Separation equations for 2D superintegrable systems on constant curvature spaces

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Abstract

Second-order conformal quantum superintegrable systems in 2 dimensions are Laplace equations on a manifold with an added scalar potential and 3 independent 2nd order conformal symmetry operators. They encode all the information about 2D Helmholtz or time-independent Schrödinger superintegrable systems in an efficient manner: Each of these systems admits a quadratic symmetry algebra (not usually a Lie algebra) and is multiseparable. We study the separation equations for the systems as a family rather than separate cases. We show that the separation equations comprise all of the various types of hypergeometric and Heun equations in full generality. In particular, they yield all of the 1D Schrödinger exactly solvable (ES) and quasi-exactly solvable (QES) systems related to the Heun operator. We focus on complex constant curvature spaces and show explicitly that there are 8 pairs of Laplace separation types and these types account for all separable coordinates on the 20 flat space and 9 2-sphere Helmholtz superintegrable systems, including those for the constant potential case. The different systems are related by Stäckel transforms, by the symmetry algebras and by Böcher contractions of the conformal algebra $so(4,\mathbb{C})$ to itself, which enables all systems to be derived from a single one: the generic potential on the complex 2-sphere. This approach facilitates a unified view of special function theory, incorporating hypergeometric and Heun functions in full generality.

1 Introduction

We show how Laplace superintegrable systems theory unifies and simplifies the theory of the special functions of mathematical physics, hypergeometric and Heun equations, and exactly solvable and quasi-exactly solvable systems, through the study of the family of separation equations for these systems. Quantum superintegrable systems are those with maximal symmetry, which permits their explicit solution. We consider here one of the simplest classes of such systems: 2nd order superintegrable systems in 2 complex (or real) variables. This is an integrable Hamiltonian system on an 2-dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_2 + V,$$

that admits 3 algebraically independent 2nd order partial differential operators L_1, L_2, H commuting with H, the maximum possible, [1],

$$[H, L_j] = 0, \qquad j = 1, 2.$$

Here $[A, B] = AB - BA$ is the operator commutator. Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically. The 2nd order 2D systems have been classified. There are 37 nondegenerate (3 linear parameter potential) systems, on a variety of manifolds, and 21 degenerate (1 parameter potential) systems. The generating symmetries of every such system form a quadratic algebra [1]. For simplicity, here we consider only the systems on flat space and the 2-sphere (listed in Appendix A); the treatment for other spaces is similar.

The fundamental system is quantum S9, the generic potential on the complex 2-sphere, $s_1^2 + s_2^2 + s_3^2 = 1$. This system has generating symmetries

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \tag{1}$$

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2},\ L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \tag{2}$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and cyclic permutations. The algebra is given by

$$\begin{aligned}
[L_i, R] &= 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),\\
R^2 &= \tfrac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2\\
&\quad + \tfrac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \tfrac{1}{3}(16 + 176a_1)L_1\\
&\quad + \tfrac{1}{3}(16 + 176a_2)L_2 + \tfrac{1}{3}(16 + 176a_3)L_3 + \tfrac{32}{3}(a_1 + a_2 + a_3)\\
&\quad + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3.
\end{aligned} \tag{3}$$

Here, $R = [L_1, L_2]$ and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$, and to simplify the structure equations we set $L_3 = H - L_1 - L_2 - a_1 - a_2 - a_3$.

The Higgs oscillator (Quantum $S3$) is the fundamental degenerate system. It is the same as $S9$ with $a_1 = a_2 = 0$, $a_3 = a$ but admits additional symmetry. The symmetry algebra is generated by

$$X = J_3, \qquad L_1 = J_1^2 + \frac{as_2^2}{s_3^2}, \qquad L_2 = \frac{1}{2}(J_1J_2 + J_2J_1) - \frac{as_1s_2}{s_3^2}.$$

The structure relations for the algebra are given by $R = [L_1, L_2]$ and

$$\begin{array}{l} [L_1, X] = 2L_2, \qquad [L_2, X] = -X^2 - 2L_1 + H - a, \\ [L_1, L_2] = -(L_1X + XL_1) - (\frac{1}{2} + 2a)X, \\ 0 = \{L_1, X^2\} + 2L_1^2 + 2L_2^2 - 2L_1H + \frac{5+4a}{2}X^2 - 2aL_1 - a. \end{array} \tag{4}$$

All these systems can be treated more conveniently as Laplace equations, [2, 3]. Since every 2D manifold is conformally flat, there always exist "Cartesian-like" coordinates x, y such that $H = \frac{1}{\lambda(x,y)}(\partial_x^2 + \partial_y^2) + V(x, y)$. Thus the Helmholtz equation $H\Psi = E\Psi$ on some conformally flat space is equivalent to the Laplace equation (with potential)

$$(\partial_x^2 + \partial_y^2 + \tilde{V}(x, y))\Psi = 0 \tag{5}$$

on flat space, where $\tilde{V} = \lambda(V - E)$, so the eigenvalue E has been incorporated as a parameter in the new potential.

More generally, we consider Laplace systems of the form

$$H\,\Psi(\mathbf{x}) \;\equiv\; (\,\Delta_2 + V(\mathbf{x})\,)\,\Psi(\mathbf{x}) \;=\; 0\;. \tag{6}$$

Here Δ_2 is the Laplace-Beltrami operator on a real or complex $2D$ Riemannian or pseudo-Riemannian manifold and V is a non-zero scalar potential. All variables can be complex, except when we impose constraints such as square integrability. A conformal symmetry of equation (6) is a partial differential operator L in the variables $\mathbf{x} = (x_1, x_2)$ such that $[L, H] \equiv LH - HL = R_L H$ for some differential operator R_L. A conformal symmetry maps any solution Ψ of (6) to another solution. Two conformal symmetries L, L' are identified if $L = L' + S\,H$ for some differential operator S, since they agree on the solution space of (6). (For brevity we will say that $L = L'$, mod (H) and that L is a conformal symmetry if $[L, H] = 0$, mod (H).) The system is *conformally superintegrable* if there exist three algebraically independent conformal symmetries, L_1, L_2, L_3 with $L_3 = H$. It is second order conformally superintegrable if L_2 can be chosen to be a 2nd order differential operator, and L_1 of at most 2nd order. (If the system admits symmetries such that L_1, L_2 can be chosen as 1st order, we say it is 1st order conformally superintegrable).

An important fact is that the mapping of a Helmholtz superintegrable system $H\Psi = E\Psi$ to the Laplace equation (5) preserves superintegrability,

i. e., system (5) is conformally superintegrable, [2, 3]. Further, Suppose we have a second order conformal superintegrable system

$$H = \partial_{xx} + \partial_{yy} + V(x, y) = 0, \quad H = H_0 + V$$

where $V(x, y) = W(x, y) - E\,U(x, y)$ for arbitrary parameter E. The potential U defines a *conformal Stäckel transform* to the (Helmholtz) system

$$\tilde{H}\Psi = E\Psi, \quad \tilde{H} = \frac{1}{U}(\partial_{xx} + \partial_{yy}) + \tilde{V}$$

where $\tilde{V} = \frac{W}{U}$. and this Helmholtz system is superintegrable, [2, 3]. The Laplace system on flat space has metric $ds^2 = dx^2 + dy^2$, and volume element $dx\ dy$, whereas the Stäckel transformed system has metric $ds^2 = U(dx^2 + dy^2)$ and volume element $U\ dx\ dy$. There is a similar definition of Stäckel transforms of Helmholtz superintegrable systems $H\Psi = E\Psi$ which take superintegrable systems to superintegrable systems, essentially preserving the quadratic algebra structure. Thus any second order conformal Laplace superintegrable system admitting a nonconstant potential U can be Stäckel transformed to a Helmholtz superintegrable system.

A crucial observation here is that each equivalence class of Laplace superintegrable systems is multiseparable, [1, 5, 6, 7], and that the separable coordinates are invariant under Stäckel tranforms. Thus all Helmholtz systems in an equivalence class are separable in *exactly the same coordinates*. The separable coordinates may look different when subjected to a Stäckel transform but they are unchanged. The equivalence classes of Laplace nondegenerate and degenerate superintegrable systems with functionally independent generators are listed in tables 1 and 2. For each Laplace system (given in Cartesian coordinates x, y) we list the possible separable orthogonal coordinates and the constant curvature Helmholtz systems that arise from it via Stäckel transforms. To identify the relevant Stäckel tranforms we use the notation $\mathbf{a} = (a_1, a_2, a_3, a_4)$ to represent the nondegenerate Laplace potential $V = \sum_i a_i V^{(i)}$ and $\mathbf{a} = (a_1, a_2)$ to represent the degenerate Laplace potential $V = a_1 V^{(1)} + a_2 V^{(2)}$. For a fixed choice of parameters $\mathbf{b} = (b_1, b_2, b_3, b_4)$, $\mathbf{b} = (b_1, b_2)$, respectively, each Stäckel transform is described by $\frac{1}{V(\mathbf{b})}\left(\partial_x^2 + \partial_y^2 + \frac{V(\mathbf{a})}{V(\mathbf{b})}\right) = 0$. In the tables we list the possible transforms that correspond to constant curvature Helmholtz systems.

