN COVERINGS.

5.1   Introduction

The classical Riemann-Roch formula is an important result that connects the dimension of a space of holomorphic functions and the dimension of a space of meromorphic $(1, 0)$-forms on a compact Riemann surface. These spaces are linked with the notion of "divisors", which provide constraints limiting where poles (i.e., singularities) and zeros can be and of what order (i.e., multiplicities). The Riemann-Roch formula gives an equality between the difference of these two dimensions (i.e., the index of the $\bar{\partial}$ operator) and the degree of the divisor, i.e. the total sum of multiplicities.

Since then, many generalizations of the Riemann-Roch formula have been discovered from various viewpoints. Among these, Gromov and Shubin established their Riemann-Roch type theorems for elliptic differential operators on compact and non-compact manifolds in [32, 33]. Namely, these results link the dimensions of spaces of solutions of elliptic equations on compact manifolds with the prescribed divisor of allowed poles and required zeros. As in the classical case, the differences between these dimensions are the same as the degrees of the corresponding divisors. Roughly speaking, one can say that the degree of a divisor is the difference between the indices of the operators with and without required zeros and singularities.

On the other hand, Liouville type theorems count the dimension of the space of solutions growing polynomially of a given order at infinity. Such solutions can be considered as ones with poles at infinity. This dimension can be interpreted as an index, since in this situation the co-dimension of the range turns out the be equal to zero.

The obvious similarity of the questions (as well as some results and proofs) suggest to one to seek a combination of these results. This is what we attempt in this chapter.

In Sections 5.2 and 5.3, we introduce the necessary preliminaries concerning Liouville and Riemann-Roch type theorems, reminding some notions and results from [49, 50] and Gromov-Shubin's works [32, 33]. In Section 5.4, we obtain the main results that combine the Liouville and Riemann-Roch statements. It is interesting that the combination is non-trivial, and the results are not easily predictable. Proofs of these main results are provided in Section 5.5. In Section 5.6, we provide the realizations of the main theorems in several specific situations. Some technical details and proofs are delegated to Section 5.7.

Besides the non-trivial combination of the Liouville and Riemann-Roch theorems, some extensions of these individual theorems are also obtained.

5.2 Some preliminaries for Liouville type results

5.2.1 Some preliminaries for periodic elliptic operators on abelian coverings

Let X be a noncompact smooth Riemannian manifold of dimension n equipped with an isometric, properly discontinuous, free, and co-compact action of a finitely generated abelian discrete group G. The compact orbit space is $M := X/G$.

Remark 5.2.1. No harm will be done, if the reader assumes that $X = \mathbb{R}^d$ and M is the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The results are new in this case as well. The only warning is that in this situation the dimension of X and the rank of the group $\mathbb{Z}^d$ coincide, while this is not required in general.

Let μ_M be the Riemannian measure of M and μ_X be its lifting to X. Thus, μ_X is a G-invariant. We also use the notation $L^2(X)$ (see Chapter 4) to denote the space of L^2-functions on X with respect to μ_X.

We consider the G-invariant bilinear [1] duality

$$\langle \cdot, \cdot \rangle : C_c^\infty(X) \times C^\infty(X) \to \mathbb{C}, \quad \langle f, g \rangle = \int_X f(x)g(x)\, \mathrm{d}\mu_X. \qquad (5.1)$$

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[1]One can also consider the sesquilinear form to obtain analogous results.

By continuity, we can extend it to a G-invariant bilinear non-degenerate duality

$$\langle \cdot, \cdot \rangle : L^2(X) \times L^2(X) \to \mathbb{C}. \tag{5.2}$$

Let A be an elliptic operator of order m on X with smooth coefficients. In this chapter, we will be assuming that A is G-**periodic**. Note that then A can be pushed down to an elliptic operator on M. We will assume in most cases (except rare non-self-adjoint considerations) that the operator A is bounded below.

The formal adjoint operator (transpose) A^* (with respect to the bilinear duality (5.1)) has similar properties. In particular, A^* is also a periodic elliptic operator of order m on X.

Note that since G is a finitely generated abelian group, G is the direct sum of a finite abelian group and $\mathbb{Z}^d$, where d is the rank of the torsion free subgroup of G. In what follows, without any effect on the results, we could replace M by the compact Riemannian manifold $X/\mathbb{Z}^d$ and thus, we can work with $\mathbb{Z}^d$ as our new deck group[2]. Therefore, we assume henceforward that $G = \mathbb{Z}^d$, where $d \in \mathbb{N}$.

We remind the reader that the reciprocal lattice G^* of the deck group $G = \mathbb{Z}^d$ is $(2\pi\mathbb{Z})^d$ and the Brillouin zone $B = [-\pi, \pi]^d$ is chosen as its fundamental domain in this chapter. The quotient $\mathbb{R}^d/G^*$ is the dual torus $(\mathbb{T}^*)^d$. So, G^*-periodic functions on $\mathbb{R}^d$ can be naturally identified with functions on $(\mathbb{T}^*)^d$.

For any **quasimomentum** $k \in \mathbb{C}^d$, let γ_k be the **character** of the deck group G defined as $\gamma_k(g) = e^{ik \cdot g}$ (for a given character, the corresponding quasimomentum is defined modulo the reciprocal lattice). If k is real, γ_k is unitary and vice versa. Abusing the notations slightly, we will sometimes identify each unitary character of $\mathbb{Z}^d$ with a quasimomentum k in the dual group $(\mathbb{T}^*)^d$.

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[2]This is exactly what we do in Chapter 4: One can always eliminate the torsion part of G by switching to a subcovering $X \to X/\mathbb{Z}^d$.

We denote by $L^2_k(X)$ the space of all γ_k-automorphic function $f(x)$ in $L^2_{loc}(X)$, i.e. such that

$$f(g \cdot x) = \gamma_k(g)f(x), \text{ for a.e } x \in X \text{ and } g \in G. \tag{5.3}$$

It is convenient at this moment to introduce, given a quasimomentum k, the following line (i.e., one-dimensional) vector bundle E_k over M:

Definition 5.2.2. Given any $k \in \mathbb{C}^d$, we denote by E_k the bundle associated with the principal G-bundle X over M with the character γ_k. Namely, we consider the free left action of G on the Cartesian product $X \times \mathbb{C}$ given by

$$g \cdot (x, z) = (g \cdot x, \gamma_k(g)z), \quad (g, x, z) \in G \times X \times \mathbb{C}.$$

Now E_k is defined as the orbit space of this action and the canonical projection $X \times \mathbb{C} \to M$ descends to the surjective mapping $E_k \to M$.

From the definition, E_k is a vector bundle over M with fiber $\mathbb{C}$ (see e.g., [54]) and any L^2-section of the bundle E_k must satisfy the quasi-periodic property (5.3). Now the space $L^2_k(X)$ can be identified with the space of all L^2-sections of the bundle E_k.

This construction can be easily generalized to Sobolev spaces:

Definition 5.2.3. For any quasimomentum $k \in \mathbb{C}^d$ and any real number s, we denote by $H^s_k(X)$ the closed subspace of $H^s_{loc}(X)$ consisting of γ_k-automorphic functions. Then $H^s_k(X)$ is a Hilbert space equipped with a natural inner product arising from the inner product of the Sobolev space $H^s(\overline{\mathcal{F}})$, where $\mathcal{F}$ is any fixed fundamental domain for the action of the group G on X. The topology of the Hilbert space $H^s_k(X)$ does not depend on the choice of the fundamental domain $\mathcal{F}$.

Equivalently, the space $H^s_k(X)$ can be identified with $H^s(E_k)$, the space of all H^s-sections of the bundle E_k.

For any k, the periodic operator A maps $H_k^m(X)$ into $L_k^2(X)$. Thus, the restrictions of A to these subspaces $H_k^m(X)$ define elliptic operators $A(k)$ on the spaces of sections of the bundles E_k over the compact manifold M. When A is self-adjoint and k is real, the operator $A(k)$, with the space $H^m(E_k)$ as the domain, is an unbounded, bound from below self-adjoint operator in $L^2(E_k)$.

In this chapter, we use a version of the Floquet transform introduced in the previous chapters as follows

$$f(x) \mapsto \mathbf{F}f(k,x) = \sum_{g \in G} f(g \cdot x)\overline{\gamma_k(g)} = \sum_{g \in G} f(g \cdot x)e^{-ik \cdot g}, \quad k \in \mathbb{C}^d. \qquad (5.4)$$

For reader's convenience, we recall some basic properties of this Floquet transform in Section 5.7. As one would expect, this transform decomposes the original operator A on the non-compact manifold X into a direct integral of operators $A(k)$ acting on sections of line bundles E_k over torus, which is compact:

$$A = \int_{(\mathbb{T}^*)^d}^{\oplus} A(k)\, \mathrm{d}k, \quad L^2(X) = \int_{(\mathbb{T}^*)^d}^{\oplus} L^2(E_k)\, \mathrm{d}k. \qquad (5.5)$$

Here the measure $\mathrm{d}k$ is the normalized Haar measure of $(\mathbb{T}^*)^d$, which can be also considered as the normalized Lebesgue measure on the Brillouin zone B. Therefore, the union of the spectra of operators $A(k)$ over the torus $(\mathbb{T}^*)^d$ is the spectrum of the periodic operator A. Finally, the notions of Bloch variety, dispersion relations, Fermi surfaces of the operator A are defined similarly as in Chapter 2 and Chapter 4 (see Definition 2.2.5). Note that most of the fundamental properties of these notions in the flat case still hold in the abelian covering case, e.g., Proposition 2.2.6.

5.2.2 Floquet-Bloch solutions and Liouville theorem on abelian coverings

In this section, we introduce the notions of Bloch and Floquet solutions of periodic PDEs and then state the Liouville theorem of [50].

Definition 5.2.4. For any $g \in G$ and quasimomentum $k \in \mathbb{C}^d$, we denote by $\Delta_{g;k}$ the "k-twisted" version of the first difference operator acting on functions on the covering X as follows

$$\Delta_{g;k}u(x) = e^{-ik \cdot g}u(g \cdot x) - u(x). \tag{5.6}$$

The iterated "twisted" finite differences of order N with quasimomentum k are defined as

$$\Delta_{g_1,\ldots,g_N;k} = \Delta_{g_1;k}\ldots\Delta_{g_N;k}, \quad \text{for} \quad g_1,\ldots,g_N \in G. \tag{5.7}$$

Definition 5.2.5. A function u on X is a **Floquet function of order** N **with quasimomentum** k if any finite difference of order $N+1$ with quasimomentum k annihilates u. Also, a **Bloch function with quasimomentum** k is a Floquet function of order 0 with quasimomentum k.

According to this definition, a Bloch function $u(x)$ with quasimomentum k is a γ_k-automorphic function on X, i.e., $u(g \cdot x) = e^{ik \cdot g}u(x)$ for any $g \in G$. Thus, if u is a continuous Bloch function with a *real* quasimomentum, then $u \in L^2_{loc}(X)$ and for any compact subset K of X, the sequence $\{\|u\|_{L^2(gK)}\}_{g \in G}$ is bounded (i.e., belongs to $\ell^\infty(G)$).

It is also known [50] that $u(x)$ is a Floquet function of order N with quasimomentum k if and only if u can be represented in the form

$$u(x) = e_k(x)\left(\sum_{|j| \leq N}[x]^j p_j(x)\right),$$

where $j = (j_1,\ldots,j_d) \in \mathbb{Z}_+^d$, and the functions p_j are G-periodic. Here for any $j \in \mathbb{Z}^d$,

we define

$$|j| := |j_1| + \dots |j_d|, \tag{5.8}$$

while $e_k(x)$ and $[x]^j$ are analogs of the exponential e^{ikx} and the monomial x^j on $\mathbb{R}^d$ (see [50] for details). For convenience, in Section 5.7, we collect some basic facts of Floquet functions on abelian coverings. In the flat case $X = \mathbb{R}^d$, a Floquet function of order N with quasimomentum k is the product of the plane wave e^{ikx} and a polynomial of degree N with G-periodic coefficients.

An important consequence of this representation is that any Floquet function $u(x) \in L^2_{loc}(X)$ of order N with a real quasimomentum satisfies the following L^2-growth estimate

$$\|u\|_{L^2(gK)} \leq C(1 + |g|)^N, \quad \forall g \in G \quad \text{and} \quad K \Subset X.$$

Here $|g|$ is defined according to (5.8), and we have used the Švarc-Milnor lemma from geometric group theory (see e.g., [57, Lemma 2.8]) to conclude that on a Riemannian co-compact covering X, the Riemannian distance between any compact subset K and its g-translation gK is comparable with $|g|$.

If u is continuous, the above L^2-growth estimate can be replaced by the corresponding L^∞-growth estimate.

We now need to introduce the spaces of polynomially growing solutions of the equation $Au = \lambda u$. To simplify the notations, we will assume from now on that $\lambda = 0$, since, as we discussed before, we can deal with the operator $A - \lambda$ instead of A (see also Definition 2.2.5).

Definition 5.2.6. Let $K \Subset X$ be a domain such that X is the union of all G-translations of K, i.e.,

$$X = \bigcup_{g \in G} gK. \tag{5.9}$$

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For any $s, N \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the vector spaces

$$V_N^p(X) := \left\{ u \in C^\infty(X) \mid \{\|u\|_{L^2(gK)} \cdot \langle g \rangle^{-N}\}_{g \in G} \in \ell^p(G) \right\},$$

and

$$V_N^p(A) := \left\{ u \in V_N^p(X) \mid Au = 0 \right\}.$$

Here $\langle g \rangle := (1 + |g|^2)^{1/2}$.

It is clear that these spaces are independent of the choice of the compact subset K satisfying (5.9). In particular, one can take as K a fundamental domain for G-action on X. Moreover, we have $V_{N_1}^{p_1}(X) \subseteq V_{N_2}^{p_2}(X)$ and $V_{N_1}^{p_1}(A) \subseteq V_{N_2}^{p_2}(A)$ whenever $N_1 \leq N_2$ and $p_1 \leq p_2$.

Definition 5.2.7. For $N \geq 0$, we say that the **Liouville theorem of order** (p, N) **holds for A**, if the space $V_N^p(A)$ is finite dimensional.

Now we can restate one of the main results in [50] as follows

Theorem 5.2.8. [50]

   (i) The following statements are equivalent:

      (1) The cardinality of the real Fermi surface $F_{A,\mathbb{R}}$ is finite modulo G^*-shifts, i.e., Bloch solutions exist for only finitely many unitary characters γ_k.

      (2) The Liouville theorem of order (∞, N) holds for A for some $N \geq 0$.

      (3) The Liouville theorem of order (∞, N) holds for A for all $N \geq 0$.

  (ii) Suppose that the Liouville theorem holds for A. Then for any $N \in \mathbb{N}$, each solution $u \in V_N^\infty(A)$ can be represented as a finite sum of Floquet solutions:

$$u(x) = \sum_{k \in F_{A,\mathbb{R}}} \sum_{0 \leq j \leq N} u_{k,j}(x), \tag{5.10}$$

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where each $u_{k,j}$ is a Floquet solution of order j with a quasimomentum k.

(iii) *A crude estimate of the dimension of $V_N^\infty(A)$:*

$$\dim V_N^\infty(A) \leq \binom{d+N}{N} \cdot \sum_{k \in F_{A,\mathbb{R}}} \dim \operatorname{Ker} A(k) < \infty.$$

Due to the relation between $F_{A,\mathbb{R}}$ and $F_{A^*,\mathbb{R}}$ (see Proposition 2.2.6), the Liouville theorem holds for A if and only if it also holds for A^*.

5.2.3   Explicit formulas for dimensions of spaces $V_N^\infty(A)$

In order to obtain explicit formulas for the dimensions of $V_N^\infty(A)$, we need to remind some notions from [50]. Recall that, for each quasimomentum k, $A(k)$ is in the space $\mathcal{L}(H_k^m(X), L_k^2(X))$ of bounded linear operators acting from $H_k^m(X)$ to $L_k^2(X)$. For a real number s, the spaces $H_k^s(X)$ are the fibers of the following analytic G^*-periodic Hilbert vector bundle over $\mathbb{C}^d$:

$$\mathcal{E}^s := \bigcup_{k \in \mathbb{C}^d} H_k^s(X) = \bigcup_{k \in \mathbb{C}^d} H^s(E_k). \tag{5.11}$$

Consider a quasimomentum k_0 in $F_{A,\mathbb{R}}$. We can trivialize [3] the vector bundle $\mathcal{E}^m$, so that in a neighborhood of k_0, $A(k)$ becomes an analytic family of bounded operators from $H_{k_0}^m$ to $L_{k_0}^2$ (see Subsection 5.7.2). Suppose that the spectra of operators $A(k)$ are discrete for any value of the quasimomentum k.

Assume now that zero is an eigenvalue of the operator $A(k_0)$ with algebraic multiplicity r. Let Υ be a contour in $\mathbb{C}$ separating 0 from the rest of the spectrum of $A(k_0)$. Due to Proposition 5.7.3, we can pick a small neighborhood of k_0 such that the contour Υ does not intersect with $\sigma(A(k))$ for any k in this neighborhood. We denote by $\Pi(k)$ the

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[3] In fact, $\mathcal{E}^m$ is analytically trivializable (see e.g., [47,82]) although we do not need this fact here.

r-dimensional spectral Riesz projector for the operator $A(k)$, associated with the contour Υ. Now one can pick an orthonormal basis $\{e_j\}_{1 \leq j \leq r}$ in the range of $\Pi(k_0)$ and define $e_j(k) := \Pi(k)e_j$. Then let us consider the $r \times r$ matrix $\lambda(k)$ of the operator $A(k)\Pi(k)$ in the basis $\{e_j(k)\}$, i.e.,

$$\lambda_{ij}(k) = \langle A(k)e_j(k), e_i(k) \rangle = \langle A(k)\Pi(k)e_j, e_i \rangle. \tag{5.12}$$

Our considerations in this part will not change if we multiply $\lambda(k)$ by an invertible matrix function analytic in a neighborhood of k_0.

Remark 5.2.9. An important special case is when $r = 1$ near k_0. Then $\lambda(k)$ is just the band function that vanishes at k_0.

Now, using the Taylor expansion around k_0, we decompose $\lambda(k)$ into the series of homogeneous matrix valued polynomials:

$$\lambda(k) = \sum_{j \geq 0} \lambda_j(k - k_0), \tag{5.13}$$

where each λ_j is a $\mathbb{C}^{r \times r}$-valued homogeneous polynomial of degree j in d variables.

For each quasimomentum $k_0 \in F_{A,\mathbb{R}}$, let $\ell_0(k_0)$ be the order of **the first non-zero term** of the Taylor expansion (5.13) around k_0 of the matrix function $\lambda(k)$.

The next result, extracted from [50], provides explicit formulas for dimensions of the spaces $V_N^\infty(A)$.

Theorem 5.2.10. [50] *Suppose that the real Fermi surface $F_{A,\mathbb{R}}$ is finite (modulo G^*-shifts) and the spectrum $\sigma(A(k))$ is discrete for any quasimomentum k.*

(a) For each integer $0 \leq N < \min\limits_{k \in F_{A,\mathbb{R}}} \ell_0(k)$, we have

$$\dim V_N^\infty(A) = \sum_{k \in F_{A,\mathbb{R}}} m_k \left[\binom{d+N}{d} - \binom{d+N-\ell_0(k)}{d} \right], \qquad (5.14)$$

where m_k is the algebraic multiplicity of the zero eigenvalue of the operator $A(k)$.

(b) If for every $k \in F_{A,\mathbb{R}}$, $\det \lambda_{\ell_0(k)}$ is not identically equal to zero, the formula (5.14) holds for any $N \geq 0$.

The reader might wonder what does the binomial coefficient $\binom{d+N-\ell_0(k)}{d}$ mean if $d + N - \ell_0(k)$ happens to be negative. We adopt the following agreement:

Definition 5.2.11. If in some formulas throughout this text one encounters a binomial coefficient $\binom{M}{N}$, where the difference $M - N$ is negative, we define its value to be equal to zero.

It is worthwhile to note that the positivity of $\ell_0(k)$ is equivalent to the fact that both algebraic and geometric multiplicities of the zero eigenvalue of the operator $A(k)$ are the same. Also, the non-vanishing of the determinant of $\lambda_{\ell_0(k)}$ implies that $\ell_0(k) > 0$.

5.2.4 A characterization of the spaces $V_N^p(A)$

Notation 5.2.12. For a real number r, we denote by $\lfloor r \rfloor$ the largest integer that is strictly less than r, while $[r]$ denotes the largest integer that is less or equal than r.

The following statement follows from Theorem 5.2.8 (ii):

Lemma 5.2.13. $V_N^\infty(A) = V_{[N]}^\infty(A)$ *for any non-negative real number N.*

The proofs of the next two theorems are delegated to the Subsection 5.7.6.

Theorem 5.2.14. *For each $1 \leq p < \infty$ such that $pN > d$, one has*

$$V^{\infty}_{\lfloor N - d/p \rfloor}(A) \subseteq V^p_N(A).$$

If, additionally, the Fermi surface $F_{A,\mathbb{R}}$ is finite modulo G^*-shifts, then

$$V^p_N(A) = V^{\infty}_{\lfloor N - d/p \rfloor}(A).$$

Corollary 5.2.15. *If the Fermi surface $F_{A,\mathbb{R}}$ is finite modulo G^*-shifts, then Liouville theorem of order (p, N) for a pair (p, N) such that $pN > d$ holds for A if and only if the Liouville theorem of order (∞, N) holds for A for some $N \geq 0$, and thus, according to Theorem 5.2.8, for all $N \geq 0$.*

The following theorem could be regarded as a version of the unique continuation property at infinity for the periodic elliptic operator A.

Theorem 5.2.16. *Assume that $F_{A,\mathbb{R}}$ is finite (modulo G^*-shifts). Then the space $V^p_N(A)$ is trivial if either one of the following conditions holds:*

(a) $p \neq \infty, N \leq d/p$.

(b) $p = \infty, N < 0$.

In fact, a more general version of these results is:

Theorem 5.2.17. *Let $p \in [1, \infty)$. Let also ϕ be a continuous, positive function defined on $\mathbb{R}^+$ such that*

$$N_{p,\phi} := \sup \left\{ N \in \mathbb{Z} : \int_0^{\infty} \phi(r)^{-p} \cdot \langle r \rangle^{pN+d-1} \, \mathrm{d}r < \infty \right\} < \infty.$$

We define $\mathcal{V}_\phi^p(A)$ as the space of all solutions u of $A = 0$ satisfying the condition

$$\sum_{g \in \mathbb{Z}^d} \|u\|^p_{L^2(gK)} \cdot \phi(|g|)^{-p} < \infty$$

holds for some compact domain K satisfying (5.9).

If $F_{A,\mathbb{R}}$ is finite (modulo G^*-shifts), then one has

- *If $N_{p,\phi} \geq 0$, then $\mathcal{V}_\phi^p(A) = V_{N_{p,\phi}}^\infty(A)$.*

- *If $N_{p,\phi} < 0$, then $\mathcal{V}_\phi^p(A) = \{0\}$.*

Note that if $\phi(r) = \langle r \rangle^N$, then $\mathcal{V}_\phi^p(A) = V_N^p(A)$.

The proofs of Theorems 5.2.14 and 5.2.16 (provided in Subsection 5.7.6) easily transfer to this general version. We thus skip the proof of Theorem 5.2.17.

Remark 5.2.18. It is worthwhile to note that results of this section do not require the assumption of discreteness of spectra of the operators $A(k)$. This is useful, in particular, when considering overdetermined problems.

5.3 The Gromov-Shubin version of the Riemann-Roch theorem for elliptic operators on noncompact manifolds

We follow here closely the paper [33] by M. Gromov and M. Shubin, addressing only its parts that are relevant for our considerations.

5.3.1 Some notions and preliminaries

Through this section, P will denote a linear elliptic differential expressions with smooth coefficients on a non-compact manifold $\mathcal{X}$ (later on, $\mathcal{X}$ will be the space of an abelian co-compact covering). We denote by P^* its transpose elliptic differential operator, defined via the identity

$$\langle Pu, v \rangle = \langle u, P^*v \rangle, \quad \forall u, v \in C_c^\infty(\mathcal{X}),$$

where $\langle \cdot, \cdot \rangle$ is the bilinear duality (5.1).

We notice that both P and P^* can be applied as differential expressions to any smooth function on $\mathcal{X}$ and these operations keep the spaces $C^\infty(\mathcal{X})$ and $C_c^\infty(\mathcal{X})$ invariant.

We assume that P and P^* are defined as operators on domains $\mathrm{Dom}\, P$ and $\mathrm{Dom}\, P^*$, such that

$$C_c^\infty(\mathcal{X}) \subseteq \mathrm{Dom}\, P \subseteq C^\infty(\mathcal{X}), \tag{5.15}$$

$$C_c^\infty(\mathcal{X}) \subseteq \mathrm{Dom}\, P^* \subseteq C^\infty(\mathcal{X}). \tag{5.16}$$

Definition 5.3.1. We denote by $\mathrm{Im}\, P$ and $\mathrm{Im}\, P^*$ the ranges of P and P^* on their corresponding domains, i.e.

$$\mathrm{Im}\, P = P(\mathrm{Dom}\, P), \quad \mathrm{Im}\, P^* = P^*(\mathrm{Dom}\, P^*). \tag{5.17}$$

As usual, $\mathrm{Ker}\, P$ and $\mathrm{Ker}\, P^*$ denote the spaces of solutions of the equations $Pu = 0$, $P^*u = 0$ in $\mathrm{Dom}\, P$ and $\mathrm{Dom}\, P^*$ respectively.

We also need to define some auxiliary spaces[4]. Namely, assume that we can choose linear subspaces[5] $\mathrm{Dom}'\, P$ and $\mathrm{Dom}'\, P^*$ of $C^\infty(\mathcal{X})$ so that

$(\mathcal{P}1)$

$$C_c^\infty(\mathcal{X}) \subseteq \mathrm{Dom}'\, P \subseteq C^\infty(\mathcal{X}), \tag{5.18}$$

$$C_c^\infty(\mathcal{X}) \subseteq \mathrm{Dom}'\, P^* \subseteq C^\infty(\mathcal{X}), \tag{5.19}$$

and

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[4]Most of the complications in definitions here and below come from non-compactness of the manifold.

[5]The notation Dom' might confuse the reader, leading her to thinking that this is a different domain of the operator. It is rather an object dual to the domain Dom.

($\mathcal{P}2$)

$$\operatorname{Im} P^* \subseteq \operatorname{Dom}' P, \quad \operatorname{Im} P \subseteq \operatorname{Dom}' P^*.$$

($\mathcal{P}3$) The bilinear pairing $\int_{\mathcal{X}} f(x)g(x) \, d\mu_X$ (see (5.1)) makes sense for functions from the relevant spaces, to define the pairings

$$\langle \cdot, \cdot \rangle : \operatorname{Dom}' P^* \times \operatorname{Dom} P^* \mapsto \mathbb{C}, \quad \langle \cdot, \cdot \rangle : \operatorname{Dom} P \times \operatorname{Dom}' P \mapsto \mathbb{C},$$

so that

($\mathcal{P}4$) The duality ("integration by parts formula")

$$\langle Pu, v \rangle = \langle u, P^* v \rangle, \quad \forall u \in \operatorname{Dom} P, \ v \in \operatorname{Dom} P^*$$

holds.

