Stack Semantics of Type Theory

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Abstract

We give a model of dependent type theory with one univalent universe and propositional truncation interpreting a type as a *stack*, generalizing the groupoid model of type theory. As an application, we show that countable choice cannot be proved in dependent type theory with one univalent universe and propositional truncation.

1 Introduction

The axiom of univalence [?, ?] can be seen as an extension to dependent type theory of the two axioms of extensionality for simple type theory as formulated by Church [?]. This extension is important since, using universe and dependent sums, we get a formal system in which we can represent arbitrary structures (which we can not do in simple type theory) with elegant formal properties. The goal of this paper is to contribute to the meta-theory of such systems by showing that *Markov's principle* and *countable choice* are not provable in dependent type theory extended with one univalent universe and propositional truncation. For simple type theory such independence results can be obtained by using *sheaf semantics*, respectively over Cantor space (for Markov's principle) and open unit interval $(0, 1)$ (for countable choice). There are however problems with extending sheaf semantics to universes [?, ?]. In order to address these issues we use a suitable formulation of *stack semantics*, which, roughly speaking, replaces *sets* by *groupoids*. The notion of stack was introduced in algebraic geometry [?, ?] precisely in order to solve the same problems that one encounters when trying to extend sheaf semantics to type-theoretic universes. The compatibility condition for gluing local data is now formulated in terms of isomorphisms instead of strict equalities. In this sense, our model can also be seen as an extension of the groupoid model of type theory [?]. One needs to formulate some strict functoriality conditions on the stack gluing operation, which seem necessary to be able to get a model of the required equations of dependent type theory.

We see this work as a first step towards the proof of independence of countable choice from type theory with a hierarchy of univalent universes and propositional truncation, which we hope to obtain by an extension of our model to an ∞-stack version of cubical type theory [?].

The paper is organized as follows. We first present a slight variation of the groupoid model that we find convenient for expressing the stack semantics. We then explain how to represent propositional truncation in this setting, and how it can be used to formulate countable choice. We then notice that, even in a constructive meta-logic where countable choice fails, the axiom of countable choice does hold in this groupoid model. The groupoid model can be refined rather directly over a Kripke structure, and we present then our notion of stacks over a general topological space together with a proof that we get a model of dependent type theory with one univalent universe and propositional truncation. Instantiating our model to the case of Cantor space and open unit interval $(0, 1)$ we obtain the results that Markov's principle and countable

$$\vdash () \qquad \frac{\Gamma \vdash A}{\vdash \Gamma.A}$$

$$\frac{\vdash \Gamma}{\vdash 1 : \Gamma \to \Gamma} \qquad \frac{\vdash \tau : \Theta \to \Delta \qquad \vdash \sigma : \Delta \to \Gamma}{\sigma\tau : \Theta \to \Gamma} \qquad \frac{\Gamma \vdash A \qquad \vdash \sigma : \Delta \to \Gamma}{\Delta \vdash A\sigma}$$

$$\frac{\Gamma \vdash A}{\Gamma.A \vdash \mathbf{q} : A\mathbf{p}} \qquad \frac{\Gamma \vdash a : A \qquad \vdash \sigma : \Delta \to \Gamma}{\Delta \vdash a\sigma : A\sigma}$$

$$A1 = A \qquad A(\sigma\tau) = (A\sigma)\tau \qquad a1 = a \qquad a(\sigma\tau) = (a\sigma)\tau$$

$$\frac{\Gamma \vdash A}{\vdash \mathbf{p} : \Gamma.A \to \Gamma} \qquad \frac{\Gamma \vdash A \qquad \vdash \sigma : \Delta \to \Gamma \qquad \Delta \vdash a : A\sigma}{\vdash (\sigma, a) : \Delta \to \Gamma.A}$$

$$1\sigma = \sigma \qquad \sigma 1 = \sigma \qquad \sigma(\tau\upsilon) = (\sigma\tau)\upsilon \qquad \mathbf{p}(\sigma, a) = \sigma \qquad \mathbf{q}(\sigma, a) = a \qquad (\mathbf{p}\sigma, \mathbf{q}\sigma) = \sigma$$

$$\frac{\Gamma.A \vdash B}{\Gamma \vdash \Pi A B} \qquad \frac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda b : \Pi A B} \qquad \frac{\Gamma \vdash f : \Pi A B \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}\,(f, a) : B\,[a]}$$

$$\mathsf{app}\,(\lambda b, a) = b\,[a] \qquad \lambda \mathsf{app}\,(f\mathbf{p}, \mathbf{q}) = f$$

Figure 1: Type theory

choice cannot be proved in dependent type theory with one univalent universe and propositional truncation.

2 Type theory

As in [?], we will use a generalized algebraic presentation of type theory that is name-free and has explicit substitutions. For instance, if we write $A \to B$ for $\Pi A(B\mathbf{p})$ then we have $\Gamma \vdash \lambda\mathbf{q} : A \to A$ since $\Gamma.A \vdash \mathbf{q} : A\mathbf{p}$. The advantage of using such a presentation is that it makes it easier to check the correctness of the model: Building such a model is reduced to defining operations such that certain equations hold. The main rules are presented in figures ??, ??, ?? and ??. We omit equivalence, congruence and substitution rules. The conversion rules assume appropriate typing premises.

We write $[a]$ for the substitution $(1, a)$ and $[a, b]$ for $([a], b)$.

3 Groupoid model

In this section, we review the *groupoid model* of [?], with a slightly different presentation inspired from [?]. We work in a set theory with a Grothendieck universe $\mathcal{U}$ (or a suitable constructive version of it if we work in a constructive set theory such as CZF [?]).

A *groupoid* is given by a set Γ of objects and for each $\rho, \rho' \in \Gamma$ a set $\Gamma(\rho, \rho')$ of paths/isomorphisms along with a composition operation $\alpha \cdot \alpha'$ in $\Gamma(\rho, \rho'')$ for α in $\Gamma(\rho, \rho')$ and α' in $\Gamma(\rho', \rho'')$ and a unit element 1_ρ in $\Gamma(\rho, \rho)$ and an inverse operation α^{-1} in $\Gamma(\rho', \rho)$ satisfying the usual unit, inverse and associativity laws. We may write $\alpha : \rho \cong \rho'$ for α in $\Gamma(\rho, \rho')$.

$$\frac{\Gamma \vdash A \text{ small} \qquad \Gamma.A \vdash B \text{ small}}{\Gamma \vdash \Pi A B \text{ small}} \qquad \frac{\Gamma.A \vdash B \text{ discrete}}{\Gamma \vdash \Pi A B \text{ discrete}}$$

$$\Gamma \vdash \mathsf{U} \qquad \frac{\Gamma \vdash A \text{ small discrete}}{\Gamma \vdash |A| : \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U}}{\Gamma \vdash \mathsf{El}\, a \text{ small discrete}}$$

$$\mathsf{El}\ |A| = A \qquad |\mathsf{El}\, a| = a$$

$$\frac{\Gamma \vdash A \text{ small}}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \text{ discrete}}{\Gamma \vdash A}$$

