Results Math (2018) 73:5

1422-6383/18/010001-18
published online January 30, 2018
https://doi.org/10.1007/s00025-018-0765-1

Results in Mathematics

On m-Subharmonic Ordering of Measures

Van Thien Nguyen

Abstract. We study an order-relation induced by m-subharmonic functions. We shall consider maximality with respect to this order and a related notion of minimality for certain m-subharmonic functions. This concept is then applied to the problem of convergence of measures in the weak*-topology, in particular Hessian measures.

Mathematics Subject Classification. 32U15, 32U99, 06F99.

Keywords. m-Subharmonic function, Cegrell classes, complex Hessian operator, ordering measures.

1. Introduction

In this paper we study an order-relation between measures on an m-hyperconvex domain Ω in $\mathbb{C}^n$. Let μ and ν be measures on Ω. We say that μ is m-subharmonically greater than ν if $\int_\Omega (-\varphi)d\mu \geq \int_\Omega (-\varphi)d\nu$, $\forall\ \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ and write $\mu \succcurlyeq \nu$, where $\mathcal{E}_{0,m}(\Omega)$ is the Cegrell class of negative m-subharmonic functions defined in Sect. 2. It is easy to see that the condition $\mu \geq \nu$ implies $\mu \succcurlyeq \nu$. But the inverse is not true (see Example 1). We also show that if u, v are functions in the Cegrell class $\mathcal{F}_m(\Omega)$ such that $u \leq v$, then their complex Hessian measures are in the relation $H_m(u) \succcurlyeq H_m(v)$ (see Proposition 2). But the inverse is not true (see Example 2).

In Sect. 4, we study maximality with respect to the $\succcurlyeq$-ordering, and a related notion of minimality for m-subharmonic functions in the class $\mathcal{F}_m(\Omega)$. A finite measure μ on Ω is said to be maximal if for any measure ν on Ω such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succcurlyeq \mu$ implies that $\nu = \mu$. The Dirac measure is a maximal measure. Theorem 9 shows that each finite measure on Ω with compact support is majorized in the $\succcurlyeq$-ordering by a maximal measure with the same total mass. A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any

Birkhäuser

function $v \in \mathcal{F}_m(\Omega)$ with the same total Hessian mass, the relation $v \leq u$ implies that $v = u$. We show that if a function $u \in \mathcal{F}_m(\Omega)$ and $H_m(u)$ is maximal measure, then u is minimal function (see Proposition 5). But the converse is still unknown. Theorem 10 shows that if $u \in \mathcal{F}_m(\Omega)$ is such that $H_m(u)$ is carried by an m-polar set, then u is a minimal function. However, there are functions in $\mathcal{F}_m(\Omega)$ whose Hessian measure are maximal and are not carried by an m-polar set. We also prove that each function in $\mathcal{F}_m(\Omega)$ is minorized by a minimal function with the same total Hessian mass.

In Sect. 5, we apply the m-subharmonic ordering to the problem of convergence in the weak*-topology. First, we prove that if $\{\mu_j\}$ is an m-subharmonically increasing sequence of measures on Ω with uniformly bounded total mass then μ_j converges to a measure μ in the weak*-topology. And finally, we use the notion of maximal measure to prove a sufficient condition of convergence in the weak*-topology for the class $\mathcal{F}_m(\Omega)$ (see Theorem 14).

2. Preliminaries

Let Ω be an open set in $\mathbb{C}^n$ and let m be a natural number $1 \leq m \leq n$.

As usual let $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, and let $\beta = dd^c|z|^2$ be the canonical Kähler form in $\mathbb{C}^n$. Denote by $SH_m(\Omega)$ the set of all m-subharmonic functions in Ω, and $SH_m^-(\Omega)$ for the set of all nonpositive m-subharmonic functions in Ω. For $u_1, \ldots, u_m \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$, the operator

$$\begin{aligned} H_m(u_1, \ldots, u_m) &:= dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m} \\ &= dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}) \end{aligned}$$

is a nonnegative Radon measure. In particular, when $u = u_1 = \cdots = u_m$, the Hessian measures

$$H_m(u) := (dd^c u)^m \wedge \beta^{n-m}$$

are well-defined for $u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ (see [4]).