We also need an appropriate notion of a polar (annihilator) to a subspace:

Definition 5.3.2. For a subspace $L \subset \operatorname{Dom} P$, its **annihilator** L° is the subspace of $\operatorname{Dom}' P$ consisting of all elements of $\operatorname{Dom}' P$ that are orthogonal to L with respect to the pairing $\langle \cdot, \cdot \rangle$:

$$L^\circ = \{u \in \operatorname{Dom}' P \mid \langle v, u \rangle = 0, \text{ for any } v \in \operatorname{Dom} P\}.$$

Analogously, M° is the annihilator in $\operatorname{Dom}' P^*$ of a linear subspace $M \subset \operatorname{Dom} P^*$ with respect to $\langle \cdot, \cdot \rangle$.

Following [33], we now introduce an appropriate for our goals notion of Fredholm type operators.

Definition 5.3.3. P is a **Fredholm operator on** $\mathcal{X}$ if the following requirements are satisfied:

(i)
$$\dim \operatorname{Ker} P < \infty, \dim \operatorname{Ker} P^* < \infty$$

and

(ii)
$$\operatorname{Im} P = (\operatorname{Ker} P^*)^\circ.$$

Then the **index** of P is defined as

$$\operatorname{ind} P = \dim \operatorname{Ker} P - \operatorname{codim} \operatorname{Im} P = \dim \operatorname{Ker} P - \dim \operatorname{Ker} P^*.$$

5.3.2 Point divisors

We need to recall the rather technical notion of a **rigged divisor** from [33]. However, for reader's sake, we start with more familiar and easier to comprehend particular case of a **point divisor**, which appeared initially in the first version of Gromov-Shubin's analog of the Riemann-Roch formula [32].

Definition 5.3.4. A **point divisor** μ on X consists of two finite disjoint subsets of X

$$D^+ = \{x_1, \ldots, x_r\}, D^- = \{y_1, \ldots, y_s\} \tag{5.20}$$

and two tuplets $0 < p_1, \ldots, p_r$ and $q_1, \ldots, q_s < 0$ of integers[6]. The **support of the point divisor** μ consists of the points $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$. We will also write $\mu = x_1^{p_1} \ldots x_r^{p_r} \cdot y_1^{q_1} \ldots y_s^{q_s}$.

---

[6]In other words, μ is an element of the free abelian group generated by points of $\mathcal{X}$.

In [32], such a divisor is used to allow solutions of an elliptic equation $Pu = 0$ to have poles up to certain orders at the points of D^+ and zeros on D^-. Namely,

(i) For any $1 \leq j \leq r$, there exists an open neighborhood U_j of x_j such that on $U_j \setminus \{x_j\}$, one has $u = u_s + u_r$, where $u_r \in C^\infty(U_j)$, $u_s \in C^\infty(U_j \setminus \{x_j\})$ and when $x \to x_j$,

$$u_s(x) = o(|x - x_j|^{m-n-p_j}).$$

(ii) For any $1 \leq j \leq s$, as $x \to y_j$, one has

$$u(x) = O(|x - y_j|^{|q_j|}).$$

5.3.3 Rigged divisors

The notion of a "rigged" divisor comes from the desire to allow for some infinite sets $D_\pm$, but at the same time to impose only finitely many conditions ("zeros" and "singularities") on the solution.

So, let us take a deep breath and dive into it. First, let us define some distribution spaces:

Definition 5.3.5. For a closed set $C \subset \mathcal{X}$, we denote by $\mathcal{E}'_C(\mathcal{X})$ the space of distributions on $\mathcal{X}$, whose supports belong to C (i.e., they are zero outside C).

Definition 5.3.6.

1. A **rigged divisor** associated with P is a tuple $\mu = (D^+, L^+; D^-, L^-)$, where $D^\pm$ are *compact nowhere dense disjoint subsets* in $\mathcal{X}$ and $L^\pm$ are *finite-dimensional* vector spaces of distributions on X supported in $D^\pm$ respectively, i.e.,

$$L^+ \subset \mathcal{E}'_{D^+}(\mathcal{X}), \quad L^- \subset \mathcal{E}'_{D^-}(\mathcal{X}).$$

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2. The **secondary spaces** $\tilde{L}^{\pm}$ associated with $L^{\pm}$ are defined as follows:

$$\tilde{L}^{+} = \{u \mid u \in \mathcal{E}'_{D^+}(\mathcal{X}), Pu \in L^+\}, \quad \tilde{L}^{-} = \{u \mid u \in \mathcal{E}'_{D^-}(\mathcal{X}), P^*u \in L^-\}.$$

3. Let $\ell^{\pm} = \dim L^{\pm}$ and $\tilde{\ell}^{\pm} = \dim \tilde{L}^{\pm}$. The **degree** of μ is defined as follows:

$$\deg_P \mu = (\ell^+ - \tilde{\ell}^+) - (\ell^- - \tilde{\ell}^-). \tag{5.21}$$

4. The **inverse of** μ is the rigged divisor $\mu^{-1} := (D^-, L^-; D^+, L^+)$ associated with P^*.

Remark 5.3.7.

- Notice that the degree of the divisor involves the operator P, so it would have been more prudent to call it "degree of the divisor with respect to the operator P," but we'll neglect this, hoping that no confusion will arise.

- Observe that P and P^* are injective on $\mathcal{E}'_{D^+}$ and $\mathcal{E}'_{D^-}$, correspondingly [7]. Thus,

$$\ell^{\pm} \geq \tilde{\ell}^{\pm}. \tag{5.22}$$

- The sum of the degrees of a divisor μ and of its inverse is zero.

Although we have claimed that point divisors are also rigged divisors, this is not immediately clear when comparing Definitions 5.3.4 and 5.3.6. Namely, we have to assign the spaces $L_{\pm}$ to a point divisor and to check that the definitions are equivalent in this case.

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[7]For example, if $u \in \mathcal{E}'_{D^+}$ and $Pu = 0$ then u is smooth due to elliptic regularity, but then $u = 0$ everywhere since the complement of D^+ is dense.

This was done[8] in [33], if one defines the spaces associated with a point divisor as follows:

$$L^+ = \left\{ \sum_{1 \leq j \leq r} \sum_{|\alpha| \leq p_j - 1} c_j^\alpha \delta^\alpha(\cdot - x_j) \mid c_j^\alpha \in \mathbb{C} \right\}$$

and

$$L^- = \left\{ \sum_{1 \leq j \leq s} \sum_{|\alpha| \leq |q_s| - 1} c_j^\alpha \delta^\alpha(\cdot - y_j) \mid c_j^\alpha \in \mathbb{C} \right\},$$

where δ and δ^α denote the Dirac delta function and its derivative corresponding to the multi-index α.

It was also shown in [33]) that the degree $\deg_P(\mu)$ in this case is $\sum_{1 \leq j \leq r} \left[\binom{p_j + n - 1}{n} - \binom{p_j + n - 1 - m}{n} \right] - \sum_{1 \leq j \leq s} \left[\binom{q_j + n - 1}{n} - \binom{q_j + n - 1 - m}{n} \right]$. Here, as before, n is the dimension of the manifold X and m is the order of the operator P.

5.3.4 Gromov-Shubin theorem on noncompact manifolds

To state (a version of) the Gromov-Shubin theorem, we now introduce the spaces of solutions of P with allowed singularities on D^+ and vanishing conditions on D^-.

Notation 5.3.8. For a compact subset K of $\mathcal{X}$ and $u \in C^\infty(\mathcal{X} \setminus K)$, we shall write that

$$u \in \mathrm{Dom}_K P,$$

if there is a compact neighborhood $\hat{K}$ of K and $\hat{u} \in \mathrm{Dom}\, P$ such that $u = \hat{u}$ outside $\hat{K}$.

Definition 5.3.9. For an elliptic operator P and a rigged divisor $\mu = (D^+, L^+; D^-, L^-)$, the space $L(\mu, P)$ is defined as follows:

$$\{u \in \mathrm{Dom}_{D^+} P \mid \exists \tilde{u} \in \mathcal{D}'(\mathcal{X}) \text{ such that } \tilde{u} = u \text{ on } \mathcal{X} \setminus D^+, P\tilde{u} \in L^+ \text{ and } (u, L^-) = 0\}.$$

---

[8]Which is not trivial.

Here $(u, L^-) = 0$ means that u is orthogonal to every element in L^- with respect to the canonical bilinear duality. The distribution $\tilde{u}$ are regularization of $u \in C^\infty(\mathcal{X} \setminus D^+)$. Now we can state a variant of Gromov-Shubin's version of the Riemann-Roch theorem.

Theorem 5.3.10. *Let P be an elliptic operator such that (5.15) and properties ($\mathcal{P}$1)-($\mathcal{P}$4) are satisfied. Let also μ be a rigged divisor associated with P. If P is a Fredholm operator on $\mathcal{X}$, then the following Riemann-Roch inequality holds:*

$$\dim L(\mu, P) - \dim L(\mu^{-1}, P^*) \geq \operatorname{ind} P + \deg_P(\mu). \tag{5.23}$$

If both P and P^* are Fredholm on $\mathcal{X}$, (5.23) becomes the Riemann-Roch equality:

$$\dim L(\mu, P) = \operatorname{ind} P + \deg_P(\mu) + \dim L(\mu^{-1}, P^*). \tag{5.24}$$

Remark 5.3.11.

1. Although the authors of [33] do not state their theorem in the exact form above, the Riemann-Roch inequality (5.23) follows from their proof.

2. If one considers the difference $\dim L(\mu, P) - \dim L(\mu^{-1}, P^*)$ as some "index of P in presence of the divisor μ" (say, denote it by $\operatorname{ind}_\mu(P)$), the Riemann-Roch equality becomes

$$\operatorname{ind}_\mu(P) = \operatorname{ind} P + \deg_P(\mu) \tag{5.25}$$

   and thus it says that introduction of the divisor changes the index of the operator by $\deg_P(\mu)$.

It is useful for our future considerations to mention briefly some of the ingredients [9] of the proof from [33]. To start, Gromov and Shubin [33] define some auxiliary spaces,

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[9]The reader interested in the main results only, can skip to Corollary 5.3.13.

which we recall now. Let

$$\Gamma(\mathcal{X}, \mu, P) := \{u \in C^\infty(X \setminus D^+) \mid u \in \mathrm{Dom}_{D^+} P, \ \exists \tilde{u} \in \mathcal{D}'(\mathcal{X}) \text{ such that}$$

$$\tilde{u} = u \text{ on } \mathcal{X} \setminus D^+, P\tilde{u} \in L^+ + C^\infty(\mathcal{X}) \text{ and, } \langle u, L^- \rangle = 0\},$$

As before, $\tilde{u}$ is a regularization of $u \in C^\infty(\mathcal{X} \setminus D^+)$. The space of all such regularizations $\tilde{u}$ is

$$\tilde{\Gamma}(\mathcal{X}, \mu, P) := \{\tilde{u} \in \mathcal{D}'(\mathcal{X}) \mid \tilde{u}_{|\mathcal{X} \setminus D^+} \in \Gamma(\mathcal{X}, \mu, P), P\tilde{u} \in L^+ + C^\infty(\mathcal{X})\}.$$

Let us also introduce the spaces

$$\Gamma_\mu(\mathcal{X}, P) = \{u \in \mathrm{Dom}\, P \mid \langle u, L^- \rangle = 0\}$$

and

$$\tilde{\Gamma}_\mu(\mathcal{X}, P) = \{f \in \mathrm{Dom}'\, P^* \mid \langle f, \tilde{L}^- \rangle = 0\}.$$

Then for any $u \in \Gamma(\mathcal{X}, \mu, P)$, one can extend by continuity the restriction of Pu on $\mathcal{X} \setminus D^+$ to a smooth function on the whole $\mathcal{X}$. Let us denote this extension by $\tilde{P}u$. Then $\tilde{P}$ is a linear map from $\Gamma(\mathcal{X}, \mu, P)$ to $\tilde{\Gamma}_\mu(\mathcal{X}, P)$. In the same manner, we can also define the corresponding extension $\tilde{P}^*$ as a linear map from $\Gamma(\mathcal{X}, \mu^{-1}, P^*)$ to $\tilde{\Gamma}_{\mu^{-1}}(\mathcal{X}, P^*)$. The important point here is that $L(\mu, P)$ and $L(\mu^{-1}, P^*)$ are the kernels of these two extensions $\tilde{P}$ and $\tilde{P}^*$, correspondingly. Let us also introduce the duality

$$(\cdot, \cdot) : \Gamma(\mathcal{X}, \mu, P) \times \tilde{\Gamma}_{\mu^{-1}}(\mathcal{X}, P^*) \to \mathbb{C} \tag{5.26}$$

as follows:

$$(u, f) := \langle \tilde{u}, f \rangle, \quad u \in \Gamma(\mathcal{X}, \mu, P), f \in \tilde{\Gamma}_{\mu^{-1}}(\mathcal{X}, P^*),$$

where $\tilde{u}$ is any element in the preimage of $\{u\}$ under the restriction map from $\tilde{\Gamma}(\mathcal{X}, \mu, P)$ to $\Gamma(\mathcal{X}, \mu, P)$. Similarly, we get the duality

$$(\cdot, \cdot) : \tilde{\Gamma}_\mu(\mathcal{X}, P) \times \Gamma(\mathcal{X}, \mu^{-1}, P^*) \to \mathbb{C} \tag{5.27}$$

As it was remarked in [33], these dualities are well-defined and non-degenerate and more-over, one has the following relation

$$(\tilde{P}u, v) = (u, \tilde{P}^*v), \quad \forall u \in \Gamma(\mathcal{X}, \mu, P), v \in \Gamma(\mathcal{X}, \mu^{-1}, P^*).$$

Gromov and Shubin then prove the following identity

$$\dim \operatorname{Ker} \tilde{P} = \operatorname{ind} P + \deg_P(\mu) + \operatorname{codim} \operatorname{Im} \tilde{P}^*. \tag{5.28}$$

by applying the additivity of Fredholm indices to some short exact sequences of the spaces introduced above (see [33, Lemma 3.1] and [33, Remark 3.2]). The assumption that P is Fredholm on $\mathcal{X}$ is important in this consideration.

Furthermore, $(\operatorname{Im} \tilde{P})^\circ = \operatorname{Ker} \tilde{P}^*$ ([33, Lemma 3.4]) and hence, $\operatorname{Im} \tilde{P} \subset (\operatorname{Ker} \tilde{P}^*)^\circ$. Here $(\operatorname{Im} \tilde{P})^\circ$ and $(\operatorname{Ker} \tilde{P}^*)^\circ$ are the annihilators of $\operatorname{Im} \tilde{P}$, $\operatorname{Ker} \tilde{P}^*$ with respect to the du-alities (5.26) and (5.27), respectively. By [33, Lemma 3.3], one gets the inequality (3.9) in [33]:

$$\dim \operatorname{Ker} \tilde{P}^* = \operatorname{codim} (\operatorname{Ker} \tilde{P}^*)^\circ \leq \operatorname{codim} \operatorname{Im} \tilde{P}. \tag{5.29}$$

Then the Riemann-Roch inequality (5.23) follows immediately from (5.28) and (5.29). If P^* is also Fredholm, one can apply (5.23) for P^* and μ^{-1} instead of P and μ to get

$$\dim \operatorname{Ker} \tilde{P} \leq \operatorname{ind} P + \deg_P(\mu) + \dim \operatorname{Ker} \tilde{P}^*. \tag{5.30}$$

Thus, the Riemann-Roch equality (5.24) can be deduced from (5.23) and (5.30). In this case, we also have $\operatorname{Im} \tilde{P} = (\operatorname{Ker} \tilde{P}^*)^\circ$ and $\operatorname{Im} \tilde{P} = (\operatorname{Ker} \tilde{P}^*)^\circ$ as a byproduct of the proof of (5.24) (see [33, Theorem 2.12]).

Remark 5.3.12. If (5.29) becomes an equality, i.e.,

$$\dim \operatorname{Ker} \tilde{P}^* = \operatorname{codim} \operatorname{Im} \tilde{P},$$

one obtains the Riemann-Roch equality (5.24) for the rigged divisor μ without assuming that P^* is Fredholm on $\mathcal{X}$. Conversely, if (5.24) holds, then $\operatorname{Im} \tilde{P} = (\operatorname{Ker} \tilde{P}^*)^\circ$.

As a result, we have the following useful corollary:

Corollary 5.3.13. *Let P be Fredholm on $\mathcal{X}$, $\operatorname{Im} P = \operatorname{Dom}' P^*$, and $\mu = (D^+, L^+; D^-, L^-)$ be a rigged divisor on the manifold $\mathcal{X}$. Then the Riemann-Roch equality (5.24) holds for P and this divisor μ.*

Moreover, the space $L(\mu^{-1}, P^*)$ is trivial, if the following additional condition is satisfied: Suppose that u is a smooth function in $\operatorname{Dom} P$ such that $\langle Pu, \tilde{L}^- \rangle = 0$. Then there exists a solution v in $\operatorname{Dom} P$ of the equation $Pv = 0$ satisfying $\langle u - v, L^- \rangle = 0$.

In particular, this assumption holds automatically if $D^- = \emptyset$ and $L^- = \{0\}$.

We end this part by recalling an application of Theorem 5.3.10, which is needed later.

Example 5.3.14. [33, Example 4.6] Consider $P = P^* = -\Delta$ on $\mathcal{X} = \mathbb{R}^d$, where $d \geq 3$ and

$$\operatorname{Dom} P = \operatorname{Dom} P^* = \{u \mid u \in C^\infty(\mathbb{R}^d), \Delta u \in C_c^\infty(\mathbb{R}^d) \text{ and } \lim_{|x| \to \infty} u(x) = 0\},$$

$$\operatorname{Dom}' P = \operatorname{Dom}' P^* = C_c^\infty(\mathbb{R}^d).$$

Then the operators P and P^* are Fredholm on $\mathbb{R}^d$, $\operatorname{Ker} P = \operatorname{Ker} P^* = \{0\}$, $\operatorname{Im} P = \operatorname{Im} P^* = C_c^\infty(\mathbb{R}^d)$, and thus $\operatorname{ind} P = 0$ (see [33, Example 4.2]).

Let

$$D^+ = \{y_1, \ldots, y_k\}, \quad D^- = \{z_1, \ldots, z_l\}.$$

with all the points $y_1, \ldots, y_k, z_1, \ldots, z_l$ pairwise distinct. Consider the following distributional spaces: L^+ is the vector space spanned by Dirac delta distributions $\delta(\cdot - y_j)$ supported at the points y_j $(1 \le j \le k)$; L^- is spanned by the first order derivatives $\frac{\partial}{\partial x_\alpha} \delta(\cdot - z_j)$ of Dirac delta distributions supported at z_j $(1 \le j \le l, 1 \le \alpha \le d)$. [10]

Consider now the rigged divisor $\mu := (D^+, L^+; D^-, L^-)$. Then $\deg_{-\Delta}(\mu) = k - dl$. Furthermore,

$$L(\mu, -\Delta) = \left\{ u \mid u(x) = \sum_{j=1}^{k} \frac{a_j}{|x - y_j|^{d-2}}, a_j \in \mathbb{C}, \text{ and } \nabla u(z_j) = 0, j = 1, \ldots, l. \right\},$$

and

$$L(\mu^{-1}, -\Delta) = \left\{ v \mid v(x) = \sum_{j=1}^{l} \sum_{\alpha=1}^{d} b_{j,\alpha} \frac{\partial}{\partial x_\alpha} \left(|x - z_j|^{2-d} \right), b_{j,\alpha} \in \mathbb{C}, \text{ and } \right.$$

$$\left. u(y_j) = 0, j = 1, \ldots, k. \right\}.$$

In this case, the Gromov-Shubin-Riemann-Roch formula (Theorem 5.3.10) is

$$\dim L(\mu, -\Delta) = k - dl + \dim L(\mu^{-1}, -\Delta). \tag{5.31}$$

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[10] Note that the secondary spaces $\tilde{L}^\pm$ are trivial.

5.4 The main results

In this section, we consider a periodic elliptic operator A on an abelian covering X. We assume that the Liouville property holds for the operator A at the level $\lambda = 0$, i.e., the real Fermi surface of A (see Definition 2.2.5) is finite (modulo G^*-shifts) (see Theorem 5.2.8). We consider the following two cases:

5.4.1 Non-empty Fermi surface

In this case, suppose that $F_{A,\mathbb{R}} = \{k_1, \ldots, k_\ell\}$ (modulo G^*-shifts), where $\ell \geq 1$. To present our main results in this case, we need to make the following assumption on the local behavior of the Bloch variety of the operator A around each quasimomentum k_j in the real Fermi surface.

Assumption $\mathcal{A}$

($\mathcal{A}$1) For any quasimomentum k, the spectrum of the operator $A(k)$ is discrete. Under this assumption, the following lemma can be deduced immediately from Proposition 5.7.3 in Section 5.7 and perturbation theory (see e.g., [39, 66]):

Lemma 5.4.1. *For each quasimomentum $k_r \in F_{A,\mathbb{R}}$, there is an open neighborhood V_r of k_r in $\mathbb{R}^d$ and a closed contour $\Upsilon_r \subset \mathbb{C}$, such that*

(a) The neighborhoods V_r are mutually disjoint;

(b) The contour Υ_r surrounds the eigenvalue 0 and does not contain any other points of the spectrum $\sigma(A(k_r))$;

(c) The intersection $\sigma(A(k)) \cap \Upsilon_r$ is empty for any $k \in V_r$.

Then, any $k \in V_r$, we can define the Riesz projector $\Pi_r(k)$ associated with $A(k)$ and the contour Υ_r. Thus, $\Pi_r(k)A(k)$ is well-defined for any $k \in V_r$. Let m_r be

the algebraic multiplicity of the eigenvalue 0 of the operator $A(k_r)$. The immediate consequence is:

Lemma 5.4.2. *The projector $\Pi_r(k)$ depends analytically on $k \in V_r$. In particular, its range $R(\Pi_r(k))$ has the same dimension m_r for all $k \in V_r$ and the union $\bigcup_{k \in V_r} R(\Pi_r(k))$ forms a trivial holomorphic vector bundle over V_r.*

($\mathcal{A}2$) We denote by $A_r(k)$ the matrix representation of the operator $\Pi_r(k)A(k)|_{R(\Pi_r(k))}$ with respect to a fixed holomorphic basis $(f_j(k))_{1 \leq j \leq m_r}$ of the range $R(\Pi_r(k))$ when $k \in V_r$. Then $A_r(k)$ is an invertible matrix except only for $k = k_r$. We equip $\mathbb{C}^{m_r}$ with the maximum norm and impose the following integrability condition:

$$\sup_{1 \leq r \leq \ell} \int_{V_r \setminus \{k_r\}} \|A_r(k)^{-1}\|_{\mathcal{L}(\mathbb{C}^{m_r})} \, dk < \infty,$$

where $\mathcal{L}(\mathbb{C}^{m_r})$ is the algebra of linear operators on $\mathbb{C}^{m_r}$.

Remark 5.4.3.

(i) Thanks to Proposition 5.7.2 in Section 5.7, Assumption ($\mathcal{A}1$) is satisfied if A is either self-adjoint or a real operator of even order.[11]

(ii) When the rank d of G is greater than 2, Assumption ($\mathcal{A}2$) holds at a generic spectral edge (see Section 5.6).

To formulate the results, we need to define appropriate spaces of solutions of polynomial growth satisfying the conditions that are associated with a rigged divisor μ.

Definition 5.4.4. Given any $p \in [1, \infty]$ and $N \in \mathbb{R}$, we define

$$L_p(\mu, A, N) := L(\mu, A_N^p),$$

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[11]Here A is real means that Au is real whenever u is real.

where the operator A_N^p stands for A with the domain

$$\operatorname{Dom} A_N^p = \{u \in V_N^p(X) \mid Au \in C_c^\infty(X)\}.$$

In other words, $L_p(\mu, A, N)$ is the space

$$\{u \in \operatorname{Dom}_{D^+} A_N^p \mid \exists \tilde{u} \in \mathcal{D}'(X) : \tilde{u} = u \text{ on } X \setminus D^+, A\tilde{u} \in L^+, (u, L^-) = 0\}.$$

We thus restrict the growth of a function at infinity, impose the divisor conditions, and require that it satisfies the homogeneous equation outside of a compact.

Remark 5.4.5. Consider $u \in L_p(\mu, A, N)$. Let K be a compact domain in X such that $\bigcup_{g \in G} gK = X$. Define $G_{K,D^+} := \{g \in G \mid \operatorname{dist}(gK, D^+) \geq 1\}$, where we use the notation $\operatorname{dist}(\cdot, \cdot)$ for the distance between subsets arising from the Riemannian distance on X. Since $Au = 0$ on $X \setminus D^+$, the condition "$u \in \operatorname{Dom}_{D^+} A_N^p$" can be written equivalently as follows:

$$\{\|u\|_{L^2(gK)} \cdot \langle g \rangle^{-N}\}_{g \in G_{K,D^+}} \in \ell^p(G_{K,D^+}).$$

By Schauder estimates (see Proposition 5.7.7), this condition can be rephrased as follows:

$$\sup_{x : \operatorname{dist}(x, D^+) \geq 1} \frac{|u(x)|}{\operatorname{dist}(x, D^+)^N} < \infty, \qquad \text{when} \quad p = \infty,$$

$$\int_{x : \operatorname{dist}(x, D^+) \geq 1} \frac{|u(x)|^p}{\operatorname{dist}(x, D^+)^{pN}} \, \mathrm{d}\mu_X(x) < \infty, \quad \text{when} \quad 1 \leq p < \infty.$$

(5.32)

So, depending on the sign of N, this condition controls how u grows or decays at infinity.

Our first main result is the next theorem.

Theorem 5.4.6. *Assume that either $p = \infty$ and $N \geq 0$ or $p \in [1, \infty)$ and $N > d/p$. Let p' be the Hölder conjugate of p. Then under Assumption $\mathcal{A}$, the following Liouville-*

Riemann-Roch inequality holds:

$$\dim L_p(\mu, A, N) \geq \dim V_N^p(A) + \deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N), \qquad (5.33)$$

where $\dim V_N^p(A)$ *can be computed via Theorems 5.2.10 and 5.2.14.*

Remark 5.4.7.

- This is an extension to include Liouville conditions of the Riemann-Roch inequality (5.23). One might wonder why in comparison with the "μ-index"

$$\dim L_p(\mu, A, N) - \dim L_p(\mu^{-1}, A^*, -N),$$

the above inequality only involves the dimension of the kernel $V_N^p(A)$, rather than a full index. The reason is that in this case, the dimension is indeed the full index (the corresponding range co-dimension being zero).

- Also, Assumption ($\mathcal{A}1$) forces the Fredholm index of $A(0)$ on M to vanish (see [47, Theorem 4.1.4]). Therefore, $\mathrm{ind}_M A(k)$ ($k \in \mathbb{C}^d$) and the L^2-index of A (by Atiyah's theorem [7]) are zero as well.

The next consequence of Theorem 5.4.6 is useful.

Theorem 5.4.8. *If* $\dim V_N^p(A) + \deg_A(\mu) > 0$*, then there exists a nonzero element in the space* $L_p(\mu, A, N)$*. In other words, there exists a nontrivial solution of the growth described in (5.32) that also satisfies the conditions on "zeros" and "singularities" imposed by the rigged divisor* μ*.*

To state the remaining results, we will need the following definition related to various types of divisors. Namely, one can consider divisors containing only "zeros" or "singularities" from a given divisor.