Figure 2: Universe in type theory

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathsf{Path}\, A\, a\, b \text{ discrete}} \qquad \frac{\Gamma \vdash A \text{ small} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathsf{Path}\, A\, a\, b \text{ small}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}\, a : \mathsf{Path}\, A\, a\, a}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \qquad \Gamma.A.A\mathbf{p}.\mathsf{Path}\, A\mathbf{pp}\, \mathbf{qp}\, \mathbf{q} \vdash C \\ \Gamma.A \vdash c : C\, [\mathbf{q}, \mathsf{refl}\, \mathbf{q}] \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash p : \mathsf{Path}\, A\, a\, b\end{array}}{\Gamma \vdash \mathsf{J}\, c\, a\, b\, p : C\, [a, b, p]}$$

$$\mathsf{J}\, c\, a\, a\, (\mathsf{refl}\, a) = c\, [a]$$

Figure 3: Equality in type theory

$$\frac{\Gamma.A \vdash B}{\Gamma \vdash \Sigma A B} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[a]}{\Gamma \vdash (a,b) : \Sigma A B} \qquad \frac{\Gamma \vdash p : \Sigma A B}{\Gamma \vdash p.1 : A} \qquad \frac{\Gamma \vdash p : \Sigma A B}{\Gamma \vdash p.2 : B[p.1]}$$

$$(a,b).1 = a \qquad (a,b).2 = b \qquad (p.1, p.2) = p$$

$$\Gamma \vdash \mathsf{N} \text{ small discrete} \qquad \Gamma \vdash 0 : \mathsf{N} \qquad \frac{\Gamma \vdash n : \mathsf{N}}{\Gamma \vdash \mathsf{suc}\, n : \mathsf{N}}$$

$$\frac{\Gamma.\mathsf{N} \vdash C \qquad \Gamma \vdash c : C[0] \qquad \Gamma.\mathsf{N}.C \vdash d : C[\mathsf{suc}\,\mathbf{q}]\,\mathbf{p} \qquad \Gamma \vdash n : \mathsf{N}}{\Gamma \vdash \mathsf{rec}\, c\, d\, n : C[n]}$$

$$\mathsf{rec}\, c\, d\, 0 = c \qquad \mathsf{rec}\, c\, d\, (\mathsf{suc}\, n) = d\,[n, \mathsf{rec}\, c\, d\, n]$$

$$\Gamma \vdash \mathsf{N}_2 \text{ small discrete} \qquad \Gamma \vdash 0 : \mathsf{N}_2 \qquad \Gamma \vdash 1 : \mathsf{N}_2$$

$$\frac{\Gamma.\mathsf{N}_2 \vdash C \qquad \Gamma \vdash c : C[0] \qquad \Gamma \vdash d : C[1] \qquad \Gamma \vdash b : \mathsf{N}_2}{\Gamma \vdash \mathsf{rec}_2\, c\, d\, b : C[b]}$$

$$\mathsf{rec}_2\, c\, d\, 0 = c \qquad \mathsf{rec}_2\, c\, d\, 1 = d$$

Figure 4: Dependent sum, natural numbers and Booleans in type theory

A map $\sigma : \Delta \to \Gamma$ between two groupoids Δ and Γ is given by a set-theoretic map $\sigma\, \nu$ in Γ for ν in Δ and a map $\sigma\, \beta$ in $\Gamma(\sigma\, \nu, \sigma\, \nu')$ for β in $\Delta(\nu, \nu')$ which commutes with unit, inverse and composition.

A family A of groupoids indexed over a groupoid Γ, written $\Gamma \vdash A$, is given by a family of sets $A\rho$ for each ρ in Γ and sets $A\alpha(u,u')$ for each α in $\Gamma(\rho,\rho')$ and $u \in A\rho$ and $u' \in A\rho'$. We may write $\omega : u \cong_\alpha u'$ for ω element of $A\alpha(u,u')$ and we may omit the subscript α if it is clear from the context. We also have unit $1_u : u \cong_{1_\rho} u$ and inverse $\omega^{-1} : u' \cong_{\alpha^{-1}} u$ and composition $\omega \cdot \omega' : u \cong_{\alpha\cdot\alpha'} u''$ also satisfying the unit, inverse and associativity laws. We furthermore should have a *path lifting structure*, which is given by two operations $u\alpha$ in $A\rho'$ and $u \uparrow \alpha : u \cong_\alpha u\alpha$ for u in $A\rho$ and $\alpha : \rho \cong \rho'$ satisfying the laws

$$u1_\rho = u \qquad (u\alpha)\alpha' = u(\alpha \cdot \alpha') \qquad u \uparrow 1_\rho = 1_u \qquad (u \uparrow \alpha) \cdot (u\alpha \uparrow \alpha') = u \uparrow (\alpha \cdot \alpha')$$

We see that $u \uparrow \alpha$ "lifts" the path $\alpha : \rho \cong \rho'$ given an initial point u in $A\rho$.

Each $A\rho$ has a canonical groupoid structure, defining $A\rho(u,u')$ to be $A1_\rho(u,u')$. If $\alpha : \rho \cong \rho'$ we can define a groupoid map $A\rho \to A\rho'$ using the lifting operation. We thereby recover the groupoid model as defined in [?].

If $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ we define $\Delta \vdash A\sigma$ by composition: $(A\sigma)\nu$ is $A(\sigma\, \nu)$ and $(A\sigma)\beta(v,v')$ is $A(\sigma\, \beta)(v,v')$.

A *section* $\Gamma \vdash a : A$ is given by a family of objects $a\rho$ in $A\rho$ together with a family of paths $a\alpha : a\rho \cong_\alpha a\rho'$ satisfying the laws $a1_\rho = 1_{a\rho}$ and $a(\alpha \cdot \alpha') = a\alpha \cdot a\alpha'$.

If $\Gamma \vdash A$, we define a new groupoid $\Gamma.A$: An object (ρ, u) in $\Gamma.A$ is a pair with ρ in Γ and u in $A\rho$ and a path $(\alpha, \omega) : (\rho, u) \cong (\rho', u')$ is a pair $\alpha : \rho \cong \rho'$ and $\omega : u \cong_\alpha u'$. We then have $\mathbf{p} : \Gamma.A \to \Gamma$ defined by $\mathbf{p}(\rho, u) = \rho$ and $\mathbf{p}(\alpha, \omega) = \alpha$ and the section $\Gamma.A \vdash \mathbf{q} : A\mathbf{p}$ defined by $\mathbf{q}(\rho, u) = u$ and $\mathbf{q}(\alpha, \omega) = \omega$.

We say that a family $\Gamma \vdash A$ is *small* if each set $A\rho$ and $A\alpha(u, u')$ is in the given Grothendieck universe $\mathscr{U}$. We say that this family is *discrete* if the lifting is *uniquely* determined: Given u in $A\rho$ and $\alpha : \rho \cong \rho'$ there is a unique u' in $A\rho'$ such that $A\alpha(u, u')$ is inhabited and this set is a singleton in this case. This notion of discrete family can be characterized in terms of the common definition of *discrete groupoid*, which says that a groupoid is discrete if the only paths are units.

Lemma 1. *$\Gamma \vdash A$ is discrete if and only if each groupoid $A\rho$, $\rho \in \Gamma$, is discrete.*

Proof. Assume $\Gamma \vdash A$ to be a discrete family and let $\omega \in A1_\rho(u, u')$ be an arbitrary path. We immediately have $u' = u$ and $\omega = 1_u$ by discreteness of $\Gamma \vdash A$ and $1_u \in A1_\rho(u, u)$.

