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Investigating the Professional Knowledge of New Zealand Primary School Teachers when Teaching Mathematics for Numeracy

A thesis
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ABSTRACT

This research investigated the relationship between teachers' espoused professional knowledge, professional knowledge in practice, and student learning, when teaching 'mathematics for numeracy' in the New Zealand primary school classroom. The focus was on teaching within the multiplicative and proportional domains, as research at the time the study commenced indicated that these areas of mathematics were problematic for many teachers. The purpose of the research was to identify strengths, weaknesses, and inconsistencies in teachers' practice; links between espoused views and actions; and to consider the usefulness of a framework for investigating teacher knowledge in practice. This study is intended to inform teacher reflection and professional development, and contribute to improvements in teaching practice and student achievement.

A multiple-case study design, underpinned by an interpretivist paradigm, was used, which included four case-study teachers from two primary schools: School A, was a central city full primary school (Years 1 to 8) and School B, a rural town primary school (Years 1 to 6). The study aligned with a social constructivist perspective on teaching and learning. The data were obtained through four main sources: (1) pre-unit and post-unit student assessment tasks; (2) recorded observations; (3) semi-structured interviews; and (4) teacher questionnaires.

Comparison between students' initial and final assessment data showed little progress in understanding of multiplication and division, with a more noticeable improvement in fractional understanding. Classroom observations were analysed under three broad categories: content knowledge, pedagogical knowledge, and pedagogical content knowledge, and highlighted important issues relating to the professional knowledge of teachers and the contribution this made to student learning. Results indicated that the mathematical content knowledge of the teachers was stronger than their content knowledge in a pedagogical context. While teaching for conceptual understanding frequently challenged the teachers, they recognised the importance of conceptual understanding prior to procedural learning for their students. They struggled with on-the-spot identification of the next steps of learning for individual students and there was little evidence of focus on questioning that extended students' thinking that might have assisted in overcoming misconceptions and confusions with concepts. There were times when the teachers' espoused theories differed from their theory-in-practice, while at times they were similar to each other.

The research concluded that in teaching practice the many facets of PCK, within the broader construct of professional knowledge, were more than topic-specific. Instead, they were person-specific and lesson-specific, with different categories coming to the fore in different proportions, for different reasons, including: lesson structure, context, problem types, the opportunities afforded students for conversation, and use of manipulatives. While not all categories of professional knowledge were evident in every lesson, they combined over a period to

underscore the complexities of teaching and ultimately have an effect on student learning.

An outcome of the study was a Wheel of Knowledge designed for teachers, identifying key areas of knowledge to be addressed in mathematics teaching. Alongside this, a more detailed Professional Knowledge Framework was created for researchers, based on categories identified from this research as important in identifying teacher professional knowledge in classroom practice. Both models have the potential to identify areas of teacher professional knowledge required to improve students' mathematical understanding.

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PREFACE

A well-known Māori proverb finishes with the words, 'He aha te mea nui o te ao? Māku e kī atu, he tangata, he tangata, he tangata.' Translated this means 'What is the most important thing in the world? I say to you, it is people, it is people, it is people.' As educators, we are fortunate to work with this most valuable resource (people) and in particular be in the luxurious position of helping to nurture their developing minds. To work with children is a privilege as they learn to appreciate the importance of lifelong learning and the value of Earth's resources and beauty.

I dedicate this research to all classroom teachers who work tirelessly to support their students' learning.

TABLE OF CONTENTS

ABSTRACT.....	i
ACKNOWLEDGEMENTS	iii
PREFACE.....	iii
TABLE OF CONTENTS.....	v
LIST OF TABLES	ix
LIST OF FIGURES	x
CHAPTER ONE INTRODUCTION	1
1.1 Introduction.....	1
1.2 Education Reforms and Changing Practice as a Teacher	2
1.3 Teachers and Teaching	3
1.4 Teachers’ Professional Knowledge.....	7
1.5 The New Zealand Context	9
1.6 Rationale for this Study	14
1.7 Researcher’s Positioning.....	16
1.8 Aim of the Study.....	18
1.9 Structure of the Thesis	18
CHAPTER TWO LITERATURE REVIEW	21
2.1 Introduction.....	21
2.2 Theories Associated with Mathematics Learning.....	22
2.3 Reforms in Mathematics Education.....	25
2.3.1 Factors Contributing to Current Mathematics Education Reforms	28
2.4 Teachers’ Professional Knowledge.....	34
2.4.1 Pedagogical Content Knowledge (PCK)	35
2.4.2 Content Knowledge and Mathematical Knowledge for Teaching.....	37
2.5 Mathematics for Numeracy.....	41
2.5.1 Numeracy Development Project and Student Achievement.....	44
2.5.2 Developing Mathematical Discussions.....	46
2.6 Teaching the Multiplicative Domain	48
2.7 Teaching the Proportional Domain.....	55
2.7.1. Constructs of Rational Number	58
2.7.2 New Zealand Teachers’ Knowledge in the Multiplicative and Proportional Domains.....	63

2.8 Frameworks of Teacher Knowledge	65
2.8.1 Frameworks of Teacher Knowledge and Classroom Practice	67
2.8.2 The Framework used in this Research	69
2.9 Summary	71
CHAPTER THREE METHODOLOGY and METHODS.....	75
Section A: Theory associated with Research Methodology and Methods.....	75
3.1 Introduction.....	75
3.2 Research Questions	75
3.3 Research Methodology.....	75
3.3.1 Epistemology and Methodological Considerations	76
3.4 Research Methods in Education.....	77
3.5 Case-study Research	78
3.5.1 Multiple-case Study Research.....	80
3.5.2 Case Studies and Generalizability.....	80
3.6 Roles of the Researcher.....	82
3.7 Ethical Considerations	83
3.8 Data Gathering Methods	84
3.8.1 Questionnaires.....	85
3.8.2 Interviews.....	86
3.8.3 Observations	87
3.8.4 Tests and Assessment Tasks	88
3.9 Transcribing and Data Analysis	89
Section B: Research Methods and Design	92
3.10 Participants.....	92
3.10.1 Case-study Schools	92
3.10.2 Case-study Teachers and their Classes	94
3.11 Ethical Considerations	95
3.12 Data Gathering Methods	96
3.12.1 Observations and Recordings.....	97
3.12.2 Questionnaires.....	97
3.12.3 Interviews and Learning Conversations.....	98
3.12.4 Documents, Tests, and Assessment Tasks	99
3.13 Transcribing and Data Analysis	101
3.14 Summary	106

CHAPTER FOUR RESULTS and ANALYSIS: TEACHER KNOWLEDGE.....	107
4.1 Introduction.....	107
4.2 Teachers' Subject Matter Knowledge and Pedagogy	107
4.2.1. Multiple-choice Questions.....	107
4.2.2 Teachers' Understanding of Multiplication and Division	107
4.2.3 Teachers' Understanding of Fractions and Decimals	111
CHAPTER FIVE RESULTS and ANALYSIS: THE MULTIPLICATIVE DOMAIN .	119
5.1 Introduction.....	119
5.2 Observed Teacher Professional Knowledge in Multiplication	120
5.2.1 Clearly PCK (A)	123
5.2.2 Content Knowledge in a Pedagogical Context (B).....	139
5.2.3 Pedagogical Knowledge in a Content Context (C).....	146
5.3 Multiplicative Domain Summary	151
CHAPTER SIX RESULTS and ANALYSIS: THE PROPORTIONAL DOMAIN.....	153
6.1 Introduction.....	153
6.2 Observed Teacher Professional Knowledge in the Proportional Domain	153
6.2.1 Clearly PCK (A)	155
6.2.2. Content Knowledge in a Pedagogical Context (B).....	167
6.2.3 Pedagogical Knowledge in a Content Context (C).....	174
6.3 Proportional Domain Summary	178
CHAPTER SEVEN RESULTS and ANALYSIS: STUDENT LEARNING	181
7.1 The Multiplicative Domain.....	181
7.1.1 The Multiplicative Domain Initial Assessment	181
7.1.2 The Multiplicative Domain Final Assessment.....	187
7.2 The Proportional Domain	191
7.2.1 The Proportional Domain Initial Assessment.....	192
7.2.2 The Proportional Domain Final Assessment	196
7.3 Assessment Tasks Summary.....	202
7.3.1 The Multiplicative Domain Assessment Tasks Summary	202
7.3.2 The Proportional Domain Assessment Tasks Summary.....	203
CHAPTER EIGHT DISCUSSION: THE MULTIPLICATIVE DOMAIN	205
8.1 Teachers' Subject Matter Knowledge and Pedagogy	205
8.1.1 Introduction.....	205
8.1.2 Multiple-choice Questions.....	205
8.1.3 Teachers' Understanding of Multiplication and Division	206
8.2 Teacher Practice in the Multiplicative Domain	210

8.2.1 Clearly PCK (A)	211
8.2.2 Content Knowledge in a Pedagogical Context (B)	219
8.2.3 Pedagogical Knowledge in a Content Context (C)	225
8.3 Student Learning in the Multiplicative Domain.....	230
8.3.1 Assessment Tasks and Student Learning in the Multiplicative Domain.....	230
8.4 Aligning Teaching Materials and Language	236
8.5 The Multiplicative Domain Summary	240
8.5.1 Teachers' Subject Matter Knowledge and Pedagogy	240
8.5.2 Teacher Practice in the Multiplicative Domain	241
8.5.3 Student Learning in the Multiplicative Domain.....	242
8.5.4 Aligning Teaching Materials and Language	243
CHAPTER NINE DISCUSSION: THE PROPORTIONAL DOMAIN	245
9.1 Teachers' Subject Matter Knowledge and Pedagogy	245
9.1.1 Teachers' Understanding of the Proportional Domain	245
9.2 Teacher Practice in the Proportional Domain	248
9.2.1 Clearly PCK (A)	248
9.2.2 Content Knowledge in a Pedagogical Context (B)	259
9.2.3 Pedagogical Knowledge in a Content Context (C)	263
9.3 Student Learning in the Proportional Domain	268
9.3.1 Assessment Tasks and Student Learning in the Proportional Domain	268
9.4 Aligning Teaching Materials and Language	275
9.5 Proportional Domain Summary	277
9.5.1 Teachers' Subject Matter Knowledge and Pedagogy	277
9.5.2 Teacher Practice in the Proportional Domain	277
9.5.3 Student Learning in the Proportional Domain	279
9.5.4 Aligning Teaching Material and Language	280
CHAPTER TEN DISCUSSION: KEY RELATIONSHIPS	283
10.1 Introduction.....	283
10.2 Professional Knowledge of Primary School Teachers of Mathematics	283
10.3 Using a Framework of Professional Knowledge.....	294
CHAPTER ELEVEN CONCLUSIONS, LIMITATIONS, and IMPLICATIONS.....	301
11.1 Introduction.....	301
11.2 Conclusions.....	302
11.2.1 Professional Knowledge in Practice	302
11.2.2 Relationships between Espoused Professional Knowledge, Professional Knowledge in Practice, and Student Learning.....	304

11.2.3 The use of Frameworks in Research.....	307
11.3 Limitations of the Study	314
11.4 Implications for Future Teaching and Further Research	317
11.5 Final Words.....	319
REFERENCES	323
APPENDICES	355
Appendix A: Principal Information and Consent	355
Appendix B: Teacher Information Sheet (All Teachers) and Consent	357
Appendix C: Questionnaire For Teachers.....	360
Appendix D: Parent/Caregiver Information and Consent.....	365
Appendix E: Student Information and Consent.....	367
Appendix F: Statement of Confidentiality for Transcribers	369

LIST OF TABLES

Table 3.1	Case-study teachers and their classes.....	95
Table 3.2	The year’s professional development programme	96
Table 3.3	Approximate comparisons between the New Zealand Curriculum Levels and Number Framework Stages.....	100
Table 4.1	Case-study teachers perceived classroom practice.....	115
Table 5.1	Observed multiplication lessons.....	119
Table 5.2	Use of one manipulative coded against different categories.....	123
Table 5.3	Frequency of each PCK category used in the introductory multiplication lesson and final lesson (in brackets) and totals (excluding Anna).....	124
Table 6.1	Observed fractions lessons.....	153
Table 6.2	Use of the WALT coded against different categories.....	155
Table 6.3	Frequency of each PCK category used in the observed fraction lesson.....	156
Table 7.1	Number (and percentage rounded to nearest whole number) of students with correct responses on the initial multiplicative assessment tasks	182
Table 7.2	Number (and percentage rounded to nearest whole number) of children with correct responses on the final multiplicative assessment tasks	187
Table 7.3	Number (and Percentage rounded to nearest whole number) of students with correct responses on the initial assessment of fractions.....	192

Table 7.4	Number (and Percentage rounded to nearest whole number) of students with correct responses on the final assessment of fractions196
Table 7.5	Number (and Percentage rounded to nearest whole number) of students with correct responses on post-unit fractions assessment (additional questions from pre-unit assessment)201
Table 11.1	Comparing coding of the inductive, grounded approach to the deductive, framework approach.....309
Table 11.2	Comparing two frameworks of mathematical knowledge for teaching.....313

LIST OF FIGURES

Figure 2.1	Framework for Analysing Pedagogical Content Knowledge (Chick, Baker, Pham, & Cheng, 2006).....70
Figure 3.1	Adapted framework for Analysing Pedagogical Content Knowledge in Action (adapted from Chick, Baker, Pham, & Cheng, 2006)103
Figure 4.1	Scenario 1: Multi-digit multiplication108
Figure 4.2	Scenario 2: Using the divisible by three rule109
Figure 4.3	Andy's recording of $516 \div 3$110
Figure 4.4	Scenario 3: Quotitive division110
Figure 4.5	Andy's solution to 56 peanuts into bags of 14.....111
Figure 4.6	Scenario 4: Addition of decimals.....111
Figure 4.7	Diagram showing Anna's recording of $1.45 + 0.9$112
Figure 4.8	Scenario 5: Addition of fractions112
Figure 4.9	Anna's diagram to explain $\frac{6}{8} + \frac{7}{8} = 1\frac{5}{8}$113
Figure 4.10	Scenario 6: Multiplication of Fractions113
Figure 5.1	Rhinoceros and Bunny animal strip cards showing 3×4 and 4×3 respectively120
Figure 5.2	Sample of multiple coding in Andy's multiplication lesson.....122
Figure 5.3	Small squares of paper were used to find area.....125
Figure 5.4	Some sides unknown and Andy's clue to finding the area of the triangle126
Figure 5.5	The difference between 6×4 and 4×6 is shown with array models129
Figure 5.6	Diagram showing 4×5 as 4 groups of 5131
Figure 5.7	Area problems on the whiteboard131
Figure 5.8	Finding the area of an irregular shape.....133

Figure 5.9	A student’s model of 3×5 as an equation	135
Figure 5.10	Models of 10×2 and 2×10	137
Figure 5.11	Recognising the multiple of three pattern	140
Figure 6.1	Sample of multiple coding in Beth’s fraction lesson.....	154
Figure 6.2	Proportional representation.....	158
Figure 6.3	Student examples of 5 apples shared among 4 people	158
Figure 6.4	Students discuss fractions represented on the deci-pipes	159
Figure 6.5	Folding and cutting paper (wafer biscuit)	160
Figure 6.6	An example of 21 shared out into 3 (uneven) pieces	161
Figure 6.7	Body Fractions showing 2 wholes and one quarter.....	162
Figure 6.8	Ten tenths are lined up alongside one whole pipe	162
Figure 6.9	Representations of $\frac{3}{8}$	163
Figure 6.10	Examples of 3 wafers shared among 4 people	165
Figure 7.1	Recording repeated addition using running on.....	183
Figure 7.2	Three students’ representations of 3×5	183
Figure 7.3	One student’s representation of 12 biscuits put into packets of 3	184
Figure 7.4	Different representations of 2×5 and 5×2	185
Figure 7.5	Using $6 \times 5 = 30$ to solve $6 \times 4 = ?$	185
Figure 7.6	Task 8: Understanding an array model of multiplication.....	185
Figure 7.7	3×6 or 6×3 ?	188
Figure 7.8	Twenty biscuits divided into packets of four	189
Figure 7.9	Quotitive division showing 12 biscuits into packets of 3.....	189
Figure 7.10	3×5 and 5×3	190
Figure 7.11	Using $6 \times 5 = 30$ to solve $6 \times 6 = ?$	190
Figure 7.12	Using 6×5 to solve 6×6	190
Figure 7.13	Examples of the double and halve strategy	191
Figure 7.14	Representations of 26 apples into 4 bags	191
Figure 7.15	Addition of unit fractions	193
Figure 7.16	(a) Faces drawn on each quarter (b) The number on each of the quarters is added.....	195
Figure 7.17	Ordering of fractions	196
Figure 7.18	Addition of numerators and denominators	197
Figure 7.19	The “add across” error.....	198
Figure 7.20	Each person has one piece left therefore they eat the same amount	198

Figure 7.21	Uneven fraction representations	198
Figure 7.22	The circle divided into quarters with a smiley face on each.....	199
Figure 7.23	Confusion between divisor and dividend when finding $\frac{1}{3}$ of 21.	199
Figure 7.24	Fractions ordered according to size of the denominator	200
Figure 7.25	Numerator and denominator ordered using whole number thinking	200
Figure 7.26	The whole number (16) added to the numerator (5)	201
Figure 7.27	Attempt to solve $4.3 - 3.89$	202
Figure 7.28	Attempt to find the number tenths in all of the number 5.23.....	202
Figure 10.1	Wheel of Professional Knowledge.....	297
Figure 10.2	Framework of Professional Knowledge.....	299

CHAPTER ONE

INTRODUCTION

1.1 Introduction

People are besieged by numbers in every aspect of their lives. It may even be said, “numbers rule our life”. Numbers are at the central core of today’s lifestyle and modern technology, beginning when the alarm goes off each morning, computers are turned on, spreadsheets created, texts messages sent, appointments fulfilled, credit cards used, bills paid through the internet, and so on. As current primary school students prepare for adult life in a world that will be technologically based beyond current imagination, there is a need for them to know and understand mathematics in a manner that is appropriate to societal changes and expectations.

Where once it was sufficient to master basic arithmetic, the requirements of mathematics and numeracy in today’s world are different. Being numerate includes thinking mathematically about situations. It is not the same as knowing how to calculate, it is about being able to think about and have an understanding of numbers using conventions (e.g., measurement systems, terminology, tools etc.), relevant to one’s own culture (e.g., in English the base-ten counting system), and use their mathematical thinking meaningfully and appropriately in different situations (Nunes & Bryant, 1996). Numeracy is thus an essential capability for individuals who wish to participate fully in a democratic society and to utilise knowledge and skills, and critical reasoning capabilities in everyday life (Perso, 2006, 2007; Skalicky, 2007). “It is not enough to learn procedures: it is necessary to make these procedures into thinking tools” (Nunes & Bryant, 1996, p. 19). Therefore, given the recent reforms in mathematics education, and the current cultural and social aspects to primary schooling, primary school teachers could now be known as, *teachers of mathematics for numeracy* (Perso, 2006, p. 40), as they prepare their students for life skills beyond the classroom.

The working definition of numeracy used in this research was the same as that used in the Effective Teachers of Numeracy Project (Askew, Rhodes, Brown, Wiliam, & Johnson, 1997): “Numeracy is the ability to process, communicate, and interpret, numerical information in a variety of contexts” (p. 6). Numeracy is

often referred to in terms of what it means to be numerate. Throughout this research, to be numerate is defined as: “To have the ability and inclination to use mathematics effectively in our lives – at home, at work, and in the community” (Ministry of Education, 2001, p. 1). Askew et al., built on their definition of numeracy, to define effective teachers of numeracy as teachers who help their pupils:

“acquire knowledge of and facility with numbers, number relations and number operations based on an integrated network of understanding, techniques, strategies and application skills; learn how to apply that knowledge of and facility with numbers, number relations and number operations in a variety of contexts.” (p. 10).

1.2 Education Reforms and Changing Practice as a Teacher

Changing the ways mathematics is taught and learned requires considerable effort, and can be both demanding and challenging (Anthony & Hunter, 2005; Lamon, 2007; Walshaw, 2014). Generally, education reform is about improving student performance and depends on what teachers and students do in their classrooms. In recent years, the role of the mathematics teacher has changed in profound ways, having become more complex and sophisticated in response to major societal, economic, cultural, technological, and political changes (Anthony, Beswick, & Ell, 2012; Bennison, 2015; Hattie, 2003, 2009; Jorgenson, 2014; Lowrie, 2015). This change in role has necessitated an on-going need for teachers to continue to reflect on their teaching practice and make changes as required. Recent reforms in New Zealand have seen a change in curriculum implementation, where the emphasis was previously on coverage of topics, to a system of expected learning outcomes for all students (Ministry of Education, 2007). The current New Zealand Curriculum (NZC) document encourages students to be “confident, creative, connected, actively involved life-long learners” (Ministry of Education, 2007, p. 4), who are able to solve problems confidently in a range of situations. This has resulted in a problem-solving approach to mathematics where there is prominence placed on teaching students *how* to solve problems, alongside teaching them *about* problem-solving, *for* problem-solving (Hunter, 2012; Lambdin & Walcott, 2007; Zevenbergen, Dole, & Wright, 2004).

The teaching of mathematics in schools has at its core three components, which are necessary for quality teaching programmes: knowledge of mathematics and

associated learning theory; mathematical and pedagogical skills; and practical wisdom and activity (Grootenboer & Edwards-Groves, 2014). In order for teachers to implement these three components, they need to understand the mathematics they teach in a way that results in them developing an identity as a teacher of numeracy (Bennison & Goos, 2013).

One contributing factor previously cited by researchers (Ball, 1992; Burns, 1998; Skemp, 1976) as part of the reason for poor mathematics proficiency, was the focus on developing procedural knowledge at the expense of conceptual understanding. A procedural approach to learning was referred to by Skemp as “instrumental understanding”, while conceptual understanding was referred to as “relational understanding”. Effective teachers of mathematics facilitate learning opportunities that emphasize conceptual understanding, strategic competence, and adaptive reasoning, alongside the traditional mathematics education focus of procedural fluency (Kilpatrick, Swafford, & Findell, 2001). Developing conceptual understanding during instructional time is essential if mathematics is to be learnt with understanding and is a key component of numeracy. Something is only understood if one can see how it is related or connected to other things that are known (Hiebert & Carpenter, 1992; Skemp, 1989). This approach allows students the opportunity to implement ways of solving problems that make sense to them by developing their number sense, reasoning, and operation sense.

Teachers are required to anticipate and understand a range of computational strategies that students might use (Hartnett, 2015), by planning and preparing for students’ responses (Smith & Stein, 2011). As a result, for many teachers there has been a shift from teaching standard algorithmic procedures for calculating, to allowing students to observe patterns and relationships, and make connections within and between concepts, in order to develop a feel for numbers (Hartnett, 2015; Mulligan, 2013; Mulligan & Mitchelmore, 2009).

1.3 Teachers and Teaching

It has been argued that the most productive option for improving classroom instruction is to focus directly on improving the *teaching* rather than the *teachers* (Hiebert & Morris, 2012; Stigler & Hiebert, 2004). Hiebert and Morris maintain that improving teaching and gaining better results, begins by improving the

instructional methods that are implemented in the classroom. However, setting up a choice between teaching and teachers and differentiating between the two can be problematic, as teaching is generally simultaneously linked with the improvement of teachers (Chick, Baker, Pham, & Cheng, 2006; Davis & Renert, 2014; Lampert, 2012; Stigler & Hiebert, 1999), and may sometimes even be used interchangeably (Hiebert & Morris, 2012).

Attempts at defining quality teaching have systematically explored the nature of teachers' work, and the relationship between teacher knowledge and student achievement (Gess-Newsome, 1999b). Recently Blazer (2015) researched the unique contribution of specific teaching dimensions to student outcomes, by focusing on teacher characteristics, practices, and skills. He argued that research, which combined background characteristics of teachers (education and teaching experience), teachers' knowledge (mathematical content knowledge and knowledge of students' performance), and non-instructional classroom behaviours (preparation for class) was new to the mathematics research and went beyond what existed in the field of previous mathematics research. Blazer reasoned that the combination of a teacher's characteristics, practices, and skills, would ultimately be reflected in their students' achievement.

If student learning is to improve then ways to improve teaching must be explored and it is classroom-based research that examines the reality of what occurs in teaching practice. An example of such research is that of Stigler and Hiebert (1999), who conducted video-recorded, classroom-based research in several countries. The teachers were asked if they had read mathematics education reform documents, (e.g., those published in the National Council of Teachers of Mathematics) and whether they used the reform ideas in their classroom practice. Stigler and Hiebert concluded (after observation of the videos) that even when teachers acknowledged reading the reforms and believed that they implemented reform ideas in practice, there was little evidence of the teaching actually reflecting the goals of the reforms. After analysing the research from several countries, Stigler and Hiebert also determined that teaching is a culturally-based activity, with much homogeneity of methods within a country, and that one of the key attributes to teaching is the relationship between teachers and students as they work through problems. It is this relationship between teachers and students that

informed Blazer's research, and became the focus for Hattie's (2003) New Zealand based research.

As with any subject area, the individual teacher makes a difference to student achievement in mathematics. Research undertaken by Hattie (2003) considered the attributes of excellence in teaching and concluded that approximately 30% of the variance in student achievement was the direct influence of the individual teacher. Hattie advocated that it was what teachers know, do, and care about, that makes the difference to student learning. Researchers have thus advocated that more attention be directed at higher quality teaching and higher expectations, so that students can meet appropriate challenges (Hattie, 2003; Rubie-Davies, 2007, 2010). Teachers require knowledge about how mathematics is learned, how topics are sequenced for learning, where conceptual blockages occur, and where misunderstandings are likely, in order to carry out quality teaching (Barton, 2009). Capturing the essence of teaching by studying what it is a teacher knows, what they do, why they do it, and what effect it might have on student learning, is therefore an on-going topic of research and discussion.

Currently there is limited research, and insufficient associated theories, to inform teachers about why it is that some highly mathematically qualified and motivated teachers are unsuccessful, and why it is that the students of some mathematically unqualified teachers receive top results (Barton, 2009). Effective teachers work hard to build trusting classroom communities and the relationships formed within those communities become the basis for developing students' mathematical competencies and identities (Walshaw, 2014). Many students struggle with mathematics as they encounter obstacles when engaging with it as a subject. It is therefore imperative that educators understand what effective mathematics teaching looks like and what can be done to break the pattern of struggle (Anthony & Walshaw, 2009a; Boaler, 2013). There is a need for teachers to have and to maintain a sound knowledge of mathematics to underpin the structural, material, and intellectual choices that they make in their classrooms (Anthony & Walshaw, 2008). Teachers must be mathematicians and must know how to make the classroom a place in which mathematics is accomplished (Barton, 2009).

Knowledge for the twenty-first century suggests a need to shift from many of the former traditional roles of teachers and learners (Bolstad & Gilbert, 2012; Gilbert,

2006; Hattie, 1999a) towards an inquiry-based programme (Ministry of Education, 2007). Bolstad and Gilbert suggested that if the purpose of schools is no longer to transmit knowledge, then the teacher's roles must be reconceived. Alongside this is the notion that if students are no longer merely to absorb and store up knowledge, then their roles also need to be reconceived. It is argued therefore, that rather than hypothetical learning, teaching should support students to engage in knowledge-generating activities, in authentic contexts (Bolstad & Gilbert, 2012; Fraivillig, Murphy, & Fuson, 1999; Ministry of Education, 2007, 2008a). Teachers need to get to know their students, their interests, their extra-curricular activities, their families, their cultural background, as these are important attributes to contextualising mathematics learning (Higgins & Averill, 2010).

Professional development has a vital role to play in changing the knowledge and practice of teachers (Borko, 2004; Cohen & Ball, 1990). The notion of professional development, which typically tells teachers what to do, has been challenged by Loughran (2010), who preferred to highlight the importance of "professional learning" which he saw as supporting teachers in their growth. As teaching comprises many competing demands, understanding teaching has often been seen as problematic (Hattie, 1999a; Loughran, 2010). Loughran argued that the term "problematic" should not be viewed in a negative sense, but rather as dilemma based. He claimed that if teaching is understood as problematic, then it stands to reason that one aspect of learning about teaching is embedded in the journey of development and growth, and is guided by what individuals see as important to their (teaching) practice through their experiences. Teaching is a never-ending process of learning and part of the professional learning must involve new ways of seeing situations, testing out alternative approaches, and learning to see practice from both a teacher's and a learner's perspective (Loughran, 2010).

Inquiry and sharing the outcomes of that inquiry are central to the teaching profession (Romberg & Carpenter, 2005). While it is recognised that classroom instruction has the most direct effect on student learning (Hattie, 1999; Hiebert & Wearne, 1993; Romberg & Carpenter, 2005), in order to develop high-quality instructional programmes in mathematics, teachers need to "share learning goals, take collective responsibility to reach them, jointly address the challenges that

arise, and share in developing methods to respond to those challenges” (Romberg & Carpenter, 2005, p. 325). Teaching and learning must be supported by opportunities for teachers to engage in dialogue within professional learning communities.

1.4 Teachers’ Professional Knowledge

When addressing the complexity of the knowledge required for teaching, Shulman (1987) proposed that if teachers’ professional knowledge was to be organized into a handbook, the categories might include: “content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners; knowledge of educational contexts; and knowledge of educational ends, purposes, and values” (p. 8). Shulman recognised that there was a balanced interaction achieved between the three categories of general pedagogical knowledge, knowledge of curriculum, and knowledge of subject matter, which would provide teachers with specialised teaching knowledge, to which he gave the phrase *Pedagogical Content Knowledge* (PCK). In Shulman’s view, PCK is a form of practical knowledge used by teachers to guide their actions in contextualised classroom settings.

Research has shown there is interplay between general pedagogical knowledge (derived from research and scholarly literature) and personal pedagogical knowledge (fuelled by personal beliefs and practical experiences) (Ball, Thames, & Phelps, 2008; Morine-Dersheimer & Kent, 1999). Subject matter and beliefs, pedagogical knowledge and beliefs, and knowledge and beliefs about context are all influences on PCK (Magnusson, Krajcik, & Borko, 1999). Efforts to help teachers make substantial changes in their teaching must allow them to acquire both new knowledge and a change in beliefs. The many competing and at times confusing ideas associated with PCK, teacher beliefs, and student outcomes, have led in recent times to greater use of the term *professional knowledge* (Gess-Newsome, 2015). Many educational researchers have thus explored PCK, its place within professional knowledge, and its influence on quality teaching practice (Anthony & Walshaw, 2009a, 2009b; Ball 2000a; Ball et al., 2008; Campbell et al., 2014; Chick & Beswick, 2013; Chick et al., 2006; Gess-Newsome, 1999a; Hill, Ball, & Schilling, 2008; Lewis, 2014; Loughran, Berry, & Mulhall, 2012; Ma, 2010; Schoenfeld, 2013; Walshaw, 2014). The combination of

PCK, teacher beliefs, and student outcomes, has resulted in a shift in thinking around PCK by researchers. Subsequently, the development of a model of Teacher Professional Knowledge and Skill (TPK&S) has recently been produced (Gess-Newsome, 2015). The TPK&S model includes PCK and the influence it has on classroom practice and student outcomes. The model acknowledges weaknesses and limitations of original PCK thinking identified by Shulman (2015), including the non-cognitive attributes (emotion, affect, feelings, and motivation) and pedagogical *action* associated with teaching.

Subject matter knowledge and PCK are inextricably intertwined and are key elements of teachers' professional knowledge (Ball et al., 2008; Shulman, 1986). However, understanding the subject matter of mathematics sufficiently well to teach it is not just about being able to do the mathematics. The difference between demonstrating content knowledge and teaching content knowledge, has been questioned along with what content knowledge and pedagogical content knowledge is necessary and sufficient, for the teaching of mathematics in the primary school classroom (Ball, Lubienski, & Mewborn, 2001; Hill, Sleep, Lewis, & Ball, 2007; Moch, 2004). Knowledge relies on the teacher's understanding of the content that they share with their students and the ability to transform that content into a context, which the students will understand (Chick & Baker, 2005; Moch, 2004). In mathematics, the term Mathematical Knowledge for Teaching (MKT) is often used alongside subject matter knowledge (Ball et al., 2008; Barton, 2009; Davis & Renert, 2014). As well as general pedagogy and pedagogical content knowledge, MKT includes pure mathematics content knowledge with a focus on both what a teacher must know, as well as how one teaches it, encompassing the mathematical environment that a teacher must create in order to combine the two dimensions. It is more than book knowledge and requires teachers to be flexible with in-the-moment responsiveness (Davis & Renert, 2014).

Frameworks of teacher knowledge

Ever since Shulman introduced PCK into research terminology, many frameworks have been developed depicting the components deemed necessary by their writers. When discussing effective science teaching, Magnusson, Krajcik, and Borko (1999) suggested that PCK could be described as the *transformation* of several types of knowledge (including subject matter knowledge). Magnusson et al., saw

PCK as including five components: (1) orientation to teaching the subject; (2) knowledge of the curricula; (3) knowledge of assessment; (4) knowledge of instructional strategies; and (5) knowledge of students' understanding of the subject (p. 99).

The original ideas of Shulman, and the five components of the science framework developed by Magnusson et al., are similar to those instigated by Ball et al. (2008) in their mathematics framework. Ball et al. developed a framework of Mathematical Knowledge for Teaching (MKT), which is divided into two main sections: subject matter knowledge and pedagogical content knowledge. Subject matter knowledge considers the overlap between common content knowledge, specialised content knowledge, and horizon content knowledge. The framework breaks PCK down by examining the relationship between knowledge of content and students, knowledge of content and curriculum, and knowledge of content and teaching.

Schoenfeld (2002) created a framework of knowledge based on theories associated with the mathematical knowledge required for teaching. He then applied his theoretically-based framework to videos he had recorded of actual teaching practice. The framework incorporated Knowledge, Orientation, and Goals (KOG). He used the framework to explain teacher actions and decisions made in the classroom according to the KOG of the teacher.

As a result of research associated with teaching mathematics, Chick et al. (2006) developed a mathematical framework based on an identified range of component knowledge areas that captured Shulman's three main aspects of PCK: subject matter knowledge, pedagogical knowledge, and knowledge of curriculum. The framework is divided into three sections: Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context. Chick et al. based their framework categories on literature with the intention that it could be applied to data collected from discussions and interviews, written data (for example questionnaires), and teaching events (Chick et al., 2006).

1.5 The New Zealand Context

The national curriculum for schools in New Zealand is the result of the 1877 New Zealand Education Act, which made primary education free, secular, and

compulsory for all (Shuker, 1987). All New Zealand state schools are now expected to provide learning and teaching programmes based on the national curriculum statements in either *The New Zealand Curriculum* (NZC) (Ministry of Education, 2007), or *Te Marautanga o Aotearoa* (Ministry of Education, 2008c).

If the learning environment is not favourable for students, the chances are that new knowledge will not be built, there will be no connections made with existing knowledge, misunderstandings will occur, and learning will not reach its full capacity. The New Zealand Tertiary Education Commission (TEC) likens learning to the koru (New Zealand fern frond):

Its natural and gradually unfolding growth pattern could be seen to reflect the process of successful learning, or ako. As fronds mature, new fronds begin to grow, nourished and sheltered by the work of existing fronds, the plant's root system and a favourable environment. (Tertiary Education Commission, 2008, p. 5).

As educators today are facing the immense challenge of meeting the needs of a rapidly changing society, the NZC endeavours to meet these demands. Schools are given autonomy to create a broad, balanced programme suited to their students' needs and interests, that covers a selection of carefully considered objectives which may be explored in depth, within a two-year period (Ministry of Education, 2007).

For the first time, the NZC document includes a section specifically related to teachers' practice, entitled "Effective Pedagogy" (Ministry of Education, 2007, p. 34). This provides guidelines for teacher actions leading to quality classroom programmes, requiring teachers to regularly reflect on, and inquire into, the impact of their teaching practice on students' learning. This is seen as a cyclical process, resulting in quality teaching and referred to as the "Teaching as Inquiry" process (Alton-Lee, 2003; Ministry of Education, 2007). This is a process where teachers become researchers of their own practice and could be described as a form of action research, where they bridge the gap between research and practice (Ball, 2000a; Cohen et al., 2000; Stigler & Hiebert, 2004). Effective curriculum implementation needs to take into account the individual teacher's mastery of the subject matter being taught, along with expertise in how to teach it. The depth of subject knowledge a teacher possesses will be directly reflected in the programme developed in the classroom.

In more recent times it has been acknowledged that a limitation of earlier work on PCK was insufficient recognition of the significance of the broader social and cultural context within teaching (Shulman, 2015). Shulman determined that PCK must not only be pedagogical content knowledge, but also pedagogical culture knowledge and pedagogical context knowledge. Aligned to the notion of pedagogical culture knowledge, the NZC has embedded Principles, relating to cultural diversity and the Treaty of Waitangi (Ministry of Education, 2007, p. 9), which teachers are expected to adhere to in their daily teaching practice. This also aligns with Stigler and Hiebert's (1999) research, which determined that teaching is a culturally-based activity, with much homogeneity of methods within a country, and that one of the key attributes to teaching is the relationship between teachers and students.

The Numeracy Development Project

On-going teacher professional development is an expectation of the New Zealand Ministry of Education and a criterion requirement of registered teachers with the New Zealand Education Council. A substantial amount of mathematics professional learning in New Zealand during the years 2000 to 2010, was related to the implementation of the Ministry of Education-led Numeracy Development Project (NDP), which saw most teachers received approximately 20 hours of support. The NDP was initiated as a result of poor performance by New Zealand students on the Third International Mathematics and Science Study (TIMSS) (Ministry of Education, 1997). The results of that study showed poor understanding by students in number (place value, fractions, and computation), measurement, and algebra concepts (Ministry of Education, 1997). As a result of the TIMSS study, the Minister of Education established the Mathematics and Science Taskforce to provide the kind and level of support classroom teachers required to implement curriculum reforms. Among the many recommendations, the report emphasised: the need for teacher support material, accompanied by some form of teacher development in mathematics; an initial focus on number concepts and then a move to other areas; and that the professional support needed to be school based and provided over a period of time (Ministry of Education, 1997).

The goal of the NDP was to promote high-quality mathematics teaching. Three key themes emerged from the findings of the Literacy and Numeracy Strategy (Ministry of Education, 2001, 2002) and the Taskforce Report (Ministry of Education, 1997) and these formed the main purposes of the NDP:

1. Improve the achievement of students in Number and Algebra and in the other strands of the mathematics and statistics learning area.
2. Develop the pedagogical and content knowledge of teachers to enable them to meet the learning needs of all their students.
3. Inform schools and communities about the significance of numeracy to the future lives of their students.

(Ministry of Education, 2001, p. 2)

A key feature of the NDP was the individual task-based diagnostic interview, known as the Numeracy Project Assessment Tool (NumPA) (Ministry of Education, 2008d). A diagnostic interview has elements of what has been termed a “smart tool” (Robinson & Timperley, 2007). Robinson and Timperley report that a smart tool incorporates a valid theory of the task for which it was designed, and that the tool itself is well designed. Although individual interviews are clearly demanding of teaching time, the experiences of teachers involved in NDP professional development, indicated that the benefits of NumPA were considerable in assessing understanding of what students knew (and did not know), and could do in mathematics (Bobis et al., 2005). The diagnostic interview, as well as the data generated from its use, contributes to implementation of the NDP into the mathematics classroom (Higgins & Bonne, 2009). NumPA results from the orally assessed tasks were often combined with those of written tasks from such tools as the Progressive Achievement Test (PAT) (New Zealand Council for Educational Research, 2006), as they both informed teachers’ planning and assisted in the development of students’ mathematical thinking.

A key theoretical component of the NDP was the introduction to teachers of a Number Framework which outlines the progressions students make through stages of strategy and knowledge application (Ministry of Education, 2008a). The strategy section of the NDP Framework, describes the mental processes students use to determine an answer to any given mathematical problem, while the knowledge section outlines what the students need to know in order to solve those problems. Both sections of the Number Framework are set out in stages of increasing complexity, requiring greater understanding of number, as learners

progress through them (Bobis et al., 2005; Higgins & Parsons, 2011; Wright, 2014). Gaps in understanding previous stages can have a compounding impact on low student achievement and create self-perpetuating cycles of underachievement as they progress through the school years (Allsopp, Kyger, & Lovin, 2007).

The teaching and learning sequence for introducing new strategies and concepts is referred to in the NDP material as the “teaching model” (Ministry of Education, 2008b; Wright, 2014). The model was based on the work of Pirie and Kieren (1989, 1994) and is derived from constructivist learning theory. The strategy teaching model is a subtle tool that should not be interpreted as a set of mechanical rules, but rather is the basis for teachers to see their teaching as experimental and always in the process of development. The critical idea is that rather than teachers telling students how to solve problems, problems are given to the students so that they grapple with them in order to construct meaning for themselves (Ministry of Education, 2008b). In engaging with new concepts the teaching model suggests that students will progress through three stages: using materials; solving problems through imaging; and using number properties. The importance of the phases of concrete, to representational, to abstract has been emphasised in research and is referred to it as the Concrete Representational Abstract (CRA) model (Flores, 2010). Moving through these phases demonstrates greater degrees of abstraction in a student’s thinking. At times students fold back to previous phases of the model, as it is critical that they attempt to connect mathematical abstraction, with the actions on materials and increased complexity of numbers (Ministry of Education, 2008b; Wright, 2014). There is a complexity associated with using number properties and the gap between the use of materials and abstraction (called Using Number Properties in this model) can at times be difficult for students to bridge (Ministry of Education, 2008b). Using visual imagery prior to the introduction of more formal procedures assists in the transition between using materials and the abstraction process (Bobis, 1996; Flores, 2010). In the NDP model this visualising process is referred to as “using imaging”.

The NDP changed the way mathematics was taught in New Zealand primary schools, putting more emphasis on encouraging students to have multiple mental strategies for solving problems with conceptual understanding (Wright, 2014).

Attention was given to developing students' mathematical thinking through targeted questioning in order to achieve a higher level of mathematical thinking (Higgins & Parsons, 2011). A modelling book was frequently used as a shared recorded history of learning, and provided the teacher and students with a means of informing discussion through linking back to previous mathematics sessions (Higgins, 2006).

Facilitators worked alongside teachers in their classrooms using a contextual approach to teaching, which focused on students' strategies, meaningful activities, and multiple representations (Higgins, 2005a; Higgins & Parsons, 2011). A school-based lead teacher appointed in each school worked under the guidance of an external facilitator (Higgins, Sherley, & Tait-McCutcheon, 2007). Lead teachers were responsible for undertaking administrative tasks (for example collecting school-wide NumPA data), communicating information throughout the school, and assisting in professional development. As time progressed, sustainability of the change in practice brought about in schools as a result of the NDP professional learning, meant the lead teacher's role became increasingly important within the school. There was a need to develop a positive learning community between the principal, lead teacher, and classroom teachers (Higgins et al., 2007).

1.6 Rationale for this Study

The purpose of this study was to further understand the relationship between the espoused professional knowledge of teachers, their professional knowledge in practice, and the contribution it makes to student learning, when teaching mathematics for numeracy in the New Zealand primary school classroom. Professional knowledge required for teaching is complex and multi-layered, covering many different aspects of knowledge required by teachers in relation to the subject they teach, the manner in which they teach, and the students they teach. This research investigated the attributes that are associated with the many categories of teachers' professional knowledge, that allow some teachers to be more successful in supporting and engaging their students to achieve the mathematical understanding required, when solving problems competently and confidently. In identifying the strengths and weaknesses, it was envisaged that these could be utilised in future reflection by teachers and in professional learning

opportunities, to improve teaching practice. An outcome of the research was to compile a model that would assist teachers to reflect on their professional knowledge in practice to assist in raising the level of achievement of their students. This study focused on teaching the multiplicative and proportional domains as set out in the Numeracy Framework, as longitudinal numeracy data at the time the study commenced, indicated that students were underachieving in these two important areas (Young-Loveridge, 2009, 2010). Young-Loveridge aggregated data from over thirty-three thousand students, from 2003 to 2007, and found that at the end of Year 6 just over half (53%) of the students were meeting the Ministry's numeracy expectations in the multiplicative domain (Young-Loveridge, 2010, pp. 21-22) as outlined in the Mathematics Standards (Ministry of Education, 2009, 2010). The longitudinal data also showed that in the proportional domain, by the end of Year 5, 29% of the students were meeting expectation (Stage 6 or higher), and by the end of Year 6, 44% of the students were achieving at this level (Young-Loveridge, 2010). This data was collected from teachers who had completed one year of the NDP, and during that time received twenty hours of professional learning and in-class support from a numeracy facilitator. Subsequent National Standards data (Ministry of Education, 2015b), confirms the earlier data, that there are still issues with achievement in mathematics, particularly for Māori and Pasifika students.

Part of the reason for the low achievement might be attributed to the relatively short length of time each teacher was given to implement the change in their teaching (Higgins & Parsons, 2011; Knight, 2005). Research has clearly shown that change takes time (Darling-Hammond, 2010; Gilbert, 2006; Ma, 2010; Schoenfeld, 2011). Another part of the reason for low student achievement might be due to the lack of teacher knowledge (Patterson, 2015). Patterson suggested that teachers' mathematics abilities are even more important today, when teaching mathematics with conceptual understanding, than in the "olden days" when they relied on rote-learned facts and procedures. Students' understanding of mathematical concepts emphasised in mathematics teaching in today's classrooms, depends on teachers understanding mathematical concepts themselves.

While Young-Loveridge's research showed that students were under-achieving in the proportional domain, other research suggested New Zealand teachers' PCK, in

particular their content knowledge was also lacking in this area (Ward & Thomas, 2007; Ward, 2009). Very little in-school, classroom-based research associated with the teaching of the multiplicative and proportional domains has been carried out in New Zealand primary schools, possibly due to the difficulty it imposes. Recent New Zealand-based research completed by Ward (2009) on teachers' knowledge of fractions, utilised data collected anonymously from questionnaires and surveys. This meant that the data was based on the teachers' opinions of their teaching rather than their actual classroom practice. While observations can be time consuming and intrusive, they put the researcher right where the action is, where the reality of what is going on can be noted (Corbin & Strauss, 2008; Stigler & Hiebert, 2004). Hence, this study focussed primarily on in-class research and combined this with questionnaires to gain a more accurate overall representation of teachers' professional knowledge. The research studied the relationship between the teachers' espoused professional knowledge, professional knowledge in action, and student achievement.

1.7 Researcher's Positioning

At the time this study commenced, I worked in a dual role as a numeracy adviser to primary school teachers, and as a mathematics education teaching fellow at a university. This meant that I was in the unique position of supporting teachers currently working within the classroom, as well as preparing pre-service students for their role as a teacher. The Ministry of Education funded NDP was nearing completion of its initial phase, during which most teachers in New Zealand primary schools had been involved in intensive support, focused on making a difference to the instructional practices of their mathematics programmes. As previously noted, the NDP data presented at the time showed that students were not achieving at the expected Standard (Ministry of Education, 2009a, 2010; Young-Loveridge, 2009, 2010). Primary school teachers around New Zealand had devoted much time and energy to their professional learning and development in mathematics over this period, many also completing university papers that had been set up alongside the NDP programme. Of particular interest and concern to me was the question, "*Why was it that after eight years of the NDP, did research data still indicate that the majority of students were not achieving at their expected curriculum level?*" I wanted to gain insight about why the progress that

students were expected to have made was not occurring as anticipated, particularly at the senior level of the primary school. The Government, schools, and teachers, had spent large amounts of money and many hours of time on professional learning and development. Therefore, another niggling question was, “*What was it that had been overlooked in the NDP process?*” These questions concerned me as I searched for ways I could further help in-service teachers and pre-service students I worked with, improve their teaching practice and raise the level of student achievement. I wished to find out more about teachers’ professional knowledge associated with the teaching of mathematics, the impact this knowledge had on their mathematics teaching practice, and how it contributes to student learning.

At the end of Year 6, many New Zealand students move from their current primary school to intermediate school. If they are to progress to intermediate school (Years 7 & 8) at the expected level of achievement, they should be ready to work at Stage 7 on the Number Framework (Ministry of Education, 2008a), or Level 4 within the NZC (Ministry of Education, 2007, 2010). With Young-Loveridge’s data indicating that less than half of the students were achieving at Stage 6 by the end of Year 6, as an advisor and researcher I was curious to understand why this was so. I discussed the data with my colleagues in an effort to find an answer. Some believed it was lack of content knowledge on the part of the teacher, and students would not progress in their mathematics if their teachers did not understand it themselves, while others argued that it was the way the teachers taught the mathematics as one can have limited subject matter knowledge and still be a good teacher. Others argued that the teachers had participated in the NDP professional development, had plenty of resources and support, but found it difficult to change the way they were currently teaching. Therefore, [friendly] arguments and discussions among my colleagues continued. All ideas seemed valid and if progress was to be made by the teachers and an increase in student achievement to be made, then constraints and enablers needed to be identified, so that constructive support could be given. This concern became the basis for this research.

1.8 Aim of the Study

The aim of this research was to investigate key factors that enable teachers to put effective teaching of mathematics for numeracy into practice in the primary school mathematics classroom, in order to improve student achievement.

Three specific questions framed this research:

- 1) What professional knowledge is evident when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?
- 2) What relationships are there between teachers' espoused professional knowledge, professional knowledge in practice, and student learning, when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?
- 3) How does the use of a framework assist in the investigation of teachers' professional knowledge in practice?

1.9 Structure of the Thesis

The thesis began with an introduction highlighting key concepts and background contexts of the study, along with the rationale and aim of the study. It continues in Chapter Two, with a review of the background literature and research relevant to developing the professional knowledge of primary school teachers. It examines theories associated with mathematics learning, the impact of education reforms on teaching and learning, and the influence these have had on current teaching practices. The study explores teachers' professional knowledge by highlighting the role of pedagogical content knowledge (PCK) and the term *numeracy* in relation to mathematics education. Relevant literature related to key knowledge required when teaching the multiplicative and proportional domains is explored. Some of the frameworks available for investigating mathematics teaching are critiqued in relation to teaching practice. Chapter Three outlines the methodological approach taken and describes the theoretical perspectives that underpin the research. It outlines the case-study teachers involved in the research, data-gathering methods, tools used in analysis of the research, along with ethical considerations including the researcher's dual roles and possible conflict of interest. Chapters Four, Five, Six, and Seven present the results and analysis of teacher espoused professional knowledge, teacher practice (in the multiplicative

and proportional domains) and student learning. Chapter Eight presents a discussion relating to the multiplicative domain results, while Chapter Nine presents a discussion of the proportional domain results. Chapter Ten discusses the relationship between espoused professional knowledge, professional knowledge in action, and student learning. The final chapter (Chapter Eleven) presents conclusions in relation to the research questions, consideration of limitations of the study, and implications of the findings for both teaching and further research.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter presents a review of literature around the topic of this research: The relationship between primary school teachers' professional knowledge, their teaching practice, and student achievement, when teaching mathematics for numeracy in the multiplicative and proportional domains. Chapter 1 set out the use of the term Professional Knowledge, based on the seminal work of Shulman (1986), to describe the ways in which teachers construct and reconstruct the knowledge required for teaching (Beattie, 1997; Bobis, Higgins, Cavanagh, & Roche, 2012; French, 2007; Ponte, 1994; Schoenfeld, 2011). Understanding what a teacher's professional knowledge really is, what it looks like, and how it might be used during classroom instruction can be exceptionally difficult (Loughran, 2010; Ma, 2010).

The literature review initially considers theories associated with mathematics teaching and learning along with mathematics education reform throughout recent decades. Following this is a critical examination of teachers' professional knowledge, including PCK, content knowledge, and mathematical knowledge required for teaching. This is followed by distinction between use of the terms *mathematics* and *numeracy* and the place of professional learning and development in the implementation of these in classroom practice.

It is evident that mathematics teaching differs somewhat from one country to another (Delaney, Ball, Hill, Schilling, & Zopf, 2008; Ma, 2010; Roche & Clarke, 2011; Stigler & Hiebert, 2004), and that the PCK required is subject specific and context specific (Gess-Newsome, 1999a, 2015; Hill et al. 2008; Shulman, 2015; Steele, 2005). The next section of this chapter examines the teaching of the multiplicative and proportional domains in the New Zealand context. The chapter concludes with an analysis of frameworks of teacher knowledge available to researchers.

2.2 Theories Associated with Mathematics Learning

While there are many theories associated with research, two main theories currently dominate mathematics education: Piaget's constructivism and Vygotsky's socio-cultural theory (Confrey & Kazak, 2006; Lambdin & Walcott, 2007). Whilst there are distinctions between the ideologies associated with these two theories, some theorists combine key aspects to form a third theory, known as social-constructivist theory (Lambdin & Walcott, 2007).

Constructivism

One of the bases for the development of constructivist theories in education can be attributed to the work of psychologist, Jean Piaget. Piaget developed ideas of children progressing through particular stages in their thinking patterns and creating knowledge in an active way. Central to constructivism is the notion that learners are not blank slates but instead creators (constructors) of their own learning. The networks within which they associate, are both the result of the already constructed knowledge and a tool from which new knowledge can be formed (Yackel & Cobb, 1996). Hence, children do not simply absorb new knowledge but create it for themselves based on networks, which are continually modified. Constructivism recognises that mathematics must make sense to the students if they are to retain it.

It has been argued that constructivism is a theory of learning, rather than a theory of teaching, and is said to have developed in mathematics education to counter the effects of behaviourism (Confrey & Kazak, 2006; Lambdin & Walcott, 2007; Simon, 1995; Steffe & Kieren, 1994). As such, constructivism has influenced teachers' understanding of how children learn mathematical concepts of numeration, quantification, space, logic, chance, and data (Confrey & Kazak, 2006). Teaching recognises constructivist theory and strategies used in the teaching process will focus on the strengths and resources children bring to tasks, that makes their involvement and participation central to the learning (Cobb, 2007; Confrey & Kazak, 2006; Lambdin & Walcott, 2007). The introduction of constructivism has thus helped educators think about what mathematical knowledge is, how it is acquired, and what the implications are for teaching.

In the modern mathematics classroom, teachers generally utilise a questioning and facilitating role, and learning is seen as an activity where shared mathematical meanings are constructed with others, and drawn from the mathematical learning environment (Fraivillig et al., 1999; Hansen, 2005; Tout & Motteram, 2006; Way, 2008). Whether a child is listening passively to a teacher or peer, or actively participating in discussion, his or her brain is applying prior knowledge to make sense of new information: it is constructing meaning to the things he or she is thinking about (Cobb, 1994).

The notion of constructivism is recognised in NZC (Ministry of Education, 2007) where “making connections to prior learning and experience” are emphasised (p.34). The NZC acknowledges that students learn best when they integrate new learning with what they already understand. This also allows teachers to make connections across learning areas.

Socio-cultural theory

In more recent years, theories related to teaching and learning in mathematics have placed an increasing importance, and emphasis on factors relating to social factors at a macro level (Cobb, 1994; Cobb & Steffe, 1983; Griffiths, 1998). This has resulted in an increase in interest on the sociocultural aspects of mathematics education and their impact on students’ learning (Jorgensen, 2010; Rubie-Davies, 2010; Rubie-Davies, Peterson, Sibley, & Rosenthal, 2015; Seah, Atweh, Clarkson, & Ellerton, 2008). From a sociocultural perspective, the classroom is comprised of students from varying cultural and social backgrounds, with varying beliefs and expectations of education. Learning is carried out through active engagement and participation in activities in particular contexts, and in the mathematics classroom is the basis of the interplay between who they are, those with whom they are working, and the mathematics context they are addressing (Bruner, 1986; Cobb & Bauersfeld, 1995; Lerman, 2006; Op’t Eynde, 2004; Op’t Eynde, De Corte, & Verschaffel, 2002).

Central to sociocultural theory is the work of Vygotsky. Vygotsky suggested that:

“Every function in the child’s cultural development appears twice: first, on the social level and, later on, on the individual level; first, between people (inter-psychological) and then inside the child (intra-psychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals.”

(Vygotsky, 1978, p. 57).

Vygotsky (1978) introduced the term Zone of Proximal Development (ZPD), as a symbolic space created through the child's interaction with more knowledgeable learners (Vygotsky, 1978). When working in the ZPD, particular attention is given to the language being used, as the language of students influences how they will interpret situations and strategies, and build understandings (Bell & Woo, 1998). This includes learning, which is guided by effective, analytic questions posed by the teacher.

Socio-cultural theorists report that collaboration and conversation is crucial to the transformation of external communication to internal thought (Cobb & Bauersfeld, 1995; Op't Eynde, 2004). This clearly exists in the mathematics classroom as students and teachers interact and work in partnership with each other in groups. When a learning environment is constructed in which students are required to talk and act like mathematicians this becomes as much of a priority as the lesson learning outcome (Askew, 2007; Hunter, 2012). Teachers encourage discussions and provide opportunities for the sharing of ideas, explanations, and opinions, so that common understanding is reached by all of the students participating in the learning (Cobb, 1994). When children present their ideas and verbally present their rationale for their actions, argumentation becomes part of the social interaction (Bruner, 1990; Cobb & Bauersfeld, 1995). Bruner refers to this as "folk psychology" of classroom learning.

As early as the first year at school, children learn to listen to other children, and to talk about their solutions to problems (Cobb, Wood, & Yackel, 1990; Wood, Cobb, & Yackel, 1995; Yackel & Cobb, 1996). Different types of interactions the students are involved in influence the learning opportunities for them. There are times when the conversation becomes univocal (when the perspective of one student dominates), and one student explains their solution to a problem while the other student(s) listen to and makes sense of the problem (Cobb & Bauersfeld, 1995). At other times the conversation is multivocal (when both [all] students voice their opinions) and both students involved express their ideas and challenge each other's thinking (Cobb & Bauersfeld, 1995). Case studies carried out by Cobb and Bauersfeld concluded that the multivocal conversations were usually the more productive and provided a basis for mathematical communication.

Social-constructivism

Some theorists posited that elements of constructivism and socio-cultural theory can be combined, and so it is unnecessary to choose between the two paradigms (Cobb, 2006; Norton & D'Ambrosio, 2008). Hence, Piaget's constructivism and Vygotsky's socio-cultural theory, morphed into enthusiasm for modern-day constructivism (Lambdin & Walcott, 2007, p. 15). The sense-making and process that children undergo as they construct their own knowledge are drawn from constructivism and combined with the social and cultural interactions of the classroom.

Mathematics education over the last few decades has been based around the alternatives to the traditional perspectives on what it means to learn and to know mathematics (Lambdin & Woolcott, 2007) and is now seen as a cognitive activity, influenced by social and cultural processes (Wood et al., 1995; Wood, Williams, & McNeal, 2006). The Numeracy Development Project (NDP) teaching model used in New Zealand primary schools, reflects a social-constructivist perspective (Wright, 2014). The teaching model acknowledges both the social setting created in the mathematics classroom and the personal construction of knowledge as children progress through the materials, to imaging, and then to the number properties phases. The NDP teaching model also reflects constructivist ideologies in that the children's prior learning practices and understandings impact on how they interpret and engage with new situations.

2.3 Reforms in Mathematics Education

Teacher education programmes, professional development, and curricula have changed the approach to teaching mathematics, as a result of on-going reforms in education. While reforms in schools may differ from country to country in content, direction, and pace, they generally have five common factors: (1) Governments intervene to change conditions under which students learn in order to accelerate improvements and raise standards of achievement. (2) They address implicit worries of governments concerning fragmentation of personal and social values in society. (3) They challenge teachers' existing practices. (4) They increase the workload for teachers. (5) Inattention is given to teachers' identities – including commitment, job satisfaction, and effectiveness (Day, 2002).

Reform in mathematics education takes considerable time due to the many steps from government policy, to changes in an individual school or classroom (Day, 2002; Lamon, 2007). The most frequently cited reasons for education reform have to do with the impact on learning outcomes for students (Levin, 2001). Although education reforms are often framed in terms of student outcomes, the approach to changing student outcomes usually involves attempts to alter the way that individual teachers work in their classrooms, and the way that schools work as institutions.

The teaching of mathematics in schools throughout the twentieth century saw six identifiable phases, each with its unique emphasis: drill and practice, meaningful arithmetic, new mathematics, back to basics, problem solving, and standards and accountability (Lambdin & Walcott, 2007). Each of these phases introduced what was seen as new and innovative practices, for that particular period of time. For example, the drill and practice era of the 1920-1930 period was based on rote memorisation of facts and frequent practice of algorithms. Following the Great Depression of the 1930s, mathematics education saw a swing from meaningless rote learning, to a new emphasis on developing mathematics in a more meaningful way. Many mathematics educators of the time struggled with the merits of incidental learning and it was soon claimed that to learn arithmetic meaningfully, it was necessary to understand it systematically. This meant that while it was acknowledged that the significance of number was functional, the meaning of number was mathematical (Lambdin & Walcott, 2007).

The 1960s saw one of the greatest changes in education, as a result of the introduction of the space age. It was thought that schools were no longer preparing students sufficiently to be capable of understanding the concepts necessary to compete in the new technology-driven workforce (Lambdin & Walcott, 2007; McQueen, 2006; Skemp, 1989). The 1960s became known as the “new mathematics” phase (Brown, Askew, Baker, Denvir, & Millett, 1998; Lambdin & Walcott, 2007). Mathematics was taught in sequential topics which became increasingly more complex and were often returned to, again and again. Before long, concerns were expressed about whether students were learning what was required for the workplace and life in general and in the 1970s mathematics

teaching returned to drill and practice. This was labelled the “back-to-basics” phase (Brown et al., 1998; Lambdin & Walcott, 2007; Perso, 2007).

The 1980s saw many educators believe that the pendulum had swung too far in returning to the basic facts drills and the “problem-solving” phase was introduced (Lambdin & Walcott, 2007). During this era, lessons on problem-solving strategies included drawing pictures, working co-operatively in groups, and verbalising thinking. The emphasis was on teaching mathematics in ways that had continuity between school and the outside world and in a manner, which enabled learners to bring their intelligence, rather than rote learning, into use when solving their mathematics problems (Skemp, 1989).

The problem-solving approach introduced during the 1980s became further refined in the 1990s, when a distinction was made between the original idea of teaching students how to solve problems, to teaching them about problem-solving and for problem solving (Hunter, 2012; Lambdin & Walcott, 2007). This resulted in action to address the standard of mathematics education, through changes in teachers’ qualifications, classroom instruction, accountability, and mandated assessments (Lambdin & Walcott, 2007). The New Zealand mathematics curriculum of this era (Ministry of Education, 1992) was written in five strands, number, algebra, geometry, measurement, and statistics. These were subsequently amalgamated into three strands: number and algebra, geometry and measurement, and statistics (Ministry of Education, 2007). Along with a reduction in the number of mathematics strands, the NZC (Ministry of Education, 2007) included sections on vision (what is wanted for the young people of New Zealand), principles (foundations of curriculum decision making, values (to be encouraged and explored), and key competencies (capabilities for living and lifelong learning).

Since the 1990s accountability and assessment expectations have come to the fore. NZC promotes assessment as a relationship between teachers and students and teaching and learning:

“The primary purpose of assessment is to improve students’ learning and teachers’ teaching, as both student and teacher respond to the information that it provides. It involves the focused and timely gathering, analysis, interpretation, and use of information, that can provide evidence of student progress”

(Ministry of Education, 2007, p. 39).

Many countries have seen the development of numeracy standards, and standards-based curricula. For example, New Zealand has *The New Zealand Curriculum Mathematics Standards for Years 1–8* that require teachers to twice-yearly report on student progress against (end of year) curriculum expectations (Ministry of Education, 2009a). Australia has *The National Assessment Program – Literacy and Numeracy (NAPLAN)* which is an annual assessment for students in Years 3, 5, 7, and 9, and has been an everyday part of the school calendar since 2008 (Australian Curriculum and Reporting Authority, 2011, 2015). The United States has the *National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) and the *Common Core State Standards (CCSSI)* (Common Core State Standards Initiative, 2010).

2.3.1 Factors Contributing to Current Mathematics Education Reforms

Mathematics lessons need to allow students to see the relevance it has to them by making connections between what they are learning inside the classroom and the things they care about in the world around them (Boaler, 2008; Davis & Renert, 2014; Skemp, 1989; Steen, 1999). This has required a change in teaching style for many teachers, with a shift from the more traditional didactic model that focused on students' proficiencies in reproducing existing solution methods and strategies, to one that encourages students to construct their own meaningful mathematical concepts through an inquiry-based model (Boaler, 2008).

The current standards-based education system supports a curriculum that emphasises concepts and meanings, rather than rote learning, and promotes integrated, rather than piecemeal usage of mathematical ideas (Howley, Larsen, Solange, Rhodes, & Howley, 2007; Stigler & Hiebert, 2004). Two key factors have been identified when solving mathematics problems: the concepts and the processes (Hull, Balka, & Miles, 2011; Ma, 2010). In today's mathematics classroom concepts are taught first and foremost. Procedures are also learnt, but not without first acquiring a conceptual understanding (Schwartz, 2008). Developing procedural knowledge at the expense of conceptual understanding has often been cited as part of the reason for poor mathematics proficiency (Ball, 1992; Burns, 1998; Kazemi & Stipek, 2001; Scharton, 2004; Skemp, 1976). As

explained in Chapter 1, the procedural approach to teaching was referred to by Skemp as “instrumental understanding” or “rules without reason”, while conceptual understanding was known as “relational understanding”. When students are drilled in methods and rules that do not make sense to them, it is not only a barrier for their mathematics understanding, but it also leaves the students frustrated, and with a negative disposition towards mathematics in the long term (Boaler, 2008; Davis & Renert, 2014; Yackel, 2001).

Conceptual understanding in mathematics develops when students see the connections between procedures and concepts, and can explain the relationships between facts, based on structures and patterns (Rittle-Johnson, Siegler, & Alibali, 2001). One of the benefits of emphasising conceptual understanding to students is that they are less likely to forget concepts than procedures, and once conceptual knowledge is gained they can use it to reconstruct a procedure they may have otherwise forgotten (Schwartz, 2008). Conceptual understanding is intertwined with procedural knowledge (Wong & Evans, 2007) and the combination is much more powerful than either one alone. Rittle-Johnson et al., concluded that conceptual understanding was a prerequisite for students to select appropriate procedures to use when solving mathematical problems, and that developing procedural knowledge had effect on conceptual understanding. However, as Schwartz has asserted, for teachers to focus on the conceptual teaching of mathematics, they must first have conceptual understanding themselves and one of the biggest challenges in moving from a procedurally oriented way of teaching to conceptually oriented teaching, has been ensuring that teachers have the necessary mathematical understandings. Once conceptual understanding is developed, it becomes conceptual knowledge to sit alongside procedural knowledge (Rittle-Johnson et al., 2001).

In order for students to progress in their mathematics learning, teachers need to engage in classroom practice that encourages students to make connections between mathematical ideas and concepts (Askew, 1999, 2007, 2013; Schwartz, 2008; Skemp, 1976; Stigler & Hiebert, 2004; Treffers, 2001). Developing conceptual understanding is essential if mathematics is to be learnt with understanding and something will only be understood if one can see how it is related or connected to other things that are known (Ball & Bass, 2003; Hiebert &

Carpenter, 1992; Kazemi & Stipek, 2001; Skemp, 1989). Connections need to be made between different aspects of mathematics (for example division and fractions), between different representations of mathematics (between symbols, words, diagrams, and objects), and with children's methods (valuing children's thinking and sharing their methods) (Askew, 1999). Skemp (1989) advocated that to understand something means to have the capacity to be able to transfer it into another similar situation, theory, or strategy (relational understanding). If mental models are to be of any use, they must be remembered not as single experiences from a range of past events, but for the commonalities of these experiences, which can be recognised on future occasions (Burns, 1998, Hiebert & Carpenter, 1992; Skemp, 1989). These commonalities form connections, or a large web of number relationships with an interaction between numbers and operations (Treffers, 2001). According to Treffers, the ability to see relationships as something tangible, in spite of their abstract nature, should make it possible for students to progress into higher levels of mathematics study at a later stage.

Alongside the experiences of their students, teachers need to consider the appropriateness of the problems and tasks they assign their students. A mathematical task may be described as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (Stein, Grover, & Henningsen, 1996, p. 460). Stein et al., categorized mathematical tasks in terms of the cognitive demand they necessitate. They defined "memorization and procedures without connections" tasks as lower-level tasks. These tasks require students to solve a problem by remembering memorized information and applying a procedure, without understanding its meaning. "Procedures with connections" and "doing mathematics" tasks were identified as higher level. These tasks require students to choose and apply suitable procedures make connections within and between multiple representations, in order to find the reasonable solution. Higher-level tasks have a greater impact on students' thinking and understanding. It is important for teachers to anticipate how students may perceive a task mathematically by examining the task from the students' perspective (Smith & Stein, 2011; Stein et al., 1996).

Associated with the importance of conceptual understanding in mathematics is the use of tools and manipulatives. A tool refers to any object, drawing, or picture,

which represents that concept (Perry & Howard, 1997; Suh, 2007; Swan & Marshall, 2010; Thompson, 1994). For example, drawings may be used as a tool for emerging ideas, as sometimes it is difficult for students to think about and understand abstract relationships if relying only on words and symbols. A mathematics manipulative is defined as, “any object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered” (Swan & Marshall, 2010, p. 14). Manipulatives are frequently used in mathematics lessons with the claim that they extend students’ learning of mathematical concepts and operations, as they make them more comprehensible (Burns, 1998; Ma, 2010; Nührenbörger & Steinbring, 2008; Ross, 1989; Schoenfeld, 2011; Swan & Marshall, 2010; Wright, 2014). Manipulatives can be used to represent the mathematical concepts underlying the procedure, and that connections need to be made between the two – the manipulative and the mathematical idea (Carbonneau, Marley, & Selig, 2013; Clement, 2004; Clements & McMillen, 1996; Fennell & Rowan, 2001; Ma, 2010; Pape & Tchoshanov, 2001; Zevenbergen et al., 2004). However, simply taking manipulatives, picking them up and using them, will not magically impart mathematical knowledge and understanding (Swan & Marshall, 2010). Appropriate discussion is required alongside the use of manipulatives to make the links to the mathematics explicit or the students may end up with misconceptions. Teachers often require professional development on the incorporation of manipulatives into their teaching, to give insights into how they can assist with children’s learning (Stein & Bovalino, 2001).

Social-constructivism occurs in the manner in which the teacher and children interact during discussions that sit alongside the use of manipulatives. It is central to the learning that teachers have a discussion with their students following the use of manipulatives, so that students can explain their solutions to problems (Gould, 2005a, 2005b; Ma, 2010). The intention for using the manipulative must be clear and the teacher needs to be aware of what interpretation the students are making of them (Yackel, 2001). If the students do not explain their use of the tools and/or manipulatives, then teachers are in jeopardy of replacing verbal rules and procedures, with rules and procedures for using them. Discussion means that understanding the link between the manipulation of the objects and the related

symbolic representation (the mathematical equation), can be established (Hiebert, 1984; Ma, 2010; Yackel, 2001). The relationship between the manipulative and mathematical understanding and insights is developed when students use the equipment to construct a model and interpret its meaning. Recent research of Flores (2010) indicated that when using the Concrete to Representational to Abstract (CRA) model (manipulatives, to pictures or drawings, to numbers only), students seldom made errors in basic mathematics computation, which resulted in improved confidence and assessment scores.

Current teaching focuses on the structure underlying numbers and number operations (Anghileri, 2006; Mulligan, 2013; Mulligan & Mitchelmore, 2009, 2013; Mulligan, English, Mitchelmore, & Crevensten, 2013). Curriculum reforms have led to a shift from teaching standard procedures for calculating, to allowing students to observe patterns and relationships, and make connections, in order to develop a feel for numbers. The classroom setting should allow for multiple solution methods, and teachers need to anticipate a wide range of possible strategies students might utilise to solve these, rather than reliance on the traditional algorithmic method (Anghileri, 2006; Smith & Stein, 2011), while simultaneously scaffolding students' use of mathematical language and knowledge (Ball, 2002; Chick, 2015).

The emphasis on teaching concepts and meanings positions mathematical knowledge as a social process (such as advocated by Vygotsky), whereby students construct mathematical ideas together, based on their understanding and experiences of the world in which they live (Ross, 2005). Teachers take a less central role, acting as facilitators for student-led exploration of mathematics, discussions, and development of mathematical ideas, allowing students to take a more active role in their learning (Boaler, 2008; Hunter, 2009; Stigler & Hiebert, 2004; Vosniadou, 2001; Wiliam & Bartholomew, 2004). Open-ended tasks based on students' interests and mathematical strengths are effective in engaging students, promoting mathematical justification, and developing conceptual knowledge (Anthony & Walshaw, 2009b; Nardi & Stewart, 2003). The teacher's role is not simply to accept the methods used by students, but to help them adopt better ones.

Productive mathematical inquiry and argumentation should be part of classroom discourse (Brown & Renshaw, 2006; Hunter, 2006; Walshaw & Anthony, 2007), as students reconstruct their thinking and build stronger explanations (Whitenack & Yackel, 2002). Discourse “reflects an enterprise and the perspective of a community of practice” (Wenger, 1998, p. 86). Individuals participate with varying levels of expertise across a range of speech genres, using an individual voice (Bakhtin, 1994). During the dialogic nature of discourse, students use the language of inquiry and argumentation and learn to question, argue, explain, justify, and generalise (Hunter, 2006).

Teachers need to develop working environments and practices that encourage students to work in groups (Vosniadou, 2001). The teacher acts as a co-ordinator providing guidance and support in mathematics content learning, alongside the development of skills that allow the students to work together. Critics of this change in approach to teaching mathematics, maintained that mathematical rigour was being threatened because students were no longer taught standard methods and they were wasting time chatting to friends in groups (Boaler, 2008). The repercussion this had among communities meant that some teachers were afraid to try new ideas and methods in their teaching and returned to the more traditional methods (Boaler, 2008). However, the ability to work together is a skill that needs to be taught. Once achieved, it allows students to help each other and utilise mathematical reasoning when explaining their ideas to others (Bragg, Herbert, Loong, Vale, & Widjaja, 2016; Hunter, 2009, 2010; Vosniadou, 2001).

Determining and specifying a mathematical goal that clearly identifies what students are to know and understand about mathematics as a result of participation in a lesson, is an important starting point when planning and teaching a lesson (Smith & Stein, 2011). A specific goal provides the teacher with a clear outcome that guides the learning and selection of activities that takes place. The goals assist in formative assessment and feedback to students (Black, Harrison, Lee, Marshall, & Wiliam, 2004; Black & Wiliam, 1998). Smith and Stein emphasised the significance of selecting appropriate activities or tasks, based on the learning goals. Goals with higher-level demands, allow students to engage with and make connections between concepts, while those with lower-level demands lead to limited opportunity for student thinking. Without specific learning goals,

determining what learning has occurred as a result of the instruction and activities can be problematic. Setting appropriate goals and associated tasks should assist in constructive discussion around the key mathematical ideas of the lesson (Hiebert, Morris, Berk, & Jansen, 2007).

2.4 Teachers' Professional Knowledge

In any profession, there is a specialised professional knowledge that makes it unique and distinct from other professions; the teaching profession is no exception (Shulman, 2010, 2015). As a result of on-going reforms, the role of the teacher has changed in profound ways in recent years and teaching in today's classrooms requires professionalism, skills, and knowledge that teachers of previous years did not require (Hattie, 2003; 2009). Linking the professional knowledge of teachers, to the relationship between classroom practice and student understandings as a result of those practices, has thus been a focus of researchers in recent times (Anthony & Walshaw, 2007, 2008, 2009a; Ball, 1991, 2002; Ball et al., 2008; Chick, 2007; Chick et al., 2006; Hattie, 2009; Hill et al., 2008; Patahuddin, 2008; Schoenfeld, 2011, 2013). Hattie acknowledged that the role of the teacher is more complex and sophisticated, and has changed in response to the major societal, economic, cultural, and political changes that have taken place. Hattie emphasised that today's teachers must be reflective, analytical, and literate; they must be creative, imaginative, knowledgeable, and sensitive to the diverse needs of the children in today's classrooms. If teachers are to create classroom experiences and conditions, which prepare the children for tomorrow, they also need to promote the skills of student inquiry and reflective thought (Alton-Lee, Hunter, Sinnema, & Pulegatoa-Diggins, 2012).

Concern over the mathematical knowledge of primary school teachers, has been expressed by researchers for many years (Askew, 2008; Baker & Chick, 2006; Ball, 1991; Ball et al., 2001; Chick & Beswick, 2013; Hill, Rowan, & Ball, 2005; Ma, 2010; Schoenfeld, 2013). It has become both politically and educationally necessary to provide evidence about why knowledge of mathematics content by itself is insufficient for effective teaching of mathematics (Bobis, Mulligan, & Lowrie, 2013). While knowledge of mathematics content is important, and specific content knowledge is required, the special type of knowledge required by teachers referred to as pedagogical content knowledge is considered to be the

most important (Shulman, 1986). Therefore, attention has moved beyond merely examining what knowledge matters, to also including why different types of knowledge are important for teaching mathematics (Ball et al., 2008; Barton, 2009).

Today, professional knowledge may be seen in terms of knowledge *for* practice, knowledge *in* practice, and knowledge *of* practice (Loughran, 2010). Schoenfeld (2011) attempted to identify what classroom interactions, and what pedagogies, result in students' "robust understanding" of important mathematics. He defined an individual's knowledge as "the information that he or she has potentially available to bring to bear, in order to solve problems, achieve goals, or perform other such tasks" (p. 25). He further noted that according to this definition, a person's knowledge is not necessarily correct in the mind of others, but it provides a part of that individual's knowledge base.

2.4.1 Pedagogical Content Knowledge (PCK)

Teachers have implicitly combined two key areas of knowledge during their teaching practice: the knowledge of the subject they teach, not only the facts, but how they know what they know; and knowledge about the practice of teaching, how to communicate the subject to the students, and how they learn (Shulman, 1986). The third area of teacher knowledge referred to as pedagogical content knowledge (PCK), became a widespread part of education terminology as a result of Shulman's seminal address to the American Educational Research Association in 1986. He described PCK as, "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987, p. 8). PCK is the particular knowledge teachers develop over time and through experience, about the teaching of content in particular ways in order to lead to enhanced student understanding. PCK is not only about *what* we teach, but *how* we teach, in order to maximise student learning and understanding (Loughran et al., 2012).