Definition 5.4.9.

- The divisor $(\emptyset, 0; \emptyset, 0)$ is called **the trivial divisor**.

- Let $\mu = (D^+, L^+; D^-, L^-)$ be a rigged divisor on X. Then the positive part μ^+ and the negative part μ^- of μ are defined as the tuples $(D^+, L^+; \emptyset, 0)$ and $(\emptyset, 0; D^-, L^-)$, respectively. Hence, for a rigged divisor μ, μ^+ (resp. μ^-) is trivial whenever $D^+ = \emptyset$ and L^+ is zero (resp. $D^- = \emptyset$ and L^- is zero).

Our next result shows that if μ^- is trivial, the Liouville-Riemann-Roch inequality becomes an equality:

Theorem 5.4.10.

Consider a rigged divisor μ and its positive part μ^+. Under the assumption of Theorem 5.4.6, the space $L_{p'}((\mu^+)^{-1}, A^*, -N)$ is trivial and

$$\dim L_p(\mu^+, A, N) = \dim V_N^p(A) + \deg_A(\mu^+). \tag{5.34}$$

In particular,

$$\dim L_p(\mu, A, N) \leq \dim V_N^p(A) + \deg_A(\mu^+). \tag{5.35}$$

In other words, the inequality (5.35) gives an upper bound for the dimension of the space $L_p(\mu, A, N)$ (see (5.33) for its lower bound) in terms of the degree of the positive part of the divisor μ and the dimension of the space $V_N^p(A) = L_p(\mu_0, A, N)$, where μ_0 is the trivial divisor. By (5.34), one can see that this upper bound is sharp for divisors containing only "singularities".

When μ^- is nontrivial, determining the triviality of the space $L_{p'}(\mu^{-1}, A^*, -N)$ is more complicated. In the next proposition, we consider two situations: if the degree of

μ^+ is sufficiently large, then the space $L_{p'}(\mu^{-1}, A^*, -N)$ degenerates to zero, while the spaces $L_{p'}(\mu^{-1}, A^*, -N)$ can have arbitrarily large dimensions if μ^+ is trivial.

Here we recall that a differential operator has the strong unique continuation property if any local smooth solution which has a vanishing order of infinity at some point vanishes everywhere.

Proposition 5.4.11.

(a) For any $N \geq 0$, $p \in [1, \infty]$, and $d \geq 3$,

$$\sup \dim L_{p'}(\mu^{-1}, -\Delta_{\mathbb{R}^d}, -N) = \infty,$$

where the supremum is taken over all divisors μ with trivial positive parts.

(b) Let the assumption of Theorem 5.4.6 be satisfied and A^* have the strong unique continuation property. Let also the covering X be connected and a point x_0 of X be given. Let us consider $p \in [1, \infty]$, $N \in \mathbb{R}$, a compact nowhere dense subset D^- such that $x_0 \notin D^-$, and a finite dimensional subspace L^- of $\mathcal{E}'_{D^-}(X)$. Then there exists $M > 0$ depending on the data (A, p, N, x_0, D^-, L^-) such that $\dim L_{p'}(\mu^{-1}, A^*, -N) = 0$ for any rigged divisor $\mu = (D^+, L^+; D^-, L^-)$ satisfying $x_0 \in D^+ \subseteq X \setminus D^-$ and $\mathcal{L}^+_M := \operatorname{span}_{\mathbb{C}}\{\partial_x^\alpha \delta(\cdot - x_0)\}_{0 \leq |\alpha| \leq M} \subseteq L^+ \subset \mathcal{E}'_{D^+}(X)$.

The second part of the above proposition is a reformulation of [32, Proposition 4.3].

Theorem 5.4.6 can be improved if an additional assumption is imposed.

Theorem 5.4.12. *Besides Assumption $\mathcal{A}$, we suppose further that for each $1 \leq r \leq \ell$, the function $k \in V_r \mapsto \|A_r(k)^{-1}\|^2_{\mathcal{L}(\mathbb{C}^{m_r})}$ is integrable [12]. If one of the following two conditions*

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[12]e.g., this occurs at generic edges when $d \geq 5$.

- $p \geq 2$ *and* $N \geq 0$

- $p \in [1, 2)$ *and* $2pN > (2 - p)d$

is satisfied, then the inequality (5.33) holds. In particular, for any rigged divisor μ, we have

$$\dim L_2(\mu^+, A, 0) = \deg_A(\mu^+) \tag{5.36}$$

and

$$\dim L_2(\mu, A, 0) = \deg_A(\mu) + \dim L_2(\mu^{-1}, A^*, 0). \tag{5.37}$$

Remark 5.4.13. When μ is trivial, the equality (5.36) means the absence of non-zero L^2-solutions, and thus generically, spectral edges are not eigenvalues. Here the condition on integrability of $\|A_r(k)^{-1}\|^2_{\mathcal{L}(\mathbb{C}^{m_r})}$ is not required.

One can show that the following L^2-solvability result (or Fredholm alternative) follows from (5.37).

Proposition 5.4.14. *Let D be a non-empty compact nowhere dense subset of X and a finite dimensional subspace $L \subset \mathcal{E}'_D(X)$ be given. Define the finite dimensional subspace $\tilde{L} = \{v \in \mathcal{E}'_D(X) \mid A^*v \in L\}$. Consider any $f \in C_c^\infty(X)$ satisfying $(f, \tilde{L}) = 0$. Under the assumptions of Theorem 5.4.12 for the periodic operator A of order m, the following two statements are equivalent:*

(i) *f is orthogonal to each element in the space $L_2(\mu, A^*, 0)$, where μ is the rigged divisor $(D, L; 0, \emptyset)$.*

(ii) *The inhomogeneous equation $Au = f$ has a (unique) solution u in $H^m(X)$ such that $(u, L) = 0$.*

We present now examples which show that when μ^+ is trivial, the Liouville-Riemann-Roch equality may hold in some cases, while failing miserably in others. The proofs of

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the propositions below can be found in Section 5.5.

Proposition 5.4.15. *Consider the Laplacian* $-\Delta$ *on* $\mathbb{R}^d$, $d \geq 3$. *For any* $N \geq 0$ *and positive integer* ℓ, *there exists a rigged divisor* μ *such that* μ^+ *is trivial and*

$$\dim L_\infty(\mu, -\Delta, N) \geq \ell + \dim V_N^\infty(-\Delta) + \deg_{-\Delta}(\mu) + \dim L_1(\mu^{-1}, -\Delta, -N). \quad (5.38)$$

Now let $A = A(x, D)$ be an elliptic constant-coefficient homogeneous differential operator of order m on $\mathbb{R}^d$ that satisfies Assumption $\mathcal{A}$. Consider two non-negative integers $M_0 \geq M_1$. We fix a point x_0 in X and we define $D^- := \{x_0\}$ as well as the following finite dimensional vector subspace of $\mathcal{E}'_{D^-}(\mathbb{R}^d)$:

$$L^- := \operatorname{span}_\mathbb{C}\{\partial^\alpha \delta(\cdot - x_0)\}_{M_1 \leq |\alpha| \leq M_0}.$$

Let μ be a rigged divisor on $\mathbb{R}^d$ of the form $(D^+, L^+; D^-, L^-)$, where D^+ is a nowhere dense compact subset of $\mathbb{R}^d$ that does not contain the point x_0 and L^+ is a finite dimensional subspace of $\mathcal{E}'_{D^+}(\mathbb{R}^d)$.[13]

Proposition 5.4.16. *Assume that one of the following two conditions holds:*

- $1 \leq p < \infty$, $N > d/p + M_0$.

- $p = \infty$, $N \geq M_0$.

Then $\dim L_{p'}(\mu^{-1}, A^*, -N) = 0$ *and the Liouville-Riemann-Roch equality holds:*

$$\dim L_p(\mu, A, N) = \dim V_N^p(A) + \deg_A(\mu).$$

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[13] A particular case is when $M_1 = 0$ and μ^+ is trivial, i.e., μ becomes the point divisor $x_0^{-(M_0+1)}$ (see Definition 5.3.4).

5.4.2 Empty Fermi surface

In this case, the Liouville theorem becomes trivial, because the emptiness of the Fermi surface implies that there is no non-zero, polynomially growing solution (see [50, Theorem 4.3]), which is an analog of the Schnol theorem (see e.g., [19, 29, 78]). One can obtain a Liouville-Riemann-Roch type result by combining the Riemann-Roch and the Schnol theorems. It is shown in [70] that a more general and stronger statement than the Schnol theorem holds for any C^∞-bounded uniformly elliptic operator on a manifold of bounded geometry and subexponential growth [14]. In this setting, a corresponding Liouville-Riemann-Roch theorem can also be proven. We consider only the case of C^∞-bounded uniformly elliptic operators [15] on a co-compact Riemannian covering.

We begin with some definitions from [70] first. Let $\mathcal{X}$ be a co-compact connected Riemannian covering and $\mathcal{M}$ be its base. The deck group G is a countable and finitely generated discrete group (not necessarily abelian). Let $d_{\mathcal{X}}(\cdot, \cdot)$ be the G-invariant Riemannian distance on $\mathcal{X}$. Due to the compactness of $\mathcal{M}$, there exists $r_{inj} > 0$ (injectivity radius) such that for every $r \in (0, r_{inj})$ and every $x \in \mathcal{X}$, the exponential geodesic mapping $\exp_x : T\mathcal{X}_x \to \mathcal{X}$ is a diffeomorphism of the Euclidean ball $B(0, r)$ centered at 0 with radius r in the tangent space $T\mathcal{X}_x$ onto the geodesic ball $B_{\mathcal{X}}(x, r)$ centered at x with the same radius r in $\mathcal{X}$. Taking $r_0 \in (0, r_{inj})$, the geodesic balls $B_{\mathcal{X}}(x, r)$, where $0 < r \leq r_0$, are called **canonical charts** with x-coordinate in the charts.

Definition 5.4.17. [70, 71]

(i) A differential operator P of order m on $\mathcal{X}$ is C^∞-**bounded** if in every canonical chart, P can be represent as $\sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$. Here the coefficient $a_\alpha(x)$ is smooth such that for any multi-index β, $|\partial_x^\beta a_\alpha(x)| \leq C_{\alpha\beta}$ where the constants $C_{\alpha\beta}$ are inde-

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[14] One could find details about analysis on manifolds of bounded geometry in e.g., [21, 70, 71].
[15] Such operators are not necessarily periodic.

pendent of the chosen canonical coordinate.

(ii) A differential operator P of order m on $\mathcal{X}$ is **uniformly elliptic** if

$$|P_0^{-1}(x,\xi)| \leq C|\xi|^{-m}, \quad (x,\xi) \in T^*\mathcal{X}, \xi \neq 0.$$

Here $T^*\mathcal{X}$ is the cotangent bundle of $\mathcal{X}$, $P_0(x,\xi)$ is the principal symbol of the operator P, and $|\xi|$ is the length of the covector (x,ξ) with respect to the metric on $T^*\mathcal{X}$ induced by the given Riemannian metric on $\mathcal{X}$.

(iii) $\mathcal{X}$ is of **subexponential growth** if the volumes of balls of radius r grow subexpone-tially as $r \to \infty$, i.e., for any $\epsilon > 0$ and $r > 0$,

$$\sup_{x \in \mathcal{X}} \operatorname{vol} B(x,r) = O(\exp(\epsilon r)).$$

Here $\operatorname{vol}(\cdot)$ is the Riemannian volume on $\mathcal{X}$.

(iv) Let x_0 be a fixed point in $\mathcal{X}$. A differential operator P on $\mathcal{X}$ satisfies **Strong Schnol Property** (SSP) if the following statement is true: If there exists a non-zero solution u of the equation $Pu = \lambda u$ such that for any $\epsilon > 0$

$$u(x) = O(\exp(\epsilon d_{\mathcal{X}}(x,x_0)))$$

then λ is in the spectrum of P.

Clearly, any G-periodic elliptic differential operator with smooth coefficients on $\mathcal{X}$ is C^∞-bounded uniformly elliptic.

We now turn to a brief discussion of growth of groups (see e.g., [62]). Let us pick a finite, symmetric generating set S of G. The **word metric** associated to S is denoted by

$d_S : G \times G \to \mathbb{R}$, i.e., for every pair (g_1, g_2) of two group elements in G, $d_S(g_1, g_2)$ is the length of the shortest representation in S of $g_1^{-1} g_2$ as a product of generators from S. Let e be the identity element of G. The volume function of G associated to S is the function $\mathrm{vol}_{G,S} : \mathbb{N} \to \mathbb{N}$ defined by assigning to every $n \in \mathbb{N}$ the cardinality of the open ball $B_{G,S}(e, n)$ centered at e with radius n in the metric space (G, d_S). Although the values of this volume function depend on the choice of the generating set S, its asymptotic growth type is independent from it. The group G is said to be of **subexponential growth** if

$$\lim_{n \to \infty} \frac{\ln \mathrm{vol}_{G,S}(n)}{n} = 0.$$

It is known that the deck group G is of subexponential growth if and only if the covering $\mathcal{X}$ is so (see e.g., [68, Proposition 2.1]). Virtually nilpotent groups clearly have polynomial growth [16]. Thus, any virtually nilpotent co-compact Riemannian covering $\mathcal{X}$ is of subexponential growth. Groups with intermediate growth, which were constructed by Grigorchuk [30], provide other non-trivial examples of Riemannian coverings with subexponential growth.

Theorem 5.4.18. [70, Theorem 4.2] *If $\mathcal{X}$ is of subexponential growth, then any C^∞-bounded uniformly elliptic operator on $\mathcal{X}$ satisfies (SSP).*

Remark 5.4.19. A Schnol type theorem can be established without the subexponential growth condition, if the growth of a generalized eigenfunction is controlled in an integral (over an expanding ball), rather than point-wise sense, see [12, Theorem 3.2.2 for the quantum graph case and more general remarks and references in Section 3.8].

Henceforth, the word length of g in G is defined as $|g| := d_S(e, g)$. Similarly, we also

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[16]The celebrated Gromov's theorem [34] shows that virtually nilpotent groups are the only ones with polynomial growth.

say that a positive function $\varphi : G \to \mathbb{R}^+$ is of **subexponential growth** if

$$\lim_{|g| \to \infty} \frac{\ln \varphi(g)}{|g|} = 0.$$

Again, this concept does not depend on the choice of the finite generating set S (see [62, Theorem 1.3.12]).

Definition 5.4.20. Let φ be a positive function defined on the deck group G such that both φ and its inverse φ^{-1} are of subexponential growth. Let us denote by $\mathcal{S}(G)$ the set of all such φ on G. Then for any $p \in [1, \infty]$ and $\varphi \in \mathcal{S}(G)$, we define:

$$\mathcal{V}_\varphi^p(\mathcal{X}) = \{u \in C^\infty(\mathcal{X}) \mid \{\|u\|_{L^2(g\mathcal{F})} \varphi^{-1}(g)\}_{g \in G} \in \ell^p(G)\},$$

where $\mathcal{F}$ is a fundamental domain for $\mathcal{M}$ in $\mathcal{X}$. Also, let P_φ^p be the operator P with the domain $\{u \in \mathcal{V}_\varphi^p(\mathcal{X}) \mid Pu \in C_c^\infty(\mathcal{X})\}$. Then we denote by $L_p(\mu, P, \varphi)$ the space $L(\mu, P_\varphi^p)$, where μ is a rigged divisor on $\mathcal{X}$. In a similar manner, we also define the space $L_p(\mu, P^*, \varphi)$, where P^* is the transpose of P. In particular, if $G = \mathbb{Z}^d$ and $\varphi(g) = \langle g \rangle^N$, $\mathcal{V}_\varphi^p(\mathcal{X})$ is the space $V_N^p(\mathcal{X})$ introduced in Definition 5.2.6, while $L_p(\mu, P, \varphi)$ coincides with the space $L_p(\mu, P, N)$ appeared in Definition 5.4.4.

We now state our result.

Theorem 5.4.21. *Consider any Riemannian co-compact covering $\mathcal{X}$ of subexponential growth with a discrete deck group G. Let P be a C^∞-bounded uniformly elliptic differential operator P of order m on $\mathcal{X}$ such that $0 \notin \sigma(P)$. Let us denote by φ_0 the constant function 1 defined on G. Then the following statements are true:*

(a) For each rigged divisor μ on $\mathcal{X}$, $L_p(\mu, P, \varphi) = L_\infty(\mu, P, \varphi_0)$, where $p \in [1, \infty]$ and $\varphi \in \mathcal{S}(G)$. Thus, all the spaces $L_p(\mu, P, \varphi)$ are the same.

(b) $\dim L_\infty(\mu, P, \varphi_0) = \deg_P(\mu) + \dim L_\infty(\mu^{-1}, P^*, \varphi_0)$.

(c) If $\mu = (D^+, L^+; \emptyset, 0)$, $\dim L_\infty(\mu, P, \varphi_0) = \deg_P(\mu)$.

Now let $D \subset \mathcal{X}$ be a compact nowhere dense subset, and L be a finite dimensional subspace of $\mathcal{E}'_D(\mathcal{X})$. Denote $\mu := (D, L; 0, \emptyset)$. We also define the space $\tilde{L} := \{u \in \mathcal{E}'_D(\mathcal{X}) \mid P^*u \in L\}$.

As in Corollary 5.4.14, we obtain:

Corollary 5.4.22. *Let the assumptions of Theorem 5.4.21 hold and a function $f \in C_c^\infty(X)$ be given such that $(f, \tilde{L}) = 0$. The following statements are equivalent:*

(i) f is orthogonal to the vector space $L_\infty(\mu, P^*, \varphi_0)$.

(ii) There exists a unique solution u of the inhomogeneous equation $Pu = f$ such that $(u, L) = 0$ and $u \in \mathcal{V}_\varphi^p(\mathcal{X})$ for some $p \in [1, \infty]$ and $\varphi \in \mathcal{S}(G)$.

(iii) The equation $Pu = f$ admits a unique solution u which has subexponential decay and satisfies $(u, L) = 0$.

(iv) The equation $Pu = f$ admits a unique solution u which has exponential decay and satisfies $(u, L) = 0$.

Remark 5.4.23.

(i) Comparing to the Riemann-Roch formula (5.24), the Fredholm index of P does not appear in the formula in Theorem 5.4.21 (b) since P is invertible in this case.

(ii) When μ is trivial, Theorem 5.4.21 (c) becomes Theorem 5.4.18 and our Corollary 5.4.22 is an analog of [47, Theorem 4.2.1] for the co-compact Riemannian coverings of subexponential growth.

5.5 Proofs of the main results

First, we introduce some notions for the proofs.

Definition 5.5.1. For each $s, N \in \mathbb{R}$ and $p \in [1, \infty]$, we denote by $V_{s,N}^p(X)$ the vector space consisting of all function $u \in C^\infty(X)$ such that for some compact subset K of X satisfying (5.9), the sequence $\{\|u\|_{H^s(gK)} \langle g \rangle^{-N}\}_{g \in G}$ belongs to $\ell^p(G)$. We put

$$V_{s,N}^p(A) := V_{s,N}^p(X) \cap \operatorname{Ker} A.$$

Moreover, let $A_{s,N}^p$ be the elliptic operator A with the following domain

$$\operatorname{Dom} A_{s,N}^p = \{u \in V_{s,N}^p(X) \mid Au \in C_c^\infty(X)\}.$$

When $s = 0$, we get back the notions of $V_N^p(X)$ and $V_N^p(A)$ in Definition 5.2.6, and the definitions of the operator A_N^p and its domain $\operatorname{Dom} A_N^p$ appeared in Definition 5.4.4.

Proof of Theorem 5.4.6

Proof. Fixing a pair (p, N) satisfying the assumption of the statement. We recall that A is an elliptic differential operator of order m on X. Let $\mathcal{F}$ be the closure of a fundamental domain for G-action on X. We also pick a compact neighborhood $\hat{\mathcal{F}}$ in X so that $\mathcal{F} \Subset \hat{\mathcal{F}}$ and $\hat{\mathcal{F}}$ satisfies the conclusion of Proposition 5.7.7. Our proof consists of several steps.

Step 1. We claim that given $p \in [1, \infty], N \in \mathbb{R}$, and any rigged divisor $\mu = (D^+, L^+; D^-, L^-)$, one should have

$$L(\mu, A_{m,N}^p) = L(\mu, A_N^p) = L_p(\mu, A, N). \tag{5.39}$$

Indeed, it suffices to show that $L(\mu, A_N^p) \subseteq L(\mu, A_{m,N}^p)$.

Consider $u \in L(\mu, A_N^p)$. Due to Remark 5.4.5, this implies that

$$\{\|u\|_{L^2(g\hat{\mathcal{F}})} \cdot \langle g \rangle^{-N}\}_{G_{\hat{\mathcal{F}}, D^+}} \in \ell^p(G_{\hat{\mathcal{F}}, D^+}), \tag{5.40}$$

where $G_{\hat{\mathcal{F}}, D^+} = \{g \in G \mid \operatorname{dist}(g\hat{\mathcal{F}}, D^+) \geq 1\}$. Let $\mathcal{O} := X \setminus D^+$, then $Au = 0$ on $\mathcal{O}$ and moreover, the set $G^{\mathcal{O}} = \{g \in G \mid g\hat{\mathcal{F}} \cap D^+ = \emptyset\}$ contains $G_{\hat{\mathcal{F}}, D^+}$. By applying Proposition 5.7.7, for any $g \in G_{\hat{\mathcal{F}}, D^+}$, one derives

$$\|u\|_{H^m(g\mathcal{F})} \lesssim \|u\|_{L^2(g\hat{\mathcal{F}})}. \tag{5.41}$$

By (5.40) and (5.41), $\{\|u\|_{H^m(g\mathcal{F})} \cdot \langle g \rangle^{-N}\}_{G_{\mathcal{F}, D^+}} \in \ell^p(G_{\mathcal{F}, D^+})$. By using Remark 5.4.5 again, this shows that $u \in L(\mu, A_{m,N}^p)$. This proves(5.39).

This claim shows that instead of dealing with A_N^p, it suffices to work with $A_{m,N}^p$. We now introduce its 'adjoint' $(A_{m,N}^p)^*$ (in the sense of Subsection 5.3.1).

Definition 5.5.2. We denote by $(A_{m,N}^p)^*$ the elliptic operator A^* with the domain

$$\operatorname{Dom}(A_{m,N}^p)^* = \{v \in V_{m,-N}^{p'}(X) \mid A^*v \in C_c^\infty(X)\},$$

where $1/p + 1/p' = 1$. In another word, $(A_{m,N}^p)^* = (A^*)_{m,-N}^{p'}$.

We also define

$$\operatorname{Dom}' A_{m,N}^p = \operatorname{Dom}'(A_{m,N}^p)^* := C_c^\infty(X).$$

Clearly, $C_c^\infty(X) \subseteq \operatorname{Dom} A_{m,N}^p$, $\operatorname{Dom}(A_{m,N}^p)^* \subseteq C^\infty(X)$ (see (5.15)).

In the next steps, we will apply Theorem 5.3.10 to the operators $A_{m,N}^p$ and $(A_{m,N}^p)^*$.

Step 2. In order to apply Theorem 5.3.10, first, we need to check properties $(\mathcal{P}1) - (\mathcal{P}4)$. Obviously, the first three properties $(\mathcal{P}1) - (\mathcal{P}3)$ hold by definition. To show $(\mathcal{P}4)$, let $u \in \operatorname{Dom} A_{m,N}^p$ and $v \in \operatorname{Dom}(A_{m,N}^p)^*$.

Note that since the operator A is G-periodic, one has

$$\|Au\|_{L^2(g\mathcal{F})} \lesssim \|u\|_{H^m(g\mathcal{F})} \text{ and } \|A^*v\|_{L^2(g\mathcal{F})} \lesssim \|v\|_{H^m(g\mathcal{F})}, \text{ for any } g \in G.$$

Now by Hölder's inequality, we have

$$\left|\sum_{g\in G}\langle Au, v\rangle_{L^2(g\mathcal{F})}\right| \leq \sum_{g\in G}\left|\langle Au, v\rangle_{L^2(g\mathcal{F})}\right| \leq \sum_{g\in G}\|Au\|_{L^2(g\mathcal{F})} \cdot \|v\|_{L^2(g\mathcal{F})}$$

$$\leq \|\{\|Au\|_{L^2(g\mathcal{F})}\langle g\rangle^{-N}\}_{g\in G}\|_{\ell^p(G)} \cdot \|\{\|v\|_{L^2(g\mathcal{F})}\langle g\rangle^{N}\}_{g\in G}\|_{\ell^{p'}(G)}$$

$$\lesssim \|\{\|u\|_{H^m(g\mathcal{F})}\langle g\rangle^{-N}\}_{g\in G}\|_{\ell^p(G)} \cdot \|\{\|v\|_{H^m(g\mathcal{F})}\langle g\rangle^{N}\}_{g\in G}\|_{\ell^{p'}(G)}$$

$$< \infty.$$

$$(5.42)$$

Similarly,

$$\left|\sum_{g\in G}\langle u, A^*v\rangle_{L^2(g\mathcal{F})}\right| \leq \|\{\|u\|_{H^m(g\mathcal{F})}\langle g\rangle^{-N}\}_{g\in G}\|_{\ell^p(G)} \cdot \|\{\|v\|_{H^m(g\mathcal{F})}\langle g\rangle^{N}\}_{g\in G}\|_{\ell^{p'}(G)} < \infty.$$

Hence, both $\langle A^p_{m,N}u, v\rangle$ and $\langle u, (A^p_{m,N})^*v\rangle$ are well-defined.