For each $\rho \in \Gamma$, assume $A\rho$ to be a discrete groupoid and let $\omega' \in A\alpha(u, u')$, $\omega'' \in A\alpha(u, u'')$ be two arbitrary paths over some $\alpha \in \Gamma(\rho, \rho')$. Then, ω'' can be expressed as the composite of ω' and $\omega'^{-1} \cdot \omega'' \in A1_{\rho'}(u', u'')$. The discreteness of $A\rho'$ forces $\omega'^{-1} \cdot \omega''$ to be a unit path so that $u'' = u'$ and $\omega'' = \omega'$. □

We define U to be the following groupoid: An object X in U is exactly an element of the given Grothendieck universe $\mathscr{U}$, and an element of $\mathsf{U}(X, X')$ is a bijection between X and X'. We can then define the small and discrete family $\mathsf{U} \vdash \mathsf{El}$ by taking $\mathsf{El}\, X$ to be the set X and $u \cong_\alpha u'$ to be the subsingleton set $\{0 \mid u' = \alpha u\}$, that is $u \cong_\alpha u'$ is inhabited and is the singleton $\{0\}$ exactly when $u' = \alpha u$.

Proposition 1. *The family $\mathsf{U} \vdash \mathsf{El}$ is a* universal *small and discrete family: If $\Gamma \vdash A$ is small and discrete, then there exists a* unique *map $|A| : \Gamma \to \mathsf{U}$ such that $\mathsf{El}\,|A| = A$ (with* strict *equality).*

For $\Gamma \vdash A$ and $\Gamma.A \vdash B$ we define $\Gamma \vdash \Pi A B$ by taking $(\Pi A B)\rho$ to be the set of functions $c\, u$ in $B(\rho, u)$ and $c\, \omega$ in $B(1_\rho, \omega)(c\, u, c\, u')$ commuting with unit and composition, and $(\Pi A B)\alpha(c, c')$ to be the set of functions $\gamma\, \omega : c\, u \cong_{(\alpha,\omega)} c'\, u'$ such that $(\gamma\, \omega_0) \cdot (c'\, \beta') = (c\, \beta) \cdot (\gamma\, \omega_1)$ if $\beta : u_0 \cong_\rho u_1$ and $\beta' : u'_0 \cong_{\rho'} u'_1$ and $\omega_0 : u_0 \cong_\alpha u'_0$ and $\omega_1 : u_1 \cong_\alpha u'_1$. There is then [?, ?] a canonical way to define a composition operation (we need the path lifting structure for $\Gamma \vdash A$) and path lifting structure for $\Gamma \vdash \Pi A B$.

Proposition 2. *If $\Gamma.A \vdash B$ is discrete, then so is $\Gamma \vdash \Pi A B$.*

Proof. In order to show that $\Gamma \vdash \Pi A B$ is a discrete family, it suffices to show that $(\Pi A B)\rho$ is a discrete groupoid for each $\rho \in \Gamma$. Assume $\Gamma.A \vdash B$ to be a discrete family and let $\gamma \in (\Pi A B)1_\rho(c, c')$ be an arbitrary path. In particular, Bu is a discrete groupoid forcing $\gamma\, 1_u \in B1_u(c\, u, c'\, u)$ to be a unit path for each $u \in A\rho$ so that $c'\, u = c\, u$ for all $u \in A\rho$. The discreteness of $\Gamma.A \vdash B$ also forces $c'\, \omega = c\, \omega$ and $\gamma\, \omega = 1_c\, \omega$ in $B\omega(c\, u', c\, u'')$ for all $\omega \in A1_\rho(u', u'')$, which concludes $c' = c$ and $\gamma = 1_c$. □

If $\Gamma \vdash A$ and $\Gamma \vdash a_0 : A$ and $\Gamma \vdash a_1 : A$ we define the *discrete* family $\Gamma \vdash \mathsf{Path}\, A\, a_0\, a_1$. We take $(\mathsf{Path}\, A\, a_0\, a_1)\rho$ for $\rho \in \Gamma$ to be the set $A1_\rho(a_0\rho, a_1\rho)$ and $(\mathsf{Path}\, A\, a_0\, a_1)\alpha(\omega, \omega')$ for $\alpha : \rho \cong \rho'$ to be the subsingleton $\{0 \mid \omega \cdot a_1\alpha = a_0\alpha \cdot \omega'\}$.

It is then possible [?, ?] to check that this defines a model of type theory as presented by the rules of figures ??, ??, ?? and ??.

3.1 Propositional truncation

We say that a groupoid is a *proposition* if and only if there exists exactly one path between two objects. So Γ is a proposition if and only if each set $\Gamma(\rho, \rho')$ is a singleton. More generally, we say that a family $\Gamma \vdash A$ is a proposition if each set $A\alpha(u, u')$ is a singleton.

Lemma 2. *$\Gamma \vdash A$ is a proposition if and only if each groupoid $A\rho$, $\rho \in \Gamma$, is a proposition.*

Proof. It is clear that each $A\rho$ is a proposition if the whole family $\Gamma \vdash A$ is a proposition.

Assume now each $A\rho$ to be a proposition and let $\alpha \in \Gamma(\rho, \rho')$ as well as $u \in A\rho$, $u' \in A\rho'$. Then, the set $A\alpha(u, u')$ is inhabited by the composite $(u \uparrow \alpha) \cdot p_{u\alpha,u'}$ of the lifting of u over α with the unique path between $u\alpha$ and u' in $A\rho'$. Furthermore, for any two paths $\omega, \omega' \in A\alpha(u, u')$ the composite $\omega^{-1} \cdot \omega'$ is forced to be the unit path at u' so that $\omega' = \omega \cdot \omega^{-1} \cdot \omega' = \omega$. □

We define as usual (where names are used for readability)

$$\mathsf{isProp}\, A = \Pi(x_0\, x_1 : A)\mathsf{Path}\, A\, x_0\, x_1$$

Proposition 3. *If $\Gamma \vdash A$, then there exists a section $\Gamma \vdash p : \mathsf{isProp}\, A$ if and only if each groupoid $A\rho$, ρ in Γ, is a proposition.*

Proof. It is enough to show that there exists a family of paths $p_{\rho,u,u'} \in A1_\rho(u, u')$, $u, u' \in A\rho$, $\rho \in \Gamma$, satisfying $p_{\rho,u,u'} \cdot \omega' = \omega \cdot p_{\rho',v,v'}$ for all $\omega \in A\alpha(u, v)$ and $\omega' \in A\alpha(u', v')$, $u, u' \in A\rho$, $v, v' \in A\rho'$, $\alpha \in \Gamma(\rho, \rho')$, $\rho, \rho' \in \Gamma$, if and only if each groupoid $A\rho$, $\rho \in \Gamma$, is a proposition.

Assume such a family p and let $\rho \in \Gamma$, $u, u' \in A\rho$, then $A1_\rho(u, u')$ is inhabited by the composite $p_{\rho,u,u'} \cdot (p_{\rho,u',u'})^{-1}$ and, moreover, any other path $\omega \in A1_\rho(u, u')$ satisfies $p_{\rho,u,u'} \cdot 1_{u'} = \omega \cdot p_{\rho,u',u'}$ so that $A1_\rho(u, u')$ is indeed a singleton.