The introduction of the term PCK by Shulman (1986) was thus responsible for the focus within teaching and research on teacher knowledge of content, pedagogy, students, and the connections between the three. Shulman expanded on these three broad groupings, and the following definition of PCK has become accepted by many educationalists:

The most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others... and understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning (p. 9).

Teachers must have a strong appreciation of the importance of pedagogical understanding, and of subject matter knowledge required for the quality teaching of mathematics (Ball et al., 2008; Chick, 2007; Hill et al., 2008; Loughran et al., 2006). PCK is not a single entity that is the same for all teachers: it may be the same or similar for some teachers, but it will be different for others (Loughran et al., 2012). The recognition of PCK may be difficult at times due to the amalgam of its inextricably linked components - knowledge of pedagogy, and knowledge of content (Magnusson et al., 1999).

PCK is about effectively communicating a subject to people for whom the content may be new (Loughran et al., 2012). While it requires knowledge of what is taught and how it is taught, it also requires knowledge of how students think and what they understand before they learn the subject matter, as well as how they think while they are learning. Hence, teachers must have knowledge about how mathematics is learned, how topics should be sequenced for learning, where conceptual blockages may occur, and where misunderstandings are likely (Barton, 2009). There is a foundation of pedagogy within PCK which is general across curriculum areas and should be developed by all teachers (Loughran et al., 2012; Shulman, 2015). These include planning, teaching methods, group work, individual work, questioning, wait time, feedback, modelling, and evaluations. However, like content knowledge in general, knowledge about how people learn the content is specific to a particular subject.

The understanding of content as it relates to one's subject area is crucial and PCK in its entirety cannot simply be 'imported' from one subject area to another (Gess-Newsome, 2015; Hill et al., 2008; Loughran et al., 2012; Magnusson et al., 1999; Shulman 2015). For example, mathematics teachers require an understanding of how and why a particular mathematical procedure works, while effective science teachers need to know how to best design and guide learning experiences under particular conditions and constraints, in order to help their students to develop

scientific knowledge and understanding (Magnusson et al., 1999). PCK is a dynamic construct that describes the process a teacher employs when teaching particular subjects, to particular learners, in particular settings (Ball et al., 2008; Shulman, 2015). More recently it has been acknowledged that PCK is domain specific and contextualised (Shulman, 2015). It is not only subject specific, but within a subject it is topic specific, and is referred to as Topic-Specific Professional Knowledge (TSPK), (Gess-Newsome, 2015). While this knowledge is specific to a particular topic being taught (for example multiplicative thinking or proportional thinking), it is also related to the students' developmental level.

More recently, Shulman (2015) has acknowledged that early research on PCK overlooked some important elements of teaching. He identified four weaknesses and limitations of his initial PCK research as: lacking in attention to emotion, affect, feelings, and motivation (the non-cognitive attributes); focussed on PCK intellectually with emphasis on teacher thinking and insufficient recognition given to pedagogical action; insufficient attention given to the broader social and cultural context; and limited in terms of outcomes of teaching, including a teacher's vision and goals for education, and the relationship of PCK to students' outcomes, including the minds and hearts of students. Originally, PCK gave meaning to the thinking that was apparent at the time. "PCK is an attribute that teachers develop, and it cannot be found among mere subject matter or among those who are good with kids. It was a policy claim about how special teachers were and how they ought to be regarded and respected" (Shulman, 2015, p. 11).

2.4.2 Content Knowledge and Mathematical Knowledge for Teaching

Research has shown that each subject and associated topics are taught differently depending on the depth and quality of a given teacher's understanding of both the content and associated pedagogy of the topic concerned (Shulman, 2015). When relating content knowledge to the practice of teaching, Ball et al. (2008) referred to this specialised content knowledge as the "distinct bodies of identifiable content knowledge that matter for teaching" (p. 389). As indicated in Chapter 1, this special knowledge is an essential requirement of the classroom and is referred to as Specialised Content Knowledge (SCK), as opposed to Common Content

Knowledge (CCK), the knowledge held or used by an average mathematically literate citizen (Ball et al., 2001, 2005, 2008; Hill et al., 2008).

As a result of research into primary school teachers' practices, the mathematical knowledge for teaching (MKT) construct was developed, combining CCK and SCK (Speer et al., 2015). MKT is often used alongside subject matter knowledge (Ball et al., 2008; Barton, 2009; Davis & Renert, 2014) and has been described as:

“a way of being with mathematics knowledge that enables a teacher to structure learning situations, interpret students' actions mindfully, and respond flexibly, in ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice” (Davis & Renert, 2014, p. 4).

As well as general pedagogy and pedagogical content knowledge, MKT includes a pure mathematics content knowledge (mastery of mathematics a minimum of four years above the level being taught) with a focus on both what a teacher must know as well as how it is taught, along with the mathematical environment that a teacher must create (Davis & Renert, 2014). It is more than book knowledge and requires teachers to be flexible with in-the-moment responsiveness, and to understand an issue from a learner's perspective (Schoenfeld, 2011). There are times when a student provides an unexpected response to a question or an unanticipated idea within a discussion and the teacher has to decide how to respond and whether to explore the idea further. The teacher must decide on their feet, whether to deviate from the planned lesson or pause and respond to the student's ideas. This knowledge required for in-the-moment actions and interactions in the classroom and the unpredictability of what follows is sometimes referred to as *contingency knowledge* (Rowland, Turner, Thwaites, & Huckstep, 2009). Contingency, or being able to “think on one's feet” (p. 33), is an important aspect of MKT and involves a combination of all seven of Shulman's categories of knowledge: subject-matter knowledge; knowledge of pedagogy; pedagogical content knowledge; knowledge of curriculum; knowledge of learners; knowledge of context; and knowledge of purpose.

The construct MKT, has become a focus of research in recent years (Barton, 2009; Ball et al., 2008; Ball et al., 2009; Clark, Clarke, & Cheeseman, 2006; Davis & Renert, 2014; Hill et al., 2008; Roche, Clarke, Clarke, & Chan, 2016; Speer et al., 2015) as understanding of the knowledge required to teach mathematics is sought.

Understanding the specific mathematics subject matter that is required for teaching is necessitated by all teachers, regardless of the level at which they teach. Teachers need to see into the subject matter through the eyes, hearts, and minds of learners, as their task is to transform the content in ways that make it accessible to the learners while maintaining its integrity (Ball, 1993; Shulman, 2015). Recent research has emphasised the content-specific nature of MKT and PCK in relation to mathematics teaching (Roche et al., 2016) and emphasised the incremental nature of teacher learning.

While knowing mathematics is essential, it is not enough to be able to teach it effectively (Moch, 2004). There is a difference between having content knowledge and being able to teach a subject. It is easy to recognise when a student makes a mistake, but more difficult to interpret why they have gone wrong (Hill et al., 2008; Ma, 2010). While a certain amount of mathematical content knowledge is required for effective teaching, having this knowledge is not sufficient on its own (Askew, 2008; Begg, 2005; Goya, 2006). Enhancing students' understanding is an ultimate goal of school education (Darling-Hammond & Ducommun, 2010) and special features associated with a teacher's MKT are connected to student learning and achievement (Hill et al. 2004, 2005, 2007; Ma, 2010). Goya questioned whether teachers who do not fully understand basic mathematical operations can be expected to help their students build understanding and reasoning skills. Hence, a key question often considered based around this notion is: "Is a teacher knowing how to do mathematics, sufficient for effective teaching of mathematics?" Researchers have agreed that being a good mathematician is insufficient, as teachers also need to understand how they can support students in their learning through their PCK (Askew, 2007; Ball et al., 2005; Goya, 2006; Moch, 2004; Shulman, 1986; Ward, 2009). Within the complexity of PCK, teachers need to demonstrate a range of mathematical knowledge, including: procedural knowledge, procedural fluency, conceptual knowledge, and mathematical connections (Ball & Bass, 2003; Ball, Sleep, & Bass, 2009).

To facilitate learning, teachers need to emphasise and promote the connections between, and among, ideas and topics (Ma, 2010). Ma described these as well-developed, interconnected, knowledge packages, made up of a thorough understanding of mathematics, having breadth, depth, connectedness, and

thoroughness, and referred to this as *profound understanding of fundamental mathematics* (PUFM). The term *profound* is often considered to mean intellectual depth with its three interconnected connotations *deep*, *vast*, and *thorough*. Understanding a topic with *depth* means connecting it with more conceptually powerful ideas of the subject, while understanding a topic with *breadth*, is to connect it with those ideas of similar or less conceptual power (Ma, 2010). Ma concluded that there is an important depth to seemingly basic concepts and teachers require this profound understanding to be truly effective in the classroom. A teacher with PUFM is aware of the conceptual structure of mathematics as PUFM goes beyond being able to compute correctly, to giving a rationale for the computational process. In planning lessons and orchestrating discussion among students, a teacher with PUFM draws on the knowledge of how to teach (pedagogy), but in understanding the student's responses, the teacher also draws on subject matter knowledge.

Within PUFM is the need for a strong number sense (Ma, 2010). Number sense is the “well-interconnected knowledge about numbers and how they operate or interact” (Baroody, 2006, p. 22) and utilises an ability to improvise and use creativity with numbers while finding sensible ways to make computation easier (Briand-Newman, Wong, & Evans, 2012). Number sense, or the capacity to make sense of numbers, is recognised as foundational knowledge required by teachers and students to understand and link quantities to numerical constructs and mathematical strategies. Number sense begins with counting when preschool children learn the one-to-one principle and includes number knowledge, number transformation, counting, estimation, and number patterns (Berch, 2005; Jordan, Kaplan, Ola’H, & Locuniak, 2006). A person with well-developed number sense has developed a meaning for numbers and their relationships (Briand-Newman et al., 2012) and can often find an answer more quickly using basic computational techniques, rather than using a calculator (Ma, 2010; Schwartz, 2008).

The significance of PUFM as described by Ma (2010) can be aligned to Hattie’s (1999, 2003) notion of surface and deep learning. Hattie suggested that surface learning is more about the content (knowing the ideas and doing what is needed), and deep learning is more about understanding (reacting and extending ideas, and an intention to impose meaning). He concluded that expert teachers are more

successful at both types of learning and exhibit these in their classroom practice. While content knowledge is important in teaching, it is the application of this within pedagogical content knowledge that is of greater importance.

The notion of *Profound Understanding of Emergent Mathematics* (PUEM) was later introduced to mathematics by Davis (2012). PUEM is a category of knowing or a way of being with mathematics, that includes elaborate formal content knowledge, specialised content knowledge, and the content knowledge required in the work of teaching (Davis, 2012; Davis & Renert, 2014). Davis argued that a teacher's disciplinary knowledge is vast, intricate, and evolving. He asserted that no teacher could be expected to be aware of the whole range of understandings included in primary school mathematics. Rather than this knowledge being thought of as a discrete body of foundational knowledge, which is a clear-cut and well-connected set of basics, it is sophisticated and emergent. Therefore, the knowledge needed by teachers is a complex mix associated with various realizations of mathematical concepts. The term *realizations* was mooted by Sfard (2008) and is used to refer to associations a learner might use to make sense of a mathematical construct.

Teaching is complex and teachers cannot be expected to attend to and respond to everything that arises in a lesson (Roche et al., 2016). However, experienced teachers have been found to be more likely to recall students' strategies than novice teachers and more likely to interpret them appropriately (Jacobs, Lamb, & Philipp, 2010; Roche et al., 2016).

2.5 Mathematics for Numeracy

Recent reforms have seen more use of the term numeracy in education (Askew et al., 1997; Bennison, 2015; Coben 2000, 2003; Goos, Dole, & Geiger, 2011; Goos, Geiger, & Dole, 2010; Perso, 2006; Skalicky, 2007). Often the terms mathematics and numeracy are used interchangeably, and yet some argue that there is a difference in meaning (Coben, 2000, 2003; Hogan, 2002; Perso, 2006; Steen, 1999). Mathematics is about the exploration and use of patterns and relationships in quantities, space and time; representing and symbolising these ideas, and eventually learning to abstract and generalise (Ministry of Education, 2007, p. 26). The definition of numeracy was described in Chapter 1 as: "the ability to process,

communicate and interpret numerical information in a variety of contexts” (Askew et al., 1997, p. 6). The development and conceptualisation of the term numeracy has been an important influence on the teaching of mathematics and was first attributed to the United Kingdom’s Crowther report in 1959, where numeracy was described as the mirror image of literacy (Tout & Motteram, 2006). Prior to the 1950s, school mathematics focussed on computation, but with the introduction of computational tools and the associated requirement for higher-order thinking skills, the need for people to be able to transfer their mathematics understandings to everyday life increased and alongside this, use of the term numeracy (Perso, 2006).

The terms mathematical literacy and quantitative literacy are used consistently in the United States when referring to mathematics education (Madison, 2007; Steen, 1999). It has been suggested that the ultimate goal of mathematics education should be the development of numeracy or mathematical literacy. Teaching should stimulate learners to develop the ability to give meaning to numbers and numerical facts in everyday life (Commonwealth of Australia, 2008; Ministry of Education, 2007, 2008c; Perso, 2006; Steen, 1999; Tertiary Education Commission, 2008; Tout & Motteram, 2006). Thus, the concept of numeracy is closely related to that of functional mathematics, where numeracy is often described as applying mathematics in context (Coben, 2003; Tout & Motteram, 2006). While Mathematics and Statistics is the name given to the learning area in the NZC (Ministry of Education, 2007), the term numeracy has more recently become a term used in schools, as well as in adult tertiary education.

Professor Gordon Stanley, who chaired the Australian National Numeracy review, reported that clarification of the numeracy/mathematics distinction is essential if national consistency in curriculum and outcomes is to be achieved (Commonwealth of Australia, 2008, preface vii). Stanley noted that an accepted interpretation of numeracy, which should inform work in education, was that described by the Australian Association of Mathematics Teachers (AAMT) as:

involving the disposition to use, in context, a combination of underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic); mathematical thinking and strategies; general thinking skills; and grounded appreciation of context (AAMT, 1997, p. 15).

Numeracy is often referred to in terms of what it means to be numerate and was defined for New Zealand schools as “the ability and inclination to use mathematics effectively – at home, at work and in the community” (Ministry of Education 2001, p. 1). Perso (2006), similarly defined numeracy as “the disposition and capacity to use mathematics to function effectively and fully at home and in society” (p. 36). She further suggested that although numeracy is seen as being about the mathematics you know, it is also about having a disposition and a confidence to use it. “Knowing some mathematics is essential but not sufficient for numeracy. However, knowing some mathematics must precede the choice to use it or not” (Perso, 2006, p. 37). The New Zealand Tertiary Education Commission (2008) argued similarly, that when it came to solving real-life problems, being able to do mathematics did not necessarily mean being able to use mathematics.

The mathematical skills one requires in life are constantly changing as societal expectations change. Coben (2000) works in the field of adult education and has defined being numerate as:

to be competent, confident, and comfortable with one’s judgement on *whether* to use mathematics in a particular situation and if so, *what* mathematics to use, *how* to do it and what *degree of accuracy* is appropriate, and *what* the answer means in relation to context (p. 35).

Coben stressed that if adults are to become more numerate, then as individuals they need to utilise their many strengths in various areas, in different ways, in different contexts, and for different purposes.

Numeracy builds on number sense, which means that working with numbers and numerical problems is based on a feeling for numbers and insight into number relationships (Dehaene, 2011; Nunes & Bryant, 1996; van den Heuvel-Panhuizen, 2001d). “It encompasses being able to make a sensible choice between using mental arithmetic, estimation, column calculation and algorithms, or using a calculator” (van den Heuvel-Panhuizen, 2001e, p. 246). “Estimation makes an important contribution to attaining the general aim of numeracy” (van den Heuvel-Panhuizen, 2001c, p. 174), with the process of rounding off providing a path to a greater understanding of our number system. Making estimations allows students to develop the ability to deal sensibly with numbers in both their daily life and in a purely mathematical context ((van den Heuvel-Panhuizen, 2001c).

Numeracy is more than mathematics, and may be seen as making sense of mathematics, as it builds bridges between mathematics and the real world. Given the current cultural and social context of schooling, Perso (2006) argued that no longer are educators purely teachers of mathematics, but instead teachers of *mathematics for numeracy*. She suggested that there needs to be a focus on mathematics as the fundamental prerequisite for numeracy for all students throughout their schooling, as they prepare for life skills in the world beyond the classroom (Perso, 2006).

Effective teachers of numeracy are those who help their pupils: “acquire knowledge of, and facility with numbers, number relations and number operations based on an integrated network of understanding, techniques, strategies and application skills; and learn how to apply this knowledge in a variety of contexts” (Askew et al., 1997, p. 10). Askew et al. found that teachers of numeracy utilise teaching approaches that: connect different areas of mathematics and different ideas in the same area, using a variety of words, symbols, and diagrams; use pupils’ descriptions of their methods and reasoning to help establish connections and address misconceptions; emphasise the importance of using mental, written, or electronic methods of calculation, that are most appropriate for the problem at hand; and particularly emphasise the importance of mental skills. This supports Perso’s notion that numeracy is about the mathematics you know and about having a disposition and a confidence to use it.

2.5.1 Numeracy Development Project and Student Achievement

Much of the Numeracy Development Project (NDP) research was centred on progress in student achievement measured against the Number Framework (Ministry of Education, 2008a) in terms of the frequency and sophistication of the numeracy strategies and mathematical knowledge (Young-Loveridge, 2005, 2006, 2007, 2008a, 2009, 2010; Thomas & Tagg, 2006, 2007, 2008). Over the period of the NDP professional development programmes, students made substantial gains in terms of progress on the Framework (Young-Loveridge, 2010). Data from the years 2003, 2005, and 2007 were analysed in terms of effect size. Effect size measures the magnitude of difference between two sets of data. A small effect size ranges from 0.0 to 0.20, a medium effect size from 0.20 to 0.50, and a large effect size is any value above 0.50 (Cohen, 1988). It has been argued that in

education, a score between 0.20 and 0.40 is considered average, and more than 0.40 above average, and greater than 0.60 excellent (Hattie, 2009, p. 17). Young-Loveridge (2010) reported that in terms of the additive domain, between the initial assessment at the beginning of the year and final assessment at the end of the year (in relation to the Number Framework stage of the students involved) average effect size gain of those students in Years 5-6 was more than half a standard deviation (0.57 and 0.52), and in Years 7-8 was close to half a standard deviation (0.48 and 0.51). On the multiplicative domain, average gains by those students in Years 5-6 were well over half a standard deviation (0.66 and 0.61) while at Years 7-8 average gains were a little over half a standard deviation (0.55 and 0.53).

As teachers reflect on student achievement and schools design and review their curriculum, they are expected to refer to the New Zealand Curriculum Mathematics Standards (Ministry of Education, 2009). The Mathematics Standards were introduced into New Zealand primary schools in 2009, the prime purpose being “to promote quality teaching and learning in every New Zealand classroom and success for all students” (Ministry of Education, 2009, p. 6). The Mathematics Standards were designed to help teachers of students in Years 1 to 8, set clear expectations for the mathematics knowledge and strategies students require to achieve Levels 1 to 4 of the New Zealand Curriculum. The Ministry of Education (2009) emphasised that the two documents (The Mathematics Standards and The New Zealand Curriculum), were designed to complement each other. The NZC drives the teaching, while The Mathematics Standards support teachers to assess their students’ achievement, in relation to the curriculum. It is intended that The Standards would assist teachers and schools to monitor student progress and the success of teaching and learning programmes. They support decisions about next steps for learning with students, enabling teachers to make judgements about their students’ progress, so that clear learning goals can be set (Ministry of Education, 2009).

Research undertaken by Young-Loveridge (2009, 2010) compared the achievement of students with the Ministry of Education curriculum expectations as indicated in the Mathematics Standards (Ministry of Education, 2009, 2010). Findings indicated that at the end of Years 6-8, student achievement was well below the expectations set by the Ministry (e.g., less than 50% of the Year 6

students were at, or above, Stage 6 on the additive domain of the Number Framework, Ministry of Education, 2010), which is well short of the majority expected to have reached the Level 3 objective. Likewise, less than half of the Year 7 students were at Stage 6 on the additive domain, while over a third were categorised as cause for concern.

2.5.2 Developing Mathematical Discussions

The New Zealand curriculum document, refers to the potential value of outcomes for student learning when interacting with each other (Ministry of Education, 2007), which aligns with Vygotsky's socio-cultural theory. A classroom's norms for determining whose turn it is to talk can follow different kinds of patterns and actions. One norm established during classroom discussions is the *teacher initiation-student response-teacher evaluation* (IRE) model (Flores, 2010), or the IRF model where the 'F' stands for feedback (Cazden, 2001). However, teachers who are more inclusive of all students in conversations create "negotiation-rich opportunities" (Waring, 2009, p. 796). It has been argued that in mathematics classrooms both explanation and justification have important roles, as students develop arguments during discussion (Ball, 1993; Forman & McPhail, 1993; Goos, 2004; Hunter, 2005, 2006, 2010; Lampert, 1990; Stein, Engle, Smith, & Hughes, 2008; Whitenack & Yackel, 2002; Wood et al., 2006; Yackel & Cobb, 1996). A key part of Piaget's notion of constructivism, is contained within the forthcoming expectation of challenge, or disagreement from listening to group members. The resulting discussion is what extends explanation of challenge to justification (Bruner, 1990; Hunter, 2006). At times, a student may explain an idea to clarify their own thinking to others, while at other times they may make an argument to validate their own thinking or to justify an activity. While explaining and justifying are important aspects of reasoning about mathematical ideas, sense-making evolves from questioning and challenging the thinking of others (Hunter, 2006, 2012).

The idea of making mathematics problematic for learners is well supported by research (Boaler, 2003; Fennema, Franke, Carpenter, & Carey, 1993; Hiebert, et al., 1996; Hunter, 2010). Hiebert et al. claimed that mathematics instruction should begin with problems, dilemmas, and question for students, allowing the students to wonder why things are and to search for solutions. The term

problematic does not mean that students become frustrated and find the situation difficult. Instead, it has been argued that students should be encouraged to problematize the mathematics they study by constructing a community of mathematical inquiry. The community of mathematical inquiry allows students to solve problems and explain, and examine their explanations, leading to the construction of understanding (Hiebert, et al., 1996; Hunter, 2006, 2010; Stein et al., 2008). From a functional perspective, understanding means participating in a community of learners and allowing the classroom activity to involve participation and discussion. These interactive processes may also be termed *visible thinking* (Hull et al., 2011). With visible thinking there is a heightened level of awareness by a participant of their own thoughts and thought processes, as well as those of the individuals with whom they are working.

The inquiry-based approach to group work when teaching numeracy and promoted in the NDP, is aligned to power-sharing interactions between teachers and students (Higgins, 2005b, Higgins & Averill, 2010). Mathematical discussions are now considered a key component of mathematical inquiry and effective mathematics teaching (Kazemi & Stipek, 2001; Hunter, 2009, 2010, 2012; Stein et al., 2008). Effective teachers are responsive, in that they constantly elicit, monitor, and respond spontaneously to their students' thinking (Franke & Kazemi, 2001). The role of the teacher in managing discourse was also emphasised in Fraivillig et al.'s (1999) Advancing Children's Thinking (ACT) model, which differentiates between eliciting, supporting, and extending concepts, in response to children's actions and explanations. The teacher has the responsibility for developing a community where students share ideas about mathematics, while searching for solutions (Hiebert et al., 1996), and develops and builds on individual and collective sense making of the students rather than merely acknowledging they are correct (Stein et al., 2008). Such discussions support students' learning of mathematical discourse practices, so that they can be guided and encouraged to construct their own mathematical ideas. The teacher needs to guide the students in discussion by eliciting methods of solution and analysing their features (Fraivillig et al., 1999; Hiebert et al., 1996).

Reasoned argument within mathematics education has implications for the knowledge teachers require and their role in a child's learning process. Teachers

need to be skilled in supporting students to describe their reason for a given answer in class discussion, and to do this effectively need to scaffold students' use of mathematical language and knowledge (Ball, 2002). The teachers also need to be able to understand and interpret their students' reasoning and be in a position to support and extend their thinking (Fraivillig et al., 1999). The more frequently teachers ask students to describe their solution strategies and explain their responses, the more students engage in class, and the higher their gains are in mathematics achievement (Hiebert & Wearne, 1993).

2.6 Teaching the Multiplicative Domain

Thinking multiplicatively encompasses many different mathematical ideas and according to the support material for New Zealand's NDP involves:

Constructing and manipulating factors (the numbers that are multiplied) in response to a variety of contexts...having key items of knowledge (for example basic facts), [and] deriving from known facts using the properties of multiplication and division [commutative, associative, distributive, inverse] (Ministry of Education, 2008f, p. 3).

Being able to recite and/or recall basic facts is insufficient to be a multiplicative thinker. It requires the capability to be able to work flexibly with concepts, strategies, and representations of multiplication and division in a wide range of contexts (Clark & Kamii, 1996; Siemon, Breed, & Virgona, 2005). Multiplicative thinking is considered a big idea of mathematics that underpins mathematical thinking beyond primary school years (Hurst & Hurrell, 2014; Siemon, Bleckley, & Neal, 2012). It is characterised by: a capacity to work flexibly and efficiently with an extended range of numbers including larger whole numbers, decimals, common fractions, ratio, and percent; an ability to solve a range of problems involving multiplication and division; and the means to communicate this effectively in a variety of ways (Siemon et al., 2005). Multiplicative thinking can be based on varying processes which can include: grouping; number line hopping; number line stretching or compressing; folding and layering; branching; grids or arrays; area, volume, and dimension; steady rise or slope; number line rotation (for integer multiplication); and proportional reasoning (Davis, 2008). Understanding multiplication and division is considered to be central to knowing mathematics and to problem solving in other mathematical areas (Carpenter, Fennema, & Franke, 1996; Carpenter, Fennema, Franke, Levi, & Empson, 2015), including algebra (Baek, 2006).

Learning mathematics in primary school includes developing an understanding about the connections and distinctions between numbers, quantities, and relations (Nunes, Bryant, Sylva, & Barros, 2009). A quantity can be represented by a number, although it does not necessarily need to be measured and represented by a number: for example, two people can compare their height with each other, without any numbers (Nunes et al., 2009). Understanding the connections between quantities and relations may be seen as part-whole relations, or additive relations. Additive reasoning is about occasions in which objects (or sets of objects) are put together or separated (Carpenter et al., 2015; Nunes & Bryant, 1996; Nunes et al., 2009). For example, when comparing the number (quantity) of books one child has to another child, if there is a difference of 12 this is seen additively. While there are links between additive and multiplicative reasoning, as multiplication and division sums can be solved through repeated addition and repeated subtraction (Carpenter et al., 2015), there may also be a one-to-many relationship between two sets (Nunes & Bryant, 1996). For example, one dog has four legs (1-to-4) or one child has two eyes (1-to-2). This reasoning leads to the idea of replication (Kieren, 1994; Greer, 1994) in the understanding of ratio, so that the one-to-many correspondence is maintained (Nunes & Bryant, 1996). For example, in a set where there are three dogs there are 12 legs, the 1-to-4 ratio has been replicated three times.

Six problem structures of practical situations that involve multiplication or division of whole numbers have been identified: equivalent groups (3 cars have 4 people in each); multiplicative comparison (Laura has 3 times as many books as Luke); rectangular area or arrays (3 rows of 4 children); rate (a car travels for 3 hours at an average speed of 80km per hour); ratio (a ratio of 3 boys to 4 girls in the class); and Cartesian product (the number of different possible combinations of 3 shirts and 4 skirts), (Baek, 2006; Nunes & Bryant, 1996). It is important young students are exposed to the different problem types, as it is the understanding of abstract relationships between numbers that will encourage the use of efficient approaches.

To be able to use multiplication and division to solve problems, students should know most of their times table facts, and be able to derive new ones from known facts (Suggate, Davis, & Goulding, 2010). Some students continue to use counting

facts to solve problems for some time and it should not be assumed that they know facts from memory simply because they obtain answers quickly (Carpenter et al., 2015). The most important aspect when learning multiplication facts is the way in which each one is related to others (Anghileri, 2006; Carpenter et al., 2015; Nunes & Bryant, 1996). Students need firstly to understand how these facts can be obtained from first principles (repeated addition or skip counting), secondly to recognise patterns, and thirdly to understand the commutative nature of multiplication (Baroody, Bajwa, & Eiland, 2009; Carpenter et al., 2015; Suggate et al., 2010). When these ideas are consolidated through repeated practice, the facts become known by heart, often referred to as “basic facts”, and are then available for rapid recall.

Number facts, or basic facts, are learned at a recall level over a much longer period of time than has been previously assumed, and develop through the experience of solving problems and reflecting on strategies (Carpenter et al., 2015; Perso, 2007). Recent research of Hurst and Hurrell (2016) indicated that students are often taught certain procedures for solving multiplication problems (including use of algorithms, using properties of multiplication including commutativity, and the inverse relationship) before they have developed a conceptual understanding of the mathematics involved. The students struggled to explain why they carried out certain strategies in their problem solving beyond a procedural level (such as switching the numbers for the commutative property).

The initial idea developed with young students is generally the groups of idea (Baek, 2006; Davis, 2008; Greer, 1994; Siemon et al., 2005; Nunes & Bryant, 1996), and the acquisition of an equal-grouping structure is at the core of multiplicative thinking (Clark & Kamii, 1996; Sullivan, Clarke, Cheeseman, & Mulligan, 2001). Mulligan and Mitchelmore (1997) found that counting strategies were integrated into repeated addition and subtraction processes, and then later generalized into multiplication and division through the use of the rectangle. Davis (2008) believed that “the most flexible and robust interpretation of multiplication is based on a rectangle” (p. 88). Rectangular arrays are important images associated with the understanding of multiplication as the image portrays a set of objects arranged in rows and columns in the shape of a rectangle (Carpenter et al., 2015; Davis, 2008; Haylock, 2010; Young-Loveridge & Mills, 2009a,

2009b, 2010). The rectangular array representation shows the link to replicated units utilised in repeated addition and skip counting, and to finding the total number of objects within the rectangle. The array is an effective model for understanding multiplication of larger numbers (Young-Loveridge & Mills, 2009a) and is also closely linked to calculating the area of a rectangular region. Students might draw a picture or construct an array model of the rectangle and then count each of the squares, thus showing the relationship between the array and area. The area-based model also shows how and why the algorithm for whole-number multiplication works and can later be extended to multiplication of decimals (Davis, 2008).

The mathematical property relating to the commutative law of multiplication ($a \times b = b \times a$) is the generalization that the multiplier and multiplicand can be interchanged without changing the result (Baroody, 1999; Carpenter et al., 2015; Steffe, 1994). However, the commutative property of multiplication is harder for children to understand than the commutative property of addition (Carpenter et al., 2015). Unlike addition, in multiplication representation of the two different forms of the equation is quite different (Carpenter et al., 2015; Steffe, 1994; Suggate et al., 2010) and children do not immediately understand that that the two numbers can be interchanged to solve problems (Carpenter et al., 2015). For example, in the context of equal groups problems, it is not always obvious why 30 groups of 2 items has the same number of items as 2 groups of 30 items. The understanding of commutativity in multiplication involves partitioning and recombining the groups of objects in a different way (Steffe, 1994). The understanding of the commutative rule is also important in establishing the flexibility needed to find the most efficient strategy for any given problem, especially at a later stage in mathematics (Anghileri, 2006; Baroody, 1985, 1999; Carpenter et al., 2015).

In English-medium classes in New Zealand schools, as with most other English-speaking systems, the first number in a multiplication expression represents the multiplier and the second number the multiplicand. Hence: in the expression 3×6 , the number 3 shows the number of sets (multiplier), while the number 6 is the size of each set (multiplicand). This is the everyday interpretation of the multiplication symbol as times, thus 3×6 could be interpreted as three times 6, or three groups of 6, or 6 replicated 3 times ($6 + 6 + 6$). It is the elementary idea interpreted by

many, that multiplication means so many sets of, or groups of (Anghileri, 2006; Haylock, 2010; Skemp, 1989; Suggate et al., 2010) and is the interpretation utilised throughout this study. However, this convention is not necessarily shared across all countries (Haylock, 2010; Suggate et al., 2010), or across all cultures or languages (Ministry of Education, 2008f). For example, in many Asian countries and in te reo Māori classes in New Zealand schools, 3×6 is regarded as 3 replicated 6 times ($3 + 3 + 3 + 3 + 3 + 3$) (Ministry of Education, 2008f).

A range of different strategies for solving multiplication and division problems is taught in many classrooms in New Zealand schools (Crooks, Smith, & Flockton, 2010; Ministry of Education, 2008f). The New Zealand National Education Monitoring Project (NEMP) assessment results indicated that between 2005 and 2009, there was a decline in performance for Year 8 students on multiplication problems (Crooks et al., 2010). Crooks et al. reported that strategy explanations when solving multiplication problems showed a major decline in vertical, algorithmic strategies, and an increase in horizontal strategies. NEMP results also indicated a lack of understanding of division problems, in particular those with remainders.

When teaching division, students need to realise that when sharing takes place the number of objects to be shared out (dividend) is not necessarily a multiple of the number between which the sharing is to take place (divisor), and so there is a remainder. How the remainder is dealt with depends on the context of the problem (Anghileri, 1999; Carpenter et al., 2015). For example, should 16 pencils be shared equally between 5 people, or 25 children put into 6 cars for a school trip, the remaining pencils or people would not be cut into smaller bits. Research has shown that students often complete the calculation to such problems correctly, but have difficulty giving a solution to the problem that is consistent with the meaning of the problem (Buys, 2001; Lamberg & Wiest, 2012; Roche et al., 2016; Silver, Shapiro, & Dentsch, 1993; Suggate et al., 2010). Understanding that the remainder of a problem can sometimes be shared out equally, leads to the world of fractions (Buys, 2001). Whole-to-part comparisons can be made that involve comparing a whole object to part of an object (continuous) or set of objects (discrete). Examples of this may be where 25 metres of rope can be cut into 4

equal lengths (each will be 6.25 metres long), or 5 pancakes can be shared out equally between 4 people (each person gets 6 and one-quarter piece).

In multiplication and division problems, there are three elements to be considered: the size of the whole, the number of parts, and the size of the parts (Nunes & Bryant, 1996). In a multiplication problem, it is generally the product (the total number, or the size of the whole) that is missing. If either the multiplier (the number of parts) or multiplicand (the size of the parts) is missing, then the problem involves division. This leads to understanding two different problem types in division: the sharing model (partitive) and the grouping model (quotitive) (Anghileri, 2006; Carpenter et al., 2015; Ministry of Education, 2008a; Mousley, 2000; Nunes & Bryant, 1996; Roche & Clarke, 2009; Roche et al., 2016; Small, 2013; Suggate et al., 2010). In the partitive model, the divisor is a number of parts or sub-collections and the quotient is the size of each part. For example, in $15 \div 3 = 5$, 15 objects are shared into 3 equal groups and there are 5 in each group. In the quotitive model, the divisor shows the size of each part or sub-collections and the quotient is the number of equal-sized groups. For example, in $15 \div 3 = 5$, 15 is divided into groups of 3, and 5 is the number of equal-sized groups. It is important that students are exposed to both types of division problems (Anghileri, 1999; Carpenter et al., 2015; Ministry of Education, 2008f; Mousley, 2000; Nunes & Bryant, 1996; Roche & Clarke, 2009; Roche et al., 2016).

Research undertaken in Australia has shown that teachers' understanding of division is poor (Roche & Clarke, 2009; Roche et al., 2016). Indications from Roche and Clarke's research, were that a little over half of the teachers, 47 out of 92 (51%), correctly represented partitive division, while a total of 23 (25%) correctly represented quotitive division. Of the 23 who represented quotitive division correctly, 10 (11%) could describe quotition but could not transfer this understanding to division with decimals. In order to help students, solve division accurately there is a need for teachers to spend time modelling with materials and pictures and to consider a range of word problems to help differentiate between partitive and quotitive problem types (Carpenter et al., 2015; Mulligan & Mitchelmore, 1997, 2013; Roche & Clarke, 2009; Roche et al., 2016). With the support of appropriately designed tasks, young students have the cognitive capacity to explain and reason solutions to division problems (Clarke, et al., 2006).

However, choosing appropriate tasks can be difficult (Chick & Harris, 2007) and modifying a task to make it simpler (or more difficult), while still illustrating the general principle required, is a critical issue for teaching (Ball, 2000b). Designing, selecting, and implementing appropriate tasks is related to a teacher's PCK and content knowledge (Charalambous, 2010).

There are many language issues and misconceptions associated with the teaching and learning of multiplication and/or division. One language issue arises in the difficulty associated with the wording of problems given to children (Nunes & Bryant, 1996). For example, the wording of a scenario in quotitive and partitive division problems, along with the context given in problems with remainders, will dictate the construct of the particular problem (Greer, 1994; Roche et al., 2016). Understanding of division in the quotitive form with whole numbers becomes important with fractions. For example, understanding $20 \div 4$ can mean 'how many groups of 4 are in 20' carries over into expressions such as $2 \div \frac{1}{4}$, which may be interpreted as, "How many groups of one quarter (usually said as how many quarters) are in two?" Similarly, understanding the language associated with the multiplication symbol (\times), in relation to use of the term "of" used in "groups of" or "sets of", becomes important when understanding multiplication of fractions. With multiplication of fractions, the "of" idea is needed to read sentences such as $\frac{1}{4} \times 8$ as "one quarter of eight" (Greer, 1994; Suggate et al., 2010). There is often a need to construct the representation of the situation described in the problem to understand a strategy required to solve the problem (Greer, 1994).

Much of the teaching in the multiplicative domain, which later extends into the development of algebraic thinking, relies heavily on recognition and a strong understanding and appreciation of basic pattern and structure of number (Haylock, 2010; Mulligan & Mitchelmore, 1997, 2009, 2013; Papic, Mulligan, & Mitchelmore, 2011; Sophian, 2007). Mulligan and Mitchelmore (2013) define pattern as "any predictable regularity involving number, space, or measure, and structure as the way in which the various elements are organised and related" (p. 30). Recent research has shown that appreciation of structure and pattern may be one of the main differences between high and low achievers in mathematics (Mulligan, 2013; Mulligan & Mitchelmore, 1997, 2013). Early experiences with number operations provide important links across many structures that ultimately

underpin children’s understanding of mathematics (Anghileri, 2006) and allow children to later use patterns to generalise rules (Hansen, 2005; Schoenfeld, 2011). Patterns such as subitized patterns, one-more patterns, numerical patterns, spatial patterns, and array structures, are typical foundational patterns used in primary school mathematics (Wu, 2007) and contribute towards the understanding related to multiplicative thinking.

2.7 Teaching the Proportional Domain

Multiplicative thinking is the basis of proportional thinking and reasoning (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007; Van Dooren, de Bock, & Verschaffel, 2010) and a necessary prerequisite for understanding algebra, ratio, and rate, interpreting statistical and probability situations, and understanding and reading scale (Harel & Confrey, 1994; Lamon, 1996; Siemon et al., 2005). Proportional reasoning means having the ability to understand the multiplicative relationship inherent in situations of comparison (Behr et al., 1992). Lamon (1993) expands on this notion when she writes, “Proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another” (p. 41). The term *deliberate* is used to clarify that proportional reasoning is more about the use of number sense than formal, procedural solving of proportions.

It is estimated that more than half of the adult population cannot be viewed as proportional thinkers (Lamon, 1993). In the early years of schooling, proportional reasoning begins with multiplication and division and is developed through the study of fractions and decimals, and later extends to ratio and proportion (Chick, 2010; Lamon, 2006, 2007; Sowder, 2007). Ratios describe a part-to-part or a part-to-whole comparison where equal parts of one “thing” are combined with parts of a different “thing” (Lamon, 2006) and build on fractional relationships understood during early fraction and decimal learning. An understanding of fractions provides a foundation for learning in mathematics, including place value, measurement, ratios, proportions, scale, algebra, probability, percentages and decimals. It is applied in real-life contexts including reading maps, calculating the best deal of purchase when shopping, increasing and decreasing the size of mixtures from recipes when baking, enlarging documents on the photocopier, predicting outcomes, and fair sharing (Dole, 2010).

Whilst proportional reasoning extends beyond the knowledge of fractions and decimals, in this study teaching in the proportional domain focused on fractions and decimals. Fractions is a topic that many teachers find difficult to understand and teach (Chick, 2010; Clarke, Roche, & Mitchell, 2007; Gould, 2005a, 2005b; Post, Cramer, Behr, Lesh, & Harel, 1993; Smith, 2002; Watson, Callingham, & Donne, 2008; Way, Bobis, & Anderson, 2015). Consequently, many students struggle with learning basic fraction concepts at the primary school level (Anthony & Walshaw, 2007; Davis, Hunting, & Pearn, 1993; Young-Loveridge, Taylor, Hāwera, & Sharma, 2007), which results in proportional reasoning causing difficulty for many middle-school students (Lamon, 2007, Watson et al., 2008). Difficulty with fractions was also exemplified in the National Education Monitoring Project assessment results, which indicated that both Years 4 and 8 students scored poorly on tasks involving fractions, especially fractions other than halves and quarters (Crooks et al., 2010).

As discussed earlier, an important consideration for teachers is ‘connectivity’ and a central feature of learning with understanding is that knowledge is connected rather than consisting of bits of isolated information. As many of the mathematical topics covered are complex, not only do they require understanding of prior domains, but they also need to be able to be related to each other (Lamon, 1994; Pitkethly & Hunting, 1996). Interconnections make knowledge less likely to be forgotten, to be accessed in many different ways, and to be used for solving unfamiliar problems (Lesh et al., 1987). In general, sense is made of new ideas by relating them to something which is already known. Difficulties with fractions are greatly reduced if instruction involves providing students with opportunity to build on known concepts, as they engage in mathematical activities that promote understanding (Olive, 2001). To connect understanding of fraction magnitude with whole number properties, students must initially have a foundation of multiplicative thinking. Students extend this knowledge to think relatively, and unitise and understand the ratio between the numerator and the denominator (Boyer & Levine, 2012; Gould, 2005b; Lamon, 1994).

Teaching fractions and proportionality, includes the exploration of concepts not accessible with whole numbers (Siegler, Fazio, Bailey, & Zhou, 2013) and at the same time contradicts many characteristics of previously learnt whole number

properties (Bailey, Siegler, & Geary, 2014, Siegler et al., 2013; Smith, 2002). This means that operations with fractions are difficult for many because the rules associated with whole numbers do not work with fractional numbers (Bailey et al., 2014; Gould, 2005a; Ma, 2010, Roche, 2005; Smith, 2002). When whole number reasoning is applied to fractions, it is known as *whole number bias* (Ni & Zhou, 2005) and common misconceptions occur, such as the *add across error* (Siegler, Thompson, & Schneider, 2011; Young-Loveridge et al., 2007). The add across error occurs when the numerator and denominator are treated as discrete whole numbers and students will incorrectly calculate expressions such as $\frac{1}{3} + \frac{2}{5}$ as $\frac{3}{8}$, their reasoning that $2 + 1 = 3$ and $3 + 5 = 8$ (Siegler et al., 2011). This misconception disregards the basic number property that the sum of two positive numbers must be greater than either addend. However, much of the confusion in teaching and learning fractions appears to arise from the many different interpretations and representations (Clarke et al., 2007). Students might overcome some of this difficulty, if they are encouraged to represent the mathematical knowledge they learn in various ways using elements such as spoken language, written language, manipulatives, pictures, and real-world situations (Chick, 2015; Lesh, Landau, & Hamilton, 1983; Lesh, Post, & Behr, 1987; Miheso-O'Connor, 2011).

There appears to be a difference between teachers' conceptual and procedural understanding of fractions (Way et al., 2015), and being able to connect intuitive knowledge and familiar contexts with symbols and formal classroom instruction (Hasemann, 1981; Smith, 2002). Conceptual knowledge of fractions includes, "understanding of the properties of fractions: their magnitudes, principles, and notations" (Siegler et al., 2013, p. 14). This knowledge underpins procedural knowledge, which Siegler et al. describe as including, "fluency with the four fraction arithmetic operations" (p. 14). Insufficient conceptual understanding of mathematics concepts and relationships increases a teacher's reliance on procedural knowledge (Way et al., 2015).

There are correlations between visualisation skills and students' abilities to understand mathematics, as visual models provide a scaffold for students to develop an image and visualisation of mathematical concepts. This correlation assists students in problem solving situations, as there is a link between the use of

static visual models and the success of written problems (Anderson-Pence, Moyer-Packenham, Westenskow, Shumway, & Jordan, 2014). Anderson-Pence et al. describe a static visual model as, “a still picture that is either printed or drawn on a page, to represent mathematical concepts” (p. 3), and concluded that the models students experience, and the ability to interpret and represent visual static models of fractions, is a precursor to understanding proportional concepts. Traditionally, instruction in fractions has not encouraged meaningful representation, but when students have been taught for understanding, they are later able to solve problems involving more complex fractions (Hansen, 2005; Suggate et al., 2010). Students need to visualise mathematical concepts and engage in real-world mathematics in order to develop a meaning of mathematics and apply new knowledge to a range of problem solving situations (Anderson-Pence et al., 2014; van Garderen, 2006).

2.7.1. Constructs of Rational Number

A rational number is a number that can be written as a ratio. That means it can be written as a fraction, in which both the numerator and the denominator are whole numbers (Kieren, 1976). Kieren originally identified seven sub-constructs of rational number: fractions, decimal fractions, equivalent fractions, ratios, multiplicative operators, quotients, and measures. Kieren identified that the part-whole construct was the foundation to students’ learning about rational number and later revised his seven sub-construct to four, based on the part-whole notion (Kieren, 1980; 1988): “multiplicative operators” (a fraction is used to act on another number e.g., $\frac{1}{3}$ of 12 or $\frac{1}{3} \times 12$); “quotients” (answers to sharing division problems e.g., 2 pizzas shared among 3 students gives $\frac{2}{3}$ each); “measures” (how many times a fraction fits into a given fraction or whole number e.g., $\frac{3}{4}$ is the same measure as 2 lots of $\frac{3}{8}$); and “ratios” (relationship between two things of the same attribute e.g., 1 litre of cordial to 4 litres of water) and “rates” (relationship between 2 different measurements e.g., 90km per hour [distance and time]). Kieren emphasised that the sub-constructs were not complete, or to be viewed in isolation or independent of each other, and there was the need for students to integrate the sub-constructs as they combined to create a generalisation of rational number.

Young students' fraction knowledge often begins with the teaching of the part-whole concept where, for example, it is understood that $\frac{3}{4}$ is three parts of a whole that has been partitioned into four equal parts (Steffe & Olive, 2010). However, research on fraction learning has identified the need for school experiences to support student conceptions beyond the part-whole construct (Steffe & Olive, 2010; Way et al., 2015). Therefore, it has been suggested that Kieren's sub-constructs of fractions need to be taught in order of conceptual challenge, for long-term understanding to occur (Hansen, 2005). Both Hansen and Kieren (1980) make explicit the core conceptual structure people use when applying rational number thinking to given situations. Each of these types of fractional understanding brings with it misunderstandings which are frequently seen in use by students, and even more disturbingly as the research of Ward and Thomas (2007, 2009), and Young-Loveridge (2008b) indicated, by some of their teachers.

The suggested order of conceptual challenge for teaching fractions advocated by Hansen (2005), in relation to Kieren's (1980) four sub-constructs is:

1. *Fractions as part of a whole*: A child requires experiences of dividing an object into equal parts where the parts can be directly compared to each other. The key idea of a whole unit being equally divided into smaller parts relates to Kieren's notion of rational number as being based on part-whole relationships. Depending on the type of unit being sub-divided, parts must be equal in number, length, area, or volume (Kieren, 1988; Lamon, 2007). Where parts are equal in number, representations can be described as discrete; where parts are equal in area or length, representations can be described as continuous (Lamon, 2006). Understanding the continuous model is important, with teachers giving students a number of different shapes for further variation, to enable them to understand the value of a fraction and its fractional unit (Ma, 2010). In some instances, when a child is asked to divide a semicircle into quarters, or a circle into thirds, they may divide them into four pieces, or three pieces, with no regards to equal-sized portions (Hansen, 2005; Ma, 2010; Smith, 2002). The child may have been used to dividing squares, rectangles, and circles into pieces, so may incorrectly generalise that the same method works for all shapes (Ma, 2010).

Fractions as part of a whole includes Kieren's (1976) original sub-construct of fraction equivalence. Fraction equivalence has often been reduced to the mastery

of the rule, “multiply or divide the numerator and denominator of a fraction by the same number” (Ni, 2001, p. 413). As a result of rule-based learning, many students are unable to identify, construct, and understand equivalent fractions (Behr, Wachsmuth, Post, & Lesh, 1984; Gould, 2005b; Pearn 2003; Siemon, Virgona, & Corneille, 2001; Way et al., 2015; Wong, 2010; Wong & Evans, 2007). To assist teachers in advancing students’ understanding of fraction equivalence, Wong developed a learning pathway. The pathway was built on research, which concluded that students with a conceptual understanding of fraction equivalence have an integrated knowledge. This knowledge includes understanding that: equivalent fractions can be constructed from manipulatives or pictorial representations by repartitioning or chunking; equivalent fractions can be constructed using symbolic notation; and a fraction quantity is part of an equivalent group in which all fraction numerals represent the same quantity (Siegler et al., 2013; Wong, 2010).

2. *Fractions as part of a set:* Following on from the continuous model of fractions, students need to become comfortable reasoning and talking about parts of discrete quantities (collections of objects) as fractions (Hansen, 2005; Smith, 2002). This idea is built on the understanding that you can have three-quarters of a bag of 20 marbles, just as you can have three-quarters of a cake. A whole is not always represented by a single object such as one cake, or a rectangle, and may be a collection of objects, such as a bag of apples (or marbles), or a class of students. Misconceptions can occur when students fail to understand the concept of whole and do not recognise the complete set of objects as the whole unit (Ma, 2010). This idea is similar to Kieren’s (1980) construct of fractions as *multiplicative operators* where a fraction is used to operate on another: in this instance $\frac{3}{4}$ of 20 (marbles) or $\frac{3}{4} \times 20$.

3. *Fractions as numbers on a number line:* When students are introduced to fractions, they usually begin with unit fractions such as one-half ($\frac{1}{2}$), or one-quarter ($\frac{1}{4}$) (Hansen, 2005; Ministry of Education, 2008a). The student may then believe that a fraction is a number smaller than one and always comes between the numbers 0 and 1, and when the number line is extended to numbers greater than one, difficulty may arise (Hansen, 2005). There may be confusion between

knowing where to place the numeral $\frac{1}{2}$ (one-half), and where to find one half of a given quantity or number (place something half-way along a given part of the number line e.g., half-way between 0 and 4). This misconception is connected to understanding the difference between counting-based understanding (in this instance recognising $\frac{1}{2}$ as a number on the number line) and collections-based conception ($\frac{1}{2}$ of a number of objects, or in this instance half of the measure of the space between 0 and 4) (Yackel, 2001). This idea also relates to Kieren's (1980) construct of fractions as a measure of a quantity, relative to one whole unit. Lamon (1996) explained that continual partitioning allows measurement with precision. The measurements are also known as "points", and the number line provides a model to demonstrate the points. The measure interpretation is different from other constructs in that the number of equal parts in a unit can vary depending on how many times you partition it (Lamon, 1996).

4. *Fractions as operators*: This is where a fraction can be used as an operator to shrink and stretch a number such as $\frac{3}{4} \times 12 = 9$, or $\frac{3}{4}$ of $12 = 9$ (Clarke et al., 2007), and is where the link between division and fractions is strongest (Clarke et al., 2007; Hansen, 2005; Lamon, 2006, 2007; Suggate et al., 2010). For example, if 20 marbles are divided equally between 4 children, each child gets 5 marbles, or each child's share is $\frac{1}{4}$ of the total marbles. This means that $20 \div 4 = 5$, or $\frac{1}{4}$ of 20 is 5. Hansen's progressions do not make a clear distinction between fractions as multiplicative operators and fractions as quotients as identified in Kieren's constructs, although the link with division is very clear in this instance. The fraction ($\frac{1}{4}$) behaves like an operator (the $\frac{1}{4}$ operates on the number 20) while at the same time may be seen as $20 \div 4 = 5$. A fraction may also be seen as equivalent to the result of division, for example $\frac{1}{4} = 1 \div 4$ (Ma, 2010). When one whole object is divided into 4 equal pieces, the result will be $\frac{1}{4}$.

5. *Fractions as ratios*: This is when numbers are used to compare one quantity with another (Hansen, 2005). Ratio is a multiplicative relationship, although a child who is at an early stage of number understanding may see it as an additive one. For example, if in a group of students at school there were 3 girls and 6 boys, the ratio of girls to boys is 1 to 2 (usually written as 1:2). However, Hansen

suggested that a student drawing on early knowledge of mathematics when comparing the two sets may say there are fewer girls than boys and the difference is 3. The relationship is seen in this instance as one of difference between the numbers, rather than one of ratio, or proportionality.

The above models, or views of fractions presented by Hansen (2005) are based on progressively increasing levels of complexity and are required to carry out operations on, and with fractions. Although these constructs can be considered separately, they are unified by three big ideas: identification of the unit, partitioning, and the notion of quantity (Carpenter et al., 1996). Equi-partitioning allows a student to share or partition a whole into a number of equal parts to create unit fractions, such as one-half, or one-quarter (Steffe, 2004). Iterative fractions result from the repetition of unit fractions, which produces a new fraction, or quantity (Olive & Steffe, 2002). For example, one-quarter can be iterated four times to create four-quarters or a whole. This precedes understanding of the multiplicative relationship between part and whole.

The operations of addition, subtraction, multiplication, and division of fractions, must ultimately be understood (Suggate et al., 2010). Addition and/or subtraction of fractions with un-like denominators cannot take place until equivalence is used to change one or both of the fractions, to make the denominators the same. Multiplying fractions appears to be much simpler than addition, but understanding the concept is not so easy. Both multiplication and division of fractions depend on understanding the concepts that have been taught with whole numbers prior to fractional representation (Ma, 2010; Suggate et al., 2010).

Decimal fractions

Fractions may also be represented in decimal form. Decimal numbers are an extension of the whole-number place-value system and are symbolic representations of units less than one (Hansen, 2005). The concepts underpinning decimals are the similar to the constructs and models outlined above for underpinning fractions. These include: part of a whole; part of a set; numbers on a number line; decimals as operators; and quotients. Student errors in the use of decimals are likely to have originated in misunderstanding of place value and fractions (Hansen, 2010; Ma, 2010). Misconceptions students have related to decimals include: the more digits there are to the right of the decimal point, the

larger the number (based on whole number thinking, e.g., 3.2468 is larger than 3.28); and conversely the fewer digits to the right of the decimal point, the smaller the number (e.g., 3.5 is less than 3.468); misunderstanding between place value columns and the number line; decimal numbers are negative numbers (less than zero e.g., 0.25 must be less than 0); the decimal point separates two numbers (e.g., 3.5 is 3 r5, or 3.50pm when used in time to separate hours and minutes), decimals are finite; and zeroes do not matter (Ma, 2010; Steinle & Stacey, 1998, 2004).

Through experience with decimals, students realise that the more digits there are after the decimal point, the more precise the number (Ma, 2010). Students also learn to realise that decimals can be rounded (to nearest tenth, or hundredth, etc.) for calculations. However, students must also be taught when it is inappropriate to round decimals in the real world, for example: in medical contexts this might cause risk to patients, or in the building industry incorrectly calibrated machines could damage equipment (Coben et al., 2010). Research undertaken by Coben et al. concluded that it is important students understand estimation as part of decimal place value, to recognise when errors are made in calculations.

2.7.2 New Zealand Teachers' Knowledge in the Multiplicative and Proportional Domains

A teacher's knowledge of mathematics makes an important contribution to his/her effectiveness as a teacher (Ball et al., 2005; Goya, 2006; Ma, 2010; Shulman, 1986). One of the goals of mathematics instruction is to enhance student learning (Baumert et al., 2010) and help students understand the structure of mathematics (Lambdin & Walcott, 2007). Coming to understand the underlying structure of the mathematics is vitally important for effective teaching and learning, as the difference between high and low achievers in mathematics may be attributed to their appreciation of structure and pattern (Bobis, Mulligan, & Lowrie, 2008; Mulligan & Mitchelmore, 1997). Multiplicative reasoning is complex, multifaceted and a pre-requisite for fractional thinking. Being in a position to respond effectively to students' fractional thinking relies on an understanding of the development of conceptual knowledge of fractions (understanding fraction properties, magnitude, and notations) and procedural knowledge of fractions (fluency in the four operations) (Siegler et al., 2013).

Research undertaken alongside the NDP showed that PCK of New Zealand teachers was not at a very advanced level (Ward & Thomas, 2007; Ward, 2009; Young-Loveridge & Mills, 2009a). There are many challenges for teachers to understand the many aspects of multiplicative thinking, in order to support students' conceptual understanding. Young-Loveridge and Mills identified a weakness in teachers' understanding related to number properties that underpin multiplication (commutative, distributive, associative, and inverse), and emphasised the importance of this aspect of subject matter knowledge to multiplicative thinking. Similarly, Ward and Thomas' research generated poor responses to the PCK questions, which emphasised some teachers' lack of depth in understanding and in associated misconceptions, related to knowledge for teaching fractions. Ward and Thomas' study contained seven questions based on scenarios involving the teaching and learning of fractions. Results of the assessment from the 44 teachers involved showed a full range of scores, with one teacher receiving the possible 17 points and one teacher scoring zero. In general, teachers scored more highly on questions based on content knowledge, than those that required a description of the key ideas involved, or the actions they would take with a student. Overall, 32% of teachers involved in the research were unable to answer proportional reasoning problems correctly, suggesting that teachers may lack knowledge in this domain.

Further research by Ward and Thomas (2008, 2009) with teachers who had participated in the NDP project, aimed to identify whether there was a link between teacher PCK and student achievement in fractions. The teachers completed a pencil-and-paper assessment, while the student achievement data was collected from results of the NumPA interview. Ward and Thomas (2009) concluded that there was a relationship between teacher scores on the assessment and the student gain scores from initial and final assessments: students of teachers with high content knowledge (CK) made 20% more than average gain, compared to that made by students of low-scoring CK teachers. The teachers' CK was more pronounced the higher the year level they taught. Students in Years seven and eight classes of high scoring CK teachers made on average 42% more progress than those of lower scoring CK teachers.

Research conducted by Young-Loveridge (2008b) found that teachers' limited CK, combined with their personal misunderstandings about fractions, meant that in some instances teachers were in fact teaching concepts incorrectly. For example, one teacher was observed asking a child, "What is half of a quarter?" The child responded, "an eighth." The teacher (supposedly) corrected the child by saying, "No, half of a quarter is two-eighths," and went on to say, "Half of a third is two-sixths". This exemplifies some of the challenges teachers have with fraction understanding and also emphasises the importance of teachers modelling correct language when expressing some of these challenging ideas (Chick, 2015; Young-Loveridge, 2008b).

2.8 Frameworks of Teacher Knowledge

The various components of teacher knowledge and their importance for the teaching process (as identified by Shulman in his seminal paper in 1986), have provided the basis for knowledge frameworks in mathematics that gauge the mathematical knowledge required for teaching (Ball et al., 2008; Bobis et al., 2012; Hill et al., 2008; Roche & Clarke, 2009). Many decades after Shulman first proposed the idea of PCK, one might think that a comprehensive theory of decision-making could provide a base for a usable classroom observation scheme. However, producing a workable framework for classroom observations that is theoretically grounded is more difficult than one would expect. As Schoenfeld (2013) explained:

although the theoretical and practical enterprise are in many ways overlapping, the theoretical underpinnings for the observation [scheme] are sufficiently different (narrower in some ways and broader in others) and the constraints of almost real-time implementations so strong that the resulting analytic scheme is in many ways radically different from the theoretical framework that gave rise to it (p. 607).

Creating a classroom-based workable framework to explore the professional knowledge of teachers has thus become an ongoing interest of many researchers (Ball et al., 2008; Chick et al., 2006; Gess-Newsome, 1999a, 2015; Magnusson et al., 1999; Roche & Clarke, 2009; Schoenfeld, 2013). Many hours have been spent formulating knowledge frameworks against which the attributes of PCK and other key knowledge areas important for teaching, can be observed. Capturing the many dimensions associated with teaching into a manageable observation scheme poses many challenges, and those utilising such a scheme seldom get to see the "twists and turns of plausible, but unworkable ideas, that precede the presentation of the

clean final product” (Schoenfeld, 2013, p. 607). Categorising teacher knowledge into suitable headings for eliciting information about teacher action in practice has often been contentious. Thus, to develop a framework that is applicable to the teaching of all mathematics content is difficult.

As a result of many years of research into mathematics teachers’ knowledge (Ball, 1990, 1991, 1993, 2000b; Ball et al., 2005), Ball et al. (2008) developed a framework of Mathematical Knowledge for Teaching (MKT), providing a basis for the different types of knowledge required of teachers. Ball et al. suggested that in order to be effective and to give meaningful feedback to students, two main types of knowledge are utilised: Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). They claimed that teachers need to incorporate the sub-constructs of these two types of knowledge into their teaching and divided SMK into three components: (1) “common content knowledge” of mathematics (knowledge of a kind used in a variety of settings, not necessarily confined to teaching); (2) “specialised content knowledge” (the mathematical knowledge, and skill, unique to teaching); and (3) “horizon knowledge” (shows how mathematics topics are connected over the span of mathematics included in the curriculum). Teachers of mathematics have to do a particular kind of mathematical work that is not necessary in other fields. This work involves an unpacking of mathematics that is not required in settings other than teaching. The PCK section in the Ball et al. (2008) framework, consists of three components originally emphasised by Shulman (1986, 1987): (1) “knowledge of content and students” (combines knowing about mathematics and knowing about students); (2) “knowledge of content and teaching” (combines knowing about mathematics and knowing about teaching); and (3) “knowledge of content and curriculum” (related to knowing about mathematics and understanding curricular knowledge).

A Recent ‘Professional Knowledge’ Model

There has been a shift in mathematical research in recent years away from “knowledge as a substance” to “knowledge as an activity” (Settlage, 2013). In recent times, a framework of teacher professional knowledge was developed as a result of a PCK summit held in Colorado, which brought together 22 active science educator researchers (Gess-Newsome, 2015). The new framework is referred to as a model of ‘Teacher Professional Knowledge and Skill’ (TPK&S)

and recognises the role of PCK within professional knowledge, while acknowledging four weaknesses and limitations of Shulman's earlier PCK understanding and models (Gess-Newsome, 2015). These included: an absence of the non-cognitive attributes including emotion, feelings, and motivation; an emphasis on the pedagogical mind rather than the pedagogical practice (including the effect of teaching on the hearts and minds of students); insufficient evidence given to the broader social and cultural context; and insufficient evidence of goals and outcomes of learning. The model of TPK&S acknowledges the connections between teacher professional knowledge bases (assessment knowledge, pedagogical knowledge, content knowledge, knowledge of students and curricular knowledge), topic-specific professional knowledge, classroom practice, and student outcomes (Gess-Newsome, 2015). Shulman (2015) acknowledged that it is in classroom practice that PCK should be examined and the TPK&S model is unique, in that it acknowledges the significance PCK as both a "knowledge skill" within topic-specific instruction and as a skill within the "act of teaching" (Gess-Newsome, 2015).

2.8.1 Frameworks of Teacher Knowledge and Classroom Practice

The theoretical underpinnings of PCK frameworks used by researchers, have led recently to the development of frameworks based on categories and dimensions used within classroom practice, (Schoenfeld, 2013), principles of practice (Smith & Stein, 2011), and powerful ideas (Askew, 2013; Schwartz, 2008). Schoenfeld focused on what teachers need to know in order to explain on a moment-by-moment basis, the decision made by a student in the midst of a well-practiced activity. As a result of Shulman's work, Schoenfeld acknowledged the three categories of knowledge generally accepted as being, content knowledge, general pedagogical knowledge, and pedagogical content knowledge. However, he suggested that they might be better used when part of a seamless whole and argued that the distinctions researchers and professional developers make between these three categories (when used in research) becomes to some extent inconsequential in classroom practice. He proposed four categories of knowledge and activity that are required to construct a model of a person's decision making in teaching as: resources (most centrally, knowledge); goals; orientations (beliefs, values, preferences); and decision-making (Schoenfeld, 2013).

The shift by mathematics education researchers towards the study of teaching practices is exemplified in the framework of five practices for productive mathematics classrooms developed by Stein et al. (2008). The five practices identified include: anticipating (teachers do the problem in as many ways as they can, in order to anticipate how students might mathematically interpret a problem); monitoring (paying close attention to students' actual responses to tasks, observing their mathematical thinking and solution strategies as they are working); selecting (particular students are selected to share their work with the class); sequencing (the teacher makes a decision about how to sequence the presentations or sharing of ideas); and connecting (the teacher helps the students draw connections between their solutions and other students' solutions, as well as the key mathematical ideas of the lesson). Stein et al. indicated that these practices were designed to help teachers use students' responses to advance the mathematical understanding of the class as a whole, by allowing teachers to have some control over what could happen in the discussion as well as providing an opportunity to shift some of the decision making to the planning of the lesson. The goal is to have student presentations and sharing of ideas build on one another, to develop powerful mathematical ideas.

Good questioning is central to quality teaching and learning (Fraivillig et al., 1999; Sullivan & Clarke, 1991; Sullivan & Lilburn, 2004; Way, 2008). The importance of questioning techniques is a category that does not appear on most frameworks of teacher knowledge and yet the results of research by Fraivillig et al., indicated that the manner in which questions are formed influences student achievement. Research has shown that most questions posed are lower-order questions and greater attention needs to be given to higher-order questions that encourage higher levels of cognition (Fraivillig et al., 1999; Francis, 2015; Hunter, 2012; Sullivan & Clarke, 1991, Way, 2013; Wimer, Ridenour, Thomas, & Place, 2001). Hence, Fraivillig et al., developed the ACT framework which focussed on the ability to establish a community of learners supporting questioning, enquiry, and elaboration by creating conversations between the teacher and students, and among students. This occurred through the facilitation of what Fraivillig et al., described as three overlapping components: eliciting, supporting, and extending. In order to advance students' thinking, teachers must find out the students' current

knowledge and the strategies they use for mathematical problem solving. Encouraging students to express themselves in a variety of ways will assist teachers to elicit information that can be used to direct learning goals and desired outcomes, as well as assessing the individual student's knowledge and thought processes (Fraivillig et al., 1999).

The teacher's actions and responses to eliciting is then used to support students' conceptual understanding within the framework of their cognitive abilities (Fraivillig et al., 1999). Supporting the student who has described his/her strategy by giving examples of similar work previously covered, is expected to assist with understanding by providing clarity to the already learned knowledge. The other students are also supported because they are being exposed to strategies and thought patterns to which they may not have previously been exposed (Siegler, 1988). In extending students' mathematical thinking, teachers should maintain high expectations by constantly shifting the aims, so that students strive to extend themselves and rise to challenges. This is best learned in a safe environment where students are able to take risks, try alternative methods and strategies, and learn from each other. The teacher facilitates this learning, by knowing just how far they are able to extend each student.

2.8.2 The Framework used in this Research

The PCK framework developed by Chick et al. (2006) (Figure 2.1) was the basis for the analysis of classroom observations of this research. It was a framework developed by mathematics researchers, for use when exploring teachers' mathematical knowledge. In-depth analysis was instrumental to this research and Chick et al.'s framework provided categories for the researcher to carry out a fine-grained analysis of the teachers' classroom practice. The breakdown of three broad areas into sub-categories, aligned to the three main areas of professional knowledge originally mooted by Shulman (1986, 1987) based on content knowledge, general pedagogical knowledge, and pedagogical content knowledge. Chick et al. incorporated the need for students' prior knowledge, student problems, relevant representations, teaching strategies, student activities, student thinking, and curriculum knowledge into three parts considered to be necessary for PCK: Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context.

PCK Category	Evident when the teacher...
<i>Clearly PCK</i>	
Teaching Strategies	Discusses or uses strategies or approaches for teaching a mathematical concept or skill.
Student Thinking	Discusses or addresses student ways of thinking about a concept, or recognises typical levels of understanding.
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept.
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity.
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams).
Knowledge of Resources	Discusses/Uses resources available to support teaching.
Curriculum Knowledge	Discusses how topics fit into the curriculum.
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used.
<i>Content Knowledge in a Pedagogical Context</i>	
Profound Understanding of Fundamental Mathematics	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics.
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept.
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts.
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident).
Methods of Solution	Demonstrates a method for solving a maths problem.
<i>Pedagogical Knowledge in a Content Context</i>	
Goals for Learning	Describes a goal for students' learning (may or may not be related to specific mathematics content).
Getting and Maintaining Student Focus	Discusses strategies for engaging students
Classroom Techniques	Discusses generic classroom practices.

Figure 2.1 Framework for Analysing Pedagogical Content Knowledge (Chick, Baker, Pham, & Cheng, 2006)

The first category, Clearly PCK, shows aspects where both pedagogy and mathematics are so intertwined that the pedagogy cannot be separated from the mathematics (Chick et al., 2006). This includes aspects such as teaching strategies used, students' thinking, varying models and representations, resources, and curriculum knowledge. Within this category are elements such as the cognitive demands of tasks, and appropriate and detailed representations of concepts. The second category, Content Knowledge in a Pedagogical Context, focuses on aspects drawn most directly from content, where the knowledge is more clearly

influenced by having to know mathematical content. Key elements here include showing a profound understanding of fundamental mathematics, deconstructing knowledge into its key components, and highlighting mathematical connections. The third category, Pedagogical Knowledge in a Content Context, covers situations where knowledge is more clearly concerned with teaching skill, but some mathematics is still needed. It includes knowledge of classroom techniques as well as strategies for getting and maintaining student focus.

By outlining the categories within each section of the Framework and specifying the criteria for when each is evident (Figure 2.1), the Framework provides a set of filters for looking at the teaching process and the associated knowledge used by teachers. While allowing for critical distinctions among types of knowledge, overlap between categories is seen as unavoidable, ensuring the many aspects of PCK are covered (see also Section 5.2, Figure 5.2; Section 6.2, Figure 6.1). The categories are not seen as hierarchical, but instead combine to form an over-all view of the teacher's knowledge (Chick, 2007).

Chick et al., had utilised the framework for research relating to teacher knowledge carried out via questionnaires and interviews, and acknowledged a true test of the framework would be the use of it in other situations, such as classroom lessons (Chick et al., 2006). This was the ideal opportunity to take a recognised framework of knowledge and implement it in observed practice.

2.9 Summary

It is generally acknowledged that it is no longer acceptable for students to leave school without the basic skills, knowledge, and dispositions they need to function effectively in life (Hattie, 2003, 2009). As a result of recent educational reforms, the New Zealand Ministry of Education has placed a priority on the teaching of numeracy (and literacy) at all levels of primary school education, and the professional knowledge required of teachers in order to be effective teachers of mathematics for numeracy is varied and complex. The shift in focus from reliance on memorisation of facts and procedures to one of conceptual understanding requires an understanding of the relationship between knowledge of subject matter, knowledge of teaching practice, and knowledge of curriculum.

Understanding the mathematics currently being taught, is more than knowing the procedures for getting the right answer. The special knowledge required of teachers is referred to as PCK, and is not a single entity that is the same for all teachers (Loughran et al., 2012). International research has shown that determining a teacher's PCK is difficult (Ball et al., 2008; Chick, 2006, 2007; Hill et al., 2008; Loughran et al., 2006) partly due to the amalgam of its intricately linked components - knowledge of pedagogy, and knowledge of content (Magnusson et al., 1999), which can be domain specific (Shulman, 2015). Determining a teacher's PCK in action in the classroom is even more difficult, due to the added complexities associated with in-the-moment actions and the time taken to gather data.

Shulman (2015) acknowledged that early research on PCK placed little emphasis on emotions, affect, feelings and motivation – non-cognitive attributes; did not attend to pedagogical action; gave little attention to social and cultural contexts; and did not recognise the outcomes of instruction – the relationship between how teachers thought and evidence of student learning. More recently, this has resulted in a shift towards examining teachers' overall professional knowledge required for teachers, a component of which is PCK.

In New Zealand, studies are limited with respect to the professional knowledge required of primary school teachers in the mathematics classroom. Long-term research carried out by Young-Loveridge (2005, 2006, 2007, 2008a, 2009, 2010) alongside the NDP, showed that in the multiplicative and proportional domains, students were achieving below the levels expected by the Ministry of Education (2009a). During this time, research carried out by Ward and Thomas (2007, 2009) via written questionnaires, indicated some lack of depth in teachers' knowledge in these domains.

The purpose of this thesis is to address the relationship between teachers' espoused professional knowledge, professional knowledge in practice, and student learning, through a critical view of the teaching of mathematics for numeracy in the multiplicative and proportional domains. When this research began, there was an identified gap in New Zealand mathematics research, exploring the relationship between teacher knowledge and classroom practice through observation of lessons. Previous research and associated literature identified a number of frameworks

based on Shulman's notion of PCK. Chick et al.'s PCK framework was used as the basis for this study, due to the explicit categories suitable for fine-grained analysis. However, the researcher soon recognised that teacher practice encompassed a wider range of professional knowledge, thus the addition of further categories at the analysis stage (Figure 3.1).

The following chapter begins by re-stating the aims and objectives of the study. It explores the methodology associated with this research and includes a section that outlines the schools and teachers who participated in the study, along with ethical considerations, including the role of the researcher. Data-gathering methods discussed include observations, questionnaires, interviews and learning conversations, and assessment tasks. Finally, the methods and theoretical framework for data analysis are outlined.

CHAPTER THREE

METHODOLOGY and METHODS

Section A: Theory associated with Research Methodology and Methods

3.1 Introduction

This chapter positions this study methodologically and describes the methods used. The aim of methodology is to help researchers understand not the products of the inquiry, but the process itself (Cobb, 2007; Cohen et al., 2000) and as such sits between the identified research questions and the collected data (Gray, 2014). Section A of this chapter provides a justification for the chosen research methodology and methods, while Section B provides details of the research design. A multiple case-study approach was utilised, as this was deemed the most appropriate for addressing the aim and questions of this research.

3.2 Research Questions

Three specific questions framed this research:

- 1) What professional knowledge is evident when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?
- 2) What relationships are there between teachers' espoused professional knowledge, professional knowledge in practice, and student learning, when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?
- 3) How does the use of a framework assist in the investigation of teachers' professional knowledge in practice?

The research was situated within the context of teaching the multiplicative and proportional domains in New Zealand upper primary school classrooms (Years 5-8).

3.3 Research Methodology

When discussing methodology in educational research Cohen, Manion, and Morrison (2000) described methods as, "that range of approaches used in educational research to gather data which are to be used as a basis for inference

and interpretation, for explanation and prediction” (p. 44). Methods of inquiry include assumptions about the nature of the reality of the topic being studied, what constitutes knowledge of that reality, and what are appropriate methods (ways) of building knowledge of that reality. These assumptions combine to make up the essential idea of what underpins the term paradigm in research methodology. The standard, everyday meaning of paradigm is exemplar, or model. In the context of research methodology, the term has also come to mean a set of philosophical assumptions about the phenomena to be studied, about how they can be understood, and even about the proper purpose and product of research (Hammersley, 2012). Cohen et al. reported that by studying paradigms (models), research can be moved from being regarded as a technical exercise, to recognising that research is concerned with understanding the world, informed by how the world is viewed, and what is seen as the purpose of the understanding.

3.3.1 Epistemology and Methodological Considerations

“*Epistemology* is the theory of knowledge and is concerned with the question of what counts as valid knowledge” (Holloway & Wheeler, 2002, p. 3). Epistemology provides a foundation for deciding on knowledge possibilities. It is a way of explaining “how we know, what we know” (Crotty, 1998, p. 3) and is inherent in the chosen methodology used in any research.

There are many epistemologies available for use. However, when studying the social aspects of the world, two key epistemologies that emerge are the positivist perspective and the interpretive perspective. *Positivism* is a scientific approach that is based on testing theories and hypotheses (Holloway & Wheeler, 2002). Positivist research is generally based on objectivity and there is distance established between the researcher and those being studied. *Interpretive* research is based on a social approach and how people make sense of their world and attach meaning to it. The interpretive view can be linked to Weber’s *Verstehen* approach, where reality is socially constructed and the research seeks to understand something in context and through the eyes of the participants (Cohen et al., 2000). Weber argued that the social sciences should treat people as human beings and access to their experiences should be gained by listening to them and observing them. At times, there may be a conflict between the positivist perspective, with an emphasis on the explanation of human behaviour, and

interpretive perspective, which emphasises the understanding of the behaviour (Cohen et al., 2000; Crotty, 1998).

Researchers who adopt a positivist (objective) perspective to the social world, choose from a range of data collection methods including surveys and experiments, thereby taking a quantitative approach to their research (Cohen et al., 2000). Cohen et al. proposed that this contrasts with researchers who favour the view that stresses the importance of the interpretive (subjective) experience of individuals and take on a qualitative approach to their research.

The *interpretive* perspective is the methodology used in this study. The research is based on the real world of the participants and focuses on understanding their mathematics classrooms by listening to them and observing their actions. It is concerned with the participants' actions and seeks to make sense of their activities and attach meaning to them.

3.4 Research Methods in Education

Empirical research is regarded as the main type of research used in education today (Phillips, 2011). Empirical research in education is usually subdivided into two main categories: qualitative research and quantitative research. However, in practice, research is rarely purely qualitative or quantitative and often becomes a combination of the two.

Qualitative Research

Qualitative research sits predominantly within the interpretive paradigm (Bogdan & Biklen, 1992) and is richer and much more open-ended than quantitative research. It is an approach to obtaining data by exploring traits of human behaviour, and social life, in natural settings where general broad questions are investigated (Charles, 1995; Creswell, 2008; Mertler, 2012; Yin, 2012). As the task of the qualitative researcher is to capture what people say and do due to how they view their world, the information is largely verbal and collated through interviewing, observation, description, and recording (Creswell, 2009; Gray, 2014; Poland & Pederson, 1998). The role of the researcher is to gain a deep and holistic overview of the context under study within the specific setting of the participants (Gray, 2014; Toma, 2011).