2 Bôcher contractions

All Laplace conformally superintegrable systems can be obtained as limits of the basic system [1111], [2, 3]. The conformal symmetry algebra of the underlying flat space free Laplace equation is $so(4, \mathbb{C})$, and these limits are described by Lie algebra contractions of this conformal algebra to

Laplace Systems	Non-degenerate potentials: $\partial_x^2+\partial_y^2+V(x,\,y)$
[1111] Laplace separable coordinates: Helmholtz superintegrable systems:	$V(x,y)=\frac{a_1}{x^2}+\frac{a_2}{y^2}+\frac{4\,a_3}{(x^2+y^2-1)^2}-\frac{4\,a_4}{(x^2+y^2+1)^2}$ $a)$ spherical: $x=\frac{\sin(\theta)\,\cos(\phi)}{1+\cos(\theta)},\ y=\frac{\sin(\theta)\,\sin(\phi)}{1+\cos(\theta)}$ $b)$ ellipsoidal: $x^2=\frac{(cu-1)(cv-1)}{(1-c)(1+\sqrt{cuv})^2},\ y=\frac{c(u-1)(v-1)}{(c-1)(1+\sqrt{cuv})^2},\ c\neq 0,1,$ $S9$: $(1,0,0,0),(0,1,0,0,),(0,0,1,0),(0,0,0,1)$ $S8$: $(1,1,1,1),(0,1,0,1),(1,0,1,0),(0,1,1,0),(1,0,0,1)$ $S7$: $(1,1,0,0),(0,0,1,1)$
[211] Laplace separable coordinates: Helmholtz superintegrable systems:	$V(x,y)=\frac{a_1}{x^2}+\frac{a_2}{y^2}-a_3\,(x^2+y^2)+a_4$ $c)$ Cartesian: $x,y,$ $d)$ polar: $x=r\cos(\theta),\ y=r\sin(\theta),$ $e)$ elliptic: $x=c\sqrt{(u-1)(v-1)},\ y=c\sqrt{-uv},$ $S4$: $(1,1,0,0)$ $S2$: $(1,0,0,0),(0,1,0,0)$ $E1$: $(0,0,0,1)$ $E16$: $(0,0,1,0)$
[22] Laplace separable coordinates: Helmholtz superintegrable systems:	$V(x,y)=\frac{a_1}{(x+i\,y)^2}+\frac{a_2\,(x-i\,y)}{(x+i\,y)^3}+a_3-a_4\,(x^2+y^2)$ $d)$ polar $f)$ hyperbolic: $x=\frac{u^2+v^2+u^2v^2}{2uv},\ y=\frac{i(u^2+v^2-u^2v^2)}{2uv}$ $E7$: $(1,0,a_3,0)$ $E8$: $(0,0,1,0)$ $E17$: $(0,0,0,1)$ $E19$: $(0,1,0,a_4)$
[31] Laplace separable coordinates: Helmholtz superintegrable systems:	$V(x,y)=a_1-a_2\,x+a_3\,(4\,x^2+y^2)+\frac{a_4}{y^2}$ $c)$ Cartesian $g)$ parabolic: $x=\xi^2-\eta^2,\ y=2\xi\eta,$ $S1$ $(0,0,0,1)$ $E2$: $(1,0,0,0)$
[4] Laplace separable coordinates: Helmholtz superintegrable systems:	$V(x,y)=a_1-a_2\,(x+i\,y)+a_3\,(3(x+i\,y)^2+2(x-i\,y))-a_4\,(4(x^2+y^2)+2(x+i\,y)^3)$ $h)$ semi-hyperbolic: $x=-(w-u)^2+i(w+u),\ y=-i(w-u)^2+(w+u),$ $E9$: $(0,1,0,0)$ $E10$: $(1,a_2,0,0)$
[0] Laplace separable coordinates: Helmholtz superintegrable systems:	$V(x,y)=a_1-(a_2\,x+a_3\,y)+a_4\,(x^2+y^2)$ $c)$ Cartesian $E3'$: $(1,0,0,0)$ $E11$: $(a_1,1,\pm i,0)$ $E20$: $(\frac{a_2^2+a_3^2}{4},a_2,a_3,1)$

Table 1: Four-parameter potentials for Laplace systems. Each of the Helmholtz non-degenerate constant curvature superintegrable (i.e., 3-parameter) eigenvalue systems is Stäckel equivalent to exactly one of these systems.

Laplace Systems	Degenerate potentials $\partial_x^2+\partial_y^2+V(x,\,y)$
A	$V(x,y)=\frac{4\,a_3}{(x^2+y^2-1)^2}-\frac{4\,a_4}{(x^2+y^2+1)^2}$
Laplace separable coordinates:	$a)$ spherical $b)$ ellipsoidal
	$i)$ horospherical: $x=\frac{1}{2}\frac{u^2+v^2-1}{u-iv},\ y=-\frac{i}{2}\frac{u^2+v^2+1}{u-iv}$,
	$j)$ degenerate elliptic: $x=\frac{u^2+v^2}{4uv}+\frac{uv}{u^2v^2+1}$,
	$y=-\frac{i(u^2+v^2)}{4uv}+\frac{iuv}{u^2v^2+1}$
Helmholtz superintegrable systems:	$S3$: $(1,0),(0,1)$ $S6$: $(1,1)$
B	$V(x,y)=\frac{a_1}{x^2}+a_4$
Laplace separable coordinates:	$c)$ Cartesian $d)$ polar
	$g)$ parabolic $e)$ elliptic
Helmholtz superintegrable systems:	$S5$: $(1,0)$ $E6$: $(0,1)$
C	$V(x,y)=a_3-a_4\,(x^2+y^2)$
Laplace separable coordinates:	$c)$ Cartesian $d)$ polar
	$f)$ hyperbolic $e)$ elliptic
Helmholtz superintegrable systems:	$E3$: $(1,0)$ $E18$: $(0,1)$
D	$V(x,y)=a_1-a_2\,x$
Laplace separable coordinates:	$c)$ Cartesian $g)$ parabolic
Helmholtz superintegrable systems:	$E5$: $(1,0)$
E	$V(x,y)=\frac{a_1}{(x+i\,y)^2}+a_3$
Laplace separable coordinates:	$d)$ polar $f)$ hyperbolic
Helmholtz superintegrable systems:	$E12$: (a_1,a_3), $a_1a_3\neq 0$ $E14$: $(0,1)$
F	$V(x,y)=a_1-a_2\,(x+i\,y)$
Laplace separable coordinates:	$c)$ Cartesian $h)$ semi-hyperbolic
Helmholtz superintegrable systems:	$E4$: $(1,0)$, $E13$: (a_1,a_2), $a_2\neq 0$

Table 2: Two-parameter degenerate potentials for Laplace systems.

itself, which can be classified. We call these Böcher contractions since they are motivated by ideas of Böcher, [4], who used similar limits to construct separable coordinates of free Laplace, wave and Helmholtz equations from basic cyclidic coordinates. (A major difference here is that our systems include potentials.) There are 4 basic Böcher contractions of 2D Laplace systems and each one when applied to a Laplace system yields another Laplace superintegrable system, [2, 3]. These in turn induce contractions of the Helmholtz systems in each equivalence class to Helmholtz systems in other classes, over 200 contractions in all. However, we can summarize the basic results for Laplace systems in Figures 1 and 2. A system can be obtained from another superintegrable system via contraction provided it is connected to the other system by directed arrows. All systems follow from [1111] for nondegenerate potentials and from A for degenerate potentials, and A is a restriction of [1111] with increased symmetry.