In the opposite direction, we can actually assume the whole family $\Gamma \vdash A$ to be a proposition. Then, defining $p_{\rho,u,u'}$ to be the unique path from u to u' satisfies $p_{\rho,u,u'} \cdot \omega' = \omega \cdot p_{\rho',v,v'}$ because there exists exactly one path from u to v' over α. □

For $\Gamma \vdash A$ we define $\Gamma \vdash \|A\|$ as follows. For each ρ in Γ we take $\|A\|\, \rho = A\rho$, and for each α in $\Gamma(\rho, \rho')$, u in $A\rho$ and u' in $A\rho'$ we take $\|A\|\, \alpha(u, u')$ to be a fixed singleton $\{0\}$. We then have sections of $\Gamma \vdash \mathsf{isProp}\ \|A\|$ and $\Gamma \vdash A \to \|A\|$, and given sections of $\Gamma \vdash \mathsf{isProp}\, B$ and $\Gamma \vdash A \to B$ there is a section of $\Gamma \vdash \|A\| \to B$. In this way, we get a model of the *propositional truncation* operation.

3.2 Countable choice

The statement of *countable choice* can be formulated as the type [?]

$$\mathsf{CC} = \Pi(A : \mathsf{N} \to \mathsf{U})(\Pi(n : \mathsf{N})\, \|\mathsf{El}\,(A\, n)\|) \to \|\Pi(n : \mathsf{N})\mathsf{El}\,(A\, n)\|$$

Notice that we can develop the groupoid model in a constructive meta-theory where countable choice may or may not hold.

Theorem 1. *The statement* CC *is valid in the groupoid model (even if countable choice does not hold in the meta-theory).*

Proof. It is enough to define $c\, A\, f = f$ and $c\, \alpha\, \omega = 0$ to get $() \vdash c : \mathsf{CC}$. □

4 Stack model

4.1 Groupoid-valued presheaf model

We suppose given a poset with elements $U, V, W, X, \ldots$. The groupoid model extends directly as a groupoid-valued presheaf model over this poset. A *context* is now a family of groupoids $\Gamma(U)$ indexed by elements of the given poset such that objects ρ and paths α in $\Gamma(U)$ can be restricted to $\rho|V$ and $\alpha|V$ in $\Gamma(V)$ if $V \subseteq U$ such that the restriction operation defines a groupoid map $\Gamma(U) \to \Gamma(V)$ which is the identity map for $V = U$ and the composite of $\Gamma(X) \to \Gamma(V)$ and $\Gamma(U) \to \Gamma(X)$ for $V \subseteq X \subseteq U$.

For a given context Γ, we define then what is a family $\Gamma \vdash A$. It is given by a family of sets $A\rho$ for each U and ρ in $\Gamma(U)$ together with a restriction $u|V$ in $A(\rho|V)$ for u in $A\rho$ satisfying $u|U = u$ and $(u|X)|V = u|V$, as well as a family of sets $A\alpha(u, u')$ for each $\alpha : \rho \cong \rho'$ in $\Gamma(U)$, u in $A\rho$ and u' in $A\rho'$ together with a restriction $\omega|V$ in $A(\alpha|V)(u|V, u'|V)$ for ω in $A\alpha(u, u')$ satisfying $\omega|U = \omega$ and $(\omega|X)|V = \omega|V$. In particular, we require the restriction operation on the sets $A\alpha(u, u')$ to commute with unit and composition. Such a family is called *small* if the sets $A\rho$ and $A\alpha(u, u')$ are elements in the Grothendieck universe $\mathcal{U}$, and it is called a *proposition* if the canonical groupoid structure on each $A\rho$ defines a proposition, or, equivalently, if each set $A\alpha(u, u')$ is a singleton. Furthermore, we should have a lifting operation $u \uparrow \alpha$ with the law $(u \uparrow \alpha)|V = (u|V) \uparrow (\alpha|V)$. A family is called *discrete* if the liftings $u \uparrow \alpha$ are uniquely determined: Given U and ρ in $\Gamma(U)$ and given u in $A\rho$ and $\alpha : \rho \cong \rho'$ there is a unique u' in $A\rho'$ such that $A\alpha(u, u')$ is inhabited, and this set is a singleton in this case.

We can extend the groupoid model to this setting.

An element c of $(\Pi A B)\rho$ for ρ in $\Gamma(U)$ is a function $c\,u$ in $B(\rho|V, u)$ for $V \subseteq U$ and u in $A(\rho|V)$ and $c\,\omega$ in $B(1_{\rho|V}, \omega)(c\,u, c\,u')$ for ω in $A1_{\rho|V}(u, u')$ commuting with unit and composition such that $(c\,a)|W = c\,(a|W)$ and $(c\,\omega)|W = c\,(\omega|W)$ if $W \subseteq V \subseteq U$.

An element in $(\Sigma A B)\rho$ for $\rho \in \Gamma(U)$ is a pair (a, b) where $a \in A\rho$ and $b \in B(\rho, a)$ with restrictions $(a, b)|V = (a|V, b|V)$. Paths in $(\Sigma A B)\alpha((a, b), (a', b'))$, where $\alpha : \rho \cong \rho'$ are pairs (ω, μ) where $\omega : a \cong_\alpha a'$ and $\mu : b \cong_{(\alpha,\omega)} b'$ with restrictions $(\omega, \mu)|V = (\omega|V, \mu|V)$.

Given sections a_0 and a_1 of A, an element in $(\mathsf{Path}\, A\, a_0\, a_1)\rho$ for $\rho \in \Gamma(U)$ is a path $\omega : a_0\rho \cong a_1\rho$ with restrictions as in A. For every element ω and path $\alpha : \rho \cong \rho'$ there is a unique path from ω over α going to $(a_0\alpha)^{-1} \cdot \omega \cdot a_1\alpha : a_0\rho' \cong a_1\rho'$.

4.2 Stack structure

We assume given a topological space with a notion of *basic open* closed under nonempty intersection and a notion of *covering* of a given basic open by a family of basic opens. We consider only coverings $(U_i)_{i\in I}$ of some basic open U where the set of indices I is *small.* To simplify the presentation we assume that each basic open set is *nonempty.* We write U_{ij} for $U_i \cap U_j$ and U_{ijk} for $U_i \cap U_j \cap U_k$ when they are nonempty.

Since basic opens form a poset, we can consider the notion of type family over this poset as defined in the previous subsection.

In the following we will define what is a *stack structure* on a type family.

We recall that a *sheaf* F is given by a presheaf, i.e. a family of *sets* $F(U)$ with restriction maps $u|V$ in $F(V)$ for $V \subseteq U$ such that $u|U = u$ and $(u|V)|W = u|W$ if $W \subseteq V \subseteq U$, which satisfies the condition that if we have a covering $(U_i)_{i\in I}$ of U and a family of compatible elements u_i in $F(U_i)$ (i.e. $u_i|U_{ij} = u_j|U_{ij}$) then there exists a unique u in $F(U)$ such that $u|U_i = u_i$ for all i.

A type family $\Gamma \vdash A$ is called a *prestack* if it satisfies the following sheaf condition on paths: If $\alpha : \rho \cong \rho'$ is in $\Gamma(U)$, u and u' are in $A\rho$ and $A\rho'$ respectively and we have a family of paths

$\omega_i : u|U_i \cong_{\alpha|U_i} u'|U_i$ which is compatible (that is $\omega_i|U_{ij} = \omega_j|U_{ij}$), then we have a *unique* path $\omega : u \cong_\alpha u'$ such that $\omega|U_i = \omega_i$ for all i.