Qualitative research, allows the researcher to explore concepts such as thoughts, feelings, ideas, and experiences, whose essence is often lost in other research approaches (Bogdan & Taylor, 1975; Corbin & Strauss, 2008; Creswell, 2008). Qualitative researchers are generally concerned with the process involved in research and not simply with the outcomes. For this reason, there is a close connection between qualitative research and teaching, and many teachers gain an insight into new educational endeavours through this method of research (Bogdan & Biklen, 1992; Charles, 1995). Because the researcher is the main means of collecting information, the researcher's personal skills and knowledge of the topic, are also important factors in the research process (Bogdan & Biklen, 1992; Miles & Huberman, 1994; Patton, 2002).

Quantitative Research

Strategies of inquiry associated with quantitative research, observe and measure information numerically, using instruments to collect statistical data - often surveys, tests, and questionnaires (Cohen et al., 2000; Creswell, 2008; Crotty, 1998). Quantitative research utilises a deductive approach to research design where specific questions are asked that the research will test (Gray, 2014). Quantitative research turns the data collected into numbers, and its function is to form the basis for a variety of statistical analyses to help make comparisons (Croll, 1986).

3.5 Case-study Research

One method of educational research is the case-study approach, which stems from the ethnographic approach (Creswell, 2008, 2009) and is now recognized as a method that has its own research design (Wellington, 2000; Yin, 2014). A case study is “a detailed examination of one setting, or one single subject, or one single depository of documents, or one particular event” (Bogdan & Biklen, 1992, p. 58). The basic idea is that one case is explored in detail, using whatever method and data seem appropriate, and attempts to understand the complex interrelationship among all that exists by focusing on an in-depth exploration of the actual case (Stake, 1995). Some researchers may identify a case as an object of study, while others consider it a procedure of inquiry (Stake, 1995). The case may be a study of an individual, a small group, an organization, a community, or a nation (Creswell, 2008, 2009; Denscombe, 2007; Lichtman, 2011; Stake, 1995). It may

be a study of a phenomenon, a programme, a person, or a process (Lichtman, 2011). All case studies begin in a place of action, and ultimately the findings contribute back to that place.

Case study has been a common research method associated with social sciences used in the fields of psychology, sociology, anthropology, social work, business, education, nursing, and community planning (Yin, 2014). In qualitative research, the case study aims to understand the case in depth in its natural setting, recognizing its complexity and its context (Creswell, 2009; Gray, 2014). It has a holistic focus aiming to understand the wholeness and unity of the case. Case studies are an examination of an instance in action (Bassegy, 1999), or a phenomenon of some sort in a bounded context (Creswell, 2008; Yin, 2014). They need not always include direct, detailed observations as a source of evidence (Yin, 2014).

Different writers have categorized case-study research in various ways. Four broad styles of case study concerned with educational action have been identified by Bassegy (1999): (1) the *ethnographic* method is usually used in the social sciences whereby a single case is studied in depth through participant observation, supported by interview; (2) the *evaluative* case study may be a single case or collection of cases, studied in depth with the purpose of providing educational actors, or decision makers, with information that will help them to judge the merit and worth of policies, programmes, or institutions; (3) the *educational* style is concerned with neither social theory, nor evaluative judgment, but with enriching the thinking of educators; and (4) *action research* case study is concerned with the development of the case, or cases, under study by feedback of information, which can guide revision and refinement of the action. A case study tends to fall into one of these four broad areas, a combination of them, or maybe even none of them (Bassegy, 1999). This then tends to form a fifth category where each may be independent of the others, or an interlocking relationship is formed between the differing case-study styles. This shows the difficulty of attempting to categorize research specifically, as there is often a tendency for the parameters to overlap.

Stake (1995) identified three main types of case study: (1) the *intrinsic case study*, where the study is undertaken because the researcher wants a better understanding of a particular case; (2) the *instrumental case study* where a particular case is

examined to give insight into an issue, or to refine a theory; and (3) the *collective case study* where the instrumental case study is extended to cover several cases, to learn more about the phenomenon. The collective method is also referred to as the *comparative case method* (Yin, 2003). The first two of these are single-case studies, while the third (collective case study, or comparative case study) involves multiple cases, and is also known as multiple-case study.

This study falls within the *educational* style as identified by Bassey (1999) and the *collective* case study of Stake, also known as multiple-case study.

3.5.1 Multiple-case Study Research

Multiple-case study design refers to the investigation of more than one participant, where the focus is both within and across the cases (Creswell, 2008). In-depth understanding requires only a few cases be investigated, because for each case examined the researcher has limited time to devote to exploring the depths of the case (Creswell, 2008). The researcher seeks to develop an in-depth understanding of the case by collecting multiple forms of data.

The evidence from multiple-cases is more compelling than one case and therefore the overall study is more robust (Yin, 2014). The idea of multiple-case design being more robust and efficient, builds on the scientific idea of replication of experiments providing accurate findings. If certain findings were uncovered in a single experiment, an ensuing priority would be to replicate this finding by conducting a second, third, and even fourth experiment (Yin, 2014). Only with such replications is the original finding robust. A similar logic is applied to the multiple-case study. The ability to conduct a number of case studies may then bring with it an opportunity to form some type of generalizability. If the aggregate provides some similarity, generalizations are made, whereas if the cases are contradictory the initial propositions must be revised and retested with another set of cases (Yin, 2012, 2014). This logic is similar to the way researchers approach conflicting experimental findings.

3.5.2 Case Studies and Generalizability

There may seem to be a paradox that lies between the study of a singular event and the search for a generalization. Whether a case study should even seek to generalize depends on the context and purposes of the particular project (Yin,

2003). Clearly, every case studied is both unique and similar to others in some way. The question is whether the focus should be on what is unique about a particular case, or on what is common with other cases (Yin, 2003). When generalizability is a goal, and focus is on the potential common elements in a case (or multiple cases), it is necessary for the analysis of the case-study data to be conducted at a sufficient level of abstraction (Bassey, 1999).

Discovering the important features, developing an understanding of them, and conceptualizing them for further study, is often achieved through case-study research. “Generalization and application are matters of judgment rather than calculation, and the task of case study is to produce ordered reports of experience which invite judgment and offer evidence to which judgment can appeal” (Bassey, 1999, p. 26). Yin (2014) however, suggests that a case study is not a singular sample of research and suggests it should be an opportunity to shed empirical light about some theoretical concepts or principles. This means that there will be interest in going beyond the specific case that will likely strive for generalizable findings, or lessons learned (analytic generalizations). Multiple-case studies should aim for analytic generalizations and the generalizations should be at a conceptual level higher than that of a specific case (Yin, 2014).

Bassey (1999) treated the idea of generalization as a statement that had to be absolutely true, such as scientific generalizations, and argued that very few generalizations in education could be categorized in that manner. He acknowledged that two other kinds of generalization could apply to the social sciences: *statistical generalizations* and *fuzzy generalizations*. Statistical generalization arose from a quantitative measure, while the fuzzy generalization came from a qualitative measure, where the study of singularities made a claim that it is possible, likely, or unlikely, that what was found in the one instance, will be found in similar situations elsewhere. For this reason, rather than use the term generalizability, Bassey often preferred to use the term *relatability* and suggested the relatability of a case is more important than its generalizability. An important criterion for judging the merit of a case study is the extent to which the details are sufficient and appropriate for a teacher to ‘relate’ the decision making to that described in the case study.

The two categories of generalization used by Bassey (1999) have similarities with the two types noted by Yin (2012, 2014), who described them as *statistical generalization* and *analytical generalization*, and noted that for case-study research the latter is the more appropriate type. When explaining analytical generalization, Yin (2012) asserted that they depend on using a study's theoretical framework, to establish a logic that might be applicable to other situations. He suggested the first step involves a conceptual claim, where researchers show how their findings have informed the relationships among a set of concepts, theoretical constructs, or sequence of events. The second step involves applying the same theoretical propositions connected with other situations outside the study. Thus, case studies tend to generalize to other situations based on analytical claims, as opposed to surveys and other quantitative methods, that tend to generalize to populations based on statistical claims. Yin (2014) also posited the need initially to decompose each case into a set of common variables. Case patterns may then be traced across the set of cases, which is referred to as qualitative comparative analysis (QCA). When carrying out analysis of this type, the unique aspects of each case are also taken into account through some form of qualitative analysis, that will complement any quantitative tallies that may have been noted (Yin, 2014).

3.6 Roles of the Researcher

In case-study research, the main source for data collection and analysis is the researcher (Creswell, 2008; Yin, 2012). The demands of a case study rely on the researcher's intellect, ego, and emotions, and are far greater than other research methods (Yin, 2014). There is an ongoing need for interaction between the theoretical issues being studied and the data being collected. The case-study researcher uses a range of techniques and multiple sources to collect relevant data. This requires a range of attributes including the ability to: ask good questions; be a good listener; stay adaptive; have a firm grasp of the issues being studied; avoid biases; knowing how to conduct research ethically including being sensitive in situations such as classroom observations and interviews; and have analytical capability to manage survey results (Creswell, 2008; Yin, 2014). The goal of the researcher is to cite relevant evidence, whether from interviews, documents,

observations, or archival evidence, in composing an adequate answer to the questions posed (Yin, 2014).

During data collections and analysis of qualitative research, it is important to consider the methodological issue relating to *reflexivity* (Corbin & Strauss, 2008). Reflexivity is self-awareness by the researcher of the relationship between the researcher, the participants, and the research environment (Lamb & Huttinger, 1989). It involves being aware of what influences the researcher's in-the-moment responses, while also being aware of the researcher's relationship to participants. The researcher's feelings or emotions may be conveyed to the participants (often on an unconscious level), and the participants may react to the researcher's responses accordingly (Corbin & Strauss, 2008). Reflexivity helps make explicit any moral dilemmas that might otherwise go unnoticed and assists in ensuring objectivity within the research process (Corbin & Strauss, 2008).

3.7 Ethical Considerations

A good case-study researcher will strive for the highest standards of ethical practice while doing research, including a responsibility to scholarship, as well as being honest, avoiding deception, and accepting responsibility for one's own work (Cohen et al., 2000). Ethical concerns in educational research can be complex and subtle and can, if not carefully dealt with, place the researcher in a range of moral predicaments. Research ethics is about being clear about the nature of the agreement you have entered into with your research subjects (Bell, 2010). Ethical research involves getting informed consent from those you are going to interview, question, observe, or take materials from; protecting those who participate in the study from harm; protecting the privacy and confidentiality of those who participate; protecting vulnerable participants (for example students); and selecting participants equitably (Yin, 2014).

Specific ethical considerations arise for all research involving human subjects (Corbin & Strauss, 2008; Yin, 2014). One main ethical concern is the balance between the demands of the researcher as a professional in pursuit of truth and the subjects' rights and values, which may potentially be threatened by the research (Cohen et al., 2000). The collection and analysis of data have customarily required objectivity, but it is generally recognised that complete objectivity is

impossible (Corbin & Strauss, 2008). The researcher always comes to the research from some position, and brings to the research their personal paradigm including perspectives, training, and knowledge. The lens of the researcher becomes entwined in all aspects of the research process including the gathering, analysing, and interpreting the data (Corbin & Strauss, 2008; Yin, 2003). Background, knowledge, and experience, allow the researcher to be sensitive to concepts in the data, to see connections between the concepts, and respond accordingly (Corbin & Strauss, 2008). It is thus important that all research be conducted with sensitivity, as the researcher puts himself/herself into the research (Corbin & Strauss, 2008). It means having insight into and being aware of relevant issues, events, and happenings in the situation (the classroom) and the data collected.

Much education research is often concerned with both teachers and children. There are important differences between research with children and research with adults, and implications for the methods of research. Researchers are usually in positions of authority over children and this raises the possibility that children may find it difficult to dissent, disagree, or say things which adults may not like (Yin, 2003). The researcher needs to be aware that children may say what they believe the researcher wants to hear. Therefore, a caring relationship is paramount when carrying out research with children and it is important for the researcher to view the research from the participants' perspectives (Noddings, 2003). The researcher is encouraged to consider collaboration with the participants and to avoid imposition on them (Noddings, 2003).

Confidentiality is an important issue when doing research. Researchers must protect the people participating in their study. Involving people as research participants carries ethical obligations, including ensuring that pseudonyms are used for real names and places (Corbin & Strauss, 2008; Yin, 2014).

3.8 Data Gathering Methods

As stated previously, case-study research is not limited to a single source of data and good case studies benefit from having multiple sources of evidence (Yin, 2012, 2014). Data may be collected from a variety of sources including: direct observations, interviews, surveys, archival records, documents, and physical artefacts. Regardless of the sources used, case-study evidence often includes both

qualitative and quantitative data (Yin, 2012). The qualitative data is primarily descriptive data and cannot always be explained numerically, as it is the quality and richness of the response which should be the focus (Basit, 2003; Cresswell, 2008, 2009; Gray, 2014). The research report frequently begins with microanalysis of data from which theories may be developed and/or descriptions of lived experiences and narrative stories created (Corbin & Strauss, 2008) and the end of process tends to be primarily descriptive in presentation.

This study is a multiple-case study that uses both qualitative and quantitative data. However, there was a leaning towards more qualitative data being collected, in line with the interpretivist perspective of social methodology.

3.8.1 Questionnaires

Questionnaires can be of value when collecting data for case-study research, as they can provide both qualitative and quantitative information. There are many types of questionnaires, which can vary in terms of their purpose, size, and appearance (Denscombe, 2007). A well-designed questionnaire can be difficult to create in order to gain the information required, while also being acceptable to those participating in the research, and later ensuring there are no problems in the analysis and interpretation stages (Bell, 2010; Denscombe, 2007).

Questionnaires allow for a large amount of information to be gathered in a relatively short space of time. There are seven question types: open or verbal, list, category, ranking, quantity, grid, and scale (Bell, 2010). Where quantitative data is required, structured closed questions are useful in that they can generate statistical information and enable comparisons across groups in the sample. Such questions may be presented using a Likert scale for the responses. They are usually, though not always, on a three-point, five-point, or seven-point range and ask the respondent to indicate or rank, order of agreement or disagreement, by circling an appropriate number. Data presented is ordinal, as opposed to interval data, and care needs to be given when interpreting the scale, as the intervals may not be the same (Bell, 2010). Multiple-choice questions also allow quantitative data to be readily collected. However, multiple-choice items, without the opportunity for participants to elaborate on their decision, have limited use (Clements & Ellerton, 1995). If qualitative information is required, then a less structured, open-ended questionnaire may be more appropriate (Cohen et al.,

2000). Open questions allow the participant to write a response in their own words and to qualify and explain their reasons for doing so.

When designing a questionnaire, many different needs and issues might be considered, as these may affect interpretation of the data (Bell, 2010; Creswell, 2008). For example, care needs to be given to the wording of questions, as words which may appear to have a common meaning to some people, may mean something totally different to others, and the writer must try to avoid confusion and take care not to make assumptions that the reader will know or understand what is asked of them. Emotive language and hypothetical questions may lead, or possibly mislead, respondents into answering in a particular way and should be avoided.

3.8.2 Interviews

When the researcher needs to gain insights into things like people's feelings, opinions, emotions, and experiences, then an interview is a method attuned to the intricacy of the subject matter (Corbin & Strauss, 2008; Denscombe, 2007). An interviewer can follow up ideas, probe responses, and investigate motives and feelings, which a questionnaire cannot do. One major advantage of the interview is its adaptability, as responses can be developed and clarified (Bell, 2010). The key idea of successful interviewing is to be friendly, but not too friendly. A balance must be struck between the warmth required to generate rapport, and the detachment necessary to see the interviewee as someone under question (Oakley, 1981).

There are many different types of interview available to the researcher, including structured interviews, semi-structured interviews, unstructured interviews (sometimes known as the ethnographic interview), one-to-one interviews (also referred to as individual interviews), and group interviews, sometimes known as focus-group interviews (Denscombe, 2007; Yin, 2012). The structured interview involves tight control over the format of the questions and sometimes the answers (Denscombe, 2007). In many ways, they are like a questionnaire where the questions are pre-planned and administered face-to-face with the interviewee. With semi-structured interviews, the interviewer is flexible in the order of questions and topics covered, allowing the interviewee to develop ideas and speak more openly. In the unstructured interview, the researcher's role is to be as un-

intrusive as possible. Interview questions are not pre-planned but are based instead on general ideas, allowing questions to emerge. They allow the interviewees to use their own words and develop their own thoughts. Denscombe argued that the most common form of semi-structured or unstructured interview is the one-to-one interview where the researcher only has one person's ideas to grasp and question, and one participant's voice to recognise when later transcribing.

Interview bias may occur in research when carried out by an individual who has a strong view about the topic they are researching. In such an instance, the researcher may find it difficult to maintain a dispassionate, objective, arm's length approach to the research situation (Bell, 2010). Instead, there is a need for the researcher to be flexible during the interview and allow the respondent to voice personal opinions. A successful interview has many of the characteristics of a prolonged and intimate conversation, minimising the difference between the interviewer and the respondent (Denscombe, 2007). An equal relationship will develop based on trust, enabling greater openness and insight.

3.8.3 Observations

Case-study research should take place in the real-world setting of the case, thus creating the opportunity for direct observation (Bell, 2010). Effective observation requires considerable skill and is not an easy option for the collection of data, but it can reveal characteristics of those studied, which would have been impossible to discover by other means. Observation involves orderly viewing of people's actions and the recording, analysis, and interpretation of their behaviour (Gray, 2014). Interviews, surveys, and questionnaires provide important data, but they only reveal how people perceive what happens, not what actually happens. Observations on the other hand can be useful in discovering what happens in reality and whether people behave in the manner, they claim to behave. Observations are time consuming and can be intrusive. However, observation is important to research in that it is not unusual for persons to say that they are doing one thing, but in reality they are doing something else (Corbin & Strauss, 2008). The only way to know the actuality of a context is through direct observation, which puts the researcher in the setting where they can see what is occurring.

Direct observations occur in a field setting and can focus on human actions, physical environments, or real-world events, and are one of the most distinctive

features of case-study research (Yin, 2012). However, the material obtained from observation can lead to imposing the researcher's own interpretation on what was observed (Bell, 2010). The collecting of observational data includes the taking of field notes and use of a recording device, in order to create a narrative based on what was seen and/or heard.

Audio-visual material

Alongside direct observation, qualitative data may also be collected with video and audio images. Audio-visual materials include images or sound that researchers collect to help them understand the study and are gathered via photographs, audio-recordings, digital images, and paintings (Creswell, 2008). Audio-visual recording may overcome the partiality of the researcher's field notes and records when based on observation alone (Cohen et al., 2000). It overcomes the possibility of recording only those events that happen frequently and provides the capacity for comprehensive material and completeness of analysis. It helps to make explicit the implicit models teachers have of their practice and how students learn (Cohen et al., 2000). Verbatim recordings are an added advantage when used for later review and data analysis.

Video-recorded material can capture conversations and expressions between the teacher and learners, as well as between and among the learners themselves. Insight is gained into the complexity of the intersection of a teacher's pedagogical content knowledge and the application of knowledge in practice (Maher, 2008). Videotapes and film images provide data and examples about real life as it occurs, and provide an opportunity for the researcher to share their perceptions of reality. Audiotape recorders record conversations within the lesson, but lack the possibility of recording visual and non-verbal factors such as a look of surprise on a child's face. However, while video recordings might provide more accurate data than the audiotape, they might also be more constraining due to the intimidation they pose to some participants.

3.8.4 Tests and Assessment Tasks

The field of testing is extensive in terms of variety, volume, scope, and sophistication (Cohen et al., 2000). By carrying out tests, researchers have available a documented method of collecting data that is quantitative as well as qualitative (Creswell, 2008). Tests in the education setting may include

commercially-produced, teacher-produced, and/or researcher-produced tests. Commercially-produced tests are not tailored to local contexts or needs, while the teacher-produced and researcher-produced tests will be tailored to a specific context. The purpose and content will deliberately fit the specific needs of the research. Cohen et al. refer to this as *fitness for purpose*. For example, a pre-test provides a measure on an attribute or characteristic before participants receive instruction, and after instruction, a post-test can be carried out on the same attribute. Pre-testing and post-testing can have a potential threat to validity in research if participants become familiar with the questions and remember responses for later testing (Cohen et al.). Hence, in mathematics tests, there is a need to change the values in each question to reduce this possibility.

Assessment used in New Zealand schools involves the “focused and timely gathering, analysis and use of information that can provide evidence of students’ progress” (Ministry of Education, 2007, p. 39). While formal tests may be used, assessment is frequently “of the moment” (Ministry of Education, 2007, p. 39). As stated in Section 1.5, one of the mathematics assessment tools used for gathering information is the Numeracy Project Assessment tool (NumPA) (Ministry of Education, 2008b). The NumPA assessment is carried out as an individual interview and contains a series of questions, which uncovers the student’s mental strategies used when solving problems and number knowledge.

3.9 Transcribing and Data Analysis

Transcription represents a translation from one system (oral) to another system (written) (Cohen et al., 2000; Creswell, 2008). Transcribing can be isolated from the dynamics of the classroom and does not necessarily capture the social, interactive dynamics of the lesson. As it is only the spoken word which is transcribed from the audio-recording, it becomes solely a record of verbal data with non-verbal information being omitted (Creswell, 2008). Hence, transcribed data is often used in conjunction with video-recorded material, which captures the non-verbal essence of the lesson.

Data analysis is complex when working with mixed-method research, especially the qualitative data (Braun & Clarke, 2006; Creswell, 2008). The variety and diversity in approaches to data collection, means that there is no single right way

to conduct qualitative data analysis and much depends on the purpose of the research. Added to this complexity, is the notion that case-study analysis takes many forms. As Yin (2012) noted, qualitative research does not necessarily follow the routine procedures that exist with other research methods. What connects the many approaches to data analysis is a central regard to transforming and interpreting qualitative data in a rigorous way, in order to capture the complexities of the social worlds researchers seek to explain.

Coding is the initial activity undertaken in qualitative analysis and the basis for what follows later (Yin, 2012). Researchers are increasingly using electronic methods of coding data and developed computer programmes are now available to support analysis of the narrative text (Basit, 2003; Lewins & Silver, 2007). They support the researcher's coding and categorizing of the notes taken, or verbatim transcripts (Basit, 2003; Yin, 2012). Computer programmes should be seen as tools, which can enhance the ability of the researcher to search for, store, sort, and retrieve materials (Yin, 2012). As Yin noted, unlike software for analysing numeric data where the computer often uses an algorithm to produce data, there is no algorithm when analysing narrative data. The qualitative analysis begins by the researcher organising the data into grouped relationships (Basit, 2003). Qualitative approaches are diverse and complex, and thematic analysis is a fundamental method utilised by qualitative researchers (Braun & Clarke, 2006). Whether using computer software to help or not, the researcher is the one who must define the codes to be used and the procedures used for piecing together the coded evidence into themes. A researcher uses the coding process to dig beneath the surface to dissect the data meaningfully, while keeping the similarities within different parts together (Corbin & Strauss, 2008).

There are different types of coding which may be used and among these is selective coding. Selective coding is where concepts are integrated around core categories (Strauss & Corbin, 1990). These may include predetermined frameworks, which inform the selected codes. A computer programme may take over the arduous tasks of marking up, cutting, sorting, and reorganising, that was once done with scissors and paper, but the analytic work to produce new insights must still be carried out by the researcher (Basit, 2003; Corbin & Strauss, 2008). NVivo is one software package for qualitative research that is designed to help

researchers analyse non-numeric data and (among other things) allows them to collect, organize, and analyse content of transcribed material from observations, interviews, and discussions (Lewins & Silver, 2007; NVivo Feature List, 2014). In NVivo the containers for storing themes or ideas are called nodes. All the information about a theme is gathered together in a node using coding stripes, which can be printed to assist in the crosschecking and analysis of themes.

While coding is often the first part of the data analysis process, the second is that of memoing. The principles with coding and memoing originated from grounded theory, and are now used widely in qualitative research. Creswell (2008) asserted that while memoing is the second important part of the process, it is not necessarily the second stage. The operations of coding and memoing are not sequential as memoing begins at the start of the analysis alongside the coding. Memos are notes the researcher writes, like an on-going dialogue with oneself throughout the research process, to elaborate on ideas about the data from field notes, transcripts, and coded categories (Creswell, 2008; Corbin & Strauss, 2008). The memos allow the researcher to explore hunches, ideas, and thoughts and take them apart searching for broader explanations (Creswell, 2008).

Once the data is coded and memos made, the material needs to be interpreted. When searching for regularities in the social world, induction is a central component (Cohen et al., 2000). As concepts are developed inductively from the data, their relationships are then mapped out. The fact that much qualitative analysis requires induction has meant that the term *analytic induction* is often used: analytic induction refers to the systematic examination of similarities between cases to develop concerns or ideas (Creswell, 2000). While induction is crucial in analysis, deduction is also required, since theory generation involves theory verification. This sort of qualitative data analysis is a series of inductive and deductive steps where data-driven inductive generation, is followed by deductive examination for the purpose of verification (Creswell, 2000).

Triangulation

Triangulation of data is frequently used in case-study research. Triangulation is the use of two or more methods of data collection in relation to the study of human behaviour and is used when both quantitative and qualitative data is considered (Cohen et al., 2000). As data collection takes place from a number of

different sources, triangulation occurs in an effort to establish trustworthiness and reduce the likelihood of misinterpretation (Creswell, 2008; Stake, 1995). Triangulation involves separate collection and analysis of multiple sources of data, which are later merged either through data transformation, or at the stage of interpretation of results (Creswell, 2008). In triangulation, the researcher values both qualitative and quantitative data and allows the researcher to gather information that uses the best features of both types. Triangulation is suitable when a more holistic view of education is sought and complementary results or inconsistencies may emerge (Cohen et al., 2000).

Section B: Research Methods and Design

Introduction

In order to answer the research questions, multiple-sources of data collection were used. Data collected from the complex real-world context of the classroom through observations and field notes, were analysed concurrently with data gathered from the questionnaire and assessment tasks. These were analysed in both a qualitative and quantitative manner. The analysis moved beyond the either-or thinking associated with qualitative or quantitative methods alone (Cohen et al., 2000; Creswell, 2008) as the process of collecting, analysing, and weaving together of data occurred. While this research is predominantly qualitative in nature and sits within the interpretivist paradigm (Bogdan & Biklen, 1992), numerical data was unpacked through a deductive process, allowing for relationships across cases to be identified (Mouly, 1978). Although a multiple case-study design was used and brought with it the features of a case study, within and across four cases, comparisons between individual cases were not at the forefront. Instead, while findings from the individual cases are acknowledged, they were ultimately combined to look for reliability (Bassegy, 1999), by identifying both similarities and differences across the cases.

3.10 Participants

3.10.1 Case-study Schools

Two New Zealand primary schools were invited to participate in the research project. Convenience sampling (Marshall, 1996) was used, as the schools were currently committed to participating in mathematics professional development

with the researcher, who was their numeracy adviser at the time (see sections 3.10.11 and 3.11), due to mathematics support allocated and funded by the Ministry of Education.

School A was a central city full primary school (Years 1 to 8) with a decile rating of 3 (Note: Decile rankings are allocated by the New Zealand Ministry of Education reflecting the socio-economic status of the community, ranging from 1 the lowest, to 10 the highest). The school had six classes ranging from Year 1 to Year 8, and a non-teaching principal. The students in the school came from a wide range of ethnic backgrounds, with less than 10% being of New Zealand European descent, approximately 30% of New Zealand Māori heritage, and 50% of the children categorised as speaking English as a Second Language (ESOL), also known as English Language Learners (ELL).

School B was a rural town primary school, with students from Years 1 to 6 and a decile rating of 5. The school had 22 classes, along with a non-teaching principal and two deputy principals. Approximately 60% of the students were of New Zealand European ethnicity, 28% were Māori, and the remaining 12% came from varying ethnic groups. The senior school syndicate (Years 5 & 6), from which two teachers participated in this study, were cross-grouped by ability across six classes for their mathematics lessons. The classes were grouped according Numeracy Project Assessment (NumPA) data and formative data, from the previous years' teachers.

Initial discussions were held with the principal and deputy principal(s) from each school, to outline the intent of the research. Copies of the principal's agreement and participation agreement were shown and discussed (Appendix A), and the school leaders were invited to consider participation in the research. Once the principal gave approval, letters of information along with an outline of the research was presented at the first meeting with staff from each school. An invitation was extended to all teachers at the Years 5 to 8 levels, to participate in the research (Appendix B). The researcher was careful to explain fully the ethical considerations of the research and extra care was taken to discuss with the teachers any conflict of interest that may arise between the work of the researcher as an adviser, and that of a researcher. Two teachers from each school along with their students, became the case-study participants of this study. The participation

agreement for data to be collected and observations made of their teaching practice was given to each teacher (Appendix B: Additional information for case-study teachers).

3.10.2 Case-study Teachers and their Classes

The case-study research focused on the senior classes of each school. Pseudonyms beginning with the letter appropriate to their school were chosen for the teachers, to show the connection between them and their school. At School A, the leadership team determined class combinations, generally based on each student's age. The overlap of Year 7 students occurred because of the spread of student numbers and class sizes school wide. Classes included Andy, the teacher of the Year 6 (n=22) and Year 7 (n=6) class and Anna, the teacher of the Year 7 (n=10) and Year 8 (n=17) class (Table 3.1).

At School B, there were six Years 5 and 6 composite classes and two teachers (Beth and Bob) volunteered to participate as case-study teachers. School B cross-grouped their classes by ability for mathematics, based on the Number Framework strategy stage, ascertained from the NumPA data at the end of the previous year. The majority of Bob's students had been identified at early Stage 5, while Beth's students were transitioning from Stage 4 to early Stage 5 (Section 3.10.1). Bob's class (Year 5, n=15; Year 6, n=13) was third in ranking (one being the top class out of the six) and Beth's class (Year 5, n=12; Year 6, n=9) the fourth class in ranking (Table 3.1).

The four case-study teachers were also involved in workshops and classroom visits alongside the rest of the teachers at their school, as part of their school-wide professional development programme (Table 3.2). The timeline highlights when various data were collected from the case-study teachers, for this research. As the researcher wished to focus on the teaching of multiplication and division, the data gathering occurred in the second half of the year. This gave the researcher, the teachers, and their students, time to familiarise themselves with each other during previous visits to the classrooms and by the time the identified lessons took place, the teachers and students were used to the researcher's presence in the room. The visits to the case-study teachers' classrooms for research purposes were clearly identified and discussed with the teachers early in the year, and the researcher reminded the teachers of these during the prior visit.

Table 3.1
Case-study teachers and their classes

Teacher	School	School Decile	Year Level (number of students)	Number Framework Stage(s)	Maths Class	Teacher's Experience (in years)
Anna	A	3	7 (n = 10) 8 (n = 17)	4 – 6 4 - 6	Regular class, ability grouping	20
Andy	A	3	6 (n=22) 7 (n=6)	3 - 5 4 - 6	Regular class, ability grouping	11
Beth	B	5	5 (n=12) 6 (n=9)	4 4 - 5	Cross-class ability grouping (4 th out of 6 classes)	2
Bob	B	5	5 (n=15) 6 (n=13)	4 – 5 5	Cross-class ability grouping (3rd out of 6 classes)	1

3.11 Ethical Considerations

This research adhered to the guidelines as set out in The University of Waikato Code of Ethics and all material associated with ethical practice was approved by the Faculty of Education Ethics Committee, prior to commencement of the research. Determining what is ethical is situational and complex (Corbin & Strauss, 2008) and I was bound by the University's requirements as well as personal high standards of ethics. At the initial meeting with principals and deputy principals, opportunity was given to discuss any issues or concerns regarding the study, with potential conflict of interest on the part of the researcher highlighted.

It was made clear that the researcher would remind the teachers in advance when data was going to be collected for the study and discuss with them any issues they may have. If the teachers did not want data included in the research, they could choose to withdraw it, and/or themselves, at any point up until the end of that year.

The researcher visited the case-study teachers' classes and discussed with the students that their teacher had agreed to involvement in the study. It was explained that in order for the study to occur, they also needed to give consent to participate. Issues with regards to anonymity were discussed with the students and they were assured that their names would not be used in the findings. The students were given letters of information and consent for their parents/caregivers (Appendix D) and for themselves (Appendix E), which they took home, discussed with their parents/caregivers, signed, and returned to school.

Table 3.2

The year's professional development programme

Date	Meeting/Observation	Brief Description of purpose
January	Staff meeting - the Number Framework	Understanding stage progressions on the Framework.
	Participation agreements discussed Initial questionnaires handed out	All teachers completed agreements to participate in research. Senior school' teachers invited to be case study teachers.
February	Staff meeting - assessment tools	An overview of some assessment tools available for use in mathematics. Introduce the use of GloSS and IKAN for diagnostic assessment. Teachers carry out assessment on all students.
	Questionnaires collected Case-study teachers identified	Initial questionnaires completed by teachers. Case-study teachers complete agreements.
March	Staff meeting - Getting started, where to now?	Teachers encouraged to analyze data and set goals for the year.
	In class visit - meet children and discuss research	Visit case-study classes and discuss with the children the research their teachers are involved in. Permission slips given to children to take home.
	In class visit - observation of teaching	Each teacher observed in class. Learning Conversation followed.
April	Staff meeting - addition & subtraction	Support understanding of progressions when teaching knowledge and strategy in addition and subtraction.
	In class visit - observation of teaching	Each teacher observed in class. Learning Conversation followed.
June	Survey about Learning and Teaching completed	Children and teachers complete survey about learning and teaching in mathematics.
	Staff meeting - multiplication & division In class visit – observation of teaching	Support understanding of progressions when teaching knowledge and strategy in multiplication and division. All teachers observed teaching multiplication lesson. Learning conversation followed.
August	In class visit - pre unit assessment tasks	All children complete the initial multiplication and division assessment tasks.
	In class visit - observation of teaching	Each teacher observed introducing multiplication and division. Learning conversation followed.
	In class visit - observation of teaching	Each teacher observed teaching towards end of unit on multiplication and division. Learning conversation followed.
September	In class visit - post unit assessment tasks	All children complete the final multiplication and division assessment tasks.
	Staff meeting - proportional thinking	Support understanding of progressions when teaching knowledge and strategy in fractions and decimals.
October	In class visit - pre unit assessment tasks	All children complete the initial fraction assessment tasks.
	In class visit - observation of teaching	Each teacher observed teaching midway through unit on fractions. Learning conversation followed.
November	In class visit - observation of teaching	Teachers observed teaching towards end of unit on fractions. Learning conversation followed.
	In class visit - post unit assessment tasks	All children complete the final fraction assessment tasks.
December	staff meeting - statistics	Ideas for teaching statistics & probability. Teachers carry out GloSS assessment on students.
	In class visit - end of year questionnaire	Interview with teachers on the year. Students complete questionnaire on how the year has gone for them.
	Staff meeting - data analysis and next	What does the data say about your teaching and the children's

(Note: Highlighted sections indicate when data were collected from the case-study participants)

3.12 Data Gathering Methods

The research focussed on the relationship between the teachers' espoused professional knowledge as evidenced in their questionnaire, their professional knowledge as observed in teaching practice, and their students' learning, in the

multiplicative and proportional domains. Hence, data collection required multiple sources and included questionnaires, classroom-based observations, field notes, interviews, assessment information, and copies of student work. Validity of findings was assisted by triangulation between the different data sources.

3.12.1 Observations and Recordings

Observations were a key part of data collection, which emphasised the professional knowledge of teachers in action, in classroom practice. Although the students were familiar with the researcher being in their room, observations always began with her sitting at a student desk in order to be as inconspicuous as possible. The intention of this was to gather data of a typical mathematics lesson and encourage the students and the teacher to continue as they would at any other time. In order to validate observations of the lessons, field notes were written, photos taken, and lessons both video-recorded and audio-recorded.

Video and Audio-recording

The video-recording and audio-recording of each observed lesson (Table 3.2), allowed the researcher to return to specifics of the lessons and crosscheck details later. Each lesson was recorded on a small video camera, while at the same time the teachers wore a small microphone that was connected to a digital voice recorder. The recordings provided a permanent and accurate record of what was said and carried out during the lessons, and were downloaded to a computer for storage. While research has found there are times when having an electronic device may change how the students and teacher react (Morrison, 1993), this did not appear to be the case in this research. After each session, the teachers were asked if they felt having the recording devices affected their teaching, and they all commented that after the initial act of attaching the recorder to their clothing, they forgot it was even there. Lessons were transcribed from the audio-recordings and cross-checked against the video-recordings for accuracy.

3.12.2 Questionnaires

A questionnaire (Appendix C) was given to the teachers as a means of gathering information related to the teachers' perceived pedagogical content knowledge (PCK). The questionnaire was originally compiled and utilised to determine teachers' PCK as part of earlier research conducted into the teaching of

multiplication and division, and use of the equal addition strategy (Young-Loveridge & Mills, 2009a, 2009b, 2010, 2011), which sat alongside the Numeracy Development Project professional development.

The questionnaire was compiled in three sections: Section A included the teachers' views about classroom mathematics practices; Section B was multi-choice questions focussed on aspects of multiplicative and proportional subject matter knowledge; and Section C provided scenarios about teaching mathematics, where judgements were required in relation to mathematical understanding and pedagogical practice. Each scenario described how a student might solve a given problem and asked the teacher to explain what action they would take next with that student. The teachers explained how they would solve the problem and were asked to draw a diagram to support their thinking.

As stated previously in relation to teacher knowledge, knowing how to solve a problem, recognising how a student solves a problem, and knowing where to take the child next, are part of a teacher's disciplinary knowledge of mathematics, or mathematics-for-teaching (Davis & Renert, 2014). For these reasons the questions given to the case-study teachers were based on the knowledge required to teach Levels 3 (Stage 6), 4 (Stage 7), and early Level 5 (Stage 8) of the curriculum (for detail of each question given and reason for its inclusion refer to Sections 4.22 and 4.23). Students at Level 3 are expected to "record and interpret additive and simple multiplicative strategies using words, diagrams and symbols" (Ministry of Education, 2007, Level Three chart) and it was felt that teachers should be able to do likewise. Hence, the teachers were asked to show how they would solve each problem and to use a diagram to support their thinking when explaining the strategy they would use to solve each scenario.

3.12.3 Interviews and Learning Conversations

At the end of each observed lesson, the researcher briefly interviewed the teacher. The interviews were one-to-one and predominantly unstructured, as they were based on the recently observed lesson. Originally, the researcher asked the teacher for an interview after each teaching session. However, during the first interview the researcher and teachers agreed to change the label of the dialogue to "learning conversation." The change in terminology came about because while it was essentially an unstructured interview, the process became a conversation between

the researcher and the teacher. The conversations were carried out in a quiet area away from the classroom, where the researcher and teacher would not be distracted. In order to keep the conversation open and honest (for both parties), it was decided not to record the conversation, but instead the researcher would make field notes. Together the researcher and the teacher reflected on the observed teaching session, the impact they thought it had had on the students' learning, and what this meant for future lessons. At the end of the year, a concluding semi-structured interview took place (that was recorded), reflecting on the journey undertaken throughout the year in relation to their teaching practice and the students' learning.

3.12.4 Tests, and Assessment Tasks

In this study, data included photocopying some of the students' follow-up exercises from their workbooks and copies of recordings in the group modelling book. As the lessons progressed, the modelling book provided evidence of the question types given to the students, the manner in which they solved the problems, and threads of discussion that took place. It contained word problems, equations, and at times the pasted in samples of the materials used to solve problems. An analysis was not carried out on these data sources, but they were used as supporting evidence to illustrate learning that was taking place in the classroom lessons.

Student achievement data from Progressive Achievement tests (PAT) and Numeracy Project Assessment (NumPA) tools were also collected. This was limited to information about the overall results for the class and did not show a breakdown of individual tasks for each student. Ethical approval had not been sought for detailed individual student's previous assessment data, but the schools were happy to share collated results. The main reason for collecting background assessment data was to provide the researcher with some information about the mathematical capabilities of each class compared to their expected levels of achievement (Table 3.3), prior to the research taking place.

Initial and final assessment tasks designed by the researcher, were administered in the multiplicative and proportional domains. The term "assessment tasks" is used throughout the thesis as the researcher wanted the students to regard them as questions given by the researcher, in order to gather data to show how their

knowledge and understanding changed (or not) throughout each unit of work. The researcher assured the students that assessment tasks data would be used as part of the “evidence of overall patterns of effectiveness” (Patton, 2002, p. 151) of the teaching that had taken place. Thus, a collation of results became the focus, rather than individual’s data, as it was the key aspects of learning in the class as a whole, that the researcher wished to focus on. The assessment tasks were to be seen as supporting evidence in the triangulation of achievement data, alongside observed lessons and modelling books.

The assessment tasks were based on key aspects of knowledge students at Years 5 to Years 8 are either expected to have, or are currently acquiring (particularly in the case of the Year 5 students), according to the NZC Levels (Ministry of Education, 2007), the Number Framework Stages (Ministry of Education, 2008a), and the National Mathematics Standards (Ministry of Education, 2009a), (Table 3.3).

Table 3.3
Approximate comparisons between The New Zealand Curriculum Levels and Number Framework stages.

Curriculum Level	Number Framework Stage	Number Framework Descriptor	Class Level
1	1-3	Count All	Year 1
1	4	Advanced Counting	Year 2
2	5	Early Additive	Years 3 & 4
3	6	Advanced Additive	Years 5 & 6
4	7	Advanced Multiplicative	Years 7 & 8
5	8	Advanced Proportional	Year 9

Ministry of Education (2010)

The Numeracy Project Assessment Tools, including NumPA and GloSS, identify a stage on the Number Framework a student is working at, based on the strategies used for solving problems. For example, if a student solved the problem: “If $3 \times 20 = 60$, what does 3×18 equal?” by derivation, such as, “ $3 \times 18 = 60 - (3 \times 2) = 54$ ” the student will be categorised as working at Stage 6, or Advanced Additive the expected working stage for students in Years 5 and Year 6 (Table 3.3).

The Numeracy Development books and planning sheets available to teachers through the New Zealand Mathematics website (Ministry of Education, *n.d.a*) present lessons as transitioning from one stage to the next, because while a student is working at one stage they are being prepared for the next stage. For example, students who are working at Stage 6 during Years 5 and 6 are preparing to move from the Advanced Additive (AA) Stage (Level 3) they are currently working at, to the Advanced Multiplicative (AM) Stage (Level 4). Similarly, during Years 7 and 8 students who are working at AM are preparing to move to the Advanced Proportional (AP) Stage (Level 5). While students should be transitioning upwards from the stage that they were assessed at, if during teaching gaps are identified, then use of the lower transition sheets may be required for a short period to support filling those gaps.

The knowledge required for solving strategies at each of the Number Framework Stages is learnt at the previous stage. For example, knowledge developed whilst at Stage 6 (e.g., equivalent fractions for halves, thirds, quarters, fifths, and tenths with denominators up to 100, and up to 1 000), is used for solving problems at Stage 7 (Ministry of Education, 2008a, p.21).

The assessment tasks given to the students, came from material developed for Levels 2 and 3 of NZC. The reasons behind inclusion of the specific initial assessment tasks (Sections 7.1.1 & 7.2.1) and final assessment tasks (Sections 7.1.2 & 7.2.2) are outlined in detail alongside the results and analysis, and are related to the “goals for learning” and the “knowledge of assessment” categories of the framework (Figure 3.1). Results of the initial assessment tasks were not made available to the students, as the final assessment tasks were based on similar question types for comparison. However, they were offered to the teachers to assist in unit planning.

3.13 Transcribing and Data Analysis

The transcripts of three selected lessons for each of the four case-study teachers were one of the key sources of data collection and analysis for this research. A contracted transcriber, who had previously signed a confidentiality agreement (Appendix F), transcribed the lessons verbatim from the audio-recorded files. All teachers carried out the lessons chosen for analysis at a similar time during the

year (Table 3.2). The timing of these was agreed to collectively by the principal, mathematics lead teacher, the case-study teachers, and the researcher, early in the school year. They included: the initial lesson at the commencement of the unit on multiplication and division, a lesson near the end of the same unit, and a lesson mid-way through the teaching of fractions. The transcribed lessons were cross-checked by the researcher who read the transcripts, listened to the audio-recordings, and watched the video recordings simultaneously to check the data for accuracy. Watching recorded lessons whilst reading transcripts allowed the researcher to capture the essence of the nature of each lesson.

The complete transcriptions were exported into the computer programme NVivo 10 for coding. The coding was managed deductively using an adapted version of the PCK framework of Chick et al. (Figure 3.1). This meant reading the data systematically, identifying the appropriate theme, and assigning the appropriate code from the “PCK category” and “the evident when the teacher” headings on the framework (Figure 3.1).

Due to the repositioning of the framework, there were times when the interpretation of Chick’s framework was subtly different for analysing PCK in action (as opposed to Chick et al.’s original research based on questionnaires). Observation of the lessons highlighted the teachers’ needs to react to situations as they occurred in unpredictable circumstances in the classroom. Therefore, slight rewording under the heading “evident when the teacher” was required in two instances in the Clearly PCK (purpose of content knowledge and curriculum knowledge) and the three original Pedagogical Knowledge in a Content Context categories (Figure 3.1). In these instances where the framework began with the words discusses or describes, this was adapted to read discusses/demonstrates, or describes/uses (terms used by Chick et al. on other descriptors within the Framework). For example, in the Pedagogical Knowledge in a Content Context section, under the Classroom Techniques sub category, the explanation of “evident when the teacher discusses generic classroom practice” was changed to read “evident when the teacher discusses/demonstrates generic classroom practices”. At times, it was not the teachers’ actions that determined a category, but rather it was the resulting actions of the students that determined which was

selected. The detailed descriptors were used to identify the subcategory during the coding process and the appropriate coding stripe was applied.

PCK Category	Evident when the teacher...
A. Clearly PCK	
# 1. Purpose of Content Knowledge	Discusses/Demonstrates reasons for content being included in the curriculum or how it might be used.
# 2. Curriculum Knowledge	Discusses/Demonstrates how topics fit into the curriculum.
3. Teaching Strategies	Discusses or uses strategies or approaches for teaching a mathematical concept or skill.
4. Cognitive Demands of Task	Identifies aspects of the task that affect its complexity.
5. Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams).
6. Knowledge of Resources	Discusses/Uses resources available to support teaching.
7. Student Thinking	Discusses or addresses student ways of thinking about a concept, or recognizes typical levels of understanding.
8. Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept.
B. Content Knowledge in a Pedagogical Context	
1. Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept.
2. Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts.
3. Methods of Solution	Demonstrates a method for solving a mathematics problem.
4. Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident).
5. Profound Understanding of Fundamental Mathematics	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics.
C. Pedagogical Knowledge in a Content Context	
# 1. Classroom Techniques	Discusses/Demonstrates generic classroom practices.
# 2. Getting and Maintaining Student Focus	Discusses/Demonstrates strategies for engaging students.
# 3. Goals for Learning	Describes/Uses a goal for students' learning (may or may not be related to specific mathematics content).
*4. Knowledge of Assessment	Discusses/Demonstrates summative and/or formative assessment practices.
*5. Questioning - Supporting	Questions asked support students' comments, assist students in clarifying thoughts, ask for group support, and/or ask others to paraphrase explanations.
*6. Questioning - Eliciting	Questions asked elicit different solution methods, encourage elaboration, and/or promote collaborative problem solving.
*6. Questioning - Extending	Questions asked encourage generalizations, consider relationships between concepts, allow for reflection on multiple solutions methods, and/or provide challenge.

Figure 3.1 Adapted framework for analysing Pedagogical Content Knowledge in action (* indicate additions to Chick, Baker, Pham, & Cheng, 2006, while # indicate adaptations).

Coding highlighted the overlap between categories on the framework, which was also acknowledged by Chick et al. (2006, p. 140). One example of this was the apparent link between the category “appropriate and detailed representations of concepts” and the category “knowledge of resources”: when teachers demonstrated an appropriate way to model or illustrate a particular concept, the use of concrete manipulatives (knowledge of resources) was essential. In these instances, when examples of the overlap become evident a double coding was applied (Figures 5.1 & 6.1).

As coding began, the researcher also considered two important aspects related to the professional knowledge required for teaching, to be missing from the framework. They were knowledge of assessment and use of questioning. Following email communication with Helen Chick (H. Chick, personal communication, September 2013), the researcher decided that for the purpose of this research each of these aspects warranted a specific heading on the framework.

The three questioning types added to the framework were *supporting*, *eliciting*, and *extending*, based on the Advancing Children’s Thinking Framework (ACT) of Fraivillig et al. (1999). Fraivillig et al. used these three headings to describe the actions of teachers and the methods they employ to promote quality teaching practice and student learning. Supporting questions were used when the teachers asked questions, which enabled students to clarify their own thinking, such as, “How do you know that?” (Section 5.2.3: C.5 & Section 6.2.3: C.5). Eliciting type questions promoted interaction among the students and collaborative problem solving, with questions such as ‘who did it another way?’ or by asking one child to explain another child’s previously explained strategy (Section 5.2.3: C.6 & 6.2.3: C.6). Extending type questions were used when the teachers challenged students to move beyond their initial efforts, often by encouraging them to make generalisations, such as, “Can you explain how that be might work for all multiplication problems?” or “Can you justify your answer for me?” (Section 5.2.3: C.7 & Section 6.2.3: C.7). The additional question type headings were added thus to the amended framework (Figure 3.1) and included in the analysis of data.

Coding was discussed with a senior academic in mathematics education who was then given a sample to cross-check. A sample of the transcript was coded blindly

three months later and coding stripes cross-checked for accuracy. Recoding samples at a later date is recognised as an accepted system for checking coding for reliability (Corbin & Strauss, 2008). The recoded sample was found to be consistent with initial coding throughout. The coding was cross-checked for a third time during the analysis process.

Once the transcribed lessons were coded, Qualitative Comparative Analysis (QCA) took place (Yin, 2014), as case patterns were traced across the set of cases. While the unique aspects of each case were taken into account, each case's combination with other cases was tallied within contexts, creating a quantitative cross-case analysis. The cross-case analysis was carried out within the three main sections of the Chick et al. framework: Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context. Quantitative tallies thus complemented the qualitative data at the forefront of analysis and together they allowed generalizations to be formed and provided information to address the research questions.

Reviewing the purpose of Chick et al.'s framework

As the Chick et al. framework (Figure 2.1) had not been used previously in the classroom, in order to determine whether it was fit for purpose, further coding of two lesson observations using alternative approaches (one in the multiplicative domain and one on the proportional domain) was carried out. The deductive analysis used against the named categories on the adapted Chick et al. framework (Figure 3.1), were compared to (i) initial analysis from an inductive approach and (ii) further deductive analysis against the established, recognised framework of Ball et al. (2008) which had been used previously to analyse classroom-based research (Section 11.2.3). Unlike the deductive approach used when analysing the data against a given framework, using an inductive approach meant that the theory (in this instance the themes to emerge) should not be presumed prior to the research, but should follow it (Glaser & Strauss, 1967). Similarly, coding against the categories on the adapted Chick et al. framework were compared to coding against the categories on the established Ball et al., framework, to see if the framework which had not been previously used in the classroom, was suitable for observations of practice.

It was envisaged that the coding of the categories on the adapted framework could be compared to themes or categories of other approaches and established frameworks, and similarities and differences that emerge could be paralleled to determine suitability of the framework for classroom use. The process of comparing the coding against the categories on the adapted framework to other approaches enabled the researcher to identify that while there are other ways to interpreting the data, the detailed framework used in this instance seemed fit for purpose for analysis of the observed lessons in this research. Refer to Section 11.2.3 for more details.

3.14 Summary

This chapter was outlined in two sections. Section A began by presenting the general rationale and specific research questions of this study. It outlined the multiple case-study method chosen for this research. The roles of the researcher and ethical considerations of the research were presented. Different data gathering methods, including questionnaires, interviews, observations, and assessments were outlined, along with a description of types of data analysis and the use of the qualitative data analysis software, NVivo 10.

In the second part of the chapter, Section B: Research Design, the methodology was described in action as it related to this classroom-based research. It detailed the settings and participants, along with ethical considerations including the dual role of the researcher. The collection, transcribing, and analysis of data were then described. Two lessons were used to compare categories used for analysis of data against the adapted Chick et al., framework to themes which emerged from an inductive, grounded approach to analysis. One lesson was coded against another established framework (Ball et al., 2008) and compared to the Chick et al., (2006) framework. These comparisons assisted in determining that the Chick et al., framework was fit for purpose.

Chapter Four presents insights into teacher professional knowledge and understanding based on responses to a questionnaire containing scenarios and position statements.

CHAPTER FOUR

RESULTS and ANALYSIS: TEACHER KNOWLEDGE

4.1 Introduction

This chapter presents teachers' responses to a questionnaire concerning subject matter knowledge and pedagogical understanding, in relation to given mathematics scenarios and position statements.

4.2 Teachers' Subject Matter Knowledge and Pedagogy

As set out in Chapter 3, the four case-study teachers completed questionnaire tasks (prior to the classroom observations) (Appendix C), based on multiplicative thinking and proportional thinking. Each scenario described how a student might solve a particular problem and asked the teacher to explain what action they would take next with that student. The teachers were asked what they thought the answer to the problem was and to describe how they had solved the problem, using a diagram to support their thinking.

4.2.1. Multiple-choice Questions

The total number of multiple-choice items was limited to eight, as they do not give teachers the opportunity to elaborate their decision-making and explain their reasoning. The teachers answered all of the multi-choice problems correctly, the only exception being Anna who when asked the best estimate for 45×105 , responded from the four choices (A: 4 000; B: 4 600; C: 5 200; D: 47 250) with A and B. Of the eight multi-choice problems, three asked for estimated answers (1, 4, & 8) and it was presumed that these questions were solved through estimation and not by finding an exact answer.

4.2.2 Teachers' Understanding of Multiplication and Division

Three of the six scenarios on the questionnaire related to the teaching of multiplication and division are presented.

Scenario 1: The teaching of multi-digit multiplication (Figure 4.1):

Jon was given the following problem:
What is $11 \times 99 =$

Jon took one away from 11 and added one to 99; he then multiplied 10 by 100 to get an answer of 1000.

(a) What action would you take next with Jon?
(b) What is the answer? Draw a diagram and explain how **you** would solve the problem.

Figure 4.1 Scenario 1: Multi-digit multiplication

Students in Years 5 and 6 are expected to be working at Stage 6 on the Number Framework (Table 3.3), and their lessons based around transitioning from Stage 6 to Stage 7 (Section 3.12.4). Scenario 1, was included in the questionnaire as one of the key ideas being developed at Stage 6 on the Number Framework is “rounding and compensating from tidy numbers” (Ministry of Education, 2008f, p. 52), which occurs alongside the learning of multiplication using multi-digit numbers.

Andy and Bob recognised Jon had used an additive strategy to solve the multiplication problem and described a place-value partitioning (PVP) strategy that they would teach: “split it into parts and teach him to do $(10 \times 99) + (1 \times 99)$ ”. Anna focussed on rounding and compensating as a strategy, and added that she would use materials to show Jon where he had gone wrong. She explained, “*One added to 99 makes one group of 100. You need 11 groups of 100, and you then take 11 away.*” Beth decided she would begin by asking Jon to explain his reason for solving the problem that way. She wrote, “*I would work through the problem with him, hoping he would see for himself why what he did, did not work. I would give him a smaller $\times 11$ problem and go from there.*”

When asked to solve the problem and describe their solution method, Andy, Anna, and Beth, gave the correct answer, while Bob gave an incorrect answer. Anna and Beth combined PVP with rounding and compensating and explained that they would calculate “ 10×99 , plus 1×100 , minus 1 ($990 + 100 - 1$).” Andy and Bob solved the problem using PVP of “ 10×99 plus 1×99 .” However, Bob multiplied “ $10 \times 99 = 990$ and $1 \times 99 = 99$,” but incorrectly gave the sum of the two numbers as 999.

Beth solved the problem in a different way from how she would show Jon ($11 \times 100 - 11 \times 1$) and while she mentioned manipulatives, did not expand on

what materials she would use and how they would be applied. Anna suggested giving Jon a smaller problem as a starting point and “going from there.” Anna was observed in class, starting with small numbers to ensure understanding of a concept, although there were few instances of the *go from there* phase and pushing students up to the higher numbers.

Two division scenarios were presented, each with a different focus although there was a connection between the two.

Scenario 2: The rule for divisibility by three (Figure 4.2):

Mere was given the following problem:

Hera owns a factory that makes tricycles. Each tricycle needs 3 wheels. She has 516 wheels. Will all the wheels be used to make tricycles, or will there be some wheels left over?

Mere added the digits together ($5 + 1 + 6 = 12$); she knew that the number was not divisible by 9, because 9 does not go into 12 evenly, and concluded that it was not divisible by three, so there would be some wheels, leftover.

(a) What action would you take next with Mere?
(b) What was the answer? Draw a diagram and explain how **you** would solve the problem.

Figure 4.2 Scenario 2: Using the divisible by three rule

At Stage 6, developing knowledge of the divisibility rules for the numbers 2, 3, 5, 9, and 10 takes place as students transition from AA to AM (Ministry of Education, 2008f, p. 41), ready to use when solving problems at Stage 7 (Years 7 and 8). This question was included in the questionnaire, as in order to develop understanding of the divisibility rules among their students the teachers require understanding of the rules themselves. However, questionnaire responses suggested that none of the teachers were familiar with the divisibility rule for 9 as presented in the NDP Book 6 (Ministry of Education, 2008f, pp. 70-73). This lesson unpacks why it is that when you add the digits of a given number together, if the sum of the digits equals nine (or a multiple of nine), then the whole number is divisible by nine. Understanding divisibility by 9 is then related to the rule for divisible by three (as three groups of 3, equals nine).

Andy and Bob gave no response as to what they would do next with Mere. Beth suggested asking Mere why she had added the numbers in that manner and continued, “*I would explain this may not work all the time. I would tell her to split the number into parts she knows, such as $300 \div 3$.*” Anna thought that she would discuss with Mere divisibility by 3, by “*first breaking the 516 into two groups of 150, plus another 150, leaving 66 to divide by three.*”

When asked how they would solve the problem, Bob and Beth said they would do a standard written algorithm; Anna recorded, “Use a 3 to one bike ratio, or else: 3 by 170 = 510; plus 6 wheels left means two more bikes; equals 172 bikes”; while Andy claimed to use place-value partitioning, although his recorded solution did not clearly show this (Figure 4.3).

$3 \times 100 = 300$ $516 - 200 = 216$ $21 \div 3 = 7$
 172 Tricycles $6 \div 3 = 2$

Figure 4.3 Andy’s recording of $516 \div 3$

While it is acknowledged that the strategies used by the teachers are all acceptable strategies to know and to implement, the teachers would be expected to recognise the number patterns of three and nine and be familiar with the associated lessons in the NDP support material.

Scenario 3: Quotitive division (Figure 4.4):

Rob was given the following problem:
*Ana was packing chocolate peanuts into bags for the school fair. She decided to put **14** chocolate peanuts into each bag. How many full bags would she get from **56** chocolate peanuts?*

Rob subtracted 14 from 56 to get 42; he then subtracted 14 three more times. Rob worked out that he could subtract 14 from 56 four times, so Ana must get four bags of 14 from 56 peanuts.

(a) What action would you take next with Rob?
 (b) What is the answer? Draw a diagram and explain how **you** would solve the problem.

Figure 4.4 Scenario 3: Quotitive division

The connection between the two division scenarios was in the use of the quotitive form of division, which is emphasised in lessons at Stages 5 and 6 (Ministry of Education, 2008f, pp. 36, 38, 54) and during proportional adjustment (p. 57). Anna was the only teacher who stated what she would do next and suggested teaching Rob about doubling and halving numbers, as it was quicker than repeated subtraction. However, she gave no indication as to how she would show the understanding of $56 \div 14$, although this did become evident when she later explained how she would solve the problem.

When asked to describe how they would solve the quotitive division problem, Andy used reversibility and answered the problem as a multiplication one. Andy’s recording did not show quotitive division and rather than 2 groups of 14 (2×14)

doubled to 4 groups, showed 14 groups of 2 (14×2). Inaccurate recording was further exemplified in representation of the steps required to solve the problem, and (mis)use of the equals (=) symbol, (Figure 4.5). Anna used a doubling strategy and recorded, “*Either halve and halve again, or double and double again. The answer would be 2 lots of 2, or 4.*” Bob recorded, “*I can see $14 + 14 = 28$ and $28 + 28 = 56$,*” while Beth wrote, “ $56 \div 14 = 4$ ” and gave no further explanation as to how she got that answer.

$14 \times 2 = 28 \times 2 = 56$
 $2 \times 2 = 4.$

Figure 4.5 Andy’s solution to 56 peanuts into bags of 14

4.2.3 Teachers’ Understanding of Fractions and Decimals

Three of the six scenarios on the questionnaire related to the teaching of fractions and decimals.

Scenario 4: Addition of decimals to two decimal places (Figure 4.6)

Jenny was given the following problem:

$$1.45 + 0.9 =$$

Jenny calculated the answer by adding $45 + 9 = 54$, so the answer is 1.54.

(a) What would you do next with Jenny?

(b) What is the answer? Draw a diagram and explain how **you** would solve the problem.

Figure 4.6 Scenario 4: Addition of decimals

Students who are moving from Stage 6 to Stage 7 are learning to add and subtract decimals to two decimal places (Ministry of Education, 2008g, pp. 45-46). When asked what the next steps of teaching would include, Anna, Beth, and Bob, mentioned that they would work with Jenny on place-value understanding. Anna recorded that she would work on tenths, hundredths, and whole numbers, while Bob suggested tenths. Beth noted tenths and hundredths and extended her explanation to include, “use a number line to show where the decimal numbers 0.45 and 0.9 belong in relation to other numbers.” Andy gave no indication as to what he would do next with Jenny.

All teachers solved the problem correctly: Andy took 0.1 off the 1.45 and added it to the 0.9 giving $1.35 + 1.0 = 2.35$; Beth used a similar strategy by firstly solving $1.45 + 1.0 = 2.45$, followed by $2.45 - 0.1 = 2.35$; Bob used a standard vertical algorithm; and Anna used place-value partitioning (Figure 4.7).

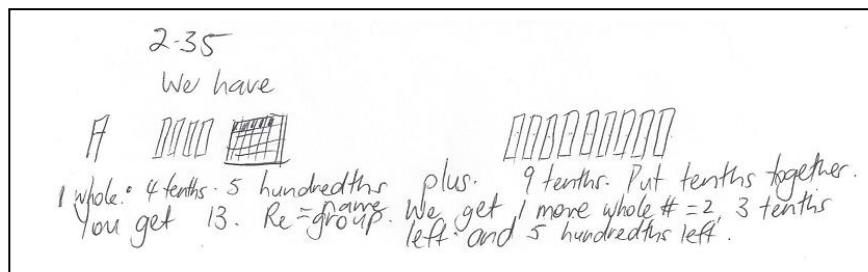


Figure 4.7 Scenario 4: Diagram showing Anna's recording of $1.45 + 0.9$

There is an expectation that place value is understood, with whole numbers up to one million and decimals to two places, when working at Stage 6 on the Number Framework (Years 5 & 6) (Ministry of Education, 2007, 2008a). In responding to Scenario 4, Anna (the teacher of a Year 7 & 8 class) mentioned the importance of place-value understanding and the relationship between the fractional pieces and the whole unit, within decimal understanding. The small sticks shown in her diagram (Figure 4.7) are similar to the deci-pipe manipulatives she was familiar with and used in her observed lesson, when she taught this concept to her students.

Scenario 5: Addition of fractions with compatible denominators (Figure 4.8).

This scenario is an extension of a similar example given to the students in their final assessment tasks.

Pete was given the following problem:

Tama and Karen buy two pizzas. Tama eats $\frac{3}{4}$ of one pizza while Karen eats $\frac{7}{8}$ of the other one. How much pizza do they eat altogether?

Pete converted $\frac{3}{4}$ to $\frac{6}{8}$ so he had $\frac{6}{8} + \frac{7}{8}$; he then added 6 and 7 to get 13, and 8 and 8 to get 16, and gave the answer as $\frac{13}{16}$.

(a) What would you do next with Pete?

(b) What is the answer? Draw a diagram and explain how **you** would solve the problem.

Figure 4.8 Scenario 5: Addition of fractions

Knowing equivalent fractions for halves, quarters, thirds, fifths, and tenths, is knowledge being developed at Stage 6 (Ministry of Education, 2008g, p. 36). All teachers solved Scenario 5 correctly. When asked what he would do next with Pete, Andy responded, "Teach him don't add the denominators," with no

explanation as to how he might teach this. Bob drew a diagram which implied that he would teach equivalent fractions, as he changed $\frac{3}{4}$ into $\frac{6}{8}$ and added them together. He gave no explanation alongside the diagram as to how he would approach the teaching of equivalent fractions in order to find a common denominator. Anna explained that she would, “*consolidate fraction understanding, beginning with fractions must be equal sizes or portions [when adding together].*” She would then, “*back track, to ensure understanding of the denominator.*” Beth noted that she would ask Pete to explain the thinking behind his answer and use this to move on. Beth also mentioned using manipulatives to consolidate understanding (she did not mention what these would be), and that she would check for understanding of the fraction (name).

When asked how they would solve the problem, Andy, Beth, and Bob, all wrote: $\frac{3}{4} = \frac{6}{8}$. This means $\frac{6}{8} + \frac{7}{8} = \frac{13}{8}$ or $1\frac{5}{8}$. Anna solved the problem the same way and supported her explanation with a diagram to explain her thinking (Figure 4.9).

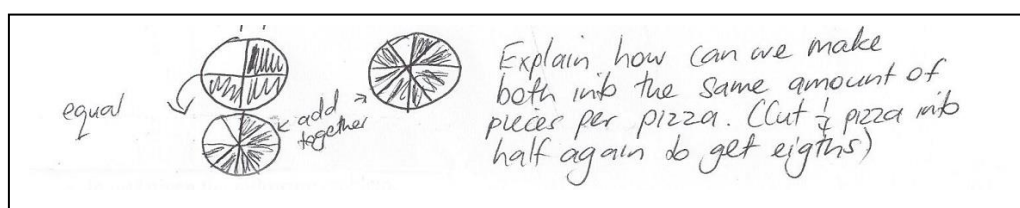


Figure 4.9 Anna’s diagram to explain $\frac{6}{8} + \frac{7}{8} = 1\frac{5}{8}$

Anna mentioned checking that the students understand that the denominators must be of equal-sized portions, before adding them together. Anna’s knowledge carried over into her classroom practice and 77% of Anna’s students solved addition of compatible fractions correctly in their final assessment tasks (Table 7.4).

Scenario 6: Multiplication of fractions (Figure 4.10):

Jo was given the following problem:

There was $\frac{3}{4}$ of a birthday cake left over after the party. Sarah took $\frac{1}{3}$ of the leftover cake home for her brother. How much cake did Sarah take home to her brother?

You hear Jo say “one third of three quarters; that’s the same as one third times three-quarters...”

(a) What would you do next with Jo?

(b) What is the answer? Draw a diagram and explain how **you** would solve the problem.

Figure 4.10 Scenario 6: Multiplication of Fractions

All teachers solved the final scenario, multiplication of fractions correctly (Figure 4.10). Multiplication of fractions begins at Stage 5 (Years 3 & 4) when students learn that six groups of one-half is the same as three whole objects, and record this as $6 \times \frac{1}{2} = \frac{6}{2} = 3$ (Ministry of Education, 2008g, p. 33). Multiplying fractions by other fractions is a strategy developed on the Stage 6 to Stage 7 (AA to AM) ratios and proportions planning sheet, which the teachers downloaded from the New Zealand mathematics website. The first lesson on the AA to AM planning sheet looks at “What would the part be called if you cut one third in to 4 pieces (recorded as $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$).” The final lesson on the AA to AM planning sheet refers to the NDP Book 8 (Ministry of Education, 2008h, p. 24) and a lesson on “fraction times a fraction,” as the curriculum elaborations (Ministry of Education, *n.d.c*) identify that understanding multiplication of fractions is to be known at Level Four. While not all of the teachers had students in their class ready for this problem, understanding multiplication of fractions (including decimal fractions) would be expected teacher knowledge, for teachers of Years 5 to 8 students.

The teachers identified that the student in the scenario had solved the problem correctly, although Anna, Andy, and Bob, were unable to provide any explanation about what the next steps of learning would be for Jo. Andy wrote, “*Nothing as she is right*”; while Anna and Bob gave no response. Beth recognised Jo was correct and would ensure Jo had not solved the problem procedurally: “*Get Jo to explain her answer. Check Jo understands and make sure she is not just remembering rules.*” No mention was made of understanding that when multiplying two fractions, the product is about finding a piece of a piece (in this instance of a cake). However, the difficulty students have in understanding multiplication of fractions that Beth would check is consistent with research, which has found that understanding multiplication of fractions is not easy and is often confused with division (Ma, 2010; Sowder, 1988). Sowder found that confusion arose for the students who noticed that when fractions or decimals are multiplied, the answer comes out smaller, so “it’s kind of like dividing” (p. 232).

When asked to explain how they would solve the problem, Anna gave no response. Andy recorded:

$$\frac{3}{4} \times \frac{1}{3} = \frac{3}{12} = \frac{1}{4}$$

Andy gave no explanation alongside his written expression and it appears he wrote the factors in the same order as they were in the scenario. Andy’s written representation would be interpreted as $\frac{3}{4}$ of $\frac{1}{3}$ and is not what Jo expressed in her explanation. The accepted practice in New Zealand schools is that the multiplication symbol for fraction multiplication, is interpreted as “of” (Ministry of Education, 2008g, p. 63). Andy’s representation contrasts with Bob and Beth who correctly recorded:

$$\frac{1}{3} \times \frac{3}{4} = \frac{3}{12} = \frac{1}{4}$$

While Bob offered no explanation as to why he ordered the factors that way, Beth was explicit in the reason for her recording and noted that the problem asked for $\frac{1}{3}$ of $\frac{3}{4}$.

4.2.4. Perceived Mathematics Practice

In the questionnaire, teachers were asked to rank five statements relating to their perceived teaching practice. All responses were within the always, often, and sometimes categories, and none of the teachers responded with hardly ever, or never (Table 4.1).

Table 4.1

Case-study teachers’ perceived classroom mathematics practices

	Andy	Anna	Beth	Bob
1. I encourage students to explain their thinking to each other	A	A	O	O
2. I encourage students to question the strategies of others	A	A	O	S
3. I encourage students to justify their choice of strategy and their thinking to others	A	O	A	S
4. I encourage students to work together on solving problems	O	O	O	O
5. I encourage students to include in their maths books drawings, diagrams, or other recording methods which represent their thinking	O	S	A	S

Note: A = Always, O = Often, S = Sometimes, HE = Hardly Ever, N = Never

Statement 1: I encourage students to explain their thinking to each other.

The first statement, “I encourage students to explain their thinking to each other” had a response of “always” from Andy and Anna, and “often” from Beth and Bob.

Statement 2: I encourage students to question the strategies of others.

When asked if they encouraged students to question the strategies of others, Andy and Anna responded “always”, Beth “often”, and Bob “sometimes” (Table 4.1).

Statement 3: I encourage students to justify their choice of strategy and their thinking to others.

When given the statement, ‘I encourage students to justify their choice of strategy and their thinking to others’, Andy and Beth replied with “always”, Anna “often”, and Bob “sometimes”.

Statement 4: I encourage students to work together on solving problems.

When presented with the statement, ‘I encourage students to work together on solving problems’, all of the teachers answered with “often”.

Statement 5: I encourage students to include in their mathematics books drawings, diagrams, or other recording methods, which represent their thinking.

When the teachers were asked if they ‘encourage students to include in their maths books drawings, diagrams, or other recording methods which represent their thinking’, Beth responded with “always”, Andy “often”, and Anna and Bob “sometimes”.

4.3 Summary

The questionnaire results provided an insight into the teachers’ espoused professional knowledge and perceptions about their classroom practice. The first part of each scenario asked the teachers to explain what they would do next with the student in each instance. This was to gauge the extent to which the teachers identified the current knowledge and problem-solving strategies held by the student in each given scenario, and what would be appropriate next teaching steps. In most instances, the teachers struggled to do this. The teachers often commented that they would teach the student the same way as the given example and provided no further explanation or identification of the next steps to learning. Questions the teachers might ask each student to check for understanding, or identify of the progression of learning to examples of increased difficulty, were not forthcoming. In particular, Bob and Andy found it challenging to identify what actions they would take in future lessons with each student.

The second part of each scenario, asked the teachers to give their answer to the problem and “draw a diagram and explain” how they solved it. Anna and Beth generally included a diagram with their explanation, while Andy and Bob struggled with this representation and frequently solved the problem using a written algorithm. The teachers seldom offered an explanation as to why they solved the scenario in the manner that they had.

Chapter Five presents insights into teacher professional knowledge and understanding based on observed teaching in the multiplicative domain.

CHAPTER FIVE

RESULTS and ANALYSIS: THE MULTIPLICATIVE DOMAIN

5.1 Introduction

This chapter presents an analysis of two lessons from each class (one for Anna) within the multiplicative domain. In line with the multiple-case study methodology adopted for this thesis, findings from the four case-study teachers are presented within their individual practice and across practices, in order to identify what professional knowledge is apparent both individually and collectively. The first lesson was the lesson taught at the commencement of a six-week unit and the second lesson was taught towards the end of the unit.

The students in each class were divided in half according to the complexity of the strategies implemented when solving multiplication problems on their GloSS assessment. In each of the observed lessons, half of the class carried out independent worksheet activities at their desks consolidating prior teaching and learning, while the other half was observed working with the teacher. The exception to this was lesson one for Bob, who worked with the whole class, for the whole lesson and the start of the second lesson for Andy (Table 5.1). Andy began his second lesson with the whole class together, a lesson format often used by teachers during their NDP professional learning (Ministry of Education, 2008b, p. 13), before working with smaller groups.

Table 5.1
Observed multiplication lessons

Teacher	Lesson Number	Whole class (or) Group	Lesson Focus
Andy	1	Group	understanding of the multiplication symbol, '×' understanding commutativity
Anna	1	Group	understanding of the multiplication symbol, '×' understanding commutativity
Bob	1	Class	understanding of the multiplication symbol, '×' building on $\times 2$, $\times 5$, $\times 10$ to solve problems
Beth	1	Group	understanding of the multiplication symbol, '×' understanding commutativity
Andy	2	Class (start of lesson)	area of composite shapes on whiteboard
Andy	2	Group	arrays in multiplication and area of rectangles
Bob	2	Group	relationship between $\times 5$ and $\times 10$
Beth	2	Group	building on $\times 2$, $\times 5$, $\times 10$ to solve problems

5.2 Observed Teacher Professional Knowledge in Multiplication

Introducing multiplication and division to the students

Each of the four case-study teachers began the introductory lesson for the multiplication and division unit in a similar manner, by gauging the students' understanding of the multiplication symbol. Anna introduced her lesson by inviting the students to write in the modelling book what they perceived to be the meaning of multiplication. Anna said, "*I just want you to write it down somewhere there (pointed to the modelling book). Just pop down, your thoughts about what you think multiplication is*". Each child later explained what it was he/she had written.

Andy began his lesson by placing manipulatives (animal strip cards [strips of card showing different animals in groups from one animal on a card to 10 animals on a card] and Unifix cubes) in front of the students. He asked a volunteer to show what *three times four* looked like. One student constructed his interpretation of the equation with cards showing three groups of four (Figure 5.1 i), while another showed four groups of three (Figure 5.1 ii). Andy then said, "*Tell the person next to you what one you think is correct? When I said that I wanted three times four, which of those two options (Andy pointed to the cards) do you think is correct?*"

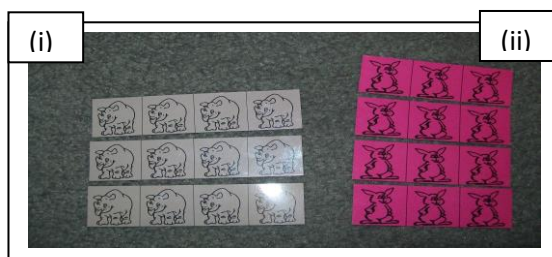


Figure 5.1 (i) Rhinoceros and (ii) Bunny animal strip cards showing 3×4 and 4×3 respectively

Bob acknowledged the learning intention (referred to as the WALT) which was written on the whiteboard, "*We are learning to solving multiplication problems from what we know about our twos, fives, and ten.*" He picked up a Slavonic abacus, moved two rows of seven beads across and said:

Bob: Can anybody tell me another way of saying [this]? Remember we are doing multiplication.

Child 1: Seven times two.

Child 2: Two times seven.

Bob thought for a brief moment and then continued:

- Bob: Two times seven. What I want you to have a look at now are the two problems. I might be a little bit off track but I want you to see even though [Child 1] said *two times seven*, and [Child 2] said *seven times two*.
- Child 2: [interrupts] Seven twos.
- Bob: OK. Remember what, we did before? Equals is also equal to. What can, (pause), what can equals be said, as?
- Child 3: Same as.
- Bob: The same as yes. OK, but what we are going to look at is that *two times seven* is actually equal to *seven times two*, but they are different. Does anybody know why they are different?

Beth began the lesson by recording the learning intention in the modelling book and the students read it together: “*WALT understand that the order of the numbers in a multiplication equation can change the meaning even though the answer is the same.*” A discussion followed, during which the meanings of the words equation, equals, and factors, were unpacked. From there, Beth moved into her lesson based on use of the multiplication symbol, as it relates to understanding commutativity.

Results and analysis of the two lessons

Observation of the teachers’ professional knowledge during the first multiplication lesson is presented in depth and correlated to a lesson towards the end of the unit. When analysed against the PCK Framework (Figure 3.1), variation in frequency of times a particular category was used in each of the two lessons (Table 5.3) was seen as a point of difference in practice, rather than improvement or decline in usage (Chapter Three Section B: Introduction). The second lesson was recorded for three of the case-study teachers, as due to unforeseen circumstances, Anna was unable to complete teaching of the multiplication unit. A relieving teacher under Anna’s guidance subsequently taught the students in her class.

The aim of the study was to see where teachers applied their professional knowledge in classroom practice, and it was deemed necessary to record their application of such knowledge against the different categories as they arose. The frequency of specific aspects of teacher professional knowledge often saw characteristics of two, or even more categories on the PCK Framework, overlap. An example of such overlap, was when Andy asked if there was a difference between *four times five* and *five times four*. The students argued amongst themselves and Andy said:

If you've got a good argument, let's hear it. I could be wrong. It wouldn't be the first time in my life I was wrong. So if you can justify your answer, let me know. Okay.