Our goal is to show that these two quantities are equal. To do this, for each $r \in \mathbb{N}$, we define

$$G_r := \{g \in G \mid |g| \geq r\},$$

and $\mathcal{F}_r$ as the union of all shifts of $\mathcal{F}$ by deck group elements whose word length do not exceed r, i.e.,

$$\mathcal{F}_r := \bigcup_{g\notin G_{r+1}} g\mathcal{F} = \bigcup_{|g|\leq r} g\mathcal{F}.$$

Obviously, $\mathcal{F}_r \Subset \mathcal{F}_{r+1}$ for any $r \in \mathbb{N}$ and the union of these subsets $\mathcal{F}_r$ is the whole covering X. Let $\phi_r \in C_c^\infty(X)$ be a cut-off function such that $\phi_r = 1$ on $\mathcal{F}_r$ and $\operatorname{supp}\phi_r \Subset$

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$\mathcal{F}_{r+1}$. Furthermore, all derivatives of ϕ_r are uniformly bounded with respect to r. In particular, the following estimates hold for any smooth function w on X and any $g \in G$:

$$\|\phi_r w\|_{H^m(g\mathcal{F})} \lesssim \|w\|_{H^m(g\mathcal{F})}, \quad \|(1 - \phi_r)w\|_{H^m(g\mathcal{F})} \lesssim \|w\|_{H^m(g\mathcal{F})}. \tag{5.43}$$

Put $u_r := \phi_r u$ and $v_r := \phi_r v$. Since u_r and v_r are compactly supported smooth functions on X, $\langle Au_r, v_r \rangle = \langle u_r, A^* v_r \rangle$. Therefore, it is enough to demonstrate

$$\langle Au_r, v_r \rangle \to \langle Au, v \rangle \quad \text{and} \quad \langle u_r, A^* v_r \rangle \to \langle u, A^* v \rangle, \tag{5.44}$$

as $r \to \infty$. By symmetry, we only need to show the first convergence of (5.44). We use the triangle inequality to reduce (5.44) to checking

$$\lim_{r \to \infty} \langle A(u - u_r), v \rangle = \lim_{r \to \infty} \langle Au_r, (v - v_r) \rangle = 0. \tag{5.45}$$

We repeat the argument of (5.42) for the pairs of functions $((1 - \phi_r)u, v)$ and $(\phi_r u, (1 - \phi_r)v)$, and then use (5.43) to derive

$$|\langle Au_r, (v - v_r) \rangle| + |\langle A(u - u_r), v \rangle|$$

$$\leq \sum_{g \in G} \left| \langle A(\phi_r u), (1 - \phi_r)v \rangle_{L^2(g\mathcal{F})} \right| + \left| \langle A((1 - \phi_r)u), v \rangle_{L^2(g\mathcal{F})} \right|$$

$$= \sum_{|g| \geq r+1} \left| \langle A(\phi_r u), (1 - \phi_r)v \rangle_{L^2(g\mathcal{F})} \right| + \left| \langle A((1 - \phi_r)u), v \rangle_{L^2(g\mathcal{F})} \right|$$

$$\lesssim \left\| \{ \|\phi_r u\|_{H^m(g\mathcal{F})} \langle g \rangle^{-N} \}_{g \in G_{r+1}} \right\|_{\ell^p(G_{r+1})} \cdot \left\| \{ \|(1 - \phi_r)v\|_{H^m(g\mathcal{F})} \langle g \rangle^N \}_{g \in G_{r+1}} \right\|_{\ell^{p'}(G_{r+1})}$$

$$+ \left\| \{ \|(1 - \phi_r)u\|_{H^m(g\mathcal{F})} \langle g \rangle^{-N} \}_{g \in G_{r+1}} \right\|_{\ell^p(G_{r+1})} \cdot \left\| \{ \|v\|_{H^m(g\mathcal{F})} \langle g \rangle^N \}_{g \in G_{r+1}} \right\|_{\ell^{p'}(G_{r+1})}$$

$$\lesssim \left\| \{ \|u\|_{H^m(g\mathcal{F})} \langle g \rangle^{-N} \}_{g \in G_{r+1}} \right\|_{\ell^p(G_{r+1})} \cdot \left\| \{ \|v\|_{H^m(g\mathcal{F})} \langle g \rangle^N \}_{g \in G_{r+1}} \right\|_{\ell^{p'}(G_{r+1})}.$$

$$\tag{5.46}$$

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Since $u \in V_{m,N}^p(X)$ and $v \in V_{m,-N}^{p'}(X)$, it follows that as $r \to \infty$, either

$$\left\| \left\{ \|u\|_{H^m(g\mathcal{F})} \langle g \rangle^{-N} \right\}_{g \in G_{r+1}} \right\|_{\ell^p(G_{r+1})}$$

or

$$\left\| \left\{ \|v\|_{H^m(g\mathcal{F})} \langle g \rangle^{N} \right\}_{g \in G_{r+1}} \right\|_{\ell^{p'}(G_{r+1})}$$

converges to zero (depending on either p or p' is finite), while the remaining term is bounded. Thus, we always have

$$\lim_{r \to \infty} \left\| \left\{ \|u\|_{H^m(g\mathcal{F})} \langle g \rangle^{-N} \right\}_{g \in G_{r+1}} \right\|_{\ell^p(G_{r+1})} \cdot \left\| \left\{ \|v\|_{H^m(g\mathcal{F})} \langle g \rangle^{N} \right\}_{g \in G_{r+1}} \right\|_{\ell^{p'}(G_{r+1})} = 0.$$

This fact and (5.46) imply (5.45). Hence, the property $(\mathcal{P}4)$ holds for $A_{m,N}^p$ and $(A_{m,N}^p)^*$.

Step 3. Clearly, $\operatorname{Ker} A_{m,N}^p = \{u \in \operatorname{Dom} A_{m,N}^p \mid Au = 0\} = V_{m,N}^p(A) = V_N^p(A)$. The latter equality is due to Schauder estimates (see (5.41) in Step 1). Also, $\operatorname{Ker}(A_{m,N}^p)^* = V_{m,-N}^{p'}(A^*) = V_{-N}^{p'}(A^*) = 0$ by Theorem 5.2.16. Hence, the kernels of $A_{m,N}^p$ and $(A_{m,N}^p)^*$ are of finite dimensional.

To prove that $A_{m,N}^p$ is Fredholm on X, we only need to show that $\operatorname{Im} A_{m,N}^p = C_c^\infty(X) = \left(\operatorname{Ker}(A_{m,N}^p)^* \right)^\circ$. Given any $f \in C_c^\infty(X)$, we want to find a solution u of the equation $Au = f$ such that $u \in V_N^p(X)$. If such a solution u is found then automatically, u is in $V_{m,N}^p(X)$ by the same argument in Step 1 and the fact that $Au = 0$ on the complement of the compact support of f. Thus, f must belong to the range of $A_{m,N}^p$ and the proof is then finished. So our remaining task is to find such a solution u. This can be done as the following way. First, we pick a cut-off function η_r such that $\eta = 1$ around k_r and $\operatorname{supp} \eta_r \Subset V_r$, where V_r is the neighborhood of k_r appearing in *Assumption* $\mathcal{A}$. Define

$$\eta = \sum_{r=1}^{\ell} \eta_r.$$

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Note that the Floquet transform $\mathbf{F}f$ is smooth in (k, x) since $f \in C_c^\infty(X)$. We decompose $\mathbf{F}f = \eta\mathbf{F}f + (1-\eta)\mathbf{F}f$. Since the operator $A(k)$ is invertible when $k \notin F_{A,\mathbb{R}}$, the operator function

$$\widehat{u_0}(k) := A(k)^{-1}((1 - \eta(k))\mathbf{F}f(k))$$

is well-defined and smooth in (k, x). By Theorem 5.7.5, the function $u_0 := \mathbf{F}^{-1}\widehat{u_0}$ has rapid decay. We recall when $k \in V_r$, the Riesz projection $\Pi_r(k)$ is defined in *Assumption A*. Clearly, $0 \notin \sigma(A(k)_{|R(1-\Pi_r(k))})$, where we use the notation $R(T)$ to denote the range of an operator T. Now the operator function

$$\widehat{v_r}(k) := \eta_r(k)(A(k)_{|R(1-\Pi_r(k))})^{-1}(1 - \Pi_r(k))\mathbf{F}f(k)$$

is also smooth and thus, the function $v_r := \mathbf{F}^{-1}\widehat{v_r}$ has rapid decay by Theorem 5.7.5 again. In particular, $u_0, v_r (1 \leq r \leq \ell)$ are in the space $V_0^\infty(X)$.

Let us fix $1 \leq r \leq \ell$. For any $k \in V_r \setminus \{k_r\}$, due to $(\mathcal{A}2)$, we can define the operator function $\widehat{w_r}(k) := \eta_r(k)(A(k)_{|R(\Pi_r(k))})^{-1}\Pi_r(k)\mathbf{F}f(k)$, which is in the range of $\Pi_r(k)$. By expanding $\Pi_r(k)\mathbf{F}f(k)$ in terms of the basis $(f_j(k))_{1 \leq j \leq m_r}$, it is easy to see that

$$\|\widehat{w_r}(k)\|_{L_k^2(X)} \lesssim \max_{1 \leq j \leq m_r} \|A_r(k)^{-1}f_j(k)\|_{L_k^2(X)} \cdot \|\mathbf{F}f(k)\|_{L_k^2(X)}.$$

From this and the integrability condition in $(\mathcal{A}2)$, we obtain

$$\int_{\mathbb{T}^d} \|\widehat{w_r}(k)\|_{L_k^2(X)} \, \mathrm{d}k \lesssim \int_{V_r \setminus \{k_r\}} \|\mathbf{F}f(k)\|_{L_k^2(X)} \cdot \|(A_r(k)^{-1}f_j(k))_{1 \leq j \leq m_r}\|_{\ell^\infty(\mathbb{C}^{m_r})} \, \mathrm{d}k$$

$$\lesssim \sup_{k \in V_r} \|\mathbf{F}f(k)\|_{L_k^2(X)} \cdot \int_{V_r \setminus \{k_r\}} \|A_r(k)^{-1}\|_{\mathcal{L}(\mathbb{C}^{m_r})} \, \mathrm{d}k < \infty.$$

Hence, $\widehat{w_r} \in L^1(\mathbb{T}^d, \mathcal{E}^0)$.

Summing up, the function $\widehat{u} := \widehat{u}_0 + \sum_{1 \le r \le \ell} (\widehat{v}_r + \widehat{w}_r)$ belongs to $L^1(\mathbb{T}^d, \mathcal{E}^0)$, and moreover, it satisfies the equation

$$A(k)\widehat{u}(k) = A(k)\widehat{u}_0(k) + \sum_{1 \le r \le \ell} A(k)(\widehat{v}_r(k) + \widehat{w}_r(k))$$

$$= (1 - \eta(k))\mathbf{F}f(k) + \sum_{1 \le r \le \ell} \eta_r(k)\mathbf{F}f(k) = \mathbf{F}f(k).$$

From the above equation, $\widehat{u}(k, x)$ is smooth in x for each quasimomentum k. We define $u := \mathbf{F}^{-1}\widehat{u}$ by using the formula (5.58). According to Lemma 5.7.6, $u \in L^2_{loc}(X)$. For any $\phi \in C^\infty_c(X)$, we can use Fubini's theorem to get

$$\langle u, A^*\phi \rangle_{L^2(X)} = \int_X \mathbf{F}^{-1}\widehat{u}(k, x) \cdot A^*\phi(x) \, d\mu_X(x)$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \int_X \widehat{u}(k, x) \cdot A^*\phi(x) \, d\mu_X(x) \, dk = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \int_X A\widehat{u}(k, x) \cdot \phi(x) \, d\mu_X(x) \, dk$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \int_X A(k)\widehat{u}(k, x) \cdot \phi(x) \, d\mu_X(x) \, dk$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \int_X \mathbf{F}f(k, x) \cdot \phi(x) \, d\mu_X(x) \, dk = \langle f, \phi \rangle_{L^2(X)}.$$

Hence, u is a weak solution of the inhomogeneous equation $Au = f$ on X. Elliptic regularity then implies that u is a classical solution and therefore, $u \in V^\infty_0(X)$ due to Lemma 5.7.6 again. If either $N \ge 0$ when $p = \infty$ or $N > d/p$ when $p \in [1, \infty)$, we always have $V^\infty_0(X) \subseteq V^p_N(X)$. Thus, this shows that $A^p_{m,N}$ is a Fredholm operator on X.

Step 4. From Step 2 and Step 3, the operator $A^p_{m,N}$ satisfies the assumption of Theorem 5.3.10. Then the Liouville-Riemann-Roch inequality (5.33) follows immediately from (5.39) and Theorem 5.3.10. $\qquad\square$

Remark 5.5.3. Assumption $(\mathcal{A}2)$ is needed to guarantee the validity of the Liouville-Riemann-Roch inequality (5.4.6) (at least when $p = \infty$ and $N = 0$). Indeed, consider $-\Delta$ in $\mathbb{R}^2$. Let μ be the point divisor $(\{0\}, L, \emptyset, 0)$, where $L = \mathbb{C}\delta_0$. It is not difficult to

see that the space $L_\infty(\mu, -\Delta, 0)$ contains only constant functions since the standard fundamental solution $u_0(x) = -\frac{1}{2\pi} \ln |x|$ is not bounded at infinity. Clearly, $L_1(\mu^{-1}, -\Delta, 0)$ is trivial. Hence, we have:

$$\dim L_\infty(\mu, -\Delta, 0) = 1 < 2 = \deg_{-\Delta}(\mu) + \dim V_0^\infty(-\Delta) + \dim L_1(\mu^{-1}, -\Delta, 0).$$

Proof of Theorem 5.4.10

Proof. From Step 3 in the proof of Theorem 5.4.6, the operator $A^p_{m,N}$ is Fredholm on X and $\operatorname{Im} A^p_{m,N} = C^\infty_c(X) = \operatorname{Dom}'(A^p_{m,N})^*$. Now we can apply Corollary 5.3.13 to finish the proof of the equality (5.34). The upper bound estimate (5.35) follows from (5.34) and the trivial inclusion $L_p(\mu, A, N) \subseteq L_p(\mu^+, A, N)$. $\square$

Proof of Proposition 5.4.11

Proof. (a) We define $\mu_0 := (\emptyset, 0; D^-, L^-)$. Now suppose for contradiction that for any $M > 0$, the space $L_{p'}(\mu_M^{-1}, A^*, -N)$ is non-trivial for some rigged divisor $\mu_M = (D^+, L_M^+; D^-, L^-)$ such that $\mathcal{L}_M^+ \subseteq L_M^+$. Note that $L_{p'}(\mu_M^{-1}, A^*, -N)$ is a subspace of $L_{p'}(\mu_0^{-1}, A^*, -N)$. It follows from Proposition 5.4.10 that $L_{p'}(\mu_0^{-1}, A^*, -N)$ is a finite dimensional vector space and thus, we equip it with any norm $\|\cdot\|$. Thus, there is a sequence $\{u_M\}_{M \in \mathbb{N}}$ in $L_{p'}(\mu_0^{-1}, A^*, -N)$ such that $\|u_M\| = 1$ and $(u_M, L_M^+) = 0$. In particular, $(u_M, \mathcal{L}_M^+) = 0$ and therefore, $\partial^\alpha u_M(x_0) = 0$ for any $0 \leq |\alpha| \leq M$. By passing to a subsequence if necessary, there exists $v \in L_{p'}(\mu_0^{-1}, A^*, -N)$ for which $\lim_{M \to \infty} \|u_M - v\| = 0$. It is clear that $\|w\|_{C^M(K)} \lesssim \|w\|$ for any w in $L_{p'}(\mu_0^{-1}, A^*, -N)$, $M \geq 0$, and compact subset $K \Subset X \setminus D^-$. Hence, for any multi-index α, $\partial^\alpha v(x_0) = \lim_{M \to \infty} \partial^\alpha u_M(x_0) = 0$. As a local smooth solution of A^*, v must vanish on $X \setminus D^-$ due to the strong unique continuation property of A^*. Consequently, $v = 0$ as an element in $L_{p'}(\mu_0^{-1}, A^*, -N)$ and this gives us a contradiction to the fact that $\|v\| = 1$.

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(b) It suffices to prove the statement for the case $p = \infty$. If $r \geq 0$, we define the corresponding point divisor $\mu_r := (\emptyset, 0; \{0\}, L_r^-)$, where

$$L_r^- := \left\{ \sum_{|\alpha| \leq r} c_\alpha \partial^\alpha \delta(\cdot - 0) \mid c_\alpha \in \mathbb{C} \right\}.$$

Let us consider the function $v_\alpha(x) := \partial^\alpha(|x|^{2-d})$ for each multi-index α such that $|\alpha| > N + 2$. It is clear that $|v_\alpha(x)| \lesssim |x|^{-|\alpha|-d+2}$ for $x \neq 0$. Therefore, we obtain

$$\sum_{g \in \mathbb{Z}^d} \|v_\alpha\|_{L^2([0,1)^d + g)} \cdot \langle g \rangle^N \lesssim \sum_{g \in \mathbb{Z}^d} \langle g \rangle^{-|\alpha|-d+2+N} < \infty.$$

Since $|x|^{2-d}$ is a fundamental solution of $-\Delta$ on $\mathbb{R}^d$ (up to some multiplicative constant), v_α belongs to the space $L_1(\mu_r^{-1}, -\Delta_{|\mathbb{R}^d}, -N)$ provided that $N + 2 < |\alpha| \leq r$. Now let us pick some multi-indices $\alpha_1, \ldots, \alpha_{r-N-2}$ such that $|\alpha_j| = N + 2 + j$ for any $1 \leq j \leq r - N - 2$. By homogeneity, the functions v_{α_j} are linearly independent as smooth functions on $\mathbb{R}^d \setminus \{0\}$. By letting $r \to \infty$, this completes our proof.

$\square$

Remark 5.5.4.

(i) There are large classes of elliptic operators that have the strong unique continuation property, e.g., elliptic operators of second-order with smooth coefficients or elliptic operators with real analytic coefficients.

(ii) Note that the finiteness of the real Fermi surface $F_{A,\mathbb{R}}$ would imply the weak unique continuation property of A^*, i.e., A^* does not have any non-zero compactly supported solution (see e.g., [47]). We do not know whether the first statement of Proposition 5.4.11 holds if one drops the strong unique continuation property of A^*.

Proof of Theorem 5.4.12

Proof. The proof is similar to the proof of Theorem 5.4.6, except for Step 3 where we have a minor modification. Let us keep the same argument and notions in Step 3. The goal here is to prove the solvability of the equation $Au = f$, where $f \in C_c^\infty(X)$ and $u \in V_N^p(X)$. Under the L^2-assumption, the functions $\widehat{w}_r$ $(1 \le r \le \ell)$, which are defined in Step 3, belong to $L^2(\mathbb{T}^d, \mathcal{E}^0)$. Thus, $\widehat{u} \in L^2(\mathbb{T}^d, \mathcal{E}^0)$ and then by Theorem 5.7.5, u is in $L^2(X)$ as well as $Au = f$. This means that $u \in V_0^2(X)$. If $p \ge 2, N \ge 0$, the inclusion $V_0^2(X) \subseteq V_N^2(X)$ is obvious while if $p \in [1,2), N > (1/p - 1/2)d$, one can use Hölder's inequality to obtain the inclusion $V_0^2(X) \subset V_N^p(X)$. This completes the proof of the first statement. In particular, when $p = 2, N = 0$, both of the operators A_0^2 and $(A_0^2)^* = (A^*)_0^2$ are Fredholm. Therefore, we obtain the equality (5.37) since it is known that $\dim V_0^2(A) = \dim V_0^2(A^*) = 0$ (see Theorem 5.2.16 (a)). $\qquad\square$

Remark 5.5.5.

(a) The integrability of $\|A_r(k)^{-1}\|_{\mathcal{L}(\mathbb{C}^{m_r})}^2$ is needed in Theorem 5.4.12. For example, let us consider $A = -\Delta$ on $\mathbb{R}^d$ $(d < 5)$ and the point divisor μ representing a simple pole at 0. Then the fundamental solution $c_d|x|^{2-d}$ does not belong to $L_2(\mu, -\Delta, 0)$ and thus, $\dim L_2(\mu, -\Delta, 0) = 0 < 1 = \deg_{-\Delta}(\mu)$. Therefore, the equalities (5.36) and (5.37) do not hold in this case.

(b) Under the assumption of Theorem 5.4.12, the Liouville-Riemann-Roch inequality (5.33) holds for **any** $N \ge 0$ if and only if $p \ge 2$. Indeed, suppose that $d \ge 5$ and $p < 2$, then $(2 - d)p \ge -d$ and therefore,

$$\int_{|x| \ge 1} |x|^{(2-d)p} \, dx = \infty.$$

This implies that $L_p(\mu, -\Delta, 0) = \{0\}$, where μ is also the point divisor 0^1. So (5.33) fails since $\dim L_p(\mu, -\Delta, 0) = 0 < 1 = \deg_{-\Delta}(\mu)$.

Proof of Proposition 5.4.14

Proof. We evoke the operators $A^2_{m,0}$ and $(A^2_{m,0})^*$ and their corresponding domains from the proof of Theorem 5.4.6. Now we recall from our discussion in Subsection 5.3 the notations of the operators

$$\widetilde{A^2_{m,0}} : \Gamma(X, \mu^{-1}, A^2_{m,0}) \to \tilde{\Gamma}_{\mu^{-1}}(X, A^2_{m,0})$$

and

$$\widetilde{(A^2_{m,0})^*} : \Gamma(X, \mu, (A^2_{m,0})^*) \to \tilde{\Gamma}_{\mu}(X, (A^2_{m,0})^*)$$

are extensions of the two operators $A^2_{m,0}$ and $(A^2_{m,0})^*$ with respect to the divisors μ^{-1} and μ, respectively. From (5.37) and Remark 5.3.12, we obtain the duality $L_2(\mu, A^*, 0)^\circ = (\text{Ker}\,\widetilde{(A^2_{m,0})^*})^\circ = \text{Im}\,\widetilde{A^2_{m,0}}$. Put in another word, f is orthogonal to $\tilde{L}$ and $L_2(\mu, A^*, 0)$ if and only if $f = Au$ for some u in the space $\Gamma(X, \mu^{-1}, A^2_{m,0}) = \{v \in H^m(X) \mid Av \in C_c^\infty(X), \langle v, L\rangle = 0\}$. This proves the equivalence of (i) and (ii). $\qquad\square$

Proof of Proposition 5.4.15

Proof. If $\ell \in \mathbb{N}$, we choose ℓ distinct points $z_1, \ldots, z_\ell$ in $\mathbb{R}^d$ and define $\mu_\ell := (\emptyset, 0; D^-, L^-)$, where $D^- = \{z_1, \ldots, z_\ell\}$ and

$$L^- = \left\{ \sum_{1 \leq j \leq \ell} \sum_{1 \leq \alpha \leq d} c_{j\alpha} \frac{\partial}{\partial x_\alpha} \delta(x - z_j) \mid c_{j\alpha} \in \mathbb{C} \right\}.$$

In terms of the notations in Example 5.3.14, $k = 0, l = \ell$.

Let us recall now the spaces $L(\mu, -\Delta)$ and $L(\mu^{-1}, -\Delta)$ from Example 5.3.14. By their definition, it is clear that $L_\infty(\mu, -\Delta, 0) = \mathbb{C}$. Hence,

$$\dim L_\infty(\mu, -\Delta, 0) = 1 = \dim L(\mu, -\Delta) + \dim V_0^\infty(-\Delta). \tag{5.47}$$

On the other hand, if $v \in L_1(\mu^{-1}, -\Delta, 0)$ then

$$\lim_{R \to \infty} \sum_{|g| \geq R} \|v\|_{L^2([0,1)^d+g)} = 0.$$

It follows that as $|g| \to \infty$, $\|v\|_{L^2([0,1)^d+g)} \to 0$. Using elliptic regularity, this is equivalent to the condition $\lim_{|x| \to \infty} v(x) = 0$. Thus, $L_1(\mu^{-1}, -\Delta, 0)$ is a subspace of $L(\mu^{-1}, -\Delta)$. Define

$$v_{j\alpha}(x) = \frac{\partial}{\partial x_\alpha} |x - z_j|^{2-d}.$$

Then $v_{j\alpha} \in L(\mu^{-1}, -\Delta)$ (see Example 5.3.14). We now claim that $v_{j\alpha} \notin L_1(\mu^{-1}, -\Delta, 0)$ for any $1 \leq j \leq \ell$ and $1 \leq \alpha \leq d$. Suppose for contradiction that

$$v_{j\alpha}(x) = (2 - d) \frac{(x_\alpha - (z_j)_\alpha)}{|x - z_j|^d} \in L_1(\mu^{-1}, -\Delta, 0).$$

This implies that for some $R > 0$, we have

$$V_{\alpha,R} := \sum_{g \in \mathbb{Z}^d, |g| \geq R} \left(\int_{[0,1)^d+g} \frac{|x_\alpha - (z_j)_\alpha|^2}{|x - z_j|^{2d}} \, \mathrm{d}x \right)^{1/2} < \infty.$$

But this gives us a contradiction, since

$$V_{\alpha,R} \gtrsim \sum_{g \in \mathbb{Z}^d, |g| \geq R} \frac{|g_\alpha|}{|g|^d} \geq \sum_{g \in \mathbb{Z}^d, g_\alpha \neq 0, |g| \geq R} \frac{1}{|g|^d} = \infty.$$

From the above claim and the linearly independence of the functions $v_{j\alpha}$ as smooth functions on $\mathbb{R}^d \setminus D^-$, it follows that

$$\dim L_1(\mu^{-1}, -\Delta, 0) \leq \dim L(\mu^{-1}, -\Delta) - d\ell. \tag{5.48}$$

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From (5.31), (5.47) and (5.48), we conclude

$$d\ell + \dim L_1(\mu^{-1}, -\Delta, 0) + \deg_{-\Delta}(\mu) + \dim V_0^\infty(-\Delta)$$

$$\leq \dim L(\mu^{-1}, -\Delta) + \deg_{-\Delta}(\mu) + \dim V_0^\infty(-\Delta) = \dim L_\infty(\mu, -\Delta, 0). \tag{5.49}$$

This yields the inequality (5.38). □

Remark 5.5.6.

(i) Note that the examples we constructed in Proposition 5.4.15 also show that the Liouville-Riemann-Roch inequality can be strict for other cases. Let us briefly mention some of them here.

Case 1. $p = \infty, N \geq 0$.

If $(d-1)\ell + 1 \geq \dim V_N^\infty(-\Delta)$, one obtains from (5.49) that

$$\dim L_1(\mu^{-1}, -\Delta, -N) + \deg_{-\Delta}(\mu) + \dim V_N^\infty(-\Delta) + \ell$$

$$\leq \dim L_1(\mu^{-1}, -\Delta, 0) + \deg_{-\Delta}(\mu) + \dim V_0^\infty(-\Delta) + d\ell \leq \dim L_\infty(\mu, -\Delta, N).$$

Case 2. $1 \leq p < \infty, N > d/p$.

Note that $\dim L_p(\mu, -\Delta, N) \geq 1$ since it contains constant solutions. Again, each function $v_{j\alpha}$ does not belong to the space $L_{p'}(\mu^{-1}, -\Delta, -N)$. In fact, for $R > 2|z_j|$ large enough and $p > 1$, we get

$$\sum_{g \in \mathbb{Z}^d, |g| > R} \|v_{j\alpha}\|_{L^2([0,1)^d + g)}^{p'} \cdot \langle g \rangle^{p'N} \gtrsim \sum_{\substack{\min_{1 \leq l \leq d} g_l > R}} \langle g \rangle^{p'(N-d)} = \infty.$$

The case $p = 1, N > d - 1$ is proved similarly. Now as in the proof of (5.49), we

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obtain the inequality

$$\dim L_{p'}(\mu^{-1}, -\Delta, -N) + \deg_{-\Delta}(\mu) + \dim V_N^p(-\Delta) + \ell$$

$$\leq \dim L(\mu^{-1}, -\Delta) - d\ell + \deg_{-\Delta}(\mu) + \dim V_N^p(-\Delta) + \ell \leq \dim L_p(\mu, -\Delta, N),$$

provided that $(d-1)\ell + 1 \geq \dim V_N^p(-\Delta)$.

In physics, the functions $v_{j\alpha}$ in $L(\mu^{-1}, -\Delta)$ are the potentials of dipoles located at the equilibrium positions z_j.