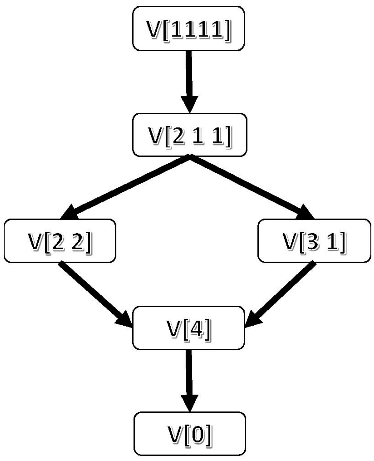

Figure 1: Contractions of nondegenerate Laplace systems

3 Stäckel transforms

A Stäckel transformed Helmholtz system from one of the tables 1 or 2 will have metric

$$ds^2 = V(\mathbf{b})(dx^2 + dy^2).$$

If the metric is flat, we need to determine coordinates $X(x, y)$,$Y(x, y)$ such that $ds^2 = dX^2 + dY^2$. Then separable coordinates $x(u, v), y(u, v)$ for

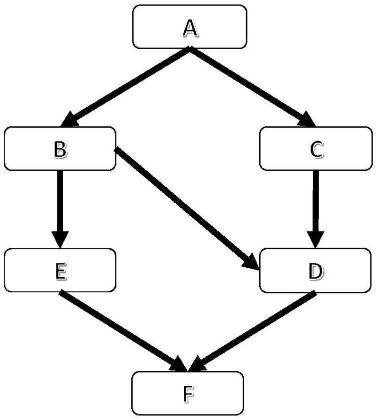

Figure 2: Contractions of degenerate Laplace systems

the Laplace system tranform to separable coordinates $X(x(u,v),y(u.v)) = X[u.v]$, $Y(x(u,v),y(u.v)) = Y[u.v]$, for the Helmholtz system. Similarly if the metric corresponds to $S_{2,\mathbb{C}}$, we need to determine coordinates $s_1(x,y)$. $s_2(x,y)$, $s_3(x,y)$ such that $s_1^2+s_2^2+s_3^2 = 1$ and $ds^2 = ds_1^2+ds_2^2+ds_3^2$. Then the separable coordinates will be expressed as $s_1[u,v], s_2[u,v], s_3[u,v]$. The possible separable coordinate systems (u,v) in flat space and $[u,v]$ on the 2-sphere are listed in Appendix B.

3.0.1 Stäckel equivalences

The relevant coordinate transformations are:

Flat space - flat space

1.
$$dX^2+dY^2 = \frac{dx^2+dy^2}{(x+iy)^2},\ X = \frac{1}{4}\frac{x^2+y^2-4}{x+iy},\ Y = \frac{i}{4}\frac{x^2+y^2+4}{x+iy}.$$

2.
$$dX^2+dY^2 = \left(\frac{x-iy}{(x+iy)^3} - a(x^2+y^2)\right)(dx^2+dy^2),$$
$$X = -ia(x+iy)^2 - \frac{i}{16}\frac{(x^2+y^2)^2+16}{(x+iy)^2},\ Y = -a(x+iy)^2 + \frac{1}{16}\frac{(x^2+y^2)^2-16}{(x+iy)^2}.$$

3.
$$dX^2+dY^2 = \left(x^2+y^2\right)(dx^2+dy^2),\ X = \frac{1}{2}(x^2-y^2),\ Y = xy.$$

4.
$$dX^2 + dY^2 = (x+iy)(dx^2+dy^2),$$
$$X = \frac{i}{2}(x+iy)^2 - \frac{i}{4}(x-iy),\ Y = \frac{1}{2}(x+iy)^2 + \frac{1}{4}(x-iy).$$

5.
$$dX^2 + dY^2 = (1 + a(x+iy))\,(dx^2+dy^2),$$
$$X = -i(x+iy)^2 + \frac{ia}{8}(x-iy) - \frac{2i}{a}(x+iy),\ Y = -(x+iy)^2 - \frac{a}{8}(x-iy) - \frac{2}{a}(x+iy) - \frac{1}{a^2}.$$

6.
$$dX^2 + dY^2 = \left(\frac{1}{(x+iy)^2} + a\right)(dx^2+dy^2),$$
$$X = -\frac{i}{4(x+iy)} + \frac{ia}{4}(x+iy) - i(x-iy),\ Y = -\frac{1}{4(x+iy)} + \frac{a}{4}(x+iy) + (x-iy).$$

7.
$$dX^2 + dY^2 = (x+iy)(dx^2+dy^2),$$
$$X = \frac{i}{2}(x+iy^2)^2 - \frac{i}{4}(x-iy),\ Y = \frac{1}{2}(x+iy)^2 + \frac{1}{4}(x-iy).$$

Flat space - sphere

1.
$$\frac{4(dx^2+dy^2)}{(x^2+y^2+1)^2} = ds_1^2 + ds_2^2 + ds_3^2,$$
$$s_1 = \frac{2x}{x^2+y^2+1}, \quad s_2 = \frac{1-x^2-y^2}{1+x^2+y^2}, \quad s_3 = \frac{2y}{x^2+y^2+1}.$$

2.
$$-\frac{dx^2+dy^2}{y^2} = ds_1^2 + ds_2^2 + ds_3^2.$$
$$s_1 = \frac{i}{2y}(1-x^2-y^2), \quad s_2 = \frac{ix}{y}, \quad s_3 = -\frac{1}{2y}(1+x^2+y^2),$$
$$x = \frac{s_2}{s_1 - is_3}, \quad y = \frac{i}{s_1 - is_3}.$$

3.
$$-\frac{x^2+y^2}{x^2y^2}(dx^2+dy^2) = ds_1^2 + ds_2^2 + ds_3^2,$$
$$s_1 = \frac{i}{8}\frac{(x^2+y^2+2)(x^2+y^2-2)}{xy},\ s_2 = \frac{i}{2}\frac{(x^2-y^2)}{xy},\ s_3 = -\frac{1}{8}\frac{(x^2+y^2)^2+4}{xy}.$$

Sphere - sphere

1.
$$-\frac{dX^2+dY^2}{Y^2} = -\frac{x^2+y^2}{x^2y^2}(dx^2+dy^2) = ds_1^2+ds_2^2+ds_3^2,$$
$$X = \frac{x^2-y^2}{2}, \quad Y = xy,$$

So
$$S_2 = \frac{1}{2}(s_2+\frac{1}{s_2}), \quad S_1+iS_3 = -\frac{(s_1-is_3)^2}{s_2}.$$

2.
$$-\frac{dX^2+dY^2}{Y^2} = \left(\frac{4}{(x^2+y^2+1)^2} - \frac{1}{y^2}\right)(dx^2+dy^2).$$
$$X = \frac{1}{8}\frac{x(x^2+y^2+1)}{x^2+y^2}, \quad Y = \frac{1}{8}\frac{y(x^2+y^2+1)}{x^2+y^2}.$$

3.
$$-\frac{dX^2+dY^2}{Y^2} = \left(\frac{4}{(x^2+y^2+1)^2} - \frac{1}{x^2}\right)(dx^2+dy^2).$$
$$X = -\frac{2iy(x^2+y^2+1)}{(x^2+y^2-2y+1)(x^2+y^2+2y+1)} \quad Y = -\frac{2ix(x^2+y^2+1)}{(x^2+y^2-2y+1)(x^2+y^2+2y+1)}.$$
$$s_1 = -\frac{(x^2+y^2+1)^2+4x^2}{4x(x^2+y^2+1)}, \ s_3 = i\frac{(x^2+y^2-1)^2-4y^2}{4x(x^2+y^2+1)},$$
$$s_2 = \frac{iy(x^2+y^2-1)}{x(x^2+y^2+1)}.$$

4.
$$4\frac{dX^2+dY^2}{(X^2+Y^2+1)^2} = -\frac{dx^2+dy^2}{y^2},$$
$$X = \frac{1}{2}\frac{1-x^2-y^2}{x-iy}, \quad Y = \frac{i}{2}\frac{x^2+y^2+1}{x-iy}.$$

5.
$$-\frac{dX^2+dY^2}{Y^2} = -16\frac{x^2+y^2}{(x^2+y^2-1)^2(x^2+y^2+1)^2}(dx^2+dy^2),$$
$$X = \frac{32xy}{(x^2+y^2+2y+1)(x^2+y^2-2y+1)}, \quad Y = -\frac{8(x^2+y^2+1)(x^2+y^2-1)}{(x^2+y^2+2y+1)(x^2+y^2-2y+1)}.$$

6.
$$-\frac{dX^2+dY^2}{Y^2} =$$
$$-\left(\frac{(x^2+y^2)([x^2+y^2+1]^2-4x^2)([x^2+y^2+1]^2-4y^2)}{x^2y^2(x^2+y^2-1)^2(x^2+y^2+1)^2}\right)(dx^2+dy^2),$$
$$X = i\frac{(y^2-x^2)([x^2+y^2]^2+1)}{(x^2+y^2)^2}, \quad Y = -2i\frac{xy([x^2+y^2]^2-1)}{(x^2+y^2)^2}.$$

Example 1 *Consider the spherical coordinates*

$$x = \frac{\sin(\theta)\cos(\phi)}{1+\cos(\theta)}, y = \frac{\sin(\theta)\sin(\phi)}{1+\cos(\theta)},$$

separable for the Laplace system [1111]*. Under the Stäckel transform* $(1,0,0,0)$ *the Laplace equation is mapped to the Helmholtz system* $S9$ *with metric* $ds^2 = -\frac{dx^2+dy^2}{x^2}$*. Using the flat space - sphere identities* 2 *we see that in terms of standard coordinates on the complex 2-sphere we have* $-ds^2 = ds_1^2+ds_2^2+ds_3^2$ *where*

$$s_1 = i\cot(\theta)\ \csc(\phi),\ s_2 = i\cot(\phi),\ s_3 = -\csc(\theta)\ \csc(\phi),$$

spherical coordinates on the 2-sphere. Under the Stäckel transform $(1,0,0,1)$ *the Laplace equation is mapped to the Helmholtz system* $S8$ *with metric* $ds^2 = -(\frac{1}{x^2} - \frac{4}{(x^2+y^2+1)^2})(dx^2+dy^2)$*. Using the sphere-sphere identities iii) we see that in terms of standard coordinates on the complex 2-sphere we have* $ds^2 = ds_1^2+ds_2^2+ds_3^2$ *where*

$$s_1 + is_3 = -\frac{\sin(\theta)}{\cos(\phi)},\ s_1 - is_3 = \frac{1}{2}\left(\sin(\theta)\ \cos(\phi) - \frac{1}{\sin(\theta)\ \cos(\phi)}\right),$$

$$s_2 = -i\cos(\theta)\ \tan(\phi),$$

degenerate elliptic coordinates of type 1.