This statement was categorised as a Teaching Strategy (Table 5.2: A.3) as Andy asked the students to justify their thinking, and it was also categorised as Getting and Maintaining Student Focus (Table 5.2: C.2), as it was important to acknowledge the use in both. An example of multiple coding is evident in the coding stripes from NVivo10 in Figure 5.2.

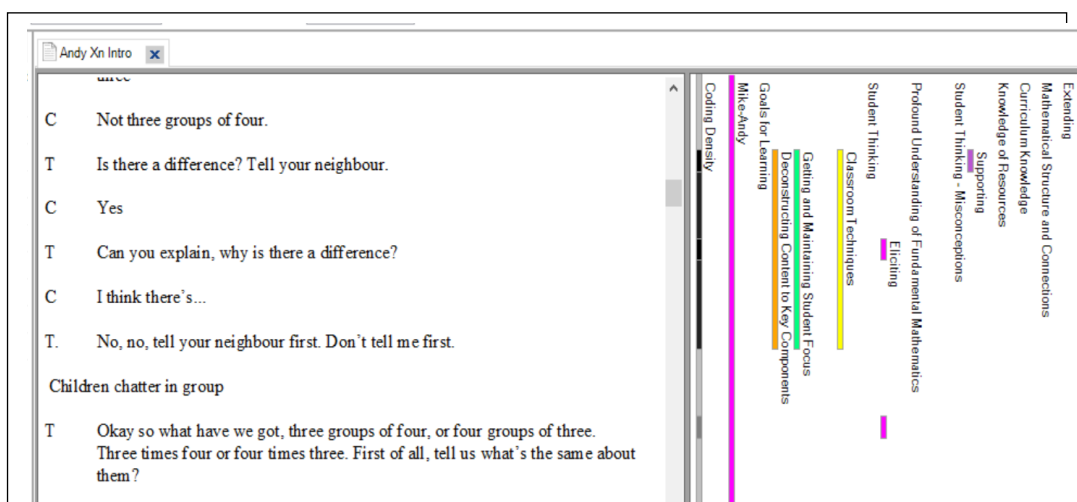


Figure 5.2 Sample of multiple coding in Andy's multiplication lesson

In the lesson, a “supporting” question was used when Andy asked the students to explain what the difference in representation was between three groups of four and four groups of three. Andy and the students were “deconstructing the content” of the mathematical concept that although 3×4 and 4×3 have the same answer, representation of the two examples is different. Asking one student to explain their thinking to others was a classroom technique, or practice, he regularly used. At the same time the sharing of ideas became a way of “getting and maintaining” the students’ attention, as they knew that at any time Andy might ask them to reword what their peers had been saying.

There were also times when observation by the researcher of particular actions, might be interpreted in different ways. For example, the use of manipulatives came to the fore many times and so were coded to various categories (Table 5.2).

Table 5.2

Use of one manipulative coded against different categories

Manipulative	Coding	Reason
Sheet of paper with small squares on top	Purpose of content knowledge (A1)	Use of estimation
	Curriculum knowledge (A2)	Connection between multiplication and area
	Student thinking – misconceptions (A8)	Misconception and difficulty associated with skip counting in hundreds
	Mathematical structure and connection (B2)	Connection between array model in multiplication and square units in area

5.2.1 Clearly PCK (A)

The Clearly PCK part of the framework highlights the difficulty in separating the pedagogical knowledge from the mathematics content. Often these aspects are intertwined to the extent that neither pedagogy nor content is the dominant aspect.

Purpose of Content Knowledge (A.1) and *Curriculum Knowledge (A.2)* Observations indicated that there was a strong overlap between the *Purpose of Content Knowledge* (an awareness of reasons for content being included in the curriculum) and *Curriculum Knowledge* (evident when a teacher shows how topics fit into the curriculum) as separated out on the PCK Framework (Figure 3.1 & Table 5.3).

Bob mentioned the curriculum during the initial lesson. Bob reminded the students they were currently at Stage 4 and Stage 5 (Curriculum Levels 1 and 2) on the Number Framework (Ministry of Education, 2008a), and that he wished to move them to their expected Stage 6 (Curriculum Level 3) by the end of the unit (Table 3.3). Although the other teachers did not specifically mention stages or levels, they commented on the capability of the students. For example, Anna stated, “*Because my knowledge of you people is that you are no longer counting them in fives, you’re doubling the fives, or some of you even triple the fives don’t you? Because you’re very quick in your calculating, aren’t you?*” She acknowledged that the current problems were easy for the students to calculate, but was gauging their understanding of multiplication prior to moving onto more complex problems.

Table 5.3

Frequency of each PCK category used in the introductory multiplication lesson and final lesson (in brackets) and totals (excluding Anna).

PCK Category	Andy	Anna	Beth	Bob	Total
A. Clearly PCK					
1. Purpose of Content Knowledge	0 (0)	0	0 (0)	1 (0)	1 (0)
2. Curriculum Knowledge	1 (5)	2	1 (1)	4 (1)	6 (7)
3. Teaching Strategies	6 (9)	6	7 (7)	6 (2)	19 (18)
4. Cognitive Demands of Task	3 (4)	1	1 (3)	2 (1)	6 (8)
5. Appropriate and Detailed Representations of Concepts	2 (6)	3	2 (4)	2 (0)	6 (10)
6. Knowledge of Resources	4 (1)	1	2 (8)	5 (3)	11 (10)
7. Student Thinking	4 (7)	6	8 (4)	10 (5)	22 (16)
8. Student Thinking - Misconceptions	2 (3)	5	11 (1)	2 (5)	15 (9)
B. Content Knowledge in a Pedagogical Context					
1. Deconstructing Content to Key Components	7 (5)	13	20 (8)	10 (5)	37 (18)
2. Mathematical Structure and Connections	1 (5)	0	1 (1)	2 (0)	4 (6)
3. Methods of Solution	2 (2)	6	1 (5)	2 (1)	5 (8)
4. Procedural Knowledge	0 (1)	4	3 (1)	4 (1)	7 (3)
5. Profound Understanding of Fundamental Mathematics	1 (1)	4	3 (1)	2 (0)	6 (2)
C. Pedagogical Knowledge in a Content Context					
1. Classroom Techniques	4 (4)	5	0 (1)	2 (1)	6 (6)
2. Getting and Maintaining Student Focus	5 (10)	5	0 (1)	2 (1)	7 (12)
3. Goals for Learning	7 (3)	8	8 (4)	8 (3)	23 (10)
4. Knowledge of Assessment	0 (0)	4	2 (3)	2 (0)	4 (3)
5. Questioning - Supporting	8 (21)	20	16 (26)	26 (21)	50 (68)
6. Questioning - Eliciting	8 (7)	14	1 (10)	18 (9)	27 (26)
7. Questioning - Extending	1 (6)	2	6 (4)	1 (0)	8 (10)

Whereas specific reference to the curriculum was seldom observed in the introductory teaching session, follow-up learning conversations with the teachers indicated relevant curriculum knowledge and associated expectations. For example, when discussing their class levels during the learning conversations, Bob and Beth mentioned that the majority of the students in their classes were below the expected Curriculum Level and comparable Number Framework Stage. Similarly, Anna and Andy mentioned that some of their students were below expectation. All of the teachers mentioned that they utilised planning sheets from the New Zealand Mathematics website (Ministry of Education, *n.d.a*) and relied

on these for lesson content guidance (relationships between NZC, the NDP books, and other published Ministry material are included on these plans).

In the final lesson, the two combined areas (A.1 and A.2) again occurred with little frequency. Andy showed the most difference in his application of curriculum knowledge (from 1 instance to 5 instances) between the two lessons. In the final lesson, multiplication in number and area in measurement, were combined when he focused on the area of shapes, using the students' understanding of multiplication. While Andy always emphasised correct answers, in the final lesson he also encouraged exploration of ideas as part of the accurate solving of problems. For example, in one instance, the students had a blank sheet of paper and a small pile of cut out squares of paper (Figure 5.3). The students began by estimating how many smaller squares were on the sheet of paper. This was the only time the word estimation was heard in any of the lessons observed. After the students estimated the number of squares, they worked out how many covered the sheet of paper by placing the small squares on top of it, to form an array (Figure 5.3). The students made the connection between the multiplication fact and area when they saw that three rows of six squares (3×6) equalled an area of 18 square units.

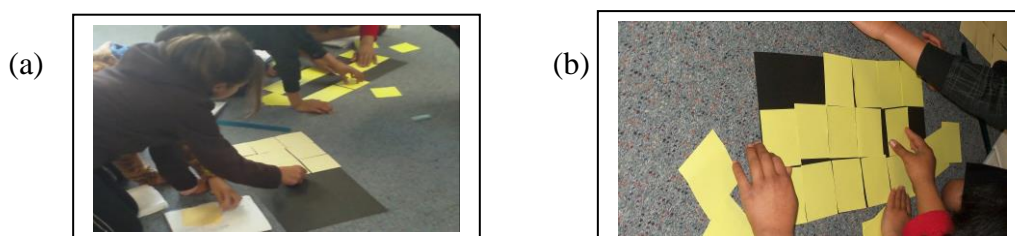


Figure 5.3 (a) Small squares of paper were used to find the area by constructing (b) three rows of six squares

In the final lesson, Andy recognised that the students were not answering questions using strategies for an appropriate stage on the NDP Framework (for their class level) and encouraged the students to use an alternative, more advanced strategy. For example, after Andy's students worked out the number of small squares covering the larger sheet of paper by individually counting or skip counting the squares (Figure 5.3), he suggested that they could have worked out the total in a smarter way using multiplication. As his students were Years 6 and 7, they were expected to solve such problems multiplicatively (Stage 6) rather than additively (Stage 5). After some prompting, the students agreed that there were

three rows of six squares totalling eighteen and each square was one hundred square centimetres. Some of the students recognised that 18 times 100 square centimetres, meant that there were 1800 square centimetres in total.

Another example of Andy's curriculum knowledge was when he worked with the class in the final lesson and challenged the students to find the area of composite shapes, when the lengths of some of the sides were unknown (Figure 5.4a). Rather than giving the students all of the required measures and implementing a formula to find the area, he extended the students' thinking by encouraging them to find the missing measures themselves. When the students were required to find the area of a triangle, he supported the students' thinking by giving them a square cut in half diagonally, so they could see that the area of the triangle was half the area of the square (Figure 5.4b).

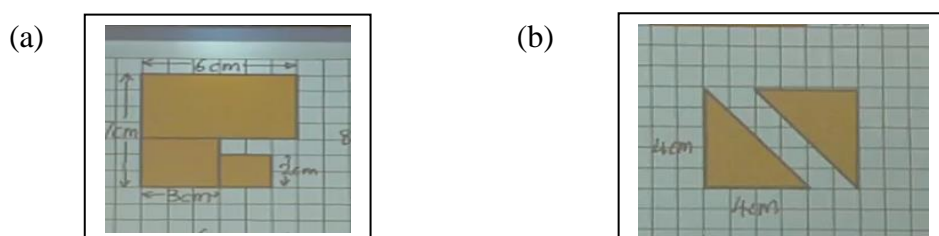


Figure 5.4 (a) some sides unknown (b) Andy's little clue to finding the area of the triangle

In the final lesson, Beth encouraged the students to see the connections between times tables, including $\times 5$ and $\times 10$, and using doubles as a means of carrying out multiplication of $2\times$. At one stage the conversation went:

- Beth: What is sixteen lots of five going to be?
 Child: Eighty.
 Beth: Eighty, why is it going to be eighty?
 Child: Because you are just, ah saying it in halves. [Then] You double ten it equals twenty and you double twenty equals forty, if you double forty it equals eighty.
 Beth: OK, so what would thirty-two times five be?
 Child: Um ... a hundred and ... sixty.

The child saw the connection between 2 groups of 5 equalling 1 group of 10 and therefore the number of groups halved, so that 16 groups of 5 became 8 groups of 10.

Teaching Strategies (A.3)

Teaching strategies, or approaches for teaching a mathematical concept or skill (Figure 3.1), were used a similar number of times by the teachers in the initial multiplication lesson ($n = 6$ or 7), (Table 5.3). However, there was a noticeable

change between the two lessons in the frequency (of teaching strategies) used by Andy (6 to 9) and Bob (6 to 2), while Beth's remained the same (7). The change in frequency was observed as being aligned to the teaching approach of each lesson, as outlined below.

In both lessons, Beth spent most of the time sitting on the floor alongside her students, with a modelling book in the centre of the group. Questions were posed for the students, who recorded their ideas and explained their thinking in the modelling book, which was referred back to throughout the lesson. Bob's introductory lesson was very teacher-directed and he taught seated on a chair with the class on the floor in front of him, while in the final lesson he sat on the floor with the students. In the initial lesson, Bob did not utilise a modelling book with all recording carried out on the class whiteboard. However, in the final lesson, Bob and the students recorded regularly in the modelling book, which was on the floor in front of them.

All of the teachers began their teaching sessions by sharing the learning intention (WALT), the exception being Andy's initial multiplication lesson. The WALT became the central focus of the lesson and was referred to regularly, especially when the teachers sensed that the students' conversations were unrelated to mathematics and the key ideas being taught. Bob was the most conscious of the WALT, especially in the initial lesson where he recorded it on the whiteboard and referred to constantly.

Throughout the initial lesson, the use of manipulatives played an important role when consolidating the students' understanding of multiplication. Bob held a Slavonic abacus and occasionally invited individual students to use it while they explained their thinking to others. In contrast to this, Andy, Anna, and Beth, had sufficient equipment for all students to model their answers simultaneously. Andy's students used animal array cards (Figure 5.1), while Anna's and Beth's students used Unifix cubes. The concrete manipulatives were an appropriate support for students in these three classes to problem solve collaboratively and discuss ideas in pairs.

In the initial lesson, the teachers asked the students to discuss ideas together, in order to gauge understanding. However, the talk was based generally around

telling each other step-by-step what had been carried out, with discussion and justification of solutions seldom occurring. Collaborative problem solving encouraged by Andy, Anna, and Beth, contrasted with Bob, whose students seldom participated in conversations as the lesson was predominantly teacher led and followed the Initiate, Respond, Evaluate (IRE) pattern. Andy, who wanted the students to solve problems by themselves, provided minimal support during discussions. This was evident when Andy checked the students' understanding of the difference between the written expressions, 3×4 and 4×3 :

Andy: Is there a difference? Tell your neighbour.

Child: Yes.

Andy: Can you explain why is there a difference?

Child: I think there's ..."

Andy: No, no. Tell your neighbour first. Don't tell me first.

In the final lesson, discussion became a little more evident in Andy's class, occurred to some extent in Beth's class, and showed little change in Bob's class (Table 5.3: A.3). Beth spoke regularly throughout the lesson and generally directed the conversation. In the final lesson, while Bob sat with the students on the floor, he maintained a structured, teacher-directed lesson. The IRE pattern generally continued with little opportunity for students to participate in thoughtful discussion together. At times the students talked, but observations showed that seldom did the students mention alternative strategies or challenge each other in their thinking.

In their final lesson, Beth and Anna used word problems as a teaching strategy (A.3). An example of Beth's use of word problems, was when her students unpacked the difference between 5×3 and 3×5 :

If you had five people in your family and you had some biscuits to distribute out, and you gave them three each, would that be the same as if you had three people in your family and you gave them five biscuits each?

While Anna acknowledged word problems early in her lesson, the students continued using their equipment to solve written equations and the word problems never eventuated:

"When we make up our word problems (pointing to 4×6 and 6×4 recorded in the modelling book) it can be quite different. And we'll make up word problems later regarding our multiplication equations."

Cognitive Demands of Task (A.4)

The teachers unpacked the *cognitive demands of the task* when they identified aspects of the task that affected its complexity for their students. In the initial lesson, all teachers identified the same difficulty the students were experiencing with understanding the multiplier as the first factor in the multiplication equation and the multiplicand as the second factor. Understanding the commutative property as it relates to multiplication was used to help reinforce this idea, by comparing the structure of the problems (as interpreted in the New Zealand classroom) as reflected in the written equation, for example 6 groups of 4 and 4 groups of 6 or, 6×4 and 4×6 (Figure 5.5). The teachers supported this understanding by focusing on array models, although Beth also allowed her students opportunities to create linear models on number lines along with cut out materials (when they showed repeated units additively). Bob modelled multiplication equations on the Slavonic abacus, while the other teachers encouraged the students to construct examples for themselves using Unifix cubes.

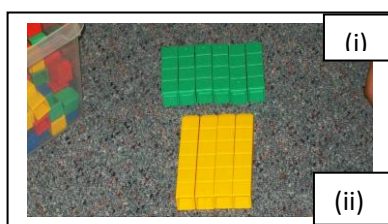


Figure 5.5 Array models showing (i) 6×4 and (ii) 4×6

The importance of using correct mathematics language associated with the multiplication symbol came to the fore. Andy emphasised the meaning of multiplication symbol as groups of and as he pointed to the written equation said:

This \times thing here [said like letter x of the alphabet], we are going to start thinking of as groups of. So whenever you see this \times thing, I want you to start thinking of groups of. So [think] 6 groups of 10 (Andy pointed to the \times symbol), or 8 groups of 20 (he recorded 8×20 and pointed to the \times symbol).

Anna stated that the size of the numbers they were multiplying would affect the complexity of the task. While consolidating the students' understanding of equations she used simple (small) numbers, like 6 times 4. She asked, "*What do you think about when you multiply 6 times 4, or 4 times 6?*" She encouraged thinking with single-digit numbers, as these were easier to model and/or image, and later moved to larger two-digit, and three-digit numbers. She moved on to problems involving a two-digit factor such as 4×99 and 99×4 , and alongside

this discussed a range of possible strategies that could be used. These strategies mainly focussed on the students' prior knowledge of commutativity and place-value, and included *place-value partitioning* and *rounding and compensating*.

There was little variance in frequency of awareness by the teachers of the cognitive demands of the tasks, between the start and end of the multiplication unit (Table 5.3). Andy was aware that some students experienced difficulty understanding the relationship between multiplication and area, such as when the students were finding the area of a triangle. Rather than tell the students how to solve the problem, he left them to see if they could use prior knowledge. Andy pointed to a diagram on the whiteboard and said, "*I've given you a little clue by taking the same triangle and flipping it over and sticking it over there* (Figure 5.4). *It might [just] be a brain teaser for you.*"

In the final lesson, Beth's awareness of the difficulties some students had understanding multiplication, meant that she referred back to key ideas taught in previous lessons (as evident in the modelling book). For example, when teaching the doubling and halving strategy using the $\times 5$ tables, she realised that some students did not recognise the pattern formed. She returned to the $\times 10$ tables which they had understood earlier and used this to reinforce the relationship with $\times 5$ tables.

Appropriate and Detailed Representations of Concepts (A.5) and Knowledge of Resources (A.6)

The two framework categories of *Appropriate and Detailed Representations of Concepts* and *Knowledge of Resources* in many instances overlapped with the *Teaching Strategies* identified above. Models constructed by the teachers and/or their students were often recorded as an equation in the modelling book. Beth was the only teacher who transferred the model made into a diagram and emphasised the multiplier and multiplicand by placing a circle around the objects within each set (Figure 5.6). Concepts being developed were sometimes put into word problems to help clarify the meaning and were aligned to the manipulatives available. For example, Beth represented the context of biscuits (4 packets of biscuits, with 5 biscuits in each packet) with Unifix cubes and Andy discussed groups of animals (4 burrows, with 3 bunnies in each) alongside the animal strips

(Figure 5.1). However, there were times when the teachers gave the students equations to solve without a context.

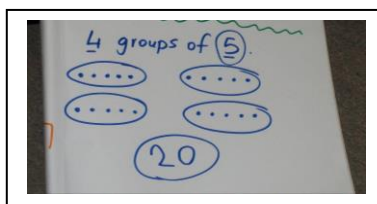


Figure 5.6 Diagram showing 4×5 as 4 groups of 5

In the final lesson, Beth and Andy supported conceptual understanding of multiplication with arrays. Beth simultaneously combined the use of Unifix cubes with the abacus to reinforce the array context on two different models, while Andy made the connection between arrays and the area of different shapes with his class, through diagrams on the whiteboard (Figure 5.7), and when teaching a group through cut-out squares of paper (Figure 5.3). As the students solved the problems together, they discussed previously learned multiplication strategies amongst themselves in a manner that had not been evident in the first lesson.

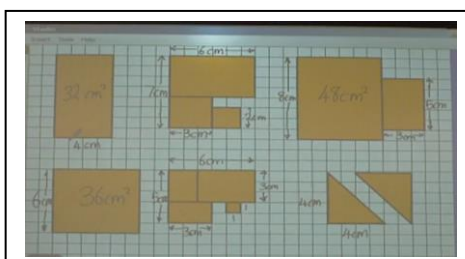


Figure 5.7 Area problems on the whiteboard

Student Thinking (A.7)

In the initial lesson, *Student Thinking* was utilised by Beth and Bob twice as frequently as that of Andy and Anna (Table 5.3: A.7). This was possibly due to the teaching style of the lessons: Bob's lesson was predominantly teacher-directed and provided opportunities for him to respond immediately to the students' ideas; Beth had a semi-structured lesson, which combined giving the students time to solve problems themselves with the IRE model. It was during these teacher-directed conversations that building on student thinking occurred. This contrasted with Andy and Anna's students who appeared less reliant on their teacher for direction, spending more time manipulating materials and solving problems in their groups.

Bob's students often mentioned ideas he had not apparently anticipated, and while he acknowledged the students' thoughts, he seldom capitalised on the opportunity to extend their thinking. An example of this, was when he showed four rows of ten beads on the abacus and asked the students what problem they represented:

- Child: [Be] cause there's two ways.
Bob: There's two ways?
Child: You can do two times twenty and [um] four times ten.
Bob: OK. Wow, this is a good one. [Child] here said two times twenty, which is right, or four times ten. Can you tell me, just by looking through it, what would that be? (Bob pointed to his model of four rows of ten beads on the abacus).
Child: This one would be four times ten.
Bob: Four times ten. OK, remember four times ten is like that. (On the abacus Bob showed four rows with ten beads on each). Two times twenty is like that. (On the abacus Bob showed two rows of beads with ten in each on one side of the abacus, and two rows of ten beads on the other side of the abacus). So we're looking at these two and those two (he pointed to the two different models). Although they give the same answer, they are different.
Child: That's quicker (the child pointed to the four rows of ten beads).
Bob: Pardon?
Child: But that way is quicker to answer it.

Rather than explore the student's response and ask for further explanation of his reasoning, Bob returned to his pre-conceived idea and moved completely away from the student's thoughts onto something else. While Bob might have capitalised further on the student's initial statement, "*There are two ways to solve the problem,*" he refocused the students on the WALT for the day. Bob's use of acknowledge a student's idea and then move on, was in contrast to Anna's way of utilising her students' thinking, where she encouraged risk-taking and the sharing of ideas:

- Anna: Which is easier: multiplying ninety-nine groups of four, or four groups of ninety-nine?
Child: Four groups of ninety-nine.
Anna: Tell us more. Share your thoughts. You've got good thinking I can tell....
Share [child's name] why you chose that one?

The student then justified her thinking. Others agreed or disagreed, with respectful discussion often leading to non-threatening debate among the students.

Capitalising on the students' thoughts was less evident in the concluding lesson, with the exception of Andy (Table 5.3: A.7). Bob continued to maintain a structured lesson and rigidly adhered to his learning intention. The planned lesson progressed and he provided little opportunity for the students to suggest strategies and ideas of their own, nor did he capitalise on opportunities that arose, such as

understanding and using place value to solve multiplication and division by ten.

An example of this is evident in the following conversation:

- Bob: How many tens are there in the number eighty?
Child: Eight.
Bob: Well done. How many tens are there in one hundred and ten?
Child: Eleven.
Bob: Eleven.
Child (2): I know how to work it out. It's in the number.
Bob: How do you work it out?
Child: It's just the first two numbers, or it's the number.
Bob: Okay.
Child: Eighty, it's eight.
Bob: Yes, well done.
Child: And with a hundred and ten, it's eleven.
Bob: Okay. Does anyone know what this piece of equipment is?

Bob praised the student for his knowledge and immediately moved on to his planned lesson. Similarly, at one stage during the final lesson, one of Beth's students mentioned adding a zero, and the comment was overlooked. This contrasted with Anna's initial lesson when a student stated, "*If you multiply by ten, you just add a zero.*" Anna paused momentarily, and discussed the moving of numbers across a place value each time ten groups of a number are made.

In the final lesson, Andy challenged the students to move beyond their comfort zone and build on their understanding of finding areas of rectangles by multiplication (Curriculum Level 3), to finding the area of composite shapes (early Curriculum Level 5). He presented problems and posed questions, giving the students opportunities to respond to these, for example when he gave the students an irregular shape to find the area of (Figure 5.8a). Some of the students panicked because the shape did not have stated measures for all sides and Andy shaded part of the shape, suggesting that this might help them to solve the problem (Figure 5.8b).

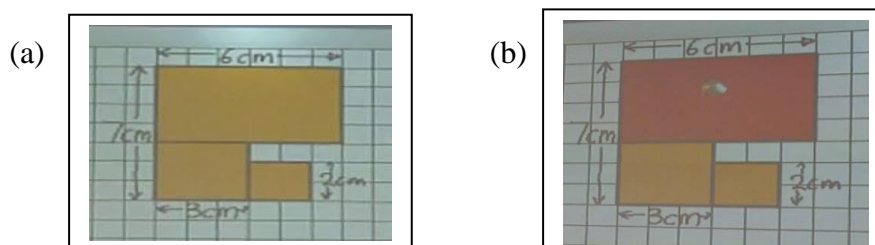


Figure 5.8 Finding the area of an irregular shape (a) as shown initially and (b) part of the diagram shaded

Later during discussion, a student came forward to share his thoughts using the squares on the whiteboard to help solve the problem. Andy asked, “*If you didn’t have the squares, how would you know what it was?*” At other times, Andy built on students’ thinking and used their ideas to ensure understanding, with comments such as: “*Can you actually explain that, because that is really awesome*”; “*Now let’s see if you guys [students] are right*”; “*But is it different? Are they different, or are they the same?*”

Student thinking – Misconceptions (A.8)

Alongside building on students’ thinking, was the recognition by the teachers of *Students’ Misconceptions*. In the initial lesson, Andy and Bob had two instances of recognising misconceptions, which contrasted with Anna (5) and Beth (11). Bob modelled multiplication problems on the Slavonic abacus, but the students did not have manipulatives available to them individually. Occasionally he invited students to share their thinking (on the abacus), but few offered and they appeared not to volunteer unless they were confident in contributing a correct response. This meant Bob was not in a position to notice possible misconceptions, because conversations were generally with individual students who had volunteered correct answers. This contrasted with Andy, who gave all the students manipulatives and allowed them time to talk and come to agreement within their groups. However, Andy sometimes answered a question for the students, rather than allowing time for their responses. This limited the opportunities for him to gauge students’ misconceptions.

Many misconceptions focused around the accepted interpretation of the multiplication symbol as “groups of”. Beth’s students struggled to understand this idea and she continually re-worded and/or modelled the concept:

- Beth: How would we write a multiplication equation?
Child 1: You would say two (um), two and... I can’t remember the rest of it.
Child 2: You’d do something like two plus...
Beth: Plus. Would it be [plus] if it was multiplication?
Child 2: Oh, no it would be...
Child 3 (interrupts): It’s a times table.
Beth: Yes, a times table. Multiplication is a big fancy word for times table.
Child 3: So two times four would equal eight?

Once Beth had reinforced the meaning of multiplication, she asked the students to use the Unifix cubes, to show what 3 times 5 (3×5) looked like. The conversation continued:

Child: Do we have to get three blocks?
 Beth: You are going to need to get enough for 3 times 5.
 Child 1: We'll need to get eight, eight blocks (adding the factors).
 Beth: Eight blocks, do you think you'll need eight?
 Child 1: Yes.
 Beth: You get out how many you think you might need to do three times five.
 Child 2: See I did it prettily [the child had constructed the equation (Figure 5.9)].
 Beth: Oh not like that. I want you to show me another way. We can actually explain times is (as), three lots of five.
 Child 2: Three lots of five? Like, five, ten, fifteen.

Confusion between addition, multiplication, and conceptual understanding of the expression 3×5 , was still evident (Figure 5.9).

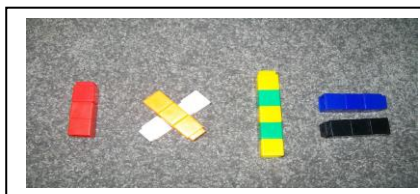


Figure 5.9 A student's model of 3×5 as an equation

Because of on-going misconceptions around the meaning of multiplication, Beth used a real-life context to help her explain the difference. She turned the multiplication problem into a division problem, to help demonstrate the difference:

You've got a total of fifteen biscuits in the cupboard. This family over here has five children okay. How many biscuits are they getting each? Over here you have a total of fifteen and you're dividing it [the biscuits] up for three people. Are these people [points to a circle drawn in modelling book] getting the same as these people?

A lengthy conversation was held around division as groups of things (biscuits) being shared out evenly, and multiplication also about groups of things. Beth recorded two equations, $3 \times 5 = 15$ and $5 \times 3 = 15$, asked half of the students to make three groups of five, and half to make five groups of three. Immediately one student said: *"They're both the same. It's pretty obvious because you have just turned it around."* Beth responded, *"Okay. Well, you show me with your blocks what each one means."* The students modelled their equations with the Unifix cubes, and eventually the conversation continued:

Beth: Okay, so we've swapped them [the numbers in the equation] around. They've got the same answer, but do they mean the same thing?
 Child 1: Yes.
 Child 2: No.

Beth used the phrase groups of, most of the time when conversing with the students. Misconceptions arose when the students were required to make

connections between groups of during discussion, use of the word times in word problems, and the symbol in the written equation.

There were times when Anna unwittingly caused confusion among her students. As Anna's class explored the difference in representation between equations such as 6×3 and 3×6 , she asked the students which expression they thought would be the easier to solve and sometimes failed to recognise when a student's explanation was the wrong way around:

Child: Ah, 6 times 3 [be]cause if you do 6 times 3 (recorded as 6×3), you just think about 6, 12, 18, and if you do 3 times 6 (recorded as 3×6) you do 3, 6, 9, 12, 15, 18.

Anna: OK. So you were adding up, you just went up, you skip counted up in sixes?

This was contrary to what Anna had been teaching as the meaning of multiplication (6×3 interpreted as 6 groups of 3, and 3×6 as 3 groups of 6). She did not notice the error and instead agreed with the student. Anna acknowledged that the student had skip counted in sixes, but this did not relate to the expression she had identified as the easier to use. This may have caused some confusion for the students.

Beth identified the greatest number of student misconceptions in her first lesson, while in the final lesson she identified the least (Table 5.3). This was possibly because her second lesson was less teacher-directed and she allowed the students more time to problem solve together in their groups. The reverse happened for Bob, whose identification of misconceptions increased from the first lesson (2) to the last (5). In the final lesson, Bob's students had their own manipulatives, which meant that he recognised more readily when they were confused. They struggled to recognise the relationship between repeated addition and multiplication, and he tried to reinforce the difference:

Bob: I have said eight plus eight, but how can we express that in multiplication?

Child: Eight plus eight equals sixteen.

Bob: How can we express that as in multiplication? You've got eight plus eight.

Child: Um.

There was still confusion with understanding the multiplication symbol, in relation to the context that had been taught throughout the unit:

Bob: Is this two times ten, or is this ten times two? (pointing to a model on the abacus)

Child: Both. It's both.

There were times when Bob’s students showed misconceptions, which went unrecognised. For example, one student modelled 2 groups of 10 (2×10) and another modelled 10 groups of 2 (10×2). Bob recorded example (i) in Figure 5.10, as having modelled 2×10 , when 10 groups of 2 are counted, and example (ii) as 10×2 , when 2 groups of 10 are shown. This contradicted the meaning of multiplication that he had previously taught the students.

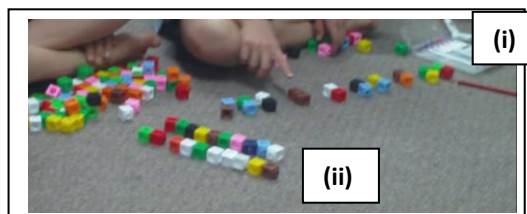


Figure 5.10 Models of (i) 10×2 and (ii) 2×10

Andy remained consistent in recognising misconceptions (2 in the initial lesson and 3 in the final lesson). In Andy’s class, the students experienced difficulty skip counting with larger numbers. At one stage, they determined that each of the smaller pieces of paper was 100 square centimetres (they measured $10\text{cm} \times 10\text{cm}$) and attempted to work out how many square centimetres were on a large sheet of paper (Figure 5.3):

- Andy: So how many square centimetres do you have now? You’ve got one hundred...
- Child 1: One hundred, two hundred, three hundred, four hundred, five hundred, six hundred, seven hundred, eight hundred, nine hundred and ten.
- Andy: What comes after nine hundred?
- Child 2: Nine hundred and ten.
- Andy: Nine hundred and the next number is? Th...
- Child 1: One thousand.
- Andy: Good girl.
- Child 1: One thousand, two thousand.
- Andy: Not two thousand. One thousand, and one [what]?
- Child 1: Hundred. One thousand, two hundred.

Beth recognised that some students found it difficult to determine what happened to the product when the multiplier was doubled, for example when they multiplied by ten. Beth modelled the pattern on the abacus, and showed: two groups of ten; four groups of ten; eight groups of ten; then Beth asked, “*What would 16 groups of ten be?*” Beth recorded the facts one-step at a time in the modelling book and discussed the pattern with the students. As they doubled the multiplier, the students initially thought the product went up in groups of two, and then they changed it to twenty. One student said, “*You just add a zero*”. Eventually they

recognised that when the multiplier doubled, the product also doubled. Beth repeated the idea with the $\times 5$ tables, as she wanted them to see and understand the pattern. The conversation continued:

Beth: Now let's go back and do it with the fives. Let's just make sure that we can see how it's doubling. Can you show me 2 lots of 5, or 2 times 5 (2×5) please? Can you write that down?

Child: (records the expression in the modelling book) There.

Beth: Now, show me 4 lots of 5, or 4 times 5. Can you write the equation for that down? [Child's name] is going to do the next one. I want you to double it to get me 8 lots of 5. Show me 8 lots of 5 on there, on the bead board [frame]. Show me how you would do 8 times 5. What are 8 groups of 5 going to be? Can you write the equation for me?

Child: What would that other one be, if we had to equal 80?

Beth: To equal 80. Let's work backwards and see if you can figure [it] out. Remember, we had groups of 5, so what might it have been? What might this number have been? (Points to the first number – the multiplier in the recorded equation 2×5). See if you can work backwards. 2 times 5 = 10, 4 times 5 is? (pauses for a brief moment) 20. 8 times 5 = 40. Something times 5 = 80.

Child: 9?

Beth then asked, "*What would sixteen lots (groups) of five be?*" Some of the students were able to double the 40 to get 80, so she said, "*Okay, what would 32 lots of five be?*" Some of the students knew that they doubled the $16 \times 5 = 80$ and attempted to answer, but could not calculate this mentally. Beth later became confused when she tried to show the connection between the $\times 5$ and $\times 10$ facts, and the connection between doubling and halving in this context. She steered towards this at one stage, but forgot what she was trying to achieve and said, "*I had it right in my head, and I'm confusing it now.*" She was confused about what she was doing in her lesson and realised that she was also confusing the students. Beth stopped the lesson and said, "*I might just leave it there for now because I don't want to confuse you anymore. I'll have a think about it and work with you again tomorrow.*"

At times Andy contradicted himself, in the final lesson. An example of this was when he was establishing 4 groups of 5 equals 20 ($4 \times 5 = 20$):

Andy: Four. OK. 4 times 5 is?

Child: 20.

Andy: 20. Let's count it. 4, 8...

Child: 12, 16, 20.

Andy: 4 times 5 is (slight pause) 20.

Andy began the count in groups of four (instead of five) and the students continued from his example. Bob created a similar confusion for his students

when he wanted them to recognise the connection between groups of five and groups of ten. He modelled 5×10 (5 groups of 10) on the Slavonic abacus. He said, “*If I split this apart, what equation have I got? 5 times 10 is the same as ...?*” As he spoke, he split the groups of ten in half. The conversation continued:

Child (1): 10 times 5

Bob: No, I don't [paused]. 5 times 10 is the same as... ?

Child (2): 10 times 5.

Bob: Yes, I know but remember we're using fives [groups of five] to solve the equation. We're halving it right? 5 times 10 (5×10) is equal to (what)?

Child (2): 50.

Bob: Fifty. 5 times 10. If we halve it?

Child (2): 5 times 5?

Bob: 5 times 5, No. 5 times 10 is equal to 10 times 5. Okay?

Two students answered 10 times 5 at different times. Bob said, “*No*” to one of the students and “*Yes*” to the other, which caused confusion among the students. Part of Bob's difficulty arose, because the numbers in the five groups of ten (recorded as 5×10) and ten groups of five (10×5) were reversed from one expression to the other. Had he used different numbers in his example, for example: $4 \times 10 = 8 \times 5$, the confusion may not have occurred.

5.2.2 Content Knowledge in a Pedagogical Context (B)

In this part of the framework, the most important part of the knowledge is to do with the mathematics, rather than how it is taught. The focus is on mathematics content knowledge as used for teaching, including subject matter knowledge.

Deconstructing Content to Key Components (B.1)

In both the initial and final lessons, *Deconstructing Content to Key Components* was the area within this section of the framework that was utilised the most frequently, by all teachers (Andy 7 [5], Anna 13, Beth 20 [8], and Bob 10 [5]). Deconstruction of content was necessary when students did not understand a particular mathematical concept, which was fundamental for applying that concept in a given situation. An example of deconstruction was, when the teachers taught multiplication as repetition of equal groups (a composite unit), and they reiterated the importance of understanding that the first factor in the equation (the multiplier), represented the number of groups, while the second factor (the multiplicand), represented the size of each group. An example of this, was when Beth unpacked the idea that 8×2 meant 8 groups of 2 (or $2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$), and Anna explored the difference between 5×8 and 8×5 . Recording both

expressions additively enabled the students to appreciate the difference between the multiplier and the multiplicand. While the teachers acknowledged that they wanted the students to move on from solving multiplication problems additively, they recognised that by representing them as repeated addition helped the students understand what occurs in the multiplication process. Later Beth asked, “*How could we use addition to show two times five?*” A child replied, “*Five plus five.*” Understanding this key idea was crucial for implementing other ideas that followed. However there were occasional times when the teachers acknowledged expressions incorrectly, such as when one of Anna’s students said that 6×3 could be skip counted 6, 12, 16, 24 (Section 5.2.1: A8). This was contrary to the meaning of the symbol as taught in the lesson and caused confusion later.

Deconstruction also occurred when the teachers used the array model to show the meaning of commutativity in relation to multiplication. Discussions were held with the students about the connection between repeated addition, which they had modelled and discussed earlier, and the array that they modelled with Unifix cubes. The students modelled representations such as 6×3 and 3×6 as arrays, and discussed the difference between appearances of the two models. Anna extended her students’ understanding of the array when she asked them to use a rounding and compensating strategy to solve 3 times 99 (3×99), by visualising a representation of 3 groups of 100 (3×100) and then subtracting one from each group (3×1).

One key component Beth highlighted was the importance of patterns in mathematics. As the students skip counted in different groupings, the patterns were recorded on the hundreds board (Figure 5.11).



Figure 5.11 Recognising the multiple of three pattern

These patterns became a focus throughout the lesson and Beth returned to them when the students were unsure about solving their multiplication problems, such

as when they became confused about the difference in representation between 4×3 and 3×4 .

Mathematical Structure and Connections (B.2)

Teachers were observed overlapping *Deconstructing Content*, and *Mathematical Structure and Connections* within their lessons. For example, as Bob and Beth highlighted the importance of the array model in understanding multiplication they made connections between $\times 5$ and $\times 10$ on the Slavonic abacus. They showed that every group of ten was comprised of two groups of five, resulting in twice as many groups with half as many objects in each. At one stage, Bob showed 40 beads on the abacus. He asked the students how many groups of 10 that was and invited someone to also show how many groups of 5 that might be. The abacus was ideal for seeing the important link between one group of ten and two groups of five, as while the beads are displayed in rows of ten, each row contains two groups of five shown in different colours.

In the final lesson, Bob recorded 4×18 on the whiteboard and drew arrows demonstrating how this could be solved by doubling the 4 to make 8, and halving the 18 to make 9. Bob mentioned he was getting off track, but continued on for the more able students in the group and demonstrated how doubling and halving could be further developed into the idea of thirding and trebling. He wrote the equation 6×12 on the whiteboard and said, “*We divide that by 3 [pointed to number 6], and times that by 3 [pointed to number 12].*” Generally, the larger number is divided and the smaller number multiplied. However, in this instance, the students recorded 2×36 and used their knowledge of doubles to solve the problem, by doubling 36 to get 72. He returned to the problem 4×18 :

So four times eighteen, even though we don't do the eighteen times table, comes out to seventy-two. We could use threes, here's the thing. We could divide that by three [18] and times that by three [4]. Whatever we have to do to one number we have to do the opposite to the other. Sort of balance it out.

Bob recorded the resulting 12×6 , which gave him the same answer as the earlier 6×12 , as well as the original equation of 4×18 .

In the initial lesson, Andy noticed that some students modelled the groups within the multiplication expression by colour co-ordinating the Unifix cubes and responded, “*Ok I'm not interested in colour co-ordination. This is not algebra - we are not doing patterning.*” He had drawn everyone's attention to algebra and

missed an opportune moment to discuss the patterns formed. However, in the final lesson, Andy made deliberate connections between multiplication and algebra. The students were finding the area of shapes and he commented at one stage, *“I’ve talked about it before and I’ve given you the formula for it, the algebra formula for it. But, I want you to tell me in your book. Give me a sentence why it works.”* At the time, Andy was discussing how to find the area of a triangle. He explained, *“Understanding a formula is a bit like doing algebra, in that a formula works every time you used it, regardless of the size of the numbers.”*

A mathematical structure and connection Andy identified in the final lesson, was the link between the formation of arrays in multiplication and square units in area. While he did not specifically state the connection, it was implicit when the students were asked to count the number of rows of square paper, the number of pieces of paper in each row, and multiply them to find the area (Figure 5.3b). He told the students that the small pieces of paper were ten centimetre squares and asked them to use their rulers to make a ten centimetre square on their paper. *“Find the ten centimetres with a ruler, and then you can rule the lines.”* Earlier, Andy had consolidated understanding of the multiplication symbol, through representation of commutativity. *“Think about it. Six groups of three are lots of little groups and three groups of six are only a few big groups, isn’t it? [So] they add up to the same, but they don’t look like quite the same thing.”* Some students were unsure, so they used Unifix cubes to construct the expressions and examine the difference between the representations that both totalled the same amount. When the students were solving the area of different shapes, Andy prompted them, *“Remember we’re dividing this into squares aren’t we?”* They had made their square unit representations on paper, they had modelled them with cubes, and as they solved equations, he pointed to the square units on the board as a reminder (Figure 5.7). He also reminded the students, *“I have given you the formula. I want to know why it works.”* His emphasis on understanding what was happening was a noticeable difference from his first lesson to his final lesson.

Methods of Solution (B3)

The teachers demonstrated methods for solving problems in varying ways. The importance of understanding some basic facts ($2\times$ [doubles], $\times 2$, $\times 5$ and $\times 10$), and using these to work out other problems, was the final lesson emphasis for Beth

and Bob. Beth used *fly flip cards* to show the connection between groups of five and groups of six. The cards had a pictorial representation of 5 flies on the front of each card, with the number 6 written beneath. The students found the difference between the 5 flies on the front of the card and the number 6 by determining the number of flies on the back of the card. This showed the connection between using what is known ($\times 5$ tables) to work out the unknown ($\times 6$ tables). For example, $8 \times 6 = (8 \times 5) [5 \text{ flies on the front of each of the } 8 \text{ cards}] + (8 \times 1) [1 \text{ fly on the back of each card}]$.

Anna also encouraged building on known facts to work out unknown facts, as a strategy to solve problems. When the students shared their procedure for solving 3×6 , one student said she had skip counted in sixes. Anna replied:

So you were adding up, you counted up, skip counted up in sixes. Anyone do it in any way different? [Paused a moment]. Everyone did it the same way? You wanted to count up, or skip count in sixes? Did anyone use the fives? And go up in 3 groups of 5, and then add on three more?

She continued:

Anna: If we had 3 groups of 5. [She counted] 1, 2, 3, 4. Oops, 3 groups of 5 [modelled with blocks], but we want 3 groups of 6. We would have to add on three more wouldn't we?

Child: Yes.

Anna: So we would have to add one more on to each row to get our three groups of six. So fifteen and three is ...?

Anna was aware that the students knew their $\times 5$ table, and encouraged them to see that $\times 6$ was about adding one more to each group, and therefore, 3×6 could be solved by doing $(3 \times 5) + (3 \times 1)$. While Anna demonstrated this strategy with known basic facts, she later encouraged the students to use the same strategy on more difficult, double-digit operations. An example used was, $3 \times 27 = (3 \times 25) + (3 \times 2)$. Anna's students also adeptly applied the use of doubles. As she solved 4×8 , one student explained that she could solve four times something, by doubling and doubling again:

Anna: Alright so when you doubled 8 what did you get?

Child: 16.

Anna: And then what did you do after that?

Child: Got the 16 and added the other 16, and got 32.

Anna: So in actual fact, you doubled 8 and got 16, and you doubled 16 and you got (what)?

Child: 32.

Anna pushed this idea a little further and continued:

So we could keep that going and get larger [number of] groups of 8 couldn't we, by doubling again and getting 8 groups of 8.

Rounding and compensation was also a solution strategy used by Anna's students, which extended the earlier taught notion of using what is known to work out the unknown. She asked the students, "*Is it easier to solve 4×99 or 99×4 ?*" They agreed that 4×99 was easier, as 99 can be rounded to 100, and the answer found by calculating $(4 \times 100) - (4 \times 1)$.

Procedural Knowledge (B4)

All teachers showed procedural knowledge early in their final lesson, when they outlined rules with no further explanation at the time. Andy mentioned the formula required to find the area of a square and suggested it be used to find the area of other shapes. He told the students the answer was in square units, but gave no reason why. Beth initially told the students, "*You find doubles by finding two lots (groups) of something,*" and then moved on to discuss other multiplication facts. Bob mentioned that two groups of five equalled one group of ten. He then said, "*And so six times five, is equal to three times ten,*" and did not capitalise on this to any extent at the time.

Profound Understanding of Fundamental Mathematics (B5)

Aspects of PUFM were demonstrated infrequently by all of the teachers, with a range of 1 to 4 in the initial lesson and 0 or 1, in the final lesson. Anna showed her understanding of place value when she discussed the importance of this knowledge with the students. The students were asked if it was easier to solve 3×100 or 100×3 . They decided 3 groups of 100 were much easier to visualise and understand than 100 groups of 3. One student said, "*I just added zeros at the end.*" Another student responded, "*Yes that's a quicker way.*" Anna replied, "*A lot of people do that but does that help with your understanding? What I am wanting is that you understand what you are doing.*" Anna reminded the students of what occurred when they added zeros to numbers for multiplication, and removed zeros from numbers when they divided. She explained:

Anna: What you are actually doing is you're putting zeros in the tens and ones [columns] as place holders, because we are shifting it [the number 3] up two place values, when multiplying by 100. We've talked about this before. The rule is you move up two place values when you multiply by 100. Similarly, when we multiply by 10, we move up one, and when we multiply by 1 000, we move them up what?

Child: Three places.

Anna: A similar thing applies when we divide by 10; everything goes down a place value. When we divide by a 100 they go down two place values, when we

divide by 1 000 they go down three place values. As long as we understand what we're doing, that is the most important thing.

As Anna and the students discussed the rule of adding zeros when multiplying by multiples of 10, she sketched columns in the modelling book and used arrows to show the students what was happening in terms of the numbers moving from one place (column) to the next. Anna reminded the student of the place-value houses that they had used in previous years, where the columns within each house were ones, tens, and hundreds, and the order of the houses were ones, thousands, and millions.

Understanding of the commutative property of multiplication was another aspect of PUFM shown by Anna when she explained, "*We can think about this type of multiplication as learning to change the order of the numbers, to make multiplication easier with big numbers.*" Similarly, Andy gave the students Unifix cubes to form arrays, so that they could model understanding of the commutative property. He asked the students to justify their explanation by questioning them about whether it was important to understand the difference between the two expressions (for example 3×4 and 4×3):

Andy: Hands up if you think it is important (that there is a difference in the representation of the problem).

Child: I think it is important because if you come to a test and it says 3×4 and you might do it the other way you will get it wrong.

Andy: But we just said it gives you the same answer, didn't we?

Child: Yes, but if it says show your working, it will be different.

Beth was the only teacher who showed PUFM with connections between multiplication and division, when she used division to show the difference between $3 \times 5 = 15$ and $5 \times 3 = 15$ (Section 5.2.1: A.8). Beth used two piles of 15 blocks, and divided one pile of blocks into 5 equal groups, and one pile into 3 equal groups, asking the students to imagine that they were biscuits. The students agreed that they were not the same, as people with 5 in their family would get 3 biscuits, while the people with 3 in their family would get 5 biscuits. Beth then unpacked the idea that the result of the division problem provided each person with a group of biscuits, and that multiplying also involved the groups of idea. Had Beth recorded the equations it might have enabled the students to see the connections between the two expressions.

5.2.3 Pedagogical Knowledge in a Content Context (C)

This part of the framework acknowledges that there are occasions when professional knowledge has a major emphasis on the skills associated with teaching. While PCK within a subject area is topic-specific knowledge, this category considers generic teaching knowledge used in specific cases of mathematics teaching.

Classroom Techniques (C.1), and *Getting and Maintaining Student Focus (C.2)*

Classroom techniques are the generic classroom practices used by the teachers (Figure 3.1), and were used by the case-study teachers in a similar manner in both the initial and final lessons (Table 5.3). To varying degrees, all of the teachers invited students to share ideas with others as they solved problems. In the initial lesson, the students struggled with the concept discussion and the importance of collaboration, and while the teachers encouraged their students to work together and talk together, this occurred infrequently. Andy regularly reminded the students, to talk with each other before explaining solutions to him:

Andy: Why is there a difference?

Child: I think there's...

Andy: No, no, tell your neighbour first. Don't tell me first.

Andy persevered and continually encouraged group talk. This was evident when he set up working groups for the lesson, and said:

So you three work together, and you two work together. In fact, you could actually work together in one big group. Help each other. Okay? Talk amongst yourselves. Work it out together then prove it to me.

At other times, Andy invited the students to challenge his thinking, and on more than one occasion commented:

If you've got a good argument, let's hear it. I could be wrong. It wouldn't be the first time in my life I've been wrong. So if you can justify your answer, let me know.

Asking the students to share ideas was also a technique used to maintain student focus (C.2). Anna frequently used this practice and instructed the students, "*Talk, pair-share with the person beside you. And listen carefully.*" Andy implemented the pair share idea in a slightly different manner. As students finished their own work, he asked them to check with their neighbours to see who had used the smartest way to solve the given problem, reminding them to stay focussed at all times and be ready to give an answer at any time. Occasionally, when students appeared inattentive, he addressed individual students by name and asked them to

share their thoughts with the class. This refocused not only the individual student concerned, but also others who were aware that they might be the target in the future. He encouraged the students to listen carefully and not merely accept what others said:

How did you know that was 24? Is she right? Don't stop thinking about it just because someone is giving you an answer. It could be wrong.

Bob suggested to the students that they raise their hands and seek support if they did not understand a concept during the lesson. However, the students seldom asked for help and at one stage during the lesson when he suspected that some students were unsure, he reminded them:

Hands up who is not getting what we're saying. OK. So we all know it. Remember don't be afraid to put your hand up, because if I do carry on and you don't understand it well then, I'll go on, and then if you put your hand up later on, then I'm going to have to go all the way back.

Andy showed a greater variety of ways for gaining student attention (C.2) in the final lesson, and one of these was setting challenges for the students. He made comments such as: "*Now these ones here are a little bit trickier*"; "*I'm going to make things a little more interesting today and you [guys] are going to have to turn your brains on*"; "*Oh this is tough*"; and "*It's a brain teaser for you.*" Observations showed that the students responded to the challenges afforded them, and set about trying to prove that they could complete the tasks correctly.

Goals for Learning (C.3) and Knowledge of Assessment (C.4)

The teachers were reliant on the WALT in both lessons, to emphasise the *goals for learning*, as well as maintaining *student focus* (C.2). All teachers began their lessons by sharing the WALT and the intended learning outcomes for the students (the only exception was Andy's initial lesson). Bob, regularly returned to the WALT to provide a focus to the lesson. Continual reference also became problematic (seven times during the final lesson), as frequently Bob stifled the students' responses when they provided solutions not directly linked to the chosen learning intention. An example of this was when discussing 2×20 and 4×10 modelled on the abacus:

Bob: 4 times 10. [OK] remember 4 times 10 is like that [pointed to 4 groups of 10 representation on abacus]. 2 times 20 is like that [demonstrated on abacus]. So we're looking at these two and those two. Although they give the same, they give the same answer, they are different.

Child: That's quicker [pointed to 2×20].

Bob: Pardon?

Child: But that way is quicker to answer it.

Bob: Yes, but remember going back to our WALT we're dealing with our twos, fives, and tens.

Beth returned to the WALT at the end of the lesson, as she wished to see whether the students understood the purpose of the lesson. Beth concluded her lesson:

Beth: Can you just explain what we've learnt about?

Child: We've learnt about factors and how grouping works.

Beth: Yes but what did we learn about those equations?

Child: The numbers can be changed around but they can still, they can still add up to the same.

Beth: But they...?

Child: But they don't look the same.

The teachers' application of the *knowledge of assessment* category was one of the less frequently observed (range 0 to 4), and aligned to the lower frequency evident in the *curriculum knowledge* category (range 1 to 5) (Table 5.3: A.2). At the start of his first lesson, Bob reminded the students that they were at Stages 4 and 5 on the Number Framework (aligned to Levels 1-2) and that he wanted them to reach Stages 5 and 6 (Levels 2-3), (Section 5.2.1: A.1 & Section 5.2.1: A.2). However, the problems given during the lessons remained at Stages 4 and 5 (based around skip-counting, doubles, $\times 2$, $\times 5$, and $\times 10$ tables), and did not challenge the students to reach the higher stages appropriate for their class levels.

Written diagnostic assessment tasks were given to the students prior to the initial lesson, but none of the teachers used these to influence their planning and teaching. Bob referred to the assessment in his first lesson, "*Ah [researcher's name] took you over last week what we know. It wasn't actually a test it's just to see where you were in multiplication. What you knew.*" He advised the students that the assessment tasks were to find out what they knew and yet like the other teachers, he did not use this information as a starting point when planning lessons. Similarly, all teachers had assessment data available from other sources including Progress and Achievement Tests (PAT) and NumPA assessments, and discussions with the teachers indicated that data available from these assessments were also not used as starting points for lessons.

In Andy and Bob's final lessons connections were not made to previous lessons, which made it difficult for the researcher to ascertain to what degree formative assessment practices were used. Andy's students carried out many tasks in their personal mathematics books and it is possible that he utilised these at other times. However, Bob's students were not observed carrying out any written tasks in

either their own books or the modelling book, providing no obvious written evidence of understanding and progress. This contrasted with Beth who began her final lesson using the modelling book to revise the previous week's learning and asked the students to use their equipment to show $2 \times 5 = 10$ as an array model. She viewed the models for evidence of understanding of the written equation. She said, "*Let's just check this for revision. If I was to turn it around, what's the difference between 2 times 5 (2×5) and 5 times 2 (5×2)?*" She was immediately aware of which students were able to recognise the difference between the two expressions.

Questioning Techniques (C.5, C.6, C.7)

As indicated in Section 3.13 and Figure 3.1, question types were subdivided into three groupings, based on the patterns of practice categories of Fraivillig et al.'s (1999), Advancing Children's Thinking Framework (ACT): supporting (C.5); eliciting (C.6); and extending (C.7).

Questioning – Supporting (C.5)

All teachers used *supporting* type questions to the greatest extent throughout their lessons (Table 5.3: Andy 8 [21], Beth 16 [26], Bob 26 [21] and Anna 20). This was particularly evident when the IRE pattern occurred, with closed questions supporting the students' current thinking. An example was in the initial lesson when some students in Beth's class had made five groups of three, while others had made three groups of five:

- Beth: Can you tell me how many blocks you have got altogether over this side?
Child (1): 15.
Beth: Have you got 15?
Child (1): Yes, 3 groups of 5.
Beth: (turns to another child). How many have you got altogether?
Child (2): 15.
Beth: 15. How might we write your equation?

Supporting questions were also evident when a question was directed at an individual or group that required a particular student to explain a method, or when a student was invited to share an idea with others by using the equipment or writing in the modelling book. Examples of these included "*So how would we write that as a multiplication equation?*", "*Can anyone give me an example?*", "*(Child's name), can you show me that with your cubes?*", "*Well done, how did you do that (Child's name)?*", "*Can anyone tell me how many beads there are?*",

and “*What was your thinking when you shared that with (Child’s name)?*” These questions supported the students’ current thinking by encouraging them to clarify their own solutions. All of the teachers used this question type regularly to reinforce a particular student’s thinking and/or explanations.

Questioning – Eliciting (C.6)

The next most frequently used questioning format, was *eliciting*. The most common form of eliciting used by all of the teachers was when they encouraged students to elaborate on a peer’s response. An example of this in Bob’s class, was when one student provided a method of solution, he asked the rest of the class, “*Can you come and explain to me what [child’s name] is actually saying?*”

Eliciting questions generally provide students with the opportunity to communicate their mathematical thinking with each other, think of different solution methods, and to think beyond the initial response of those in their group. They encourage students to wait and listen to the description others gave, as part of forming their own opinion. This was evident when Bob said, “*So if we’re doing multiplication, and [child’s name] said seven plus seven, what’s another way of solving this problem?*” Anna often asked if others wanted to share their thoughts in order to seek different solution methods. Anna said, “*Talk, pair share with the person beside you, is it the same? And listen carefully.*” She wanted the students to discuss ideas together, to listen to what each other said, and later be in a position to explain why they decided on their solution. When different solution methods were given, the students were often asked to decide which might be the smarter strategy to use given a particular problem. Similarly, Andy instructed the students: “*Okay so what have we got here? Work it out with others in your group*” or, “*Is there a difference? Tell your neighbour*”. The students were encouraged to share the ideas back with the group.

Questioning – Extending (C.7)

The question type used the least often was that of *extending*. This was when students were challenged to move beyond initial efforts, were pushed to attain learning goals, and encouraged to consider and discuss interrelationships among mathematical concepts. The discussion that followed this type of questioning was indicative of high-level thinking and theorizing. One example of extending used by Andy was when he did not accept a solution to a problem and instead said to

the students concerned, *“Can you justify your answer for me please?”* At this point, Andy expected the students to go beyond initial solution methods and reason about why they solved the problem in that particular way. In an effort to draw on ideas for discussion, Andy challenged students to reconceptualise problems by asking class members for different solution methods. For example, he said, *“Can someone tell me how we could work out...?”* or *“If you didn’t have the squares (background to the diagram on the board), how would you prove the answer was six square centimetres?”* Andy wanted the students to justify their thoughts and made comments such as: *“Can you explain your theory to everyone else, please?”* At one stage, a student pointed to other students’ work and said, *“They’re right.”* Andy replied, *“They’re right, are they? Tell me, why?”* Andy also extended the students’ thinking by encouraging reflection on a previously used formula:

But what I also want to see is this. I’ve left this one in here as a special one (Andy pointed to an area problem on the whiteboard) and I wonder if you can work it out. I’ve talked about it before and I’ve given you the formula for it, the algebra formula for it. But I want you to tell me in your own words. Give me a sentence why it works.

5.3 Multiplicative Domain Summary

A comprehensive analysis of teacher knowledge and student learning from two observed multiplication lessons (the initial lesson and a lesson toward the end of the unit), was presented using a detailed framework (Figure 3.1).

In the Clearly PCK category, the teachers used a range of strategies to encourage and support their students’ learning. They utilised their students’ thinking to promote further learning with varying frequency (Table 5.3) depending on whether the students were left to problem solve among themselves, or whether discussion was more teacher directed. Manipulatives were used to represent concepts and support students in their conceptual understanding. Initially not all students had materials available for their use (e.g., Bob’s first lesson), but by the final lesson all students used them. Estimation was seldom used as a method to check for reasonableness of answers, inconsistency in the use of mathematical language caused confusion, and repeated reference to the learning intention (WALT) often stifled potential opportunities to utilise unanticipated teachable moments.

In the Content Knowledge in a Pedagogical Context category, the teachers regularly deconstructed the content of what they were teaching in an effort to assist the students with their understanding of multiplication. However, the small number of times the teachers made connections with the mathematical structure of the problems, was highlighted by the few instances in which they exhibited what is referred to as, profound understanding of mathematics. There were times when the teachers recognised student misconceptions and corrected these. For example, when students were confused over the meaning of the multiplication symbol, Beth unpacked the written expression additively to show the equal groupings. However, at other times the teachers caused misconceptions due to in-the-moment responses. For example, when Anna overlooked a child's response to 6×3 as skip counting 6, 12, 18, which was contrary to the meaning of the times symbol as taught in the lesson.

In the Pedagogical Knowledge in a Content Context category, the teachers did not utilise the assessment data available to them to plan their lessons, and instead relied heavily on the lesson sequence in the NDP books and associated planning sheets for direction and content. The majority of questions were the lower-level supporting type, and the teachers seldom extended the students with higher expectations, going beyond initial solution methods to solve problems.

The findings presented in this chapter are discussed and critiqued in Chapter Eight, in relation to research literature. The following chapter (Chapter 6) presents the results of observations of the teachers' professional knowledge when teaching a lesson from the proportional domain.

CHAPTER SIX

RESULTS and ANALYSIS: THE PROPORTIONAL DOMAIN

6.1 Introduction

This chapter presents a detailed analysis of four observed lessons, one by each of the case-study teachers, mid-way through a unit within the proportional domain. Each lesson was taught towards the end of the school year.

6.2 Observed Teacher Professional Knowledge in the Proportional Domain

One lesson was observed in each of the four case-study teachers' classes to assist in understanding the teachers' professional knowledge in relation to teaching fractions and decimals. Anna's lesson with her Years 7 and 8 students, focused on understanding decimal numbers, while Andy, Beth, and Bob, focused on fractions (Table 6.1). Andy began the lesson with all of the students working at their desks (a practice commonly used by teachers because of their NDP participation), using coloured rods to identify how many fractional pieces equalled one whole. Half of the students continued to complete worksheet problems individually, while he worked with the other half on the floor. Anna, Beth, and Bob, were observed teaching half of the class, while the other half worked at their desks, on independent tasks consolidating prior fractions learning.

Table 6.1
Observed fraction lessons

Teacher	Whole Class (or) Group	Lesson Focus
Andy	Class (start of lesson)	relationship between fractional pieces and one whole
Andy	Group	find fractions of a lengths, including seeing when a fraction is greater than one whole
Anna	Group	place value understating in decimal numbers
Bob	Group	using multiplication to find the fraction of a set
Beth	Group	find fractions of lengths, including seeing when the fraction is greater than one whole

As for the lessons for the multiplicative domain, the observations were analysed utilising the PCK Framework (Figure 3.1). The extent to which each category was observed in classroom practice is presented in Table 6.3.

As reported in Chapter 5 (Section 5.2), there were times when parts of the lessons could be coded against multiple categories on the Framework (Figure 3.1). In these instances, observations were coded against the various classifications (Figure 6.1) as it was deemed necessary to record the teachers' applications of professional knowledge against the different categories as they arose.

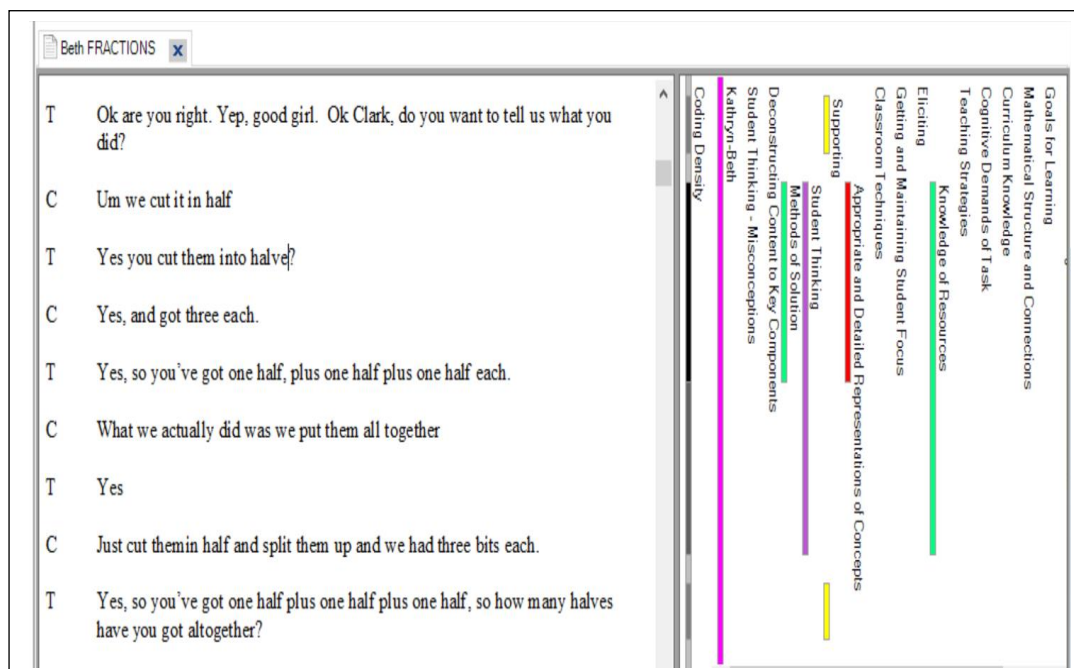


Figure 6.1 Sample of multiple coding in Beth's fraction lesson

Multiple coding is seen in Figure 6.1 when Beth showed aspects of four different categories simultaneously. The teacher demonstrated "Knowledge of Resources" through the use of cut up strips of paper to create wafer biscuits and showed "Appropriate Representation of Concepts" as she discussed how the paper had been cut into halves which might then be added together. The teacher supported the "Student's Thinking" discussing with the student how the three individual half pieces are combined to equal three halves, and "Method of Solution" is shown when the teacher supported the student to solve one half, plus one half, plus one half is equal to three halves.

There were also times when the observation of a particular action might be interpreted in different ways. For example, the use of the learning intention, or WALT arose frequently and was coded to various categories (Table 6.2).

Table 6.2
Use of the WALT coded against different categories

Action	Coding	Reason
Use of the Learning intention (WALT)	Purpose of content knowledge (A.1)	WALT copied from NDP Book 7 showing relationship to the curriculum
	Classroom techniques (C.1)	Sharing the WALT, recording in the modelling book, and unpacking what this meant for the lesson.
	Getting and maintaining student focus (C.2)	Reminding the students of the WALT when they wandered off task

Another time when an action might have been coded in a different manner was in the category Cognitive Demands of the Task (A.4). However, in all instances the researcher coded based on what was deemed the most appropriate, depending on the current teaching situation combined with student responses and actions. An example of this, was when Andy and Beth discussed that “if a number of objects are shared out evenly among a number of people, and there are more people than objects, everyone receives a part of a whole.” While understanding this idea is dependent on the word problem given, the prior knowledge of the students, and the manipulatives being used, the main issue here is that the teacher stopped and unpacked the aspects of the task that affected the level of complexity for the students (Figure 3.1), hence the A.4 coding.

6.2.1 Clearly PCK (A)

The first category examined in detail was that of Clearly PCK where it was difficult to separate the pedagogy from the mathematics, because of the overlapping links between the two concepts.

Purpose of Content Knowledge (A.1) and Curriculum Knowledge (A.2)

On analysing the observed fraction lesson, *Purpose of Content Knowledge* (an awareness of reasons for content being included in the curriculum and how it might be used), and *Curriculum Knowledge* (evident when a teacher shows how topics fit into the curriculum) were two of the categories addressed least frequently in classroom practice. The teachers did not specifically mention the curriculum and/or associated Number Framework within their lessons, although these were implicit in the chosen lesson objectives. Andy’s and Beth’s group lessons were based on a lesson when transitioning from the advanced counting to early additive stage (AC to EA) (Ministry of Education, 2008g, p. 16), while Bob’s lesson was moving from early additive to advanced additive (EA to AA)

(Ministry of Education, 2008g, p. 26), all below the appropriate curriculum level for the year level of the students.

Table 6.3

Frequency of each PCK category used in the observed fraction lesson

PCK Category	Andy	Anna	Beth	Bob	Total
<i>A. Clearly PCK</i>					
1. Purpose of Content Knowledge	0	0	0	0	0
2. Curriculum Knowledge	1	1	2	0	4
3. Teaching Strategies	5	1	2	2	10
4. Cognitive Demands of Task	5	0	3	0	8
5. Appropriate and Detailed Representations of Concepts	4	3	2	1	10
6. Knowledge of Resources	2	1	4	1	8
7. Student Thinking	4	4	5	0	13
8. Student Thinking - Misconceptions	6	2	8	1	17
<i>B. Content Knowledge in a Pedagogical Context</i>					
1. Deconstructing Content to Key Components	10	7	9	3	29
2. Mathematical Structure and Connections	0	5	2	3	10
3. Methods of Solution	1	3	8	9	21
4. Procedural Knowledge	0	2	2	7	11
5. Profound Understanding of Fundamental Mathematics	0	3	1	0	4
<i>C. Pedagogical Knowledge in a Content Context</i>					
1. Classroom Techniques	7	4	8	3	22
2. Getting and Maintaining Student Focus	10	5	4	3	22
3. Goals for Learning	1	4	3	3	11
4. Knowledge of Assessment	3	5	1	1	10
5. Questioning - Supporting	18	17	28	11	74
6. Questioning - Eliciting	6	5	6	3	20
7. Questioning - Extending	1	3	0	2	6

Anna's lesson was for students moving from AA to AM (Ministry of Education, 2008g, p. 38). The teachers admitted during follow-up conversations, that they found NZC (Ministry of Education, 2007) provided limited guidance as to exactly what the students were expected to know and understand in relation to fractions and decimals at different levels of the curriculum. Hence, there was a reliance on examples in the NDP Book 7 (Ministry of Education, 2008g) when moving from Stage 4 to Stage 5 (Andy and Beth), Stage 5 to Stage 6 (Bob), and Stage 6 to Stage 7 (Anna) and associated planning sheets, to guide their teaching.

The only observed connection to curriculum knowledge was via the chosen learning intention (WALT) which Andy, Beth, and Bob, copied directly from NDP Book 7. Andy and Beth's WALT was, "I am learning to find fractions of lengths, including when seeing a fraction is greater than one" (although they varied the lesson and included circles), while Bob's was "I am learning to use multiplication to find a fraction of a set." Anna began her lesson by acknowledging a gap identified in student knowledge from an earlier lesson and adapted a WALT in the NDP book. She recorded in the modelling book, "WALT determine how many tenths there are in all of a number."

Teaching Strategies (A.3)

All of the teachers used manipulatives to reinforce conceptual understanding, although Andy admitted during his post-research interview he had often struggled with this concept:

"I never really understood why children needed to understand what they were doing in maths when they could just give me an answer. I've never been a particularly heavily equipment-focused person so actually; you know having [lead teacher's name] hand me a whole load of equipment at the start of the year didn't actually thrill me very much. It's like, what am I going to do with this?"

At the start of lesson, Andy worked with the whole class. Andy decided to utilise manipulatives into his fraction lesson, and used different-coloured rods (Figure 6.2) to show the relationship between fractional pieces and the number of those pieces that equalled one whole. The students modelled two halves (black rods) make one whole (brown rod), four quarters (green rods) make one whole, and two eighths (yellow rods) are in each quarter. The difficulty with using rods as manipulatives was that while the number of pieces equalled one whole, the proportional difference in the size was not evident. For example, the model indicated that four green rods (Figure 6.2 [ii]), equalled one whole, but each rod was not one-quarter the size of the whole (brown rod) (Figure 6.2 [i]); or eight yellow rods (Figure 6.2 [iii]) equalled one whole, but each was not one-eighth of the size of the brown rod. The importance of proportionality in fractions was overlooked in this model (Figure 6.2).

As their lessons progressed, Andy (now working with a group), Beth, and Bob, used manipulatives to allow the students to exemplify fraction word problems in context. Andy represented apples with fraction tiles, Beth used paper strips to resemble wafer biscuits, and Bob used paper circles to represent cakes.

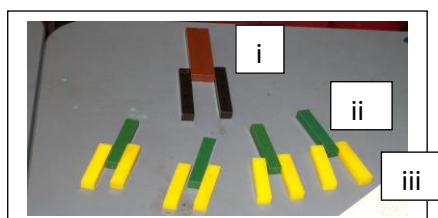


Figure 6.2 Coloured rods showed number of pieces in one whole, without proportional representation.

The real-life context in the word problem was exemplified using equipment, to make the mathematics learning more meaningful for the students. For example, Beth said, “*I want you to imagine these strips of paper are wafer biscuits. I want you to take three wafers each [indicated to pair of students], and I want you to divide them up between two people so that you get the same amount.*” Similarly, Bob picked up a paper circle and said, “*We’ve got four people at [Child’s] party. If you look at this, it represents a cake. Your birthday cake [Child]. You have 16 candles to put on the cake so that each person gets the same number of candles on their piece of cake. How many candles will each person get?*” Bob continued with further examples using a number of students’ names, and included himself and the researcher in other examples.

Andy’s students used fraction tiles to share five apples among four people. After discussion and manipulation of the equipment, the students gave each person a whole apple (tile) and re-unitised the other whole into four equal pieces, giving a further one-quarter to each person (Figure 6.3a). While the students solved the problem, Andy reminded them that they were to prove the five apples were shared evenly. One student placed a whole tile over the top of the four quarters (Figure 6.3b) in order to prove that there were five whole apples altogether.

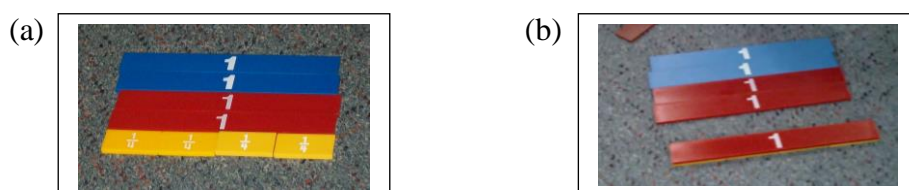


Figure 6.3 Student examples of 5 apples shared among 4 people (a) showing five whole apples gives each person 1 whole and $\frac{1}{4}$, while (b) proving that 4 whole apples and four quarters (beneath one whole) equal 5 wholes.

All of the teachers asked the students to discuss their ideas to consolidate understanding (Section 5.2.1: A.3). For example, Anna said, “*Share your thinking with someone else and say what number you are showing [on your deci-pipe] and why you’ve done what you’ve done*”. Of the four classes, Anna’s students were

more inclined to converse with each other about their solution methods. However, while they told each other what they did, they struggled with justification of their procedures (Figure 6.4).



Figure 6.4 Students discuss fractions represented on the deci-pipes.

Andy encouraged discussion and observation of his facial expressions and tone of voice (evident on the video recording) indicated that he was frustrated when the students continued to work in isolation. While the students were seated in groups, he realised that they solved problems independently and directed them towards more co-operative learning:

Let's give it a try and see if it works. So [Child 1], you make one of those cakes into quarters please and [Child 2] you do the other one because [Child 3] has already done the eighths. So you're going to need the quarter tiles. Work out together how many quarter tiles you are going to need.

At another point during the lesson, Andy said to one of the students, "*Talk to the boys. See if you can work it out. Convince him.*" He wanted the students to consolidate understanding of the mathematics through discussion with each other. However, observations showed the students continued to tell each other the steps of their problem solving, rather than discuss their ideas together.

Cognitive Demands of Task (A.4)

The teachers unpacked the *Cognitive Demands of the Task* when they identified aspects of the task that affected its complexity for their students (Figure 3.1 and Table 6.3). This meant that when the students were confused or uncertain of the nature of the task, the teachers provided a simplified explanation of what was required. For example, Andy and Beth discussed that when a number of objects (for example, biscuits or pies) were shared out (or cut up) among a number of people, and there were more people than objects, no-one received a whole: instead each person received a fraction (of the whole). This generalisation was emphasised and unpacked accordingly with equipment. However, the students often struggled with what was the whole and what was the piece to be found. For example, when Andy's students shared four pies among three people, they were

unsure initially whether they wanted quarters or thirds, what the size of the piece was relative to a whole, and what the whole was. The conversation went:

Child: Because there's three people and you put them [cut them] into six and give everybody a sixth each. I know quarters won't do it.

Andy: Why won't quarters do?

Child: Oh wait, because there's only going to be four of them [pies] and there's only three people.

Andy: Okay. So what do you need then?

In Beth's class, the students had three wafer biscuits and four people. Some of the students were not convinced that they could do the problem with more people than biscuits:

Child: Ah there's going to be an odd one, so there's three people, is there three people? Mrs [teacher's name] is there three people?

Beth: No, four people and three wafers. We've already said that each person isn't going to get a whole one.

[After a slight pause and observation around the room Beth continued]

Remember that when we're cutting into fractions, each piece has to be the same [size].

The cognitive demands and understanding the complexities associated with the fraction tasks were unpacked through manipulatives representing the contexts. Andy began the lesson using paper circles, but realised that the students had difficulties identifying the fractional names of the pieces they created, once the circles were cut. The students put the paper circles aside and used commercial fraction tiles, which they found easier for identifying the name of the piece (the fraction name was on each tile). However, the students became confused again later, when the fraction tiles represented cakes, pies, and apples and they were unable to be cut into different sized pieces.

Beth used paper circles (pies) and paper strips (wafers) according to the different contexts of the problems given, which enabled the students to construct the scenario. The folding and cutting of the strips of paper (Figure 6.5) proved to be an important part of unpacking the problem, and when mistakes were made the students often took another piece of paper and had another attempt.