Definition 1. Let E be a subset of Ω. The m-relative extremal function $h_{m,E,\Omega}$ is defined by

$$h_{m,E,\Omega}(z) = \sup\{u(z) \colon u \in SH_m(\Omega), u \leq 0 \text{ and } u \leq -1 \text{ on } E\}.$$

By [11, Proposition 1.5], we have that $h^*_{m,E,\Omega}$ is m-subharmonic on Ω.

Definition 2. Let Ω be an open set. A function $u \in SH_m(\Omega)$ is called m-maximal if $v \in SH_m(\Omega), v \leq u$ outside a compact set subset of Ω implies that $v \leq u$ in Ω.

Theorem 1 [4]. *Assume that $u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$. Then $H_m(u) = 0$ in Ω if and only if u is m-maximal.*

Now let us recall the definition of m-hyperconvex domain.

Definition 3. A bounded domain $\Omega \subset \mathbb{C}^n$ is called an m-hyperconvex if there exists an m-subharmonic function $\rho\colon \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega\colon \rho(z) < c\}$ is compact in Ω for every $c \in (-\infty, 0)$. In other words, the sublevel set $\{z \in \Omega\colon \rho(z) < c\}$ is relatively compact in Ω. Such a function ρ is called the exhaustion function.

Theorem 2 [9, Proposition 1.4.11]. *Let Ω be an m-hyperconvex bounded domain and $K \Subset \Omega$ is compact. Then $h^*_{m,K,\Omega}$ is m-maximal in $\Omega \backslash K$.*

Let us recall the definition of m-polar sets.

Definition 4. A set $E \subset \mathbb{C}^n$ is called m-polar if for any $z \in E$ there exists a neighbourhood V of z and $v \in SH_m(V)$ such that $E \cap V \subset \{v = -\infty\}$.

The following theorem was proved by Lu.

Theorem 3 [9, Theorem 1.6.5]. *If E is m-polar set, then there exists $u \in SH^-_m(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$.*

Throughout this paper Ω will denote a bounded m-hyperconvex domain in $\mathbb{C}^n$. Now we recall the definitions of the Cegrell classes.

Definition 5. (1) We let $\mathcal{E}_{0,m}(\Omega)$ denote the class of bounded functions in $SH_m(\Omega)$ such that

$$\lim_{z \to \partial\Omega} u(z) = 0 \text{ and } \int_\Omega H_m(u) < +\infty.$$

(2) A function $u \in SH_m(\Omega)$ belongs to $\mathcal{E}_m(\Omega)$ if for each $z_0 \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of z_0 and a decreasing sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ such that $u_j \downarrow u$ in U and $sup_j \int_\Omega H_m(u_j) < +\infty$.

(3) Denote $\mathcal{F}_m(\Omega)$ be the class of functions $u \in SH_m(\Omega)$ such that there exists a sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ decreases to u in Ω and $\sup_j \int_\Omega H_m(u_j) < +\infty$.

We have the following inclusions

$$\mathcal{E}_{0,m} \subset \mathcal{F}_m \subset \mathcal{E}_m \text{ and } SH^-_m(\Omega) \cap L^\infty_{loc}(\Omega) \subset \mathcal{E}_m.$$

Below we present some of the basic properties of the Cegrell classes.

Theorem 4 [2,9]. *For each $u \in SH^-_m(\Omega)$, there exists a sequence $\{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ such that $u_j \downarrow u$ in Ω.*

Proposition 1. *Let $\mathcal{K}$ be one of the classes $\mathcal{E}_{0,m}, \mathcal{F}_m, \mathcal{E}_m$. Then K is a convex cone. Moreover, if $u \in \mathcal{K}$ and $v \in SH^-_m(\Omega)$ then $\max\{u, v\} \in \mathcal{K}$.*

The following lemma explains why the functions in $\mathcal{E}_{0,m}(\Omega)$ are sometimes called test functions.

Theorem 5 [2,9]. *For $\varphi \in C^\infty_0(\Omega)$, there exist two functions u, v in $\mathcal{E}_{0,m} \cap C(\overline{\Omega})$ such that $\varphi(z) = u(z) - v(z), \forall z \in \Omega$.*

Following Cegrell's idea Lu proved that the Hessian operator is well-defined for the functions in the class $\mathcal{E}_m(\Omega)$.