For each basic open U and ρ in $\Gamma(U)$ we define what is the set of *descent data* $D(A)\rho$. A *descent datum* is given by a covering $(U_i)_{i\in I}$ of U and a family of objects $u_i \in A(\rho|U_i)$ with paths $\varphi_{ij} : u_i|U_{ij} \cong_{\rho|U_{ij}} u_j|U_{ij}$, when U_i meets U_j, satisfying the *cocycle* conditions[1]

$$\varphi_{ii} = 1_{u_i} \qquad \varphi_{ij}|U_{ijk} \cdot \varphi_{jk}|U_{ijk} = \varphi_{ik}|U_{ijk}$$

This forms a set since the index set is restricted to be small (otherwise this might be a proper class in general).

If $d = (u_i, \varphi_{ij})$ is an element of $D(A)\rho$ and $V \subseteq U$ we define its restriction $d|V$, element of $D(A)\rho|V$, which is the family $(u_i|V \cap U_i, \varphi_{ij}|V \cap U_{ij})$ restricted to indices i such that V meets U_i. A *gluing operation* $\mathbf{glue}\, d = (u, \varphi_i)$ gives an element u in $A\rho$ together with paths $\varphi_i : u|U_i \cong u_i$ such that $\varphi_i|U_{ij} \cdot \varphi_{ij} = \varphi_j|U_{ij}$ *and* satisfies the law $(\mathbf{glue}\, d)|V = \mathbf{glue}(d|V)$, that is $\mathbf{glue}(d|V)$ should be $(u|V, \varphi_i|V \cap U_i)$ where we restrict the family to indices i such that V meets U_i. This functoriality property will be crucial for checking that we do get a model of type theory with dependent product.

A *stack structure* on a prestack $\Gamma \vdash A$ is given by a gluing operation[2].

Consider a prestack $\Gamma \vdash A$ and descent datum $d = (u_i, \varphi_{ij}) \in D(A)\rho$ with $\mathbf{glue}\, d = (u, \varphi_i)$ as above. Let $v \in A\rho$ with paths $\vartheta_i : v|U_i \cong u_i$ satisfying $\vartheta_i|U_{ij} \cdot \varphi_{ij} = \vartheta_j|U_{ij}$. We remark that while it is not necessarily true that $u = v$, the prestack condition implies that we have a path $v \cong u$.

Note that while it is sufficient to define the notion of sheaf as a *property* because of the uniqueness part of the sheaf condition, it is crucial that our notion of stack is in general a *structure*, i.e. given with an explicit operation fixing a particular choice of glue.

A stack is not the same as a groupoid object in the sheaf topos. A prime example of a stack whose presheaf of objects is not a sheaf is the universe of sheaves: If we define $F(U)$ to be the collection of small sheaves over U then there is a natural restriction operation $F(U) \to F(V)$ for $V \subseteq U$, and one can check that the gluing of a compatible family of elements is not unique up to strict equality in general (but it is unique up to isomorphism). Notice that if we try to define the stack structure using global choice as in [**?**, 3.3.1, page 28] then the functoriality condition $(\mathbf{glue}\, d)|V = \mathbf{glue}(d|V)$ will not hold. There is however a more canonical definition of gluing which satisfies this condition, which will provide the interpretation of a univalent universe.

There is also a simple example of a prestack that is not a stack but whose presheaf of objects is a sheaf. Consider the topological space given by basic opens U_1, U_2, U_{12} with $U_1 \wedge U_2 = U_{12}$ and the groupoid-valued presheaf G given by the propositions on the sets $G(1) = \emptyset$, $G(U_1) = \{x_1\}$, $G(U_2) = \{x_2\}$ and $G(U_{12}) = \{x_1, x_2\}$. There are no matching families of objects or morphisms in G so that both the presheaf of objects and the presheaf of morphisms trivially satisfy the sheaf property. However, the descent datum given by $d_1 = x_1$, $d_2 = x_2$ and d_{12} the unique path between $x_1|U_{12}$ and $x_2|U_{12}$ cannot have a glue because $G(1)$ is empty.

Taking as objects the subset $D(A)(\rho, C) \subseteq D(A)\rho$ of descent data on a covering $C = (U_i)_{i\in I}$ of U and a path between two descent data (u_i, φ_{ij}) and (v_i, ψ_{ij}) to be a family of paths $\omega_i : u_i \cong v_i$ satisfying $\omega_i \cdot \psi_{ij} = \varphi_{ij} \cdot \omega_j$ has a natural groupoid structure. Moreover, the canonical restriction from $A\rho$ to $D(A)\rho$ extends to a functor from the canonical groupoid structure on $A\rho$ to $D(A)(\rho, C)$. The prestack condition for A then says that this functor is fully faithful and a gluing operation witnesses that it is essentially surjective. If A is a stack, then the canonical functor $A\rho \to D(A)(\rho, C)$ is an equivalence of groupoids.

[1] The first condition is not logically necessary.

[2] Notice that we shall not require the context Γ to be a prestack or have a stack structure.

4.3 Dependent product

The collection of types with a stack structure is closed under dependent product.

Theorem 2. *If* $\Gamma.A \vdash B$ *has a stack structure then* $\Gamma \vdash \Pi A B$ *has a stack structure.*

Proof. Let $(u_i, \varphi_{ij}) \in D(\Pi A B)\rho$ be a descent datum on a covering $(U_i)_{i \in I}$ of U. We construct a glue (u, φ_i) that commutes with restriction.

Given $x, x' \in A(\rho|V)$ and $v : x \cong x'$ on $V \subseteq U$, we construct $(u\,x, \varphi_i\,x)$ as the glue of $d_x = (u_i\,x, \varphi_{ij}\,x)$ and $u\,v : u\,x \cong_v u\,x'$ as the unique path matching $u\,x \cong u_i\,x \cong_{v|V \cap U_i} u_i\,x' \cong u\,x'$ given by the composite of $\varphi_i\,x$, $u_i\,v$ and the inverse of $\varphi_i\,x'$ on $V \cap U_i$. If in particular $V \subseteq U_i$, then this completely determines $\varphi_i : u|U_i \cong u_i$. The uniqueness of $u\,v$ is needed to show that u respects units and composites as well as restriction of paths. For u to also respect restriction of objects we need the fact that $(\mathbf{glue}\,d_x)|W = \mathbf{glue}\,d_x|W = \mathbf{glue}\,d_{x|W}$ for $W \subseteq V$.

Let $\omega_i : u|U_i \cong_{\alpha|U_i} u'|U_i$ be a matching family of paths. We show that there is a unique glue $\omega : u \cong_\alpha u'$. It is uniquely determined by the glues $\omega\,v : u\,x \cong_{(\alpha|V,v)} u'\,x'$ of $\omega_i\,v : u\,x \cong_{(\alpha|V \cap U_i, v|V \cap U_i)} u'\,x'$ for $v : x \cong_{\alpha|V} x'$. In particular, $\omega\,v = \omega_i\,v$ if $V \subseteq U_i$. Again, the uniqueness of $\omega\,v$ lets us show that ω respects composites and restrictions. □

4.4 Universe of sheaves

We define $\mathsf{U}(V)$ to be the collection of all *small* sheaves over V. There is a natural restriction operation $\mathsf{U}(V) \to \mathsf{U}(W)$ if $W \subseteq V$.