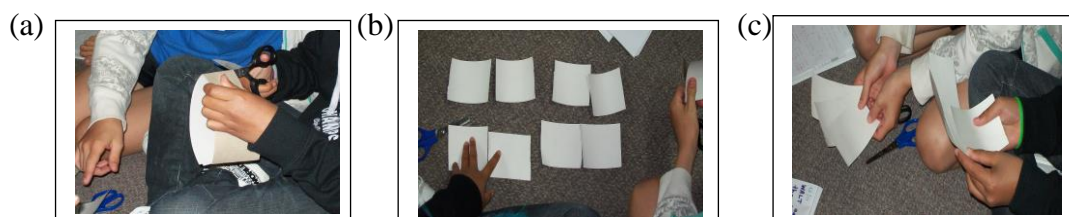


Figure 6.5 (a) Folding and cutting paper (wafer biscuit) to solve (b) how many pieces are required and (c) re-cutting if required.

Bob's students did not always recognise the importance of having equal-sized portions when representing fractions. When they divided their paper circles (cakes) into thirds they drew dividing lines to show three pieces, but these were often uneven-sized pieces (Figure 6.6). This caused problems when objects were placed onto the pieces to solve problems, such as 21 candles spread out evenly on top of the birthday cake. The uneven-sized pieces meant that the students did not always recognise the importance of even distribution.

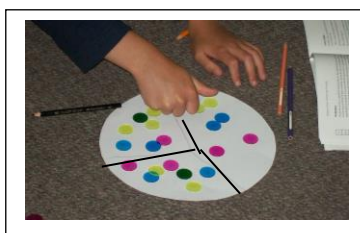


Figure 6.6 An example of 21 shared out into 3 (uneven) pieces

Bob also modelled and discussed the concept part-to-whole, on paper circles. Most of the cake was covered and the portion visible, was the fractional portion related to the whole. An example of this was when he asked the students, “*If I have some candles to spread out evenly over the top of my birthday cake and there are five candles on one quarter of my cake, how many candles would I have on my whole cake?*” Bob presented other similar problems to the students, to develop the relationship between the two concepts whole-to-part and part-to-whole.

Appropriate and Detailed Representations of Concepts (A.5) and Knowledge of Resources (A.6)

Using *Appropriate Representations* aligned to the teacher's *Knowledge of Resources* for the concept they wished their students to understand (Section 5.2.1: A.5 & Section 5.2.1: A.6). Anna's lesson focused on decimal fraction understanding and she commenced her lesson using a body fraction activity (Figure 6.7). Each student used their arm spans and folded arms to represent one whole, one half, or one quarter. As the students participated in the activity and formed the fractional representation, they related halves and quarters to the equivalent decimal fraction, which allowed them to show an understanding of common fractions and decimal equivalence.



Figure 6.7 Body Fractions showing 2 wholes and one quarter.

Anna's Year 7 and 8 students exhibited difficulties with decimal place-value understanding and she emphasised the importance of correct mathematical language, to ensure that the students were clear in distinguishing between tens and tenths, or hundreds and hundredths:

It is really important that you start using that [decimal] language and that you actually say what you're talking about. Otherwise you can be talking about any number, or any measurement, or something like that. So be specific.

The students utilized deci-pipes to model decimal place-value (Figure 6.4), with one whole pipe used for comparison of the decimal portions. The students proved that 10 tenths equalled one whole (Figure 6.8), then showed why 10 smaller pieces (hundredths) equalled one of the tenth-sized pieces. The deci-pipes allowed the students to establish that 10 hundredths equalled one tenth, and because 10 tenths equalled one whole, there would be 100 little pieces (hundredths) in that whole.

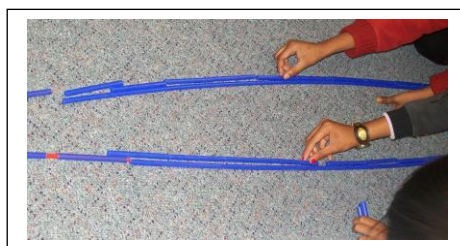


Figure 6.8 Ten tenths are lined up alongside one whole pipe.

Student Thinking (A.7)

Utilising *Student Thinking* to further consolidate understanding, was evident in the lessons of Andy (4), Anna (4), and Beth (5), and less evident in Bob's (1) lesson (Table 6.3). Andy's students were sharing three cakes among eight people when one student cut each cake (paper circle) into eight pieces. Another student disputed the necessity of cutting every cake into eight pieces:

- Andy: So you're saying instead of cutting every cake into eighths and giving people one bit from each cake, you could cut it...
- Child: You can cut all of them in different sizes.
- Andy: Let's give it a try and see if it works.

Andy realised that the student's idea of cutting each of the three cakes into different-sized pieces was a possibility, but what they were about to do would pose difficulties. However, he appreciated that making mistakes was part of the learning process and thought the students needed to realise this also:

So now you're happy. You've cut one cake into thirds (pointed to Child 1), you've cut one cake into quarters (pointed to Child 2) and you've cut one cake into eighths (pointed to a Child 3). Now you've got to share them with eight people so make eight even groups for me please. And I don't want to end up with anyone getting a small[er] amount of cake so you better make them all equal.

Andy observed the students as they shared out the pieces among the eight people. The students gave everyone an eighth, then four people a quarter, and when they began to give the rest a third they realised they could not share the pieces evenly. Andy left them to unravel how each person would get the same amount. Another group solved the same problem (sharing three cakes among eight people) with fraction tiles. Eventually some students received one eighth and one quarter, while some got three pieces of one-eighth (Figure 6.9a). Andy asked the students to prove to him that everyone had an equal amount. They did this by placing a one-quarter tile over the top of two one-eighth tiles (Figure 6.9b).

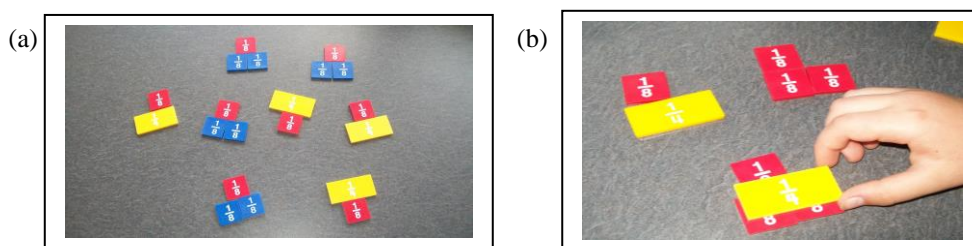


Figure 6.9 Representations of $\frac{3}{8}$ (a) as $3 \times \frac{1}{8}$ and $\frac{1}{8} + \frac{1}{4}$ and (b) Proving $\frac{1}{4}$ is the same as $\frac{1}{8} + \frac{1}{8}$

At times, Anna and Andy checked for understanding by determining whether the rest of the group agreed (or disagreed) with a student's idea. For example, a student in Anna's class used the deci-pipe to explain how she had modelled the number 0.62:

Child: I put six of the tenth pipes and I put two of the hundreds [hundredth] ones, because there's two in the hundreds [hundredths] column.
 Anna: Alright, is everyone in similar thinking as [child's name]?

Occasionally each teacher became involved in an IRE pattern of discussion using a student's thinking as part of checking for understanding. An example of this was in Beth's class when the students were sharing three cakes among four people:

Child: We cut them in half.
 Beth: You cut them in half. Did you give everyone a half?

Child: Yes.
Beth: So everyone got a half/? And what did you have left over when everyone got a half?
Child: One whole. We cut [it] into four pieces.
Beth: Then you cut it into four pieces. If you cut it into four pieces what have you cut it into? What's the fraction you've cut it into?
Child: Quarters.
Beth: Quarters, OK. So each person got what?
Child: A half and a quarter.

This type of conversation allowed Beth to check whether the student understood the physical process carried out when they cut the paper circles (cakes) into pieces.

Student Thinking – Misconceptions (A.8)

The nature of Andy's and Beth's group lessons allowed them to notice the misconceptions of the students, more readily than Anna and Bob (Table 6.3). Andy began his lesson by saying, "*You can work in pairs today, because pairs are always sharing the learning.*" The scenario was:

"I have five wholes, five whole apples. And I need to share them with [among] four people. How many apples is each person going to get? So, you guys [students] work it out. Use the tiles, these tiles and those tiles (pointed to tiles). Discuss it amongst yourselves, think about it. Five apples. You can move the tiles around, pick them up, you can swap them with the other tiles. You can do what you like, but you have got five. I want to see what you can do. You've got five apples, and you've got four people."

As Andy observed the students and listened to their conversations, he noticed misconceptions and discussed these with the students concerned. An example was, when one pair of students began to share the five apples (fraction tiles) among four people they were unsure whether to have four or five whole tiles (apples), and whether to divide each whole into four or five pieces. Andy said:

"Do you understand, now put them together, no, no, no, no, don't do that. Put them together and show me you've got five apples. Go. Prove to me that those are still five apples. No, no, no, I didn't say get another one out, I said put your stuff together and prove to me you've still got five apples. Anyone can just get another one out and stick it in there. Is that five?"

Andy recognised the students' misconception and immediately responded to it, which enabled them to continue problem solving in a meaningful way.

Beth was aware of students' misconceptions early in the lesson and while she sometimes corrected their misunderstandings, in other instances she allowed them time to work these out themselves. The first anomaly Beth observed was, regardless of how many whole objects there were to share and how many people there were to share the objects among, some of the students always began by

cutting the whole (objects) in half. While in the example of sharing three objects among four people the action gave the correct answer, the students were unaware mathematically of the size of the piece each person received:

- Beth: You've cut them all into quarters?
Child: No, I halved it and halved it again.
Beth: You halved it and halved it again? Yes. So you cut them into quarters, now put them into four groups. How many does each person get?

Beth recognised that the students did not see the connection between their action and the mathematical concept that, one-half of one-half is equal to one-quarter. Beth gave them the solution, and then guided them into understanding behind this important idea.

Another misconception some of Beth's students had was, regardless of how many people the objects (in this instance the biscuits) were shared among, each whole object must be cut up into the number of pieces that equalled the number of people. For example, three wafers among four people meant each whole wafer must be cut into four equal pieces (Figure 6.10a), or three wafers among eight people must be cut into eight equal pieces. While this is mathematically possible and always gives a correct answer, this is not always necessary and Beth wanted the students to explore the notion that there are times when problems can be solved using other more efficient strategies (Figure 6.10b).

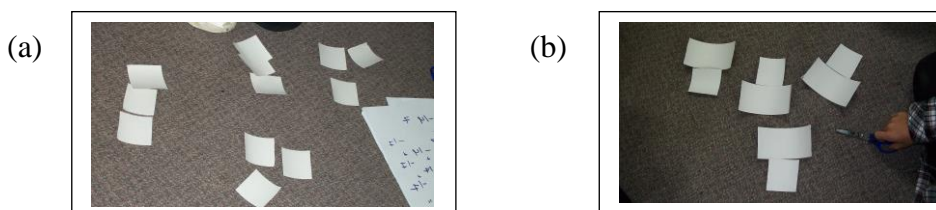


Figure 6.10 Examples of 3 wafers shared among 4 people (a) 4 people means each wafer is cut into 4 pieces (quarters) and (b) an alternative representation shown as $\frac{1}{2} + \frac{1}{4}$

Misconceptions also occurred in Beth's class when the students had difficulty deciding which number in the word problem assisted in determining into how many pieces the wholes (wafers) might be cut. For example, when there were three wafers to share among four people the students were unsure whether to cut the wafers into three equal pieces, or four equal pieces (Section 6.2.1: A.4). Initially, because they had three wafers they decided to cut them into three equal pieces, but they soon realised there were four people to share the pieces among.

After giving each person two pieces (using 8 of the 9, $\frac{1}{3}$ sized pieces), they had a one-third piece left:

- Beth: So you've cut those into thirds then?
Child: Yes but see they've [all] got two each. They've got two each, they've got two each, they've got two each, and they've got two each (pointed to each of the four people). Then we'll cut this [piece] like this (the child took the scissors and indicated cutting the third in half).
Beth: So if you cut that one, if it's a third, and you're going to cut it in half...
Child: Yes then we're going to...
Beth: What are you going to do?
Child: We're going to go like that [cut in half] and then that [cut in half again] so they'll be little-er ones, so there'll be...

The student cut the one-third in half (now two, $\frac{1}{6}$ sized pieces) and in half again (now four, $\frac{1}{12}$ sized pieces) and gave each person one of the small pieces ($\frac{1}{12}$). This meant that each person had $\frac{2}{3}$ and $\frac{1}{12}$. The students did not know the name of the smaller piece and Beth quickly explained why it was $\frac{1}{12}$. The students were still confused and Beth said:

You're giving this person two thirds plus this one here aren't you? Which is a twelfth. So you've given them two thirds and a twelfth. Shall we look at what other people did and we'll come back to you. I don't want to leave that but we'll come back to it.

The students had two misconceptions here: three wafers and four people, meant each whole wafer must be cut into one-third sized pieces; and the need to continually cut pieces in half, until there were enough (pieces) to share out evenly. The halving and halving again (something they had learned earlier in their multiplication lessons), was successful in this instance. However, the students were unaware of the name of the size of the pieces they had made, along with what portion of the original whole each person had received.

Andy recognised his students had difficulties (similar to Beth's students) in determining the number of pieces to cut the whole into (Section 6.2.1: A. 7), when they shared three pies among eight people:

- Andy: Try one of the different ones.
Child: Like, we could just try fifths.
Andy: Maybe go bigger. Why would you try fifths when you've got to give it to eight people?
Child: OK, one third. Oh, yeah one third.

Like Beth's students, thirds came to mind because there were three pies. The students did not recognise two important proportional relationships: the more pieces the whole is cut into, the smaller the piece; and the inverse relationship

between the number of pieces the whole is cut into (in this instance 8), and the size of each piece (eighths). Andy's students (like Beth's) also struggled to understand what the names of the pieces were that had been shared:

- Andy: How much cake did everybody get?
Child: One and a half. I don't know.
Andy: It can't be one and a half because one and a half would look like that (modelled one whole and one half). So it's not one and a half. What is it that everyone's got here? Anyone?
Child: So everybody gets three pieces of cake eh?
Andy: Yes, but how much of a cake does everyone get? Because it's making it even isn't it? And three pieces of cake could be anything. Okay here's three pieces of cake. Half of that cake, half of that cake and a third of that cake. I've got three pieces, and then what do the other people get?
Child: One over twenty-four?
Child 2: No, not that much.
Andy: Not one over twenty-four. You ['ve] got something of the right idea but... how many of these have you got?

The students continued to work out the number of pieces and the size of each piece. After a while, one of the students thought that maybe the pieces were eighths. Others were not sure:

- Andy: So how many eighths have you got?
Child: Three.
Andy: So how much of each cake have you got? Three ...?
Child: Sixths.
Andy: Is it one sixth? (Andy points to one piece). Why one sixth?
Child: Because there's three people and put them into six and give everybody a sixth each. I know quarters won't do it.

The student knew about quarters and eliminated those but could not determine whether the pieces were eighths or sixths.

6.2.2. Content Knowledge in a Pedagogical Context (B)

This part of the framework focuses on the mathematics content as used for teaching.

Deconstructing Content to Key Components (B.1)

Andy and Beth deconstructed the meaning of fractional numbers early in their lesson, when they discussed that the denominator relates to the number of equal pieces that make up the whole, while the numerator tells how many of those pieces there are. While Andy and Beth explored improper fractions, Beth was the only teacher who emphasised that it can be mathematically correct to write a fraction with a numerator larger than the denominator. Beth told the students, "*Improper fractions, are actually proper,*" and discussed the notion, "*if you add*

three quarters to another three quarters, you end up with six quarters.” She deliberately left converting these to a mixed-numeral, until the students accepted that mathematically it was correct to have improper fractions.

Deconstruction of content was also evident when the teachers made connections between whole-number multiplication and division, and the need to share items equally when finding a fraction of a whole (object, or set). However, at one stage during Bob’s lesson, the sharing idea dominated to such an extent that it might be questioned whether the lesson was a whole-number division lesson or fraction lesson, as the notion of proportionality associated with fractional representation did not arise. This was evident when they solved problems such as spreading 15 candles out evenly onto a cake which was cut into 3 pieces. They knew there were 5 candles on each piece, but did not unpack the idea that the cake was cut into thirds and that 5 is one-third of 15. This contrasted with Andy who often showed the connection between the two domains with comments such as, “*So you shared it between four [people]. And you told me that was one quarter.*” Similarly, Beth made the connection between fractional representation and division, when she asked the students to consider the two ideas simultaneously:

What about if you cut all of them into quarters would that help? If you cut them all into quarters, because remember there are four people.

Andy often unpacked key ideas to assist in student understanding and rather than telling the students the answer, he encouraged them to deconstruct the ideas for themselves. An example of this was when the students shared three cakes among eight people and Andy suggested that every cake need not be cut into eighths:

Andy: So you’re saying instead of cutting every cake into eighths and giving people one bit from each cake, you could cut it...

Child: You can cut all of them in different sizes?

Andy: Let’s give it a try and see if it works. Sounds [like] a good idea to me. So [Child 1], you make one of those cakes into quarters please, and [Child 2] you do the other one because [Child 3] has already done [one into] eighths.

Other students attempted to cut each cake into different-sized pieces:

Child: We could try one of the different ones. Like, we could just try fifths.

Andy: Why would you try fifths when you’ve got to give it to eight people?

Child: OK, one third. Oh, yeah one third.

Andy was overheard saying (to himself), “*This could be interesting.*” He observed the students exploring different fractions and they soon realised that thirds and

eighths were not compatible. Andy consolidated equivalent fractions to enable the students to see the relationship between thirds and sixths:

One and two sixths. Very good. [Child's name], I want you to find me a fraction in here (pointed to the box of fraction tiles) that is the same as two sixths (pause). A fraction that is the same as two sixths, please.

Beth deconstructed equivalent fractions with her students, by giving them paper strips to cut. Some students went straight to the eighths by halving their paper, halving it again, and then again. Others made halves, then quarters, then stopped. They looked at the paper pieces and thought about the problem momentarily, before folding the paper again, to make eighths.

Andy and Beth spent time unpacking the meaning of the fractional symbol. When checking that the students understood the symbol $\frac{3}{4}$, Beth reminded the students:

If you've got three on the top that tells you how many [pieces] you've got altogether. The bottom number is telling us how many pieces you've cut each whole into.

When the students constructed fractional representations later, they frequently returned to the fraction number and reminded themselves of the meaning of the numerator and denominator. Andy and Beth wanted the students to realise that fractions can be greater than one whole and again Beth reminded the students that is mathematically correct to have an improper fractions and record a fraction with the numerator greater than the denominator. At one stage, Beth asked the students, "*So can we say that one and one-half is the same as three halves?*" As the students unpacked the model they had constructed, Beth again consolidated the meaning of the numerator and denominator:

Beth: When we cut things into halves what's the number on the bottom?

Child: Two.

Beth: And how many have you got altogether?

Child: Three.

Beth: So you've got three halves haven't you?

Anna's lesson focussed on decimal fractions and the students explored how many tenths were in a given number (e.g., 3.6), using deci-pipes (Figure 6.8). Anna asked the students:

Anna: Can anyone see a pattern or a rule that you can also use, when you're doing this? If you're thinking about it, is there a rule, a mathematical rule perhaps that you could use?

Child: You're just taking the decimal point away?

Anna: Not quite. But, I can see why you might think that because it does disappear.

After an explanation by Anna, the students recognised that they had multiplied the 3.6 by 10. One child mentioned that the numbers had gone up a place value, and Anna reinforced this idea:

Every time we multiply a number by ten it goes up a place value. So, the six went up to the ones, and the three went up to the tens, and because we multiplied it by ten, because we know there are groups of ten. Ten lots of tenths are in one whole. That meant we ended up with 36 tenths in our number three point six.

As the students deconstructed place value, Anna emphasised the importance of calling the decimals (and pieces of equipment) by their correct name:

Anna: Thirty-six what?

Child: Thirty-six tenths.

Anna: Tenths. All right so you must say, the measure of the number.

Later Anna again emphasised the use of correct language:

Child: Oh, because there are ten hundredths in one whole tenth. And there are four of these.

Anna: What are these? Be specific.

Child: Um, four of the hundredths.

On one occasion, Bob presented problems on part-to-whole thinking, and asked the students to consider the relationship between the number of candles on a piece of cake, and how many would be on the whole cake. Questions posed included: “*Three-quarters of the cake have nine candles [on them]. How many candles are on the whole cake?*” These problems created discussion among the students, who initially wanted to find one-quarter of the nine candles, noticed this could not be solved evenly, and struggled to answer the problem. Realising the difficulty the students were having, Bob put that problem aside (he said for another day), and gave a problem involving a unit fraction, “*Here are 9 candles to go on one-third of the cake. How many will be on the whole cake?*” The students found this easier to solve using their manipulatives to support them.

Mathematical Structure and Connections (B.2)

Making connections between structure and concepts were observed in Anna’s (5), Beth’s (2), and Bob’s (3) lessons (Table 6.3). Anna made connections between place-value understanding of whole numbers and decimal numbers. She referred to the notion of “ten groups of” when unpacking how many tenths, or hundredths, are in a number. For example, she reiterated what a student said, “*So I did hear someone say there are ten hundredths in one tenth?*” and showed the value of each digit according to its place value, by moving from one column to another

diagrammatically in the modelling book. Anna ensured it was not a rote-learned procedure and that connection with place value was made.

Beth related division with remainders to fractions, which began by asking the students how many whole pieces everyone would receive when they shared out six wafers among four people:

- Beth: [What happens] If we've got six wafers, and four people?
Child: Ah, they get more [than a whole].
Beth: More than. How do we know that they're going to get more?
Child: Because there's more wafers than people.

Beth's students explored what would happen to the two left-over wafers after everyone had received one whole wafer. However, while the students recognised the left-overs were less than a whole, they did not describe the size of the piece in terms of a fraction of the whole.

Having explored mixed fractions and improper fractions, Andy and Beth gave the students a problem that involved sharing 3 pizzas among 8 people. While the students cut their pizzas and divided them up, or shared them out equally, again they saw the amount of pizza each person received in terms of whole numbers, or pieces of pizza. At one stage in Andy reminded the students of the need to determine how much pizza each person received (relative to the original whole):

- Andy: How much cake did everybody get?
Child: One and a half. I don't know.
Andy: It can't be one and a half because one and a half would look like that (picked up one whole fraction tile, or representation of cake, and a one-half tile). So it's not one and a half. What is it that everyone's got here? Anyone? Anyone got a guess?
Child: So everybody gets three pieces of cake.
Andy: Yes, but how much of a cake does everyone get? Because it's making it even [equal amount] isn't it? And it, three pieces of cake could be anything. It could be... Ok, here's three pieces of cake. Half of that cake, half of that cake and a third of that cake. I've got three pieces and all that [pointed to the pieces of cake], and then what do the other people get?

After re-constructing the problem and discussing the names (size) of the different pieces, eventually the students decided that each person received $\frac{3}{8}$ of a cake.

Methods of Solution (B.3)

Beth and Bob discussed *Methods of Solution* to problems more regularly (8 and 9 times respectively), than Andy (1) and Anna (3), due to their participatory role within their lessons. When Beth observed the students having difficulties, she frequently intervened and discussed procedures, rather than leaving them to solve

the problems themselves. For example, as the students shared three pies among four people, she stopped them and commented:

Ok so these people over here, you divided yours into thirds and then you divided the last piece into twelfths. You gave each person two thirds and a twelfth. You've all done it differently, you've divided yours into halves first [looks at another group]. And then you've given them a quarter, so you've given them a half and quarter. You've divided them all into quarters [another group] and you have given them three each. So, let's add up how many they've got, you've given them a quarter, plus a quarter plus a quarter. So what is, what is that equal [to]?

Bob controlled the discussion that took place during the lessons to a greater extent than the other teachers, and the on-going intervention meant he was more aware of the students' different solution methods. For example, when Bob's students solved the number of candles required on the fractional portions of birthday cakes, he was aware that two solution methods were being used - known multiplication facts and equal sharing of objects. When asked how the 16 candles were distributed over quarters, one student had replied: "*I knew there were four people so what I did was, four times four equals sixteen so four people, they get four each.*" Another student replied, "*There are sixteen counters so we just put one on each [piece] until they're all gone.*" Bob then explored these two solution methods.

Anna's reinforcement of place value meant that her students used $\times 10$ as a method to find the number of tenths in all of a number:

Anna: And what's our rule if we multiply something by ten? What happens to our digits?

Child: It goes up a place value.

Anna: It goes up a place value, doesn't it? So, here, you were correct when you were thinking in groups of ten.

After learning about $\times 10$ for tenths, the students transferred this solution method to $\times 100$ for hundredths:

Anna: What did you do [child's name]?

Child: 'Timesed' it by a hundred.

Anna: And when you timesed it by a, a hundred ...?

Child: It goes up by two place-values.

Anna: So you used the rule and you went up your two place values, so that went from the ones up to the hundreds, and similarly this went up too.

Anna's students knew the procedure for $\times 10$ as adding a zero and adding two zeros for $\times 100$, but also understood that in adding the zeros the numbers shifted in place value. They used this understanding of shift in place value, to find the number of tenths, and the number of hundredths, in any given number. At one

stage their understanding was questioned and a child explained the number of hundredths in 1.14 as:

There's a one whole and one whole has a hundred hundredths and the fourteen [indicates the point 14] has fourteen hundredths so one hundred plus fourteen hundredths is one point fourteen [one, four]?

Procedural Knowledge (B.4)

Bob focused on *Procedural Knowledge*, as opposed to conceptual understanding, more frequently than the other teachers (Table 6.3). He accepted answers given by the students and seldom checked for understanding of their responses. There were times when if a student did not answer a question immediately, another student responded spontaneously, or else Bob redirected the question, with explanation and justification of answers seldom evident. For example, when Bob wanted to know how the paper circle (being used to solve a problem) had been divided into thirds:

Bob: I would like for [Child 1] to show us this time. Twenty-one candles, there are only three people at the party. Three people, so I want you to show me how you would divide that into thirds.

Child 1: I don't how to get that.

Another child interrupted:

Child 2: I do. You go along there along there and along there [indicated on the circle where lines would be drawn]. Draw a peace sign, in other words.

Bob: Ok, that's a third. I'm happy with that.

While it is recognised that dividing a shape into an odd number (thirds, fifths, sevenths) of even sized pieces is difficult for students, Bob failed to check whether the students understood that in drawing a peace sign, they needed to show 3 equal-sized parts. Some of the students divided their circles into obviously uneven-sized pieces (Figure 6.6), which had further repercussions, including incorrect answers to given problems.

The multiplicative relationship between fractions when numbers are halved, (finding a fraction of a fraction), was the basis of Beth's teaching. She asked, "*If you cut a half in half you get quarters, so if you cut a third in half what are you going to get?*" Beth emphasised doubling the number and taught it procedurally, which over-rode the important understanding that each time you cut something in half, it results in twice as many pieces. This idea also linked to the doubling and halving strategy she had taught earlier in multiplication, but that connection was not made.

Profound Understanding of Fundamental Mathematics (PUFM) (B.5)

A *Profound Understanding of Fundamental Mathematics* (Figure 3.1), was seldom observed through the teaching of fractions by the teachers (Table 6.3: frequency range 1 to 3). An exception was, when Anna showed her understanding of place value in both whole numbers and decimal numbers, and she passed this on to the students through the manipulation of the deci-pipes. The students' representations showed ten-tenths made one whole, ten-hundredths equalled one-tenth, and one hundred-hundredths equalled one whole. She did not allow the students to say that they had taken the decimal point away and checked that they understood the numbers were changing place value, when multiplied by powers of ten. Once understanding had occurred, she allowed the procedural use of adding zeros, when solving problems included $\times 10$ and $\times 100$.

Beth utilised her understanding of fractional numbers, when she encouraged her students to understand that a fraction could be greater than one, equal to one, or less than one. She regularly emphasised that the denominator represented the number of equal pieces that made up one whole, and the numerator was the number of those pieces.

6.2.3 Pedagogical Knowledge in a Content Context (C)

This section explored knowledge drawn directly from observation of pedagogical practice in the teaching of fractional concepts (Table 6.2). Analysis of lessons indicated that some over-lap occurred between findings in the Clearly PCK and Pedagogical Knowledge in a Content Context categories, and the relationship between the multiplication lessons and fraction lessons.

Classroom Techniques (C.1)

Varying *Classroom Techniques* were observed in the lessons of each of the four case-study teachers. However, one common technique was to begin the fraction lessons, as they did their multiplication lessons (Section 5.2.1: A3 and Section 5.2.3: C.3), by sharing the learning intention (WALT). Bob and Andy recorded the WALT in the modelling book at the start of the lesson, but seldom utilised the book for any other purpose, which contrasted with Beth and Anna who were mindful of recording students' mathematical thinking alongside their use of manipulatives, and often used the modelling book for this purpose.

The students in all classes sat on the floor for their fractions lesson, as it was easier to manipulate the equipment from this position. One point of difference between the teachers was the manner in which they grouped students to share the materials. Andy's students worked in groups of four, Anna's and Beth's students worked in pairs, while Bob's students chose whether to work alone, in pairs, or in small groups.

As lessons progressed from multiplication to fractions, the teachers encouraged the students to discuss ideas together as they solved problems. However, Anna, Beth, and Bob, often interrupted the students if they suspected there was a difficulty and talked them through steps to support the finding of a correct solution, asked someone else in the class to explain their answer, or answered the question themselves. This contrasted with Andy who often allowed the students to make mistakes. He reminded the students that making mistakes was part of the learning process and that if they were unsure of their answer it would be unpacked when they shared their solution methods. It proved to be a valuable learning experience for the students, because an incorrect answer often led to understanding the strategy for finding a correct one. He said, "*Well try it. See what you get. You try something different.*" There were times when Andy intentionally challenged the students to re-consider their answers. Conversations went along the lines of the following example:

Andy: Then, how many pieces would each person get?
Child: Oh, eight.
Andy: Would they get eight?
Child: [possibly now doubting himself] Oh, one.
Andy: OK, alright, find the eighths for me and prove it.

Getting and Maintaining Student Focus (C.2)

One way the teachers maintained student focus was to create word problems in real-life contexts, which included the students' names. For example, Bob used the names of students when he presented birthday cake problems, such as, "*John (pseudonym for a child in his class) has fifteen candles to put on the cake so that each of the three people at the party gets the same number of candles on their piece of cake. How many candles will each person get?*"

All four teachers also used names to direct questions to specific students. If the teacher observed a student paying insufficient attention, or was unsure whether

the student understood the lesson, a question was targeted at him/her by putting his/her name on the end of it. This quickly brought everyone back into focus, because if the named student was unable to respond to the question, the students knew the question might be redirected towards them.

With fractions, the context usually involved food and the students were encouraged to visualise the whole object (e.g., cake or pie). Andy emphasised the importance of establishing the whole:

So, yes it's a one, it's a whole. So, you could think of this [picked up a paper circle] like a whole pizza, or a whole cake, or a whole block of chocolate, or a whole apple. It's up to you.

At one stage when the students had insufficient “wholes” (cakes) to work with, Andy said, “*Well, come on guys, let's bake the other one [cake]. You better bake that other one*”. The real-life situations also meant that the students appreciated the significance of equal-sized portions. Andy's students shared three cakes among eight people and each group cut their cake into different-sized fractional pieces:

So now you're happy. You've cut one cake into thirds [pointed to one group], you've cut one cake into quarters [pointed to a group] and you've cut one cake into eighths [pointed to a group]. Now you've got to share them with eight people so make eight even groups for me please. And I don't want to end up with anyone getting a small[er] amount of cake, so you better make them all equal.

Goals for Learning (C.3) and Knowledge of Assessment (C.4)

Andy used formative assessment practices to gauge prior learning. Early in the fraction teaching, one of Andy's students responded to a given question and he recognised that the student was more advanced than others in the group. He said, “*[Child's name], I'm going to suggest that you're in the wrong group. That you shouldn't be in triangles, would that be a fair suggestion? Do you want to go and move yourself to another group, please?*” Similarly, Anna began her lesson with a quick warm-up activity, called Body Fractions (Figure 6.7). As the students created decimal fractions, such as 0.25 and 2.75 with their arm-spans, Anna watched and listened to group conversations. At this time, Anna observed whether the students made the connection between the earlier created fractions and the decimal fractions (Section 6.2.1: A.5).

Anna regularly asked students to explain or justify how they had solved a particular problem, using manipulatives to support their reasoning:

Child: Because the six was in the tens [tenths] column, so I put six as the tenth pipe and I put two of the hundreds [hundredths] ones cause there's two in the hundreds [hundredths] column.

Anna: Alright, is everyone in similar thinking as [Child's name]?

As she checked whether everyone had similar ideas, she was aware of which students used correct terminology and which students had solved the problem correctly. When students wrote in the modelling book, they initialled their contribution and Anna used this recording as part of her formative assessment.

Beth utilised formative assessment to a lesser extent. However, at one stage during the lesson, she observed that many of the students had solved the problem (three wafers shared among four people) in different ways and asked them to explain to each other their solution methods. This made her aware of the different strategies used by the students and whether any extra support was required.

Questioning Types – Supporting (C.5), Eliciting (C.6), and Extending (C.7)

The data analysis indicated that the emphasis on different question types remained the same in the fractions lessons (Table 6.3), as for the multiplication lessons (Table 5.3). The teachers utilised *supporting* type questions to the greatest extent (74 overall), with many of these following the back-and-forth IRE model. Beth carried out this type of questioning most regularly (28), with questions similar to the following. “*If we add them up, how many halves have we got altogether?*”, “*How do you write that you have three halves?*”, “*How do you know that?*”, “*If you cut that in half what are you making?*”, and “*How many pieces did you cut your whole into?*”

The second most frequently used question type was that of *eliciting* (20). Bob used eliciting questions when he checked students' explanations of their actions: “*So can anyone tell me what that actually means?*” Beth encouraged elaboration of ideas, such as, “*If we've got six wafers and four people, is each person going to get more than a whole, or less than a whole? How do we know?*” and “*Let's see, who else did it another way?*” Andy promoted collaborative problem solving and asked the students, “*Now what are we going to do here?*” or, “*So how we going to do that?*” Anna sought different solution methods and asked, “*How do we know that? [Child's name], what's your thought?*”

Challenging and *extending* the students' thinking with higher-order questions was observed the least in all classes (1 to 3). Anna asked for generalisations, to check

the students' depth of understanding: *“Is there a rule, a mathematical rule perhaps that you could use? And how can we apply the same rule that we've done here [points to the tenths], with hundredths?”*

6.3 Proportional Domain Summary

The professional knowledge of teachers when teaching fractions and decimals for understanding, was analysed against the framework presented in Figure 3.1 (Table 6.3). This Framework was also used for analysis of the multiplication lessons (Table 5.3) and similarities and differences were observed between the two domains as indicated throughout this chapter.

In the Clearly PCK category, the teachers relied on the NDP planning sheets and lesson progressions in the NDP Book 6 (Ministry of Education, 2008g) to plan their lessons, with little reference made to the NZC (Ministry of Education, 2007). All teachers encouraged their students to use manipulatives to support their understanding of concepts. Both commercial materials (e.g., fractions tiles, rods, and deci-pipes) and paper strips and circles, allowed the students to form representations of word problems (e.g., birthday cake scenarios), which would seem to support their learning. However, there were times when the manipulatives available to the students did not represent the fraction key idea being unpacked and misconceptions occurred. For example, Andy's use of coloured rods showed the number of equal pieces in one whole, but because the rods (e.g., those representing one-half, one-quarter, and one-eighth) were all the same length, the proportional size of pieces was overlooked.

In the Content Knowledge in a Pedagogical Content Context category, the teachers regularly deconstructed the content of what they were teaching in an effort to assist the students, in this case, with their understanding of fractional number. The three teachers who focused on fractions (Andy, Beth, and Bob) spent time discussing the meaning of the fractional number, while Beth extended this to improper fractions. Equivalent fractions were explored using multiplication basic facts learnt earlier. However, when the students shared pieces of cake, pies, or biscuits among friends, they often focused on division and seldom recognised what portion of the original piece each person received. For example, during Bob's lesson the students spread 15 candles evenly onto a cake which was cut into

3 pieces. They knew there were 5 candles on each piece, but did not unpack the idea that the cake was cut into thirds and that 5 is one-third of 15. Similarly, when Andy's students shared 3 pizzas among 8 people answers focussed on the number of pieces of pizza each person got, rather than each person having three-eighths of the original amount. Anna's lesson focused on decimal fractions with an emphasis on place-value understanding.

In the Pedagogical Knowledge in a Content Context category, all teachers encouraged more discussion among the students (than the multiplication lessons). Andy stood back and observed his students allowing them to learn from their mistakes, while the other three teachers were more inclined to intervene when they observed the students experiencing difficulties. All teachers used word problems based on real-life experiences applicable to the students (usually around food) in an effort to support their learning. There was an emphasis on the lower-level supporting type questions and rarely extended the students' thinking and understanding.

The findings presented in this chapter are discussed and critiqued in Chapter Nine, in relation to research literature. The following chapter (Chapter 7) presents the results and analysis of students' learning in the multiplicative and proportional domains.

CHAPTER SEVEN

RESULTS and ANALYSIS: STUDENT LEARNING

7.1 The Multiplicative Domain

The students completed assessment tasks prior to teaching a unit of work on the multiplicative domain, with similar tasks given at the conclusion. The tasks focussed on key items of knowledge to be understood at Levels 2 and 3 of The New Zealand Curriculum (Ministry of Education, 2007). Given that 25% of the students were in latter part of Year 5 and the remaining 75% were in Years 6, 7, and 8, the majority were expected to be capable of solving the tasks correctly (Table 3.3), using an appropriate range of strategies developed at Stage 6 (Level 3) of the Number Framework (Ministry of Education, 2008a, 2008f, 2009).

7.1.1 The Multiplicative Domain Initial Assessment

The initial multiplication and division assessment comprised ten paper-and-pencil tasks designed to ascertain current knowledge. Alongside written instructions on the assessment sheets, the students were given oral instructions explaining that they were to complete each task, explain how they worked it out and when asked, draw a diagram to show their thinking. The researcher discussed with the students the importance of them completing all parts of each question and read them to the students, to ensure they understood the requirements. The initial assessment results are presented on Table 7.1 and show that there was no task, where more than 50% of the students in every class gave a complete and correct response.

Task-by-task analysis of the initial assessment is outlined in detail below. The actual task (and task description) given to the students is presented following each numbered sub-heading, along with examples of responses written by the students on their recording sheets. Both the number of students and corresponding percentages are recorded for comparison purposes, as the number of participating students changed from the initial assessment (Table 7.1) to the final one (Table 7.2). Results of each task were collated as class cohorts rather than for individual students, as the researcher was interested in gathering the data as part of identifying the overall change that had occurred in student understanding, due to the teaching and learning that had taken place throughout each unit of work. The

answers to the tasks involved the drawing of diagrams and writing of explanations to support their understanding, rather than requiring computational answers alone, as NZC requires students to use words, diagrams, and symbols to record and interpret additive and multiplicative strategies (Ministry of Education, 2007, Level 3 fold out chart). Therefore some of the following tasks also asked for these recordings. A response was deemed correct if the student gave a correct numerical answer and also recorded mathematical thinking that matched the word problem.

Table 7.1

*Number (and percentage rounded to nearest whole number) of students with correct responses on the **initial** multiplicative assessment tasks*

School Teacher	Task 1 Mult. as repeated addition $4+4+4+4+4+4 = 6 \times 4$	Task 2 Diagram of $3 \times 5 =$	Task 3 Division Partitive $20 \div 4$	Task 4 Division Quotitive $12 \div 3$	Tasks 5 & 6 Commutative Property 2×5 5×2	Task 7 Using the 6×5 basic fact to solve 6×4	Task 8 Understanding the array 6×5	Task 9 Using 4×7 to derive 4×14	Task 10 Division with remainders $(30 \div 4)$ $7 \text{ r } 2$ & $7 \frac{1}{2}$ or 7.5
A (total) n = 53	19 (36)	9 (17)	5 (9)	3 (6)	14 (26)	20 (38)	31 (58)	31 (58)	18 (34) & 3 (6)
Andy Y5/6 n = 25	8 (32)	7 (28)	3 (12)	2 (8)	10 (40)	9 (36)	15 (60)	10 (40)	1 (4) & 2 (8)
Anna Y7/8 n = 28	11 (39)	2 (7)	2 (9)	1 (4)	4 (14)	11 (39)	16 (57)	21 (75)	17 (61) & 1 (4)
B (total) n = 50	15 (30)	8 (16)	6 (12)	1 (2)	12 (24)	22 (44)	24 (48)	17 (34)	3 (6) & 0
Bob Y5/6 n = 28	8 (29)	4 (14)	5 (18)	1 (4)	10 (36)	10 (36)	12 (43)	12 (43)	1 (4) & 0
Beth Y5/6 n = 22	7 (32)	4 (18)	1 (5)	0	5 (23)	12 (55)	12 (55)	5 (23)	2 (9) & 0
Overall Total n = 103	34 (33)	17 (17)	11 (11)	4 (4)	26 (25)	42 (41)	55 (53)	48 (47)	21 (20) & 3 (3)

Task 1: Understanding multiplication as repeated addition

$4 + 4 + 4 + 4 + 4 + 4 = 24$. How would you write this as a multiplication fact?

As students transition from the Advanced Counting (AC) stage to the Early Additive (EA) stage, they are learning the language of multiplication, the connection between multiplication and repeated addition, and that repeated addition problems can be recorded as multiplication facts e.g., $5 + 5 + 5$ can be recorded as 3×5 (Ministry of Education, 2008f, pp. 11, 15). It is suggested this knowledge is taught in context using word problems and manipulatives at the AC

and EA stages (Table 3.3). Task 1 was intended to determine if these students, in Years 5 to 8, were able to present their thinking in this way.

The standard practice in New Zealand schools is to interpret the multiplication symbol (\times) as groups of, or sets of, with the first number in a multiplication expression representing the multiplier and the second number, the multiplicand (Ministry of Education, 200f, p. 12). Therefore, it was expected that the addition equation would be written showing 6 groups of 4, as $6 \times 4 = 24$. In this task, 34 (33%) students recorded this correctly. Others created different forms of addition equations, with recording errors including the *running on error* (Figure 7.1).

A rectangular box containing the handwritten equation: $4+4=8+4=12+12=24$. The numbers and symbols are written in a cursive, slightly slanted style.

Figure 7.1 Recording repeated addition using running on

Task 2: Understanding the multiplication symbol

Draw a picture of what 3×5 would look like.

For Task 2, a total of 17 (17%) students drew an accurate picture showing 3 sets of 5 (in line with the groups of notion as outlined in Task 1). Many students were unable to draw a picture, while other difficulties included representing 3×5 by sketching 3 objects, times (symbol inserted) 5 objects (Figure 7.2).

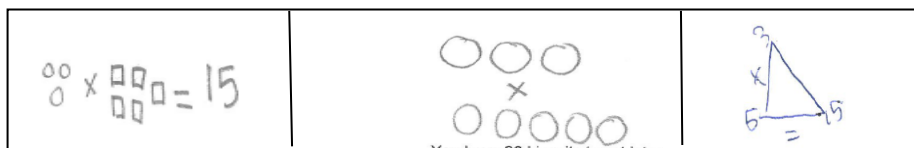


Figure 7.2 Three students' representations of 3×5

Task 3: Partitive division

You have 20 biscuits to put into 4 equal packets. How many biscuits will go into each packet? Draw a diagram to show how you worked this out. How would you write this as a mathematics equation?

Eleven of the 103 students (11%) solved Task 3 correctly accompanied by a correct partitive division diagram. Some students drew the representation, but were unable to write the division equation solved, while others wrote the equation, but unable to show a correct representation. The most common mistake was to draw 20 divided into packets of 4, while others saw the numbers 20 and 4, and used these to draw the multiplication equation 4 groups of 20 (4×20).

Task 4: Quotitive division

You have 12 biscuits to put into packets, with 3 biscuits in each packet. How many packets can you make? Draw a diagram to show how you worked this out. How would you write this as a mathematics equation?

Students are expected to understand the two different contexts for division: sharing and grouping/measuring. Understanding of the two contexts begins at Stage 4, when the students are transitioning from AC to EA (Ministry of Education, 2008f, pp. 11, 17, 19).

Task 4, quotitive division, appeared to be the most difficult task for the students to represent, with four (4%) students correct. There was confusion between multiplication and division, and responses included $12 \div 3 = 36$ along with $3 \div 12 = 36$. Many other incorrect answers showed 12 divided into 3 packets (rather than packets of 3). One student drew 12 biscuits, with each one divided into three (uneven) parts (Figure 7.3).

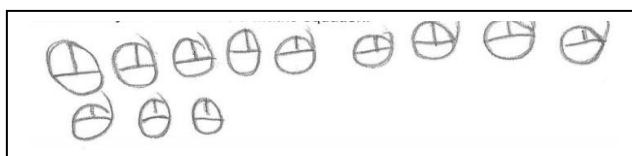


Figure 7.3 One student's representation of 12 biscuits put into packets of 3

Tasks 5 & 6: Understanding the commutative property of multiplication

Task 5: What is the answer to $2 \times 5 = ?$ Draw a diagram to show what 2×5 looks like.

Task 6: What is the answer to $5 \times 2 = ?$ Draw a diagram to show what 5×2 looks like.

As the students move from EA to AA (Stage 5 to Stage 6) they learn to understand the commutative property of multiplication (Ministry of Education, 2008f, p. 24) and that while the answer is the same, the representation is different.

Overall, 26 (25%) students interpreted the written expressions and drew their diagrams around the correct way. The most common error was representing the equations around the opposite way to the accepted interpretation in New Zealand schools of the multiplication symbol as 'sets of' or 'groups of'. Other errors included: lack of recognition of the difference between the two problems, with answers given the same in both instances, either $5 + 5$, or $2 + 2 + 2 + 2 + 2$; and showing one group of two objects and one group of five objects, with the multiplication symbol in between (Figure 7.4).

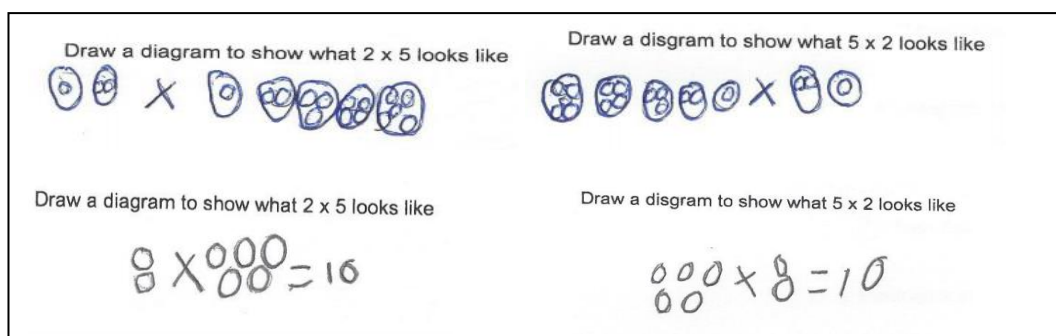


Figure 7.4 Different representations of 2×5 and 5×2

Task 7: Using known facts to derive unknown facts (Figure 7.5)

I have 6 groups of 5 cubes and know to write this as $6 \times 5 = 30$. How could I use this to work out $6 \times 4 = ?$ Show how you can do this on the diagram below.

What is the answer to $6 \times 4 =$

Figure 7.5 Using $6 \times 5 = 30$ to solve $6 \times 4 = ?$

This built on a key idea (introduced at Stage 5 of the NDP Framework), that knowledge of the $\times 5$ facts can be used to derive other facts (Ministry of Education, 2008f, pp. 29, 32). At School A, 20 (38%) students were correct, while at School B, 22 (44%) were correct. Mistakes included either shading in the last column, shading in the first 24 blocks (first 4 columns and 4 more blocks), or leaving one block blank, and recording the answer to the equation as numbers other than 24.

Only one student from each school supported their diagram with a correct explanation.

Task 8: The array model of multiplication (Figure 7.6)

1. How many smiley faces are in each row?
2. How many rows are there?
3. How many smiley faces are there altogether?
4. How did you work this out?

Figure 7.6 Task 8: Understanding an array model of multiplication

As students transition from AC to EA (Stage 4 to Stage 5) on the framework, “students are introduced to the array model that visually lends itself to identifying equal rows or columns. The convention in New Zealand is to regard 4×3 as four sets of three” (Ministry of Education, 2008f, p. 12). “Given an expression such as

5×4 , the student is able to create an array and then derive other multiplication facts from this, e.g., 6×4 by adding on a row of 4 and 4×4 by subtracting a row of 4” (Ministry of Education, 2008f, p. 15).

Overall, 55 (53%) students solved this task correctly. When answering Parts 1 to 3, the most common error was confusion between the terms column and row. Incorrect responses for Part 3 were mainly due to computational errors. When asked how they found the total number of faces, responses were both additive ($5 + 5 + 5 + 5 + 5 + 5 = 30$) and multiplicative ($6 \times 5 = 30$).

Task 9: Using a known fact to work out the unknown

If you know $4 \times 7 = 28$, what does 4×14 equal? Show how did you worked this out.

Students at Stage 6 and transitioning from AA to AM (Stage 6 to Stage 7), solve multiplication problems by deriving from other known multiplication facts and division by proportional adjustment. For example, if you know that $5 \times 8 = 40$, then that can be used to work out 5×16 , by doubling 40 (Ministry of Education, 2008d, pp. 17, 58).

At School A, 31 (58%) students correctly derived 4×14 from the known fact $4 \times 7 = 28$, while at School B, 17 (34%) students did this. While some gave a correct response, only a small number of students said that they derived the answer by doubling the 28, recognising 4×14 as $4 \times 7 \times 2$.

Task 10: Division with remainders

There are 30 apples to put into 4 equal sized bags. How many apples will there be in each bag?

Students at Stage 6 who are transitioning from AA to AM are learning to solve division problems that have remainders. They are learning to understand that the amount left (the remainder) can be expressed as a whole number, or a fraction, or as a decimal (Ministry of Education, 2008f, p. 60).

At School A, 18 (34%) students gave an answer of 7 remainder 2, while at School B, 3 (6%) students gave that same answer. One student from School A gave an answer of $7\frac{1}{2}$ and two gave 7.5. Seven students suggested 7 apples in two bags, with 8 apples in two bags.

The initial assessment results were offered to the teachers to assist in the planning of their unit of work. However, none of the teachers requested the data preferring

to plan directly from the NDP planning sheets and Book 6 (Ministry of Education, 2008f).

7.1.2 The Multiplicative Domain Final Assessment

Assessment tasks, similar to those prior to the teaching of multiplication and division, were given at the end of the unit as an indication of the students' learning throughout the lessons. The collated results are presented in Table 7.2.

The percentage of students correct on final assessment tasks indicated an improvement in some areas and a decline in others (Tables 7.1 & 7.2). However, it must be noted that some students were absent for the final assessment and the total student number decreased by close to 10%, from 103 to 93.

Table 7.2

*Number (and percentage rounded to nearest whole number) of students with correct responses on the **final** multiplicative assessment tasks*

School Teacher	Task 1 Mult. as repeated addition $5+5+5+5 = 4 \times 5$	Task 2 Diagram of $3 \times 6 =$	Task 3 Division Partitive $20 \div 4$	Task 4 Division Quotitive $12 \div 3$	Tasks 5 & 6 Commutative Property 3×5 5×3	Task 7 Using the 6×5 basic fact to solve 6×6	Task 8 Using the $3 \times 10 = 30$ basic fact to derive $\square \times 5 = 30$	Task 9 Division with remainders $(26 \div 4)$ $6 \text{ r } 2$ & $6 \frac{1}{2}$ or 6.5
A (total) n = 47	13 (28)	10 (21)	3 (6)	3 (6)	20 (43)	24 (51)	6 (13)	10 (21) & 3(6)
Andy Y5/6 n = 25	6 (24)	4 (16)	3 (12)	2 (8)	12 (48)	10 (40)	1 (4)	1 (4) & 3 (12)
Anna Y7/8 n = 22	7 (32)	6 (27)	0	1 (5)	8 (36)	14 (64)	5 (23)	9 (41) & 0
B (total) n = 46	30 (65)	12 (26)	6 (13)	3 (7)	8 (17)	18 (39)	1 (2)	7 (15) & 7 (15)
Bob Y5/6 n = 25	19 (76)	5 (20)	4 (16)	3 (12)	4 (16)	11 (44)	1 (4)	6 (24) & 7 (28)
Beth Y5/6 n = 21	11 (52)	7 (33)	2 (10)	0	4 (19)	7 (33)	0	1 (5) & 0
Overall Total n = 93	43 (46)	22 (24)	9 (10)	6 (6)	28 (30)	42 (45)	7 (8)	17 (18) & 10 (11)

Task-by-task analysis of the final assessment is presented following each numbered sub-heading, along with comparisons made to similar initial assessment tasks.

Task 1: Understanding multiplication as repeated addition

$5 + 5 + 5 + 5 = 20$. How would you write this as a multiplication fact?

At School A, there was a decrease in the total number of students who solved this task correctly from 19 (36%) to 13 (27%), while at School B, there was a substantial increase with the number of students correct doubling from 15 (30%) to 30 (65%). The most common response of the remaining students at both schools was to write the equation as $5 \times 4 = 20$, which according to their taught lessons would be interpreted as 5 groups of 4.

Task 2: Understanding the multiplication symbol

Draw a picture of what 3×6 would look like.

There was a small increase in the number of students at School A, from 9 (17%) to 10 (21%), and a slightly greater increase at School B from 8 (16%) to 12 (26%), who correctly drew a representation of 3×6 by drawing 3 sets of 6 (as taught in their lessons). Some students drew their interpretation of the equation but did not clearly show which group within the representation was the multiplier, and which was the multiplicand, leaving their understanding of the difference questionable (Figure 7.7a). However, Figure 7.7 (b) shows the circles connected in groups of three, suggesting that the representation of 6×3 was shown.



Figure 7.7 (a) lack of clarity between 3×6 and 6×3 whereas (b) connected circles suggest 6 groups of 3.

Task 3: Understanding partitive division

You have 20 biscuits to put into 4 equal packets. How many biscuits will go into each packet? How would you write this as a mathematics equation? Draw a diagram to show how you worked this out.

At School A, there was a small decrease from 5 (9%) to 3 (6%) students who showed partitive division correctly. At School B, 6 (12%) students drew the division problem accurately on both assessments. Of the remaining students the majority used reversibility to interpret the problem and wrote a multiplication equation alongside their diagram showing 5 packets (groups) of 4, rather than 4 packets (groups) of 5 (Figure 7.8).

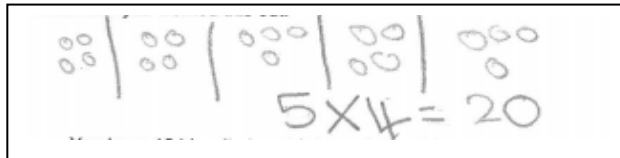


Figure 7.8 Twenty biscuits divided into packets of four instead of four equal packets

Task 4: Quotitive division

You have 12 biscuits to put into packets, with 3 biscuits in each packet. How many packets can you make? How would you write this as a mathematics equation? Draw a diagram to show how you worked this out.

When comparing the initial and final assessments at School A, there was no change in the number of students (3, or 6%) who wrote the equation correctly and drew a correct diagram showing quotitive division, while at School B, there was a small increase from 1 (2%) to 3 (7%) students. Figure 7.9 shows a student's example of the quotitive form of representation. The main reasons for incorrect responses included recording the problem using multiplication with the multiplier and multiplicand around the wrong way, and sketching a representation of partitive division rather than quotitive division (12 biscuits into 3 packets, as opposed to packets of 3).

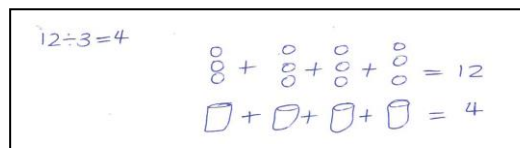


Figure 7.9 Quotitive division showing 12 biscuits into packets of 3

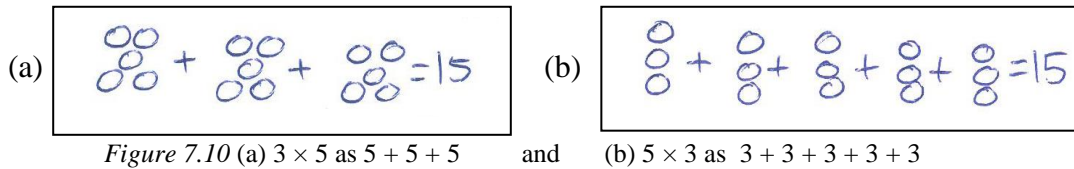
In some instances, the students solved the two different division types the same way, not interpreting the change in wording of the scenario accurately. This meant both problems were solved as either quotitive division, or both as partitive division.

Tasks 5 & 6: Understanding the commutative property of multiplication

Task Five: What is the answer to $3 \times 5 = ?$ Draw a diagram to show what 3×5 looks like.

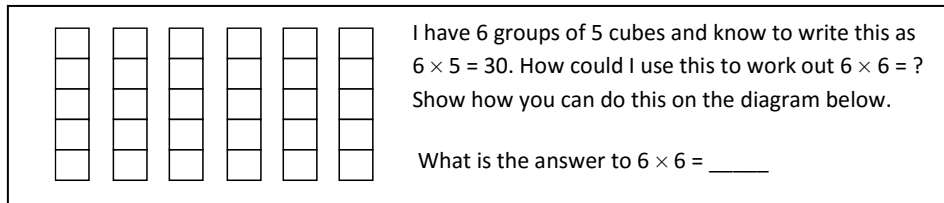
Task Six: What is the answer to $5 \times 3 = ?$ Draw a diagram to show what 5×3 looks like.

At School A, there was an increase of correct responses from 14 (26%) to 20 (43%) and at School B, a slight decrease from 12 (24%) to 8 (17%). The most common representation of correct responses was repeated addition (Figure 7.10).

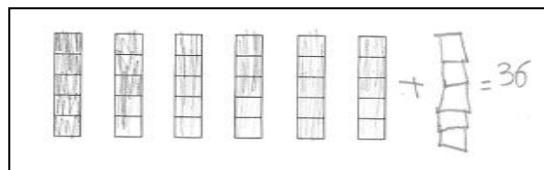


Task 7: Using known facts to derive unknown facts

The initial assessment required one less object in each set, while in the final assessment one object was added to each set (Figure 7.11).



At School A, there was an increase from 20 (38%) to 24 (51%) in the number of students who used $\times 5$ facts to solve other problems, while at school B, there was a slight decrease from 22 (44%) to 18 (39%). The most common error was adding one more set, rather than adding one item to each set (Figure 7.12).



Task 8: Using the known to work out the unknown

I know that $3 \times 10 = 30$.
How can I use this to work out $\square \times 5 = 30$?

The final assessment showed a large decrease in the number of students who used a given basic fact to solve an unknown one (Task 9 on the initial assessment). At School A, there were 31 (58%) correct on the initial assessment and 6 (13%) on the final assessment, while at School B the number of correct responses decreased from 17 (34%) to 1 (2%). Some students possibly knew the basic fact, $6 \times 5 = 30$ and solved the problem correctly by merely inserting the number 6 into the box, with no explanation about how they solved the problem. However, they did not show whether they had an understanding of the relationship between the $\times 5$ and $\times 10$ tables, to explain their answer. The six children at School A, who derived

from the known fact, used the double and halve strategy (Figure 7.13), while only one child at School B, used derivation.

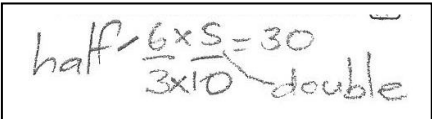
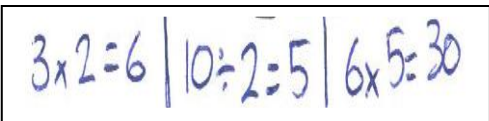
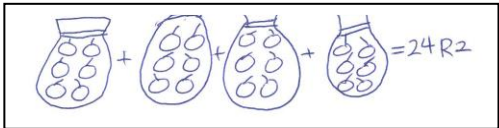
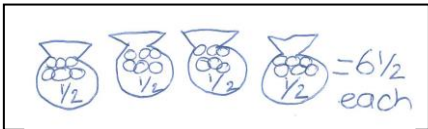
(a)  (b) 

Figure 7.13 (a) and (b) Using the ‘double and halve’ strategy

Task 9: Division with remainders

There are 26 apples to put into 4 equal sized bags. How many apples will there be in each bag?

Task 9 on the final assessment was similar to Task 10 on the initial assessment and overall there was a slight increase from 24 (23%) to 27 (29%) students who solved division with remainders correctly. However, a breakdown of results showed School A, with a decrease in the number of students correct from 21 (40%) to 13 (27%), while at School B, there was an increase from 3 (6%) to 27 (30%). The final scenario given to the students meant that the correct responses included 6 remainder 2, $6\frac{1}{2}$, and 6.5 (Figure 7.14). In reality, not many people would want to buy a bag with one-half of an apple in it. However, if the apples were shared out evenly that would be the case.

(a)  (b) 

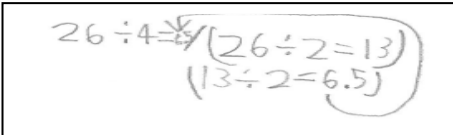
(c) 

Figure 7.14 Representations of 26 apples into 4 bags (a) bags of 6 apples with 2 remainder, (b) each bag has $6\frac{1}{2}$ apples and (c) each bag has 6.5 apples

7.2 The Proportional Domain

The students completed a short paper-and-pencil assessment prior to the commencement of the unit on fractions and decimals, with a similar assessment given at the conclusion. As with the multiplication and division assessment, instructions given to the students included: solve each problem; explain how you worked it out; and where possible draw a diagram to show your thinking.

The two assessments, assisted in ascertaining the students' progress in the understanding of fractional number.

7.2.1 The Proportional Domain Initial Assessment

Six tasks were included in the students' initial assessment, based on Level 3 in the New Zealand Curriculum. As the class levels ranged from Years 5 to Year 8, it was expected the majority of students would have sufficient knowledge to solve these problems correctly (Ministry of Education, 2007; 2009a), (Table 3.2). The initial assessment results are presented on Table 7.3, followed by a description of example responses.

Table 7.3

*Number (and Percentage rounded to nearest whole number) of students with correct responses on the **initial** assessment of fractions*

School: Teacher	Task 1 Addition of unit fractions $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} =$	Task 2 Addition of fractions with compatible denominators $\frac{1}{10} + \frac{2}{5} =$	Task 3 Word problem comparing fractions $\frac{1}{2}$ & $\frac{4}{8}$	Task 4 Part-to-whole - unit fractions If $\frac{1}{4} = 3$ 1 whole = ?	Task 5 Whole-to-part - fraction of a set $\frac{1}{3}$ of 18 = ?	Task 6 Ordering fractions smallest to largest
A (total) n = 45	30 (67)	2 (4)	12 (27)	34 (76)	16 (36)	11 (24)
Andy Y6/7 n = 25	12 (48)	0	3 (12)	16 (64)	3 (12)	1 (4)
Anna Y7/8 n = 20	18 (90)	2 (10)	9 (45)	18 (90)	13 (65)	10 (50)
B (total) n = 44	17 (39)	0	9 (20)	30 (68)	17 (39)	2 (5)
Bob Y5/6 n = 23	15 (65)	0	7 (30)	20 (87)	13 (57)	1 (4)
Beth Y5/6 n = 21	2 (10)	0	2 (10)	10 (48)	4 (19)	1 (5)
Overall Total n = 89	47 (53)	2 (2)	21 (24)	64 (72)	33 (37)	13 (15)

Task 1: Addition of unit fractions (same denominator)

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} =$$

As students move from AC (Stage 4) to EA (Stage 5) (early curriculum Level 2) they learn to add unit fractions, and begin to recognise equivalent fractions (Ministry of Education, 2008g, pp. 17, 21). At Level Two they learn that, "fractions are iterations (repeats) of a unit fraction, for example, $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

and $\frac{4}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ ” (Ministry of Education, *n.d.c*). Addition of unit fractions are frequently taught in real-life food contexts, such as pizzas and cakes. “The students have a good understanding of coordinating the numerator and denominator of fractions when they demonstrate that they do not need materials and images to make comparisons” (Ministry of Education, 2008g, p. 21). Therefore, Task 1 was an addition of unit fractions example using symbols, to see if the expressions and contexts learnt in earlier years were understood.

At School A, 30 (67%) of the students added the unit fractions correctly, while at School B, 17 (39%) were correct. The most common reason for an incorrect response was the adding across error (18%), resulting in the answer $\frac{3}{12}$ (Figure 7.15a). Another error recorded was interpreting the numerators as one, while adding the denominators to get 12 (Figure 7.15b).

(a) $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{12}$

(b) $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{12}$

Figure 7.15 Addition of unit fractions: (a) adding across error, (b) the numerator is recognised as one, while the denominator is added to get 12.

Task 2: Addition of fractions with compatible denominators

$$\frac{1}{10} + \frac{2}{5} =$$

As students progress from AC (Stage 4) to EA (Stage 5), they “need to have a good understanding about what the numerator and denominator represents in a fraction symbol” (Ministry of Education, 2008g, p. 16) and are “developing knowledge of the symbols for halves, quarters, thirds, fifths and tenths” (p. 15). Through models comparing the size of fractions, they begin to recognise equivalent fractions such as two-quarters is the same as one-half. When the students are transitioning from AA (Stage 6) to AM (Stage 7) they are learning the equivalent fractions for halves, quarters, thirds, fifths, and tenths with dominators up to 100, and that in order to add fractions together, they must be of the same denominator.

At School A, two students (4%) solved this equation correctly, while at School B, no students were correct. The most common mistake was the add across error, where 38% gave the answer $\frac{3}{15}$. Another frequent mistake was adding the

numerators, while maintaining one of the denominators (realising they should have a common denominator), with answers either $\frac{3}{5}$, or $\frac{3}{10}$.

Task 3: Comparing fractions

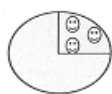
Judith eats $\frac{1}{2}$ a pizza and Jenny eats $\frac{4}{8}$ of a pizza. Who eats the most?

Draw a diagram to show how you worked this out.

As noted in Task 2, students moving from AC to EA are beginning to learn about equivalent fractions. By Level Three (AA) students should be able to find simple equivalent fractions related to doubling and halving, for example $\frac{3}{4} = \frac{6}{8}$, to add and subtract fractions with the same denominators, for example $\frac{3}{4} + \frac{3}{4} = \frac{6}{4} = 1\frac{2}{4}$ (Ministry of Education *n.d.c*). “Halves, quarters and eighths are particularly suited to the students at the early stages (e.g., AC to EA) because they easily relate to doubling, and halving (Ministry of Education, 2008g, p. 21). Task 3 was about understanding the relationship between quarters and eighths.

At School A, 12 (27%) of the students recognised the two fractions as equivalent, while at School B, 9 (20%) of the students recognised the equivalent fractions. Incorrect responses included: Judith got the most because $\frac{1}{2}$ is the biggest size fraction you can have; and incorrect attempts at drawing the two pizzas, resulting in the pizza cut into eighths with unequal-sized portions. The decision as to whether Judith or Jenny had the larger amount was then based on the inaccurately drawn diagrams.

Task 4: Part-to-whole thinking



If $\frac{1}{4}$ of my circle has 3 smiley faces, how many are there on the whole circle? How do you know?

Part-to-whole thinking is a key idea taught as students transition from EA (Stage 5) to AA (Stage 6) (Ministry of Education, 2008g, pp. 26-27).

At School A, 34 (76%) of the students solved Task 4 correctly, while at School B, 30 (68%) were correct. Correct results included: drawing the fraction pieces onto the circle, putting the faces onto it, and counting them (Figure 7.16a); identifying three faces on one-quarter, then adding four quarters to total 12 (Figure 7.16b); and using the multiplication basic fact, $3 \times 4 = 12$.

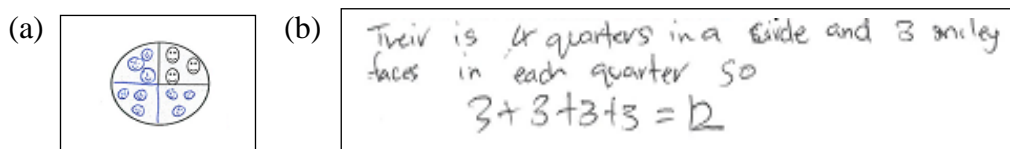


Figure 7.16 (a) Faces drawn on each quarter (b) The number on each of the quarters is added

Task 5: Whole-to-part thinking

What is $\frac{1}{3}$ of 18? How did you know?

Whole-to-part thinking is a key idea taught as students transition from EA to AA (Ministry of Education, 2008g, pp. 26-27).

Similar numbers of students at both schools solved this correctly (School A: 16 [36%]; School B: 17 [39%]). Of those who were correct, the majority converted the problem to a multiplication fact ($3 \times 6 = 18$ or $6 \times 3 = 18$), while others recorded it as division ($18 \div 3$). A common error at School B, was misinterpreting the $\frac{1}{3}$ as $\frac{1}{4}$, then finding $\frac{1}{4}$ of 18 using a halving strategy (learned earlier in the multiplication unit): the students found a quarter by halving and halving again ($\frac{1}{2}$ of 18 is 9 and $\frac{1}{2}$ of 9 is 4.5).

Task 6: Ordering fractions from smallest to largest

Order these fractions from smallest (on the left) to largest (on the right):

$$\frac{1}{3} \quad \frac{6}{4} \quad \frac{1}{4} \quad \frac{3}{4} \quad \frac{1}{2}$$

Students at AA (Stage 6) are expected to be able to order unit fractions and as they transition from AA to AM (Stage 7), are learning to order all fractions involving halves, thirds, quarters, fifths and tenths. Students working at the AA to AM stage are learning that fraction equivalence is critical to understanding the order of numbers, along with the notion that there are infinite names for a given point on the number line (Ministry of Education, 2008g, p. 37).

At School A, while 11 (24%) of the students ordered the fractions correctly, there was one student from Andy's class (Years 6 & 7) with 10 from Anna's class (Years 7 & 8). At School B, 2 (5%) students were correct, with one student coming from each class. Incorrect responses included: (i) ordering the unit fractions from smallest to largest according to the numeral in the denominator, then those with the same denominator (the number 4) according to the numerator

(Figure 7.17a). (ii) ordering the denominators from largest to smallest, and when the denominators were the same, ordering the numerator from largest to smallest (Figure 7.17b). (iii) $\frac{1}{2}$ was thought to be the largest fraction as it had a denominator with the smallest numeral, therefore regardless of how all the other fractions were ordered, $\frac{1}{2}$ was placed as the largest.

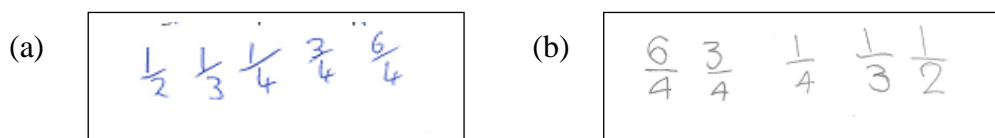


Figure 7.17 Ordering of fractions with (a) denominators ordered as whole numbers, then numerators and (b) fractions with the same denominator ordered first.

7.2.2 The Proportional Domain Final Assessment

There were ten tasks in the final assessment. The first six tasks (Table 7.4) were similar to those of the initial assessment (Table 7.3).

Table 7.4

Number (and percentage rounded to nearest whole number) of students with correct responses on the **final** assessment of fractions.

School Teacher	Task 1 Addition unit fractions $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} =$	Task 2 Addition of fractions with compatible denominators $\frac{1}{10} + \frac{2}{5} =$	Task 3 Word problem comparing fractions $\frac{3}{4}$ & $\frac{7}{8}$	Task 4 Part-to-whole unit fraction If $\frac{1}{4} = 5$ 1 whole = ?	Task 5 Whole- to-part unit fraction $\frac{1}{3}$ of 21 = ?	Task 6 Order fractions smallest to largest
A (total) n = 49	34 (69)	22 (45)	28 (57)	30 (61)	21 (43)	5 (10)
Andy Y6/7 n = 23	9 (39)	2 (9)	13 (57)	10 (43)	3 (13)	1 (4)
Anna Y7/8 n = 26	25 (96)	20 (77)	15 (58)	20 (77)	18 (69)	4 (15)
B (total) n = 41	23 (56)	1 (2)	19 (46)	33 (80)	29 (71)	3 (7)
Bob Y5/6 n = 21	10 (48)	1 (5)	10 (48)	20 (95)	20 (95)	1 (5)
Beth Y5/6 n = 20	13 (65)	0	9 (45)	13 (65)	9 (45)	2 (10)
Overall Total n = 90	57 (63)	21 (23)	47 (52)	63 (70)	50 (56)	8 (9)

An additional four tasks (Table 7.5) were added to the final assessment, from Level 4 of the New Zealand Curriculum. Two of the extra tasks were more difficult fraction problems, while two related to decimal understanding, as this

area of proportional reasoning had been included in some of the class learning sessions.

The final assessment Tasks 1 to 6, are presented along with comparisons to similar initial assessment tasks.

Task 1: Addition of unit fractions

$$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} =$$

At School A, nearly all of Anna’s class were correct (25 or 96%), with a decrease from 12 (48%) to 9 (39%) in Andy’s class. At School B, there was a reversal in terms of the number of students correct, between the two classes. In Bob’s class the number of correct responses lessened from 15 (65%) to 10 (48%), while Beth’s class they increased from 2 (10%) to 13 (65%). The most common error (18%) was the add across error (Figure 7.18), where the numerator numbers were added together as well as the denominators. Another error was recording $\frac{1}{5}$ as the answer (students were unsure whether the numerator or denominator be kept the same, so did not change either).

$$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{1+1+1}{5+5+5} = \frac{3}{15}$$

Figure 7.18 Addition of numerators and denominators

Task 2: Addition of fractions with compatible denominators

$$\frac{1}{10} + \frac{2}{5} =$$

At School A, correct responses in Anna’s class increased from 2 (10%) students correct on the initial Task 2, to 20 (77%) students correct on the final task, while Andy’s class increased from zero correct to 2 (9%). At School B, one child (2%) from Bob’s class was correct. As in the initial assessment, the most common mistake was the add across error (35%), which gave the same answer for Tasks 1 and 2 (Figure 7.19a). The students did not realise that while $\frac{1}{5} + \frac{1}{5}$ in Task 1 equalled the $\frac{2}{5}$ in Task 2, the other addend was different in each question, so the answer could not possibly be the same. Other incorrect answers included: the fractions $\frac{1}{5}$, and $\frac{3}{4}$; adding the denominators and ignoring the numerators; and

solving Task 1 correctly, then after using the “add across” error in Task 2, returned to Task 1 and changed their answer accordingly (Figure 7.19b).

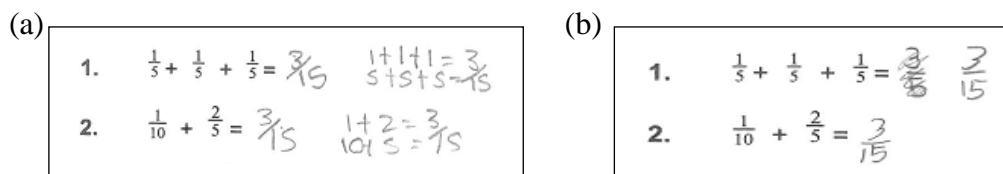


Figure 7.19 The “add across” error (a) gave the same answer for different questions, (b) was used in Task 2, resulting in a change to Task 1.

Task 3: Comparing fractions

Judith eats $\frac{3}{4}$ a pizza and Jenny eats $\frac{7}{8}$ of a pizza. Who eats the most?

Draw a picture to show how you worked this out.

More than twice as many students were correct on Task 3 in the final assessment (47 or 52%), compared to the initial assessment (21 or 24 %). At School A, the largest increase was in Andy’s class where correct responses went from 3 (12%), to 13 (57%). At School B, the biggest increase was in Beth’s class from two (10%), to nine (45%). Some students saw Judith and Jenny having the same amount of pizza because they each had one piece remaining (Figure 7.20).

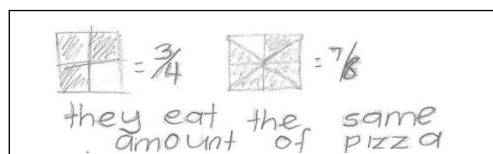


Figure 7.20 Each person has one piece left therefore they eat the same amount

One student gave a correct answer for the wrong reason: Jenny ate the most because both numbers (numerator and denominator) were bigger (i.e., the 7 and the 8 were larger numbers, than the 3 and the 4), therefore Jenny got the most. Inaccurate sketches also showed Jenny to get the most (Figure 7.21)

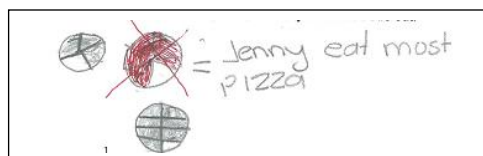


Figure 7.21 Uneven fraction representations

Task 4: Part-to-whole thinking

A picture showing smiley faces did not accompany the task in the final assessment:

If $\frac{1}{4}$ of my circle has 5 smiley faces, how many are there on the whole circle?
Show how you worked this out.

At School A, there was a slight decrease in the number of correct responses from 34 (76%) in the initial assessment to 30 (61%), in the final assessment. The opposite occurred at School B, where there was a slight increase from 30 (68%) to 33 (80%). The majority of students divided a circle into quarters, sketched the faces on each quarter and either solved the problem using addition ($5 + 5 + 5 + 5 = 20$) or multiplication ($4 \times 5 = 20$). Incorrect responses included dividing a circle into quarters, and placing one smiley face (confusion with one-quarter) in each quarter (Figure 7.22).

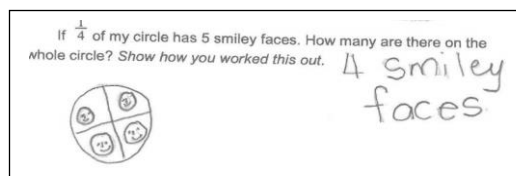


Figure 7.22 The circle is divided into quarters with a smiley face on each

Task 5: ‘Whole-to-part’

What is $\frac{1}{3}$ of 21? How do you know?