(ii) We can also modify our example in Proposition 5.4.15 to obtain examples for (5.38) in the case of point divisors. For instance, we could take the point divisor $\mu = (\emptyset, 0; D^-, L^-)$, where $D^- = \{z_1, \ldots, z_\ell\}$ and $L^- = \operatorname{span}_{\mathbb{C}}\{\partial^\alpha \delta(x-z_j)\}_{1 \leq j \leq \ell, 0 \leq |\alpha| \leq 1}$. Similarly, $L_\infty(\mu, -\Delta, 0) = L_1(\mu^{-1}, -\Delta, 0) = \{0\}$ and moreover, $\deg_{-\Delta}(\mu) = \ell(d+1)$. Hence, $\dim L_\infty(\mu, -\Delta, 0) = (\ell(d+1)-1) + \deg_{-\Delta}(\mu) + \dim V_0^\infty(-\Delta) + \dim L_1(\mu^{-1}, -\Delta, 0)$.

Additionally, our method can also be adapted without much difficulty to provide more examples of the inequality (5.38) when the positive parts μ^+ and negative parts μ^- of the rigged divisors μ are required to be both non-trivial.

Proof of Proposition 5.4.16

Proof. By translation invariance of A, we can suppose without loss of generality that $x_0 = 0$. Now we fix a pair (p, N) as in the assumption of the statement. We recall from Definition 5.5.1 the notations of the operator $A_{m,N}^p$ and its corresponding domain $\operatorname{Dom} A_{m,N}^p$. We will apply Corollary 5.3.13 to the operator $P := A_{m,N}^p$ in order to prove the statement. From our assumption on the operator A and the pair (p, N) and from the conclusion of Step 3 of the proof of Theorem 5.4.6, we only need to show the following claim: If u is a smooth function on $\mathbb{R}^d$ such that $|u(x)| \lesssim \langle x \rangle^N$ and $\langle Au, \tilde{L}^- \rangle = 0$, then

there is a polynomial v of degree M_0 satisfying $Av = 0$ and $\langle u - v, L^- \rangle = 0$. Indeed, if this claim holds true, v will belong to the space $\mathrm{Dom}\, A^p_{m,N}$ due to our condition on p and N. This will fulfill all the necessary assumptions of Corollary 5.3.13 in order to apply it.

To prove the claim, we first introduce the following polynomial:

$$v(x) := \sum_{M_1 \leq |\alpha| \leq M_0} \frac{\partial^\alpha u(0)}{\alpha!} x^\alpha.$$

Then, $\langle v - u, g \rangle = 0$ if $g = \partial^\alpha \delta(\cdot - 0)$ and $M_1 \leq |\alpha| \leq M_0$. Let $a(\xi)$ be the symbol of the constant-coefficient differential operator $A(x, D)$, i.e.,

$$A = A(x, D) = \sum_{|\alpha| = m} \frac{1}{\alpha!} \partial_\xi^\alpha a(0) D^\alpha.$$

Define $\tilde{M}_j := \max\{0, M_j - m\}$ for $j \in \{0, 1\}$. A straightforward calculation gives us:

$$
\begin{aligned}
A(x, D)v(x) &= i^m \sum_{|\alpha|=m} \sum_{|\beta|=\tilde{M}_1}^{\tilde{M}_0} \frac{1}{\alpha!} \frac{1}{\beta!} \partial_\xi^\alpha a(0) \partial_x^{\alpha+\beta} u(0) \cdot x^\beta \\
&= \sum_{|\beta|=\tilde{M}_1}^{\tilde{M}_0} \frac{1}{\beta!} \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \partial_\xi^\alpha a(0) \cdot D^\alpha (\partial_x^\beta u)(0) \right) \cdot x^\beta \\
&= \sum_{|\beta|=\tilde{M}_1}^{\tilde{M}_0} \frac{1}{\beta!} A \partial^\beta u(0) \cdot x^\beta.
\end{aligned}
$$

Because $\partial^\beta \delta(\cdot - 0) \in \tilde{L}^-$ when $\tilde{M}_1 \leq |\beta| \leq \tilde{M}_0$, we obtain $A \partial^\beta u(0) = \partial^\beta A u(0) = 0$ for such multi-indices β. Now we conclude that $Av = 0$, which proves our claim. $\qquad \square$

Remark 5.5.7.

(i) In Proposition 5.4.16, if $d > m$ then any elliptic real-constant-coefficient homogeneous differential operator A of order m on $\mathbb{R}^d$ satisfies Assumption $\mathcal{A}$. Notice that

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m must be even. Since $F_{A,\mathbb{R}} = \{0\}$, it is not hard to see from Theorem 5.2.10 that $\dim V_N^p(A) = \dim V_N^p((-\Delta)^{m/2})$. In particular, if μ is the point divisor $x_0^{-(M_0+1)}$, the Liouville-Riemann-Roch formula becomes

$$\dim L_p(\mu, A, N) = \begin{cases} h_{d,[N]}^{(m)} - h_{d,M_0}^{(m)}, & \text{if } p = \infty. \\ h_{d,\lfloor N-d/p\rfloor}^{(m)} - h_{d,M_0}^{(m)}, & \text{otherwise,} \end{cases}$$

though this can be proven elementarily. Here for $A, B, C \in \mathbb{N}$, we denote by $h_{A,B}^{(C)}$ the quantity $\binom{A+B}{A} - \binom{A+B-C}{A}$, where we adopt the agreement in Definition 5.2.11.

(ii) As a special case of Proposition 5.4.16, the Liouville-Riemann-Roch equality for the Laplacian operator could occur when μ^- is not trivial (compare with Theorem 5.4.10). As we have seen so far, the corresponding spaces $L_{p'}(\mu^{-1}, -\Delta, -N)$ in this situation are trivial too. It is worth mentioning that it is possible to obtain the Liouville-Riemann-Roch equality in certain cases for which the dimensions of the spaces $L_{p'}(\mu^{-1}, -\Delta, -N)$ are as large as possible. For instance, let $p = \infty$ and $r \geq N+3$, we define $\mu := (\emptyset, 0; D^-, L^-)$ with $D^- = \{0\}$ and $L^- = \text{span}_{\mathbb{C}}\{\partial^\alpha \delta(\cdot - 0)\}_{|\alpha|=r}$. Then clearly, $L_\infty(\mu, -\Delta, N) = V_N^\infty(-\Delta)$. From the proof of the second part of Proposition 5.4.11,

$$L_1(\mu^{-1}, -\Delta, -N) = \text{span}_{\mathbb{C}}\{\partial^\alpha(|x|^{2-d})\}_{|\alpha|=r}$$
$$= \{u \in C^\infty(\mathbb{R}^d \setminus \{0\}) \mid -\Delta u \in L^-, \lim_{|x|\to\infty} |u(x)| = 0\}.$$

By Theorem 5.3.10, it is easy to see that the dimension of this space is equal to the degree of the divisor μ^{-1} (see also Example 5.3.14). Thus, $\dim L_\infty(\mu, -\Delta, N) =$

$$\dim L_1(\mu^{-1}, -\Delta, -N) + \deg_{-\Delta}(\mu) + \dim V_N^\infty(-\Delta) \text{ and as } r \to \infty,$$

$$\dim L_1(\mu^{-1}, -\Delta, -N) = \binom{d+r-1}{d-1} - \binom{d+r-3}{d-1} \to \infty.$$

Proof of Theorem 5.4.21

Proof. The key lemma of our proof is the following statement:

Lemma 5.5.8. *Let us consider* $p_1, p_2 \in [1, \infty]$ *and two positive functions* φ_1 *and* φ_2 *in* $\mathcal{S}(G)$ *such that one of the following two conditions holds:*

- $p_1^{-1} + p_2^{-1} \geq 1$ *and* $\varphi_1 \varphi_2$ *is bounded on* G.

- $p_1^{-1} + p_2^{-1} \leq 1$ *and* $\varphi_1^{-1} \varphi_2^{-1}$ *is bounded on* G.

Then the following Riemann-Roch formula holds:

$$\dim L_{p_1}(\mu, P, \varphi_1) = \deg_P(\mu) + \dim L_{p_2}(\mu^{-1}, P^*, \varphi_2),$$

where μ *is any rigged divisor on* $\mathcal{X}$.

Proof of Lemma 5.5.8.

Proof. We follow the strategy of the proof of Theorem 5.4.6. As in Definition 5.5.1, for each $s \in \mathbb{R}$, $\varphi \in \mathcal{S}(G)$ and $p \in [1, \infty]$, let us introduce the following space

$$\mathcal{V}_{s,\varphi}^p(\mathcal{X}) := \{u \in C^\infty(\mathcal{X}) \mid \{\|u\|_{H^s(g\mathcal{F})} \cdot \varphi(g)\}_{g \in G} \in \ell^p(G)\}.$$

and we denote by $P_{m,\varphi}^p$ the operator P with the domain

$$\operatorname{Dom} P_{m,\varphi}^p := \{u \in \mathcal{V}_{m,\varphi}^p(\mathcal{X}) \mid Pu \in C_c^\infty(\mathcal{X})\}.$$

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For the elliptic differential operator P^*, we also use the corresponding notations $(P^*)^p_{m,\varphi}$ and $\mathrm{Dom}(P^*)^p_{m,\varphi}$.

Now let us fix a pair of two real numbers (p_1, p_2) and a pair of two functions (φ_1, φ_2) satisfying the conditions of Lemma 5.5.8. From now on, we will consider the operator $P^{p_1}_{m,\varphi_1}$ and its "adjoint" $(P^{p_1}_{m,\varphi_1})^* := (P^*)^{p_2}_{m,\varphi_2}$. As before, we define $\mathrm{Dom}' P^{p_1}_{m,\varphi_1} = \mathrm{Dom}'(P^*)^{p_2}_{m,\varphi_2} = C_c^\infty(\mathcal{X})$.

Our goal is to verify the assumptions of Theorem 5.3.10 for the operator $P^{p_1}_{m,\varphi_1}$ and its adjoint $(P^*)^{p_2}_{m,\varphi_2}$. The proof also goes through four steps as in the proof of Theorem 5.4.6. We consider two cases.

Case 1. $p_1^{-1} + p_2^{-1} \geq 1, \varphi_1\varphi_2 \lesssim 1$. The proof of Step 1 stays exactly the same as before (see Remark 5.7.8). For Step 2, the first three properties $(\mathcal{P}1) - (\mathcal{P}3)$ are also obvious. For the property $(\mathcal{P}4)$, we want to show that whenever $u \in \mathrm{Dom}\, P^{p_1}_{m,\varphi_1}$ and $v \in \mathrm{Dom}(P^*)^{p_2}_{m,\varphi_2}$,

$$\langle Pu, v \rangle = \langle u, P^*v \rangle. \tag{5.50}$$

Because P and P^* are C^∞-bounded, we can repeat the approximation procedure and obtain similar estimates from the proof from Theorem 5.4.6 for showing the identity (5.50) whenever $u \in \mathrm{Dom}\, P^{p_1}_{m,\varphi_1}$ and $v \in \mathrm{Dom}(P^*)^{p_1'}_{m,\varphi_1^{-1}}$. On the other hand, $\mathcal{V}^{p_2}_{m,\varphi_2}(\mathcal{X}) \subseteq \mathcal{V}^{p_1'}_{m,\varphi_1^{-1}}(\mathcal{X})$ and hence, $\mathrm{Dom}(P^*)^{p_2}_{m,\varphi_2} \subseteq \mathrm{Dom}(P^*)^{p_1'}_{m,\varphi_1^{-1}}$. With this inclusion, it is enough to conclude the property $(\mathcal{P}4)$ in this case and this finishes Step 2. For Step 3, first, it is clear that the kernels of $P^{p_1}_{m,\varphi_1}$ and $(P^*)^{p_2}_{m,\varphi_2}$ are both trivial since P and P^* satisfy (SSP) (see Theorem 5.4.18). So the rest is to verify the Fredholm property for both operators $P^{p_1}_{m,\varphi_1}$ and $(P^*)^{p_2}_{m,\varphi_2}$, i.e., to prove that

$$\mathrm{Im}\, P^{p_1}_{m,\varphi_1} = \mathrm{Im}(P^*)^{p_2}_{m,\varphi_2} = C_c^\infty(\mathcal{X}). \tag{5.51}$$

Let us prove (5.51) for the operator $P_{m,\varphi_1}^{p_1}$ since the other identity is proved similarly. We denote by $G_P(x, y)$ the Green's function of P at the level $\lambda = 0$, i.e., $G_p(x, y)$ is the Schwartz kernel of the resolvent operator P^{-1}. It is known that $G_P(x, y) \in C^\infty(\mathcal{X} \times \mathcal{X} \setminus \Delta)$, where $\Delta = \{(x, x) \mid x \in \mathcal{X}\}$. Moreover, all of its derivatives have exponential decay off the diagonal (see [71, Theorem 2.2]). However, it is more convenient for us to use its L^2-norm version, i.e., [71, Theorem 2.3]: there exists $\varepsilon > 0$ such that for every $\delta > 0$ and every multi-indices α, β, one can find a constant $C_{\alpha\beta\delta} > 0$ such that

$$\int_{x:d_\mathcal{X}(x,y) \geq \delta} |\partial_x^\alpha \partial_y^\beta G_P(x, y)|^2 \exp\left(\varepsilon d_\mathcal{X}(x, y)\right) \mathrm{d}\mu_\mathcal{X}(x) \leq C_{\alpha\beta\delta}. \tag{5.52}$$

Here the derivatives $\partial_x^\alpha, \partial_y^\beta$ are taken with respect to canonical coordinates and the constants $C_{\alpha\beta\delta}$ do not depend on the choice of such canonical coordinates. Note that these decay estimates (5.52) still work in the case of exponential growth of the volume of the balls on $\mathcal{X}$. Let $f \in C_c^\infty(\mathcal{X})$ and K be its compact support in $\mathcal{X}$. We introduce

$$u(x) := P^{-1} f(x) = \int_\mathcal{X} G_P(x, y) f(y) \, \mathrm{d}\mu_\mathcal{X}(y),$$

where $\mu_\mathcal{X}$ is the Riemannian measure on $\mathcal{X}$. Thus, $u \in L^2(\mathcal{X})$ since P^{-1} is a bounded operator on $L^2(\mathcal{X})$. It is clear that u is a weak solution of the equation $Pu = f$, and hence, by regularity, u is a smooth solution. We only need to prove that $u \in \mathcal{V}_{m,\varphi_1}^1(\mathcal{X}) \subseteq \mathcal{V}_{m,\varphi_1}^{p_1}(\mathcal{X})$. Let us consider any g in $G_{\bar{\mathcal{F}},K} := \{g \in G \mid \mathrm{dist}\,(g\bar{\mathcal{F}}, K) > 1\}$. Since $\mathcal{X}$ is quasi-isometric to the metric space (G, d_S) via the orbit action by the Švarc-Milnor lemma, it is not hard to see that there are constants $C_1, C_2 > 0$ such that for every $g \in G_{\bar{\mathcal{F}},K}$ and every $(x, y) \in g\mathcal{F} \times K$, one has

$$2C_1|g| - C_2 \leq d_\mathcal{X}(x, y) \leq (2C_1)^{-1}|g| + C_2. \tag{5.53}$$

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Taking $\delta = 1$, we can find $\varepsilon > 0$ so that the decay estimates (5.52) satisfy. Now using Hölder's inequality, (5.52) and (5.53), we derive

$$\|u\|_{H^m(g\mathcal{F})} \lesssim \sup_{g \in G_{\bar{\mathcal{F}},K}} \max_{|\alpha| \leq m} \left(\int_{g\mathcal{F}} \left| \int_K |\partial_x^\alpha G_P(x,y)| \cdot |f(y)| \, d\mu_{\mathcal{X}}(y) \right|^2 d\mu_{\mathcal{X}}(x) \right)^{1/2}$$

$$\lesssim \|f\|_{L^2(\mathcal{X})} \cdot \max_{|\alpha| \leq m} \left(\int_{g\mathcal{F}} \int_K |\partial_x^\alpha G_P(x,y)|^2 \, d\mu_{\mathcal{X}}(y) \, d\mu_{\mathcal{X}}(x) \right)^{1/2}$$

$$\lesssim \|f\|_{L^2(\mathcal{X})} \cdot \exp\left(-2C_1 \varepsilon |g|\right) \max_{|\alpha| \leq m} \sup_{y \in K} \left(\int_{g\mathcal{F}} |\partial_x^\alpha G_P(x,y)|^2 \exp\left(\varepsilon d_{\mathcal{X}}(x,y)\right) d\mu_{\mathcal{X}}(x) \right)^{1/2}$$

$$\lesssim \|f\|_{L^2(\mathcal{X})} \cdot \exp\left(-2C_1 \varepsilon |g|\right) \lesssim \|f\|_{L^2(\mathcal{X})} \cdot \varphi_1(g) \cdot \exp\left(-C_1 \varepsilon |g|\right).$$

Note that the above estimates hold up to multiplicative constants which are uniform with respect to $g \in G_{\bar{\mathcal{F}},K}$. Therefore, u belongs to $\mathcal{V}^{p_1}_{m,\varphi_1}(\mathcal{X})$, and this proves (5.51). In particular, the Fredholm indices of the operators $P^{p_1}_{m,\varphi_1}$ and $(P^*)^{p_2}_{m,\varphi_2}$ vanish. Now we are able to apply Theorem 5.3.10 to finish the proof of Lemma 5.5.8 in this case.

Case 2. $p_1^{-1} + p_2^{-1} \leq 1, \varphi_1^{-1}\varphi_2^{-1} \lesssim 1$. Consider a rigged divisor μ on $\mathcal{X}$. By assumptions, $L_{p_2'}(\mu, P, \varphi_2^{-1}) \subseteq L_{p_1}(\mu, P, \varphi_1)$ and $L_{p_1'}(\mu^{-1}, P^*, \varphi_1^{-1}) \subseteq L_{p_2}(\mu^{-1}, P^*, \varphi_2)$. From these inclusions and Case 1, we obtain

$$\dim L_{p_1}(\mu, P, \varphi_1) = \deg_P(\mu) + \dim L_{p_1'}(\mu^{-1}, P^*, \varphi_1^{-1}) \leq \deg_P(\mu) + L_{p_2}(\mu^{-1}, P^*, \varphi_2)$$

$$= \dim L_{p_2'}(\mu, P, \varphi_2^{-1}) \leq \dim L_{p_1}(\mu, P, \varphi_1).$$

Since all of the above inequalities must become equalities, this yields the corresponding Riemann-Roch formula in this case. $\qquad \square$

We use Lemma 5.5.8 to prove all of the statements now. First, one can get the identity in the second statement of Theorem 5.4.21 by taking $p_1 = p_2 = \infty$ and $\varphi_1 = \varphi_2 = \varphi_0$ in Lemma 5.5.8. Also, due to Theorem 5.4.18, there is no non-zero solution of P^* with subexponential growth. This implies that if $\mu^{-1} = (\emptyset, 0; D^+, L^+)$, the space

$L_\infty(\mu^{-1}, P^*, \varphi_0)$ is trivial. Thus, the third statement follows immediately from the second statement. For the first statement, let us consider $p \in [1, \infty]$ and a function $\varphi \in \mathcal{S}(G)$. Now from Lemma 5.5.8 and the second statement, one gets:

- If φ is bounded,

$$\dim L_1(\mu, P, \varphi) = \deg_P(\mu) + \dim L_\infty(\mu^{-1}, P^*, \varphi_0) = \dim L_\infty(\mu, P, \varphi_0). \quad (5.54)$$

- If φ^{-1} is bounded and $1 \le p \le \infty$,

$$\dim L_p(\mu, P, \varphi) = \deg_P(\mu) + \dim L_\infty(\mu^{-1}, P^*, \varphi_0) = \dim L_\infty(\mu, P, \varphi_0). \quad (5.55)$$

We consider three cases.

Case 1. If φ is bounded, the two spaces $L_1(\mu, P, \varphi)$ and $L_\infty(\mu, P, \varphi_0)$ are the same since their dimensions are equal to each other by (5.54). Moreover, $L_1(\mu, P, \varphi) \subseteq L_p(\mu, P, \varphi) \subseteq L_p(\mu, P, \varphi_0) \subseteq L_\infty(\mu, P, \varphi_0)$. This means all these spaces are the same.

Case 2. If φ^{-1} is bounded, $L_\infty(\mu, P, \varphi_0) \subseteq L_\infty(\mu, P, \varphi)$. Using (5.55) with $p = \infty$, we have $L_\infty(\mu, P, \varphi_0) = L_\infty(\mu, P, \varphi)$. Moreover, (5.55) also yields that all the spaces $L_p(\mu, P, \varphi)$, where $1 \le p \le \infty$, must have the same dimension and therefore, they are the same space, which coincide $L_\infty(\mu, P, \varphi_0)$.

Case 3. If neither φ nor φ^{-1} is bounded, we can consider the function $\phi := \varphi + \varphi^{-1}$. Clearly, ϕ is in $\mathcal{S}(G)$ and $\phi \ge 2$. Then according to Case 1 and Case 2,

$$L_p(\mu, P, \phi) = L_\infty(\mu, P, \varphi_0) = L_p(\mu, P, \phi^{-1}).$$

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Also, $L_p(\mu, P, \phi^{-1}) \subseteq L_p(\mu, P, \varphi) \subseteq L_p(\mu, P, \phi)$ since $\phi^{-1} \leq \varphi \leq \phi$. This means that $L_p(\mu, P, \varphi) = L_\infty(\mu, P, \varphi_0)$.

$\square$

Proof of Corollary 5.4.22

Proof. As in the proof of Corollary 5.4.14, the equivalence of the first three statements is an easy consequence of Theorem 5.4.21 and Remark 5.3.12. It is obvious that (iv) implies (iii). To see the converse, one can repeat the argument in the proof of Lemma 5.5.8 to show that the solution $u = P^{-1}f$ has exponential decay due to (5.52). By the unique solvability of the equation $Pu = f$ in $L^2(\mathcal{X})$, (iii) implies (iv). $\square$

5.6 Applications of Liouville-Riemann-Roch theorems to specific operators

In this part, we will try to look at some examples where one can apply the results in Section 3.2.5.

Let us begin with self-adjoint operators. Suppose that A is a bounded from below and self-adjoint periodic elliptic operator of order m on an abelian co-compact covering X. We know that the symmetric operator A with the domain $C_c^\infty(X)$ is essentially self-adjoint in $L^2(X)$. Unless confusion can arise, we always use the same notation A for the unique self-adjoint extension [17] of A in $L^2(X)$.

To apply the results from Section 5.4, we will reformulate *Assumption $\mathcal{A}$* in Section 5.4.

Assumption $\mathcal{A}'$

Suppose that $F_{A,\mathbb{R}} = \{k_1, \cdots, k_\ell\}$ (modulo G^*-shifts). Let $\{\lambda_{r,j}\}_{j=\overline{1,m_r}}$ be the set of dispersion branches that equals to 0 at the quasimomentum k_r $(1 \leq r \leq \ell)$. There exists

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[17] The domain of this extension is the Sobolev space $H^m(X)$.

a family of pairwise disjoint neighborhoods V_r of k_r such that the function $k \in V_r \mapsto$ $\max_{1 \leq j \leq m_r} |\lambda_{r,j}(k)|^{-1}$ is L^1-integrable.[18]

Clearly, *Assumption $\mathcal{A}$* and *Assumption $\mathcal{A}'$* are equivalent if A is self-adjoint.

Notation 5.6.1. Let N be a natural number.

- We denote by $h_{d,N}$ the dimension of harmonic polynomials of order at most N in d-variables, i.e.,

$$h_{d,N} := \dim V_N^\infty(-\Delta_{\mathbb{R}^d}) = \binom{d+N}{d} - \binom{d+N-2}{d}.$$

- We also denote by $c_{d,N}$ the dimension of the space of all homogeneous polynomials of degree N in d variables, i.e.,

$$c_{d,N} := \binom{d+N}{d} - \binom{d+N-1}{d} = \binom{d+N-1}{N}.$$

5.6.1 Periodic operators with nondegenerate spectral edges

Let λ be an energy level that coincides with one of the spectral edges of a gap in the spectrum of A. By shifting, we can suppose that $\lambda = 0$.

One can expect the Fermi surfaces at the spectral edges to be "normally" finite and hence, Liouville type results are applicable in these situations. We make the following assumption.

Assumption $\mathcal{B}$. There exists a band function $\lambda_j(k)$ such that for each quasimomentum k_r in the real Fermi surface $F_{A,\mathbb{R}}$, one has

($\mathcal{B}$1) 0 is a simple eigenvalue of the operator $A(k_r)$.

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[18]Note that for each $k \in V_r \setminus \{k_r\}$, $\lambda_{r,j}(k) \neq 0$ since $V_r \cap F_{A,\mathbb{R}} = \{k_r\}$.

(B2) The Hessian matrix $\operatorname{Hess} \lambda_j(k_0)$ is non-degenerate.

For self-adjoint second-order periodic elliptic operators, it is commonly believed in mathematics and physics literature (see e.g, [48]) that generically (with respect to the co-efficients and other free parameters of the operator), extrema of band functions are isolated, attained by a single band and have non-degenerate Hessians.

Suppose now that the free abelian rank d of the deck group G is greater than 2. The non-degeneracy assumption (B2) implies the integrability of function $|\lambda_j(k)|^{-1}$ over a small neighborhood of $F_{A,\mathbb{R}}$. Hence, *Assumption $\mathcal{A}$* follows from *Assumption $\mathcal{B}$*.

Due to Theorem 5.2.10, the dimension of the space $V_N^\infty(A)$ is equal to $\ell h_{d,[N]}$ (see Notation 5.6.1). Applying the results in Section 5.4, we obtain the following results for a 'generic' self-adjoint periodic elliptic operator A of second-order:

Theorem 5.6.2. *Suppose $d \geq 3$ and $N \in \mathbb{R}$. Let $\mu = (D^+, L^+; D^-, L^-)$ be a rigged divisor on X. Recall that $\mu^+ = (D^+, L^+; \emptyset, 0)$.*

(a) If $N \geq 0$ then

$$\ell h_{d,[N]} + \deg_A(\mu) + \dim L_1(\mu^{-1}, A^*, -N) \leq \dim L_\infty(\mu, A, N)$$
$$\leq \ell h_{d,[N]} + \deg_A(\mu^+),$$

> *and*

$$\dim L_\infty(\mu^+, A, N) = \ell h_{d,[N]} + \deg_A(\mu^+).$$

(b) If $p \in [1, \infty)$ and $N > d/p$ then

$$\ell h_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N)$$
$$\leq \ell h_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu^+),$$

and

$$\dim L_p(\mu^+, A, N) = \ell h_{d, \lfloor N - d/p \rfloor} + \deg_A(\mu^+)$$

(c) For $d \geq 5$, the inequality

$$\deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N) \leq \deg_A(\mu^+)$$

holds if we assume one of the following conditions:

- *$p \in [1, 2)$, $N \in (d(2 - p)/(2p), d/p]$.*

- *$p \in [2, \infty)$, $N \in [0, d/p]$.*

Example 5.6.3.   1. Let $A = A^* = -\Delta + V$ be a periodic Schrödinger operator on a co-compact abelian cover X with real-valued electric potential V. The domain of A is the Sobolev space $H^2(X)$, and thus, A is self-adjoint. Suppose that 0 is the bottom of its spectrum. It is well-known [43] that *Assumption $\mathcal{B}$* holds in this situation. Hence, all the conclusions of Theorem 5.6.2 hold with $\ell = 1$ since $F_{A,\mathbb{R}} = \{0\}$ (modulo $2\pi\mathbb{Z}^d$-shifts).