Theorem 3. U *has a stack structure.*

Proof. Let $F_i \in \mathsf{U}(U_i)$ with $\varphi_{ij} : F_i|U_{ij} \cong F_j|U_{ij}$ be a descent datum on a cover $(U_i)_{i \in I}$ of U. We construct a glue $F \in \mathsf{U}(U)$ and $\varphi_i : F|U_i \cong F_i$. We define $F(V)$ for $V \subseteq U$ as the set of families $(x_i)_i$ where $x_i \in F_i(V \cap U_i)$ and $\varphi_{ij}(x_i) = x_j$. Furthermore, we define $F(V) \to F(W)$ for $W \subseteq V$ component-wise by the restriction $F_i(V \cap U_i) \to F_i(W \cap U_i)$ and φ_i by the projection to the i-th component. For φ_i to be an isomorphism we need the fact $\varphi_{ii} = 1$ and $\varphi_{ij} \cdot \varphi_{jk} = \varphi_{ik}$.

We claim that the presheaf F satisfies the sheaf property. Indeed, let $v_k \in F(V_k)$ be a matching family for F on a cover $(V_k)_{k \in K}$ of V. The i-th components of v_k are a matching family for F_i on the induced cover $(V_k \cap U_i)_{k \in K}$ of $V \cap U_i$ and the gluing operation $D(F_i)(V_k \cap U_i) \to F_i(V \cap U_i)$ of a discrete stack is a bijection so that we obtain a glue $v \in F(V)$ of v_k by gluing component-wise. This glue is unique because it is component-wise unique.

Let now $\omega_i : G|U_i \cong H|U_i$, $G, H \in \mathsf{U}(U)$ be a matching family of paths on a cover $(U_i)_{i \in I}$ of U. For $x \in G(V)$, $V \subseteq U$ the family $\omega_i\,x$ in $H(V \cap U_i)$ is compatible because $\omega_i = \omega_j$ on $V \cap U_{ij}$. We define $\omega\,x$ to be the unique glue in $H(V)$ such that $(\omega\,x)|V \cap U_i = \omega_i\,x$. The uniqueness of glues allows us to verify that ω respects restriction and that the such defined ω is the unique path that agrees with ω_i on U_i. □

We define $\mathsf{U} \vdash \mathsf{El}$ by taking $\mathsf{El}F$ to be the small set $F(V)$ if F is in $\mathsf{U}(V)$ and $\mathsf{El}\alpha(a, a')$ to be the set $\{0 \mid \alpha a = a'\}$ if α is an isomorphism between F and F' in $\mathsf{U}(V)$ and a is in $F(V)$ and a' is in $F'(V)$.

Theorem 4. *The family* $\mathsf{U} \vdash \mathsf{El}$ *is a* universal *small and discrete stack: If* $\Gamma \vdash A$ *is small and discrete stack, there exists a* unique *map* $|A| : \Gamma \to \mathsf{U}$ *such that* $\mathsf{El}\,|A| = A$ *(with* strict *equality).*

4.5 Dependent sums

Theorem 5. *If* $\Gamma \vdash A$ *and* $\Gamma.A \vdash B$ *have stack structures then we can glue descent data and paths in* $\Gamma \vdash \Sigma A B$*.*

Proof. Let $((u_i, v_i), (\omega_{ij}, \mu_{ij})) \in D(\Sigma A B)\rho$ be a descent datum on a covering $(U_i)_{i\in I}$ of U. We construct a glue $((u, v), (\omega_i, \mu_i))$ that commutes with restriction.

Let (u, ω_i) to be the glue of the datum $(u_i, \omega_{ij}) \in D(A)\rho$. We describe a descent datum in $D(B)(\rho, a)$. The object part of this descent datum is given by $v_i\omega_i^{-1} \in B(\rho|U_i, u|U_i)$. We have then paths $(v_i\omega_i^{-1} \uparrow \omega_i) : v_i\omega_i^{-1} \cong_{\omega_i} v_i$ and thus paths

$$(v_i\omega_i^{-1} \uparrow \omega_i)|U_{ij} \cdot \mu_{ij} \cdot (v_j\omega_j^{-1} \uparrow \omega_j)^{-1}|U_{ij} : v_i\omega_i^{-1}|U_{ij} \cong v_j\omega_j^{-1}|U_{ij}$$

These satisfy the cocycle condition. Thus we have a descent datum in $D(B)(\rho, u)$. Let (v, μ_i') be the glue of this datum. We have paths $\mu_i \coloneqq \mu_i' \cdot (v_i\omega_i^{-1} \uparrow \omega_i) : v|U_i \cong_{\omega_i} v_i$

We then take the glue of $((u_i, v_i), (\omega_{ij}, \mu_{ij}))$ to be given by $((u, v), (\omega_i, \mu_i))$. Since

$$\mu_i'|U_{ij} \cdot (v_i\omega_i^{-1} \uparrow \omega_i)|U_{ij} \cdot \mu_{ij} \cdot (v_j\omega_j^{-1} \uparrow \omega_j)^{-1}|U_{ij} = \mu_j'|U_{ij}$$

we have that $\mu_i|U_{ij} \cdot \mu_{ij} = \mu_j|U_{ij}$.

Let $\alpha : \rho \cong \rho'$. Given a matching family of paths $(\omega_i, \mu_i) : (u, v)|U_i \cong_{\alpha|U_i} (u', v')$. Since A have a stack structure we have a unique $\omega : u \cong_\alpha u'$ with $\omega|U_i = \omega_i$. But then $\mu_i : v|U_i \cong_{(\alpha,\omega)|U_i} v'|U_i$ is a matching family for the stack B and thus have a unique $\mu : v \cong_{(\alpha,\omega)} v'$ where $\mu|U_i = \mu_i$. □

4.6 Paths

Descent data for the *discrete* family $\Gamma \vdash \mathsf{Path}\, A\, a_0\, a_1$ correspond to matching families of paths for A and they have unique glues if A is a prestack. Unique choice then gives us a function from descent data to glues for $\mathsf{Path}\, A\, a_0\, a_1$ which, also by uniqueness, necessarily commutes with restriction.

Proposition 4. *If* $\Gamma \vdash A$ *has a stack structure and* $\Gamma \vdash a_0 : A$ *and* $\Gamma \vdash a_1 : A$ *then* $\Gamma \vdash \mathsf{Path}\, A\, a_0\, a_1$ *has a discrete stack structure.*

4.7 Univalence

An equivalence between two types A and B is a map $f : A \to B$ such that for each $y : B$ the fiber of f above y is contractible. If both A and B are discrete stacks, then f being contractible means that f is an isomorphism. As in [?, 5.4], we have then a one-to-one correspondence between the type of equivalences $\mathsf{El}\, a \simeq \mathsf{El}\, b$ and the type of paths $\mathsf{Path}\, \mathsf{U}\, a\, b$.