Example 2 *Consider the Cartesian coordinates* x, y*, separable for the Laplace system* [211]*. Under the Stäckel transform* $(0,0,0,1)$ *the Laplace equation is mapped to the Helmholtz system* $E1$ *with metric* $ds^2 = dx^2+dy^2$*, separable in Cartesian coordinates. Under the Stäckel transform* $(1,1,0,0)$ *the Laplace equation is mapped to the Helmholtz system* $S4$ *with metric* $ds^2 = \left(\frac{1}{x^2}+\frac{1}{y^2}\right)(dx^2+dy^2)$*. Using the flat space - sphere identities* 3 *we see that in terms of standard coordinates on the complex 2-sphere we have* $-ds^2 = ds_1^2+ds_2^2+ds_3^2$ *where*

$$s_1 = \frac{i}{8}\frac{(x^2+y^2+2)(x^2+y^2-2)}{xy},\ s_2 = \frac{i}{2}\frac{(x^2-y^2)}{xy},\ s_3 = -\frac{1}{8}\frac{(x^2+y^2)^2+4}{xy}.$$

These are degenerate elliptic coordinates of type 2, separable on the 2-sphere.

The complete identification of Laplace separable systems with Helmholtz separable systems in constant curvature spaces is presented in tables 3 and 4. The labels *a-f* designate the 6 Laplace separable coordinate systems defined in tables 1 and 2. Note that every separable coordinate system for the zero potential Helmholtz equation $(\partial_x^2+\partial_y^2-E)\Psi = 0$ on flat space or $(\partial_{s_1}^2+\partial_{s_2}^2+\partial_{s_3}^2-E)\Psi = 0$ on the 2-sphere is also separable for some Helmholtz or Laplace superintegrable system.

	S1	S2	S3	S4	S5	S6	S7	S8	S9
Spherical		d	a	d	d	a	a		a
Horospherical	c	c	i		c				
Ellipsoidal			b			b		b	b
Degenerate elliptic 1		e	j	e	e	j	b	a	
Degenerate elliptic 2	g			c	g	i			

Table 3: Separating coordinate systems for superintegrable potentials on the two-dimensional complex sphere.

	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
Cartesian	c	c	c	c	c	c				
Polar	d		d			d		d		
Semi-Hyperbolic				h						h
Hyperbolic			f				d	f		
Parabolic		g			g	g			h	
Elliptic	e		e			e	f			

	E11	E12	E13	E14	E16	E17	E18	E19	E20
Cartesian									
Polar				d	d	d	d		
Semi-Hyperbolic	c		c						
Hyperbolic		d		f		f	f	d	
Parabolic			h		c		c		c
Elliptic		f			e		e	f	

Table 4: Separating coordinate systems for superintegrable potentials in complex two-dimensional Euclidean space. The exceptional system $E3'$ separates only in the family of Cartesian coordinates c).

4 Limits of separable coordinate systems

The separable coordinates for nondegenerate Laplace systems are related by Bôcher contractions. A Bôcher contraction applied to the commuting symmetry operators $H, L = \sum_{j,k} a^{jk}\partial_{x_j,x_j} + \cdots$, characterizing a Laplace separable coordinate system will yield another pair of symmetry operators H', L'. The target symmetries will define a new set of separable coordinates provided the target matrix $a'^{jk}(x'_\ell)$ is diagonalizable. The basic results are summarized in Figure 3. A separable coordinate system in a Laplace superintegrable system contracts to a system on a lower level provided there is a sequence of directed arrows connecting the source system to the target. This means that there exists a Bôcher contraction (or composition of Bôcher contractions) that takes the symmetry operators for the source system to the symmetry operators for the target. Note that at the top, $V[1111]$, level we can let $c \to 0$ or $c \to 1$ in the ellipsoidal coordinates to obtain spherical coordinates. In that sense we see that all of the separable coordinates for nondegenerate Laplace systems are contractions of ellipsoidal coordinates.

The corresponding results for degenerate Laplace systems are summarized in Figure 4. Here there is a difference because of the appearance of reduced coordinate systems that can only be obtained by type II limits, [8]. These limits are not pure Bôcher contractions in x, y but also involve limits of the separated coordinates. This is accomplished by allowing the parameter c in the generic separable coordinates to be appropriate functions of ϵ, usually $c = 1 + \epsilon^2$ or $c = \frac{1}{\epsilon^2}$. At the top level A there is a Bôcher contraction that maps this system into itself. By choosing appropriate functions $c(\epsilon)$ in the elliptic 1 coordinates we can obtain the spherical, horospherical and degenerate elliptic coordinates of type 1 in the limit. It follows that all separable coordinates for degenerate Laplace systems are contraction/limits of elliptic 1 coordinates.

5 The separation equations

Each family of separated solutions for a Laplace or Helmholtz superintegrable system is characterized as the family of eigenfunctions of a 2nd order symmetry operator. Each family determines an eigenbasis of separated solutions of the 2D superintegrable system. An eigenbasis of one family can be expanded in terms of a eigenbasis for another family and the quadratic structure algebras help to derive the expansion coefficients; see e.g., [9], (with some minor errors), [10, 11, 12, 13, 14]. The compete list of separation equations is given in Figures 5 and 6. (The notation (2) means that

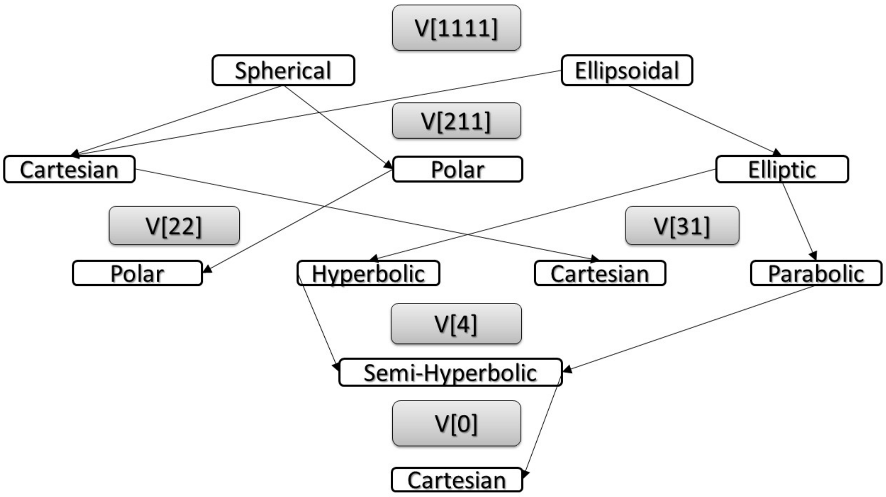

Figure 3: Bôcher contractions of separation equations for nondegenerate Laplace systems

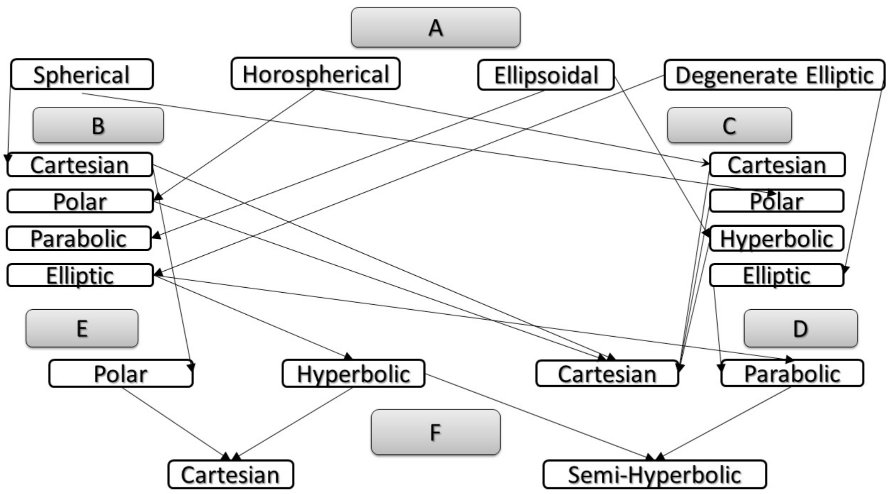

Figure 4: Limits of separation equations for degenerate Laplace systems

the separation equations for the corresponding coordinates are both of the same, except that the separation constant occurs as c in one equation and $-c$ in the other.) The definitions of the separation equations are as in [15].