2. In [13, 14], a deep analysis of the dispersion curves at the bottom of the spectrum was developed for a wide class of periodic elliptic operators of second-order on $\mathbb{R}^d$. Namely, let Γ be a lattice in $\mathbb{R}^d$ and Γ^* be its dual lattice, then these operators admit the following *regular factorization*:

$$A = \overline{f(x)}b(D)^*g(x)b(D)f(x),$$

where $b(D) = \sum_{j=1}^{d} -i\partial_{x_j} b_j : L^2(\mathbb{R}^d, \mathbb{C}^n) \to L^2(\mathbb{R}^d, \mathbb{C}^m)$ is a linear homogeneous differential operator whose coefficients b_j are constant $m \times n$-matrices of rank n

(here $m \geq n$), f is a Γ-periodic and invertible $n \times n$ matrix function such that f and f^{-1} are in $L^\infty(\mathbb{R}^d)$, and g is a Γ-periodic and positive definite $m \times m$-matrix function such that for some constants $0 < c_0 \leq c_1$, $c_0 \mathbf{1}_{m \times m} \leq g(x) \leq c_1 \mathbf{1}_{m \times m}$, $x \in \mathbb{R}^d$. The existence of this factorization implies that the first band function attains a simple and nondegenerate minimum with value 0 at *only* the quasimomentum $k = 0$ (modulo Γ^*-shifts). This covers the previous example since it was noted from [13] that for a periodic metric $g(x)$ and a periodic potential $V(x)$, the periodic Schrödinger operator $D(g(x)D) + V(x)$ admits a regular factorization.

3. Consider a self-adjoint periodic magnetic Schrödinger operator in $\mathbb{R}^n$ ($n > 2$)

$$H = (-i\nabla + A(x))^2 + V(x),$$

where $A(x)$ and $V(x)$ are real-valued periodic magnetic and electric potentials, respectively. Using a gauge transformation, we can always assume the following normalization condition on A (without effecting our results):

$$\int_{\mathbb{T}^n} A(x)\, dx = 0.$$

Note that the transpose of H is the magnetic Schrödinger operator

$$H^* = (-i\nabla - A(x))^2 + V(x).$$

From the discussion of [50] (see also [47, Theorem 3.1.7]), the lowest band function of H has a unique nondegenerate extremum at a single quasimomentum k_0 (modulo G^*-shifts) if the magnetic potential A is small enough, e.g., $\|A\|_{L^r(\mathbb{T}^n)} \ll 1$ or some $r > n$. Thus, we obtain the same conclusion as the case without magnetic potential.

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It is crucial that one has to assume the smallness of the magnetic potential since there are examples [69] showing that the bottom of the spectrum are degenerate if the magnetic potential is large enough.

We end this part by proving the following upper-semicontinuity property for families of periodic elliptic operators with non-degenerate spectral edges.

Corollary 5.6.4. *Let A_0 be a periodic elliptic operator on a co-compact abelian covering X with the deck group $G = \mathbb{Z}^d$, where $d \geq 5$, such that its real Fermi surface $F_{A_0, \mathbb{R}}$ consists of finitely many simple and non-degenerate minima of the jth-band function of A_0 ($j \geq 1$). Let B be a symmetric and periodic differential operator on X such that B is A_0-bounded. We consider the perturbation $A_z = A_0 + zB$, $z \in \mathbb{R}$. By standard perturbation theory, there exists a continuous function $\lambda(z, k)$ defined for small z and all quasimomenta k such that $k \mapsto \lambda(z, k)$ is the jth band function of A_z and $\lambda_j(z, k)$ is analytic in z. Let λ_z be the minimum value of the band function $\lambda(z, k)$ of the perturbation A_z. Then for any rigged divisor μ, there exists $\varepsilon > 0$ such that*

$$\dim L_2(\mu, A_z - \lambda_z, 0) \leq \dim L_2(\mu, A_0, 0),$$

for any z satisfying $|z| < \varepsilon$.

The corresponding statement for non-degenerate maxima also holds.

Proof. In a similar manner to the proof of Proposition 5.4.14, for the rigged divisor μ, the corresponding extension operator $\widetilde{(A_z)^2_{m,0}} : \Gamma(X, \mu, (A_z)^2_{m,0}) \to \tilde{\Gamma}_\mu(X, (A_z)^2_{m,0})$ is Fredholm. As in the proof of [79, Theorem 2], we can deduce the upper-semicontinuity of $\dim \mathrm{Ker}\, \widetilde{(A_z)^2_{m,0}}$ by using [79, Theorem 1] and [79, Theorem 3]. Since $\mathrm{Ker}\, \widetilde{(A_z)^2_{m,0}} = L_2(\mu, A_z, 0)$, this finishes our proof. $\qquad\square$

5.6.2 Periodic operators with Dirac points

An important situation in solid state physics and material sciences is when two branches of the dispersion relation touch to form a conical junction point, which is called a Dirac point. Two-dimensional massless Dirac operators or $2D$-Schrödinger operators with honeycomb symmetric potentials are typical models of periodic operators with conical structures [11, 23, 35, 51].

Let us consider a self-adjoint periodic elliptic operator A such that there are two branches λ_+ and λ_- of the dispersion relation of A that meet at 0 and form a Dirac cone. Equivalently, we can assume that locally around each quasimomentum k_r in the Fermi surface $F_{A,\mathbb{R}}$, for some $c_r \neq 0$, one has

$$\lambda_+(k) = c_r|k - k_r| \cdot (1 + O(|k - k_r|)),$$
$$\lambda_-(k) = -c_r|k - k_r| \cdot (1 + O(|k - k_r|)).$$

Then, the functions $|\lambda_+|^{-1}$ and $|\lambda_-|^{-1}$ are integrable over a small neighborhood of $F_{A,\mathbb{R}}$ provided that $d > 1$. Hence, we conclude:

Theorem 5.6.5. *Suppose $d \geq 2$. Assume that the Fermi surface $F_{A,\mathbb{R}}$ consists of ℓ Dirac conical points. Let us recall the notation $c_{d,N}$ in Notation 5.6.1. Then as in Theorem 5.6.5, we have*

(a) If $N \geq 0$ then

$$2\ell c_{d,[N]} + \deg_A(\mu) + \dim L_1(\mu^{-1}, A^*, -N) \leq \dim L_\infty(\mu, A, N) \leq 2\ell c_{d,[N]} + \deg_A(\mu^+),$$

and

$$\dim L_\infty(\mu^+, A, N) = 2\ell c_{d,[N]} + \deg_A(\mu^+).$$

(b) If $p \in [1, \infty)$ and $N > d/p$ then

$$2\ell c_{d, \lfloor N-d/p \rfloor} + \deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N)$$

$$\leq 2\ell c_{d, \lfloor N-d/p \rfloor} + \deg_A(\mu^+),$$

and

$$\dim L_p(\mu^+, A, N) = 2\ell c_{d, \lfloor N-d/p \rfloor} + \deg_A(\mu^+).$$

(c) For $d \geq 3$ and a pair (p, N) satisfying the condition in Theorem 5.6.5 (c), the same conclusion of Theorem 5.6.5 (c) also holds.

Example 5.6.6. In this example, we consider Schrödinger operators with honeycomb lattice potentials in $\mathbb{R}^2$. Let us recall briefly some notions from [23]. The triangular lattice $\Lambda_h = \mathbb{Z}\mathbf{v}_1 \oplus \mathbb{Z}\mathbf{v}_2$ is spanned by the basis vectors:

$$\mathbf{v}_1 = a \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)^t, \quad \mathbf{v}_2 = a \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right)^t \quad (a > 0).$$

The dual lattice is $\Lambda_h^* = \mathbb{Z}\mathbf{k}_1 \oplus \mathbb{Z}\mathbf{k}_2$, where

$$\mathbf{k}_1 = \frac{4\pi}{a\sqrt{3}} \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)^t, \quad \mathbf{k}_2 = \frac{4\pi}{a\sqrt{3}} \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right)^t.$$

We define

$$\mathbf{K} = \frac{1}{3}(\mathbf{k}_1 - \mathbf{k}_2), \quad \mathbf{K}' = -\mathbf{K}.$$

The Brillouin zone $\mathcal{B}_h$, a fundamental domain of the quotient $\mathbb{R}^2/\Lambda_h^*$, can be chosen as a hexagon in $\mathbb{R}^2$ such that all six vertices of this hexagon fall into two groups:

1. $\mathbf{K}$ type-vertices: $\mathbf{K}, \mathbf{K} + \mathbf{k}_2, \mathbf{K} - \mathbf{k}_1$.

2. $\mathbf{K}'$ type-vertices: $\mathbf{K}', \mathbf{K}' - \mathbf{k}_2, \mathbf{K}' + \mathbf{k}_1$.

Note that these groups of vertices are invariant under the clockwise rotation $\mathcal{R}$ by $2\pi/3$.

Let potential $V \in C^\infty(\mathbb{R}^2)$ be real, Λ_h-periodic, and such that there exists a point $x_0 \in \mathbb{R}^2$ such that V is inversion symmetric (i.e., even) and $\mathcal{R}$-invariant with respect to x_0 (see [23, Remark 2.4] for some construction of such potentials). Now consider the Schrödinger operator $H^\varepsilon = -\Delta + \varepsilon V$ ($\varepsilon \in \mathbb{R}$). One of the main results of [23] is that except possibly for ε in a countable and closed set $\tilde{\mathcal{C}}$, the dispersion relation of H^ε has conical singularities at each vertex of $\mathcal{B}_h$. Assume that λ_j^ε, $j \in \mathbb{N}$, are the band functions of the operator H^ε for each $\varepsilon \in \mathbb{R}$. Then according to [23, Theorem 5.1], when $\varepsilon \notin \tilde{\mathcal{C}}$, there exists some $j \in \mathbb{N}$ such that the Fermi surface $F_{H^\varepsilon, \lambda_j^\varepsilon(\mathbf{K})}$ of the operator H^ε at the level $\lambda_j^\varepsilon(\mathbf{K})$ contains (at least) two Dirac points located at the quasimomenta $\mathbf{K}$ and $\mathbf{K}'$ (modulo shifts by vectors in the dual lattice Λ_h^*). Now our next corollary is a direct consequence of [23, Theorem 5.1] and our previous discussion:

Corollary 5.6.7. *Let μ be a rigged divisor on $\mathbb{R}^2$ and V be a honeycomb lattice potential such that the following Fourier coefficient $V_{1,1}$ of the potential V*

$$V_{1,1} := \int_{\mathbb{R}^2/\Lambda_h} e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot x} V(x)\, \mathrm{d}x$$

is nonzero. Then for $\varepsilon \notin \tilde{\mathcal{C}}$, there exists $j \in \mathbb{N}$ such that the following inequalities hold:

1. $p = \infty, N \geq 0$:

$$\dim L_\infty(\mu, H^\varepsilon - \lambda_j^\varepsilon(\mathbf{K}), N) \geq 4([N]+1) + \deg_{H^\varepsilon - \lambda_j^\varepsilon(\mathbf{K})}(\mu)$$
$$+ \dim L_1(\mu^{-1}, H^\varepsilon - \lambda_j^\varepsilon(\mathbf{K}), -N),$$

$$\dim L_\infty(\mu^+, H^\varepsilon - \lambda_j^\varepsilon(\mathbf{K}), N) \geq 4([N]+1) + \deg_{H^\varepsilon - \lambda_j^\varepsilon(\mathbf{K})}(\mu^+).$$

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2. $1 \leq p < \infty, N > 2/p$:

$$\dim L_p(\mu, H^\varepsilon - \lambda_j^\varepsilon(\boldsymbol{K}), N) \geq 4(\lfloor N - 2/p \rfloor + 1) + \deg_{H^\varepsilon - \lambda_j^\varepsilon(\boldsymbol{K})}(\mu)$$
$$+ \dim L_{p'}(\mu^{-1}, H^\varepsilon - \lambda_j^\varepsilon(\boldsymbol{K}), -N),$$

$$\dim L_p(\mu^+, H^\varepsilon - \lambda_j^\varepsilon(\boldsymbol{K}), N) \geq 4(\lfloor N - 2/p \rfloor + 1) + \deg_{H^\varepsilon - \lambda_j^\varepsilon(\boldsymbol{K})}(\mu^+).$$

Moreover, there is some $\varepsilon_0 > 0$ so that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, we have

- $\varepsilon V_{1,1} > 0$: *the above j can be chosen as $j = 1$.*

- $\varepsilon V_{1,1} < 0$: *the above j can be chosen as $j = 2$.*

Remark 5.6.8. Other important results on the existence of Dirac points in the dispersion relations of Schrödinger operators with periodic potentials on honeycomb lattices are established for instance in [51] for quantum graph models of graphene and carbon nanotube materials and in [11] for many interesting models including both discrete, quantum graph, and continuous ones.

5.6.3 Non-self-adjoint second order elliptic operators

We now consider a class of possibly non-self-adjoint second-order elliptic operators arising from probability theory. Let A be a G-periodic linear elliptic operator of second-order acting on functions u in $C^\infty(X)$ such that in any coordinate system $(U; x_1, \ldots, x_n)$, the operator A can be represented in the form

$$A = \sum_{1 \leq i,j \leq n} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{1 \leq j \leq n} b_j(x)\partial_{x_j} + c(x), \tag{5.56}$$

where the coefficients a_{ij}, b_j, c are *real*, smooth, and periodic. We notice that the coefficient $c(x)$ of zeroth-order of A is globally defined on X since it is the image of the constant function 1 via A.

Definition 5.6.9. [3, 56, 64]

(a) A function u on X is called G-multiplicative with exponent $\xi \in \mathbb{R}^d$ if it satisfies

$$u(g \cdot x) = e^{\xi \cdot g} u(x), \quad \forall x \in X, g \in G.$$

In other words, u is a Bloch function with quasimomentum $i\xi$.

(b) The *generalized principal eigenvalue* of A is defined by

$$\Lambda_0 = \sup\{\lambda \in \mathbb{R} \mid (A - \lambda)u = 0 \text{ for some positive solution } u\}.$$

The principal eigenvalue is a generalized version of the bottom of the spectrum in the self-adjoint case (see e.g., [2]).

For an operator A of the type (5.56) and any $\xi \in \mathbb{R}^d$, it is known from [3, 47, 56, 64] that there exists a unique real number $\Lambda(\xi)$ such that the equation $(A - \Lambda(\xi))u = 0$ has a *positive* G-multiplicative solution u. We list some known properties of this important function $\Lambda(\xi)$. The reader can find proofs in [3, 56, 64] (see also [50, Lemma 5.7]).

Proposition 5.6.10. *(a)* $\Lambda_0 = \max\limits_{\xi \in \mathbb{R}^d} \Lambda(\xi)$.

(b) The function $\Lambda(\xi)$ is strictly concave, real analytic, bounded from above, and its gradient vanishes at only its maximum point. The Hessian of $\Lambda(\xi)$ is nondegenerate at all points.

(c) $\Lambda(\xi)$ is the principal eigenvalue with multiplicity one of the operator $A(i\xi)$.

(d) $\Lambda_0 \geq 0$ if and only if A admits a positive periodic (super-) solution, which is also equivalent to the existence of a positive G-multiplicative solution of the equation $Au = 0$.

(e) $\Lambda_0 = 0$ *if and only if there is exactly one normalized positive solution* u *of the equation*

$$Au = 0.$$

We are interested in studying Liouville-Riemann-Roch type results for such operators A satisfying $\Lambda(0) \geq 0$, which implies that A has a positive solution.

Example 5.6.11. 1. Divergence type operators with no zeroth-order coefficient satisfy $\Lambda_0 = \Lambda(0) = 0$.

2. If the zeroth-order coefficient $c(x)$ is nonnegative, $\Lambda(0)$ is also nonnegative.[19] Indeed, let u be a positive and periodic solution of the equation $Au = \Lambda(0)u$. If $\Lambda(0) < 0$, it follows from the equation that u is a positive and periodic subsolution of A on X. By the strong maximum principle, u must be constant. This means that $0 \leq cu = Au < 0$, which is a contradiction!

Before stating the main result of this subsection, let us provide a key lemma.

Lemma 5.6.12. *(a) If* $\Lambda(0) > 0$ *then* $F_{A,\mathbb{R}} = \emptyset$.

(b) If $\Lambda(0) = 0$ *then* $F_{A,\mathbb{R}} = \{0\}$ *(modulo* G^*-*shifts). In this case, there exists an open strip* V *in* $\mathbb{C}^d$ *containing the imaginary axis* $i\mathbb{R}^d$ *such that for any* $k \in V$, *there is exactly one (isolated and nondegenerate eigenvalue) point* $\lambda(k)$ *in* $\sigma(A(k))$ *that is close to* 0. *The dispersion function* $\lambda(k)$ *is analytic in* V *and* $\lambda(ik) = \Lambda(k)$ *if* $k \in \mathbb{R}^d$. *Moreover,*

- *When* $\Lambda_0 > 0$, $k = 0$ *is a noncritical point of the dispersion* $\lambda(k)$ *in* $V \cap \mathbb{R}^d$ *as well as of the function* Λ *in* $\mathbb{R}^d$.

- *When* $\Lambda_0 = 0$, $k = 0$ *is a nondegenerate extremum of the dispersion* $\lambda(k)$ *in* $V \cap \mathbb{R}^d$ *as well as of the function* Λ *in* $\mathbb{R}^d$.

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[19]In general, the converse is not true: the zeroth-order coefficient of the transpose A^* is not necessarily nonnegative while the principal eigenvalue of the operator $A^*(0)$ is the same as $\Lambda(0)$.

Proof. The statements of this lemma are direct consequences of [50, Lemma 5.8], Kato-Rellich theorem (see e.g., [66, Theorem XII.8]), and Proposition 5.6.10. $\square$

Theorem 5.6.13. *Let A be a periodic elliptic operator of second-order with real and smooth coefficients on X such that $\Lambda(0) \geq 0$. Let μ be a rigged divisor on X and μ^+ be the positive part of μ.*

(a) If $\Lambda(0) > 0$, $\dim L_\infty(\mu^+, A, \varphi) = \deg_A(\mu^+)$ and $\dim L_\infty(\mu, A, \varphi) = \deg_A(\mu) + \dim L_\infty(\mu^{-1}, A^*, \varphi^{-1})$ for any function $\varphi \in \mathcal{S}(G)$ (see Definition 5.4.20).

(b) If $\Lambda_0 > \Lambda(0) = 0$ and $d \geq 2$, then we have:

- *For any $N \geq 0$, $\dim L_\infty(\mu^+, A, N) = c_{d,[N]} + \deg_A(\mu^+)$ and*

$$c_{d,[N]} + \deg_A(\mu) + \dim L_1(\mu^{-1}, A^*, -N) \leq \dim L_\infty(\mu, A, N)$$
$$\leq c_{d,[N]} + \deg_A(\mu^+).$$

- *For any $p \in [1, \infty)$, $N > d/p$, $\dim L_p(\mu^+, A, N) = c_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu^+)$,*

$$c_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N)$$
$$\leq c_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu^+).$$

- *For $d \geq 3$ and a pair (p, N) satisfying the condition in Theorem 5.6.5 (c),*

$$\deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N) \leq \deg_A(\mu^+).$$

(c) If $\Lambda_0 = \Lambda(0) = 0$ and $d \geq 3$, then we have:

- *For any $N \geq 0$, $\dim L_\infty(\mu^+, A, N) = h_{d,[N]} + \deg_A(\mu^+)$,*

$$h_{d,[N]} + \deg_A(\mu) + \dim L_1(\mu^{-1}, A^*, -N) \leq \dim L_\infty(\mu, A, N)$$
$$\leq h_{d,[N]} + \deg_A(\mu^+).$$

- *For any $p \in [1, \infty)$, $N > d/p$, $\dim L_p(\mu^+, A, N) = h_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu^+)$,*

$$h_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N)$$
$$\leq h_{d,\lfloor N-d/p \rfloor} + \deg_A(\mu^+).$$

- *For $d \geq 5$ and a pair (p, N) satisfying the condition in Theorem 5.6.5 (c),*

$$\deg_A(\mu) + \dim L_{p'}(\mu^{-1}, A^*, -N) \leq \dim L_p(\mu, A, N) \leq \deg_A(\mu^+).$$

Proof. To compute the dimensions of the spaces $V_N^p(A)$, we use Lemma 5.6.12 to apply Theorem 5.2.10 and Theorem 5.2.14. Then Theorem 5.6.13 would follow immediately from Theorem 5.4.21, Lemma 5.6.12, Theorem 5.4.6, Proposition 5.4.10, and Theorem 5.4.12. □

5.7 Some auxiliary statements and proofs of technical lemmas

5.7.1 Some properties of Floquet functions on abelian coverings

We recall briefly another construction of Floquet functions on X. As we discussed in Subsection 5.2.2, it suffices to define the Bloch function $e_k(x)$ with quasimomentum k and the powers $[x]^j$ on X, where $j \in \mathbb{Z}_+^d$. To do this, we first recall the notion of additive functions on abelian coverings (Definition 4.2.7). We fix an additive function h. Write

$h = (h_1, \ldots, h_d)$. Let $j = (j_1, \ldots, j_d) \in \mathbb{Z}_+^d$ be a multi-index. We define

$$[x]^j := h(x)^j = \prod_{m=1}^{d} h_m(x)^{j_m},$$

and

$$e_k(x) := \exp\left(ik \cdot h(x)\right).$$

Clearly, $e_k(g \cdot x) = e^{ik \cdot g} e_k(x)$. Then a Floquet function u of order N with quasimomentum k is of the form

$$u(x) = e_k(x) \sum_{|j| \leq N} p_j(x)[x]^j,$$

where p_j is smooth and periodic. Observe that the notion of Floquet functions is independent of the choice of h. Namely, u is also a Floquet function with the same order and quasimomentum with respect to another additive function $\tilde{h}$. Indeed, the difference $w := h - \tilde{h}$ between two additive functions h and $\tilde{h}$ is periodic. Hence, one can rewrite

$$
\begin{aligned}
u(x) &= e^{ik \cdot \tilde{h}(x)} \sum_{|j| \leq N} e^{ik \cdot w(x)} p_j(x) \prod_{m=1}^{d} (\tilde{h}_m(x) + w_m(x))^{j_m} \\
&= e^{ik \cdot \tilde{h}(x)} \sum_{|j| \leq N} e^{ik \cdot w(x)} p_j(x) \sum_{j' \leq j} \binom{j}{j'} w(x)^{j-j'} \tilde{h}(x)^{j'} \qquad (5.57) \\
&= e^{ik \cdot \tilde{h}(x)} \sum_{|j| \leq N} \tilde{p}_j(x) \tilde{h}(x)^j,
\end{aligned}
$$

where $\tilde{p}_j(x) := \sum_{j \leq j'} \binom{j'}{j} e^{ik \cdot w(x)} p_{j'}(x) w(x)^{j'-j}$ is periodic. The following simple lemma is needed later.

Lemma 5.7.1. *Let K be a compact neighborhood in X. Then for any multi-index $j \in \mathbb{Z}_+^d$,*

there exists a constant C such that for any $x \in K$ and $g \in \mathbb{Z}^d$, one has

$$\left| [g \cdot x]^j - g^j \right| \leq C \langle g \rangle^{|j|-1}.$$

Proof.

$$\left| [g \cdot x]^j - g^j \right| = \left| \prod_{m=1}^{d} (h_m(x) + g_m)^{j_m} - \prod_{m=1}^{d} g_m^{j_m} \right| \leq C \langle g \rangle^{|j|-1},$$

for some $C > 0$ depending on $\|h\|_{L^\infty(K)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5.7.2 Some basic facts about the family $\{A(k)\}_{k \in \mathbb{C}^d}$

In this part, let us always identify elements in $L^2(M)$ with their corresponding periodic functions in $L^2_{loc}(X)$ and vice versa. We discuss briefly how the two definitions of the fiber operators $A(k)$ in this chapter and Chapter 4 (see Definition 4.2.14) are indeed equivalent. Note that the corresponding definition in Chapter 4 requires the introduction of additive functions on abelian coverings. However, each of these definitions has its own advantage.

Consider an additive function h on the abelian covering X (see Definition 4.2.7). Let $\mathcal{U}_k$ be the mapping that multiplies a function $f(x)$ in $L^2_k(X)$ by $e^{-ik \cdot h(x)}$. Thus, $\mathcal{U}_k$ is an invertible bounded linear operator in $\mathcal{L}(L^2_k(X), L^2(M))$ and its inverse is given by the multiplication operator by $e^{ik \cdot h(x)}$. Note that the operator norms of $\mathcal{U}_k$ and $\mathcal{U}_k^{-1}$ are bounded by $e^{|\Im k| \cdot \|h\|_{L^\infty(\overline{\mathcal{F}})}}$, where $\mathcal{F}$ is the fundamental domain appearing in the definition of the inner product of the Hilbert space $L^2_k(X)$ in Definition 5.2.3.

Define the following elliptic operator $\widehat{A}(k) := \mathcal{U}_k A(k) \mathcal{U}_k^{-1}$. The operator $\widehat{A}(k)$, with the Sobolev space $H^m(M)$ as the domain, is a closed and unbounded operator in $L^2(M)$. For each complex quasimomentum k, the two linear operators $\widehat{A}(k)$ and $A(k)$ are similar and thus, their spectra are identical. In terms of spectral information, it is no need to distinguish $A(k)$ and its equivalent model $\widehat{A}(k)$. One of the benefits of working with the later model is that $\widehat{A}(k)$ acts on the k-independent domain of periodic functions on X.

The following proposition gives a simple sufficient condition on the principal symbol of the operator A so that the spectra of $A(k)$ are discrete. More general criteria on the discreteness of spectra can be found in the book [4].

Proposition 5.7.2. *If A has real principal symbol, then for each $k \in \mathbb{C}^d$, $A(k)$, as an unbounded operator on $L^2(E_k)$, has discrete spectrum, i.e., its spectrum consists of isolated eigenvalues with finite (algebraic) multiplicities.*

Proof. Let B be the real part of the operator A. Since A has real principal symbol, the principal symbols of A and B are the same. By pushing down to operators on M, the differential operator $\widehat{A}(0) - \widehat{B}(0)$ is of lower order. Also, the principal symbols of the operators $\widehat{A}(k)$ and $\widehat{A}(0)$ are identical. Thus, we see that $\widehat{A}(k)$ is a perturbation of the self-adjoint elliptic operator $\widehat{B}(0)$ by a lower order differential operator on the compact manifold M. It follows from [4] that the spectrum of $\widehat{A}(k)$ must be discrete. This finishes the proof. $\qquad\square$

Now if one assumes that $\sigma(A(k))$ is discrete, then the family of operators $\{\widehat{A}(k)\}_{k \in \mathbb{C}^d}$, with compact resolvents, is analytic of type (A) in the sense of Kato [39]. Therefore, this family satisfies the upper-semicontinuity of the spectrum (see e.g., [39, 66]). We provide this statement here without a proof.

Proposition 5.7.3. *Consider $k_0 \in \mathbb{C}^d$. If Γ is a compact subset of the complex plane such that $\Gamma \cap \sigma(A(k_0)) = \emptyset$, then there exists $\delta > 0$ depending on Γ and k_0 such that $\Gamma \cap \sigma(A(k)) = \emptyset$, for any k in the ball $B_{k_0}(\delta)$ centered at k_0 with radius δ.*

Remark 5.7.4. (i) The Hilbert bundle $\mathcal{E}^m$ becomes the trivial bundle $\mathbb{C}^d \times H^m(M)$ via the holomorphic bundle isomorphism defined from the linear maps $\mathcal{U}_k$, where $k \in \mathbb{C}^d$.