4.8 Propositional truncation

We define the family of sets $\|A\|\, \rho$ inductively. For every basic open U, $\rho \in \Gamma(U)$ and $u \in A\rho$ let $u \in \|A\|\, \rho$. Moreover, for every covering $(U_i)_{i\in I}$ of a given basic open U, $\rho \in \Gamma(U)$ and $u_i \in \|A\|\, (\rho|U_i)$ let $(U_i, u_i)_{i\in I} \in \|A\|\, \rho$. Notice that this forms a set since the index set I is restricted to be small. (Without this restriction, we will get a *class* and not a *set* in general.) Then, we define the family of functions $\|A\|\, \rho \to \|A\|\, (\rho|V)$ recursively. For every pair of basic opens $V \subseteq U$, $\rho \in \Gamma(U)$ and $x \in \|A\|\, \rho$ let $x|V \coloneqq u|V$ if $x = u$ with $u \in A\rho$ and $x|V \coloneqq (U_j \cap V, u_j|U_j \cap V)_{j\in J}$, where $J \subseteq I$ is the restriction to indices $i \in I$ such that U_i meets V, if $x = (U_i, u_i)_{i\in I}$ with $u_i \in \|A\|\, (\rho|U_i)$. Lastly, we define the type family $\|A\|$ to be the proposition on the family of sets and functions just defined. The collection of discrete families $\Gamma \vdash A$ $\mathsf{discrete}$ is *not* closed under propositional truncation: Given $\alpha : \rho \cong \rho'$ and $x \in \|A\|\, \rho$, then there is a unique path $x \cong_\alpha x'$ that connects x to each element $x' \in \|A\|\, \rho'$.

The family $\Gamma \vdash \|A\|$ always has a stack structure, even without assuming one on $\Gamma \vdash A$. If we have a covering $(V_l)_{l\in L}$ of U and for each l in L we have an element x_l of $\|A\|\, (\rho|V_l)$, then this family x_l always defines in a unique way a descent datum and we can consider the family $(V_l, x_l)_{l \in L}$, which defines a gluing of the family x_l. This operation furthermore satisfies the functoriality condition.

4.9 Example: One-point space

One of the simplest examples of the notion of stack is that where the poset of basic opens has exactly one object V. In that case, a covering of V is a nonempty finite family $(V_i)_{i \in I}$ where each $V_i = V$ and a stack is a single groupoid G with a gluing operation. That is, for any family $(u_i)_{i \in I}$ of elements in G and paths $\varphi_{ij} : u_i \cong u_j$ satisfying $\varphi_{ij} \cdot \varphi_{jk} = \varphi_{ik}$ we have $\mathbf{glue}(u_i, \varphi_{ij}) = (u, \varphi_i)$ such that $\varphi_i : u \cong u_i$ and $\varphi_i \cdot \varphi_{ij} = \varphi_j$. We can use this example to motivate the definition of propositional truncation given above. Suppose we naively truncate the groupoid G to get the proposition $\|G\|$ on the objects of G, then we have no way of defining the gluing operation on $\|G\|$ since we lack a particular choice of glue for a given descent datum.

5 Countable choice

5.1 A stack model where countable choice does not hold

We write $U, V, W, \ldots$ nonempty open rational intervals included in the open unit interval $(0, 1)$. For each n, and $i = 1, \ldots, n$ we let U_i^n be $((i-1)/(n+1), (i+1)/(n+1))$ so that $(U_i^n)_{i=1,\ldots,n}$ is a covering of $(0, 1)$.

We let $|\mathsf{N}|$ be the constant presheaf where each $|\mathsf{N}|(V)$ is the set $\mathbb{N}$ of natural numbers and $\mathsf{N} = \mathsf{El}\,|\mathsf{N}|$. We have [?]

Lemma 3. $|\mathsf{N}|$ *is a (small) sheaf.*

It is also well-known that in the sheaf model over $(0, 1)$, there are Dedekind reals that are not Cauchy reals [?]. It is simple to transform this fact to a counter-example to our type-theoretic version of countable choice.

We define $A : \mathsf{N} \to \mathsf{U}$ by letting $A\,n$ be the subsheaf of the (small) constant sheaf $|\mathsf{Q}|(V) = \mathbb{Q}$ of rational numbers

$$(A\,n)(V) = \left\{ r \in \mathbb{Q} \,\middle|\, \forall (x \in V)\ |x - r| < \frac{1}{n+1} \right\}$$

Notice that each $i/(n+1)$ is an element of $(A\,n)(U_i^n)$.

Proposition 5. *In this model*

1. *the type* $\Pi(n : \mathsf{N})\,\|\mathsf{El}\,(A\,n)\|$ *is inhabited*
2. *the type* $\Pi(n : \mathsf{N})\mathsf{El}\,(A\,n)$*, and hence also the type* $\|\Pi(n : \mathsf{N})\mathsf{El}\,(A\,n)\|$*, is empty*

Proof. For each open set V, we let $s_V\,n$ be the family $(V \cap U_i^n, i/(n+1))$, i such that V and U_i^n meet, in $\|\mathsf{El}\,(A\,n)\|\,(V)$. Since we have $(s_V\,n)|W = s_W\,n$ if $W \subseteq V$, this defines a section of $\Pi(n : \mathsf{N})\,\|\mathsf{El}\,(A\,n)\|$.

For the second point, it is enough to notice that, for each given V, the set

$$(A\,n)(V) = \left\{ r \in \mathbb{Q} \,\middle|\, \forall (x \in V)\ |x - r| < \frac{1}{n+1} \right\}$$

is empty for n large enough. □

Corollary 1. *In this model, the principle of countable choice* CC *does not hold.*

Corollary 2. *One cannot show countable choice in type theory with one univalent universe and propositional truncation.*

6 Markov's principle

The interpretation of the type N was especially simple on the space $(0,1)$ using the fact that its basic opens are *connected*. We will now consider the "dual" case where the space is *totally disconnected*. We assume from now on that the basic opens are nonzero elements $e, e', \ldots$ of a Boolean algebra with decidable equality. We consider only coverings of e given by a finite partition e_i, $i \in I$, of e, that is a *finite set* of disjoint elements $e_i \leqslant e$ such that $e = \bigvee_{i \in I} e_i$.

Given a type family $\Gamma \vdash A$ and $\rho \in \Gamma(e)$, a descent datum $d \in D(A)\rho$ for this family is now simply given by a partition e_i, $i \in I$, of e and a family $u_i \in A\rho|e_i$.

We can now *strengthen* the notion of stack structure by further imposing that we have $(\mathbf{glue}\, d)|e_i = u_i$ for $d = (u_i) \in D(A)\rho$. This *strict gluing condition* states that the required equalities between $(\mathbf{glue}\, d)|e_i$ and u_i are *strict* equalities. In fact, it is enough to require $\mathbf{glue}(u) = u$ for partitions consisting of exactly one element.

This refinement is needed for the elimination of natural numbers and Booleans in the universe.

Proposition 6. *If $\Gamma.A \vdash B$ satisfies the strict gluing condition, then so does $\Gamma \vdash \Pi AB$.*

Proposition 7. *If $\Gamma \vdash A$ and $\Gamma.A \vdash B$ satisfy the strict gluing condition, then so does $\Gamma \vdash \Sigma AB$.*

If $\mathsf{U}(e)$ is the collection of sheaves on e, we can refine the stack structure on U in order to satisfy the strict gluing condition: If e_i is a partition of e and F_i is a sheaf on e_i we define $F = \mathbf{glue}(e_i, F_i)$ by taking $F(e')$, for $e' \leqslant e$, to be the product of all $F_i(e' \wedge e_i)$ if e' meets strictly more than one e_i, and to be *exactly* $F_i(e')$ if $e' \leqslant e_i$. This defines a sheaf, and the functoriality law $\mathbf{glue}(e_i, F_i)|e' = \mathbf{glue}(e_i \wedge e', F_i|e_i \wedge e')$ is satisfied.