We first consider the non-degenerate systems in Figure 5. Except for the special case of the extended isotropic oscillator [0], each superintegrable system corresponds to exactly one set of coordinates for which the separation equations are of Heun type, i.e., they have 4 regular singular points. (They are listed in the right-hand column.) Moreover, system [1111] corresponds to the general Heun equation, [211] corresponds to the confluent Heun equation, [22] to the double-confluent Heun equation, [31] to the bi-confluent Heun equation, and [4] to the triconfluent Heun equation, all in full generality with 5 parameters. (The parameters are supplied by the 4 parameters a_i in the non-degenerate potential and the separation constant c.) Thus any solution of one of these Heun equations determines an eigenfunction of a 2D Laplace or Helmholtz superintegrable system. Similarly the separation equations in the left-side columns are all of hypergeometric type, 3 regular singularities. System [1111] corresponds to the general hypergeometric equation, system [211] to the hypergeometric and confluent hypergeometric equations, [22] to the confluent hypergeometric equation, [31] to the confluent hypergeometric and parabolic cylinder equations, and [0] to the parabolic cylinder equation, all in full generality (including all parameters).

The degenerate systems in Figure 6 are parameter restrictions of non-degenerate systems for which the restricted system has additional symmetry. Except for the special case of the isotropic oscillator F, a restriction of [0], each superintegrable system corresponds to at least one set of coordinates for which the separation equations are of Heun type. These equations do not have the full number of parameters, but additional symmetry properties. Except for the spheroidal wave equation, these special Heun systems do not appear to have individual names but we suggest that they deserve special attention. The separation equations in the left-side columns are all of hypergeometric type. System A corresponds the hypergeometric equation and to the Gegenbauer equation, system B to the confluent hypergeometric and to Bessel's equation, as well as the equation for exponential functions, C to the confluent hypergeometric equation and the parabolic cylinder equation, as well as the equation for exponential functions, D to the Airy equation, as well as the equation for exponential functions, E to Bessel's equation, and F to the parabolic cylinder equation and the Airy equation, all in full generality.

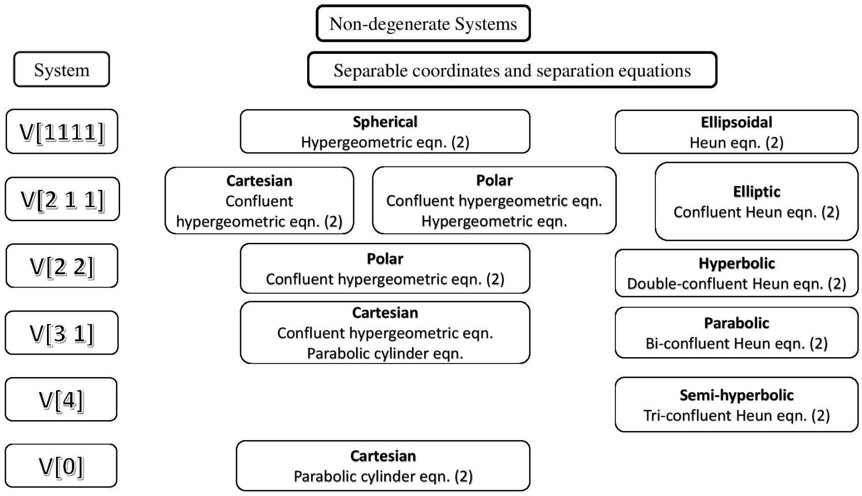

Figure 5: Separation equations for nondegenerate Laplace systems

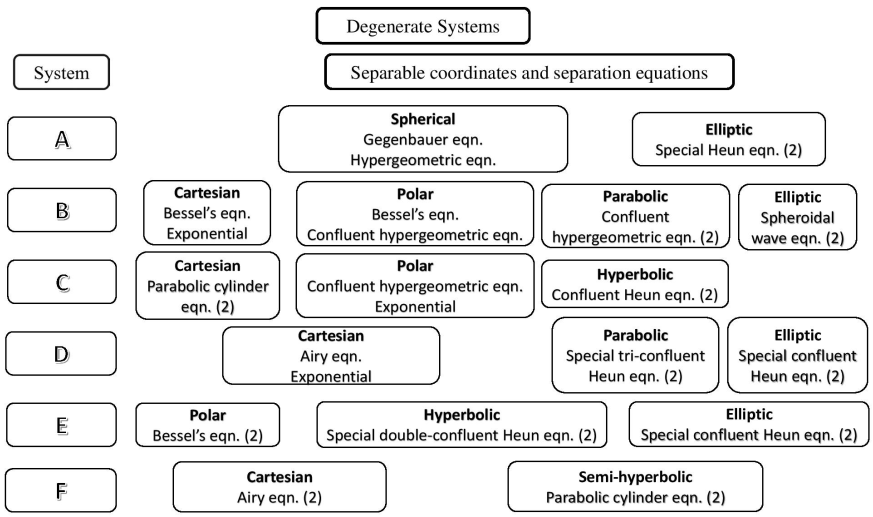

Figure 6: Separation equations for degenerate Laplace systems

6 Hypergeometric type equations

1. Hypergeometric equation: $z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\,\frac{dw}{dz} - abw = 0.$
2. Confluent hypergeometric equation: $z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0.$
3. Parabolic cylinder equation: $\frac{d^2w}{dz^2} + (az^2+bz+c)w = 0.$
4. Gegenbauer equation: $(1-z^2)\frac{d^2w}{dz^2} - 2(\mu+1)z\frac{dw}{dz} + (\nu-\mu)(\nu+\mu+1)w = 0.$
5. Bessel's equation: $z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2-\nu^2)w = 0.$
6. Airy's equation: $\frac{d^2w}{dz^2} - zw = 0.$

7 Heun type equations

1. Heun equation: $\alpha+\beta+1 = \gamma+\delta+\epsilon$,

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}w = 0,$$

2. Confluent Heun equation

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)}w = 0.$$

3. Doubly-confluent Heun equation:

$$\frac{d^2w}{dz^2} + \left(\frac{\delta}{z^2} + \frac{\gamma}{z} + 1\right)\frac{dw}{dz} + \frac{\alpha z - q}{z^2}w = 0.$$

4. Biconfluent Heun equation: $\frac{d^2w}{dz^2} - \left(\frac{\gamma}{z} + \delta + z\right)\frac{dw}{dz} + \frac{\alpha z - q}{z}w = 0.$
5. Triconfluent Heun equation: $\frac{d^2w}{dz^2} + (\gamma+z)z\frac{dw}{dz} + (\alpha z - q)w = 0.$
6. Spheroidal wave equation

$$\frac{d}{dz}\left((1-z^2)\frac{dw}{dz}\right) + \left(\lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right)w = 0.$$

8 Special functions

Special functions associated with these systems arise in two distinct ways:

- As separable eigenfunctions of the quantum Hamiltonian. Second order superintegrable systems are multiseparable.
- As interbasis expansion coefficients relating distinct separable coordinate eigenbases. These are often solutions of difference equations, [1]

Most of the special functions in the DLMF appear one of these ways. Böcher contractions of $S9$ to other superintegrable systems induce limits of the eigenfunctions and expansion coefficients to corresponding functions for the contracted superintegrable systems, [18, 19]. This includes a reinterpretation of the Askey Scheme relating the possible hypergeometric orthogonal polynomials via limits.

9 Exact and Quasi-exact solvability

Let $H = \frac{d^2}{dx^2} + V(x)$. We are concerned with the eigenvalue problem $H\Psi = E\Psi$. The operator H is said to be **exactly solvable**, **(ES)** if there exists an infinite flag of subspaces of the domain of H: $\mathcal{P}_N$, $N = 1, 2, 3, \cdots$, such that $n_N = \dim\mathcal{P}_N \to \infty$ as $N \to \infty$ and $H\,\mathcal{P}_N \subseteq \mathcal{P}_N \subseteq \mathcal{P}_{N+1}$ for any N. In this case, for each subspace $\mathcal{P}_N$ the n_N eigenvalues and eigenfunctions of H can be obtained by pure algebraic means.

The operator H is called **quasi-exactly solvable**, **(QES)** if there exist a single subspace $\mathcal{P}_k$ of dimension $n_k > 0$ such that $H\,\mathcal{P}_k \subseteq \mathcal{P}_k$. In this case, again we can find n_k eigenvalues and eigenfunctions of $\mathcal{H}$ by algebraic means, but we have no information about the remaining eigenvalues and eigenfunctions. See [20, 21, 22, 23, 24].