At School A, there was a small increase in the number of correct responses from 16 (36%) to 21 (43%), mostly from Anna’s class. At School B, the increase almost doubled from 17 (39%) to 29 (71%) with a large increase in both classes. Most students said that they knew that $3 \times 7 = 21$, while others wrote the equation incorrectly by mixing the divisor and dividend (Figure 7.23).

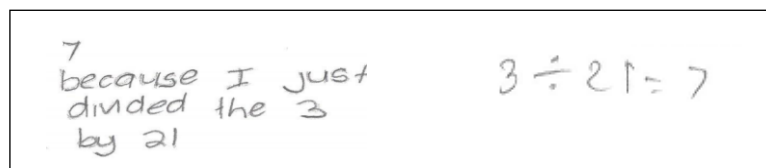


Figure 7.23 Confusion between divisor and dividend when finding $\frac{1}{3}$ of 21

Task 6: Ordering fractions from smallest to largest

Order these fractions from smallest (on the left) to largest (on the right):

$$\frac{1}{3} \quad \frac{6}{4} \quad \frac{1}{4} \quad \frac{3}{4} \quad \frac{1}{2} \quad \frac{7}{16}$$

The fractions in this task were the same as the initial assessment, with the addition of $\frac{7}{16}$. While this introduced another denominator, sixteenths are compatible with quarters and halves. At School A, one student in Andy’s class was correct (the

same as the initial assessment), with a decrease of those correct in Anna's class from 10 (50%) to 4 (15%). At School B, there was one student (5%) from Bob's class and two (10%) from Beth's class, who were correct. Incorrect responses included: (i) Disregard of the numerator and writing $\frac{7}{16}$ as the smallest fraction, because it had the largest numeral for the denominator (Figure 6.24). (ii) Recording $\frac{7}{16}$ as the largest fraction, because the number seven was the largest numeral for numerator, therefore it had the most number of pieces. (iii) Placing $\frac{1}{2}$ as the largest, as two on the denominator means it must be the biggest piece. (iv) Ordering the fractions by looking at the denominator with whole number thinking. (v) When there was more than one fraction of the same denominator, ordering the numerators from smallest to largest (Figure 7.25).

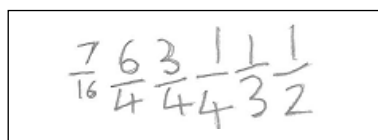


Figure 7.24 Fractions ordered according to size of the denominator

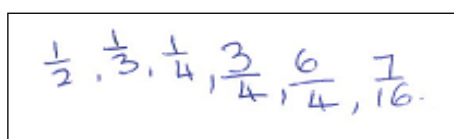


Figure 7.25 The numerator and denominator ordered using whole number thinking

Additional final assessment tasks

The final assessment included four additional tasks based on lessons taught when moving from AA to AM (Levels 3 to 4) (Ministry of Education, 2008g, pp. 35, 36, 38). These tasks were to further assess the Years 7 and 8 students in Andy's and Anna's classes, who should be familiar with these problems, along with the more able students from the other classes. The students in Andy's class had difficulty solving these tasks correctly, while none of the students in Beth's class were able to do so (Table 7.5).

Table 7.5

Number (and percentage rounded to nearest whole number) of students with correct responses on final fractions assessment (additional questions from initial assessment).

School Teacher	Task 7 Whole-to-part non unit fraction $\frac{3}{4}$ of 32	Task 8 Part-to-whole word problem $\frac{5}{9}$ were eaten. 16 were left. What was the whole?	Task 9 Decimal subtraction $4.3 - 3.89 = ?$	Task 10 Decimal Place value How many tenths in 5.23?
A (total) n = 49	15 (29)	6 (12)	7 (14)	11 (22)
Andy Y6/7 n = 23	0	0	1 (4)	4 (17)
Anna Y7/8 n = 26	15 (58)	6 (23)	6 (23)	7 (27)
B (total) n = 41	15 (37)	3 (7)	0	10 (24)
Bob Y5/6 n = 21	15 (71)	3 (14)	0	10 (48)
Beth Y5/6 n = 20	0	0	0	0
Overall Total n = 90	30 (33)	9 (10)	7 (8)	21 (23)

Task 7: Whole-to-part (non-unit fraction)

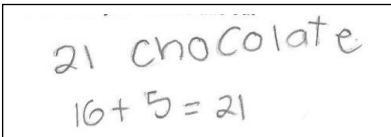
What is $\frac{3}{4}$ of 32? How did you work this out?

At School A, 15 (58%) of Anna's students were correct, while at School B, 15 (71%) of Bob's students were correct. The main error, was finding $\frac{1}{4}$ of 32, rather than $\frac{3}{4}$ of 32.

Task 8: Part-to-whole

Leah ate five-ninths of a box of chocolates. That left only 16 chocolates for Blake. How many chocolates were in the box at the start? *Show how you worked this out.*

At School A, six (23%) of Anna's students solved this correctly, while at School B, three (14%) of Bob's students were correct. Incorrect responses included the remaining 16 chocolates being added to the five (numerator of the fraction, $\frac{5}{9}$) to give a total of 21 (Figure 7.26). The students did not recognise 5 was the numerator in the fraction number, and not the number of chocolates.



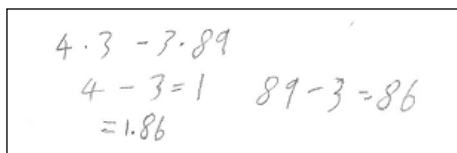
21 chocolate
 $16 + 5 = 21$

Figure 7.26 The whole number (16) added to the numerator (5)

Task 9: Decimal subtraction

The school record for the long jump was 4.3 metres. Kathryn jumped 3.89 metres. How far short of the school record was Kathryn? *Show how you worked this out.*

At School A, one (4%) student from Andy's class, and six (23%) from Anna's, were correct. At School B, no student solved this correctly. One of the incorrect responses was 1.14, obtained by rounding 3.89 up to 4 to get 0.11, and adding the 0.3 from 4.3 onto the last digit (as they would in whole number thinking) solving $0.11 + 0.3$ incorrectly to get 0.14. The 3 metres was taken away from the 4 metres by solving $4 - 3 = 1$, and lastly added the 1 to the 0.14 to get an answer of 1.14. Other students took the 3 metres away from the 4 metres leaving 1 metre, then reversed the numbers to subtract 0.3 from 0.89 to get a difference of 1.86 (Figure 7.27).



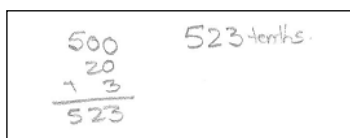
The image shows a student's handwritten attempt to solve the decimal subtraction problem $4.3 - 3.89$. The work is written in a box and consists of three lines: the first line is $4.3 - 3.89$, the second line is $4 - 3 = 1$ followed by $89 - 3 = 86$, and the third line is $= 1.86$.

Figure 7.27 Attempt to solve $4.3 - 3.89$

Task 10: Place-value understanding

How many tenths are in all of this number? 5.23

At School A, four (17 %) students from of Andy's class and seven (27%) students from Anna's class solved this correctly, while at School B, 10 (48%) from Bob's class were correct. Incorrect responses included the answers 20, 23, 523 (Figure 7.28) and 2 (the number in the tenths place).



The image shows a student's handwritten attempt to find the number of tenths in the number 5.23. The work is written in a box and consists of two parts: on the left, a vertical addition showing 500 , 20 , and 3 with a horizontal line above the 3 and the result 523 below; on the right, the text "523 tenths."

Figure 7.28 Attempt at finding the number tenths in all of the number 5.23

7.3 Assessment Tasks Summary

7.3.1 The Multiplicative Domain Assessment Tasks Summary

The results of the assessment tasks suggested that, generally there was little progress in students' understanding of multiplication and division as a result of the teaching that had taken place. Over the 19 tasks (the combined number in both the initial and final assessment) given to the students, there was only one task

where more than half of the students overall were correct (Task 8 on the initial assessment). On the final assessment, on the majority of tasks the percentage of students with correct responses either remained the same, or decreased.

When it came to recognising that repeated addition can be written as multiplication (Task 1), there was an overall increase of students correct from 34 (33%) to 43 (46%). However, analysis of the results indicated that School A, decreased from 19 (36%) to 13 (27%), while School B, increased from 15 (30%) to 30 (65%). There was also an increase in awareness and understanding of the commutative property, from 26 (25%) students to 28 (30%) (Tasks 5 & 6). However, on closer analysis of the results, the school break-down indicated the reverse of Task 1, with an increase at School A, from 14 (26%) to 20 (43%), while at School B, there was a decrease from 12 (24%) to 8 (17%).

On both the initial and final assessments, division posed the greatest difficulty, with 11 (11%) on the initial assessment and 9 (10%) on the final assessment understanding partitive division (Task 3), and 4 (4%) on the initial assessment and 6 (6%) on the final assessment understanding quotitive division (Task 4).

7.3.2 The Proportional Domain Assessment Tasks Summary

The assessment results indicated a general improvement in the percentage of students correct on each task from the initial assessment to the final assessment. However, further analysis showed Beth's class was the only one where the percentage of students either remained the same or improved on all tasks. Andy's, Anna's, and Bob's students had at least one question, where there was a decrease in the percentage correct from the initial assessment to the final assessment.

On the initial assessment, the greatest number of students solved Task 4 correctly (64 or 72%). This task was based on knowing how many objects were on part of a whole ($\frac{1}{4} = 3$), therefore determining the resulting whole. Task 4 provided a similar number of correct responses on the final assessment with 63 (70%) correct responses.

Initial assessment showed that addition of fractions with compatible denominators (Task 2: $\frac{1}{10} + \frac{2}{5}$) was the task which posed the most difficulty, with two students (2%) from the total cohort of 89 correct. Addition of compatible fractions (Task 2) showed a gain in the number of correct responses in the final assessment, with an increase from two (2%) correct to 21 (23%). However, this represented one child from School B, while at School A, there were only two (9%) from Andy's class, and 20 (77%) from Anna's Years 7 and 8 class. On the final assessment, the task with the fewest correct responses (n=8 or 9%), was ordering fractions from smallest to largest.

In the following chapter (Chapter Eight) the results related to the multiplicative domain are further analysed and discussed in relation to the research literature.

CHAPTER EIGHT

DISCUSSION: THE MULTIPLICATIVE DOMAIN

This chapter discusses results from the three preceding chapters: Chapter Four (Section 4.2.1 & Section 4.2.2); Chapter Five; and Chapter Seven (Section 7.1) associated with the teaching of multiplicative thinking.

8.1 Teachers' Subject Matter Knowledge and Pedagogy

8.1.1 Introduction

The teachers at both schools noted on their questionnaire responses and in discussions that they had learnt mathematics content in a purely procedural manner, and like many students the experience of school mathematics was not always positive, and perceived to be difficult and irrelevant (Beswick, 2005; Burns, 1998; Carroll, 1994; Grootenboer, 2001; Grootenboer et al., 2008). They had learnt to memorise facts and used these mechanically with little, or no understanding of what was occurring in the strategy involved in solving a mathematics problem. In Chapter 2, it was proposed that in order to teach mathematics competently teachers need to have a profound, flexible, and adaptive knowledge of mathematics content (Ma, 2010) and many primary school teachers lack the required conceptual understandings to select worthwhile tasks for strengthening students' mathematical understandings (Ball, 1993; Ball & Bass, 2003; Philippou & Christou, 2002; Scharton, 2004).

8.1.2 Multiple-choice Questions

The questionnaire contained eight multiple-choice items designed to give insight into the teachers' subject matter knowledge and pedagogical practice, in relation to the teaching of mathematics for understanding, in the multiplicative and proportional domains. The number of multiple-choice items was limited, as research had shown that they do not give teachers the opportunity to elaborate their decision-making and explain their reasoning, and there are no ways of checking how the answers were found (Clements & Ellerton, 1995). Of the eight multi-choice problems given to the teachers (Appendix C, Section B), three asked for estimated answers (1, 4, & 8). Mathematics goes beyond the requirements of teaching written calculating procedures, to involve both mental calculation and estimation as efficient processes for calculating (Anghileri, 2006, p. 2), and is one

of the fundamentals of numeracy (van den Heuvel-Panhuizen, 2001c). While the teachers answered the problems correctly (Section 4.2.1) on the questionnaires (and one can only presume they were solved through estimation and not by finding an exact answer), the use of estimation was not observed in any of the classrooms, by the teachers or their students.

Many opportunities for estimating answers were available in the observed multiplication and fractions lessons. Estimation requires a good number sense and assists in all computation situations, becoming a checking mechanism that can be compared to a calculated answer to gauge the plausibility of the answer (Ma, 2010). Students do not easily grasp the concept of estimate, or approximate, and the skills and strategies of estimation need to be taught to students (Bobis et al., 2013; Ministry of Education, 2007). When Andy's students shared three cakes among eight people, they might have considered as a group approximately how much cake each person would get, before they solved the problem accurately. If there were four cakes, each person would get one half. However, as there were only three cakes (one cake less), each person would get a bit less than one half. Teaching estimation and encouraging its use in a range of situations, helps students see reasonableness in their answers (Jorgensen & Dole, 2011; Ministry of Education, 2007; van den Heuvel-Panhuizen, 2001c). As these students rely more on technology, there is a need to teach them how to compute mentally and how to estimate in order to understand the numbers they should obtain when using such devices as a calculator (Perso, 2006).

8.1.3 Teachers' Understanding of Multiplication and Division

This section of the chapter discusses part of research Question 2, concerning *the teachers' espoused professional knowledge*.

There were three problems on the teachers' questionnaire related to the teaching of multiplication and division (Section 4.2.2). They were to solve each problem and discuss what they would do to progress the student in each given scenario (Figures 4.1, 4.2, & 4.4). In the first scenario, all teachers used place value partitioning (PVP) to solve 11×99 , as they recorded $(10 \times 99) + (1 \times 99)$. While Bob used PVP to solve the problem, his number sense should have told him that his given answer of 999 was incorrect. Number sense includes having a good

intuition about numbers and their relationships (Howden, 1989), or having a feel for numbers (Anghileri, 2006). Number sense is an integral part of what Ma (2010) described as PUFM and the lack of reference to number sense in the teachers' responses may have impacted on their teaching practice (Tables 5.1 & 6.1).

Anna mentioned supporting the students' understanding through the use of concrete materials, although she did not expand on what materials she would use, or how they would be implemented. Concrete materials, or manipulatives, are often used in mathematics lessons with the claim that they extend students' learning of mathematical concepts and operations, as they make them more comprehensible (Ma, 2010; Nührenbörger & Steinbring, 2008; Schoenfeld, 2011; Swan & Marshall, 2010; Wright, 2014). Beth also mentioned manipulatives and suggested giving Jon a smaller problem as a starting point and "going from there." Knowledge building, or extending the students current capabilities, is considered a conceptual shift in mathematical learning and key to improving student knowledge and understanding (Davis & Renert, 2014) and while Beth started with smaller numbers, the go from there phase to push the students up to the higher numbers, seldom eventuated.

Earlier problems using jumps on the number line, might have helped the teachers explain to the student (Jon) the steps involved in the PVP strategy. One of the early ideas taught in multiplication is relating repeated lots of, or groups of, to jumps on the number line (Anghileri, 2006). The link with the jumps, often carried out previously in repeated addition problems, can support a mental strategy that is helpful for calculating. Understanding the structure of multiplication including grouping, number-line hopping, and arrays, has been found to lead to an appreciation of multi-digit multiplication at later stages (Davis, 2008; Mulligan & Mitchelmore, 2009; Young-Loveridge & Mills, 2010).

While researchers have stressed the importance of teaching multiplication and division simultaneously for students to see the relationship between the two (Anghileri, 1999; Beak, 2006; Bobis et al., 2013; Clark & Kamii, 1996), many teachers struggle with the conceptual understanding associated with division constructs (Roche et al., 2015). In the questionnaire, two division scenarios were presented, each with a different focus, although there was a connection between the two (Section 4.2.2: Scenarios 2 & 3). The first drew on the understanding of

divisibility rules, knowledge developed as students move from AA (Stage 6) to AM (Stage 7) (Ministry of Education, 2008f). The scenario related to dividing a number by three (Figure 4.2). The clue given to the teachers was in the phrase “Mere knew the number was not divisible by 9, because 9 does not go evenly into 12, and concluded that it was not divisible by 3.” Neither Andy nor Bob knew what they would do next with Mere, while Beth and Anna broke the number 516 up into smaller chunks. The familiarity with number patterns of this nature and the notion that multiplication and division are related to the same number patterns, was explored in the research of Mulligan and Mitchelmore (2009, 2013), who found that these number relationships can provide the key to successful calculations. Mulligan and Mitchelmore (2013) found that students with an understanding of the construct referred to as Awareness of Mathematical Pattern and Structure (AMPS), learn that the first rule of structural understanding is that of generality. There is always a general rule telling you how a pattern is constructed and understanding those generalisations is the beginning of algebra. However, the generalised association between patterns in groups of three and groups of nine was not evident in this example.

The connection between the two division scenarios was in the use of the quotitive form of division. All of the teachers admitted to teaching division predominantly through partitive division and that problem scenarios used in class generally focussed on the equal sharing out of items. This meant that the groups of idea used in understanding quotitive division was not recognised and used intuitively when they saw a problem, or taught a problem.

While none of the teachers referred to quotitive division, there were some links made between equal groups in multiplication and the strategy they applied to solve the division problem. This was possibly due to the wording of the problem which asked for “14 chocolate peanuts to be put into each bag” (Figure 4.4). This scenario is based on understanding required as students transition from AA to AM (Ministry of Education, 2008f, p. 57), where division problems can be solved through proportional adjustment. As the teachers did not identify the next steps of teaching, it is suggested that they struggled to fully understand the grouping idea behind quotitive division. This is consistent with the research of Roche and Clarke (2009) and Roche et al. (2015), who found that generally quotitive division was

not understood, or taught well by teachers. In identifying the next steps of learning, the teachers might have made connections between the structures associated with multiplication and division as emphasised in the research of Mulligan and Mitchelmore (1997; 2013). The wording of the quotitive division problem should have been the clue to how to solve the problem. Division word problems given to students, similarly dictate the structure of the problems when solving them (Roche & Clarke, 2009).

8.1.3 Perceived Mathematics Practice

The teachers indicated that they either always, or often, encouraged their students to explain their thinking to others (Table 4.1). Research, has also shown that as teachers become more aware of their students' thinking, they need to generate ways the thinking can be clarified within classroom interactions (Fennema et al., 1993; Franke & Kazemi, 2001) and this is often begins through group discussion. However, the teachers often appeared to have difficulty getting their students to carry out meaningful discussions, and it is questionable whether the skills associated with group discourse (Bakhtin, 1994; Wenger, 1998) had been taught. Hunter's research (2005, 2006, 2009, 2012) established that children need to be taught specifically how to carry out meaningful discussions within their mathematics lessons and that effective discussions develop over time.

When the teachers were asked if they encouraged their students to justify their choice of strategy and thinking to others, two teachers (Andy and Beth) replied "always", one (Anna) noted "often", and one (Bob) wrote "sometimes" (Table 4.1). Observations indicated that while their students talked with each other, they struggled with discussion that involved argumentation and explanation. Cobb and Bauersfeld (1995) identified that the conversation is multivocal (when both [all] students voice their opinions), if all of the students involved are given opportunity to express their ideas and challenge each other's thinking. The difficulties the students in this study had challenging their peers, also came to the fore in the research of Hunter (2009, 2012), who found that a great deal of time was needed when creating a safe environment where students were comfortable with questioning each other. As Hunter (2009) concluded, teachers need to foster an environment where it is not considered judgemental to disagree with others and

where students can learn polite ways to disagree, or challenge each other and actively listen to what is said.

The teachers were asked if they encourage students to include in their maths books drawings, diagrams, or other recording methods which represent their thinking, and Beth responded with always, Andy with often, and Anna and Bob with sometimes (Table 4.1). While the teachers encouraged the students to use manipulatives as representations of equations and scenarios, removing the concrete representation and moving to the implementation of diagrams and sketches did not eventuate in any of the classes. The use of manipulatives, then the use of diagrams and sketches during representational phases of the CRA model, allows for greater gains in student learning (Flores, 2010). Flores asserted that the improvement was even greater among students who had learning difficulties, or were identified as lower achieving (such as in Beth's class). At School A, there were many English Language Learners (ELLs) and diagrams could have been utilised more effectively in classes to consolidate mathematical understanding. Research has found that ELLs students need opportunities to speak, write, talk, and listen in nonthreatening situations, and often a diagram and/or manipulatives may be an effective way to overcome the language barrier (Bautista Verzosa, 2011; Fernandes, 2011).

8.2 Teacher Practice in the Multiplicative Domain

This section of the chapter discusses the research question, "*What professional knowledge is evident when teaching mathematics for numeracy in the multiplicative domain?*" It incorporates part of Question 2, "*and how does it (the teachers professional knowledge) contribute to student learning?*"

The discussion in this section aligns with the results of the seven observed multiplication lessons and accompanying field notes outlined in Chapter 5 and critically examines these in relation to the relevant literature. The lessons were analysed using a comprehensive framework (Figure 3.1), which supported Question 3, "*How does the use of a framework assist in the investigation of teachers' professional knowledge in practice?*"

8.2.1 Clearly PCK (A)

Purpose of Content Knowledge (A.1) and Curriculum Knowledge (A.2)

The Purpose of Content Knowledge and Curriculum Knowledge categories overlapped and were identified as two of the less frequently observed, when the seven lessons were analysed against the PCK Framework (Table 5.3). All four teachers acknowledged during discussions that some of their students were performing below the appropriate level of NZC (Ministry of Education, 2007) for their class year. However, observations within the classroom, later analysis of the lessons, and assessment data, suggested that the teachers' interpretation of what was required at each stage on the Number Framework, and the associated links with the NZC Levels and Standards was not comprehensive. This may have contributed towards the teachers not recognising that the majority of students were below suggested expectations (Ministry of Education, 2010), and lessons taught would not have met the Curriculum level guidelines for the appropriate year levels of the classes (Ministry of Education, 2007). In relation to multiplication and division, the NZC notes that a Level 3 student should be able to "use a range of simple multiplicative strategies with whole numbers, fractions, decimals and percentages" and "know basic multiplication and division facts" (Ministry of Education, 2007, Level 3 chart). Lessons observed fell short of meeting these expected outcomes.

While the long-term goal for the teachers was for the students in their classes to have the knowledge and skills expected to solve problems appropriate for moving from AA (Stage 6) to AM (Stage 7) (Level 3 to Level 4 of the Curriculum), observations showed that problem samples given to the students did not support progress towards this expectation. Bob mentioned to the students that they were currently at Stage 4 and Stage 5 on the Number Framework and that he hoped to move them to their expected Stage 6 (see 5.2.1., A.1 & A.2). However, he continued to give them low-level problems to solve (e.g., $3 \times 5 = \square$). As these students were Years 5 and 6, they were expected to experience multiplication with a range of strategies utilising a combination of single-digit and double-digit whole numbers (Ministry of Education, 2007). Students who are not given opportunities to learn challenging and high-level work, may not achieve at high levels (Boaler,

2008; Rubie-Davies, 2007, 2010; Solomon, 2007; Zevenbergen, 2005) and this became evident in the final assessment tasks.

Beth had one of the lower achieving groups in her school (her school cross-grouped classes by ability for mathematics) and her general acceptance of a lower level of attainment by her students, was evident in the slow pace at which she moved the students. Establishing a classroom community where students develop a sense of belonging is essential if they are to engage in mathematical learning (Anthony & Walshaw, 2008) and when teachers have lower expectations for students, their achievement is adversely affected (Boaler, 2008; Rubie-Davis, 2007, 2010, 2015; Zevenbergen, 2005). Zevenbergen's research found that lower stream students believed that their mathematical experiences were more restrictive and less enriching for having been in a class of this nature. This practice can also lead to long-term problems, as while school achievement is constrained by the ability grouping, lower-streamed students are also set up for low achievement in later life skills, required for employment (Boaler, 2008; Solomon, 2007; Wiliam & Bartholomew, 2004).

The few observations of purpose of content knowledge (defined in Figure 3.1 as discusses or demonstrates reasons for content being included in the curriculum or how it might be used) may have contributed to the difficulty students had in understanding some of the key concepts taught, with the ensuing repercussions reflected in their final assessment results (Table 7.2). For example, lesson observations indicated that the teachers encouraged the students to construct models of multiplication using an array to solve problem examples (e.g., 3×5), with little discussion around patterns that the structure formed. This meant that the students grappled with the problems presented to them in the final lesson (area problems in Andy's class, and using known multiplication facts to solve unknown basic facts, in Beth's and Bob's classes). Understanding the array model has often been emphasised as the most effective representation for understanding multiplication and the basis for further multiplication applications (Anghileri, 2006; Davis, 2008), along with an emphasis on the appreciation of pattern and structure in mathematics learning (Mulligan & Mitchelmore, 1997, 2009, 2013; Mulligan et al., 2013). Teaching multiplication and division provides many opportunities for unpacking associated patterns and the structure of number and

this relationship was only mentioned in Beth's class, when they discussed the patterns of the $\times 3$ tables, and the doubling strategy.

Teaching Strategies (A.3)

The framework used for analysis described teaching strategies as being evident when, different strategies or approaches are used for teaching a mathematical concept or skill (Figure 3.1). One approach all teachers used was to identify a learning intention at the start of each lesson, recorded as a WALT (Section 5.2.1: A.3). While this was the intended purpose of the lesson, it created a narrow focus and in some instances observed to constrain the lesson. As the teachers allowed the WALT to dominate their lessons, they often missed opportunities for students who brought their own thinking to problem solving. For example, Anna and Andy briefly listened to students' explanations and rather than ask further questions to extend their ideas, refocused on the WALT, while Bob remained single-minded about the need to continually remind students of the specifics of the WALT. Observations suggested there may have been a two-fold reason for this: management of the students; and apprehension about coping with something mathematical that may arise, to which the teacher may not have known the answer. As long as they maintained focus on the WALT, they were equipped and ready to answer any questions.

While the similar number of teaching strategies (6 or 7) used in the initial lesson (Table 5.3) might suggest all teachers had similar approaches to their lessons, they varied considerably and resulted in differing classroom dynamics. For example, Anna and Beth sat on the floor with their students, maintaining accessibility to manipulatives, and a modelling book to share and record their thinking. The strength of a modelling book is in reinforcing the complexities of hands-on learning, that kinaesthetic learning alone tends to gloss over (Higgins, 2006). Anna's and Beth's usage contrasted with Bob, who sat on a chair recording on the whiteboard and rarely used the modelling book available. Andy's usage of the modelling book sat somewhere between the extremes of the openness and flexibility of Anna and Beth, and the domination of Bob. The modelling book was utilised a little more in Bob's final lesson, although at one point, he flicked back through the few pages to find an example of multiplication as repeated addition, only to realise that there was no record of the earlier discussion that had taken

place. The written examples had been on the whiteboard and had Bob, or his students recorded in the modelling book, evidence of prior learning could have been referred to directly. This would align with Higgin's research, which concluded the modelling book becomes a shared recorded history of previous learning, and provides the teacher and students with a means of informing discussion through linking back to previous mathematics sessions. The modelling book provides students with information in discussion that can be used later in building their mathematical understanding.

In the initial session, all lessons were very teacher-directed and while the teachers asked the students to share ideas, there was little sign of genuine discussion. When the students were asked to pair-share their thoughts, they described steps taken, with little evidence of conversation and no justification of solutions. As discussed in Chapter 2, this aligns to Cobb and Bauersfeld's (1995) notion of a univocal discussion, where one student explains his solution to a partner who attempts to make sense of the explanation and accepts his answer without interaction or question.

As the final lessons unfolded, there was a particularly notable change in Andy's approach to his lesson (Section 5.2.1: A.3). It was less structured (than the initial lesson) and he recognised the value of explaining and justifying ideas as a powerful strategy for supporting student learning, resulting in his students attempting to discuss and argue amongst themselves. Explanation and justification of ideas is advantageous to student learning and understanding, as when students explain an answer they build a stronger mathematical argument or find a new way of looking at the problem (Boaler, 2008; Fraivillig et al., 1999; Whitenack & Yackel, 2002). The importance of students' problem-solving together, justifying their methods of solution, and feeling comfortable about disagreeing with a peer's ideas, is a crucial part of the learning process (Hunter, 2009, 2010, 2012). Case studies have shown that the multivocal approach (Cobb & Bauersfeld, 1995) results in more productive interactions, whilst at the same challenging the mathematical learning of those involved.

Cognitive Demands of Task (A.4)

In an effort to reduce some of the cognitive demands of the tasks associated with understanding multiplication, the teachers kept initial problems relatively simple

for the students' year levels. In the initial lesson, all teachers focussed on understanding the multiplication symbol (Section 5.2.1: A.4), and the meaning of the multiplier and the multiplicand (Davis, 2008). Anna, Andy, and Beth, consolidated understanding when they encouraged their students to manipulate materials to form arrays. Bob modelled multiplication representations on a Slavonic abacus, but gave few opportunities for the students to construct arrays for themselves.

Anna's students should have been able to use a range of strategies for multiplication and division with whole numbers, as this is an expectation as students transition from AA to AM on the Number Framework (Ministry of Education, 2008f, p. 52). However, Anna's examples did not include problems where both factors were double-digit numbers, an expectation at this level. The reluctance to nudge numbers up in magnitude to create more challenging problems, and encourage the students to attempt more difficult tasks, meant they may have had less opportunity to gain insight, reason, and understanding of the important principles of solving multiplication problems (Ma, 2010).

The notion of increasing complexity of problems is supported by the strategy teaching model used by the teachers. Progression from using materials, to imaging, to using number properties, is promoted by increasing the complexity or size of numbers involved, thus making reliance on representations with materials cumbersome and inefficient (Ministry of Education, 2008b; Wright, 2014). The reluctance of the teachers to support the students through the challenging process of developing further understanding, contrasts with Japanese and Chinese classrooms, where teachers want their students to struggle with problems (Ma, 2010). Students need challenging work that results in mistakes (Boaler 2008, 2013; Ma, 2010) and the mistakes should be valued for the opportunities they provide for learning (Boaler, 2013). When students successfully unpack problems that they initially find difficult, it consolidates their understanding of concepts (Boaler 2008).

Appropriate and Detailed Representation of Concepts (A.5) and Knowledge of Resources (A.6)

All teachers represented concepts and showed their knowledge of resources through the use of manipulatives, including Unifix cubes, animal strips, the

Slavonic abacus, fly-flip cards, the number line, and a large hundreds board. While there were differences in how the teachers utilised materials in the initial lesson (e.g., Bob held an abacus, Beth's students used cubes), by the final lesson the manipulatives had become important resources for the students to construct representations of their ideas, alongside discussion about their problem solving. When teachers utilise a conceptual and dialogical approach to mathematics teaching, their choice of equipment is framed by the discussion of mathematical ideas they wish to foster among students (Higgins, 2005c).

Word problems, were also used when representing concepts. While, Anna and Bob (in the initial lesson) seldom utilised word problems, Andy and Beth regularly used them to demonstrate the term 'groups of' in relation to multiplication. Beth's students had been identified as low-achieving and she regularly explained unfamiliar concepts by comparing them to familiar objects and experiences. When teachers create scenarios and word pictures that appeal to their students, conceptual understanding is acquired by aligning mathematics to their real-life world (Anderson-Pence et al., 2014; Carpenter et al., 2015; Ma, 2010; Miheso-O'Connor, 2011; Mulligan & Mitchelmore, 1997; Schwartz, 2008). Metaphors, analogies, and models are important components of effective explanations, and the ability to transform new mathematics ideas through explanations is a necessity for teachers, if students are to understand them (Miheso-O'Connor, 2011).

Beth was observed using multiplication and division alongside each other. It is important that students see the relationship between multiplication and division and the structure of the two problem types (Anghileri, 1999; Clark & Kamii, 1996) and to show the connection. Beth used a word problem to dictate the model created (for understanding), by turning the multiplication expressions of 3×5 and 5×3 into division:

You've got a total of 15 biscuits in the cupboard. This family over here has 5 children okay. How many biscuits are they getting each? ... Over here you have a total of 15 and you're dividing it [the biscuits] up for 3 people. Are these people [points to a circle drawn in modelling book] getting the same as these people?

One of the major conceptual difficulties students have in working with multiplication structures is appreciating that groups of items are composite units, while also understanding that a group contains a given number of items (Clark &

Kamii, 1996). While the students in all classes struggled with the idea of composite units alongside multiple representations, Beth was the only teacher who attempted to unpack this.

Student Thinking (A.7) and Student Misconceptions (A.8)

The two categories of Student Thinking and Student Misconceptions are discussed simultaneously here, as when the data was analysed and compared, some interesting anomalies were revealed. In the introductory lesson, the frequency of occurrences (Table 5.3) might suggest that Bob capitalised to the greatest extent on the students' thinking (A.7 = 10), while at the same time his students had fewer misconceptions (A.8 = 2). This contrasted with Beth, whose students had the greatest number of misconceptions (11), and her recognition of misconceptions outnumbered the number of times she utilised students' thinking (8). However, the frequency data in these two sections must be combined with classroom observations in order to gain an accurate interpretation of the data in relation to the classroom teaching, as the number of times the teachers responded to the students' thinking and misconceptions was directly related to the manner in which the lessons were conducted.

Bob's initial lesson was a very teacher-directed, structured lesson, following a back and forth conversation between himself and one student at a time, suggesting the IRE model (Flores, 2010), which meant he was in a position to respond immediately to a student's ideas. Similarly, Beth's lesson began with back-and-forth conversation between the students and herself, which was where she addressed their thinking and many misconceptions. However, as the lesson progressed Beth became less directly involved and allowed the students to solve problems, manipulate materials, and discuss ideas amongst themselves. During this time, while student misconceptions occurred Beth, was not in a position to utilise students' thinking to the same degree, as the students now steered the thinking and reasoning process themselves. However, when students participate in productive discussions, they listen to and made sense of each other's solution methods (Hunter, 2009, 2010; Yackel, 2001). By allowing the students to scaffold with each other, Beth was also advancing their thinking, utilising what Fraivillig et al. (1999) referred to as 'supporting techniques'. Beth assisted the students in

clarifying their solution methods and often gave them other similar problems, to consolidate their understanding.

In the initial multiplication lesson, Anna had similar frequencies for student thinking (A.7 = 6), and student misconceptions (A.8 = 5), while Andy had the lowest frequencies for both categories (4 & 2 respectively) (Table 5.3). Andy regularly asked the students questions but, before they had time to respond, he would either ask another question, or answer it himself (Section 5.2.1: A.8). Consequently, the students had little time to show any misunderstandings or misconceptions, as they seldom had time to contribute their own ideas to the discussion. The lack of wait time displayed by Andy is consistent with the research of Black et al. (2004), who found that after asking a question, many teachers wait less than one second and if no answer is forthcoming, either ask another question or answer the question themselves. Like students in Black et al.'s study, students in Andy's class ultimately avoided answering a given question and instead waited, knowing that he would eventually answer it for them. Thus, they appeared unresponsive to questions asked of them. This strategy was upheld in Bibby's research (2002), who found that many students believed that the main characteristics of school mathematics was that it should be quick, efficient, rule-based, and full of right answers. Students upheld the notion that school mathematics was full of questions, where the person asking the question (the teacher) already knew the answer and therefore the opinions of others did not really matter (Bibby, 2002; Black et al., 2004).

Nevertheless, Andy showed an increase in the frequency of times that student thinking was utilised from the initial multiplication lesson to the final lesson (Table 5.3), which aligned to the changes made in his teaching style. Andy's final lesson was less teacher-centred (than the first lesson), allowing the students more time to respond to questions and converse among themselves. He capitalised on the students' responses and misconceptions acknowledged by giving responses such as: "now let's see if you are right? Are they (the answers) different or the same? (Andy pointed to others in the group)". These questions stimulated mathematical thinking (Way, 2008) and allowed flexibility among the students to recognise the difference between their answer and those of others, without a feeling of failure because their solution was different (Hunter, 2005, 2009). It also

provided opportunity for ‘friendly disagreement’ among group members, which is an important component of discussion, and takes time to develop and implement in a positive and comfortable manner (Hunter, 2009).

During post-lesson conversations, Bob commented that his own mathematics ability was strong as he was able to solve problems accurately (generally using an algorithm). However, while Bob was able to calculate answers to problems accurately (his questionnaire supported this), at times he caused confusion among the students which may have impacted on their learning. As explored in Chapter 2, knowing mathematics and knowing mathematics for teaching are not the same (Ball, 1991; Ball et al., 2005; Ma, 2010; Schoenfeld, 2013; Shulman, 1986, 1987, 2010). Ball et al., argued that in teaching, there is more “to knowing the subject than meets the eye” (p. 20) and that it requires “a kind of depth and detail that goes well beyond that which is needed to carry out the algorithm” (p. 22). Bob’s reliance on procedural knowledge meant there were times when conceptual understanding was not explored, resulting in student misconceptions.

8.2.2 Content Knowledge in a Pedagogical Context (B)

Deconstructing Content to Key Components (B.1)

The deconstructing content to key components category was the most frequently observed in use within Section B, identified 50 times (including Anna) in the initial lessons (Table 5.3). Deconstructing content is evident when the teacher identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept (Figure 3.1). All introductory lessons began with a focus on understanding the difference in representation when an array model of multiplication is rotated. With examples such as 5×3 being understood as 5 groups of 3, the teachers unpacked the importance of the first number in the expression representing the number of groups (multiplier), and the second number representing the number of objects in each group (multiplicand). Anna and Beth used repeated addition to explain the difference between multiplication expressions (for example 3×5 and 5×3) This derived operation (Ma, 2010, p. 113) shows that multiplication is an operation derived from addition, and allows certain types of complicated addition problems to be solved in an easier way.

The repeated addition context the teachers modelled is frequently taught as an initial representation of multiplication. However, teachers could make greater distinction between repeated addition and multiplication in order to develop further understanding of various constructs (Carpenter et al., 2015; Confrey, 1994; Nunes & Bryant, 1996; Steffe, 1994). Steffe's research concluded that when students were able to see a composite unit (in Beth's case, a unit of 3) as a countable unit, and used their number sense to track the iterations, they formed an iterating concept of multiplication. Teaching multiplication should not be solely taught as repeated addition, or as the distribution of a composite unit across the elements of another composite unit, but instead alternative constructs for developing children's multiplication understanding should be encouraged. Multiplication may also be a one-to-many correspondence between two sets, such as one child has two eyes (1-to-2), which may be replicated a number of times (Confrey, 1994; Nunes & Bryant, 1996).

Beth investigated patterns formed when the products (of factors) were modelled, or recorded. She explored patterns with groups of three (Figure 5.11), as well as patterns created through repeated doubling when she unpacked the relationship between two times, four times, eight times, sixteen times, etc. Andy alluded to the importance of pattern in relation to the teaching of algebra, when some of the students colour coordinated Unifix cubes. However, he did not explicitly unpack pattern as it related to the construct of multiplication and it is important to recognise that different kinds of patterns are embedded within different branches of mathematics (Devlin, 2000). Beth's exploration of patterns in multiplication may help to support the students' awareness of pattern and its underlying structure (Mulligan & Mitchelmore, 2009, 2013; Papic et al., 2011). When students are encouraged to recognise and understand the underlying structure of one concept, they generally do so for other concepts, and learn to generalise in a range of situations (Mulligan & Mitchelmore, 2013).

All teachers emphasised the importance of knowing certain basic facts and how they may be used to work out other unknown multiplication facts. For example, Bob used the abacus to show how $\times 6$ tables, can be derived from knowing the times $\times 5$ tables, and Anna unpacked how place-value knowledge can help solve 3×99 , by first solving 3×100 . Exploration of the $\times 5$ and $\times 10$ connections with

other factors, such as when the multiplicand is doubled or halved, also contributes to deeper understanding of the associative property (grouping) of multiplication (Wright, 2014). There are many ways in which the various structures of concepts, and elements of mathematics are organised and related to each other, including the connection between repeated addition and multiplication (Ball et al., 2008; Mulligan & Mitchelmore, 2013).

Andy's final lesson incorporated multiplication with geometry and focussed on finding the area of shapes. One group of students constructed a rectangular array model, and found the number of small square units that fitted into the allotted rectangular area. Some students initially one-to-one counted or skip-counted the number of small squares covering their paper (Figure 4.16) and Andy suggested counting the number of squares in each row, the number of rows, and then multiplying them. This highlighted the significance of the one-to-many correspondence (Confrey, 1994; Nunes & Bryant, 1996), along with the relationship between the array model in multiplication and area in geometry.

Mathematical Structure and Connections (B.2)

In the initial lesson, focus on mathematical structure and connections was minimal for all four teachers (0 to 2). By the final lesson, Andy made five connections (Table 5.3). As well as making links between arrays and area, Andy linked number and algebra. He identified rules and formulae as examples of the connection between understanding number and applying it to rules in geometry, which leads to algebraic expressions. For example, a triangle is half of a rectangle so the formula for finding the area of the triangle is connected to the rule for finding the area of a rectangle. He explained to the students that the rules (e.g., for finding the area of triangles and rectangles) were algebraic expressions, because the rule works for all triangles and all rectangles.

Bob and Beth, used the Slavonic abacus (each row of beads was in two groups of five contrasting colours) to encourage students to see the structure and connections between groups of five and groups of ten. The groups of 5 and 10 relationship, was later extended to other factor relationships where the multiplicand could be doubled or halved (e.g., the relationship between $\times 4$ and $\times 8$) (Section 5.2.2: B. 2). This led to the doubling and halving strategy, where the students were shown that 4×18 could be solved by multiplying 8×9 by drawing

arrows between the two expressions recorded on the whiteboard. Both Beth and Bob became confused in their explanation of doubling the multiplier and halving the multiplicand, when they attempted to demonstrate this idea with the abacus, so placed the equipment aside, and moved on to something else. Manipulatives are generally used in mathematics lessons with the claim that they extend students' learning of mathematical concepts, and operations, by making them more comprehensible (Burns, 1998; Ma, 2010; Nührenbörger & Steinbring, 2008; Ross, 1989; Schoenfeld, 2011; Swan & Marshall, 2010; Wright, 2014). However, this was not the case when it came to the doubling and halving strategy in Beth's and Bob's lessons. When manipulatives are utilised to represent the mathematical concepts underlying the procedure, and connections made between the two – the manipulative and the mathematical idea – mathematical understanding should be greater (Carbonneau et al., 2013; Clements & McMillen, 1996; Fennell & Rowan, 2001; Ma, 2010; Pape & Tchoshanov, 2001; Zevenbergen et al., 2004). However, teachers must model the use of manipulatives accurately or students will not make the connections between them (the manipulatives) and the mathematical concept being explored (Carbonneau et al., 2013).

Methods of Solution (B.3) and Procedural Knowledge (B.4)

Analysis of the lessons indicated that there were times when all teachers were procedural in their approach to teaching (Table 5.3: B.4 range 1 to 4; also B.2 above). Bob introduced doubling and halving procedurally (B.2 above) and when the notion of thirding and trebling arose, he recorded 3×27 on the whiteboard. He said, "*What you do to one side, you do the opposite to the other. If you times that by 3 (pointed to the number 3), you divide that by 3 (pointed to 27). Okay.*" There was no explanation about why thirding and trebling can be a strategy applied to solve multiplication problems and no modelling of the concept with equipment.

Andy's final lesson focussed on formulae for the area of rectangles, triangles, and irregular shapes. However, understanding the structure of the array was not embedded as knowledge for Andy's students, and this was reflected in the manner in which they solved area problems in class and in their final assessment results. It was a major jump for the students to move from counting rows and columns, to conceptual understanding of a formula for finding area (Clark and Kamii, 1996).

The students needed to spend time consolidating understanding of the formula for the area of a rectangle, by understanding that a multiplication array consists of finding the total amount in the structure of rows and columns of squares. Focussing on the structure of multiplication, rather than memorizing individual facts by rote makes the transfer of knowledge more likely, and is necessary for the students to understand area (Baroody et al., 2009; Clark and Kamii, 1996).

In the final lesson, Beth encouraged the students to use knowledge of doubles to find the answer to 16×5 . Beth explained this procedurally, by firstly finding 2×5 , then 4×5 , then 8×5 , and finally 16×5 . The students observed the procedure and later in the lesson when use of doubles arose again, they could not explain what to do. The students understood that 2 groups of 5 might be doubled to 4 groups of 5, but struggled to see how this procedure might be extended further to 4 groups doubled to 8 groups, then 8 groups doubled to 16 groups. The use of manipulatives enables students to see for themselves what a procedural explanation cannot always show (Swan & Marshall, 2010; Thompson, 1994; Van Garderen, 2006) and had Beth allowed students to manipulate materials as she demonstrated the repeated doubling procedure, they may have been more likely to retain this strategy for further use.

All of the teachers acknowledged during follow-up conversations, that they had learned mathematics procedurally, often revert to their known methods during teaching, and have moments when they struggle to explain concepts conceptually. Once a teacher becomes fluent and automatic in the use of elementary mathematics concepts and procedures, it may make it difficult for them to teach those conceptually to students (Ball et al., 2001; Schoenfeld, 2011, 2013; Schwartz, 1996). With automaticity, it is possible to lose recollection of how fundamental and basic mathematics understandings are constructed. Mathematical procedures that are automatic for teachers are far from obvious for students, and there needs to be a distinction made between the procedural application of knowledge and conceptual understanding (Ball et al., 2001).

Profound Understanding of Fundamental Mathematics (PUFM) (B.5)

Profound Understanding of Fundamental Mathematics (PUFM) appeared to be one of the least evident of the framework categories observed in use by the teachers (excluding Anna, 6 in the first lesson and 2 in the final lesson), (Table

5.3). Observations indicated that the teachers taught mathematics as isolated or fragmented pieces, with limited application of what Ma (2010) referred to as subject matter knowledge that is deep, broad, and thorough. This was evident in their continued reference to the WALT during teaching, and reluctance to deviate from the planned lesson, which ultimately restricted opportunities to make connections between mathematical ideas and within concepts. Ma emphasised that a teacher must make a special effort within lessons to emphasise the fact that key ideas are connected. If the focus is on one idea at a time, it can create the illusion that mathematics consists of a number of separate ideas used individually. Depth and breadth depend on a thoroughness which glues knowledge of mathematics into a coherent whole (Ma, 2010). In order for teachers to display profound understanding of fundamental mathematics (PUFM) they needed to exhibit deep and thorough conceptual understanding of identified aspects of mathematics (Chick et al., 2006).

One area of PUFM unpacked by the teachers was representation of the commutative property within the context of multiplication (Section 5.2.2: B.5). The teachers challenged the students to justify why understanding the difference in the rotated representations of expressions (e.g., 3×5 and 5×3) is important. The difficulty in understanding the difference in representation was identified in previous research (Baroody, 1999; Carpenter et al., 2015; Steffe, 1994; Suggate et al., 2010), which found that the commutative principle as it relates to multiplication, involves an awareness of the reorganization of the multiplicative construct. Because, representation of the two different forms of the equation is quite different (Carpenter et al., 2015; Steffe, 1994; Suggate et al., 2010), the students did not immediately understand that that the two numbers are interchangeable when solving problems (Carpenter et al., 2015). In this instance, it was the groups of construct, which the teachers emphasised to support this understanding.

Beth discussed the connection between multiplication and division, as she unpacked the commutative property. Making connections within a topic, has been described as understanding the topic with depth by connecting it with more conceptually powerful ideas of the subject (Chick et al., 2006; Howley et al., Kazemi & Stipek, 2001; 2007; Ma, 2010; Stigler & Hiebert, 2004). Once the

students constructed their representations of the multiplication expression, Beth demonstrated the idea of the multiplier and the multiplicand using a scenario that involved division and the sharing of biscuits. This helped to clarify the difference between 3×5 and 5×3 . However, the explanation was oral and had the representations been recorded alongside the expressions, she could have further explored the connections and shown the thoroughness Ma refers to within PUFM. The value of simultaneously teaching the relationship between multiplication and division as inverse operations, has been stressed by researchers who have advocated that understanding how these problem structures are connected, can help students generalise when they solve problems later (Ma, 2010; Mulligan, 1998; Roche & Clarke, 2009; Roche et al., 2016; Young-Loveridge, 2011).

There were many opportunities for the teachers to show an awareness of number sense (or feel for number) within the lessons. There were instances when the teachers avoided unanticipated student questions, accepted one strategy to solve a problem as opposed to encouraging students to seek alternative strategies, and often relied on rote learnt explanations of concepts (as opposed to unpacking understanding of key ideas). In these instances, a sound number sense would have seen the teachers recognise the importance of encouraging their students to understand and explain their solutions to problems. Teachers with number sense are aware of the conceptual structure of mathematics, and look at the numbers to see ways of solving an expression using a range of strategies and techniques to find an answer (Briand-Newman et al., 2012; Ma, 2010; Nunes & Bryant, 1996; van den Heuvel-Panhuizen, 2001d).

8.2.3 Pedagogical Knowledge in a Content Context (C)

The Pedagogical Knowledge in a Content Context (PKCC) section refers to the knowledge which has been drawn most directly from pedagogy (Chick et al., 2006). It contains special reference to those broad principles and strategies of classroom management and organisation that appear to transcend subject matter (Shulman, 1987).

Classroom techniques (C.1)

This research ascertained that while the teachers attempted to use particular techniques in the classroom, their desire often differed from actuality. For example, there was discrepancy between encouragement by the teachers for

discussion among the students, for them to share their thinking, to discuss, and justify their ideas, and evidence of the discussions. The students needed to develop a confidence in their ability to think and reason mathematically, and to explain and defend those reasons (Fraivillig et al., 1999; Hunter, 2009, 2012; Whitenack & Yackel, 2002). It is often in the explanation of a correct answer that a student gains a deeper understanding of a mathematical concept (Kazemi & Stipek, 2001). In mathematics classrooms, both explanation and justification have important roles as students develop arguments during the discussion process (Ball, 1993; Goos, 2004; Forman & Ansell, 2002; Hunter, 2006, 2010; Lampert, 1990; Stein et al., 2008; Whitenack & Yackel, 2002; Wood et al., 2006; Yackel & Cobb, 1996). The forthcoming expectation of challenge, and at times disagreement, from listening to other group members is what extends explanation of challenge, to justification (Hunter, 2006), and this did not occur within the observed lessons.

The teachers frequently asked the students questions and then provided the answers, and at times the students asked questions that were not addressed. The teachers' avoidance of questions, may have been partly due to uncertainty about how to progress the students (as shown on the written scenarios) and was one way to avoid being put in a situation where they were unable to provide confident answers. Similar responses to questions were identified in the research of Black and Wiliam (1998), who reported that teachers often lack the flexibility or confidence to deal with the unexpected and so they direct students towards giving the expected answer. Black and Wiliam suggested that when teachers manipulate the discussion this way, the students soon realise they do not need to work out their own answers and it often becomes a case of guessing what it is the teacher wants to hear. It would seem that guessing what was in the teacher's head was particularly evident in Andy's and Bob's classes.

Getting and maintaining student focus (C. 2) and Goals for learning (C. 3)

All teachers gained the focus of students when contextualising word problems using the names of the students, and creating scenarios relevant to their everyday lives. The students were excited when Beth made the mathematics more meaningful to them and began a word problem along the lines of, "Let's imagine Jane's (a child in the class) family was ..." Personalising problems by using the names of students in the class, was also emphasised in the research of Fraivillig et

al., (1999), who concluded that when teachers used the names of the students in their class, it helped to give the students a feeling that the mathematics was accessible, and that the classroom was participatory and their learning relevant. It recognised the importance of student at the heart of the learning. Other researchers have also emphasised the importance of relational understanding during mathematics lessons and advocated word problems as one of the most effective ways to help develop the conceptual understanding that is needed (Schoenfeld, 2011; Skemp, 2006).

A technique often used by Andy to maintain focus was to surprise, or challenge the students with statements or questions. He made comments including, “*So, what can you two do to try and solve this problem?*”; “*So what can you do to solve this conundrum?*”; “*If you don’t stop doing things you won’t learn.*” As part of his challenge to the students, he sometimes deliberately gave incorrect answers to questions, or made statements like, “*I could be wrong you know.*” At times, he challenged them a little further: “*Now these ones here, are a little bit trickier,*” or “*I’ve left this one in here as a special [one], and I wonder if you can work it out? It’s a brain teaser for you.*” Andy’s recognition of the need to challenge students aligns to other researchers who have acknowledged the importance of using students’ thinking, and at the same time challenging their thinking (Black & Wiliam, 1998; Black et al., 2004; Smith & Stein, 2011). The importance of reasoning, and challenging oneself and each other, helps students to reason mathematically (Whitenack & Yackel, 2002). The students clarified their thoughts when the teachers challenged them to extend their thinking beyond an initial response, and justify and reason their answers. Students’ mathematical thinking further advanced in an environment where there was a level of respect between the teacher and the students (Fraivillig et al., 1999), and a superficially tough, no nonsense manner was combined with affection for the students, such as was the case in Andy’s class.

Knowledge of assessment (C.4)

Capitalizing on the information gained from the initial assessment, could have highlighted general patterns of students’ uncertainties by the teachers, in readiness for teaching. However, written diagnostic assessment tasks given to the students were not accessed (teacher choice), nor was reference made to assessment data

available from other sources, including PAT (New Zealand Council for Educational Research, 2006) and NumPA (Ministry of Education, 2008d). The teachers later indicated (during interviews) that the PAT and NumPA scores were generally used in a summative manner, were included in individual student profiles when reporting to parents, and used for reporting to the Ministry of Education. While data for summative purposes is recognized, the NZC states that, “the primary purpose of assessment is to improve students’ learning and teachers’ teaching, as both student and teacher respond to the information that it provides” (Ministry of Education, 2007, p. 39). More in-depth analysis of the assessment data available may have assisted the teachers in determining a wider range of appropriate learning experiences.

Questioning Techniques

Supporting type questions (C.5) were used to the greatest extent (excluding Anna: 50 times in the initial lesson and 68 in final lesson), from the three different question types of supporting, eliciting, and extending, explored in this study (Table 5.3). Supporting questions were generally lower-order, back-and-forth or one-on-one questions, and did not encourage in-depth discussion and argumentation among the students. The high frequency, with which supporting questions were used in this research, concurred with other researchers’ findings, which identified that supportive-type questions are the most prevalent type used in classrooms and is a skill that most teachers feel comfortable with (Fraivillig et al., 1999; Waring, 2009; Way, 2008). The structure of this classroom discussion reflects the “teacher initiation, student response, teacher evaluation” (IRE or IRF) model (Cazden, 2001; Waring, 2009). This included what Cazden referred to as revoicing, which is frequently used as part of the IRE pattern of discussion to govern the conversation that followed.

Elicitation of students’ thinking (C.6) is an important part of the learning process and questioning based around eliciting ideas was found to be the second most commonly used by teachers in this study (a total of 27 times in the initial lesson and 26 in the final lesson, Anna excluded). Ascertaining what students know, and how they think about a problem, is critical to further their understanding of that mathematical concept. In the initial lesson, Bob appeared to elicit ideas to the greatest extent, within his structured lesson (18, compared to 8, 14, and 1 of the

other teachers). However, the type of elicitation he used was restricted to questions that allowed the students to explain their thinking (generally using the Slavonic abacus which Bob was holding), or share their solution methods with others. Fraivillig et al.'s (1999) research also identified the eliciting questions evidenced in this study of probing students, promoting collaborative problem-solving and using students' names, as the second most commonly used by teachers.

This study also showed a direct relationship between the eliciting type questions used and the manner in which each teacher involved a number of students in explanations within the lesson. This concurred with the research of Fraivillig et al., who concluded that by eliciting students' responses, teachers orchestrated learning opportunities for all students, while assessing individual children's thinking.

By the final lesson, the teachers extended the thinking of the students (C. 7) by using questions that encouraged individual students to go beyond their initial solution methods. At one stage, Andy said to a student, "*Can you explain your theory to the rest of the group?*" At times, Andy accepted a student's response to an answer, then to extend the thinking of the group asked others to explain mathematically the reason why the given answer was correct, or give an alternate solution. An example, was when he asked, "*What do you think this shape would be? Talk to your neighbour. Can you find different ways of working the answer out?*" The approach Andy used of questioning all students, to consider and discuss a possible range of methods for problem-solving, is consistent with the findings of Fraivillig et al. (1999), which concluded that extending students' thinking requires mathematical reflection and challenges all students to try difficult problems and arrive at different solution methods. The practice of extending a student's thinking and reasoning requires a teacher to understand the capability of each student, yet be realistic and hopeful of each student's potential.

The total frequency of usage of the different question types explored in this study (138 supporting, 67 eliciting, and 20 extending), is consistent with the research of Fraivillig et al., (1999) who found that teachers were very supportive about students' thoughts, elicited ideas to a limited degree, but when it came to extending the students' thinking, this was seldom evident. The teachers required more support themselves in the art of extending the thinking of their students,

which embraces a shift away from a focus on students acquiring proficiency in merely regurgitating their existing knowledge, towards the idea that students should be supported to construct their own mathematical ideas. This is accomplished, to some extent, by anticipating the responses of students and preparing in advance suitable questions, in readiness to extend their mathematical thinking and understanding (Smith & Stein, 2011).

8.3 Student Learning in the Multiplicative Domain

This section of the chapter discusses part of research Question 2, concerning “*the relationship between teachers’ professional knowledge and student learning*”.

Pre-unit assessment tasks were designed to ascertain the students’ current mathematics problem-solving skills, alongside their understanding of the processes involved, while similar tasks were repeated at the end of the unit, as an indication of the students’ learning. Students’ understanding of the mathematics involved, through justification of their answers, was deemed as important as the answer itself.

Students participating in the research included Beth’s and Bob’s Years 5 and 6 classes, Andy’s Year 6 and 7 students, and Anna’s students who were Years 7 and 8. For this reason the assessment tasks were all within the expected capability of students working at New Zealand Curriculum Levels 2 and 3 (Ministry of Education, 2007, 2009a), or Stages 5 and 6 of the Number Framework (Ministry of Education, 2008a).

8.3.1 Assessment Tasks and Student Learning in the Multiplicative Domain

The mathematics inherent in Task 1 (understanding multiplication as repeated addition: $4 + 4 + 4 + 4 + 4 + 4 = 24$. How would you write this as a multiplication fact?), would be taught when transitioning from AC to EA and aligns to New Zealand Curriculum (early) Level 2 (Ministry of Education, 2007) (Section 7.1.1: Task 1). As these students were Years 5 to 8, it is suggested that all students should have been able to complete this task correctly. However, the small number of students, who initially got this task correct (33%), showed that a clear majority did not understand the concept of multiplication as iterations of a unit. Awareness of multiplication as iteration of a unit, has been stressed as an important

component of students' understanding of the basic structure of multiplication, as it forms the foundation of applying simple multiplicative strategies to combine (or partition) whole numbers (Mulligan & Mitchelmore, 1997, 2009, 2013; Sophian, 2007).

On the final assessment, there was a decline in the number of students who solved Task 1 correctly at School A, from 36% to 27%, while at School B, there was an increase from 30% to 65%. These percentages appeared to be related to the amount of emphasis that the teachers placed on understanding the multiplication symbol as presenting equal groups of. Each of the four teachers began the lessons by discussing the commutative property and the fact that both 3×4 and 4×3 gave an answer of 12, while the representation of each expression was different (Section 5.2). While acknowledging that both equations equalled 12, the teachers emphasised the meaning of the multiplier and multiplicand in each written expression and highlighted the notable difference between the two, when constructed as models. At this time, Beth and Bob explicitly mentioned repeated addition of equal groups (Section 5.2.2: B.1) as the teaching of repeated addition and skip counting consolidates the concept of equal groups in multiplication (Davis, 2008; Mulligan, 1998; Mulligan & Mitchelmore, 1997). Skip counting develops concepts of multiplication around the notion of a 'composite' unit and is often seen as a fundamental principle of understanding multiplication through equal grouping (Davis, 2008; Mulligan, 1998; Mulligan & Mitchelmore, 1997; Sophian, 2007). While there are different problem structures (for example multiplicative comparison, array, Cartesian product) and contexts for teaching multiplication, understanding the structure of the groups of idea, such as when Andy and Anna referred to 3 groups of 4, and 4 groups of 3, is recognised as the basis of all other structures (Davis, 2008; Hansen, 2005; Mulligan & Mitchelmore, 1997, 2009).

Task 2 (understanding of the multiplication symbol: Draw a picture of what 3×5 would look like), was connected to the multiplication understanding implicit in Task 1. On the initial assessment, less than 20% of the students at either school correctly sketched a representation of 3×5 , although this increased slightly on the final assessment (School A: 17% to 21%, and School B: from 16% to 26%) (Tables 7.1 & 7.2). Results suggested that the students had little understanding of

the meaning of the expression “ $a \times b$ ” as interpreted in New Zealand schools as “ a groups of b ” (Ministry of Education, 2008f, p. 12) and closer consideration of the assessment results indicated that this was exemplified in their sketches of the representation. While it is acknowledged that an array structure may be represented either way (Carpenter et al., 2015), it appeared that the majority of the students had not recognised the connections between the problems they solved, the models they created in class, and were unfamiliar with presenting their ideas diagrammatically. Past research has also revealed that when students are initially given the opportunity to represent their models and/or their ideas on paper, they often struggle to do so (Anderson-Pence et al., 2014; Ma, 2010; 2010; Suh, 2007; Thompson, 1994; van Dijk, van Oers, & Terwel, 2003). Drawing of mathematical representations was seldom observed in any of the classrooms, and yet the importance of sketching has been emphasised as a means of consolidating students’ understanding of a concept (Anderson-Pence et al., 2014; Ma, 2010; Mulligan & Mitchelmore, 1997; Suh, 2007; Thompson, 1994; Young-Loveridge & Mills, 2009a). As has been suggested elsewhere, when problem solving a diagram is a useful tool as it can serve to represent the structure of the problem, as well as allow students to demonstrate abstract problems, which they may find difficult using only words and symbols (Mulligan & Mitchelmore, 1997; Nührenbörger & Steinbring, 2008; Pantziara, Gagatsis, & Ella, 2009; Suh, 2007; Thompson, 1994).

Few students understood the difference between partitive and quotitive division on either the initial or the final assessment (Tables 7.1 & 7.2)). When asked to sketch a partitive division problem (Section 7.1.1: Task 3), 11% of the students were able to demonstrate the equal sharing idea correctly on the initial assessment, with 10% correct on the final assessment (Section 7.1.2: Task 3). When asked to draw a representation of a given quotitive division word problem, thus demonstrating the notion of equal grouping, the total number of correct responses lessened to 4% on the initial assessment and 6% on the final assessment (Tables 7.1 & 7.2., Task 4). In relation to the teaching of the different division types, the teachers admitted that the equal sharing context continues to dominate their teaching, which is consistent with the research of Roche and Clark (2009) and Roche et al., (2016). The teachers’ lack of awareness of representational

difference between the two division types, resulted in quotitive division seldom being taught, and ultimately impacted on students' learning and understanding of the structure of division. As Mulligan and Mitchelmore (1997) have suggested, it would seem that more attention needed to be given by the teachers during lessons to the presentation of word problems, to strengthen understanding of the division types. Given that the word problem in Task 4, informed the students how many (biscuits) were required in each group (20 biscuits to put into packets with 3 biscuits in each packet), it seems strange that more students did not give a correct response.

On the initial assessment 25% of the students were correct on Tasks 5 and 6 (draw a diagram to show representation of 2×5 and another to show 5×2), an indication of their understanding of the commutative property related to multiplication (Table 7.1). Understanding the number of items in a group (size of the group) and the number of iterations (number of groups), is a key idea that needs to be understood in the context of multiplication (Anghileri, 2006; Baek, 2006; Clark & Kamii, 1996; Mulligan & Mitchelmore, 2009). The results on Tasks 5 and 6 suggested that the majority of the students had not recognised that there was a difference in representation between the two expressions. In the observed teaching sessions, the students often modelled repetition of groups and found the total number of items correctly, by skip counting or multiplication. However, the number of students correct at School B on the final assessment Tasks 5 and 6 lessened from 24% to 17%, suggesting that connections were not made between the models constructed in class and the assessment tasks given. Previous studies have also shown that there is a difficulty in students' understanding of commutativity as it relates to multiplication, in that the representation of the two different forms of the equation is quite different, while the product remains the same (Carpenter et al., 2015; Steffe, 1994; Suggate et al., 2010).

Task 7 (using known facts to derive unknown facts), built on a key idea taught at Stage 5 of the Number Framework (Ministry of Education, 2008f, p. 32), that when certain basic facts are known (e.g., $2 \times$ [doubles], $\times 2$, $\times 5$, $\times 10$), they can be used to work out other basic facts. In the initial assessment (Section 7.1.1: Task 7), the students were asked to utilise a diagram representing 6×5 , to determine the

answer to 6×4 . At School A, 38% of the students made this connection, while at School B, 36% of Bob's students and 55% of Beth's students were able to do so. This might suggest that while Beth's lower-ability students could not instantly recall all basic facts, they were learning to think mathematically (Confrey, et al., 2009) by making use of known facts to solve problems. Learning to think mathematically has been identified as an important element in early learning experiences, which can later be built on and extended when consolidating mathematical thinking and reasoning (Anghileri, 2006; Confrey et al., 2009; Lampert, 1990; Siemon et al., 2005).

Understanding a diagram of an array of smiley faces (Task 8), was the only initial assessment task where more than half (53%) of the overall students were correct. The model given in the assessment was of a rectangular array and emphasised the understanding of groups of, which is used as a key representation of multiplication in New Zealand schools, including the terms rows and columns (Ministry of Education, 2008f, p. 15). It is acknowledged there are many contexts to be explored when teaching multiplication. However, the equal grouping idea portrayed in the array model in Task 8, is generally thought to precede other contexts (Carpenter et al., 2015; Ministry of Education, 2008f; Mulligan & Mitchelmore, 1997; Nunes & Bryant, 1996), and its usage has been stressed in prior research (Davis, 2008; Mulligan & Mitchelmore, 1997; Young-Loveridge & Mills, 2009a, 2009b).

Task 9 on initial assessment (using a known fact [e.g., $4 \times 7 = 28$] to work out the unknown [e.g., 4×14]), was an extension of the idea presented on Task 7. This idea related to a key idea for students transitioning from EA (Stage 5) to AA (Stage 6) of the Number Framework of being able to “work out unknown multiplication facts from those already known” (Ministry of Education, 2008f, p. 24). In the initial assessment the students were required to show how if they knew $4 \times 7 = 28$, the factor 7 can be doubled to solve 4×14 . While some students gave a correct answer to 4×14 using place-value partitioning, only a small number saw the connection between the two expressions and wrote that they doubled the 28, recognising that 4×14 was two groups of 4×7 . Research has shown that instruction can be effective when it directly builds on what the student already knows, and when the strategy used connects with known knowledge it

becomes more meaningful to the student (Baroody et al., 2009). It would appear that half of the students used a known doubling strategy and built on their basic facts knowledge to solve an unknown equation, with a double-digit factor.

Task 8 (I know that $3 \times 10 = 30$. How can I use this to work out $\square \times 5 = 30$?), on the final assessment, also entailed using known facts to find an unknown fact (Section 7.1.1: Task 9 on the initial assessment). The task given on the final assessment (Section 7.1.2: Task 8) posed much more difficulty than the corresponding Task 9 on the initial assessment. This may have been because the initial assessment task represented a result unknown problem structure, while the final task was a start unknown. This problem structure difficulty was also identified in the research of Carpenter et al., (2015), who found that students struggled to understand and solve problems where the start was unknown, as opposed to problems where to result was unknown.

On Task 8 in the final assessment, the students were expected to understand the connection between the $\times 10$ tables and the $\times 5$ tables, and the doubling and halving strategy: because the multiplicand was halved, the multiplier should be doubled. However, the total number of students who solved this correctly was very few (7 from 93 students), (Table 7.2): one student in Andy's class, five in Anna's class, one in Bob's, and no-one in Beth's class. Andy, Beth, and Bob, had all mentioned in their final lesson that two groups of 5 equalled one group of 10, while Bob and Beth had modelled the relationship between $\times 5$ and $\times 10$ on the abacus with the students. The importance of the relationship between $\times 5$ and $\times 10$ explored by the teachers in this instance was emphasised in the research of Young-Loveridge (2011). Encouraging students to generate new facts using what is known, such as the $\times 5$ and $\times 10$ relationship in this instance, also minimises the number of facts which need to be memorised (Baroody, 1985; Butterworth et al., 2003; Young-Loveridge, 2011). The doubling and halving strategy discussed by the teachers, also exemplifies the connectedness of mathematics concepts outlined by Ma (2010). While these connections had been discussed and modelled in class, in all instances the teachers had demonstrated the concept to the students, rather than the students manipulating materials themselves. Beth and Bob had been muddled as they explained the double and halve strategy to the students, which might also have contributed to the difficulty their students had solving this task.

The initial assessment Task 10 (Task 9 on the final assessment), was a division problem with a remainder. “Interpreting division remainders in meaningful contexts” is part of the learning at the AA to AM stage (Ministry of Education, 2008f, pp. 42, 60). If Anna’s Year 7 and 8 students were removed from the 21 students who were correct (Table 7.1), only four students from the other three classes combined, identified remainders and explained what the leftovers meant relative to the divisor. A further three students were mathematically correct, providing an answer of $7\frac{1}{2}$ or 7.5, although given the context of the problem being solved (30 apples put into 4 equal-sized bags) they were conceptually incorrect. What happens when there are leftovers, depends very much on the context of the problem. Other research has similarly found that while students might complete the calculation correctly, they were often unable to provide a solution that was consistent with the meaning of the problem (Roche et al., 2016; Silver et al., 1993; Treffers & Buys, 2001). The result of division can generally be tested against reality, and must be guided by reality (Treffers & Buys, 2001) and what to do when division doesn’t go, provides a special complication not found with multiplication. One of the most common errors students make on division assessment problems is to divide procedurally, not paying attention to the context when addressing what to do with remainders (Lamberg & Wiest, 2012; Ma, 2010; Roche & Clarke, 2009; Roche et al., 2016).

8.4 Aligning Teaching Materials and Language

During their observed lessons, the teachers relied on the Numeracy Development Project (NDP) supporting books (Ministry of Education, 2008a, 2008b, 2008d, 2008e, 2008f, 2008g) and associated planning sheets (Ministry of Education, *n.d.a*), to implement the New Zealand Curriculum (NZC) (Ministry of Education, 2007). At the time of the classroom observations for this research, the Mathematics Standards (Ministry of Education, 2009, 2010, *n.d.b*) were in their early period of implementation. To support teachers in understanding the requirements of The Standards, Curriculum Elaborations were later developed (Ministry of Education, *n.d.c*), although these were not available to the teachers at the time this research data was gathered. However, some of the difficulties mentioned earlier in this thesis that the teachers and students experienced, may have been attributed to two key factors involving: (i) the inconsistency of

expectations; and (ii) the language used, within and across the above teaching materials and documents. The expectations and language used are interdependent for teachers when teaching for understanding and some of the discrepancies found are outlined below.

Teaching materials

The recognised relationship between the New Zealand Curriculum Level, the Number Framework Stage, and the Class Level, was outlined in Table 3.3. However, when the associated teaching documents were cross-referenced the descriptors did not always align, causing some confusion in expectations and requirements associated with the teaching of multiplication and division. For example: Students at Years 5 and 6 are generally working from Level Three in NZC. As a result of a GloSS (or NumPA) assessment, the strategies used for solving problems suggest that the students should be working at Stage 6 on The Number Framework, and thus be taught lessons from the AA (Stage 6) to AM (Stage 7) planning sheet (Section 3.12.4). Both the planning sheets and NDP Book 6 (Ministry of Education, 2008f), are written as transitioning from one stage (e.g., Stage 6) to the next stage (e.g., Stage 7), (Section 3.12.4) and in some instances the expectations of the students in these lesson are not aligned to the associated curriculum level (e.g., Level Three) expectations. For example, the NZC (Ministry of Education, 2007) says that students should be able to, “use a range of additive and simple multiplicative strategies with whole numbers... know basic multiplication and division facts... record and interpret additive and simple multiplicative strategies, using words, diagrams, and symbols, with an understanding of equality” (Level Three chart). However, the AA to AM planning sheets move beyond simple multiplicative strategies and basic multiplication facts and includes using proportional adjustment to solve division problems, using place value units to solve multiplication and division problems (including written multiplication algorithms, , 34×26), and using divisibility rules for 2, 3, 4, 5, 6, 8, & 9. This means that while teachers are consolidating the learning and understanding of basic facts up to and including the $\times 10$ tables, an expectation at NZC Level Three, if they plan from the aligned AA to AM planning sheets they are also teaching the students lessons involving double-digit times double-digit multiplication, and understanding divisibility rules of numbers such as 3 and 9.

Lessons associated with these key mathematics concepts are found in the NDP Book 6 (Ministry of Education, 2008f, p. 67 & p. 70) and the knowledge required to understand them is beyond students who are still learning to understand and implement basic multiplication and division facts.

Inconsistency of language

The inconsistency in the language associated with the teaching of the times tables as written in many of the lessons, was one of the main difficulties shown by the teachers in their teaching practice and the students in their learning (Section 5.2.1: A.4, A.7, A.8; & Section 5.2.2: B.3). For example, one of the key lessons as students move from EA (Stage Five) to AA (Stage Six) is, “learning to work out [my] times six, seven and eight tables from my times five tables” (Ministry of Education, 2008f, p. 28). This was part of the lesson taught by Beth and Bob (Table 5.1) based on the twos, fives, and tens (Ministry of Education, 2008f, p. 21). The lesson on “times six, times seven, times eight tables”, is consistent with the standard New Zealand practice of times, as groups of (Ministry of Education, 2008f, pp. 12, 15). However, the language in the lesson on “twos, fives, and tens” (taken by Beth and Bob), refers to the five times tables (5 groups of), and then carries out times five (groups of 5) problems.

Inconsistency with the language began with one of the first lessons in NDP Book 6, “Three’s Company” (Ministry of Education, 2008f, p. 12), which has a learning intention of “I am learning to solve three times problems” and proceeds to get the students to make groups of three. The Threes Company learning intention, contrasts with the following lesson called “Animals Arrays” (Ministry of Education, 2008f, p. 15), which shows repeated addition such as $4 + 4 + 4 + 4 + 4$ as 5×4 , or 5 sets of 4. Following the Animal Arrays is a lesson called, “Pirate Crews” (Ministry of Education, 2008f, p. 17). Within the diagnostic snapshot for Pirate Crews it notes, “Students who solve the problem by multiplication, e.g., $5 \times \square = 20$, by skip counting 5, 10, 15, 20, or by using repeated addition $5 + 5 + 5 + 5 = 20$...” However this is not “ $5 \times \square = 20$,” but “ $\square \times 5 = 20$,” and makes three consecutive lessons where the use of times tables is inconsistent. There are many instances of inconsistency in the use of times as groups of, and the associated recording of $\times 5$ or $5 \times$ throughout Book 6, which meant that when the lessons were taught by the teachers in this research, as written in NDP Book 6, there was

inconsistency in the language used by the teachers and their students. This inconsistency might explain why the teachers and students confused the language in their lessons (Section 5.2.1: A.4, A.7, A.8; and Section 5.2.2: B.3), which was then evident for many in their final assessment tasks (Section 7.1.2: Tasks 1, 2, 5, 6, 7, & 8).

The NDP Book 6 supports the teaching of two division types: quotitive and partitive. The explanation begins on page 4 (Ministry of Education, 2008f) where the two types of division are described, using one diagram and the context “24 marbles and 4 bags.” The examples are: (1) “Kayla has 24 marbles. She shares them equally into 4 bags. How many marbles are in each bag?” and (2) “Kayla has 24 marbles. She puts them into bags of 6 marbles. How many bags can she make?” Example 1, should be recorded as $24 \div 4 = \square$ and example 2, as $24 \div 6 = \square$. However, the recording of the division equation alongside each of the given scenarios is not given, which can be confusing given that there is only one diagram in the book. It might have been more helpful for teachers, had two separate diagrams and two explanations been given in Book 6 for one equation (e.g., $24 \div 4 = \square$). This would support the Level Three elaborations, which note that “ $56 \div 7$ can mean fifty-six shared among seven, or how many sevens are in fifty-six,” (Ministry of Education, *n.d.c*), and later lessons in NDP Book 6 (Ministry of Education, 2008f, pp. 17, 19, 36, 38).

Division was not observed in teaching practice in this research, although the two division types were discussed with the teachers during learning conversation after their initial lesson, as division related to the commutative property understanding which had been observed in unpacking the times tables. Understanding division is an important part of teaching the multiplicative domain at Levels 2, 3, and 4, and during learning conversations after their second observed lesson, this was raised once again. The teachers all reported that they had included division in their units of work, but admitted it had mainly focussed in partitive division. The lack of clarity in the recording of an equation, alongside the writing of a word problem and drawing of a diagram, in the initial instance in Book 6, may have contributed to the teachers’ teaching and the students’ uncertainties in understanding the difference between partitive and quotitive division as evidenced in their final assessment tasks (Section 7.1.2: Tasks 3 & 4).

8.5 The Multiplicative Domain Summary

8.5.1 Teachers' Subject Matter Knowledge and Pedagogy

In order to examine the professional knowledge of teachers, this chapter explored the teachers' subject matter knowledge in relation to their pedagogical practice. The first part of each scenario asked the teachers to explain what they would do next with the student in each instance. In most instances, the teachers struggled to do this. The teachers often commented that they would teach the student the same way as the given example, and provided no further explanation or identification of the next steps to learning.

Questions the teachers might ask each student to check for understanding, or identify of the progression of learning to examples of increased difficulty, were not forthcoming. In particular, Bob and Andy found it challenging to identify what actions they would take in future lessons with each student. The difficulty of knowing the next steps of learning parallels with the uncertainty shown by the teachers during their teaching practice, in terms of curriculum knowledge. The progressions on the Number Framework (Ministry of Education, 2008a) were not understood well, which reflected in the teachers' knowledge of how to move students into the next phase of learning. Not knowing what questions to ask of the students, or what steps to take next, is consistent with research findings of Watson et al. (2008), who found that knowing what questions to ask of students, or what cognitive conflict to generate, was a difficulty for teachers. They found that teachers did not recognise an appropriate zone of proximal development in which to extend and challenge their students. This area of knowledge also falls within Ball et al.'s (2008) notion of horizon content knowledge, where teachers are able to identify how current understanding may be built on in future teaching. According to Ball et al., having knowledge of the mathematical horizon, allows connections to be made to later ideas.