(ii) In general, one can use the analytic Fredholm theorem to see that the essential spectrum[20] of $A(k)$ is empty for any $k \in \mathbb{C}^d$, but this is not enough to conclude that these spectra are discrete in the non-self-adjoint case. For example, if we consider the $\mathbb{Z}$-periodic elliptic operator $A = e^{2i\pi x} D_x$ on $\mathbb{R}$, a simple argument shows that

$$\sigma(A(k)) = \begin{cases} \mathbb{C}, & \text{if } k \in 2\pi\mathbb{Z} \\ \emptyset, & \text{otherwise.} \end{cases}$$

A similar example for the higher-dimensional case $\mathbb{R}^d$, $d > 1$ can be cooked up easily from the above example.

5.7.3 Some properties of Floquet transforms on abelian coverings

We now state some useful properties of the Floquet transform $\mathbf{F}$ on abelian coverings. First, due to (5.4), one can see that the Floquet transform $\mathbf{F}f(k, x)$ of a nice function f, e.g., $f \in C_c^\infty(X)$, is periodic in the quasimomentum variable k and moreover, it is a quasiperiodic function in the x-variable, i.e.,

$$\mathbf{F}f(k, g \cdot x) = \gamma_k(g) \cdot \mathbf{F}f(k, x) = e^{ik \cdot g} \cdot \mathcal{F}f(k, x), \quad \text{for any } (g, x) \in G \times X.$$

It follows that $\mathbf{F}f(k, \cdot)$ belongs to $H_k^s(X)$, for any k and s. Therefore, it is enough to regard the Floquet transform of f as a smooth section of the Hilbert bundle $\mathcal{E}^s$ over $(\mathbb{T}^*)^d$ (identified with the Brillouin zone B).

Let $K \Subset X$ be a domain such that $\bigcup_{g \in G} gK = X$. Then given any real number s, we denote by $\mathcal{C}^s(X)$ the Frechet space consists of all functions $\phi \in H_{loc}^s(X)$ such that for any $N \geq 0$, one has

$$\sup_{g \in G} \|\phi\|_{H^s(gK)} \cdot \langle g \rangle^N < \infty.$$

---

[20]Here we use the definition of the essential spectrum of an operator T as the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not Fredholm.

In terms of Definition 5.2.6, $\bigcap_{N \geq 0} V_N^\infty(X) = \mathcal{C}^0(X) \cap C^\infty(X)$. The following theorem contains a Paley-Wiener type result for the Floquet transform. The details can be found in [47–50].

Theorem 5.7.5. *(a) The Floquet transform $\boldsymbol{F}$ is an isometric isomorphism between the Sobolev space $H^s(X)$ and the space $L^2((\mathbb{T}^*)^d, \mathcal{E}^s)$ of L^2-integrable sections of the vector bundle $\mathcal{E}^s$.*

(b) The Floquet transform $\boldsymbol{F}$ expands the periodic elliptic operator A of order m in $L^2(X)$ into a direct integral of the fiber operators $A(k)$ over $(\mathbb{T}^*)^d$.

$$\boldsymbol{F} A \boldsymbol{F}^{-1} = \int_{(\mathbb{T}^*)^d}^{\oplus} A(k)\, \mathrm{d}k.$$

Equivalently, $\boldsymbol{F}(Af)(k) = A(k)\boldsymbol{F}f(k)$ for any $f \in H^m(X)$.

(c) The Floquet transform

$$\boldsymbol{F} : \mathcal{C}^s(X) \to C^\infty((\mathbb{T}^*)^d, \mathcal{E}^s)$$

is a topological isomorphism, where $C^\infty((\mathbb{T}^*)^d, \mathcal{E}^s)$ is the space of smooth sections of the vector bundle $\mathcal{E}^s$. Furthermore, under the Floquet transform $\boldsymbol{F}$, the operator $A : \mathcal{C}^m(X) \to \mathcal{C}^0(X)$ becomes the corresponding morphism of sheaves of smooth sections arising from the holomorphic Fredholm morphism $A(k)$ between the two holomorphic Hilbert bundles $\mathcal{E}^m$ and $\mathcal{E}^0$ over the torus $(\mathbb{T}^*)^d$, i.e., it is an operator from $C^\infty((\mathbb{T}^*)^d, \mathcal{E}^m)$ to $C^\infty((\mathbb{T}^*)^d, \mathcal{E}^0)$ such that it acts on the fiber of $\mathcal{E}^m$ at k as the fiber operator $A(k) : H_k^m(X) \to L_k^2(X)$.

(d) The inversion $\boldsymbol{F}^{-1}$ of the Floquet transform is given by the formula

$$f(x) = \frac{1}{(2\pi)^d} \int\limits_{(\mathbb{T}^*)^d} \boldsymbol{F}f(k, x)\, \mathrm{d}k, \tag{5.58}$$

provided that one can make sense both sides of (5.58) (as functions or distributions).

We prove a simple analog of the Riemann-Lebesgue lemma for the Floquet transform.

Lemma 5.7.6. *(a) Let $\widehat{f}(k, x)$ be a function in $L^1((\mathbb{T}^*)^d, \mathcal{E}^0)$. Then the inverse Floquet transform $f := \boldsymbol{F}^{-1}\widehat{f}$ belongs to $L^2_{loc}(X)$ and*

$$\sup_{g \in G} \|f\|_{L^2(g\mathcal{F})} < \infty.$$

Here $\mathcal{F}$ is a fixed fundamental domain. Moreover, one also has

$$\lim_{|g| \to \infty} \|f\|_{L^2(g\mathcal{F})} = 0.$$

(b) If $f \in V_0^1(X)$ then $\boldsymbol{F}f(k, x) \in C((\mathbb{T}^*)^d, \mathcal{E}^0)$.

Proof. We recall that $\mathcal{E}^0$ can be considered as $L^2(\mathcal{F})$. To prove the first statement, we use the identity (5.58) and the Minkowski's inequality to obtain

$$\|f\|_{L^2(g\mathcal{F})} = \frac{1}{(2\pi)^d} \left\| \int_{(\mathbb{T}^*)^d} \boldsymbol{F}f(k, \cdot)\, \mathrm{d}k \right\|_{L^2(g\mathcal{F})} = \frac{1}{(2\pi)^d} \left\| \int_{(\mathbb{T}^*)^d} e^{ik \cdot g} \boldsymbol{F}f(k, \cdot)\, \mathrm{d}k \right\|_{L^2(\mathcal{F})}$$

$$\leq \frac{1}{(2\pi)^d} \int_{(\mathbb{T}^*)^d} \|\boldsymbol{F}f(k, \cdot)\|_{L^2(\mathcal{F})}\, \mathrm{d}k = \frac{1}{(2\pi)^d} \|\boldsymbol{F}f(k, x)\|_{L^1((\mathbb{T}^*)^d, \mathcal{E}^0)} < \infty.$$

For the fact $\lim_{|g| \to \infty} \|f\|_{L^2(g\mathcal{F})} = 0$, one can modify easily the usual proof of the Riemann-Lebesgue lemma, i.e., by using Theorem 5.7.5 (a) and then approximating $\boldsymbol{F}f$ by a sequence of functions in $L^2((\mathbb{T}^*)^d, \mathcal{E}^0)$.

The second statement follows directly from (5.4) and triangle inequalities. □

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5.7.4 A Schauder type estimate

For convenience, we state a well-known Schauder type estimate for solutions of a periodic elliptic equation $Au = 0$, which we needed to refer to several times in this text. We also sketch its proof for the sake of self-containedness.

Proposition 5.7.7. *Let K be a compact neighborhood of a non-empty domain in the covering X. Then there exists a compact neighborhood $\hat{K}$ in X such that $K \Subset \hat{K}$ and the following statement holds: Suppose that $\mathcal{O}$ is an open subset of X such that $\hat{K} \subset \mathcal{O}$. Define $G^{\mathcal{O}} := \{g \in G \mid g\hat{K} \subset \mathcal{O}\}$. Then for any $\alpha \in \mathbb{R}^+$, there exists $C > 0$ depending on $\alpha, K, \hat{K}$ such that*

$$\|u\|_{H^\alpha(gK)} \leq C \cdot \|u\|_{L^2(g\hat{K})}, \tag{5.59}$$

for any $g \in G^{\mathcal{O}}$ and any $u \in C^\infty(\mathcal{O})$ satisfying the equation $Au = 0$ on $\mathcal{O}$.

Proof. Let B be an almost local [21] pseudodifferential parametrix of A such that B commutes with actions of the deck group G (see e.g., [47, Lemma 2.1.1] or [71, Proposition 3.4]). Hence, $BA = 1 + T$ for some almost-local and periodic smoothing operator T on X. This implies that for some compact neighborhood $\hat{K}$ (depending on the support of the Schwartz kernel of T and the subset K) and for any $\alpha \geq 0$, one can find some $C > 0$ so that for any smooth function v on a neighborhood of $\hat{K}$, one gets

$$\|Tv\|_{H^\alpha(K)} \leq C \cdot \|v\|_{L^2(\hat{K})}.$$

In particular, for any $g \in G^{\mathcal{O}}$ and $u \in C^\infty(\mathcal{O})$,

$$\|Tu^g\|_{H^\alpha(K)} \leq C \cdot \|u^g\|_{L^2(\hat{K})}, \tag{5.60}$$

---

[21]I.e., for some $\varepsilon > 0$, the support of the Schwartz kernel of B is contained in an ε-neighborhood of the diagonal of $X \times X$.

where u^g is the g-shift of the function u on $\mathcal{O}$. Since T is G-periodic, from (5.60), we obtain

$$\|Tu\|_{H^\alpha(gK)} \leq C \cdot \|u\|_{L^2(g\hat{K})}. \tag{5.61}$$

The important point here is the uniformity of the constant C with respect to $g \in G^\mathcal{O}$.

Assuming that $Au = 0$ on $\mathcal{O}$. Thus, $u = BAu - Tu = -Tu$ on $\mathcal{O}$. This identity and (5.61) imply the following estimate

$$\|u\|_{H^\alpha(gK)} = \|Tu\|_{H^\alpha(gK)} \leq C \cdot \|u\|_{L^2(g\hat{K})}, \quad \forall g \in G^\mathcal{O}.$$

$\square$

Remark 5.7.8. We would like to emphasize that Proposition 5.7.7 holds in a more general context. Namely, it is true for any C^∞-*bounded uniformly elliptic operator* P on a co-compact Riemannian covering $\mathcal{X}$ with a discrete deck group G. In this setting, P is invertible modulo a *uniform smoothing operator* T on $\mathcal{X}$ (see [71, Definition 3.1] and [71, Proposition 3.4]). Now the estimate (5.61) follows easily from the uniformly boundedness of the derivatives of any order of the Schwartz kernel of T on canonical coordinate charts and a routine argument of partition of unity. Another approach is to invoke uniform local apriori estimates [71, Lemma 1.4].

5.7.5 A variant of Dedekind's lemma

It is a well-known theorem by Dedekind (see e.g., [58, Lemma 2.2]) that distinct unitary characters of an abelian group G are linearly independent as functions from G to a field $\mathbb{F}$. The next lemma is a refinement of Dedekind's lemma when $\mathbb{F} = \mathbb{C}$. We notice that a proof by induction method can be found in [65, Lemma 4.4]. For the sake of completeness, we will provide our analytic proof using Stone-Weierstrass's theorem.

Lemma 5.7.9. *Consider a finite number of distinct unitary characters $\gamma_1, \ldots, \gamma_\ell$ of the abelian group $\mathbb{Z}^d$. Then there are vectors $g_1, \ldots, g_\ell$ in $\mathbb{Z}^d$ and $C > 0$ such that for any $v = (v_1, \ldots, v_\ell) \in \mathbb{C}^\ell$, we have*

$$\max_{1 \leq s \leq \ell} \left| \sum_{r=1}^{\ell} v_r \cdot \gamma_r(g_s) \right| \geq C \cdot \max_{1 \leq r \leq \ell} |v_r|.$$

Proof. By abuse of notation, we can regard $\gamma_1, \ldots, \gamma_\ell$ as distinct points of the torus $(\mathbb{T}^*)^d$.

For each tuple $(g_1, \ldots, g_\ell)$ in $(\mathbb{Z}^d)^\ell$, let $W(g_1, \ldots, g_\ell)$ be a $\ell \times \ell$-matrix whose (s, r)-entry $W(g_1, \ldots, g_\ell)_{s,r}$ is $\gamma_s^{g_r}$, for any $1 \leq r, s \leq \ell$. We equip $\mathbb{C}^\ell$ with the maximum norm. Then the conclusion of the lemma is equivalent to the invertibility of some operator $W(g_1, \ldots, g_\ell)$ acting from $\mathbb{C}^\ell$ to $\mathbb{C}^\ell$.

Suppose for contradiction that the determinant function $\det W(g_1, \ldots, g_\ell)$ is zero on $(\mathbb{Z}^d)^\ell$, i.e., for any $g_1, \ldots, g_\ell \in \mathbb{Z}^d$, one has

$$0 = \det W(g_1, \ldots, g_\ell) = \sum_{\sigma \in S_\ell} \text{sign}(\sigma) \cdot \left(\gamma_{\sigma(1)}^{g_1} \cdots \gamma_{\sigma(\ell)}^{g_\ell} \right),$$

where S_ℓ is the permutation group on $\{1, \ldots, \ell\}$. Thus, the above relation also holds for any trigonometric polynomial $P(\gamma_1, \ldots, \gamma_\ell)$ on $(\mathbb{T}^*)^{d\ell}$, i.e.,

$$\sum_{\sigma \in S_\ell} \text{sign}(\sigma) \cdot P(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(\ell)}) = 0.$$

By using the fact that the trigonometric polynomials are dense in $C((\mathbb{T}^*)^{d\ell})$ in the uniform topology (Stone-Weierstrass theorem), we conclude that

$$\sum_{\sigma \in S_\ell} \text{sign}(\sigma) \cdot f(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(\ell)}) = 0, \tag{5.62}$$

for any continuous function f on $(\mathbb{T}^*)^{d\ell}$.

Now for each $1 \leq r \leq \ell$, let us select some smooth cutoff functions ω_r supported on a neighborhood of the point γ_r such that $\omega_r(\gamma_s) = 0$ whenever $s \neq r$. We define $f \in C((\mathbb{T}^*)^{d\ell})$ as follows

$$f(x_1, \ldots, x_\ell) := \prod_{r=1}^{\ell} \omega_r(x_r), \quad x_1, \ldots, x_\ell \in (\mathbb{T}^*)^d.$$

Hence, $f(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(\ell)})$ is non-zero if and only if σ is the trivial permutation. By substituting f into (5.62), we get a contradiction, which proves our lemma. $\qquad\square$

5.7.6 Proofs of technical statements

In this part, we will use the notation $\mathcal{F}$ for the closure of a fundamental domain for G-action on the covering X.

Proof of Theorem 5.2.14.

Proof. If $u \in V^\infty_{\lfloor N - d/p \rfloor}(A)$ then $u \in V^\infty_{N_0}(A)$ for some nonnegative integer N_0 such that $N_0 < N - d/p$. Thus,

$$\sum_{g \in G} \|u\|^p_{L^2(g\mathcal{F})} \langle g \rangle^{-pN} \lesssim \sum_{g \in G} \langle g \rangle^{p(N_0 - N)} < \infty.$$

Thus, $V^\infty_{\lfloor N - d/p \rfloor}(A) \subseteq V^p_N(A)$.

Now suppose that $F_{A, \mathbb{R}} = \{k_1, \ldots, k_\ell\}$ (modulo G^*-shifts), where $\ell \in \mathbb{N}$. It suffices for us to show that

$$V^p_N(A) \subseteq V^\infty_{\lfloor N - d/p \rfloor}(A). \tag{5.63}$$

A key ingredient of the proof of (5.63) is the following statement.

Lemma 5.7.10. *Suppose that $\mathcal{N} > d/p$.*

(i) *If* $u \in V_{\mathcal{N}}^p(A) \cap V_{\mathcal{M}}^\infty(A)$ *for some* $0 \leq \mathcal{M} < \mathcal{N} + 1 - d/p$ *then* $u \in V_{\mathcal{M}'}^\infty(A)$ *for some*

   $\mathcal{M}' < \mathcal{N} - d/p$. *In particular,* $u \in V_{\mathcal{N}-d/p}^\infty(A)$.

(ii) $V_{\mathcal{N}}^p(A) \cap V_{\mathcal{N}+1-d/p}^\infty(A) \subseteq V_{\mathcal{N}-d/p}^\infty(A)$.

Instead of proving Lemma 5.7.10 immediately, first, let us assume its validity. We now

prove (5.63). Consider $u \in V_N^p(A)$.

   Case 1. $p > 1$.

   We prove by induction that if $0 \leq s \leq d - 1$, then

$$u \in V_{N+d/p-(s+1)/p}^p(A) \cap V_{N-s/p}^\infty(A). \tag{5.64}$$

The statement holds for $s = 0$ since clearly, $V_N^p(A) \subseteq V_N^\infty(A)$ and $N + d/p - 1/p \geq N$.

Now suppose that (5.64) holds for s such that $s + 1 \leq d - 1$. Since $1 - 1/p > 0$, we can

apply Lemma 5.7.10 (i) to u and the pair $(\mathcal{N}, \mathcal{M}) = (N + d/p - (s+1)/p, N - s/p)$ to

deduce that $u \in V_{N-(s+1)/p}^\infty(A)$. Therefore, (5.64) also holds for $s + 1$. In the end, we have

$u \in V_N^p(A) \cap V_{N-(d-1)/p}^\infty(A)$. Again, by Lemma 5.7.10 (i), u belongs to $V_{\mathcal{M}'}^\infty(A)$ for some

$\mathcal{M}' < N - d/p$. In other words, u is in $V_{\lfloor N-d/p \rfloor}^\infty(A)$.

   Case 2. $p = 1$.

   As in Case 1, we apply Lemma 5.7.10 (ii) and induction method to prove that $u \in$

$V_{N+d-1-s}^1(A) \cap V_{N-s}^\infty(A)$ for any $0 \leq s \leq d-1$. Hence, $u \in V_N^1(A) \cap V_{N+1-d}^\infty(A)$. Due to

Lemma 5.7.10 (ii) again, one has $u \in V_N^1(A) \cap V_{N-d}^\infty(A)$. Applying Lemma 5.7.10 (i) to

u and the pair $(\mathcal{N}, \mathcal{M}) = (N, N - d)$, we conclude that $u \in V_{\lfloor N-d \rfloor}^\infty(A) = V_{N-(d+1)}^\infty(A)$.

   Therefore, Theorem 5.2.14 follows from Lemma 5.7.10. Our proof of Lemma 5.7.10

consists of several steps.

(i) **Step 1.** The lemma is trivial if $u = 0$. So we assume that u is non-zero. Since

   $V_{\mathcal{N}}^p(A) \subseteq V_{\mathcal{N}}^\infty(A)$, we can apply Theorem 5.2.8 (ii) to represent $u \in V_{\mathcal{N}}^p(A)$ as a

finite sum of Floquet solutions of A, i.e.,

$$u = \sum_{r=1}^{\ell} u_r.$$

Here u_r is a Floquet function of order $M_r \leq \mathcal{N}$ with quasimomentum k_r. Let N_0 be the highest order among all the orders of the Floquet functions u_r appearing in the above representation. Without loss of generality, we can suppose that there exists $r_0 \in [1, \ell]$ such that for any $r \leq r_0$, the orders M_r of u_r is maximal among all the orders of these Floquet functions. Thus, $M_r = N_0 \leq \mathcal{M}$ when $r \in [1, r_0]$. To prove our lemma, it suffices to show that $N_0 < \mathcal{N} - d/p$.

Step 2. According to Proposition 5.7.7, we can pick a compact neighborhood $\hat{\mathcal{F}}$ of $\mathcal{F}$ such that for any $\alpha \geq 0$,

$$\|u\|_{H^\alpha(g\mathcal{F})} \leq C \cdot \|u\|_{L^2(g\hat{\mathcal{F}})}$$

for some $C > 0$ which is independent of $g \in G$. Let $\alpha > n/2$, the Sobolev embedding theorem yields the estimate

$$\|u\|_{C^0(g\mathcal{F})} \lesssim \|u\|_{L^2(g\hat{\mathcal{F}})}, \quad \forall g \in G. \tag{5.65}$$

From (5.65) and the fact that $u \in V_{\mathcal{N}}^p(A)$, we obtain

$$\sup_{x \in \mathcal{F}} \left(\sum_{g \in G} |u(g \cdot x)|^p \langle g \rangle^{-p\mathcal{N}} \right) \lesssim \sum_{g \in G} \|u\|_{L^2(g\hat{\mathcal{F}})}^p \langle g \rangle^{-p\mathcal{N}} < \infty. \tag{5.66}$$

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Step 3. By definition (see Subsection 5.2.2), one can write

$$u(x) = \sum_{r=1}^{\ell} u_r(x) = \sum_{r=1}^{r_0} e_{k_r}(x) \sum_{|j|=N_0} a_{j,r}(x)[x]^j + O(|x|^{N_0-1}).$$

Here each function $a_{j,r}$ is G-periodic and the remainder term $O(|x|^{N_0-1})$ is an exponential polynomial with periodic coefficients of order at most $N_0 - 1$. Hence, for any $(g, x) \in G \times \mathcal{F}$, we get

$$u(g \cdot x) = \sum_{r=1}^{r_0} e^{ik_r \cdot g} \sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)[g \cdot x]^j + O(\langle g \rangle^{N_0-1}).$$

Since $N_0 - 1 \leq \mathcal{M} - 1 < \mathcal{N} - d/p$, the series

$$\sum_{g \in \mathbb{Z}^d} O(\langle g \rangle^{p(N_0-1)}) \langle g \rangle^{-p\mathcal{N}}$$

converges. From this and (5.66), we deduce

$$\sup_{x \in \mathcal{F}} \sum_{g \in G} \left| \sum_{r=1}^{r_0} e^{ik_r \cdot g} \sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)[g \cdot x]^j \right|^p \langle g \rangle^{-p\mathcal{N}} < \infty.$$

By Lemma 5.7.1, $|[g \cdot x]^j - g^j| = O(\langle g \rangle^{N_0-1})$ for any multi-index j such that $|j| = N_0$. This implies that

$$\sup_{x \in \mathcal{F}} \sum_{g \in G} \left| \sum_{|j|=N_0} \sum_{r=1}^{r_0} e_{k_r}(x) a_{j,r}(x) e^{ik_r \cdot g} g^j \right|^p \langle g \rangle^{-p\mathcal{N}} < \infty. \tag{5.67}$$

Step 4. We will use Lemma 5.7.9 to reduce the condition (5.67) to the one without exponential terms $e^{ik_r \cdot g}$, i.e., we can assume that $F_{A,\mathbb{R}} = \{0\}$ (modulo G^*-shifts). Indeed, let $\gamma_1, \ldots, \gamma_{r_0}$ be distinct unitary characters of G, which are defined via the

identities $\gamma_r(g) = e^{ik_r \cdot g}$, where $r \in \{1, \ldots, r_0\}$ and $g \in G$. Now due to Lemma 5.7.9, there are $g_1, \ldots, g_{r_0} \in G$ and a constant $C > 0$ such that for any vector $(v_1, \ldots, v_{r_0}) \in \mathbb{C}^{r_0}$, we have the following inequality

$$C \cdot \max_{1 \leq s \leq r_0} \left| \sum_{r=1}^{r_0} v_r \cdot e^{ik_r \cdot g_s} \right| \geq \max_{1 \leq r \leq r_0} |v_r|. \tag{5.68}$$

Hence, for any $(g, x) \in G \times \mathcal{F}$ and $1 \leq s \leq r_0$, we apply (5.68) to the vector

$$(v_1, \ldots, v_{r_0}) := \left(\sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)(g + g_s)^j \langle g + g_s \rangle^{-\mathcal{N}} e^{ik_r \cdot g} \right)_{1 \leq r \leq r_0}$$

to deduce that

$$
\begin{aligned}
&\max_{1 \leq r \leq r_0} \left| \sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)(g + g_s)^j \right|^p \langle g + g_s \rangle^{-p\mathcal{N}} \\
&= \max_{1 \leq r \leq r_0} \left| \sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)(g + g_s)^j \langle g + g_s \rangle^{-\mathcal{N}} e^{ik_r \cdot g} \right|^p \\
&\lesssim \max_{1 \leq s \leq r_0} \left| \sum_{r=1}^{r_0} \left(\sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)(g + g_s)^j \langle g + g_s \rangle^{-\mathcal{N}} e^{ik_r \cdot g} \right) e^{ik_r \cdot g_s} \right|^p \\
&\lesssim \sum_{s=1}^{r_0} \left| \sum_{r=1}^{r_0} \sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x)(g + g_s)^j \cdot e^{ik_r \cdot (g+g_s)} \right|^p \cdot \langle g + g_s \rangle^{-p\mathcal{N}}
\end{aligned}
\tag{5.69}
$$

Summing the estimate (5.69) over $g \in G$, we derive

$$
\max_{1 \leq r \leq r_0} \sup_{x \in \mathcal{F}} \sum_{g \in G} \left| \sum_{|j| = N_0} e_{k_r}(x) a_{j,r}(x) g^j \right|^p \langle g \rangle^{-p\mathcal{N}}
$$

$$
= \max_{1 \leq r,s \leq r_0} \sup_{x \in \mathcal{F}} \sum_{g \in G} \left| \sum_{|j| = N_0} e_{k_r}(x) a_{j,r}(x) (g + g_s)^j \right|^p \langle g + g_s \rangle^{-p\mathcal{N}}
$$

$$
\lesssim \sup_{x \in \mathcal{F}} \sum_{s=1}^{r_0} \sum_{g \in G} \left| \sum_{r=1}^{r_0} \sum_{|j| = N_0} e_{k_r}(x) a_{j,r}(x) (g + g_s)^j \cdot e^{i k_r \cdot (g + g_s)} \right|^p \cdot \langle g + g_s \rangle^{-p\mathcal{N}}
$$
(5.70)

$$
\lesssim \sup_{x \in \mathcal{F}} \sum_{g \in G} \left| \sum_{r=1}^{r_0} \sum_{|j| = N_0} e_{k_r}(x) a_{j,r}(x) g^j \cdot e^{i k_r \cdot g} \right|^p \cdot \langle g \rangle^{-p\mathcal{N}} < \infty.
$$

From (5.67) and (5.70), we get

$$
\sum_{g \in G} \left| \sum_{|j| = N_0} e_{k_r}(x) a_{j,r}(x) g^j \right|^p \cdot \langle g \rangle^{-p\mathcal{N}} < \infty,
$$
(5.71)

for any $1 \leq r \leq r_0$ and $x \in \mathcal{F}$.

Step 5. We prove the following claim: If P is a non-zero homogeneous polynomial of degree N_0 in d-variables such that $N_0 < \mathcal{N} + 1 - d/p$ and

$$
\sum_{g \in \mathbb{Z}^d} |P(g)|^p \cdot \langle g \rangle^{-p\mathcal{N}} < \infty,
$$
(5.72)

then $N_0 < \mathcal{N} - d/p$.