There is an intimate connection between these two concepts of solvability, the theory of separation of variables, and 2nd order superintegrability. To illustrate our treatment we present the motivating example for QES theory: the anharmonic oscillator with 6th order potential term:

$$H = \frac{d^2}{dx^2} + [-\frac{k_1^2}{4\omega^2} + (4n+5)\omega]x^2 - k_1x^4 - \omega^2x^6, \tag{7}$$

where n is a parameter. Here the eigenvalue equation $H\Psi = \lambda\Psi$ is a Heun equation and explicit series or integral solutions cannot be found. However, for n a fixed positive integer, there are $n+1$ eigenfunctions

$$\Psi_i = P_n^{(i)}(x^2)e^{-\frac{k_1}{4\omega}x^2 - \frac{\omega}{4}x^4},$$

$i = 0, 1, \cdots, n$ where P is a polynomial of order at most n in x^2, and the eigenvalues and eigenfunctions can be computed by algebraic means. Several theories were developed to treat this and similar examples, the most prominent being the embedding of the operator H in the enveloping algebra of $s\ell(2,\mathbb{C})$ or $gl(2,\mathbb{C})$, [22, 25].

However, this example can be understood easily in terms of superintegrability theory, as pointed out in [26]. Consider the singular anisotropic oscillator potential defining the 2nd order superintegrable system $E2$:

$$V(x,y) = \frac{1}{2}\omega^2(4x^2+y^2) + k_1x + \frac{k_2^2 - \frac{1}{4}}{2y^2}.$$

The Schrödinger equation has the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\left[2E-\omega^2(4x^2+y^2)-2k_1x-\frac{k_2^2-\frac{1}{4}}{y^2}\right]\Psi=0. \tag{8}$$

The Schrödinger equation separates in two systems: Cartesian and parabolic coordinates. Separation of variables in Cartesian coordinates leads to two independent one-dimensional Schrödinger (separation) equations with Laguerre and Hermite polynomial solutions. These separation equations are ES and one easily computes the energy spectrum

$$E=\lambda_1+\lambda_2=\omega[2n+2+k_2]-\frac{k_1^2}{8\omega^2},\qquad n=n_1+n_2=0,1,2,...$$

The Schrödinger equation in parabolic coordinates is

$$\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2\Psi}{\partial\xi^2}+\frac{\partial^2\Psi}{\partial\eta^2}\right)+\left[2E-\omega^2(\xi^4-\xi^2\eta^2+\eta^4)-k_1(\xi^2-\eta^2)-\frac{k_2^2-\frac{1}{4}}{\xi^2\eta^2}\right]\Psi=0.$$

Upon substituting $\Psi(\xi,\eta)=X(\xi)Y(\eta)$ and introducing the parabolic separation constant λ, we find the two separation equations:

$$\frac{d^2X}{d\xi^2}+\left(2E\xi^2-\omega^2\xi^6-k_1\xi^4-\frac{k_2^2-\frac{1}{4}}{\xi^2}\right)X=\lambda X, \tag{9}$$

$$\frac{d^2Y}{d\eta^2}+\left(2E\eta^2-\omega^2\eta^6+k_1\eta^4-\frac{k_2^2-\frac{1}{4}}{\eta^2}\right)Y=-\lambda Y, \tag{10}$$

each a form of the bi-confluent Heun equation, in full generality. Substituting $E=\omega[2n+2+k_2]-\frac{k_1^2}{8\omega^2}$, and setting $k_2=\frac{1}{2}$, $\xi=x$ in (9) we get the QES equation for the anharmonic oscillator with 6th order potential term, where now eigenvalue is the separation constant. Thus the 1D anharmonic oscillator (7) is a special case of the separation equation for $E2$ in parabolic coordinates.

We can verify that the separation equation admits polynomial solutions for these special values of E. First we make the change of variable $\xi=\sqrt{u}$ in equation (9) to obtain

$$\left[4u\frac{d^2}{du^2}+2\frac{d}{du}+\frac{1}{4u}\left(-4\omega^2u^4-4k_1u^3+8Eu^2-4k_2^2+1\right)\right]X=\lambda X.$$

Our aim is to modify the operator on the left so that it takes polynomials in u to polynomials. We do this with a gauge transformation: $X=u^{\frac{1}{4}+\frac{k_2}{2}}e^{-\frac{1}{4}\omega u^2-\frac{1}{4}\frac{k_1u}{\omega}}P(u)$. This leads to the equation

$$\left[4u\frac{d^2}{du^2}-2(2\omega^2u^2+k_1u-2k_2\omega-2\omega)\frac{d}{du}\right.$$

$$+\frac{1}{4\omega^2}\left(-8k_2\omega^3 u+8E\omega^2 u-16\omega^3 u+k_1^2 u-4k_1k_2\omega-4k_1\omega\right)\Big] P=\lambda P$$

where the operator on the left now maps polynomials to polynomials, However in general the oparator doesn't admit invariant subspaces, because it increases the degree of a polynomial. The only adjustable parameter here is E, so we introduce a new parameter n such that $E=\omega(2n+k_2+2)-\frac{k_1^2}{8\omega^2}$. Now the preceding equation takes the form

$$LP\equiv\left[4u\frac{d^2}{du^2}-2(2\omega^2u^2+k_1u-2k_2\omega-2\omega)\frac{d}{du}+\frac{1}{2\omega}\left(8n\omega^2u-3k_1\right)\right]P=\lambda P. \tag{11}$$

Note that if n is a nonnegative integer the operator maps the $(n+1)$-dimensional space of polynomials of order $\leq n$ into itself. Thus for each n we can use algebraic methods to determine $n+1$ eigenvalues of L and the corresponding eigenvectors.

In papers [23] a connection was made between the problem of finding QES systems in 1D and 2nd order superintegrability in nD. There it was shown that "most" generic superintegrable systems in n variables were exactly solvable in some coordinate system, which allowed the explicit determination of the discrete bound state energy eigenvalues E_i. These systems were all multiseparable. In some cases the 1D separation equations were exactly solvable. However, some equations corresponded to QES systems. In these cases the energy eigenvalue E became a parameter in the 1D potential and the $1D$ energy was the separation constant c. It was shown that the values E_i were exactly those such that the 1D equation $H\Psi=c\Psi$ had QES polynomial solutions.

In two recent papers [25, 27], Turbiner has studied and reported on the classification of QES systems in 1D. His emphasis is on QES systems that are special cases of the Heun equation and its confluent forms, and exactly solvable systems which are special cases of the hypergeometric equation. We see now that all of these systems correspond to separation equations for the 2D 2nd order superintegrable systems as given in Figures 1 and 2. Thus all of these solutions determine solutions of the 2D superintegrable systems.

9.1 A superintegrable non-Heun example

Second order nD superintegrable systems provide many examples of QES equations for $n>2$ that are not Heun equations [23]. We give an example for $n=4$. The flat space superintegrable system is, in Cartesian coordinates x_j,

$$H=\sum_{j=1}^{4}\partial_{x_j}^2+\left[\frac{1}{4}(x_2^2+x_3^2+x_4^2)+x_1^2-2ix_1-3\right]a_5-(ix_1-2)a_4+\frac{4b_1}{(x_3+ix_4)^2}$$

$$-\frac{8x_3b_2}{(x_3+ix_4)^3}+\frac{4c_1}{x_2^2}.$$

We introduce separable coordinates z_j, defined by

$$x_1 = -i(S_1 - 2),\ x_2^2 = 4S_4,$$

$$(x_3 + ix_4)^2 = 4(S_4 - S_3 + S_2 - S_1 + 1),\ x_3^2 + x_4^2 = 12 - 8S_1 + 4S_2 - 4S_4,$$

$$S_1 = z_1 + z_2 + z_3 + z_4,\ S_2 = z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4,$$

$$S_3 = z_1z_2z_3 + z_4z_2z_3 + z_1z_2z_4 + z_1z_4z_3,\ S_4 = z_1z_2z_3z_4,$$

and

$$\sum_{j=1}^{4} dx_j^2 = \sum_{k=1}^{4} \Pi_{\ell\neq k}(z_k - z_\ell)\frac{dz_k^2}{z_k(z_k-1)^2}.$$

The separation equations are:

$$\left(\partial^2_{z_j} + \sum_{k=0}^{5} a_k z_j^k + \frac{b_1}{z_j - 1} + \frac{b_2}{(z_j-1)^2} + \frac{c_1}{z_j}\right) Z(z_j) = 0,$$

for $j = 1, \cdots, 4$. Modulo a gauge transformation of the form $Z(z_j) = R_j(z_j)F_j(z_j)$ with $R_j = \exp(c_1z_j^2 + c_2z_j + \frac{c_3}{z_j-1})(z_j-1)^{c_4}z_j^{c_5}$ each of these equations is QES in polynomial variables z_j,[23], but not of Heun type.