The second part of each scenario asked the teachers to give their answer to the problem and draw a diagram and explain how they solved it. The difficulty the teachers had drawing diagrams on the questionnaires was also evident in their classroom practice, where diagrams were seldom observed being used by the students or themselves. When teachers use drawn representations, there is a

noticeable change in their PCK and conceptual understanding (Way et al., 2013) along with students' conceptual understanding (Flores, 2010; Gould 2005a; Way, et al., 2013). Similarly, when teachers use the concrete-representational-abstract (CRA) model, pictures and/or diagrams provide the intermediary step between the use of manipulatives and the use of numbers (the abstract phase) (Flores, 2010). The teachers reliance on procedural knowledge meant that drawing diagrams to explain thinking did not come naturally to them and therefore the benefits to the students of drawn representations was not articulated.

8.5.2 Teacher Practice in the Multiplicative Domain

This chapter discussed the observed professional knowledge of teachers during the teaching of multiplication and the contribution this made to student learning. It concluded that teaching multiplication for understanding is not as simple as it may appear and brings with it many complexities to be addressed within classroom lessons. Current reforms in education have placed an emphasis on teaching conceptually prior to procedural methods, which requires strength in subject matter knowledge and pedagogical practice (Ball, 1992; Howley et al., 2007; Hull et al., 2011). As these teachers acknowledged, this is a shift in practice from the procedural manner in which they were taught and, as previous research has shown, takes time to implement.

Detailed analysis of the observed multiplication lessons, was made against the modified framework of Chick et al. (Figure 3.1). Using this framework enabled the identification of commonalities and differences used by the teachers and their students, across many PCK categories. While the frequency of categories gave an indication of what occurred in each lesson, as data on their own they did not give the total representation of what occurred in practice. Observation of lessons backed up with repeated viewing of video-recordings and detailed analysis of transcripts, combined to show the importance of classroom observation as a means of gathering data about professional knowledge.

Underutilisation by the teachers of the various types of available assessment data meant the existing knowledge of the students was not well known. This is a fundamental aspect of knowledge of the learners and was especially important at School B, where they cross-group classes for mathematics and the teachers saw

the students for a short time each day. The teachers appeared not to use on-the-spot knowledge of the content, and progressions of learning in the curriculum and its associated expectations, to progress their students to Year level expectations. This limited use of such knowledge contributed towards links and connections within and between mathematical concepts seldom occurring, particularly when unplanned opportunities arose within a lesson.

The teachers were aware of the importance for students of conceptual understanding prior to procedural computation and at some point all utilised manipulatives, to support the learning and understanding process. While all of the teachers recognised some of the students' misunderstandings, there were times when these were either not recognised and therefore not corrected, or not unpacked sufficiently to ensure the students did not continue to have the misconception. Recognising the appropriate questioning type applicable to different situations is crucial for extending students' mathematical thinking, and this research showed that teachers were effective in the supporting of their students by breaking down problems and asking questions requiring simple explanations. The teachers were less inclined to extend the mathematical understanding of the students by challenging responses, encouraging discussion, and helping them form generalisations.

8.5.3 Student Learning in the Multiplicative Domain

Comparison of results between initial and final assessment tasks suggested that there was little gain in the understanding of multiplication by the students as a consequence of the teaching that had taken place. While it is acknowledged that assessment data can be misleading because it is a snapshot of students' learning at a particular point in time, the results of these assessment tasks is of concern.

Task 8 (interpreting an array of smiley faces), was the only question on the initial assessment where more than half of the students were correct and was omitted from the final assessment. On the initial assessment, less than half of the students solved any of the remaining nine tasks correctly, which is an indication that the majority of the students in these classes were categorised as below the Mathematics Standard expected of students in Years 5 to 8 (Ministry of Education, 2009). These data could have provided the teachers with suggestions for content in their teaching lessons.

Final assessment results suggest that overall, the students did not achieve as well as might have been expected relative to expected achievement levels, with all tasks showing less than half of the students correct (Table 4.2). Stage 6 on the Number Framework aligns to Level 3 in NZC (Years 5 & 6), while Stage 7 relates to Level 4 (Years 7 & 8), (Ministry of Education, 2009a, 2010). The tasks given in the assessment were based on the understanding expected at Stages 4 to 6 of the Number Framework, or Levels 2 and 3 in NZC, and the majority of students would be expected to be able to solve the problems correctly by the latter part of the year. The teachers had put a large amount of time and effort into preparation and delivery of their lessons and the number of correct responses was not indicative of the observed teaching and learning. The students did not always demonstrate understanding of key ideas expected for their class level according to national expectations.

When these results were discussed with the teachers, they were surprised at what was found. Beth was aware that her students struggled with basic-fact retention. However, Anna, Andy, and Bob, were unaware of the large number of students who did not have basic multiplication understanding and were unable to answer the tasks correctly.

8.5.4 Aligning Teaching Materials and Language

There appears to be many inconsistencies across the materials available for use by the teachers in their classrooms, in terms of Curriculum Level expectations and language used. The teachers in this research were reliant on the NDP planning sheets and associated books, for guidance in implementing NZC. Since the collection of data for this research, the Curriculum Elaborations were introduced, which align closely to the planning sheets and NDP books and have clarified many of the uncertainties of the broad achievement objectives of NZC. However, inconsistency in the language used in the NDP books caused some confusion in teaching practice, and was evidenced in the students' learning and associated assessment tasks.

In the following chapter (Chapter Nine) the results related to the proportional domain are further analysed and discussed in relation to the research literature.

CHAPTER NINE

DISCUSSION: THE PROPORTIONAL DOMAIN

9.1 Teachers' Subject Matter Knowledge and Pedagogy

This chapter discusses results associated with the preceding chapters: Chapter Four (the espoused professional knowledge of the four case-study teachers); Chapter Six (observations of the teachers' classroom practice); and Chapter Seven (evidence of student learning).

9.1.1 Teachers' Understanding of the Proportional Domain

This section of the chapter discusses part of research Question 2, concerning “*the teachers' espoused professional knowledge*”, with links made to the impact on “*teaching practice and student learning.*”

The teachers identified the student in Scenario 4 (Section 4.2.3) on the questionnaire (Figure 4.6: a decimal fraction addition problem), as having difficulty with place-value understanding. There is an expectation that place value is understood with whole numbers up to one million, and decimals to two places, when working at Stage 6 on the Number Framework, or by the end of Year 6 (Ministry of Education, 2007, 2008b). Understanding place-value requires an understanding of a part-whole concept. The basis of place-value numeration begins with understanding that “the quantities represented by the individual digits are determined by the position they hold in the whole numeral” (Ross, 1989, p. 47). The teachers recognised that providing students with this place-value knowledge allows for accurate computation of decimal fractions. Beth also suggested using a number line to show where the decimal numbers came in relation to other numbers. The use of the number line clearly shows the ordinality of the numbers (Bobis, 2007) and is encouraged as a tool for imagery and mental computation.

As Anna's students were Years 7 and 8, she placed a greater emphasis on decimals in her observed lesson and the importance of place-value understanding, when relating the fractional pieces to the whole unit. In responding to Scenario 4, Anna discussed the relationship between fractions, decimals, and place-value, which was identified by Steinle and Stacey (1998) as a difficulty for many

students. Anna carried this place-value relationship over into her observed lessons, by ensuring that when the students converted between fractions and decimals, they regarded the decimal as part of a whole through the use of deci-pipes. This meant the students were not confused between fraction and decimal conversions, like those of Steinle and Stacey's study, which found many students viewed 0.4 as $\frac{1}{4}$ and concluded 0.4 is bigger than 0.5, because $\frac{1}{4}$ is bigger than $\frac{1}{5}$. Anna was aware of this misconception, which can lead students to further place-value misunderstanding, such as when converting between fractions and decimals and confusing $\frac{83}{100}$ for 0.083.

All teachers solved addition of fractions with different but compatible denominators, correctly (Section 4.2.3: Figure 4.8, Scenario 5). They all recognised that the student (Pete) had added the numerators together and the denominators together, and that this misconception needed further exploration. However, Andy and Bob gave little information about what their next learning steps would be with the student who used this common computational error, which was also identified as a teacher difficulty in the research of Ward and Thomas (2007). Ward and Thomas, found that while the majority of teachers did not make the add across error (Smith, 2002) when adding fractions, less than 10% described clearly the key understanding that students need to acquire to add unlike fractions correctly. While Anna mentioned checking that the student understood that the denominators must be of equal-sized portions before adding them together, key ideas related to the teaching of equivalent fractions for conceptual understanding were not mentioned by any of the teachers. Other research has found that understanding equivalent fractions is cognitively demanding (Gould, 2005b; Way et al., 2015; Wong, 2010; Wong & Evans, 2007) and teachers who understand how students develop this knowledge, will help them see the links between various representations (Wong & Evans, 2007). Anna's knowledge carried over into her classroom practice as on the final assessment 77% of her students solved addition of compatible fractions correctly (Section 7.2.2: Task 2), while the combined total of students with correct answers in the other three classes was 14% (Table 7.4).

Beth mentioned manipulatives to show the meaning of the denominator, although she did not state the specifics of what manipulatives, or how they would be used. Beth's emphasis on the use of manipulatives to assist with conceptual understanding is consistent with research, which has found that mathematics manipulatives have the potential to lead to an awareness and development of concepts and ideas (Ball, 1992; Gould, 2005b; Swan & Marshall, 2010). However, it is how a teacher uses manipulatives that is important: if students merely mimic the teacher's directions or modelling, it may look as though they understand, but they could be just copying what they see (Ball 1992; Swan & Marshall, 2010). If the students do not understand what the mathematical learning is about, the manipulative does not assist in developing that concept. Beth's use of manipulatives to support students' understanding, possibly contributed to her students making substantial improvements on Task 1 (addition of unit fractions) from 10% correct in the initial assessment to 65% correct in the final assessment.

All teachers solved the final scenario, multiplication of fractions (Section 4.2.3: Figure 4.10, Scenario 6) correctly. However, understanding that when multiplying two fractions, the product is about finding a piece of a piece (in this instance of a cake), was not mentioned in their responses. Although teaching multiplication of fractions was not observed in classroom practice in this study, it is accepted practice in New Zealand schools, that when multiplying fractions the multiplication symbol is interpreted as "of" (Ministry of Education, 2008f, p. 63), and would be expected teacher knowledge. Anna, Andy, and Bob, did not identify how they could use the student's (Jo) current knowledge to extend her understanding of multiplication of fractions. Beth was the only teacher who recognised the complexity of the understanding behind multiplication of fractions and the confusion that is often made with division (Ma, 2010). Ma suggested that confusion arose because when you want to find a part of something you would expect to divide it into pieces, but when you want to find a portion of a unit with fractions, you use multiplication. For example, if you want to find $\frac{2}{3}$ of a 2 kilogram bag of apples you multiply $\frac{2}{3}$ by 2, ($\frac{2}{3} \times 2$), resulting in $1\frac{1}{3}$ kilograms.

9.2 Teacher Practice in the Proportional Domain

This section of Chapter 9 discusses the research question, “*What professional knowledge is evident when teaching mathematics for numeracy in the proportional domain?*” It incorporates part of Question 2, “*and how does it (the teachers professional knowledge) contribute to student learning?*”

This section presents a discussion of the findings of proportional teaching outlined in Chapter 6 and critiques these in relation to the relevant literature. The teaching practice analysed in this research was from a lesson mid-way through the unit on fractions. It was envisaged that during the unit on fractions, students would build on knowledge gained from the earlier unit on multiplication and division. Understanding fractions relies on certain multiplicative knowledge and connections need to constantly be made between the two domains (Ma, 2010; Pearn & Stephens, 2004). The fraction lesson was thus analysed with a focus on aspects of knowledge specific to the teaching of fractions, along with comparisons to the multiplication lesson, in order to ascertain similarities and differences in the categories of teacher knowledge used, between the multiplicative and proportional domains.

9.2.1 Clearly PCK (A)

The results presented within the Clearly PCK category exposed many similarities related to teacher knowledge in classroom practice, and the effect this had on students’ learning, within and between the multiplication (Table 5.3) and fraction (Table 6.3) lessons.

Purpose of Content Knowledge (A.1) and Curriculum Knowledge (A.2)

The teachers demonstrated similar gaps in understanding and knowledge of the curriculum, and the associated expectations of achievement by the students within the proportional domain, as that shown earlier in the multiplicative domain. The learning intention (WALT) of three of the observed fraction lessons (Table 6.1) were at Number Framework Stages pitched below the expected curriculum level for the year level of the classes and there was little emphasis on providing more difficult problems in an effort to raise the students’ levels of achievement. This was evident in Beth’s lesson, which was identified in the NDP material (Ministry of Education, 2008g, p. 16) as suitable for moving students from AC (Stage 4) to

EA (Stage 5) on the Number Framework (suitable for Year 2-3 students) (Ministry of Education, 2007, 2008a, 2010). While it is acknowledged that Beth's class had been identified through school-wide assessment as below year level expectations, as Years 5 and 6 students they could have been given opportunities to solve problems considerably more complex than those at the AC to EA stages. Researchers have consistently found that one of the most important factors in school success is the opportunity to learn (Boaler, 2008; Hattie, 2003; Loughran, 2010), and have identified that when students are in lower-ability groups, their progress is limited as their teachers often have lower expectations of them (Allsop et al., 2007; Boaler, 2008; Steffe & Olive, 2010). Indeed, few opportunities were observed that afforded the students in all classes to work on challenging and higher-level problems, in order to achieve expectations of their class levels (Ministry of Education, 2009, 2010). More opportunities within lessons for extending students' mathematical thinking by encouraging the activities of analysing, comparing, and generalizing, recognised as key components of advancing student thinking and achievement in mathematics (Fraivillig et al., 1999; Gould, 2005a), might have resulted in greater student learning.

The NDP Book 7 (Ministry of Education, 2008e) guided the teachers on the sequence of the fractions lessons. Each activity in the NDP book is based on a specific learning outcome (WALT) and the teachers (other than Anna) relied on the WALT in the books, to direct the learning (Section 6.2.1: A.1 and A.2). What the teachers seemed to lose sight of was that while the NDP material provided suggestions to supplement the curriculum, lesson planning was to meet the aims and objectives at the expected level of the New Zealand Curriculum (Ministry of Education, 2007). However, the teachers admitted in conversation that they found the wording in the curriculum document vague, making interpretation of the achievement objectives confusing and that the Number Framework provided them with a more detailed learning trajectory. For example, the Level 3 Curriculum objective, expects students to, "use a range of additive and simple multiplicative strategies with whole numbers, fractions, decimals and percentages" (Ministry of Education, 2007, Level 3 chart).

Teaching Strategies (A.3)

While all of the teachers used manipulatives as a teaching strategy during the teaching of fractions, the change in how Andy used them (from the multiplication lessons, and as the fraction lesson progressed) was noticeable. In the initial multiplication lesson, Andy introduced manipulatives for the first time and the novelty of the new equipment (Unifix cubes and animal arrays) meant that the students were observed playing with it much of the time. As his fraction lesson progressed, Andy's conscious use of manipulatives, alongside his word problems based on scenarios applicable to the students' lives, was a change in teaching strategy. Although there were times when he caused some confusions, as he became more comfortable with the use of manipulatives and the associated talk it generated among the students, he recognised the difference it was making to the students' conceptual understanding (Nührenbörger & Steinbring, 2008; Ross, 1989; Schoenfeld, 2011; Wright, 2014). Andy recognised that while gains can be made using mathematics manipulatives, "simply placing one's hands on the manipulative materials will not magically impart mathematical understanding" (Swan & Marshall, 2010, p. 19). Therefore, he encouraged more discussion among the students because, as Swan and Marshall identified, appropriate discussion makes links between the manipulatives and the mathematics concepts being taught, more explicit.

In the fraction lesson, Andy encouraged the students to explain their ideas to each other when solving problems. When the students used fraction tiles to represent the whole and parts of objects, initially the students placed one whole tile on top of the fractional pieces (representing the portions the food was cut into) to show they were equal in size (Figure 6.9). This strategy is similar to that used by Way et al. (2015), when they found that overlaying fractions on printed transparencies allowed students to compare equivalent fractions. However, Andy acknowledged that while the pieces appeared to be the same size he wanted the students to explain the equivalence mathematically. While some of the students in Andy's class took some time to adapt to the use of manipulatives followed by explanation of the results, he persevered and adapted his teaching strategies accordingly. This observed difference in approach is consistent with research, which found that a deliberate hands-on exploration of fraction (and other mathematics) concepts,

results in the dual effect of increasing teacher pedagogical knowledge, while supporting conceptual understanding in their students (Way, et al., 2015). Andy's allowing the students to explore different solution methods and discuss their findings is consistent with Yackel's (2001) and Wong's (2010) research, which found that students' explaining what they have modelled is important, if the manipulatives are not to become a tool used in a procedural manner, similar to verbal rules.

Allowing students to manipulate equipment themselves was one of the key differences also observed from Bob's multiplication lessons to his fraction lesson. While the fraction lesson tended to be teacher directed, Bob allowed the students to utilise their own manipulatives as part of their learning (Section 6.2.1: A.3). The individual's use of materials is connected to the importance of conceptual understanding during instructional time, that has been emphasised in research (Ma, 2010; Schoenfeld, 2013; Swan & Marshall, 2010; Way, et al., 2013; Wong & Evans, 2007) and aligns to relational understanding (Skemp, 1976). The use of manipulatives helped emphasise the importance of relating mathematics to everyday life, in order to relate word problems to actions (Perso, 2006).

Encouraging the students to work in pairs, to discuss ideas together, and justify their solutions, was a strategy employed by all of the teachers who regularly asked the students to talk with their buddy about their mathematics problems (Section 6.2.1: A.3). There was a noticeable change from the multiplication lessons to the fractions lessons, in the manner in which the students talked together when solving problems. For example, in Beth's class once the students had worked in pairs to share three wafer biscuits (strips of paper) between two people, they explained their solution method to others in the larger group. They then discussed and compared the different solutions and recorded the symbols representing the various strategies in the modelling book. This was a change from Beth's multiplication lessons, where she asked the students to discuss ideas but they struggled to do so. Earlier explanations were confined to a description of the steps that had been taken and while the students were still reluctant to challenge each other's thinking, the fact that they now had the confidence to share ideas meant that Beth would be in a position to build on this in the future. This aligns to the theories of learning associated with socio-culturalism where collaboration and

conversation is crucial to changing external communication to internal thought (Cobb & Bauersfeld, 1995; Rubie-Davies, 2010; Vygotsky & Luria, 1993). When a learning environment is constructed in which students are required to talk and act like mathematicians this becomes as much of a priority as the lesson learning outcome (Askew, 2007; Hunter, 2012).

The change in mathematical discussion among the students in all classes as the year progressed (from the multiplication unit to the fractions unit), may have resulted in greater understanding of the concepts taught, as evident in many of the final fractions assessment tasks. Greater understanding as a result of questioning, discussion, and justification is also evident in other research (Hunter, 2006, 2012; Kazemi & Stipek, 2001; Ma, 2010; Schoenfeld, 2011). Furthermore, the teachers in this research recognised that establishing classroom cultures that promote argumentative elements of discussion and justification is a challenging task and takes time to bring about (Hunter, 2005; 2006; 2010). The teachers persevered with this process and were seen to scaffold discussions by suggesting the students “talk with each other” about their strategies before telling the teacher.

Alongside the challenge of effective group discussions, one of the difficulties some of the students in Andy’s class encountered was a strategy he employed of questioning the students about their answers (whether they were correct or incorrect). This provocation caused uncertainty among Andy’s students who appeared confused and unsure about whether they should change their answer, or their way of thinking. At one point, he provided a solution to a problem and followed it with the statement, “*I may be wrong. It wouldn’t be the first time in my life that was the case.*” The students believed that Andy’s questions and statements were designed to elicit certain responses, and therefore instead of giving an answer they thought was correct, tried to give an answer they thought that he wanted to hear. This led the students to what has been referred to as “guess what is in the teacher’s head” (Loughran, 2010, p. 61). Like the students in Loughran’s research, Andy’s students often presumed his questions were designed to elicit a particular response and when they were challenged about a response that was correct, immediately doubted their answers.

Another focal point used by the teachers was the modelling book. The modelling book was used in a range of ways, to complement the learning taking place. For

example, in Beth's class, after the students modelled two different ways three wafer biscuits (strips of paper) might be shared between two people, the pieces of paper were glued into the modelling book. Pasting the representations into the modelling book provided a permanent record of the problem solved and was referred to during discussions about whether students with three-halves, had the same amount of wafer as those with one whole and one-half. This use is similar to that identified by Higgins (2006), who found that a modelling book enhances the hands-on approach to teaching mathematics through linking conversations and modelling of concepts. As concluded by Higgins, the visual support provided by the modelling book in Beth's class suggests that students may have known what they were learning about, and provided connections to previous knowledge when new learning took place. The students and the teacher, as suggested by Ell et al. (2010), made recordings in the modelling book and when students' names are recorded alongside their work, it may also be utilised for assessment purposes (Higgins, 2006).

Real-life scenarios, relevant to the students, were also used as an approach to learning. An example of this was Bob who used circles to represent birthday cakes and scenarios with the names of students in the class. When he increased the size of the numbers, he used more candles and one time modelled a birthday cake for himself. The difficulty was that the cakes and parties theme was a novelty to the students and at times, this distracted from the mathematics problems being solved. This was consistent with the research of Treffers and Buys (2001), who advocated for the importance of real-life scenarios, but cautioned that at times the context might get in the way as students often interpreted scenarios literally. Real-life scenarios can be particularly difficult in both division and fractions, when things do not necessarily divide evenly (Treffers & Buys, 2001).

Rather than allowing the students to become reliant on recognising the representation of fractions on one particular shape, Beth used a variety of shapes to consolidate understanding of the fractional number. For example, she used pies as a context with paper circles as the representation, and then used wafer biscuits represented by strips of paper, thus exploring one-half as both a semi-circle and rectangular shape. It is important to change shapes to ensure the students'

understand the meaning of a fraction (Gould, 2005b; Ma, 2010), and are not reliant on the visual representation.

Cognitive Demands of Task (A.4)

Andy, Bob, and Beth recognised that the students had difficulties in establishing the whole and related fractional pieces (Section 6.2.1: A.4). The students struggled to understand that one whole is not always represented by a single object (e.g., a pie, a cake, a biscuit, or a pizza), but may also be represented by a collection of objects such as a bag of apples, packet of biscuits, or several pizzas. Added to this was the complexity that sometimes the fractional piece found may be less than one whole object, for example one-quarter of two pizzas, while sometimes the fractional piece might be greater than one whole object, for example when four children each received one-quarter of six biscuits. Misconceptions occurred when, as Ma (2010) found in her research, the students failed to understand the concept of whole and did not recognise the complete set of objects as the whole unit. Fractions have many meanings and the meanings are all about relationships between numbers (Ma, 2010).

While the students were struggling to see the relationship between the whole and the piece they were finding in their problem solving, Andy, Bob, and Beth identified that the students did not fully understand the notion of equivalence. Apart from placing the fraction tiles on top of each other, Andy's students' were unsure why two-sixths and one-third were equivalent, while the difficulty Bob's students had with equivalence was compounded because many were inaccurate in their representations (Figure 6.6) and the uneven portions created an added complexity. Challenges the students had associated with understanding the relationship between the part and the whole in the first instance, and later with equivalence, is consistent with Suggate et al.'s (2010) findings, that understanding fractions as part of a whole must be strongly in place before equivalence can be fully understood. As Suggate et al. reasoned, the students needed to understand the resulting fractional portion after finding a piece of a set (e.g., 3 cakes shared among 4 people gives $\frac{3}{4}$ each), before they understood that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$.

Operations with fractions can be confusing for students and difficulties associated with identifying the names of the fractional representations came to the fore

during the observed lessons. The emphasis in both Andy's and Beth's lesson was on sharing: for example, the students shared three cakes among eight people, and four pies among three people. After cutting paper circles to share out the pieces, one student in Andy's class used whole number thinking and reported, "*You get three bits of cake.*" Another student cut two circles (cakes) into quarters and one circle into eighths, and said, "*You get one and a half pieces.*" Without knowing the fractional name of each piece and the relationship with the original whole cakes (paper circles), the students became confused. One student, looked at the cut pieces of paper and said, "*Oh, I don't know.*" Operations with fractions can be problematic for students, because the rules associated with whole numbers do not work with fractional numbers (Bailey et al., 2014; Gould, 2005a; Ma, 2010, Roche, 2005; Smith, 2002), and giving the result of the sharing problem, as a proportional representation of the whole caused difficulties.

Appropriate and Detailed Representation of Concepts (A.5) and Knowledge of Resources (A.6)

Detailed representation of concepts within the fraction lessons, relied on the use of concrete manipulatives including paper circles, counters, paper strips, fraction tiles, and deci-pipes. However, while the manipulatives supported conceptual understanding, misrepresentation sometimes occurred with paper circles, when the whole was divided into an 'odd' number of pieces (for example thirds or fifths), as these were difficult to model accurately. The difficulty in this instance associated with equal portions represented on the manipulatives, and resulted in incorrect answers to problems. This notion was also identified in past research, which found that teachers' understanding of the use of varying manipulatives needs to be strengthened (Perry & Howard, 1997; Swan & Marshall, 2010; Way et al., 2013), to ensure correct representations. Ma (2010) similarly acknowledged that many teachers struggle to come up with representations of fractions that are appropriate, and as computation becomes more complex, both the teachers and the misuse of representations may lead to misconceptions about the meaning of fractions.

Anna emphasised the importance of place-value understanding with decimals and unpacked this while the students modelled the representations on their commercially produced deci-pipes (Figure 6.8). Anna's attention to the notion

that one-tenth is one piece of a whole that has been divided into ten equal parts, is a necessary concept to understand in relation to the meaning of decimals, and is supported in the research of Roche (2010). Andy often relied on commercially produced fraction equipment (e.g., fraction tiles), which provided pieces of a consistent size, although they deprived the students of the opportunity to learn from the construction of their own material. An example of consistency with commercially produced equipment was when the students showed two-sixths as equivalent to one-third, relying on the visual alignment of the materials, and colour of the different fractional pieces. Andy pushed the students for explanations based on the understanding of equivalence. A critical skill in the development of understanding of fractions is appreciating that while fractions can have many names, there are times when one particular version of the alternate name might be more useful in a given situation (Small, 2013).

Beth's lessons indicated that she was an advocate for the use of manipulatives to reinforce conceptual understanding. The need for conceptual understanding has been emphasised in the teaching of mathematics and has been strongly advocated by many researchers (Anghileri, 2006; Ma, 2010; Skemp, 2006; Swan & Marshall, 2010; Wong & Evans, 2007; Young-Loveridge & Mills, 2009a), as conceptual understanding generally allows a person to reconstruct a procedure that they may forget through procedural learning alone (Anghileri, 2006). When teaching fractions, Beth used paper circles and rectangular strips of paper primarily as an area measure, although she also compared the length of the strips of paper, showing an awareness of Kieren's (1980) fractions as a measure construct. However, as Way et al. (2013) asserted, moving from recognising part of a rectangular strip as an area measure, to focussing on its length, was difficult for the students as they came to terms with the two different interpretations.

Student Thinking (A.7)

One notable difference from the initial multiplication lesson (Table 5.3) to the fraction lessons (Table 6.3) was the change in frequency of using student thinking (from 28 instances to 13). A possible interpretation is that the students were less secure in their understanding of fractions (than multiplication), and therefore not so confident to ask questions. On the other hand, it might be that on occasions, the teachers were insecure in their ability to teach fractions and did not give the

students the opportunity to ask questions, for fear of being unable to answer them accurately. One opt out for utilising student thinking, was the emphasis placed on the learning intention, which allowed the teachers to refocus the students on what they had planned for the lesson, and at the same time stay within their personal comfort zone. However, possibly the main reason for the lesser frequency was the change in the lesson delivery. In the fraction lessons, all teachers allowed all students to use manipulatives as part of their problem solving and to build conceptual understanding. The teachers were less dominant in the fractions lesson and while the students were thinking mathematically (possibly more so than in the multiplication lessons which were more teachers directed) this was through their participation in the hands-on activities with limited questioning by the teachers.

There were times when the teachers might have taken opportunities to delve more beneath the students' initial thinking. For example, when Anna discussed the total number of tenths in the number 3.6 one student said, "*You just take away the decimal point.*" Anna responded, "*The decimal point disappeared, because the numbers went up in place value.*" She then asked, "*What's our rule in mathematics when we multiply something by a hundred?*" A student replied, "*Go up two place values.*" However, developing procedural knowledge such as this, at the expense of conceptual understanding has often been cited as part of the reason for poor mathematics proficiency (Ball, 1992; Burns, 1998; Kazemi & Stipek, 2001; Scharton, 2004; Skemp, 1976).

Bob also overlooked responses and ideas contributed by the students. For example, when the students were finding one-quarter of sixteen, a student said, "*I know two times eight is sixteen.*" Another student responded, "*But then only two people get cake. Oh maybe you could cut the half in half.*" Instead of discussing further the size of the resulting fractional pieces if the cake was cut in half, and in half again, Bob ignored the comment and moved on to another problem. Bob's brushing-over of ideas, can be common among teachers and many teachers conduct their lessons without ever explicitly focussing on student's mathematical thinking (Fraivillig et al., 1999) for extended periods of time.

Student Thinking – Misconceptions (A.8)

The regularity with which the teachers addressed their students' misconceptions (17), was greater than the frequency they utilised their students' thinking (13)

(Table 6.3). At one stage when one pair of students began to share the five apples (fraction tiles) among four people, they were unsure whether to have four or five whole tiles (apples), and whether to divide each whole into four or five pieces (Section 6.2.1: A.8). Some of the confusion the students had may have been attributed to earlier in the lesson, when Andy used materials to consolidate fractional representations, and did not emphasise the proportional relationship between the fraction and the whole to which it was compared. For example, the students knew that one whole could be divided into two pieces (halves), or four pieces (quarters). However, they did not fully grasp that when four equal-sized pieces (quarters) are combined, they are equal in size or quantity to one whole. This is consistent with research which has found teachers need to emphasise that fractions are about the relationship between two numbers (the numerator and denominator), as well as about the relationship between the fractions itself and the whole to which it is compared (Ma, 2010; Pearn & Stephens, 2004; Siegler et al., 2011).

Some of Beth's students had a similar misconception, when they shared three wafer biscuits among four people and immediately cut their wafers into thirds (getting confused between the number of wafers and the number of people). Unlike Andy, Beth observed the students momentarily to see if they recognised their mistake, but when she realised they were having difficulty, she brought the group back together and discussed their dilemma. When Beth acknowledged the misconception her students were having, she invited a pedagogical response described as *required guidance* (Loughran, 2010) and steered the students towards arriving at the correct solution. Research has found that understanding common misunderstandings that students have, and knowing how to address these in order to remedy them, is an important component of teacher knowledge (Ball et al., 2001). When to step in and out of significant moments in students' discussions is important for teachers to recognise, in order to make a difference to students' learning (Walshaw & Anthony, 2007).

Early in her lesson, Anna recognised one of the students had difficulty saying decimal numbers correctly (for example hundredths and thousandths) and emphasised that it is important not to read decimals like whole numbers. Anna was aware of the necessity for using correct language with decimals, which was

also emphasised by Ma (2010), who found that the issue of correctness and appropriateness in decimal numbers when read and spoken, must be clearly understood and articulated. When working with decimals, it is important that they are not treated like whole numbers as this can lead to misunderstandings later. Teachers need to model (almost by over emphasising the phonic blends) the difference between whole numbers and fractional numbers such as between *tens* and *tenths*, and *hundreds* and *hundredths* (Ma, 2010).

9.2.2 Content Knowledge in a Pedagogical Context (B)

The teachers in this study generally carried out their mathematical computation procedurally, as this was how they were taught. For Andy and Bob, this sometimes became an issue in their teaching practice and was reflected in the difficulty they had when explaining content with conceptual understanding to their students. Similar difficulties were evident in the research of Ward and Thomas (2009), who explored the relationship between teacher knowledge and student achievement in the New Zealand context. They found that teachers who were unfamiliar with specific content knowledge were unlikely to know effective ways to teach that content, while teachers who were familiar with content, may or may not know how to teach it effectively.

Deconstructing content to key components (B.1)

The category Deconstructing Content to Key Components, occurred the most frequently (29) within the content knowledge section of the framework (Table 6.4). Alongside the students' misconceptions concerning fractional understanding, the teachers found a need to deconstruct concepts regularly. With the three of the observed lessons pitched at transitioning from Levels 1 to 3 in the curriculum (Andy and Beth, AC to EA; Bob, EA to AA), the concepts being developed built on what Hansen (2005) referred to as, *basic fundamental concepts*. It was crucial that the students grasped these concepts well, so that teaching could move on to higher levels, allowing the students to achieve at the appropriate level of expectation.

Deconstructing key components, provided opportunity for the teachers to hand content-related thinking over to the students. Andy often guided the students, by challenging their thoughts and making further suggestions. For example, he encouraged the students to look at various ways they could approach the dividing

of the cakes (paper circles) into equal portions, which led to a discussion about equivalent fractions. In the fraction lesson, the willingness of Andy to wait for the students to arrive at their own conclusions was a key to enabling students to find solutions themselves (Black et al., 2004), and required Andy to increase wait time, resisting the urge to jump in and either reword or answer the original question. As outlined in Chapter 2, teachers often wait less than one second, before either asking another question, or answering the question themselves (Black & Wiliam, 1998). Allowing the students time to unpack solutions to problems, is a vital stage in mathematical thinking process (Schoenfeld, 2013). Schoenfeld, established that allowing students time to solve problems themselves, broadens participation while allowing the teacher to learn more about their knowledge and misconceptions, so that future lessons can better address their needs.

Beth explored improper fractions, emphasising that a fraction can be written with a numerator larger than the denominator. Beth explained to the students, “*Improper fractions, are actually proper,*” and discussed the notion, “if you add three quarters to another three quarters, you end up with six quarters.” She left converting these to a mixed-numeral until the students accepted that mathematically it was correct to have an improper fraction. Beth’s earlier emphasis on understanding the fractional name prior to the addition of fractions, aligns with other studies, which have found that it is important students understand the fractional name, prior to using equivalent fractions for finding common denominators in order to solve problems (Hansen, 2005; Kieren, 1980; Wong, 2010). Teaching the students in later lessons that an improper fraction can be renamed into a mixed numeral, is consistent with Ma’s (2010) research, which found that students should be taught to turn the improper fraction into a mixed numeral as the final step in the process of understanding the fractional name.

Many of the fractional problems the teachers gave the students related to the discrete model of division, and involved the sharing out of items into equal portions. This provided an opportune time to deconstruct the fundamental relationship between division and fractions within proportional reasoning, but was not emphasised by the teachers. The students used multiplication and division strategies as they solved problem examples, such as 24 candles going evenly onto six pieces of cake. Frequently, emphasis on the sharing of items meant that the

lesson became purely a multiplication and division one, as no connection was made between the piece and the whole, demonstrating the proportional relationship between the items. Instead, each portion was seen as a piece of cake. Understanding that dividing is finding a part of something, or a fraction of a whole, is a fundamental idea of proportional reasoning (Shulman, 2010). Shulman acknowledged that often teaching this fundamental knowledge can be misconstrued as remedial, instead of recognising the rigour attached to it.

Mathematical Structure and Connections (B.2) and Methods of Solution (B.3)

Mathematical structure and connections were less evident in the teachers' fractional content knowledge (Table 6.3: B.2, 10 times), as the teachers made minimal connections between, and within concepts. This finding is similar to Ma's (2010) research, which found that teachers require specific subject matter knowledge that allows them to make explicit connections between and among mathematical topics. Students' recognition of division and the procedure of solving division problems are not the same thing. Mathematical ideas are intimately interconnected and interrelated, and learning and understanding a small number of ideas and the ways in which they are related, is more powerful than learning a large number of facts and procedures, in a disconnected way (Ma, 2010).

When Anna's students began problem solving with decimals, the key idea that decimal numbers are a unit being partitioned (or decomposed) into smaller and smaller pieces, possibly infinitesimally small pieces, was not made. Anna reinforced the idea that decimals are a special construct of fractions and revisited place-value understanding with whole numbers and the connection to decimal place-value understanding. While Anna made some comparisons between common fractions and decimal fractions in this lesson, this was minimal and they were generally seen as separate representations. This was contrary to findings in Ma's (2010) research, which emphasised the importance of developing and understanding some of the underlying concepts that form the inter-relationship between both systems.

Procedural Knowledge (B.4)

There were times when all of the teachers relied on rules and procedures to demonstrate solutions and in return, the students sometimes returned to a

procedural approach to explain solutions to problems. Although the teachers encouraged the use of manipulatives to assist in conceptual understanding, it is questionable as to whether the students always understood the concepts they were modelling. There were times when the teachers modelled the equipment procedurally and the students later replicated their teacher's modelling. For example, during the lesson Beth modelled the cutting of a piece of paper in half, and then in half again. At the same time, she told the students "*If you cut a half in half you get quarters, so if you cut a third in half what are you going to get?*" Beth emphasised doubling the number, which may have over-ridden the important understanding that each time you cut something in half it results in twice as many pieces that are half the size. The students replicated Beth's use of materials procedurally which may have meant that verbal rules and procedures were replaced with procedures and rules for using tools. As outlined in Chapter 2, mathematical procedures should not be learnt without conceptual understanding (Ball, 2002; Kazemi & Stipek, 2001; Ma, 2010; Perso, 2006; Yackel, 2001). If conceptual understanding is gained, it gives students tools to reconstruct a procedure they may otherwise forget (Ma, 2010), rather than using the language of fractions, without fully understanding their nature (Nunes & Bryant, 1996).

Profound Understanding of Fundamental Mathematics (B.5)

As stated in Chapter 2, the ability of teachers to address student thinking may be reliant on them having profound understanding of fundamental mathematics (PUFM) (Ma, 2010) In this study, the teachers seldom made connections between fractional concepts and what had been taught previously in multiplication and division. For example, candles (counters) were spread out evenly on top of cakes (paper circles) with no relationship made between fractional quantities and quotitive division, and understanding the size of each piece relative to the whole. This may have indicated their understanding was not always profound. Mathematics needs to be seen as a unified body of knowledge (Ma, 2010), and whilst the teachers endeavoured to lay the foundation for what the students would learn later, they did not display the skill of providing the students with underlying connections among different operations and subdomains. Whilst Ma used the term *profound* as "an intellectual depth that consists of a deep, vast and thorough connectedness" (Ma, 2010, p. 120), Loughran (2010) preferred to use the term

linking. Linking makes connections across ideas so that prior knowledge and new knowledge can interact in ways that will further develop understanding of the topic being studied. The teachers in this study did not always make mathematical connections or links across key ideas in the multiplicative and proportional domains.

At one time, when Andy's students were sharing three whole cakes among eight people, one child said, "each person will get $1\frac{1}{2}$ ", while another child said, " $\frac{1}{24}$." Andy did not acknowledge either of these answers and refocused the students on the manipulatives in an effort to decide the proportion of cake each person received. The student who gave the response of $\frac{1}{24}$ recognised the three groups of $\frac{1}{8}$ on his manipulatives but added the denominators to get an answer of $\frac{1}{24}$. A common misconception among Andy's students on Task 1 ($\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$) of the final assessment (Section 7.2.1: Task 1), was the add across error: 10 students gave an answer of $\frac{3}{15}$, while two students added the denominators leaving 1 on the numerator, an answer of $\frac{1}{15}$. If during his lesson Andy had stopped and asked the child how he arrived at the answer $\frac{1}{24}$ and reiterated addition of the numerators only, it may have helped correct this common misconception, prior to the final assessment. Shulman (1987) asserted that it was important that teachers take advantage of the opportunities to consolidate students' new learning and to use that knowledge immediately to consolidate understanding. Expert teachers often spontaneously react to what is happening in the classroom and respond to the needs of the students by predicting what types of error might be made (Loughran, 2010).

9.2.3 Pedagogical Knowledge in a Content Context (C)

During observation of the teachers' practice, there were moments when their content knowledge was at the forefront and times when the manner in which they imparted subject matter became more important. This section acknowledges that content and pedagogy are not mutually exclusive and the focus is sometimes primarily on the pedagogical strategies required for teaching fractions. While the teachers' PCK was the focus, how the students reacted to the teaching and instruction became a large part of determining this context.

Classroom Techniques (C.1)

One teaching technique the teachers continued to implement (from their multiplication lessons to their fraction lesson), was keeping the learning intention (WALT) as a focus. Determining goals for learning and sharing these with the students, is an important part of the planning and teaching process (Hiebert et al, 2007; Smith & Stein, 2011). The teachers placed importance on the students being aware of the WALT and ensured they understood what was required to meet the specified learning outcome.

Another teaching technique applied by the teachers, was encouraging students to solve problems together and to discuss ideas among themselves. This important pedagogical shift of allowing students to discuss ideas together was promoted through the NDP, which emphasised the importance of manipulating materials and the ensuing explanation by students of their actions, in terms of the underlying mathematical ideas being investigated (Higgins, 2006). However, what occurred with regularity in this study was that when the students shared their solution methods they resorted to telling procedurally how they got their answers. Many researchers have found that engagement in productive discussions, ultimately enables learners to re-look at problems and build strong arguments (Boaler, 2008; Kazemi & Stipek, 2001; Vosniadou, 2001; Walshaw & Anthony, 2007; Way, 2008; Whitenack & Yackel, 2002). This was also evidenced in Fraivillig et al.'s (1999) findings that ascertained what students knew, and how they thought about mathematical concepts, were critical components for advancing children's thinking.

Andy acknowledged the value of making mistakes. He often discussed a solution and then said, "*I could be wrong. It wouldn't be the first time in my life I was wrong.*" This put doubt in the minds of his students who were left questioning, "Were they right, or were they wrong? Or, was Andy (their teacher) right, or was he wrong?" This may have been a technique to encourage the students to be certain of their strategies and justify their solutions. While Andy's approach appeared to frustrate the students, it may have been an attempt to develop a sense of the value of argumentation in the mathematics lesson, that could provide the foundation for clarification of ideas and understanding (Fraivillig et al., 1999; Goos, 2004; Hunter, 2006, 2009, 2012; Whitenack & Yackel, 2002). Andy also

displayed anticipation of the students' responses, one of five effective practices described by Smith and Stein (2011) as being essential for orchestrating productive discourse during the mathematics discussions. Anticipating students' questions in advance assists teachers in developing questions that are bound to the context, rather than just telling students answers in-the-moment (Cobb & Bauersfeld, 1995; Smith & Stein, 2011).

Another example of a classroom technique is the use of manipulatives combined with every-day scenarios (Section 6.2.1: A.3). As discussed in Section 9.2.1: A.3, this would have ensured that the mathematics was meaningful and relevant to the students.

Getting and maintaining student focus (C.2)

Getting and maintaining student focus was frequently intertwined with the classroom techniques used by the teachers. For example, using the real-life contexts for problem solving (Section 9.2.1: A3, & Section 9.2.3: C1) also helped to gain the attention of the students. The teachers often used their students' names when creating word problems, such as Bob's word problem that started, "*We've got four people at [Child's name] party. And if you look at this (pointed to a paper circle), this represents a cake. Your birthday cake [Child]*" (Section 6.2.1: A.3 & Section 6.2.3: C.2). The student whose name was used, along with his or her peers, immediately became involved and focussed on the problem, which often generated further discussion. While this could be temporarily distracting, it certainly gained the attention of the students. For example, following on from the scenario above Bob said, "*Now you are coming to Mr [used his own name] birthday party.*" One of the students laughed and interrupted, "*So are you turning 21?*" (Section 6.2.1: A.3). Bob thought that was a good idea and gave the students a problem sharing out the 21 candles on his cake. The students participated in the fun of the occasion and became focussed on their learning as they solved problems determining the ages of different teachers. These examples suggest that scenarios with direct meaning to the students' lives maintained their focus and may have supported their mathematical learning (Smith, 2002).

Fractional representation was often presented in relation to the context of food, which was another way of ensuring the focus of the students. Contexts included cakes, biscuits, pizzas, apples, and similar foods that the students were familiar

with, and could be cut up and shared out evenly. Understanding that fractions relate to equal-sized portions is paramount (Ma, 2010), and the students definitely understood the unfairness if someone received a larger portion of cake than others.

Goals for learning (C.3) and Knowledge of assessment (C.4)

Observation of the fraction lessons along with the assessment tasks given, indicated the classes were achieving below expectation. The teachers appeared to be uncertain of what was expected at their respective curriculum level. Therefore, there was a reliance on the WALT being copied (or in Anna's case adapted) from the NDP book on fractions and decimals (Ministry of Education, 2008g), which aligned to the Number Framework Stage (Ministry of Education, 2008a), and the NZC Level (Ministry of Education, 2007).

The teachers at both schools gathered assessment information, but did not appear to use analysis of the data to inform their teaching practice. None of the teachers used the results of the initial assessment tasks to check the students' fractional understanding and the reliance on NDP books suggested that little attention was given to the results of other available assessment material (e.g., NUMPA and PAT results from earlier in the year). The teachers could have responded to the difficulties identified in these assessments and utilised them to improve the students' learning as suggested in NZC (Ministry of Education, 2007, p. 39).

During interviews, the teachers admitted that when preparing their lessons they had not taken into account the errors on specific questions the students in their mathematics class had made in these assessments, and had not used these to inform their planning.

Anna used a short activity at the start of her fraction lesson as an assessment task. The initial body fractions activity (Mills, 2011) was limited to quarters, halves, and wholes, because representations were made using each student's arm span as a unit of measure (Figure 6.7). This visual representation allowed Anna to see immediately whether the students knew equivalence between fractions and decimals, such as $\frac{1}{4}$ being equivalent to 0.25 and $\frac{3}{4}$ equivalent to 0.75. The students then used deci-pipe equipment to further explore decimal place-value understanding of tenths and hundredths (Figure 6.8), and was an example of how utilising manipulatives can stimulate students' thinking during explanations

(Swan & Marshall, 2008). It is important that students explain their findings to others, as this process encourages deeper mathematical thinking and consolidates understanding (Boaler, 2008).

Questioning – Supporting (C.5)

Supporting students by asking questions, was a technique used on many occasions (74 in total) by the teachers, within the fraction lessons. An example of this was when Anna said, “*Is everyone in similar thinking as Phillip?*” or when Beth said, “*So how many halves have we got?* (Section 6.2.3: C. 5). Supporting questions were used to the greatest extent because the students often used strategies that had been suggested by their teacher. For example, Andy directed the students’ thinking when he suggested they cut ‘the cakes’ into quarters, rather than eighths. Beth and Bob regularly participated in back and forth conversations, such as when Beth’s students shared wafer biscuits among people. While supporting questions may be thought of as simplistic, the conversation generated assisted the teachers in determining what the students knew. They typified Fraivillig et al.’s (1999) research findings, which found how students think about mathematical concepts is a critical component for advancing children’s thinking. Either the teacher, or another student, may carry out supporting a student’s thinking by restating, or explaining, what the student has said.

Questioning – eliciting (C.6)

Eliciting type questions were the second most frequently used (20), which is consistent with the findings of Fraivillig et al. (1999). These questions elicited several ideas and/or strategies for solving problems. For example, during Beth’s lesson, as the students shared three biscuits (represented by strips of paper) among four people, she discussed one group’s solution then said, “*Let’s see, who else did it another way?*” (Section 6.2.3: C.6). This allowed other groups to contribute their solution methods, which the research of Fraivillig et al. suggested, is an important component of classroom discussions and a key to mathematical success, as it provides students with opportunities to promote their mathematical thinking.

While the teachers in this study elicited a range of strategies, they did not necessarily utilise them as a teaching point. For example, one group in Beth’s class constructed one-quarter of one-third to determine the answer to a given problem. They shared the four pieces they created by cutting the one-third into

four pieces among the three people, but did not recognise or name the piece as one-twelfth of the original whole. This would have introduced multiplication of fractions and the concept that multiplying a fraction by a fraction, is about finding a piece of a piece. Discussing the strategies and questioning the students' thinking about the fractional representation of each could have led to greater understanding of equivalent fractions.

Questioning – extending (C.7)

Extending responses as a result of questioning, was observed the least number of times (6) by the teachers. One example, was when a group of Andy's students presented their solution for sharing three cakes among eight people. He challenged them by saying, "*Can you explain your theory to everyone else please?*" (Andy pointed to one of the students). *And you all* (points to everyone else) *have to discuss his theory, and decide how it is similar or different to yours.*" This meant the students had to listen carefully to the ideas of their peers and rationalise these in comparison to their own.

The findings of this research compared to Fraivillig et al.'s (1999) who found that teachers' use of the different types of questioning, increases discussion around scenarios in a way that and may increase student learning. Using the right type of questioning at different stages within a lesson can support and stimulate students during the problem-solving process (Way, 2008).

9.3 Student Learning in the Proportional Domain

This section of the chapter discusses part of research Question 2, concerning "*the relationship between teachers' professional knowledge and student learning*".

9.3.1 Assessment Tasks and Student Learning in the Proportional Domain

Tasks 1 to 3 focused on understanding the meaning of the fractional number and the relationship between the fraction and the size of the whole. These tasks were placed first on the assessment, because it has been argued that interpretations of fractions should be taught in a particular order of conceptual challenge, and that the first type of fraction teaching relates to "part of a whole" understanding (Hansen, 2005).

Task 1 ($\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$), was an early Level 2 problem (Section 7.2.1: Task 1) which means that all students should have been capable of adding the unit fractions. The NZC, states that at Level 2 students are expected to “use simple additive strategies with [whole numbers and] fractions” (Ministry of Education, 2007, Level Two chart). The curriculum elaborations for Level Two states that students are expected to understand that “fractions are iterations (repeats) of a unit fraction, for example, $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ and $\frac{4}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ (Ministry of Education, *n.d.c*). On the initial assessment only 48% of Andy’s Year Six and Year Seven students and 10% of Beth’s Year Five and Year Six students, added the fractions accurately.

In her teaching session, Beth explored the addition of unit fractions with paper circles and the number of correct responses by her students on Task 1 in the final assessment ($\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$) (Section 7.2.2: Task 1), increased from 10%, to 65%. In contrast, the number of students correct on Task 1 in Andy’s class, decreased from 48% to 39%. The students struggled with understanding the meaning of the numerator and denominator in the symbol $\frac{1}{5}$, along with the connections between repeated addition, skip counting, and multiplication. Nine (39%) of Andy’s students gave an answer of $\frac{3}{15}$ having added the numerators and then the denominators. They did not understand that when they have $\frac{1}{5}$, and added another $\frac{1}{5}$, and another $\frac{1}{5}$, they had 3 representations of $\frac{1}{5}$, or 3 times $\frac{1}{5}$. This important part-to-whole construct and the connection between repeated addition and early understanding of multiplication, has also been found by other researchers to be a difficulty for students (Behr et al., 1992; Kieren, 1988; Lamon, 2007; Van Dooren et al., 2010). Many students continued to apply additive thinking (some incorrectly) when multiplicative thinking would be more appropriate (Van Dooren et al., 2010).

Task 2, involved addition of fractions with different denominators. While the denominators were different they were compatible ($\frac{1}{10} + \frac{2}{5}$), and it was expected that the students would recognise the relationship between tenths and fifths (Section 7.2.1: Task 2). On the initial assessment, two students (4%) from Anna’s class added the fractions together accurately, while at School B, no-one solved this correctly, the main error being addition of the numerators and the

denominators. Often students treat the numerators and denominators as separate numbers rather than seeing the connection between them (Hart et al., 1981; Young-Loveridge et al., 2007), which results in what is referred to as the ‘add across error’ (Carpenter et al., 2015; Ma, 2010).

When teaching the fraction lessons, the teachers made few connections to the learning that had previously occurred in the multiplication and division lessons. All teachers were observed previously teaching the relationship between twos, fives, and tens, in their multiplication lessons and recognition was not made in Task 2, between two-fifths also being represented as four-tenths, which meant that together with another one-tenth made five-tenths, equivalent to one-half. The issue of mathematics ideas being taught in isolation has often been explored in research, which has found that students and teachers frequently see their mathematics as isolated learning experiences and do not connect different concepts together (Howley et al., 2007; Kazemi & Stipek, 2001; Ma, 2010; Steffe & Olive, 2010; Stigler & Hiebert, 2004). However, teachers will not make connections between related ideas, or between manipulatives and ideas, unless they have a clear understanding of the mathematics ideas themselves (Ma, 2010), and this knowledge appeared to be problematic for the teachers in this study.

Students’ responses in the assessment tasks also indicated some lack of number sense in relation to fractional numbers. For example, when answers to the problem $\frac{1}{10} + \frac{2}{5}$ included $\frac{1}{5}$, $\frac{3}{4}$, and $\frac{3}{15}$, if number sense had prevailed the students would have realised that when adding $\frac{1}{10}$ and $\frac{2}{5}$ together, they could not possibly get an answer of $\frac{1}{5}$. This result disregards one of the basic properties of number: that the sum of two positive numbers must be greater than either addend (Olive & Steffe, 2002; Siegler, et al., 2011). One of the reasons that connections were not made, may have been because of a lack of fundamental understanding (PUFM) by the teachers, including number sense (Ma, 2010).

Task 3 was a word problem comparing two fractions. On the initial assessment the fractions were equivalent: “Judith eats $\frac{1}{2}$ of a pizza and Jenny eats $\frac{4}{8}$ of a pizza. Who eats the most? Draw a picture to show how you worked this out.” Almost one-quarter (24%) of the students solved this problem correctly. However, while some students drew the pictures reasonably accurately, they gave an incorrect

answer. They believed $\frac{1}{2}$ was the largest piece anyone could get, because when ordering unit fractions by magnitude $\frac{1}{2}$ is the largest. Other students drew incorrect representations of the fractions that did not depict equal partitioning of the whole, which led to them giving an incorrect answer. This is consistent with the findings of Gould (2005b) who found diagrams were sometimes created in proportion to the size of the denominator, resulting in incorrect answers to given problems. In order to gain competence in solving fraction problems, greater conceptual understanding is required (Pearn 2003; Siemon et al., 2001; Way et al., 2013; Wong, 2010).

On the final assessment, Task 3 was: “Judith eats $\frac{3}{4}$ a pizza and Jenny eats $\frac{7}{8}$ of a pizza. Who eats the most?” While a little over one-half (52%) of the students solved this correctly, the difficulties in comparing fractions of different denominators is consistent with other research. Such research has found that students learned fraction equivalence through the mastery of the rule, “multiply or divide the numerator and denominator by the same number” (Pearn 2003; Siemon et al., 2001). It cannot be claimed that the students in this study were taught in this way, but this illustrates the importance of students understanding that a fraction is a member of an equivalent group in which all symbolic representations denote the same quantity (Wong, 2010). Wong’s research found that in order to advance students’ understanding of fraction equivalence, their teachers require a greater insight into the paths (trajectories) students follow. While her research focused on fractional representations (as opposed to computation), the understanding required of the area model emphasised by Wong could have provided a sound knowledge basis for addition of fractions as required in Task 2, as well as comparing of fractions in Task 3.

Task 4 on the initial assessment was, “If $\frac{1}{4}$ of my circle has 3 smiley faces, how many are there on the whole circle?” On the final assessment, the number of smiley faces was changed to five. Both classes at School A, showed a percentage decrease (Anna’s from 90% to 77% and Andy’s from 64% to 43%), while at School B, both classes showed an increase (Bob’s 87% to 95% and Beth’s 48% to 65%) in the number who solved the part-to-whole task correctly. It was expected that the task on the final assessment would be easier than that of the initial

assessment, given that the final task was based on Level 2 (Ministry of Education, 2007) knowledge (students are expected to know their $\times 5$ tables prior to $\times 3$). The increase at School B, appears to be related to Bob and Beth having given students part-to-whole problems, during their observed lessons. The percentage of students who solved this task correctly, is similar to the findings of Clarke et al. (2007) who found in Australia that at Grade Six (Year 7), 64% of students were able to move from the part of a shape, to the whole.

Task 5 on the initial assessment was, “Find $\frac{1}{3}$ of 18”. More than one-third (37%) of the students solved this correctly (Table 7.3), although they did not see the connections between whole number division ($\div 3$), and proportional understanding ($\frac{1}{3} \times$). Half of the students either recorded that they guessed an answer, or omitted to give one. Of the remaining students the most common solution method was, “halve the 18 to get 9, and halve the 9 to get 4.5.” The students did not understand what it meant to find one-third of a set and instead, procedurally utilised a halving strategy that they had learnt in their multiplication unit. However, as fraction teaching followed on from the multiplication unit, the students should have been able to divide by three or use reversibility and solve $3 \times 6 = 18$. This question related to Hansen’s (2005) and Kieren’s (2007) notion of fractions as operators, an important sub-construct in developing an understanding of fractions (Lamon, 2006).

On the final assessment, Task 5 was, “What is $\frac{1}{3}$ of 21?” On this task, Beth’s students increased from 19% correct, to 45%. Beth’s class was cross-grouped for mathematics based on school assessment at the start of the year and the identified low-achieving group of students had made a major shift in their understanding. The research of Boaler (2008) found that cross grouping often resulted in poor achievement of the students concerned, due to lower expectations by their teachers. However, in this instance, Beth had an expectation that the students could successfully solve operator-type problems and as the final assessment results indicated, worked hard to ensure this happened. Beth’s lessons focused on the relationship between visual models and the use of manipulatives to support students’ understanding. In an effort to develop conceptual understanding before procedural application, Beth also related problems to the students using real-life

contexts including their names, which may have made the problems more relevant and meaningful when carrying out their computations. Her student numbers subsequently showed the greatest improvement on the majority of fraction tasks. Beth worked towards establishing what is referred to as “a culture of success” (Black and Wiliam, 1998, p. 142), as she believed that all students could achieve.

Task 5 (whole-to-part) also saw a large increase in correct responses in Bob’s class, from 57% to 95%. This task related to scenarios solved in the observed lesson in Bob’s class, where the students used manipulatives and worked in pairs to solve birthday cake problems, which had involved the students’ names (Section 6.2.3: C.2). The increase in understanding the students gained as a result of using manipulatives is consistent with the findings of many researchers (Ball, 1992; Ma, 2010; Skemp, 2006; Swan & Marshall, 2010).

Task 6 on the initial assessment related to the magnitude of fractions, and is often related to the positions of fractions on the number line. Hansen (2010) suggests this concept is third in the order of difficulty and is within Kieren’s (1980) construct of fractions as measure. The students were asked to order fractional numbers ($\frac{1}{3}$ $\frac{6}{4}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{1}{2}$) from smallest to largest (Section 7.2.1: Task 6). Further strategies for solving this task could have included considering the fractions as part of a set and finding equivalent fractions. Research has found that using the number line portrays an understanding of the underlying conceptual difference between recognising the fractional number in terms of a counting unit and its ordinal placement (Way et al., 2013; Yackel, 2001), as opposed to a collection of objects. However, in determining the order of the fractions in this task, the students may have visualised a set of objects (e.g., 12) and ordered the fractions according to the relevant proportion of the set (e.g., $\frac{1}{4}$ is smaller than $\frac{1}{3}$, because $\frac{1}{4}$ of 12 is equal to 3, and $\frac{1}{3}$ of 12 is equal to 4).

Over three-quarters of the students solved Task 6 with similar errors. The first was ordering both numerators and denominators from larger to smaller ($\frac{6}{4}$ $\frac{3}{4}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$). This suggests that the students had previously learnt that the larger the denominator the smaller the size of the piece, which would be true if the fractions were all unit fractions. Other students did the reverse and ordered denominators from smallest to largest (using

whole-number thinking), and likewise the numerator when there was more than one fraction with the same denominator ($\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{6}{4}$). Previous research has also found that many students use their knowledge of whole numbers to solve fraction problems (Behr et al., 1984; Gould, 2005; Wong, 2010; Wong & Evans, 2007), and that when students focus on whole-number numerators, whole-number denominators, or both, they often get problems right for the wrong reasons (Gould, 2005b).

One extra fraction ($\frac{7}{16}$) was added to Task 6 ($\frac{1}{3}$ $\frac{6}{4}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{1}{2}$ $\frac{7}{16}$), for the final assessment (Section 7.2.2: Task 6). The fractions in the initial assessment were either unit fractions or had compatible denominators, and by including $\frac{7}{16}$ the students required a greater understanding of the relationship between numerators and denominators. On the initial assessment, 50% of Anna's Years 7 and 8 students ordered the fractions accurately and on the final assessment, this lessened to 15%. While the addition of the extra fraction made the task more difficult, it showed that Anna's students did not have the understanding of working with fractions expected of them at Level 4 of the curriculum, which requires students to understand equivalent fractions and be able to order them accurately (Ministry of Education, 2007).

Four additional fraction tasks relating to Levels 3 and 4 of the curriculum were included in the final assessment, primarily for Andy's Years 6 and 7 students, and Anna's Year 7 and 8 students, along with more capable students from the other classes (Section 7.2.2: Tasks 7 to 10). The additional tasks included non-unit fractions, fractions of a set, and decimals. Andy had one student (4%) who completed the decimal subtraction problem accurately, while Anna had six (23%) (no student at School B could do this). When asked how many tenths were in all of the number 5.23, 17% from Andy's class and 27% from Anna's were correct. While Anna had discussed place-value in her lesson and the students had used deci-pipes to model groupings of units within decimal fractions, they did not recognise the connection between the models they had made in class and the task in the final written assessment. A generalised understanding of fractional numbers is required when transferring between representations and contexts (Lamon, 2006), which was not apparent by Anna's students.

9.4 Aligning Teaching Materials and Language

As found in the multiplicative domain (Section 8.4), there were some inconsistencies between teaching materials used in the proportional domain which may have contributed to the difficulties the teachers and students experienced in their fractions lessons.

Level Three Curriculum objectives state that students be able to “use a range of additive and simple multiplicative strategies with whole numbers, fractions, decimals, and percentages... know fractions and percentages in everyday use.” (Ministry of Education, 2007, fold out chart). The understanding for this objective is expanded on in the curriculum elaborations, which state that students should be able: “to find fractions of quantities, for example two-thirds of 24 as $24 \div 3 \times 2 = 16$, and find simple equivalent fractions related to doubling and halving, for example $\frac{3}{4} = \frac{6}{8}$. To add and subtract fractions with the same denominators, for example $\frac{3}{4} + \frac{3}{4} = \frac{6}{4}$ and to convert improper fractions to mixed numbers, for example $\frac{17}{3} = 5\frac{2}{3}$. Students should know the decimals and percentage conversions of simple fractions (halves, quarters, fifths, tenths) and use these to solve simple percentage of amount problems, for example 50% is fifty out of one hundred. 50% is one half, so 50% of 18 is 9” (Ministry of Education, *n.d.c*). As noted in the multiplicative domain findings (Section 8.4), while the elaborations now help teachers to understand what the curriculum requires, they were unavailable to teachers at the time the data was gathered for this research. Without the elaborations, interpreting the requirements of the curriculum was often difficult for the teachers, who relied on their NDP Books and planning sheets for clarification. For example, Bob incorporated birthday cakes into his lessons (Ministry of Education, 2008g, p. 26) as he worked from the part to the whole (Section 6.2.1: A4). Beth used a lesson on wafer biscuits (Ministry of Education, 2008g, pp. 16, 17) as a guide to her lesson and both she and Andy, incorporated addition of fractions that went over one whole (Section 6.2.2: B1).

While parallels can be seen between, the NDP books, planning sheets, and elaborations, there are times when they appear to vary from NZC. For example, fraction equivalence is emphasised in the NDP Book 7, for students at Stage 6 (Ministry of Education, 2008g, p. 37), who are transitioning from AA to AM. The

idea of equivalence is further emphasised in the NDP book at AA to AM in naming fractions as decimals (p. 41) and ratios and proportions (p. 50), neither of which is mentioned at Curriculum Level Three (Ministry of Education, 2007). However, the first strategy being developed on the AA to AM planning sheets, is “find equivalent fractions by splitting, e.g., $\frac{3}{4} = \frac{15}{20}$, by splitting each quarter into fifths” and the second strategy is “order fractions using equivalence and benchmarks”, a strategy more suited to Level Four.

Curriculum Level Four achievement objectives state that students be able to, “use a range of multiplicative strategies when operating on whole number... understand addition and subtraction of fractions, decimals, and integers... find fractions decimals, percentages of amounts expressed as whole numbers, simple fractions and decimals... know the equivalent decimal and percentage forms for everyday fractions” (Ministry of Education, 2007, fold out chart). Nowhere is multiplication of fractions mentioned specifically in the curriculum at Level Four, although as the curriculum elaborations maintain, multiplication is necessary to find fractions decimals, percentages of amounts expressed as whole numbers, simple fractions and decimals. Therefore, the elaborations expand on what is required in terms of teaching multiplication of fractions at Level Four, by stating that “students will understand that finding a decimal or percentage of an amount involves finding a fraction of that amount, for example 40% of 56 = $0.4 \times 56 = 4 \times 5.6 = 22.4$... Students should be able to multiply fractions with understanding, for example $\frac{2}{3} \times \frac{3}{5} = \square$ as two-thirds of four-fifths, and use their multiplicative understanding of place value to solve multiplication and division problems with simple decimals, for example: $1.6 \times 0.4 = \square$ as $16 \times 4 \div 100 = 0.64$ and $24 \div 0.3 = \square$ as $24 \div 3 \times 10 = 80$ ” (Ministry of Education, *n.d.c*). However, without the availability of the elaborations, understanding what was required when teaching the multiplicative relationship of fractions as a requirement of the Level Four curriculum was an uncertainty for the teachers. Therefore, the teachers relied on the NDP material to guide them when planning for their Years 7 and 8 students. Solving problems that involve multiplying by fractions, is a strategy being developed on the AM to AP planning sheets and lessons in the NDP Book 6. For example, “I am learning to multiply fractions and decimals” (Ministry of Education, 2008g, p. 63). Therefore, Anna’s lesson on understanding the place

value of fractions and how many tenths and hundredths are in decimal numbers, was a foundation for teaching multiplication of decimal numbers.

9.5 Proportional Domain Summary

9.5.1 Teachers' Subject Matter Knowledge and Pedagogy

In order to examine the professional knowledge of teachers, this chapter explored the teachers' subject matter knowledge in relation to their perceived pedagogical practice when teaching fractions. Each scenario asked the teachers to identify what the student had done in order to solve the problem and explain what they would do next with the student in each instance. The results in this section paralleled those of the multiplication scenarios and in most instances, the teachers struggled to identify what the next steps of learning might be. The teachers often commented that they would teach the student the same way as the given example, and provided no further explanation or identification of the next steps to learning.

9.5.2 Teacher Practice in the Proportional Domain

There were observable differences in some areas of the teachers' professional knowledge as they moved from teaching in the multiplicative domain to teaching in the proportional domain. The teachers became more aware of the importance of allowing for different methods of solution using a range of classroom techniques, including all students having manipulatives available to reinforce understanding of concepts; standing back and allowing the students to participate in meaningful discussions, including greater justification of solution methods; and using different question types in different situations. These changes in teacher practice were considered to contribute to a general improvement in the number of correct responses on the final assessment tasks.

All of the teachers were seen to encourage the students to use manipulatives more readily during the teaching of fractions (than the earlier unit on multiplication). What was particularly noticeable was the change in how the manipulatives were used. For example, in the initial multiplication lesson, Bob modelled with equipment while the students watched, while Andy introduced manipulatives to the class for the first time and the novelty of the new equipment meant that the students played with it. During the unit on fractions, all of the students in Bob's class had manipulatives available to them, while in Andy's class the

manipulatives were no longer seen as an added fixture and incorporated into the lesson in a more purposeful manner.

Lesson observations also showed differences between teaching multiplication and fractions, in the manner in which the students talked together when left to solve problems on their own. One noticeable change in mathematical discussion among the students included greater justification of ideas, which might explain the greater understanding of the concepts taught, as evidenced by the final assessment tasks.

The teachers appeared to have some uncertainty in interpreting the curriculum and progressions of learning. This may have contributed to uncertainty in associated expectations of student achievement within the proportional domain and of the progression of key ideas to be taught. This study revealed factors outside the teachers' control that may also have contributed towards this uncertainty. Uncertainty in curriculum expectations, were possibly compounded by inconsistency in the wording between NZC, the Number Framework, and the Standards, which meant that teachers were unsure of exactly what should be taught at their class level. Some of these inconsistencies were outlined in Chapter 8 (Section 8.4) and Chapter 9 (Section 9.4).

Fractions are complex, and the cognitive demands of the problems, meant the teachers did not always identify students' difficulties. Word problems designed to develop fractional understanding were often treated as whole number division, with little recognition of the proportional representation (for example, placing candles on a cake was presented as sharing rather than finding a fraction of a set). Connections between what had been taught in multiplication and what was being taught in fractions were not made.

Learning to establish what was the whole, and what was the fraction to be found, provided challenges for both teachers and students. The students had difficulty in understanding that one whole is not always represented by a single object (e.g., in these lessons, a pie, biscuit, or pizza), but may also be represented by a collection of objects, such as a bag of apples, packet of biscuits, or several pizzas. This caused confusion when finding the fractional representation of a set.

The teachers did not always delve beneath the initial thinking of the students. They sometimes overlooked responses and ideas from the students and preferred not to stop and develop these. Instead, they continued with the planned lesson, which meant that they may not have extended the students beyond their comfort zone, or allowed them to be challenged with higher-level problems. Questioning techniques continued to focus on back-and-forth supporting type questions, at the expense of extending the students' learning through more complex questioning and problems to solve.

The results above suggest that greater emphasis could be placed on connecting ideas if the students are to remember how to solve problems. While connections were made between real-life contexts and problems solved, they were not made consistently between fractional number and whole number, fractions and decimals, and connections with manipulatives.

9.5.3 Student Learning in the Proportional Domain

The assessment tasks were based on Levels 2 and 3 in NZC, with Level 3 tasks designed for achievement by the end of Year 6. The initial tasks suggested the students were below, the expected levels of understanding in the proportional domain as outlined in the Curriculum Standards (Ministry of Education, 2009a). Improvement was made on many of the tasks, the most notable being comparing the size of fractions (Tables 7.3 & 7.4: Task 3) and finding fractions of a set (Tables 7.3 & 7.4: Task 5). However, overall there were still many areas where the students did not meet the expected levels of knowledge and understanding. For example, in the final assessment in Andy's Years 6 and 7 class, there was only one task where more than half of the students were correct. In three of the four classes (the exception being Anna's Year 7 and 8 class) fewer than three students could add fractions with compatible denominators, while a range of one to four students in all classes ordered fractions accurately from smallest to largest. Anna's lesson focussed on decimal number understanding and yet approximately one-quarter of her students carried out a decimal subtraction task accurately and only one-quarter could say how many tenths were in a number.

As was the case for multiplication, no attention was given by the teachers to assessment information available to guide their teaching practice. Successful teachers utilise assessment data to target concepts to be taught, and use these to

plan sequenced lessons (Allsopp et al., 2007). Prior knowledge is an important factor in student learning, and students learn much more effectively when they are building on what is already known (Allsopp et al., 2007; Loughran, 2010). Skilful teachers who promote learning with understanding, draw on students' current knowledge, understand how their knowledge develops, and map out learning trajectories as a result of the information gained (Confrey et al., 2009; Wilson et al., 2015). Loughran explained that mistaken or flawed prior knowledge can hinder progress and teachers must be aware of inaccuracies and address these throughout their teaching, if progress is to be made in student learning.

The students at School A, were in Years 6 to 8 and the expectation was that they would be achieving late Level 3 and early Level 4 achievement objectives of the curriculum. The above assessment results indicated that this was not the case and later discussions with the teachers showed that they were unaware of the degree to which the students' knowledge was below expectation. Earlier participation in the NDP professional development, gave the teachers prior opportunities to unpack many aspects related to the teaching of proportional reasoning. The professional development was followed closely by the introduction of the Mathematics Standards, alongside a range of support material to assist teachers in recognising whether students were achieving expected outcomes appropriate to their year level (Ministry of Education, 2009a, 2010). Learning trajectories provided the basis for the Number Framework (Ministry of Education, 2008a), which identified progressions of learning through various stages. The requirements of the progressions and incremental steps of learning shown in the Number Framework and Mathematics Standards was complex and this study suggested that more time may be required for the teachers to fully comprehend and understand the expectations of these documents.

9.5.4 Aligning Teaching Material and Language

Some of the difficulties teachers had in interpreting what was required when teaching fractions and decimals, related to interpretation of the objectives at each level of NZC. The teachers relied on the NDP books and planning sheets for clarification, but these did not always align. More recent availability of the curriculum elaborations (not written at the time of this research), has supported

teachers in understanding the curriculum requirements, with the NDP books and planning sheets providing a supporting role.

In the following chapter, Chapter Ten, the different data sets from previous chapters are combined and presented as a discussion, based on themes that emerged across the findings.

CHAPTER TEN

DISCUSSION: KEY RELATIONSHIPS

10.1 Introduction

This chapter bridges across the different data sets outlined in the previous chapters and presents a discussion based on themes that emerged from the results and analysis. The data sets began with teacher responses to questionnaires, which included written scenarios designed to identify the teachers' espoused professional knowledge, based primarily on subject matter knowledge and perceptions of desirable pedagogical practice. This was followed by a detailed analysis of professional knowledge in practice, made against a comprehensive PCK framework specifically designed for use in mathematics teaching. Classroom observations were complemented by student progress gauged through initial and final assessment data, combined with contributions from observed classroom lessons. Field notes relating to teachers' comments made during informal follow-up learning conversations, supplemented these sources.

10.2 Professional Knowledge of Primary School Teachers of Mathematics

This section discusses themes from data gathered when investigating Question 2: *What relationships are there between professional knowledge, teaching practice, and student learning, when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?*

Teachable moments and on-the-spot recognition of student learning

Uncertainties in the teachers' curriculum knowledge were identified in response to the questionnaire scenarios (Sections 4.2.2 & 4.2.3), when there were times that they were unable to identify what might be appropriate next learning steps for the students and left that section of the question blank (Andy 4; Bob 5; Anna 1). It is generally recognised that on-the-spot recognition of next steps, relies on teachers having confidence in their subject matter knowledge, in order to amend previously planned teaching and learning progressions (Clements & Samara, 2014; Confrey et al., 2014). For the teachers to determine next steps of learning, they required termed horizon knowledge (Ball et al., 2008). The teachers' horizon knowledge could have provided them with understanding of the students' current knowledge

and provided them with the vision to see how they could build on that knowledge to take the students' learning to the next level.

When observed in classroom practice, the teachers showed evidence of indecision about next steps, such as when students provided unexpected responses to a question, or unanticipated ideas within discussions, which resulted in potential on-the-spot teachable moments being overlooked or confused. When the notion doubling and halving arose, Bob struggled to explain the strategy conceptually, while Beth got herself confused mid-way through her explanation. Bob, recorded 4×18 on the whiteboard, and procedurally drew arrows to show how it could be changed to 8×9 (Section 5.2.2: B.2). Beth provided examples of what happened to the product when doubling the multiplier in multiplication, from 2 times a number, to 4 times a number, to 8 times a number (Section 5.2.1: A.7). However, when doubling and halving arose, during the exploration of the relationship between the $\times 5$ and $\times 10$ facts, Beth also got muddled and confused (Section 5.2.1: A.7). The knowledge required by Bob and Beth, for this in-the-moment action, is referred to by Rowland et al. (2009) as “contingency knowledge” and assists teachers in deviating from the planned lesson, by pausing and responding to students' ideas. Contingency knowledge encompasses Smith and Stein's notion of anticipation, and the importance of teachers actively envisioning how students might mathematically approach a task, in order to be ready to respond to answers. In these instances, neither Bob nor Beth, displayed the contingency knowledge that might have provided an understanding of the doubling and halving process when it arose. Had they anticipated this approach in advance of the lesson, they might have, as suggested by Smith and Stein (1998), been more prepared to provide mathematical explanations for the understanding behind the strategy that they showed procedurally.

The teachers' subject matter knowledge and associated confidence to explain the key concepts associated with understanding rules came to the fore in both their espoused professional knowledge and classroom practice. When given a problem on the questionnaire related to the divisibility rule for nine, the teachers were unable to show an understanding of the rule (Section 4.2.2: Scenario 2). While Beth and Anna, provided an explanation as to what they would do next with the student concerned, this was not associated with the rule for divisibility by nine.

The challenges the teachers faced unpacking something that was not familiar to them (in this case the divisibility rule) and identifying the teaching that should follow, was also seen in class where they only occasionally gave their students open-ended problems to solve, or exposed them to the experiences associated with more difficult problems. When the students struggled to solve a problem, instead of taking the time to unpack the difficulty and allowing them the opportunity to learn from the process, the teachers would sometimes ignore the problem (Section 9.2.1: A.7) or leave the problem (for another day), and give the students a simpler example (Section 6.2.2: B.1). Continually giving the students simpler problems that did not challenge them, meant that they were not given the opportunity to problematize (Hiebert et al., 1996) their mathematics. The teachers' uncertainties with on-the-spot problem solving, meant that the students were seldom put into situations where they were required to delve deeper into their knowledge to seek out solutions, and see this as a positive aspect of mathematics learning. The idea of making mathematics problematic for learners and requiring students to delve deeper is well supported by research (Boaler, 2003; Fennema et al., 1993; Hiebert, et al., 1996; Hunter, 2010). Had the students been encouraged to problematize the mathematics within a community of learners, it would have allowed them to solve problems, explain, and examine their explanations, which then leads to the construction of understanding (Hiebert, et al., 1996; Hunter, 2006, 2010; Stein et al., 2008). The development of purposeful reasoning through engagement in problematizing and later in the reasoning process is highly beneficial to the students' learning.

Difficulties associated with identification of next steps of learning may have been due to insufficient awareness of the students' current knowledge. When planning their lessons, the teachers could have taken into account the students' existing understandings, alongside their gaps in knowledge. These understandings and gaps were indicated in the results of the initial assessment tasks. The initial assessment tasks were a combination of *procedures with connections*, and *procedures without connections* (Stein et al., 1996) within a range of curriculum levels. Understanding common misconceptions students have and how to resolve them is an important component of teacher knowledge (Ball et al., 2001) and closer analysis of individual and collective data from the assessment tasks, could

have assisted the teachers in recognising specific problem difficulties associated with progressions of learning.