Our idea is to approximate the series in (5.72) by the integral

$$
\mathcal{I} := \int_{\mathbb{R}^d} |P(z)|^p \cdot \langle z \rangle^{-p\mathcal{N}} \, dz.
$$

In fact, for any $z \in [0,1)^d + g$, one can use the triangle inequality and the assumption

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that the order of P is N_0 to achieve the following estimate

$$|P(z)|^p \le 2^{p-1} \left(|P(g)|^p + |P(z) - P(g)|^p \right) \lesssim |P(g)|^p + O(\langle g \rangle^{p(N_0-1)}).$$

Integrating the above estimate over the cube $[0,1)^d + g$ and then summing over all $g \in \mathbb{Z}^d$, we deduce

$$\mathcal{I} = \sum_{g \in \mathbb{Z}^d} \int_{[0,1)^d+g} |P(z)|^p \cdot \langle z \rangle^{-p\mathcal{N}} \, \mathrm{d}z \lesssim \sum_{g \in \mathbb{Z}^d} |P(g)|^p \cdot \langle g \rangle^{-p\mathcal{N}} + \sum_{g \in \mathbb{Z}^d} \langle g \rangle^{p(N_0-1-\mathcal{N})} < \infty,$$

where we have used (5.72) and the condition $(N_0 - 1 - \mathcal{N})p < -d$.

We now rewrite our integral $\mathcal{I}$ in polar coordinates as follows

$$\mathcal{I} = \int_0^\infty \int_{\mathbb{S}^{d-1}} |P(r\omega)|^p \langle r \rangle^{-p\mathcal{N}} r^{d-1} \, \mathrm{d}\omega \, \mathrm{d}r = \int_0^\infty \langle r \rangle^{-p\mathcal{N}} r^{d-1+pN_0} \, \mathrm{d}r \cdot \int_{\mathbb{S}^{d-1}} |P(\omega)|^p \, \mathrm{d}\omega.$$

Suppose for contradiction that $(N_0 - \mathcal{N})p \ge -d$. Then it follows that

$$\int_0^\infty \langle r \rangle^{-p\mathcal{N}} r^{d-1+pN_0} \, \mathrm{d}r = \infty.$$

Thus, the finiteness of $\mathcal{I}$ implies that

$$\int_{\mathbb{S}^{d-1}} |P(\omega)|^p \, \mathrm{d}\omega = 0.$$

Hence, $P(\omega) = 0$ for any $\omega \in \mathbb{S}^{d-1}$. By homogeneity, P must be zero (contradiction). This shows our claim.

Step 6. Since u is non-zero, there are some $r \in \{1, \ldots, r_0\}$ and $x \in \mathcal{F}$ such that the

following homogeneous polynomial of degree N_0 in $\mathbb{R}^d$

$$P(z) := \sum_{|j|=N_0} e_{k_r}(x) a_{j,r}(x) z^j$$

is non-zero. Thanks to (5.71) and the condition $N_0 \leq \mathcal{M} < \mathcal{N} + 1 - d/p$ (see Step 1), the inequality $N_0 < \mathcal{N} - d/p$ must be satisfied according to Step 5. This finishes the proof of the first part of the lemma.

(ii) Consider $u \in V_{\mathcal{N}}^p(A) \cap V_{\mathcal{N}+1-d/p}^\infty(A)$. In particular, for any $\varepsilon > 0$, $u \in V_{\mathcal{N}+\varepsilon}^p(A) \cap V_{\mathcal{N}+1-d/p}^\infty(A)$. Using Lemma 5.7.10 (i), u is in the space $V_{\mathcal{M}_\varepsilon}^\infty(A)$ for some $\mathcal{M}_\varepsilon < (\mathcal{N} + \varepsilon) - d/p$. By letting $\varepsilon \to 0^+$, we conclude that $u \in V_{\mathcal{N}-d/p}^\infty(A)$. This yields the second part of the lemma.

$\square$

Proof of Theorem 5.2.16.

Proof. (a) Consider $u \in V_N^p(A)$. Due to Theorem 5.2.14 and the condition that $N \leq d/p$,

$$V_N^p(A) \subseteq V_{d/p+1/2}^p(A) = V_0^\infty(A).$$

Using Theorem 5.2.8 (ii),

$$u(x) = \sum_{r=1}^{\ell} e_{k_r}(x) a_r(x), \tag{5.73}$$

for some periodic functions $a_r(x)$.

Using (5.65) and the assumption that $u \in V_N^p(A)$, we derive

$$\sup_{x \in \mathcal{F}} \sum_{g \in G} \left| \sum_{r=1}^{\ell} e_{k_r}(x) a_r(x) e^{ik_r \cdot g} \right|^p \cdot \langle g \rangle^{-pN} < \infty. \tag{5.74}$$

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Then one can modify the argument in Step 4 of the proof of Lemma 5.7.10 (from the estimate (5.67) to (5.71)) to get

$$\max_{1 \leq r \leq \ell} \sup_{x \in \mathcal{F}} |e_{k_r}(x)a_r(x)|^p \cdot \sum_{g \in \mathbb{Z}^d} \langle g \rangle^{-pN} < \infty.$$

Hence, the assumption $-pN \geq -d$ forces that $\max_{1 \leq r \leq \ell} \sup_{x \in \mathcal{F}} |e_{k_r}(x)a_r(x)| = 0$. Thus, u must be zero.

(b) Let u be an arbitrary element in $V_N^\infty(A)$. Since $N < 0$, we can assume that u has the form (5.73). To prove that $u = 0$, it is enough to show that $e_{k_r}(x)a_r(x) = 0$ for any $x \in \mathcal{F}$ and $1 \leq r \leq \ell$. One can repeat the same argument of the previous part to prove this claim. However, we will provide a different proof using Fourier analysis on the torus $(\mathbb{T}^*)^d$.

For each $x \in \mathcal{F}$, we introduce the following distribution on $(\mathbb{T}^*)^d$

$$f(k) := \sum_{r=1}^{\ell} e_{k_r}(x)a_r(x)\delta(k - k_r),$$

where $\delta(\cdot - k_r)$ is the Dirac delta distribution on the torus $(\mathbb{T}^*)^d$ at the quasimomentum k_r. Taking Fourier series, we obtain

$$\hat{f}(g) = \sum_{r=1}^{\ell} e_{k_r}(x)a_r(x)e^{-ik_r \cdot g}.$$

As in (5.74), the assumption $u \in V_N^\infty(A)$ is equivalent to

$$\sup_{g \in \mathbb{Z}^d} \left| \hat{f}(g) \right| \cdot \langle g \rangle^{-N} < \infty.$$

Let ϕ be a smooth function on $(\mathbb{T}^*)^d$. Using Parseval's identity and Hölder's inequality,

we have

$$\left| \sum_{r=1}^{\ell} e_{k_r}(x) a_r(x) \phi(k_r) \right| = |\langle f, \phi \rangle| = \left| \sum_{g \in \mathbb{Z}^d} \hat{f}(g) \hat{\phi}(-g) \right| \lesssim \sum_{g \in \mathbb{Z}^d} |\hat{\phi}(g)| \cdot \langle g \rangle^N. \quad (5.75)$$

We pick $\delta > 0$ small enough such that $k_s \notin B(k_r, 2\delta)$ if $s \neq r$. Then we choose a cut-off function ϕ_r such that $\operatorname{supp} \phi_r \subseteq B(k_r, 2\delta)$ and $\phi_r = 1$ on $B(k_r, \delta)$. For $1 \leq r \leq \ell$, we define functions in $C^\infty((\mathbb{T}^*)^d)$ as follows:

$$\phi_r^\varepsilon(k) := \phi_r(\varepsilon^{-1} k), \quad (0 < \varepsilon < 1).$$

To bound the Fourier coefficients of ϕ_r^ε in terms of ε, we use integration by parts. Indeed, for any nonnegative real number s,

$$\begin{aligned}
(2\pi)^d |\widehat{\phi_r^\varepsilon}(g)| &= \left| \int_{(\mathbb{T}^*)^d} \phi_r^\varepsilon(k) e^{-ik \cdot g} dk \right| = \varepsilon^d \cdot \left| \int_{B(k_r, 2\delta)} \phi_r(k) e^{-i\varepsilon k \cdot g} dk \right| \\
&= \varepsilon^d \langle \varepsilon g \rangle^{-2[s]-2} \cdot \left| \int_{B(k_r, 2\delta)} (1 - \Delta)^{[s]+1} \phi_r(k) \cdot e^{-ik \cdot \varepsilon g} dk \right| \\
&\leq \varepsilon^d \langle \varepsilon g \rangle^{-2[s]-2} \cdot \sup_k \left| (1 - \Delta)^{[s]+1} \phi_r(k) \right| \lesssim \varepsilon^{d-s} \langle g \rangle^{-s}.
\end{aligned}$$

In the last inequality, we make use of the fact that $\langle \varepsilon g \rangle^{-2[s]-2} \leq \langle \varepsilon g \rangle^{-s} \leq \varepsilon^{-s} \langle g \rangle^{-s}$ whenever $\varepsilon \in (0, 1)$. In particular, by choosing any $s \in (\max(0, N + d), d)$, one has

$$|\widehat{\phi_r^\varepsilon}(g)| \cdot \langle g \rangle^N \lesssim \varepsilon^{d-s} \cdot \langle g \rangle^{N-s}. \quad (5.76)$$

Now we substitute $\phi := \phi_r^\varepsilon$ in (5.75), use (5.76), and then take $\varepsilon \to 0^+$ to derive

$$|e_{k_r}(x) a_r(x)| \lesssim \lim_{\varepsilon \to 0} \varepsilon^{d-s} \sum_{g \in \mathbb{Z}^d} \langle g \rangle^{N-s} = 0,$$

which completes the proof.

$\square$

Proof of Corollary 5.3.13.

Proof. It suffices to prove that $\operatorname{Im}\tilde{P} = \tilde{\Gamma}_\mu(\mathcal{X}, P)$ since from (5.29), $\dim\operatorname{Ker}\tilde{P}^* = \operatorname{codim}\operatorname{Im}\tilde{P} = 0$ and then the conclusion of Corollary 5.3.13 holds true as we remarked in Section 5.3. Now given any $f \in \tilde{\Gamma}_\mu(\mathcal{X}, P)$, one has $\langle f, \tilde{L}^-\rangle = 0$ and $f = Pu$ for some $u \in \operatorname{Dom}P$ since $\operatorname{Im}P = \operatorname{Dom}'P^*$. According to the assumption, we can find a solution $w = u - v$ in $\operatorname{Dom}P$ of the equation $Pw = Pu - Pv = f$ such that $\langle w, L^-\rangle = 0$. Let w_0 be the restriction of w on $\mathcal{X} \setminus D^+$. Clearly, w_0 belongs to the space $\Gamma(\mathcal{X}, \mu, P)$. Since Pw is smooth on $\mathcal{X}$, $\tilde{P}w_0 = Pw = f$ by the definition of the extension operator $\tilde{P}$. This shows that $f \in \operatorname{Im}\tilde{P}$, which finishes the proof. $\square$

Remark 5.7.11. In the special case $D^- = \emptyset$, $L^- = \{0\}$, one can prove the Riemann-Roch equality (5.24) directly, i.e., without referring to the extension operators $\tilde{P}$ and $\tilde{P}^*$. For reader's convenience, let us present this short proof following from [32]. We define the space $\Gamma(\mu, P) := \{u \in \mathcal{D}'(\mathcal{X}) \mid u \in \operatorname{Dom}P \setminus D^+, Pu \in L^+\}$. Then it is easy to check that the following sequences are exact:

$$0 \to \tilde{L}^+ \xrightarrow{i} \Gamma(\mu, P) \xrightarrow{r} L(\mu, P) \to 0$$

$$0 \to \operatorname{Ker}P \xrightarrow{i} \Gamma(\mu, P) \xrightarrow{P} L^+ \to 0,$$

where i and r are natural inclusion and restriction maps. Here the surjectivity of P from $\Gamma(\mu, P)$ to L^+ is a consequence of the existence of a properly supported pseudodifferential parametrix of P (modulo a properly supported smoothing operator) and $C_c^\infty(\mathcal{X}) \subseteq \operatorname{Dom}'P^* = \operatorname{Im}P$. Note that $\operatorname{Ker}P^* = \{0\}$. Hence, it follows that $\dim L(\mu, P) = \dim\Gamma(\mu, P) - \dim\tilde{L}^+ = \dim\operatorname{Ker}P + \dim L^+ - \dim\tilde{L}^+ = \operatorname{ind}P + \deg_P(\mu)$.

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5.8 Concluding remarks

- Let $\{A_z\}_z$ and $\{\mu_z\}_z$ be given families of periodic elliptic operators A_z satisfying the assumptions of Theorem 5.4.12 and of rigged divisors μ_z. Under appropriate conditions on these families depending "nicely" on the parameter z, the upper-semicontinuity in z of the two functions $\dim L_2(\mu_z, A_z, 0)$ and $\dim L_2(\mu_z^{-1}, A_z^*, 0)$ can be deduced from the Liouville-Riemann-Roch equality (5.37).

- From Remark 5.5.3, we know that Assumption $(\mathcal{A}2)$ cannot be dropped in Theorem 5.4.6. Besides the example in Remark 5.5.3, we give a heuristic explanation here. It is known that if $\{A_t\}$ is a family of Fredholm operators that is continuous with respect to the parameter t, the kernel dimension $\dim \operatorname{Ker} A_t$ is upper semicontinuous. The idea in combining Riemann-Roch and Liouville theorems by considering dimensions of spaces of solutions with polynomial growth as some Fredholm indices would imply that in this direction, the upper-semicontinuity property should hold for these dimensions. On the other hand, in [69], there exists a continuous family $\{M_t\}$ of periodic operators on $\mathbb{R}^2$ such that for each $N \geq 0$, $\dim V_N^\infty(M_t) = 2 \dim V_N^\infty(M_{2\sqrt{3}})$ if $2\sqrt{3} < t < 2\sqrt{3} + \varepsilon$ for some $\varepsilon > 0$ and thus, $\dim V_N(M_t)$ is not upper-semicontinuous at $t = 2\sqrt{3}$ (see [50]). In this example, the minimum 0 of the lowest band $\lambda_1(k)$ of the operator $M_{2\sqrt{3}}$ is degenerate [69]. This explains why our approach requires the "non-degeneracy" type condition $(\mathcal{A}2)$ for avoiding some intractable cases like the previous example. Notice that in general, Assumption $\mathcal{A}$ and Liouville type results are not stable under small perturbations.

- The results in this chapter extend to linear elliptic systems and operators between vector bundles.

6. SUMMARY AND CONCLUSIONS

In Chapter 3, we obtained the Green's function asymptotics of periodic elliptic operators of second-order on Euclidean spaces. Namely, we computed explicitly the leading term of the Green's function $G_\lambda(x, y)$ of a "generic" periodic elliptic operator of second-order when λ is close enough to a gap edge located at a high symmetry point of the Brillouin zone. The case when λ is below the bottom of the spectrum was known before [9, 60]. However, the techniques used there could not apply directly to our situation here. We took an approach based on [52] and the calculation in the discrete version [81].

In Chapter 4, we generalized the results of Chapter 3 and [52] to the setting of abelian coverings. One of the main tools here is the use of additive functions on abelian coverings (Definition 4.2.7) for describing such asymptotics. These asymptotics capture a well-known idea (due to Gromov): a co-compact regular cover and its deck transformation group look similar at large scales.

In Chapter 5, we studied some possible combinations between the Riemann-Roch and the Liouville type results for periodic elliptic operators on noncompact abelian coverings with compact bases. Our idea is to employ a very general version of the Riemann-Roch formula for elliptic operators on noncompact manifolds developed in [32, 33]. Riemann-Roch type theorems allow the solutions to have prescribed zeros or poles at a given divisor. On the other hand, Liouville type results address the finite-dimensionality and structure of the space of solutions with a prescribed polynomial growth on noncompact manifolds. It is thus natural to try to combine the Riemann-Roch formula and the Liouville type results. In this direction, we were able to achieve two different groups of results:

(1) Liouville-Riemann-Roch formulas for uniformly bounded elliptic operators on co-compact Riemannian coverings outside of the spectra.

(2) Liouville-Riemann-Roch inequalities and their applications for periodic elliptic operators on co-compact abelian Riemannian coverings at the edges of the spectra.

We have also obtained "an L^p-analog" of the Liouville type results of [49, 50].

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APPENDIX A

A TOPOLOGICAL APPROACH TO DEFINING ADDITIVE FUNCTIONS ON CO-COMPACT RIEMANNIAN NORMAL COVERINGS

A.1 Introduction

Let X be a connected smooth Riemannian manifold of dimension n equipped with an isometric, properly discontinuous, free, and co-compact action of a discrete group G. Notice that G is finitely generated due to the Švarc-Milnor lemma and hence, $\mathrm{Hom}(G, \mathbb{R})$ is finite dimensional. Furthermore, the orbit space $M := X/G$ is a compact Riemannian manifold when equipped with the metric pushed down from X. The action of an element $g \in G$ on $x \in X$ is denoted by $g \cdot x$. Let π be the covering map from X onto M. Thus, $\pi(g \cdot x) = \pi(x)$ for any $(g, x) \in G \times X$.

Following [56], we define the class of additive functions on the covering X as follows:

Definition A.1.1.

- A real smooth function u on X is said to be **additive** if there is a homomorphism $\alpha : G \to \mathbb{R}$ such that

$$u(g \cdot x) = u(x) + \alpha(g), \quad \text{for all} \quad (g, x) \in G \times X.$$

- We denote by $\mathcal{A}(X)$ the space of all additive functions on X.

- A map h from X to $\mathbb{R}^m$ ($m \in \mathbb{N}$) is called a vector-valued additive function on X if every component of h belongs to $\mathcal{A}(X)$.

We remark that additive functions on co-compact coverings appeared in various results such as studying the structure of positive G-multiplicative type solutions [3, 56], describ-

ing the off-diagonal long time asymptotics of the heat kernel [46] and the Green's function asymptotics of periodic elliptic operators [41] on a noncompact abelian covering of a compact Riemannian manifold.

A direct construction of additive functions on X can be found in either [50, Section 3] or [56, Remark 2.6]. However, this construction depends on the choice of a fundamental domain for the base M in X. A more invariant approach to defining additive functions on coverings was mentioned briefly in [3, 50]. Our goal in this note is to present the full details of this approach for any co-compact covering.

A.2 Additive functions on co-compact normal coverings

We begin with the following notion (see [3, 50]):

Definition A.2.1. Let $H^1_{DR}(M), H^1_{DR}(X)$ be De Rham cohomologies of M and X, respectively. We denote by $\Omega^1(M; G)$ the image in $H^1_{DR}(M)$ of the set of all closed differential 1-forms ω on M (modulo the exact ones) such that their lifts ω to X are exact. In other words, $\Omega^1(M; G)$ is the kernel of the homomorphism

$$\pi^* : H^1_{DR}(M) \to H^1_{DR}(X),$$

where π^* is the induced homomorphism of the covering map $\pi : X \to M$.

By De Rham's theorem, $\Omega^1(M; G)$ is a finite dimensional vector space. Indeed, more is true:

Lemma A.2.2. $\Omega^1(M, G) \cong \mathrm{Hom}(G, \mathbb{R})$.

Proof. By Hurewicz's theorem (see e.g., [55]), the homologies $H_1(M)$ and $H_1(X)$ are isomorphic to the abelianizations of fundamental groups $\pi_1(M)$ and $\pi_1(X)$, respectively. We can identify De Rham cohomologies $H^1_{DR}(M)$ and $H^1_{DR}(X)$ with $\mathrm{Hom}(\pi_1(M), \mathbb{R})$

and $\text{Hom}(\pi_1(X), \mathbb{R})$, correspondingly. Since X is a normal covering of M, $\pi_1(X)$ is a normal subgroup of $\pi_1(M)$ and moreover, the following sequence

$$0 \to \pi_1(X) \to \pi_1(M) \to G \to 0$$

is exact. Because $\text{Hom}(\cdot, \mathbb{R})$ is a contravariant exact functor, we deduce the exactness of the following sequence of vector spaces:

$$0 \to \text{Hom}(G, \mathbb{R}) \to H^1_{DR}(M) \to H^1_{DR}(X) \to 0.$$

Hence, $\Omega^1(M, G)$ is isomorphic to $\text{Hom}(G, \mathbb{R})$. $\qquad\square$

Fixing a base point $x_0 \in X$. For any closed 1-form ω on M such that its lift to X is exact, there exists a unique function $f_\omega \in C^\infty(X, \mathbb{R})$ such that $\pi^*\omega = df_\omega$ and $f_\omega(x_0) = 0$. Equivalently,

$$f_\omega(x) = \int_{x_0}^x \pi^*\omega, \quad \forall x \in X.$$

Lemma A.2.3. *For such 1-form ω, we have*

  i) *Fix any $g \in G$, then $f_\omega(g \cdot x) - f_\omega(x)$ is independent of $x \in X$.*

  ii) *If $\pi^*\omega = 0$ then $\omega = 0$.*

Proof.

  i) For each $g \in G$, let L_g be the diffeomorphism of X that maps any element x to the element $g \cdot x$. Since $\pi \circ L_g = \pi$, we get $df_\omega = \pi^*\omega = L_g^*\pi^*\omega = L_g^*df_\omega = dL_g^*f_\omega = d(f_\omega \circ L_g)$. Thus, $d(f_\omega - f_\omega \circ L_g) = 0$ and so, $f_\omega \circ L_g - f_\omega$ is constant since X is connected.

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ii) Fix any point $p \in M$. We pick an evenly covered open subset U of M such that it contains p. Then there is a smooth local section $\sigma : U \to X$, i.e., $\pi \circ \sigma = id_{|U}$ (see e.g., [55, Proposition 4.36]). Hence, $\omega(p) = \sigma^* \pi^* \omega(p) = 0$.

$\square$

On $\mathcal{A}(X)$, we introduce an equivalent relation $\sim$ as follows: $f_1 \sim f_2$ in $\mathcal{A}(X)$ if and only if $f_1 - f_2 = f \circ \pi$ for some function $f \in C^\infty(M, \mathbb{R})$.

By Lemma A.2.3 (i), the map $\omega \mapsto f_\omega$ induces the following linear map

$$\Lambda : \Omega^1(M, G) \to \mathcal{A}(X)/\sim$$
$$[\omega] \mapsto [f_\omega],$$

(A.1)

where the notation $[\omega]$ ($[f_\omega]$) stands for the equivalent class of ω (f_ω) in $\Omega^1(M, G)$ ($\mathcal{A}(X)/\sim$), correspondingly. We now claim that $[\omega] = 0$ if and only if $[f_\omega] = 0$, and hence Λ is an injective linear map. Indeed, due to Lemma A.2.3 (ii), the condition that ω is exact is equivalent to $\pi^* \omega = d(f \circ \pi)$ for some $f \in C^\infty(M, \mathbb{R})$. But this is the same as $df_\omega = d(f \circ \pi)$, or $[f_\omega] = 0$.

Consider an additive function f on X. According to the definition, there exists a unique group homomorphism $\ell_f : G \to \mathbb{R}$ such that $f(g \cdot x) = f(x) + \ell_f(g)$ for any $g \in G$, $x \in X$. Then the map $f \mapsto \ell_f$ induces the linear map

$$\Upsilon : \mathcal{A}(X)/\sim \to \mathrm{Hom}\,(G, \mathbb{R})$$
$$[f] \mapsto \ell_f,$$

(A.2)

which is injective.

Then the composition $\Upsilon \circ \Lambda$ is also injective. By Lemma A.2.2, $\dim_\mathbb{R} \Omega^1(M, G) = \dim_\mathbb{R} \mathrm{Hom}\,(G, \mathbb{R}) < \infty$. These facts together imply that the linear maps Υ and Λ are

isomorphism. We conclude:

Theorem A.2.4. *The three vector spaces* $\Omega^1(M, G)$, $\mathcal{A}(X)/\sim$ *and* $\mathrm{Hom}\,(G, \mathbb{R})$ *are iso-morphic.*

In particular, we obtain:

Corollary A.2.5. *Assume that* $G = \mathbb{Z}^d$. *Then there is a smooth* $\mathbb{R}^d$-*valued function* h *on* X *such that for any* $(g, x) \in \mathbb{Z}^d \times X$,

$$h(g \cdot x) = h(x) + g. \tag{A.3}$$

The following proposition says that given any additive function u on X, one can pick a harmonic additive function f such that $f - u$ is G-periodic.

Proposition A.2.6. *For any* ℓ *in* $\mathrm{Hom}(G, \mathbb{R})$, *there exists a unique (modulo a real constant)* **harmonic** *function* f *on* X *such that for any* $(g, x) \in G \times X$, *we have*

$$f(g \cdot x) = f(x) + \ell(g). \tag{A.4}$$

Proof. First, we show the existence part. Due to Theorem A.2.4, let $\tilde{f}$ be a function on X satisfying $\tilde{f}(g \cdot x) = \tilde{f}(x) + \ell(g)$ for any $(g, x) \in G \times X$. We recall the isomorphism Λ defined in (A.1). We put $\alpha := \Lambda^{-1}([\tilde{f}]) \in \Omega^1(M, G)$. By the Hodge theorem, there exists a unique harmonic 1-form ω on M such that $[\omega] = \alpha$ in $H^1_{DR}(M)$. Let f be a smooth function such that $f \in \mathcal{A}(X)$ and $\pi^*\omega = df$. Then f satisfies (A.4) since $[f] = [\tilde{f}]$ in $\mathcal{A}(X)/\sim$. Thus, it is sufficient to show that f is harmonic on X. We denote by δ_X, Δ_X and δ_M, Δ_M the codifferential and Laplace-Beltrami operators on X and M, respectively. Since the covering map π is a local isometry between X and M, its pullback π^* intertwines the codifferential operators, i.e.,

$$\delta_X \pi^* = \pi^* \delta_M.$$

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Since $\Delta_M \omega = 0$, it follows that $\delta_M \omega = 0$. Thus,

$$\Delta_X f = \delta_X df = \delta_X \pi^* \omega = \pi^* \delta_M \omega = 0.$$

For the uniqueness part, let f_1 and f_2 be any two harmonic functions on X such that (A.4) holds for each of these functions. Since $f_1 - f_2$ is G-periodic, it can be pushed down to a real function f on M. Moreover, $\pi^* \Delta_M f = \Delta_X \pi^* f = \Delta_X (f_1 - f_2) = 0$. Therefore, f must be constant since it is a harmonic function on a compact, connected Riemannian manifold M. Thus, $f_1 - f_2$ is constant. $\qquad\square$

Corollary A.2.7. *Fixing a base point x_0 in X. Then to each $\alpha \in \mathrm{Hom}(G, \mathbb{R})$, there exists a unique harmonic function f_α defined on X such that $f_\alpha(x_0) = 0$ and $\Upsilon([f_\alpha]) = \alpha$, where Υ is introduced in (A.2). Consequently,*

$$\mathcal{A}(X) = \bigsqcup_{\alpha \in \mathrm{Hom}(G, \mathbb{R})} \{ f_\alpha + \varphi \mid \varphi \text{ is periodic} \}.$$

Remark A.2.8. When $G = \mathbb{Z}^d$, the Albanese pseudo-metric d_G introduced in [46, Section 2] is just the pseudo-distance arising from any harmonic vector-valued additive function h satisfying (A.3), i.e., $d_G(x, y) = |h(x) - h(y)|$ for any $x, y \in X$.