10 Relation of an explicitly solvable QES 1D system to the superintegrable system E2

Some special cases of Heun equations reduce to hypergeometric equations, see the impressive work of Maier, [28]. Moreover, in recent papers it has been shown that some special cases of the Heun equations have explicit solutions that are expressible in terms of derivatives of hypergeometric functions. These also yield explicit solutions of 1D QES Schrödinger equations, e.g.[29]. We can observe that all such special solutions lead to eigenfunctions of 2D superintegrable systems which also have separable ES hypergeometric eigenfunctions. The quadratic algebras of the 2D systems allow us to relate the QES and ES systems. Moreover a knowledge of the possible ES systems for a 2nd order superintegrable system gives important clues about the structure of the QES systems

As an example consider inverse square root system [29], written in the form

$$\frac{d^2f(x)}{dx^2} + (\frac{a}{x^{1/2}} + \frac{b}{x} + \frac{c}{x^{3/2}} - E)f(x). \tag{12}$$

With $y = \sqrt{x}$ we have

$$y\frac{d^2f(y)}{dy^2} - \frac{df(y)}{dy} + (4ay^2 + 4by + 4c - 4Ey^3)f(y). \tag{13}$$

Changing notation, we note that the superintegrable system $E2$, (8), in Cartesian coordinates y_1, y_2 is

$$\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+(-A(4y_1^2+y_2^2)+By_1+\frac{C}{y_2^2}-E')\right)f(y_1,y_2)=0. \qquad (14)$$

In these coordinates the separable solutions $f(y_1,y_2)=g_1(y_1)g_2(y_2)$ are eigenfunctions $L_1 f=\lambda_1 f$ of the symmetry operator $L_1=\partial_{y_1^2}+(-4Ay_1^2+By_1)$, with separation equations

$$\frac{d^2g_1}{dy_1^2}+(-4Ay_1^2+By_1+\lambda_1-E')g_1=0,\ \frac{d^2g_2}{dy_2^2}+(-Ay_2^2+\frac{C}{y_2^2}-\lambda_1)g_2=0. \quad (15)$$

Here, λ_1 is the separation constant.

In parabolic coordinates η, ξ with $y_1=\xi+\eta, y_2=2i\sqrt{\xi\eta}$, (14), and with a gauge transformation for convenience, $f(\eta,\xi)=(\eta\xi)^{-3/4}f_1(\eta)f_2(\xi)$, the separation equations (9),(10), reduce to the bi-confluent Heun separation equation

$$\eta\frac{d^2f_1(\eta)}{d\eta^2}-\frac{df_1(\eta)}{d\eta}+\left(\frac{15}{16\eta}+\frac{C}{4\eta}-E'\eta+B\eta^2-4A\eta^3+\lambda_2\right)f_1(\eta)=0, \quad (16)$$

for $f_1(\eta)$ with a similar equation for $f_2(\xi)$ (with the separation constant λ_2 replaced by $-\lambda_2$). Now note that, with the restriction to the superintegrable system $E2$ with $C=-15/4$ in the potential, equations (13) and (16) are the same, provided we make the identifications

$$y=\eta, \quad 4b=-E', \quad 4a=B, \quad 4c=\lambda_2, \quad E=A. \qquad (17)$$

Thus this 1D inverse square root potential system corresponds to a special case of the separation equation for the 2D superintegrable system $E2$ in parabolic coordinates. We have already shown that the separation equation is QES, but we will now show that it has explicit solutions which are a consequence of the exactly solvable $E2$ separation equation (15) in Cartesian coordinates.

Making the substitutions (17) where appropriate and setting $y_1=y$, we can solve (15) directly to get $g_1(y)=\exp(-y(-a+Ey)/\sqrt{E})\ G(y)$ where $G(y)$ is an arbitrary linear combination of

$${}_1F_1\left(\begin{matrix}\frac{1}{8}(\frac{2E^{3/2}+(4b-\lambda_1)E-a^2}{E^{3/2}}) \\ 1/2\end{matrix};\frac{(2Ey-a)^2}{2E^{3/2}}\right) \qquad (18)$$

and

$$(2Ey-a)\ {}_1F_1\left(\begin{matrix}\frac{1}{8}(\frac{6E^{3/2}+(4b-\lambda_1)E-a^2}{E^{3/2}}) \\ 3/2\end{matrix};\frac{(2Ey-a)^2}{2E^{3/2}}\right). \qquad (19)$$

Here $G(y)$ is the general solution of the equation $S_1\, G(y) = 0$, equivalent to (15), where

$$S_1 = \frac{d^2}{dy^2} - \frac{2(2Ey - a)}{\sqrt{E}}\frac{d}{dy} - \frac{(2E^{3/2} + 4bE - \lambda_1 E - a^2)}{E}.$$

The following construction works for $b = -4c^2, \lambda_1 = -2\sqrt{E} - 32c^2$, which we now assume. Then equation (13) for the inverse square root system is equivalent to $S_2 f = 0$ where

$$S_2 = y\frac{d^2}{dy^2} - \frac{d}{dy} + 4(y^2 a - 4c^2 y + c - Ey^3).$$

We define operators K and Q by

$$K = \exp(\frac{ay}{\sqrt{E}} - y^2\sqrt{E})\left(\frac{d}{dy} - 4y\sqrt{E} + 4c + \frac{a}{\sqrt{E}}\right),$$

$$Q = \exp(\frac{ay}{\sqrt{E}} - y^2\sqrt{E})\left(y\frac{d}{dy} - 4y^2\sqrt{E} + 4cy + \frac{ay}{\sqrt{E}} - 1\right).$$

Then it is tedious but straightforward to verify the operator identity $S_2 K = QS_1$. This shows that K maps the solution space of the restricted Cartesian separation equation (15) to the solution space of the restricted parabolic separation equation and provides explicit solutions for the 1D inverse square root potential.

An analogous treatment can be given for the singular Lambert potential

$$V = \frac{V_0}{1 + 1/W(-e^{-(x+\sigma)/\sigma})}, \quad We^W = x,$$

considered in [30, 31]. Here, separation in elliptic coordinates for the Laplace system [211] are involved, and the mapping is from the solution space of the restricted Cartesian separation equation (confluent hypergeometric equation) to the solution space of the restricted confluent Heun equation.

11 Conclusions and Outlook

The theory of 2D 2nd order superintegrable Laplace systems encodes all the information about 2D Helmholtz or time-independent Schrödinger superintegrable systems in an efficient manner: there is a 1-1 correspondence between Laplace superintegrable systems and Stäckel equivalence classes of Helmholtz superintegrable systems. Each of these systems admits a quadratic symmetry algebra and is multiseparable. The separation equations are identically the same for all Helmholtz systems in an equivalence class, up to a permutation of the energy eigenvalue and the parameters in the potential. The separation equations comprise all of the various types of hypergeometric and Heun equations in full generality. In particular, they coincide with all of the 1D Schrödinger exactly solvable (ES) and quasi-exactly solvable (QES) systems related to the Heun operator and its limits. The separable solutions of these equations are the special functions of mathematical physics. The different systems are related by Stäckel transforms, by their symmetry algebras and by Böcher contractions of the conformal algebra $so(4, \mathbb{C})$ to itself, which enables all of these systems to be derived from a single one: the generic potential on the complex 2-sphere. The ES separation equtions are intimately related to the QES equations: They are Bôcher contractions of QES-type systems and can be used to determine the values of the parameters for the QES systems and, sometimes to provide a basis of solutions for the QES systems, not just a single solution. This approach facilitates a unified view of special function theory, incorporating hypergeometric and Heun functions in full generality. We have shown that all Heun equations are associated with 2D superintegrable systems and quadratic symmetry algebras, and this added structure can be used to obtain new results for Heun functions. Whereas hypergeometric special functions have been studied intensively, relatively little attention has been paid to Heun functions. The theory of 2nd order superintegrable systems places the two types of separation equations on the same level and interrelates them via symmetry algebras and contractions.

All of our considerations generalize to 2nd order superintegrable systems in 3D and higher dimensions, but the details are more complicated and much remains to be done. In 3D there is already a classification of nondegenerate systems and their contractions, [32, 33], but not yet of the degenerate cases. The separation equations in nD for $n \geq 4$ involve, in general, non-Heun equations. However, the QES theory described in [23] is still applicable.