The teachers found the achievement objectives in NZC (Ministry of Education, 2007) were not always clear in terms of requirements and relied on lesson progressions outlined in NDP books and on the associated planning sheets for clarification. The NZC, NDP books, and planning sheets, did not always align in terms of requirements and this caused confusion for the teachers. For example, the Level Three requirement is for the learning and understanding of basic facts up to and including the $\times 10$ tables. However, if lessons are planned from the aligned AA to AM planning sheets, they are also teaching the students double-digit times double-digit multiplication. Uncertainties in determining the next steps of learning meant that the lessons were often taught in sequential order as presented in the books (Sections 8.4 & 9.4).

The use of manipulatives and visual representations

The teachers often used manipulatives to support the explanation and justification of a particular strategy during the teaching practice. In the questionnaires, Anna and Beth were the only two teachers to mention that they would use manipulatives to check the students' understanding of their answer given to the scenarios. Similarly, in the multiplication lessons, Anna and Beth were observed making more use of manipulatives to represent the problem at hand, when reinforcing the importance of conceptual understanding, than did Bob and Andy. Anna's and Beth's students all had materials available to them. In contrast, Bob used the abacus at the front of the class, demonstrating show-and-tell type scaffolding (Baxter & Williams, 2010). Manipulatives are frequently used in mathematics lessons with the claim that they extend students' learning of mathematical concepts and operations, as they make them more comprehensible (Nührenbörger & Steinbring, 2008; Schoenfeld, 2011; Wright, 2014) and for Beth's lower ability students this was the case. Both of these ideas had been key parts of her observed lessons.

As the lessons moved from multiplication and division to fractions, all of the teachers provided more opportunities for their students to use manipulatives. Paper circles, paper strips, scissors, animal strips, and fraction tiles are examples of manipulatives available to all students, in the classes. The teachers used the

manipulatives to deconstruct content and to make connections between the manipulative and mathematical idea (Carbonneau et al., 2013; Fennell & Rowan, 2001) as they endeavoured to identify critical mathematical components that were fundamental for understanding particular concepts (Ma, 2010). The students use of manipulatives may have contributed towards improvement in the final fractions assessment tasks. Such a relationship would be consistent with previous research, which has established that when conceptual understanding of fractions is intertwined with procedural knowledge, a greater understanding of concepts is evident (Gould, 2005a, 2005b; Way et al., 2015; Wong & Evans, 2007), and this knowledge is often formed through the initial use of manipulatives (Swan & Marshall, 2010).

The teachers' procedural knowledge meant they were comfortable with algorithmic procedures and symbolic representations, but they appeared to be less comfortable when explaining mathematical concepts, with diagrams and non-symbolic representations. The teachers' difficulties with drawing diagrams were evident in the questionnaire scenarios, where they were asked to "give their answer to each problem" and "draw a diagram and explain" how they solved it. This contradicted the espoused encouragement of diagrams as stated by the teachers in the questionnaires, and was also not evident in their teaching practice. The exceptions were a quick sketch by Anna of the roll-over of groups of ten units in place-value understanding and Beth's drawing of groups of objects when reinforcing understanding of the multiplication symbol.

Research has shown that drawn representations (of fractions) supports understanding of mathematical concepts (Gould, 2005b; Way et al., 2015). Drawing and then sharing diagrams is important, as the ensuing discussion provides support to the students' thinking, and clarification of ideas, which would have assisted the students in understanding the concepts taught in class (Clement, 2007; Flores, 2010; Gould, 2005a, 2005b). This understanding may then have been evident in the assessment tasks, where the students were asked to draw a diagram to show how they worked their answers out. However, diagrams were used only occasionally and when they did, many were inaccurate which resulted in incorrect responses. For example, on the multiplication Task 4 (12 biscuits put into packets of 3, Figure 7.3) and fraction Task 3 (comparing the fractions $\frac{3}{4}$ and

$\frac{7}{8}$, Figure 7.20) inaccurate diagrams led to incorrect answers. When teachers use drawn representations, a noticeable change in their PCK and conceptual understanding is evident (Way et al., 2013), which results in improved student conceptual understanding (Flores, 2010; Gould 2005a; Way, et al., 2013).

Use of Learning Intentions in the lessons

In the classroom, all teachers outlined the learning intention (WALT) for the day at the start of each lesson and discussed this with their students (the exception being Andy in the initial multiplication lesson). Presenting a specific goal for learning can provide teachers and students with a clear outcome that guides the learning and selection of activities that takes place (Smith & Stein, 2011; Stein et al., 1996). Without specific learning goals, determining what learning has occurred as a result of the instruction and activities can be problematic (Hiebert et al., 2007).

However, this study found that sharing the WALT often led to teachers directing the learning, rather than facilitating the learning, as they focussed on success criteria for achieving the learning intention. In directing the learning back to the learning goal for the day, there were times when the teachers failed to capitalize on students' ideas that arose. Fostering students into lesson contributions means accepting ideas whether they are right or wrong (Fielker, 1997). Even if the students had come to a wrong conclusion, the important thing is that they participated in discussion and learnt to make their own judgements. In the observed lessons, seldom did the opportunity arise for deviation from the WALT, which meant they relied on the teachers' judgements and authority (Fiekler, 1997), rather than learning from their own ideas (and possibly mistakes).

When the results of the final multiplicative domain assessment tasks were shown to the teachers, they were disappointed. For example, Andy was surprised to see the minimal progress made by his students from the initial tasks (4 tasks showed slight improvement; in 3 out of the 9 tasks less students were correct; while in 2 tasks, correct responses remained the same). His comment was that he thought that he had "taught them" what "times" meant. However, within a Vygotskian sociocultural orientation to lessons, a primary belief is that learning in mathematics is based on open dialogue between students, and between teacher

and students, in order for the students to gain further knowledge. By constantly returning to the WALT to re-focus the learning, dialogue was directed rather than open, which resulted in limited multiplicative learning. While Andy may have encouraged discussion amongst his students, multivocal conversations were not evident. However, in the later fractions lessons, the WALT was less prominent which provided opportunities for the students to discuss ideas more openly, and possibly contributed to greater improvement from the initial assessment tasks to the final assessment tasks.

Teachers' use of discussion in the classroom

Quality discourse during discussions has been acknowledged as having a positive outcome on student learning (Hunter, 2006, 2009, 2012). The teachers, when identifying the next steps of learning in the written scenarios, did not mention encouraging discussion among students. When asked if they encouraged their students to justify their choice of strategy and thinking with others (Section 4.2.4: Statement 3), the teachers' responses included always, often, and sometimes (Table 4.1). However, such quality discussion seldom eventuated in the classroom, where the students were unaccustomed with challenging the thinking of others. The teaching was generally based on the use of a single strategy (generally due to the wording of the WALT) when solving problems, which meant the students often limited responses to describing to others in their group their procedure for solving the equation. The students shared their solution methods step-by-step, seldom explaining the reason why they solved problems in the manner they had, or justifying their strategies.

Observations suggest that as the students progressed from the multiplicative to the proportional domain, they became a little more conversant. When students create new knowledge for themselves based on networks within the classroom (their peers), they are more likely to retain their knowledge (Yackell & Cobb, 1996). The more frequently the teachers asked the students to describe their solution strategies and justify their responses the more they were engaged, which as Hiebert and Wearne (1993) maintained, resulted in higher gains in mathematics achievement. The group discussions during the students' fraction learning were more co-operative and resulted in greater overall improvement on the final proportional tasks than was evident on the earlier multiplicative tasks. During this

process, the students helped each other learn in what Vygotsky referred to as their *zone of proximal development*. The students shared their thinking, which enhanced their problem-solving skills.

In generating quality discussions, teachers need to consider their use of questioning strategies and techniques. Beth was the only teacher who in response to the questionnaire scenarios, asked students to explain their thinking. For example, when multiplying fractions she responded, “*Get (ask) Jo to explain her answer. Check Jo understands and make sure she is not just remembering rules*” (Section 4.2.4: Scenario 6). Effective teachers are responsive teachers, in that they constantly elicit, monitor, and respond spontaneously and effectively to their students’ thinking (Anthony & Walshaw, 2009; Franke & Kazemi, 2009), and in her questionnaire responses Beth showed recognition of the need to probe students’ thinking to ensure understanding.

During teaching sessions, the teachers often used lower-order questions similar to, “Why did you do that?” and there were some missed opportunities to extend the students’ thinking by asking higher-order questions along the lines of, “What would happen if we changed this [number] to?” or, “If we changed this [number], how might that affect this [number]?” When questioning, the teachers were very much focussed on what Fraivillig et al. (1999) referred to as “supporting” question types and seldom used higher-order questions that extended or advanced, the mathematical thinking of their students. As found in other research (Hallman-Thrasher, 2015), the teachers generally relied on supporting students towards correct solutions through leading questions as these frequently took less time to answer. However, when a teacher extends the thinking of students, the resulting discussion allows them to compare their own ideas with those of others and to generalise and draw conclusions (Fraivillig et al., 1999). Research has shown that a crucial factor in mathematical achievement is the manner in which students interact with their teachers through engaging in questions which require development of higher-order thinking skills (Way, 2008). Further use of questions to extend students’ thinking may have supported performance in the assessment tasks.

Making connections across mathematics strands

Being numerate involves being both efficient and effective, and includes a connectionist orientation involving an awareness of different methods of calculation, in order to choose an appropriate method (Askew, 1999). In this study, limited attention to connections meant that both the teachers and their students may not have seen mathematics in the broader sense, but rather viewed it as compartmentalised into separate unconnected ideas within topics. For example, the relationship between groupings of 5 and groupings of 10 made in the multiplication lessons, were not connected to fraction problems, such as when something is halved there are twice as many pieces. Past researchers have found that in order to facilitate and strengthen learning, teachers need to emphasise and promote the connections between and among ideas and topics (Hiebert & Carpenter, 1992; Howley et al., 2007; Kazemi & Stipek, 2001; Ma, 2010; Stigler & Hiebert, 2004; Treffers, 2001).

In the written scenarios, the teachers seldom made connections between understanding of concepts associated with prior learning and identification of what to include in the next phase of the learning process. There were times when the next step might have included returning to previously learnt concepts. For example, when decimal place-value understanding was an apparent difficulty in fractions (Section 4.2.3: Scenario 4), the teachers might have recapped whole number place-value understanding, then returned to decimal numbers. In the observed lessons, the teachers also missed many opportunities to connect the current learning contexts to past learning, both within number as well as between number and other strands of mathematics. The application of number is important to the connectionist emphasis (Askew, 1999), such as when finding equivalent fractions and finding fractions of a set. In these instances, the students require a strong foundation and understanding of multiplicative thinking, to understand the magnitude of the fraction. Students can later extend this knowledge to think relatively, as they maintain the ratio between the numerator and denominator (Lamon, 1994). However, this research found that when exploring equivalence the understandings and connections between concepts were not emphasised, which meant the students may not have recognised the connection between the key mathematical ideas they had learnt in class and examples in the assessment tasks.

Multiplication and division lessons focussed heavily on multiplication and very little on division, which may have had repercussions for the teaching of fractions later. When whole number division understanding is limited to a sharing model, it can impede progress and limit strategies available, when alternative approaches may be more effective (Anghileri, 1999). This was exemplified during the fractions lessons, when problems were presented with a focus on equal sharing, resulting in lessons based on partitive division. Emphasis was given to the number of objects that resulted when something was shared out equally, rather than the proportional relationship between the pieces and the whole. Part of the difficulty in understanding the relationship between division and proportional reasoning, may be attributed to the teachers' lack of awareness of the distinction between different division problem structures (Section 4.2.2: Scenarios 2 & 3). This meant that the students were also unaware of these differences. This became noticeable in tasks on the students' assessment, when they were given an example of each of the two division types (Tasks 3 and 4) and many were unable to do so. Other research has also shown that teachers place a large emphasis on partitive division and that teaching quotitive division seldom occurs effectively (Roche & Clarke, 2009; Roche et al., 2016). Giving the students multiplication and division problems together, and de-emphasising sharing procedures, assists in developing number sense (Anghileri, 1999).

Conceptual understanding, number sense, and mathematics for numeracy

The teachers' intuitive understanding and reliance on procedural knowledge meant that they often found unpacking the underlying structure of a problem required for conceptual understanding difficult to model to their students. The difficulty the teachers had explaining some of the mathematical concepts aligns to previous research, which has shown that a teacher will rely on procedural understanding, when they have insufficient conceptual understanding of mathematics content (Way et al., 2015). Limited conceptual understanding was initially evident on the questionnaires for Bob and Andy when they relied on rule-based vertical algorithms to show how they would solve most of the scenarios. This contrasted with Anna and Beth who responded by showing a variety of different strategies, including rounding and compensating, place-value partitioning, equal adjustment, and using ratios. The development of conceptual

understanding includes an awareness of the connections between concepts and procedures (Wong & Evans, 2007) and there were times when the relationships between ideas were overlooked. Although the teachers sometimes struggled to model the reason why things occurred mathematically, they persevered with this practice and indicated an awareness of the need for students to understand this relationship, such as when they deconstructed content as part of understanding the mathematics involved in the problems (Table 5.3: B.1 & Table 6.3: B.1). Current reforms and research emphasise the benefit of initial conceptual understanding, prior to or alongside procedural knowledge, if students are to make sense of problems (Hiebert & Carpenter, 1992; Kazemi & Stipek, 2001; Skemp, 1989; Treffers, 2001).

Teachers' number sense, as a component of Profound Understanding of Fundamental Mathematics (PUFM), seldom came to the fore in the observed lessons on multiplication and division, and fractions. Emphasising addition of fractions as the repeated adding of units of the same size, and unpacking this notion (with manipulatives) alongside the meaning of the numerator and denominator in a fractional number, would be of benefit to understanding fraction computation (Ma, 2010). During the observed fraction lessons, little attention was given to equivalent fractions. This may have helped address students' confusions and misunderstandings, including their use of the "add across" error. For example, many students in their assessment tasks, did not recognise that $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ could not possibly equal $\frac{3}{12}$, which is equivalent to $\frac{1}{4}$, nor could it equal $\frac{1}{12}$, which is less than $\frac{1}{4}$ (Section 7.2.1: Task 1 & Section 9.3.1: Task 1). Developing number sense, reasoning and operation sense is reflected in Skemp's (2006) notion of relational understanding, which allows students the opportunity to implement ways of solving problems that make sense to them. Something is only understood if one can see how it is related or connected to other things that are known (Hiebert & Carpenter, 1992; Skemp, 1989).

All of the teachers presented problems using situations or contexts that the students would probably experience in real life. These contexts included cookies, sandwiches, pizzas, money, and birthday parties. Tasks based on students' interests and mathematical strengths are an effective way to engage students,

promote mathematical justification, and develop conceptual understanding (Jane & Fey, 2000; Jennison & Beswick, 2010). However, the connections made in class between life experiences and mathematics, were not always transferred into their assessment tasks. School A, was a multi-cultural school with the greatest proportion of students from non-English speaking backgrounds and maybe the contexts of pizzas (in the assessment tasks) were not so familiar to these students.

Associated with conceptual understanding and number sense is estimation (Jordan et al., 2006; van den Heuvel-Panhuizen, 2001c). Of the eight multiple-choice questions on the questionnaires, three asked for estimated answers. However, during their teaching the teachers placed an emphasis on correct answers as they strove for accuracy, and estimation was not mentioned, or used as a self-checking mechanism. Approximation and estimation involve understanding the order of the magnitude of numbers and are important tools to understand and utilise when carrying out mathematics in the real world (van den Heuvel-Panhuizen, 2001c). During the final assessment tasks, the students could have used estimation as a self-checking mechanism on a number of tasks to guide their answers, which may have assisted them in recognising incorrect answers.

10.3 Using a Framework of Professional Knowledge

This section presents conclusions drawn from data gathered when investigating Question 3: *How does the use of a framework assist in the investigation of teachers' professional knowledge in practice?*

This study used a modified version (Figure 3.1) of the PCK framework of Chick et al. (2006), (Figure 2.1), for the in-depth analysis of the eleven classroom lessons. As noted earlier, the Chick et al. framework was chosen as it was designed with mathematics teaching at the forefront and included particular categories of knowledge that provided a thorough basis for conclusions to be made, concerning commonalities in professional knowledge.

The use of a framework to investigate teachers' professional knowledge in practice, provided a structured range of categories for detailed analysis of observed lessons. The framework highlighted the importance of content knowledge for teachers, while acknowledging that there is a difficulty separating it from pedagogy in classroom practice. As a result of analysis of the classroom

lessons within the three framework categories (Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context), the themes outlined in the preceding section (Section 10.2) emerged. On closer analysis of the teachers' espoused professional knowledge and student achievement, similar themes also emerged.

The Clearly PCK category showed that there were many inseparable links with content knowledge and pedagogy. It allowed the researcher to identify that during the mathematics lessons the teachers used a range of approaches for teaching a mathematical concept or skill, in order to encourage and support their students' learning. These approaches included using modelling books, establishing a learning outcome, using manipulatives, encouraging discussion, and using word problems.

The sub-categories within the Clearly PCK category indicated a difficulty teachers had in understanding the requirements and expectations of the curriculum, which may have contributed to uncertainty related to further teaching and learning progressions. The framework identified that one of the teacher's strengths, was her recognition of the importance of students' conceptual understanding alongside procedural knowledge. Whilst the teachers initially struggled with teaching mathematics conceptually, they persevered with this approach, emphasising to their students the importance of understanding mathematics as well as how to carry out mathematical computations accurately, in order to become numerate. In many instances, the focus on conceptual understanding required the teachers to unpack for themselves the mathematics they were teaching, as without conceptual understanding teachers will not be able to explain why a procedure works, or even why it is necessary (Way et al., (2013).

The Content Knowledge in a Pedagogical Context category, emphasised the importance of subject matter knowledge in relation to teacher practice. This section of the framework indicated to the researcher that the teachers regularly deconstructed the content of what they were teaching, in an effort to assist the students with their understanding of the mathematics. However, the few times the teachers made connections with the mathematical structure of the problems, was highlighted when they exhibited profound understanding of mathematics. The teachers did not encourage the use of estimation, a key component of number

sense (Jordan et al., 2006; van den Heuvel-Panhuizen, 2001e)), and it was not used as a method to check for the reasonableness of answers. There were times when the teachers recognised student misconceptions and corrected these, and occasions when they may have contributed to these, due to in-the-moment responses.

In the Pedagogical Knowledge in a Content Context, emphasis was placed on the requirements of teaching practice. The use of assessment data available to the teachers was underutilised when planning lessons. The majority of questions were the lower-level supporting type, and the teachers seldom extended the students with higher expectations, going beyond initial solution methods to solve problems.

New frameworks of professional knowledge

This research established that a framework provided the researcher with specific categories of knowledge to carry out fine-grained analysis of teachers' practice. After using the framework to analyse professional knowledge in practice, the researcher concluded that teachers could possibly benefit from the use of a framework to reflect on their personal professional knowledge and to support their teaching practice. Therefore, building on the findings of this research, two frameworks of professional knowledge were developed: one for teachers to reflect on their practice, and one for researchers to investigate the professional knowledge of teachers in practice.

While it is acknowledged use of the adapted framework (Figure 3.1) of Chick et al. (2006) provided a basis for the coding of data in this study, during the analysis process the researcher recognised that categories relating to knowledge of students, including their attitudes and beliefs about mathematics and relationship with their cultural identity, were missing categories. Shulman (2015) also acknowledged these notions as missing components of his original PCK thinking, which was based on theoretical underpinnings, rather than teacher practice. Having this knowledge is an important component of teachers' professional knowledge, and so the researcher included these into the establishment of a *Wheel of Knowledge* for classroom teachers. The wheel of knowledge (Figure 10.1) is a representation of the categories and sub-categories of professional knowledge, which teachers might consider within their classroom practice.

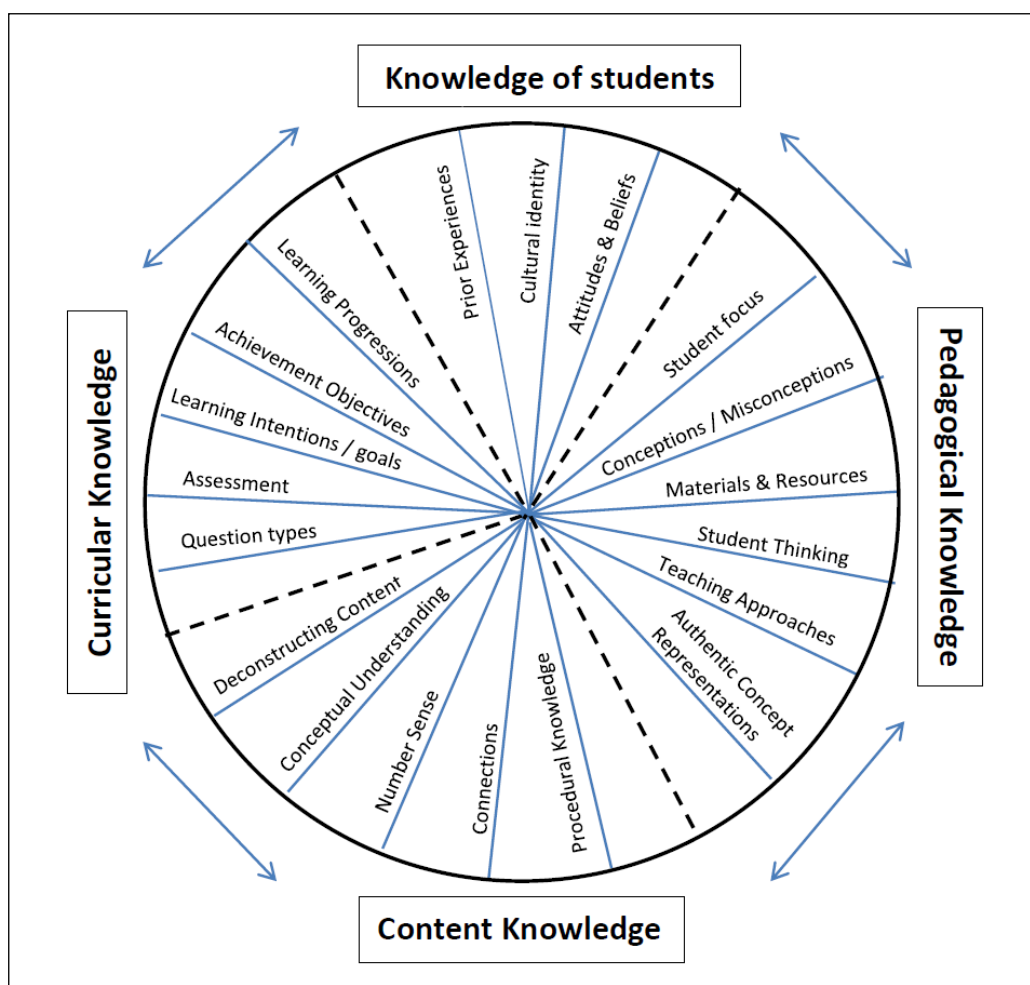


Figure 10.1 Wheel of Professional Knowledge

The wheel is based on four key categories of professional knowledge identified throughout this research including knowledge of students, pedagogical knowledge, content knowledge, and curriculum knowledge, and was created specifically for use by teachers. A wheel was chosen to represent the cyclic, on-going nature of acquisition and implementation of knowledge, and to sit alongside implementation of the “Teaching as Inquiry Cycle” (Ministry of Education, 2007, p. 35). The spokes of the wheel are not intended to represent all areas of professional knowledge required to teach mathematics effectively. However, they are a combination of aspects of professional knowledge that were identified as particularly important in the classroom in this research. The sectors might widen, or narrow, depending on the characteristics of specific mathematical topics, the students concerned, or lesson styles. While the sectors were separated within the diagram, the dynamic inter-relationship between the components is reflected in the meeting together at the centre of the wheel.

It is not expected that teachers would focus on all categories in every lesson, nor on any one category (spoke of wheel) giving it greater or lesser importance. Instead, recognition of each component should be evident over a period of time (for example, a complete unit of work). The wheel of professional knowledge could be utilised by school leaders, classroom teachers, and pre-service teachers. It might be used in staff meetings, as part of the attestation and appraisal process, or for teachers to refer to as a reminder of their professional knowledge in action.

The adapted PCK framework of Chick et al., (Figure 3.1) has been similarly adapted for researchers to use during classroom-based research. The new framework (Figure 10.2) was compiled for use by researchers when focussing on the professional knowledge of teachers, based on comparable attributes to those of The Wheel, with more detailed explanation of categories. Again, the categories were identified through this study as important components of professional knowledge, with some later found to be acknowledged by Shulman as missing components of his original PCK material.

The new framework for researchers is purposely called a Professional Knowledge Framework, as opposed to a PCK Framework (as used in this study). Many of the original research frameworks (Chick et al., 2006; Schoenfeld, 2013) were designed with Shulman's original PCK thinking at the forefront. Shulman has since acknowledged in his more recent writings (Shulman, 2015) that the idea of PCK was put forward at a particular point in time, as a result of the discussions that were happening around education as theory, practice, policy, and action. Shulman acknowledged that teachers are professionals who require a special type of professional knowledge unique to their situation and that original PCK thinking was devoid of affective characteristics, with insufficient attention given to broader social and cultural contexts. Hence, the new framework presented here is a professional knowledge framework based on findings of this research and Shulman's more recent contribution.

Professional Knowledge Category	Evident when the teacher...
A. Knowledge of Students	
*1. Prior Experiences	Acknowledges an individual student's prior personal experiences, in relation to mathematics learning.
*2. Cultural Knowledge	Demonstrates understanding of all students' identities and heritage, and the impact of these on mathematics contexts.
*3. Attitudes and Beliefs	Demonstrates an understanding of the impact of individual students' prior mathematical experiences on their current learning.
B. Curricular Knowledge	
*1. Achievement Objectives	Demonstrates understanding of expectations of curriculum objectives.
2. Goals for Learning	Uses a goal for students' learning (may or may not be related to specific mathematics content).
*3. Learning Progressions	Demonstrates understanding of learning progressions such as The Number Framework in teaching practice.
4. Knowledge of Assessment	Demonstrates summative and/or formative assessment practices.
5. Questioning - Supporting	Questions asked support students' comments, assist students in clarifying thoughts, ask for group support, and/or ask others to paraphrase explanations.
6. Questioning - Eliciting	Questions asked elicit different solution methods, encourage elaboration, and/or promote collaborative problem solving.
7. Questioning - Extending	Questions asked encourage generalizations, consider relationships between concepts, allow for reflection on multiple solutions methods, and/or provide challenge.
C. Content Knowledge	
1. Purpose of Content Knowledge	Demonstrates reasons for content being included in the curriculum or how it might be used.
2. Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept.
3. Mathematical Structure and Connections	Draws attention to mathematical structure and makes connections between concepts and topics
4. Methods of Solution	Demonstrates a method(s) for solving a mathematics problem.
5. Procedural Knowledge	Displays skills for solving mathematical problems.
*6. Conceptual Understanding	Exhibits a thorough conceptual understanding of identified aspects of mathematics.
*7. Number Sense	Demonstrates a meaning for numbers and their relationships.
D. Pedagogical Knowledge	
1. Classroom Techniques	Demonstrates positive classroom practices
2. Getting and Maintaining Student Focus	Demonstrates strategies for engaging students.
*3. Teaching Approaches	Demonstrates strategies or approaches for teaching a mathematical concept or skill.
4. Cognitive Demands of Task	Identifies aspects of the task that affect its complexity.
*5. Authentic Representations of Concepts	Demonstrates ways to model or illustrate a concept (can include materials or diagrams).
6. Knowledge of Resources	Uses resources available to support teaching.
7. Student Thinking	Addresses students' ways of thinking about a concept, or recognizes typical levels of understanding.
*8. Technology Inclusion	Demonstrates inclusion of a range of technology sources into classroom practice.

Figure 10.2 Framework of Professional Knowledge (Acknowledging: Chick et al., 2016; Shulman, 2015). Note; * denotes new categories, or changes to existing categories, on the framework (Figure 3.1) used in this research.

CHAPTER ELEVEN

CONCLUSIONS, LIMITATIONS, and IMPLICATIONS

11.1 Introduction

This chapter builds on the results and key themes discussed in relation to existing literature outlined in Chapter 10. As indicated in Chapter One, there has been limited research around the professional knowledge of New Zealand primary school teachers, and while some research has been carried out via anonymous questionnaires (Ward, 2006), which focused on the espoused views of teachers' professional knowledge, there has been little which included the investigation of actual classroom teaching practice.

This study, contributes to an awareness and understanding about the professional knowledge of primary school teachers that emerged from analysis of data from multiple sources. Knowledge of teaching and learning should inform the practice setting (the classroom) and addressing the theory-practice gap is essential in progressing teaching and learning in productive ways (Loughran, 2010). Therefore, while data were gathered from multiple sources, the research primarily focussed on teachers' professional knowledge in practice in the mathematics classroom (during the teaching of the multiplicative and proportional domains). The professional knowledge in practice was explored using a detailed framework (Figure 3.1) adapted from the framework of Chick et al., (Figure 2.1), which identified categories of knowledge associated with teaching practice. The professional knowledge evident in practice was equated to the teachers' espoused professional knowledge, and student learning based on assessment data, related to curriculum levels and expected levels of achievement. As a result of analysis across the three data sources (espoused professional knowledge, professional knowledge in practice, and student learning), themes emerged in relation to the professional knowledge of New Zealand primary school teachers (Chapter Ten).

11.2 Conclusions

Conclusions are presented based on the results, analysis, and discussions of the themes that emerged from the multiple-data sources presented in Chapters Four to Ten. They address the three specific questions, which framed this research:

- 1) What professional knowledge is evident when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?
- 2) What relationships are there between teachers' espoused professional knowledge, professional knowledge in practice, and student learning, when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?
- 3) How does the use of a framework assist in the investigation of teachers' professional knowledge in practice?

11.2.1 Professional Knowledge in Practice

This section presents conclusions drawn from the analysis of data gathered, when investigating Question 1: *What professional knowledge is evident when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?*

Teaching multiplication and fractions for understanding is not as simple as it may appear, and brings with it many complexities to be addressed within classroom lessons. The following concluding points emerged from the analysis of data and discussions in relation to the teachers' observed practice (Chapters 5, 6, 8, & 9).

The teachers used manipulatives to support the explanation and justification of a particular strategy, and emphasised to the students the importance of conceptual understanding. As the lessons moved from multiplication and division to fractions, opportunity was provided for all students to use the manipulatives. Relationships between the manipulatives and understanding of mathematical concepts and operations is developed when students use equipment to construct a model and develop its meaning (Flores, 2010; Nührenbörger & Steinbring, 2008), and this was evident when solving fractional word problems which were usually based around food. Manipulatives such as paper circles, paper strips, and fraction tiles, were used alongside real-life experiences that were meaningful to the students (Section 11.2.2). Manipulatives came to the fore in problems such as sharing three

cakes among eight people, when the students used circles and scissors to explore the different sized portions each person might receive. Manipulatives allowed both the teachers and the students to deconstruct ideas in the different problems, as they endeavoured to identify critical mathematical components that were fundamental for understanding particular concepts, such as equal sharing.

Different sub-categories of knowledge on the framework came to the fore, between the multiplication lessons (Table 5.3), and the fraction lessons (Table 6.3), depending on the, lesson structure, teaching style, context, the problem types, opportunities afforded to students for conversation, and use of manipulatives. When the lessons were teacher dominated, the problem samples were structured and deconstruction of content was prominent, with goals for learning featuring prominently. For example, Bob's initial multiplication lesson was teacher dominant and he could immediately respond to students' ideas because he was on-the-spot to do so. Whenever the lesson started to deviate from given scenarios, he refocussed students on the WALT to keep the lesson progressing as planned. This contrasted with the fraction lesson where the teachers were less directly involved in conversations and the students spent more time problem solving together. Less teacher involvement meant that deconstruction of content was less (e.g., Bob was not so directly involved so was unable to immediately respond to students' thoughts) and the goals for learning were not addressed as frequently (the students were more focussed on their lesson due to greater involvement).

This study found that sharing the WALT resulted in both positive and negative consequences. The teachers used the intended outcome of the lesson positively to refocus the students, when discussion became more social and less centred on mathematics (e.g., when Andy's students used pie scenarios and the discussion moved to different pies, then to parties and food in general). However, a negative consequence of the WALT was when it was used to direct the learning, rather than facilitate the learning, which meant that teachable moments were overlooked. For example, directed learning was evident when Bob discussed with the students, groupings of beads on the abacus and what mathematical expression the beads represented. A child gave a response (that there were two ways to solve the problem), and rather than capitalise on the response, Bob simply acknowledged it, refocused on the WALT, and moved on. A specific goal is intended to provide the

teacher with a clear outcome that guides the learning and selection of activities that take place and highlights the key mathematical ideas to develop the students' conceptual understanding (Smith & Stein, 2011). However, as this research found, the WALT should not dictate the learning.

The teachers did not capitalise on the wide range of assessment data available to them. Recognition of the current difficulties and misconceptions of the students could have provided ideal next steps of learning in the subsequent teaching sessions. Identification of gaps in understanding in previous learning is necessary, as they can have a compounding effect on low student achievement if not addressed (Allsopp et al., 2007). Closer analysis of individual and collective results on the initial tasks associated with this research, along with data from Numeracy Project Assessment (NumPA) tools and Progressive Achievement tests (PAT), could have assisted the teachers in recognising specific problem difficulties, as well as identifying the progressions of learning within Number Framework stages (Ministry of Education, 2008a, 2008f, 2008g).

Tasks selected by the teachers were generally low-level (Smith & Stein, 1998), relying on procedures, such as the sharing out of candles on cakes. These tasks meant that the teachers' questioning of students about the problem solving strategies utilised, was predominantly around low-level supportive questions (Fraivillig et al., 1999), such as "Why did you do that?" Questions that challenged and extended the students thinking were seldom presented. Insufficient opportunities presented by the teachers for students to extend and justify their thinking, became evident when the students discussed strategies together in their groups. Students tended to give systematic accounts of procedures and their peers seldom questioned these, or asked for further justification of steps taken.

11.2.2 Relationships between Espoused Professional Knowledge, Professional Knowledge in Practice, and Student Learning.

This section presents conclusions drawn from data analysis and themes identified, when investigating Question 2: *What relationships are there between teachers' espoused professional knowledge, teaching practice, and student learning, when teaching mathematics for numeracy in (a) the multiplicative domain and (b) the proportional domain?*

Two teaching strategies used by the teachers combined to provide a relationship between their teaching practice and an increase in student achievement. These were the simultaneous use of manipulatives, and the presentation of word problem examples using scenarios and contexts relevant to the students' real-life experiences. When the teachers created scenarios that appealed to the students, they enabled conceptual understanding by relating the mathematics to their real-life world (Anderson-Pence et al., 2014; Carpenter et al., 2015). Conceptual understanding was further enabled through the use of manipulatives (Stein & Bovalino, 2001). For example, when finding fractions of sets in class lessons, scenarios related to birthday parties and the number of candles on cakes, or cookies shared among class members, were solved using paper circles and counters. Not only were the names of students in the class used, but so were the names of teachers at the school – and even the researcher. The combination of real-life contexts using their own names in fraction problems in class, possibly contributed to an increase in the number of students who recalled the mathematics associated with finding fractions of sets, in assessment tasks. For example, in the initial assessment, 39 % were correct on Task 5 (find $\frac{1}{3}$ of 18), while in the final assessment 71% were correct on a similar Task 5 (find $\frac{1}{3}$ of 21).

There was an emphasis during teaching on the multiplication symbol as groups of, with 6×3 being interpreted as 6 groups of 3. Recognising how the structure in the number patterns that are formed relates to tables facts, has long-term significance in multiplicative understanding (Anghileri, 1999). Understanding the structure of the groups of idea is the basis for using known facts to solve unknown facts, as well as other structures of multiplication (Davis, 2008; Mulligan & Mitchelmore, 1997, 2009) and it appears that the multiplication models created in class (often using Unifix cubes), were not always connected to final assessment tasks. This became evident, when the students were asked to solve and draw a diagram for 3×6 (Task 2), and only 24% were able to do so correctly and when asked to use the 6×5 fact to solve 6×6 (Task 7) and less than half were able to do so.

There was a prominence within the observed lessons on teaching multiplication in isolation, without mention of links to division. Teaching multiplication and division provides some challenges for teachers when encouraging students to

develop conceptual understanding (Anghileri, 1999; Ball et al., 2001). The teachers acknowledged during learning conversations that division taught within the unit of work focussed on equal sharing problem types. However, working on multiplication and division facts together, with less emphasis on sharing procedures in division, helps develop number sense and an awareness of the links between operations (Anghileri, 1999, 2006). As with other research (Roche & Clarke, 2009, 2011; Roche et al., 2016), understanding the grouping, or sets of idea associated with quotitive division was not so well understood or addressed by the teachers. The minimal teaching time afforded division, with focus on partitive division, was reflected in the teachers' questionnaire responses (Section 4.2.2: Scenarios 2 & 3) and in the students' final assessments. Students' assessment results showed that only 10% of the students could draw and explain a partitive division problem (Section 7.1.2: Task 3, "You have 20 biscuits to put into 4 equal packets, how many biscuits will be in each packet?"). Furthermore, only 6% could explain and draw a quotitive division problem (Section 7.1.2: Task 4, "You have 12 biscuits to put into packets with 3 biscuits in each packet. How many packets can you make?").

The teachers' uncertainties of the specifics of what was expected at each curriculum level was reflected in the difficulties they had in identifying next learning steps for their students. During their NDP participation, all teachers had been introduced to the Number Framework, which is set out in stages of increasing complexity requiring greater understanding of number, as students progress through the stages (Bobis et al., 2005; Higgins & Parsons, 2011). A clearer understanding of the progressions could have assisted the teachers in identifying students' next steps of learning. The teachers had difficulty identifying where to next in the given scenarios on their written questionnaires and often left this section blank, while during teaching time the uncertainty of the subsequent progression of problem types and associated strategies, possibly contributed towards the teachers reluctance to utilise in-the-moment opportunities. There were times when the teachers may have diverted from planned lessons to discuss ideas that arose incidentally, but instead reminded the students of the current learning intention and left the idea for another time. This in-the-moment responsiveness, or ability to understand an issue from a learner's perspective

(Schoenfeld, 2011), also known as contingency knowledge (Rowland et al., 2009), is an important component of teachers' professional knowledge in classroom practice.

11.2.3 The use of Frameworks in Research

This section presents conclusions drawn from the analysis of data when investigating Question 3: *How does the use of a framework assist in identification of teachers' professional knowledge in practice?*

The subject-matter-specific professional knowledge which Shulman (1986) referred to as pedagogical content knowledge, links content knowledge and the practice of teaching and the use of a framework provided a systematic representation of categories, which provided the researcher with detailed data associated with teachers' professional knowledge in practice in the classroom. The framework used was built on one originally presented by Chick and colleagues (Figure 2.1), based on Shulman's original PCK categories of pedagogical knowledge, content knowledge, and pedagogical content knowledge. Chick et al., referred to their broad categories as Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context, to show the connections between content knowledge and the practice of teaching. Sub-constructs within each of the three broad categories, assisted the researcher in coding and analysing the data associated with the teachers' practice in detail, in order to gain insight into the attributes of classroom practice.

Identifying whether the framework used was fit for purpose for classroom observations.

In order to identify the professional knowledge of teachers during teaching practice, the Chick et al. framework (Figure 2.1) was adapted slightly for the analysis of data (Figure 3.1), as previously it had been used via questionnaires (Section 3.13). To assist in determining if the adapted Chick et al. framework could be considered fit for purpose in relation to this thesis, it was decided to compare the manner in which these data were coded to other methods of coding (Section 3.13). The deductive approach and adapted Chick et al. framework was compared to (i) a grounded, inductive approach (Glaser & Strauss, 1967), and (ii) the established model of Ball et al. (2008), which was already identified as suitable for classroom use.

(i) Comparing the analysis of data against the framework used, to a grounded theory approach:

Andy's first multiplication lesson and Bob's fraction lesson were used for this analysis. In order to make the framework comparison process valid and credible from a research perspective, a senior colleague (referred to as Jill) who was experienced in the grounded approach of data analysis, was given transcripts of the two lessons and asked to code them accordingly. After an initial coding of the lessons from a grounded approach, Jill identified a number of emerging themes and identified the contexts on which these were based. These contexts, along with other instances identified by the researcher, were identified within the lessons and traced back to categories within the framework (Table 11.1). In order to determine whether the framework used was fit for purpose a judgement was then made about how well the key idea behind each of Jill's themes was reflected in The Framework categories and descriptors (Figure 3.1).

a) *The use of manipulatives – sharing the purpose of using manipulatives with the students and drawing out the mathematics.* During the multiplication lesson Andy used animal strips and Unifix cubes, and in the fraction lesson Bob used paper circles representing birthday cakes and counters for candles. These manipulatives were used to draw out the mathematics and became evident within the Framework, when the teachers unpacked the cognitive demands of the tasks (Figure 3.1: A.4), or they deconstructed content (Figure 3.1: B.1). These categories showed a comparative relationship with the theme, along with the hands-on use of the manipulatives as the students explained their understanding of the concepts to each other (Figure 3.1: A.7). For example, using manipulatives alongside an explanation was important when Andy wanted the students to determine which of the constructed animal arrays (Figure 5.1) represented the expression three times four, and said, "Talk to the person next to you".

b) *Using students' responses – to assist in drawing out understanding of the mathematics.* This theme was traced back and clearly identified in the categories of student thinking (Figure 3.1: A.7) and student misconceptions (Figure 3.1: A.8). For example, when Andy's students were finding fractions of a set (e.g., $\frac{3}{4}$ of 12), he used the responses of students to determine the solution to the problem, and how it might be recorded as a mathematical expression.

Table 11.1

Comparing coding of the inductive, grounded approach to the deductive, framework approach.

Jill's Theme	Context	Framework Category	Fit for Purpose? Comparisons identified...
Use of manipulatives (sharing the purpose of using manipulatives with the students and drawing out the mathematics).	Animal Strips Paper Circles and Counters (for birthday cakes and candles)	A.4 Cognitive demands of task B.1 Deconstructing content A.7 Student thinking	The theme was clearly identified on the framework when Andy deconstructed the content of the problems (B.1), through the arrays formed by animal strips and cubes. They assisted him in explaining the meaning of the multiplication symbol. As the students discussed their constructions (A.7), they explained the difference between the array of 3x4 and 4x3.
Using students' responses (to assist in drawing out understanding of the mathematics)	Fractions as an operation (e.g., sharing out candles on the cake such as $\frac{3}{4}$ of 12) Commutativity in multiplication	A.7 Student thinking A.8 Student misconceptions	This theme came to the fore when, as the students shared their thinking (A.7) how they solved $\frac{3}{4}$ of 12 Andy supported them to explain their solution method to others. At times, he turned to other students to expand on a student's idea and/or explain their own.
Asking students to justify their thinking	Using animal strips (e.g., to determine 3 groups of 4)	A.7 Student thinking Questioning Techniques C.5 supporting C.6 eliciting C.7 extending	This theme was evident through utilising student thinking (A.7) and the added categories related to 'questioning techniques'. Depending on a student's thoughts and responses and how the following question was formed, the specific questioning category was determined.
Awareness of 'big ideas' in mathematics (to make suitable connections, or inappropriate connections)	Bob talked about using multiplication to solve problems in fractions and never mentioned the inverse (division). Andy talked about commutativity in multiplication (4x3 and 3x4 have the same answer but different representations). Looking at patterns , i.e. we are not doing algebra.	B.2 Mathematical structure and connections B.1 Deconstructing content to key components	This theme showed an awareness of 'big ideas' alongside missed opportunities to consolidate other big ideas. The category B.1 showed that when Andy consolidated understanding of the multiplication symbol, he looked at the structure of the animal arrays and noted that while 4x3 and 3x4 gave the same answer the formation was different. B.1 also indicated Andy later made an inappropriate connection between multiplication and algebra, and patterns.
Teacher coherence in elaborating learning goals	Bob stated a learning goal 'We are learning to use multiplication to find a fraction of a set'.	A.1 Purpose of content knowledge B.1 deconstructing content to key components.	This theme was identified in A.1 and B.1 showing the teachers' awareness of the purpose of the lesson. While Andy did not specifically state a WALT for this lesson, a strong theme of understanding the meaning of multiplication emerged and was coded frequently.
Connecting language and symbols to the mathematics in manipulatives	Andy used animal strips and arrays – and related the multiplication symbol to the arrays and emphasised 'groups of'. Bob used paper circles to emphasise multiplication and fractions of sets and covered sections for part-to whole comparisons.	A.3 cognitive demands of task A.8 student thinking - misconceptions B.1 deconstructing content to key components	This theme was evident when the teaching and learning was coded against A.3, A.8 and B.1. As Andy's students used their manipulatives (animal strips and Unifix cubes) correct language associated with the meaning of multiplication symbol and the use of the term 'groups of'. Bob used paper circles to connect the use of multiplication of fractions with 'groups of'.

Bob later gave students a part-to-whole problem (if 6 candles [counters] equal one quarter, how many are on the whole cake). A child explained the steps he used to solve the problem with the support of Bob, who then capitalised on the child's response to get the students to solve another, similar problem (if 7 candles [counters] are on one third of the cake, how many are on the whole cake?).

c) *Asking students to justify their thinking.* This theme was identified on the framework categories depending on how a student's thinking was utilised (Figure 3.1: A.7) and the following associated questioning techniques (Figure 3.1: C.5, C.6, C.7). How the teachers asked the students to justify their thoughts, determined whether it was part of the supporting (C.5), eliciting (C.6), or extending (C.7) section. For example, in Andy's class as the students discussed which of the presented array models (using animal strips) represented three times four. Andy supported the students in justifying their solution and said, "*When I wanted three times four, which of those two options do you think is correct?*" (Section 5.2.3: C.5). Bob promoted collaborative problem solving when he checked the students explanations of their actions, "*So can anyone tell me what that actually means?*" (Section 6.2.3: C.6). Andy extended the thinking of students when he asked them to justify their thoughts to others, "*Can you explain your theory to everyone else please?*" (Section 5.2.3: A.7).

d) *Awareness of big ideas in mathematics, to make connections, or inappropriate connections.* An example of this theme emerging against the framework was in Andy's on-going reference to a key idea in understanding commutativity of multiplication: while the multiplier and multiplicand are reversed in representation of the array, (e.g., 4×3 looks different on a model from 3×4), the product remains the same (they both equal 12). This along with other examples of understanding the multiplication symbol, was coded against deconstructing content to key components (Table 5.3: B.1) and mathematical structure and connections (Table 5.3: B.2). However, the theme Jill identified suggested there were times when big ideas were only partly mentioned and opportunities missed to explain them fully. An example of this, was when Bob mentioned using multiplication to find fractions of a set, and made no mention of the important relationship between fractions and division. The fact that Bob did

not mention division was not coded against a category, showing one instance when deductive coding might miss important ideas.

e) *Teacher coherence in elaborating learning goals.* In the two lessons Jill coded, Andy did not state a learning goal while early in his lesson Bob shared with his students, “We are learning to use multiplication to find a fraction of a set.” His WALT shared a lesson focus on using multiplication (which may be why he did not mention division in the big ideas theme) and examples of elaboration were coded against purpose of content knowledge (Table 5.3: A.1) and deconstructing content to key components (Table 5.3: B.1). At one stage, Bob mentioned, “*The most important thing about fractions is we have to know our times tables*” and encouraged the use of multiplication during problem solving e.g., finding $\frac{3}{4}$ of 12. While Andy did not mention a specific learning goal and therefore was not recognised in Jill’s thematic coding, his clear intention of ensuring the students understood the meaning of the multiplication symbol came through in categories on the framework.

f) *Connecting language and symbols to the mathematics in manipulatives.* This theme came to the fore in a number of the categories on the framework. For example, in the cognitive demands of the task category (Table 5.3: A.3) Andy emphasised the meaning of the multiplication symbol as times or groups of as problems were discussed. He began the lesson using animal strips to form arrays (Figure 5.1) and later used Unifix cubes for the students to construct arrays using singular units. Language and symbols were and again discussed alongside manipulatives when student misconceptions (Table 5.3: A.8 & Table 6.3: A.8) were identified. For example, when Andy noticed his students were still unsure of the meaning of the multiplication symbol, he recorded the expression in the modelling book, unpacked with them what it meant and then asked them to use their blocks to construct the correct array.

Overall, while the comparison between the grounded, inductive approach to identifying themes and the deductive approach of the Framework used, showed that the Framework was suitable for this research, it also highlighted that there are advantages and disadvantages associated with the different methods of analysis. As Glaser and Strauss (1967) emphasised, identification of themes from a critical

analysis of the data using a grounded approach meant that the researcher was open to ideas, which can sometimes be overlooked when allocating data to a predetermined set of categories (Section 11.3). This was evident in the “identification of big ideas” category. However, as Andy did specify a learning goal, Jill did not mention it in the contexts associated with her themes, but a clear lesson intention came through in the Framework categories. The Framework provided a list of categories, which enabled the researcher to acknowledge that certain aspect of PCK occurred in the classroom, which meant that it was fit for purpose for this research. However, had there been a focus on the degree to which PCK occurred and comparison between teachers made, then using a grounded approach to identify the quality of the teaching might have been more suitable.

(ii) Comparing the analysis of data against the Framework used, to an established, recognised framework (model) used in classroom research:

Ball introduced the phrase, “knowledge about mathematics” (Ball, 1990) and Ball and colleagues spent many years researching what they referred to as “the teaching of mathematics and the mathematics used in teaching” (Ball et al., 2008, p. 390). As a result of their research, Ball et al. developed a framework identifying domains of mathematical knowledge for teaching, to be used alongside teaching practice. The framework contained two broad sections Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK included Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialised Content Knowledge (SCK), while PCK included Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC) (Section 2.8 explains detail of categories).

After coding Bob’s fraction lessons using Ball et al.’s framework (Ball et al., 2008), a comparative analysis was made with coding used against the adapted Chick et al. framework (Figure 3.1). Research and associated literature, had suggested there may be similarities behind the purpose and positioning of the two frameworks, and the researcher wish to compare the recognised framework of Ball et al. which had been used for coding classroom practice, to the framework used in this research (which had not been used previously in the classroom). The comparison showed that all categories within the adapted framework of Chick et

al., could be placed within the broad categories of Ball et al.’s framework. Some of these are outlined on Table 11.2.

Table 11.2

Comparing two frameworks of mathematical knowledge for teaching

Ball et al.’s category	Chick et al.’s category	Context	Fit for purpose? Comparisons identified...
Knowledge of content and curriculum (knowing about mathematics and understanding curricular knowledge)	A.2 Curriculum knowledge B.1 Deconstructing content to key components	Sharing the learning intention, ‘WALT use multiplication to find a fraction of a set’. Teaching about whole-to-part and part-to whole alongside the use of manipulatives	The identified WALT and use of manipulatives alongside key ideas taught, directly related to ‘knowledge of content and curriculum’.
Knowledge of content and students (combines knowing about mathematics and knowing about students)	C.2 Getting and maintaining student focus	Using a student’s name in a word problem, e.g., Jock has 16 candles to put on the birthday cake....	Bob kept students focussed on the mathematics he was teaching by including them in scenarios presented for problem solving – a link to ‘knowledge of content and students’.
Specialised content knowledge (the mathematical knowledge and skill unique to teaching)	B.3 Methods of solution A.7 Student thinking	Bob recognised both multiplication and equal sharing were used to solve problems. He built on a student’s explanation, that if there were 9 candles on one third of the cake there would be 27 on the whole cake.	As a teacher, Bob noticed two different methods were used for finding fractions of a set and explored both methods. As students explained their ideas, he used these for future scenarios. These are examples of specialised content knowledge.
Horizon knowledge (shows how mathematics topics are connected over the span of mathematics included in the curriculum)	C.3 Goals for learning	Reminded the students they were using multiplication to find a fraction of a set’ and discussed further examples	Bob emphasised the importance of knowing the ‘times tables’ as unpacked in the multiplication lessons, for solving fractions of sets. This was a deliberated connection with the previous topic taught.
Common content knowledge (knowledge of a kind used in various settings, not necessarily confined to teaching)	B.4 Procedural knowledge	The students were asked how many candles would be on each piece of cake if there were 21 candles and 3 children at the party. Bob ‘told them’ to start by dividing the paper circle (cake) into thirds.	Bob capitalised on one of the children’s recent birthday and used candles and cakes as a context for scenarios. This real-life setting supported their learning and was related to the students’ common content knowledge.

There were times when both frameworks saw an overlap of categories (Section 3.13). For example, Bob asked, “*If three quarters of the cake had nine candles on it, how many candles are on the whole cake?*” One of the students gave an incorrect answer and on Ball et al.’s framework, Bob’s response could have been coded against “knowledge of content and students” (based on previous problems) or “specialized content knowledge” (based on the mathematical error). Similarly, on the Chick et al., framework there was an identified overlap between student thinking (Figure 3.1: A7), appropriate and detailed representation of concepts

(Figure 3.1: A5) and the need to deconstruct content to key components (Figure 3.1: B1). This reinforced the notion that the detailed categories of Chick et al., aligned to the broad categories of Ball et al., and were useful as a set of filters for looking at the teaching process and identifying the kind of knowledge that was being used in the classroom.

Future frameworks of knowledge

As noted in Chapter Ten, while there is a need to connect subject matter knowledge with pedagogical practice, some of the affective aspects and issues relating to broader social and cultural contexts of teaching practice (Shulman, 2015), were missing from the Framework (Figure 3.1). These aspects missing from the Framework included students' prior experiences with mathematics learning and the impact this has on current learning, along with students' cultural background and heritage and the impact of these identities on mathematical contexts. Therefore, the three categories of the framework used in this research became a foundation for the forming of a new framework for future research based on four key categories of professional knowledge, which included these contexts: knowledge of students, curricular knowledge, content knowledge, and pedagogical knowledge (Figure 10.2).

An issue that arose out of this research was the suitability of a framework for teachers to use when reflecting on their professional knowledge, compared to one for researchers to use when identifying the professional knowledge of teachers in action in their classrooms. A framework required for teachers as reflective practitioners has subtle differences from the detail required of subcategories of knowledge on a framework for researchers. Therefore, as a result of this research, a wheel of professional knowledge (Figure 10.1) was created for use by teachers, to sit alongside the professional knowledge framework (Figure 10.2) for use by researchers.

11.3 Limitations of the Study

While this research contributes to the field of teacher professional knowledge, like any research it had some limitations. Multiple-case study research was used to examine the classroom practice of teachers both within and across, four cases. The schools participated in this study as a result of convenience sampling.

Convenience sampling involves the selection of the most accessible subjects to the researcher (Marshall, 1996) and may be seen to provide an incomplete picture of general classroom teaching. However, both schools were involved in the sustainability phase of the NDP, which was part of a nation-wide professional development programme, and were characteristic of many other classes in New Zealand schools during this time.

The results of this research are based on a small number of schools, teachers, and students, to allow for manageable collection of data during the school year. Originally, it was decided to focus on teaching the multiplicative and proportional domains at Years 5 and 6, as longitudinal data at the time (Young-Loveridge, 2009a, 2010) showed that students were underachieving in these domains, at these levels. However, due to the convenience sampling of schools, limited classes were available at these levels at each school and the upper end of primary school, including Years 7 and 8 became the focus. Student absences on days the initial and final assessment tasks meant that participatory numbers varied. The variance in student numbers became particularly noticeable in the multiplicative analysis and comparisons, where initial data represented 103 students, while the final multiplication data was from 93 students. The variance in numbers was thus reflected in numbers and percentages, which made direct comparisons difficult. For example, 42 students correct on Task 7 of the initial multiplication assessment equalled 41%, while 42 correct on Task 7 of the final assessment, was 45%.

Two multiplication lessons (one for Anna) and one fraction lesson from each of the four teachers, provided eleven lessons which were transcribed and provided a large amount of data. The data of the each teachers' actions were coded and analysed individually, as well as being combined to identify consistencies, discrepancies, and trends, across identified categories of professional knowledge. An in-depth analysis of more lessons and/or more teachers would have provided a wider range of evidence of practice. However, as Bassey (1999) emphasised, generalisation is a matter of judgement and he preferred to use the term *relatability*. Bassey suggested that the merit of a case study is the extent to which the details are sufficient and appropriate for a teacher to relate to what is described in the case(s).

This research highlighted the complexities associated with teaching and classroom practice found in the research of Roche et al., (2016) noting that teachers cannot be expected to attend to and respond to everything that happens in every lesson. The research found that while the strengths and weaknesses of individual teachers professional knowledge in practice might be partly identifiable through the quantitative data (for example the number of times they utilised students' thinking), the frequency of occurrence recorded against a category only told part of what actually occurred during teaching. Frequency indicated patterns of performance and the increase or decrease in frequency, did not necessarily indicate improvement or waning in practice, but rather a change due to many factors, including the teaching-style implemented during the lesson. Instead, the qualitative data showed what happened in the classroom sessions, and provided the actuality of the impact of the teachers' knowledge on their students' learning, in varying situations.

The research also highlighted the complexities associated with data coding and analysis. While the framework used for the coding of data, was seen as fit for purpose for this research, comparison with another established framework and identifying themes using a grounded approach indicated that different methods of analysis might have highlighted different key findings. For example, the deductive approach when using the range categories on the given framework, allowed for the detailed analysis of a number of lessons. The data was then used to identify occurrences within individual lessons and trends across all of the lessons. However, the grounded approach allowed for interpretations and identification of phenomena, which might be missed when the researcher places findings against predetermined categories on an established framework.

Researcher Positioning

Consideration needs to be given to the influence of this research on the schools and the classrooms of those involved. Being a researcher and an adviser at the same time could have involved a conflict of interest and provided difficulties for the researcher, the teachers, and students participating in the research. However, the researcher was well aware of this ethical predicament and potential conflict of interest at all times. Prior to the commencement of the research, careful consideration was given to possible difficulties, which may arise due to the

different roles. Discussion had taken place with the principals, the teachers, and the students involved at both schools. The teachers were well aware of which lessons would be used for research purposes as they were reminded at the previous visit and were wearing microphones for audio-recording. The students were also informed at the start of each session. Audio-recording and video-recording might have been viewed as intrusive, although during discussion following each lesson the teachers said that after the first couple of minutes they forgot the devices were there.

One advantage of the dual role, was that it allowed the researcher to get to know the teachers and students well, as she visited their rooms regularly during the year of data gathering. The teachers participated in professional learning throughout the year that this study took place. The teachers commented that the frequent visits and participation in the research added a positive dimension to their professional learning, rather than detracting from it. The extra conversations and lesson analysis may have helped to strengthen their teaching practice in both the short term and the long-term. Participating in regular reflection and professional development is an important component of teacher's professional learning. The professional development may have contributed to improvement in student learning in the proportional domain that was taught towards the end of the year, as opposed to the students' minimal improvement in the earlier taught multiplicative domain.

11.4 Implications for Future Teaching and Further Research

This study is the first in New Zealand, which focuses on teachers' professional knowledge through multiple-data sources simultaneously: including observations of classroom practice, alongside espoused knowledge through questionnaires, student achievement data, learning conversations, and field notes. An implication of sharing the findings is that classroom teachers will be provided with a way of identifying strengths and gaps in teachers' professional knowledge. This could assist in their ongoing professional development, and improve teaching practice and student achievement.

Evidence from this study showed a potential relationship between teachers' professional knowledge in classroom practice and students' learning. It showed that although teachers may have taught a concept, for many students, learning and understanding does not occur within one or two lessons. Ensuring student understanding, had many repercussions for the teachers who strove for a balance between time taken and knowledge acquired, before moving to the next steps of learning. A suggestion for teachers is that some of the discrepancies between what occurs in classroom practice and the desired outcome may be overcome if they observe themselves in practice (via video-recording) and critique this both independently and/or collaboratively. The Wheel of Knowledge (Figure 10.1) designed for teachers, could provide a focus for reflection on their teaching practice. Reflecting on evidence of practice alongside the categories of knowledge on the wheel, would assist in identifying the teachers' strengths and provide suggestions for further learning and development.

Within this study, the complexities associated with the mathematical knowledge required for teaching (MKT) came to the fore. The relationship between the teachers' espoused professional knowledge and mathematical knowledge evident in their pedagogical practice became a noticeable factor. The teachers in this study had sufficient mathematical content knowledge for the class level they were teaching (as evident in their questionnaire responses). However, they required more support with the challenges related to the change from a procedurally-based orientation (the way they had been taught mathematics), to a conceptually-oriented approach to teaching. The teachers knew how to solve mathematical problems beyond the year levels they taught (they all had a minimum of bursary level mathematics), but they often struggled to support their students in unpacking strategies used and understanding mathematics behind the concepts.

This study found that the teaching of multiplication takes priority over division, as the teachers' MKT related to different division types was minimal. Associated with the teaching of division is the writing of effective word problems, as it is the way that the problem is structured, which dictates whether it is a quotitive or partitive problem. The teachers provided many real-life, word problem examples that the students related to during their teaching of fractions, although these tended to be partitive problems based on sharing out objects. More experience

with whole number quotitive division problems, could strengthen students' understanding fractions. More research is required in New Zealand schools into the teaching and understanding of division and the distinction between partitive and quotitive division word problems and the influence this has on the conceptual approach to teaching of fractions.

The findings of this study suggest that manipulatives helped to support students' conceptual understanding in classes at the upper end of primary school (Years 6 to 8). Having manipulatives available to all students assisted the students in generalising about concepts.

In terms of the teachers' professional knowledge, particular themes emerged from this study, which could become foci of future professional learning and development for teachers. Themes include: (i) Interpreting the requirements of the curriculum, alongside the writing of specific learning outcomes or Learning Intentions in relation to strand achievement objectives. This may include the teacher and students co-constructing a learning intention based on an identified need. (ii) Recognising when, and how, to make connections within and between concepts taught, including connections across strands. Connections also need to be made to authentic life experiences of the students, in order to make the mathematics learning more meaningful. (iii) Knowing how to judge an appropriate length of time spent on content and strategies taught at a particular stage, before moving to the next learning steps. In conjunction with the time issue is, when might unpredictable opportunities presented as teachable moments be addressed, and when are they better left for later. (iv) Analysing available assessment data, so that it is used constructively to personalise the teaching and learning for the students. (v) Learning to advance the mathematical thinking of students by asking questions which require high-order thinking and encouraging students to justify their solutions and question each other in an open friendly manner during discussions.

11.5 Final Words

The intention of this research was to investigate the professional knowledge of New Zealand primary school teachers, and the contribution this knowledge makes to student learning, when teaching mathematics for numeracy in the multiplicative

and proportional domains. This research is unique in that while most of the data was gathered through the privileged opportunity to observe teachers in their day-to-day classroom practice, the research was strengthened by the addition of teachers' espoused professional knowledge through written questionnaires, student assessment data, field notes, and learning conversations. Acquiring data from multiple sources highlighted the relationships between espoused professional knowledge, professional knowledge in practice, and student learning. The use of a detailed framework assisted in carrying out analysis of such a complex investigation. However, as with much research in education, 'the job is never complete' and out of this research comes much thinking and discussion, to inform future research.

The findings show that gaining and improving professional knowledge is a complex, dynamic process, which is person-specific, context-specific, and changes from lesson-to-lesson. Despite the theoretical distinctions among the different knowledge types (content [subject matter] knowledge, general pedagogical knowledge, pedagogical content knowledge, knowledge of the curriculum, knowledge of the learners), the research showed that they are also interwoven and interdependent. The differences in professional knowledge from teacher to teacher and from lesson to lesson, depended on many factors (for example: lesson structure, teaching style, context, tasks, word problems, use of manipulatives, and opportunities afforded to students for conversation), and reflected in students' learning. Such variability may also be impacted by the relationship between teachers' conceptual knowledge and procedural knowledge.

There is a need to pay attention to the requirements of teachers' professional knowledge, while at the same time, to better appreciate their professional practice. As stated at the beginning of this thesis, teacher professional knowledge is often not viewed as specialist knowledge in the same way that practitioner knowledge is valued in other professions (Loughran, 2010). Conceptual understanding and the shift in practice from the procedural manner in which teachers may have been taught, can provide challenges. When knowledge of subject matter and teaching are combined into practice, teachers' professional knowledge becomes noticeable and significant. "This is something which needs to be understood, more highly prized, and specifically valued, within the profession" (Loughran, 2010, p. 56).

This study supports Loughran's claim that teachers require a specialised professional knowledge to provide classroom programmes that allow for a balance between the conceptual understandings required for mathematics problem solving and the procedural knowledge required to do mathematics. Recent advances in technology, along with associated societal expectations and requirements, have meant that the need for people to transfer their mathematics understandings to everyday life has become greater. While it is acknowledged that the meaning of number is mathematical, the significance of number is functional (Lambdin & Walcott, 2007). Therefore, all (primary school) teachers should see themselves as teachers of 'mathematics for numeracy' if they are to provide their students with a disposition and a confidence to understand mathematics in order to use it effectively in their current daily lives, as well as beyond schooling and into the years ahead.

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APPENDICES

Appendix A: Principal Information and Consent

My name is Judith Mills and I am currently doing doctoral research focusing on the role of teachers' knowledge of mathematics for teaching. I recently visited you and discussed the possibility of your school being involved in my research. I am now formally inviting your school to participate in my research by allowing me to use information gathered from observations and discussions taking place in the classroom and/or at staff meetings, and student data that your teachers, have gathered. You, and the teachers from your school, are also invited to complete the enclosed questionnaire.

As noted in our discussions, all teachers will be invited to complete a questionnaire. In addition to this, up to two teachers will be invited to participate in in-depth case-studies. This will involve a maximum of one extra classroom visit per term and extra support for these teachers around their mathematics knowledge for teaching and classroom practice. It is difficult to anticipate at this point what this may involve. However, I will keep you informed of this process at frequent intervals throughout the year.

All project data will be stored in a secure location and on the laptop of myself (the researcher) which is only accessible via a protected password. The data will be used only for the purpose of this research and any publications and conference presentations arising from this research. The name of the school and participants will remain confidential to the researcher, and pseudonyms will be used in any written material and presentations to maintain anonymity.

You have the following rights in response to my request for your school to participate in the study:

- decline to participate
- decline to answer any particular question
- ask any questions about the study at any time
- provide information knowing that anonymity will be maintained in any publication
- be given access to a summary of the research findings

If you have any further questions about this project, feel free to contact me (judith@waikato.ac.nz or phone (07) 8384466 Extn 7240; mobile 027 4937224), or my supervisor, Jenny Young-Loveridge (jenny.yl@waikato.ac.nz or phone (07) 8384353 (direct) at any time.

Principal Consent:

It has been explained to me that data gathered (as outlined above) and completion of the questionnaire is for the purpose of a doctoral study. I know that, all information gathered, as well as my name, the name of the school, and the names of the teachers and pupils, will remain confidential.

I give/do not give permission for the research study to be conducted in my school.

I give/do not give permission for data gathered from classroom and staff meeting discussions and student data gathered to be used in the research.

I am willing/not willing to complete the questionnaire as part of the research

Signed:Name:

School: Date:

Appendix B: Teacher Information Sheet (All Teachers) and Consent

My name is Judith Mills and I am currently doing Doctoral research focusing on the role of teachers' knowledge of mathematics for teaching. I am inviting you to participate in my research by allowing me to use information gathered from observations and discussions taking place in the classroom and/or at staff meetings, and student data that you, the teacher, have gathered. I will be sure that you are fully aware at all times of any information that is collected and used as part of my research. This will ensure that my role as an adviser is separated out from my role as a researcher.

You are also invited to complete the enclosed questionnaire

All project data will be stored in a secure location and on the laptop of myself (the researcher) which is only accessible via a protected password. The data will be used only for the purpose of this research and any publications and conference presentations arising from this research. The name of the school and participants will remain confidential to the researcher, and pseudonyms will be used in any written material and presentations to maintain anonymity.

You have the following rights in response to my request for you to participate in the study:

- decline to participate
- decline to answer any particular question
- ask any questions about the study at any time
- provide information knowing that anonymity will be maintained in any publication
- be given access to a summary of the research findings

If you have any further questions about this project, feel free to contact me (judith@waikato.ac.nz or phone (07) 8384466 Extn7240.) or my supervisor, Jenny Young-Loveridge (jenny.yl@waikato.ac.nz or phone (07) 8384353 (direct) at any time.

Teacher consent:

It has been explained to me that data gathered (as outlined above) and completion of the questionnaire is for the purpose of a doctoral study. I know that everything I write and say will be kept private, and that my name will not be used in any report or presentation.

I give/do not give permission for data gathered from classroom and staff meeting discussions and student data gathered to be used in the research.

I am willing/not willing to complete the questionnaire as part of the research

Signed:Name:

School: Room:

Date:.....

Years of Teaching: Years of Teaching using the Numeracy Project:

Additional Information and Consent Form for Case-study teachers

I have read the information sheet and had the details of the study explained to me. It has been explained to me that this research is for the purpose of a Doctoral study.

I have agreed to participate in this study under the conditions set out in the information sheet for all teachers. I understand that in addition to these conditions I will also be involved in classroom observations that may involve audio/video recording, one-to-one discussions with the researcher, and professional learning related to subject matter knowledge, if/when required.

I know that everything I write and say will be kept private, and that my name will not be used in any report or presentation.

Signed: Name:

School: Room:

Date:

Appendix C: Questionnaire for Teachers

Name:

Section A. Teachers' Views about Mathematics

For each of the statements below, please tick the box that best represents what you think:

SA	A	U	D	SD
Strongly Agree	Agree	Undecided; neither agree nor disagree	Disagree	Strongly Disagree

#	Mathematics	SA	A	U	D	SD
1.	There is always a best way to do a maths problem					
2.	Maths is about searching for patterns					
3.	In maths, things are either right or wrong					
4.	It is important for students to be able to work out their answers quickly					
5.	It is important for students get the answer right					

Comments:

A	O	S	H	N
Always	Often	Sometimes	Hardly ever	Never

#	<i>Classroom Mathematics Practices</i>	A	O	S	H	N
1.	I encourage students to explain their thinking to each other					
2.	I encourage students to question the strategies of others					
3.	I encourage students to justify their choice of strategy and their thinking to others					
4.	I encourage students to work together on solving problems					
5.	I encourage students to include in their maths books drawings, diagrams, or other recording methods which represent their thinking					

Comments:

Section B. Questions for Teachers

1. *Without calculating the exact answer, circle the best estimate for:* **$29 \times 0.98 =$**
 - A. more than 29
 - B. less than 29
 - C. impossible to tell without working it out

2. How many different decimals are there between **1.52 and 1.53**?
 - A. None
 - B. One. What is it? _____
 - C. A few. Give two. _____ and _____
 - D. Many. Give two. _____ and _____

3. Which is the largest number?
 - A. $29 + 0.8$
 - B. 29×0.8
 - C. $29 \div 0.8$
 - D. $29 - 0.8$

4. *Without calculating the exact answer, circle the best estimate for:* **$54 \div 0.09 =$**
 - A. much less than 54
 - B. a little less than 54
 - C. a little more than 54
 - D. much more than 54

5. 0.5×840 is the same as:
 - A. $840 \div 2$
 - B. 5×840
 - C. 5×8400
 - D. $840 \div 5$
 - E. 0.50×84

6. Maia had \$426 and spent 90% of the money on clothes. *Without calculating the exact answer, circle the best estimate for how much she spent.*
 - A. slightly less than \$426
 - B. much less than \$426
 - C. slightly more than \$426
 - D. much more than \$426
 - E. impossible to tell without calculating

7. A student increased his exam score from 40 to 50. What percentage increase was this?
 - A. 10%
 - B. 20%
 - C. 25%
 - D. 50%
 - E. 90%
 - F. 100%

8. *Without calculating the exact answer, circle the best estimate for:* **$45 \times 105 =$**
 - A. 4000
 - B. 4600
 - C. 5200
 - D. 47250

Section C. Scenarios for Teachers

1. Jon was given the following problem:

What is $11 \times 99 =$

Jon took one away from 11 and added one to 99; he then multiplied 10 by 100 to get an answer of 1000.

- a) What action would you take next with Jon?
- b) What is the answer? Draw a diagram and explain how you would solve the problem?

2. Mere was given the following problem:

Hera owns a factory that makes tricycles. Each tricycle needs 3 wheels. She has 516 wheels. Will all the wheels be used to make tricycles, or will there be some wheels left over?

Mere added the digits together ($5+1+6=12$); she knew that the number was not divisible by 9 because 9 doesn't go into 12 evenly, and concluded that it was not divisible by three, so there would be some wheels leftover.

- a) What action would you take next with Mere?
- b) What is the answer? Draw a diagram and/or explain how you would solve the problem

3. Rob was given the following problem:

*Ana was packing chocolate peanuts into bags for the school fair. She decided to put **14** chocolate peanuts into each bag. How many bags would she get from **56** chocolate peanuts?*

Rob subtracted 14 from 56 to get 42; he then subtracted 14 three more times. Rob worked out that he could subtract 14 from 56 four times so Ana must get four bags of 14 from 56 peanuts.

a) What action would you take next with Rob?

b) What is the answer? Draw a diagram and explain how **you** would solve the problem?

4. Jenny was given the following problem:

$$1.45 + 0.9 =$$

Jenny calculated the answer by adding $45 + 9 = 54$, so the answer is 1.54.

a) What action would you take next with Jenny?

b) What is the answer? Draw a diagram and explain how **you** would solve the problem?

5. Pete was given the following problem:

Tama and Karen buy two pizzas. Tama eats $\frac{3}{4}$ of one pizza while Karen eats $\frac{7}{8}$ of the other one. How much pizza do they eat altogether?

Pete converted $\frac{3}{4}$ to $\frac{6}{8}$ so he had $\frac{6}{8} + \frac{7}{8}$; he then added 6 and 7 to get 13, and 8 and 8 to get 16, and gave the answer as $\frac{13}{16}$.

a) What action would you take next with Pete?

b) What is the answer? Draw a diagram and explain how **you** would solve the problem?

6. Jo was given the following problem:

There was $\frac{3}{4}$ of a birthday cake left over after the party. Sarah took $\frac{1}{3}$ of the leftover cake home for her brother. How much cake did Sarah take home to her brother?

You hear Jo say “one third of three quarters; that’s the same as one third times three-quarters...”

a) What action would you take next with Jo?

c) What is the answer? Draw a diagram and explain how **you** would solve the problem?

Appendix D: Parent/Caregiver Information and Consent

My name is Judith Mills and I am currently doing Doctoral research focusing on mathematics teaching. Your child's teacher has indicated a willingness to be included in my research and I am now writing to request your permission for your child to be included in this research also. I will be observing mathematics lessons in your child's class from time to time throughout the year, and using the data gathered from the observations as part of my research. Your child's involvement will be no more than that which occurs in normal daily classroom mathematics lessons, as my main focus is on the teaching that is taking place.

I would also like permission to audio- or video-record lessons as part of the teacher's classroom practice. In addition, I may wish to make copies of, or photograph, your child's written work as evidence of the learning taking place.

All project data will be stored in a secure location and on the laptop of myself (the researcher) which is only accessible via a protected password. The data will only be used for the purpose of this research and any publications and conference presentations arising from this research. The name of the school and participants will remain confidential to the researcher and pseudonyms will be used in any written material and presentation to maintain anonymity.

You have the following rights in response to my request for your child to participate in the study:

- decline your child's participation
- withdraw your child from the study at any point up until data is processed
- withdraw any video or audio recording of your child, photographs taken and any copies of their work, up until the time data is processed.
- you may ask any questions about the study at any time
- your child provides information knowing that anonymity will be maintained in any publication
- be given access to a summary of the research findings
- decline your child being audio-/video-recorded
- decline to allow your child's work being photocopied

If you have any further questions about this project feel free to contact me at any time at: judith@waikato.ac.nz or phone (07) 8384466 Extn7240 or my supervisor, Jenny Young-Loveridge (jenny.yl@waikato.ac.nz or phone (07) 8384353 (direct) at any time.

Consent Form: Parents/Caregivers of Student Participants

I have read the information sheet and have had the details of the study explained to me.

It has been explained to me that this research is for the purpose of a Doctoral research study.

I agree/do not agree for my child to participate in this study under the conditions set out in the information sheet.

I agree/do not agree to my child being audio-recorded.

I agree/do not agree to my child being video-recorded.

I agree/do not agree to my child's work being photo-copied.

I know that everything written and said will remain anonymous, and that my child's name will not be used in any report, or presentation.

Signed: Name:

School: Room:

Date:

Appendix E: Student Information and Consent

My name is Judith Mills and I am currently doing research focusing on mathematics in the classroom.

As part of this research I will need to visit your classroom and watch mathematics lessons. Your teacher has agreed to take part in my research and now I am writing to ask your permission also. I am interested to know what happens in mathematics lessons in your classroom.

I would like permission to audio-record or video-record lessons sometimes. In addition, I may wish to make copies, or take photos, of your written work to help with my observations.

All project data will be stored in a secure place. The data will only be used for the purpose of this research and any publications and conference presentations arising from this research. The name of the school and your name will remain confidential to me and my supervisor.

You have the following rights in response to my request for you to participate in the study:

- Choose not to participate
- ask questions about the study at any time
- provide information knowing that your name will not be used when the study is shared with other people
- ask for the audio-/video-recording to be turned off at any time
- not to allow your work to be photo-copied or photographed

If you have any further questions about this project, feel free to contact me at any time at: judith@waikato.ac.nz or phone (07) 8384466 Extn7240 or ask your teacher.

Consent Form: Student Participants

I have read the information sheet and have had the details of the study explained to me.

It has been explained to me that this research is for the purpose of a Doctoral study. I agree to participate in this study under the conditions set out in the information sheet.

I agree/do not agree to being audio-recorded.

I agree/do not agree to being video-recorded.

I agree/do not agree to my work being photo-copied or photographed.

I know that everything written and said will remain anonymous, and that my name will not be used in any report or presentation.

Signed: Name:

School: Room:

Date:

Appendix F: Statement of Confidentiality for Transcribers

Please read this document carefully, and fill in the required information in the spaces provided. Indicate your consent by signing and dating in the spaces provided at the end of the document.

I _____ (PRINT NAME)
of _____
_____ (ADDRESS)
_____ (TELEPHONE NUMBER)

consent to transcribing from audiotape to a computer disk the contents of interviews supplied to me by Judith Mills. I consent that all the information that I hear on the tapes will remain confidential and I will not discuss the contents with anyone. I also consent to having my name released to the people whose audio-recorded interviews I am transcribing, if requested.

_____ (Signature)

_____ (Date)