6.1 Natural numbers and Booleans

We define the sheaf $|\mathsf{N}|$ by taking $|\mathsf{N}|(e)$ to be the set of families (e_i, n_i) where e_i is a partition of e and $n_i \neq n_j$ if $i \neq j$. We define similarly $|\mathsf{N}_2|$ where n_i can only take the values 0 or 1, and $|\mathsf{N}_1|(e) = \{0\}$, and $|\mathsf{N}_0|(e)$ is the empty set. We define then $\mathsf{N} = \mathsf{El}\,|\mathsf{N}|$ and similarly for N_2, N_1 and N_0.

We define $\mathsf{suc}(e_i, n_i)$ to be $(e_i, n_i + 1)$ and $0(e)$ is the element $(e, 0)$.

The rec operator is then defined as a section of $\Gamma.\mathsf{N} \vdash C$

$$
\begin{aligned}
(\mathsf{rec}\, c\, d)(\rho, 0) &= c\rho \\
(\mathsf{rec}\, c\, d)(\rho, n+1) &= d(\rho, n, (\mathsf{rec}\, c\, d)(\rho, n)), \text{ where } n \in \mathbb{N} \\
(\mathsf{rec}\, c\, d)(\rho, (e_i, n_i)) &= \mathbf{glue}(e_i, (\mathsf{rec}\, c\, d)(\rho|e_i, n_i))
\end{aligned}
$$

given sections $\Gamma \vdash c : C\,[0]$ and $\Gamma.\mathsf{N}.C \vdash d : C\,\left[\mathsf{suc}\,\mathbf{q}\right]\,\mathbf{p}$.

We remark that the strict gluing condition is needed to make the above definition work, i.e. so that for $m \in \mathsf{N}(e)$ and $e' \leqslant e$ we have $((\mathsf{rec}\, c\, d)\rho\, m)|e' = (\mathsf{rec}\, c\, d)\rho|e'\, m|e'$.

6.2 A stack model where Markov's principle does not hold

We can express Markov's principle in type theory by the type:

$$
\mathsf{MP} := \Pi(h : \mathsf{N} \to \mathsf{N}_2)(\neg\neg(\Sigma(x : \mathsf{N})\mathsf{El}\,\mathsf{isZero}\,(h\,x)) \to \Sigma(x : \mathsf{N})\mathsf{El}\,\mathsf{isZero}\,(h\,x))
$$

where $\mathsf{isZero} : \mathsf{N}_2 \to \mathsf{U}$ is defined by $\mathsf{isZero} := \lambda y.\mathsf{rec}_2\, \left|\mathsf{N}_1\right|\, \left|\mathsf{N}_0\right|\, y$ and the type $\neg A$ by $A \to \mathsf{N}_0$.

We could also consider the version where we use weak existential $\exists(x : A)B = \|\Sigma(x : A)B\|$ instead of sigma type, but the two versions are logically equivalent [**?**, Exercise 3.19].

Take a countably infinite set of variables $p_0, p_1, \ldots$. Consider the free Boolean algebra generated by the atomic formulae p_n. We write $p_n = 0$ for $\neg p_n$ and $p_n = 1$ for p_n. An object e in this algebra represents then a compact open in Cantor space $\{0,1\}^{\mathbb{N}}$, where a conjunctive formula $\bigwedge p_i = b_i$ represents the set of sequences in $\{0,1\}^{\mathbb{N}}$ having value b_i at index i. A formula e in the algebra is then a finite disjunction of these.

We have an interpretation of type theory in stacks over this algebra, and we are going to see that Markov's principle is not valid in this interpretation. We define f in $\mathsf{N} \to \mathsf{N}_2$ by taking $\mathsf{f}\ n$, $n \in \mathsf{N}(e)$, at e_i to be $((e_0, 0), (e_1, 1))$ where e_b is $e_i \wedge (p_{n_i} = b)$ if e_i meets both $(p_{n_i} = 0)$ and $(p_{n_i} = 1)$, and to be (e_i, b) if $e_i \leqslant (p_{n_i} = b)$.

Proposition 8. *In this model*

1. $\neg\neg(\Sigma(x : \mathsf{N})\, \mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x))$ *is inhabited.*
2. $\Sigma(x : \mathsf{N})\, \mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x)$ *is not inhabited.*

Proof. To show that $\neg\neg(\Sigma(x : \mathsf{N})\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x))$ is inhabited it is sufficient to show that for all e the set $(\neg(\Sigma(x : \mathsf{N})\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x)))(e)$ is empty. For that it will be sufficient to show that for some $e' \leqslant e$ we have that $(\Sigma(x : \mathsf{N})\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x))(e')$ is not empty. But given any e we can simply choose $e' = (p_n = 0) \wedge e$ for some n big enough. Thus $\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ n)$ at e' is $\{0\}$ and $(\Sigma(x : \mathsf{N})\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x))(e')$ is not empty.

We now show that $\Sigma(x : \mathsf{N})\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ x)$ is not inhabited. For any $n = (e_i, n_i)$ in $\mathsf{N}(1)$ where (e_i) is a partition of 1, we can find exactly one e_i which contains (as a compact open subset of Cantor space) the constant function 1. This element e_i meets $p_{n_i} = 1$ so that $\mathsf{El}\, \mathsf{isZero}\, (\mathsf{f}\ n)$ is the empty set at $(p_{n_i} = 1) \wedge e_i$ and hence also at 1. □

Corollary 3. *In this model Markov's principle does not hold.*

Corollary 4. *One cannot show Markov's principle in type theory with one univalent universe.*

The situation however is different from the one of countable choice. The following provides an alternative argument that Markov's principle cannot be proved in type theory with one univalent universe[3].

Proposition 9. *Markov's principle does not hold in the groupoid model in a set theory where Markov's principle does not hold (for instance in suitable sheaf models of CZF [?]).*

In [?] it was shown that Markov's principle is independent from type theory with one (non-univalent) universe. The paper describes an extension of type theory where the principle does not hold and proves the consistency of that extension with a normalization argument. We note however that the model given here does not give an interpretation of the extended type theory in [?]. In particular the universe (inductively defined) in that extension satisfies the sheaf property.

7 Conclusion

One special case of sheaf models are Boolean-valued models, for instance as in the work [?], and it would be interesting to formulate a stack version of these models as well.

We expect that essentially the same kind of models can be defined over a *site* and not only over a topological space. In particular, it should be possible to extend the sheaf model in [?] to a stack model of type theory with an algebraic closure of a given field, where existence of roots is

[3]This argument gives also a proof that Markov's principle is independent of a hierarchy of univalent universes by considering the cubical set model [?] in a set theory where Markov's principle does not hold.

formulated using propositional truncation (as explained in the cited work, this existence cannot be stated using strong existence expressed by sigma types). Another example could be a stack version of Schanuel topos used in the theory of nominal sets [?].

As stated in the introduction, the argument should generalize to an ∞-stack version of the cubical set model [?]. The coherence condition on descent data will be infinitary in general, but it will become finitary when we restrict the homotopy level (and empty in particular in the case of propositions).

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