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13 Appendix A: Superintegrable systems on $E_{2\mathbb{C}}$ and $S_{2\mathbb{C}}$

Nondegenerate functionally linearly independent $E_{2\mathbb{C}}$ systems: $H\Psi = (\partial_x^2 + \partial_y^2 + V)\Psi = E\Psi$.

$$1)\, E1:\ V = \alpha(x^2+y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2},\ 2)\, E2: V = \alpha(4x^2+y^2) + \beta x + \frac{\gamma}{y^2},$$

$$3)\, E3':\ V = \alpha(x^2+y^2) + \beta x + \gamma y,$$

$$4)\, E7:\ V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-1}} + \frac{\beta(x-iy)}{\sqrt{(x+iy)^2-1}\,\left(x+iy+\sqrt{(x+iy)^2-1}\,\right)^2} + \gamma(x^2+y^2),$$

$$5)\, E8:\ V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2),\ 6)\, E9:\ V = \frac{\alpha}{\sqrt{x+iy}} + \beta y + \frac{\gamma(x+2iy)}{\sqrt{x+iy}},$$

$$7)\, E10:\ V = \alpha(x-iy) + \beta(x+iy-\frac{3}{2}(x-iy)^2) + \gamma(x^2+y^2-\frac{1}{2}(x-iy)^3),$$

$$8)\, E11:\ V = \alpha(x-iy) + \frac{\beta(x-iy)}{\sqrt{x+iy}} + \frac{\gamma}{\sqrt{x+iy}},$$

$$9)\, E16:\ V = \frac{1}{\sqrt{x^2+y^2}}(\alpha + \frac{\beta}{y+\sqrt{x^2+y^2}} + \frac{\gamma}{y-\sqrt{x^2+y^2}}),$$

$$10)\, E17:\ V = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta}{(x+iy)^2} + \frac{\gamma}{(x+iy)\sqrt{x^2+y^2}},$$

$$11)\, E19:\ V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-4}} + \frac{\beta}{\sqrt{(x-iy)(x+iy+2)}} + \frac{\gamma}{\sqrt{(x-iy)(x+iy-2)}},$$

$$12)\, E20:\ V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \beta\sqrt{x+\sqrt{x^2+y^2}} + \gamma\sqrt{x-\sqrt{x^2+y^2}}\right),$$

Degenerate $E_{2\mathbb{C}}$ systems: $H\Psi = (\partial_x^2 + \partial_y^2 + V)\Psi = E\Psi$.

$$1)\, E3:\ V = \alpha(x^2+y^2),\ 2)\, E4:\ V = \alpha(x+iy),\ 3)\, E5:\ V = \alpha x,$$

$$4)\, E6:\ V = \frac{\alpha}{x^2},\ 5)\, E12:\ V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2+c^2}},\ 6)\, E13:\ V = \frac{\alpha}{\sqrt{x+iy}},$$

$$7)\, E14:\ V = \frac{\alpha}{(x+iy)^2},\quad 8)\, E18:\ V = \frac{\alpha}{\sqrt{x^2+y^2}}.$$

Nondegenerate systems on the complex 2-sphere: $H\Psi = (J_{23}^2 + J_{13}^2 + J_{12}^2 + V)\Psi = E\Psi, \quad J_{k\ell} = s_k\partial_{s_\ell} - s_\ell\partial_{s_k}, \quad s_1^2+s_2^2+s_3^2 = 1.$

$$1)\, S1:\ V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{(s_1+is_2)^2} + \frac{\gamma(1-4s_3^2)}{(s_1+is_2)^4},$$
$$2)\, S2: V = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1+is_2)^2} + \frac{\gamma(s_1-is_2)}{(s_1+is_2)^3},$$
$$3)\, S4:\ V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\gamma}{(s_1+is_2)\sqrt{s_1^2+s_2^2}},$$
$$4)\, S7:\ V = \frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\beta s_1}{s_2^2\sqrt{s_1^2+s_2^2}} + \frac{\gamma}{s_2^2},$$
$$5)\, S8:\ V = \frac{\alpha s_2}{\sqrt{s_1^2+s_3^2}} + \frac{\beta(s_2+is_1+s_3)}{\sqrt{(s_2+is_1)(s_3+is_1)}} + \frac{\gamma(s_2+is_1-s_3)}{\sqrt{(s_2+is_1)(s_3-is_1)}},$$
$$6)\, S9:\ V = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2},$$

Degenerate systems on the complex 2-sphere: $H\Psi = (J_{23}^2 + J_{13}^2 + J_{12}^2 + V)\Psi = E\Psi, \quad J_{k\ell} = s_k\partial_{s_\ell} - s_\ell\partial_{s_k}, \quad s_1^2+s_2^2+s_3^2 = 1.$

$$1)\, S3:\ V = \frac{\alpha}{s_3^2},\ 2)\, S5:\ V = \frac{\alpha}{(s_1+is_2)^2},\ 3)\, S6:\ V = \frac{\alpha z}{\sqrt{s_1^2+s_2^2}}.$$

14 Appendix B: $S_{2\mathbb{C}}$ and $E_{2\mathbb{C}}$ separable coordinates

14.1 Orthogonal separable coordinates in $S_{2\mathbb{C}}$

As for Euclidean space, coordinates separating the Hamilton-Jacobi equation on the two-sphere correspond to constants that are quadratic in the elements of the Lie algebra of its symmetry group $O(3,\mathbb{C})$. Coordinates belong to the same family if one can be transformed to the other by a rotation or reflection. On the complex two-sphere, unlike complex Euclidean space, every quadratic constant, other than a multiple of the Hamiltonian, corresponds to a separating coordinate system.

The separable coordinates on the complex two-sphere and their characterizing symmetry operators are:

1. Spherical coordinates: $s_1 = \sin\theta\cos\varphi\,, \quad s_2 = \sin\theta\sin\varphi\,,$
$$s_3 = \cos\theta\,, \qquad L = J_3^2\,.$$
2. Horospherical coordinates: $s_1 = \frac{i}{2}\left(v + \frac{u^2-1}{v}\right)\,, \quad s_2 = \frac{1}{2}\left(v + \frac{u^2+1}{v}\right)\,,$
$$s_3 = \frac{iu}{v}\,, \qquad L = (J_1 - iJ_2)^2\,.$$
3. Ellipsoidal coordinates: $s_1^2 = \frac{(cu-1)(cv-1)}{1-c}\,, \quad s_2^2 = \frac{c(u-1)(v-1)}{c-1}\,,$
$$s_3^2 = cuv\,, \qquad L = J_1^2 + cJ_2^2\,.$$

4. Degenerate Elliptic coordinates of type 1: $s_1+is_2=\frac{4cuv}{(u^2+1)(v^2+1)}$, $s_1-is_2=\frac{(u^2v^2+1)(u^2+v^2)}{cuv(u^2+1)(v^2+1)}$, $s_3=\frac{(u^2-1)(v^2-1)}{(u^2+1)(v^2+1)}$, $L=c^2(J_1+iJ_2)^2-J_3^2$.
5. Degenerate Elliptic coordinates of type 2. $s_1+is_2=-iuv$, $s_1-is_2=\frac{1}{4}\frac{(u^2+v^2)^2}{u^3v^3}$,
$$s_3=\frac{i}{2}\frac{u^2-v^2}{uv}\,, \qquad L=\{J_3,J_1-iJ_2\}\,.$$

14.2 Orthogonal separable coordinates in $E_{2\mathbb{C}}$

Each coordinate system in which the flat space Laplace-Beltrami eigevalue equaton (or the classical Hamilton-Jacobi equation) is separable on $E_{2\mathbb{C}}$ is characterized by a 2nd order symmetry operator. Coordinate systems that are related by Euclidean group motions belong to the same family and hence a given family of coordinates (e.g. polar coordinates) is associated with an equivalence class of quadratic elements in the enveloping algebra of $e(2,\mathbb{C})$. Two elements are equivalent if one can be transformed into to other by a combination of scalar multiplication, addition of multiples of $\partial_x^2+\partial_y^2$ and Euclidean motions (including reflections). The following can be taken as a representative list of coordinate systems and corresponding constants. (Here $M=x\partial_y-y\partial_x$ and $\{A,B\}=AB+BA$.)

1. Cartesian coordinates: $x\,,y\,,\qquad L=\partial_x^2\,.$
2. Polar coordinates: $x_S=r\cos\theta\,,\quad y_S=r\sin\theta\,,\qquad L=M^2\,.$
3. Semi-hyperbolic coordinates: $x_{SH}=ic(w-u)^2+2ic(w+u)\,,\quad y_{SH}=-c(w-u)^2+2c(w+u)\,,\ L=\frac{1}{2}\{M,(\partial_x+i\partial_y)\}+c(\partial_x-i\partial_y)^2\,.$
4. Hyperbolic coordinates: $x_H=\frac{r^2+s^2+r^2s^2}{2crs}\,,\quad y_H=i\frac{r^2+s^2-r^2s^2}{2crs},\quad L=M^2+c^{-2}(\partial_x+i\partial_y)^2\,.$
5. Parabolic coordinates: $x_P=\xi\eta\,,\ y_P=\frac{1}{2}(\xi^2-\eta^2)\,,\qquad L=\{M,\partial_x\}\,.$
6. Elliptic coordinates: $x_E=c\sqrt{(u-1)(v-1)}\,,\quad y_E=c\sqrt{-uv}\,,\qquad L=M^2+c^2\partial_x^2\,.$

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