Theorem 6 [9, Theorem 1.7.14]. *Let $u^k \in \mathcal{E}_m(\Omega), k = 1, \dots, m$ and $\{u_j^k\}_j$ be sequences in $\mathcal{E}_{0,m}(\Omega)$ such that $u_j^k \downarrow u^k$, for each $1 \le k \le m$. Then the sequence of measures*

$$dd^c u_1^1 \wedge \cdots \wedge dd^c u_j^m \wedge \beta^{n-m}$$

converge to a Radon measure in weak-topology independent to the choice of sequences $\{u_j^k\}$. We define $dd^c u^1 \wedge \cdots \wedge dd^c u^m \wedge \beta^{n-m}$ to be this limit.*

Integration by parts formula is true for the function from the Cegrell class $\mathcal{F}_m(\Omega)$.

Theorem 7 [9, Theorem 1.7.18]. *Assume that $u, v, w_1, \dots, w_{m-1} \in \mathcal{F}_m(\Omega)$. Then we have*

$$\int_\Omega u dd^c v \wedge T = \int_\Omega v dd^c u \wedge T,$$

where $T = dd^c w_1 \wedge \cdots \wedge dd^c w_{m-1} \wedge \beta^{n-m}$ and the equality means that if one of the two terms is finite then they are equal.

The following theorem is sometimes called the Cegrell decomposition theorem.

Theorem 8. *Let μ be a finite, positive measure on Ω. Then there exist $\varphi \in \mathcal{E}_{0,m}(\Omega)$ and $0 \le f \in L^1(H_m(\varphi))$ such that*

$$\mu = f H_m(\varphi) + \nu,$$

where ν is carried by a m-polar set.

Proof. By the proof of [10, Theorem 4.14], we can find a function $u \in \mathcal{E}_{1,m}(\Omega)$ and $0 \le f \in L^1(H_m(u))$ such that $\mu = f H_m(u) + \nu$, where ν is charged by an m-polar subset of Ω. The rest of the proof goes verbatim as the proof of [10, Theorem 5.3]. □

3. The m-Subharmonic Ordering

Let μ_j, μ be measures on Ω. By Theorem 5, we can see that following conditions are equivalent

(1) $\lim_{j\to\infty} \int_\Omega \varphi d\mu_j = \int_\Omega \varphi d\mu, \ \forall \varphi \in C_0(\Omega)$;
(2) $\lim_{j\to\infty} \int_\Omega \varphi d\mu_j = \int_\Omega \varphi d\mu, \ \forall \varphi \in C_0^\infty(\Omega)$;
(3) $\lim_{j\to\infty} \int_\Omega \varphi d\mu_j = \int_\Omega \varphi d\mu, \ \forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C_0(\overline{\Omega})$.

If one of above assertion is satisfied, we say that μ_j tends to μ on Ω in the weak*-topology.

Remark 1. (1) If $\mu_j \to \mu$ in the weak*-topology on Ω, then

$$\mu(\Omega) \leq \liminf_{j\to\infty} \mu_j(\Omega).$$

(2) Assume that $\{\mu_j\}_j$ is a sequence measures on Ω and $\sup_j \mu_j(\Omega) < \infty$, then there exists a subsequence $\{\mu_{j_k}\}_k \subset \{\mu_j\}_j$ such that μ_{j_k} converges to a measure μ in the weak*-topology as $k \to \infty$.

Definition 6. Let μ and ν be measures on Ω. We write $\mu \succcurlyeq \nu$ if and only if

$$\int_\Omega -\varphi d\mu \geq \int_\Omega -\varphi d\nu, \ \forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega}).$$

And we say that μ is m-subhamonically greater than ν.

Remark 2. (1) If $\mu \succcurlyeq \nu$, then $\int_\Omega -\varphi d\mu \geq \int_\Omega -\varphi d\nu$, $\forall \varphi \in SH_m^-(\Omega)$ by Theorem 4. In particular, $\mu(\Omega) \geq \nu(\Omega)$.

(2) If $\mu \geq \nu$, then $\mu \succcurlyeq \nu$. But Example 1 shows that the opposite implication is not true.

Example 1. For $a \in \Omega$, let δ_a be the Dirac measure at a. Let σ_r be the normalized measure on the sphere $\partial B(a,r)$, where r enough small such that $B(a,r) \subset \Omega$. Then for each $\varphi \in SH_m^-(\Omega)$, by the subharmonicity we have

$$\int_\Omega \varphi d\delta_a = \varphi(a) \leq \int_{\partial B(a,r)} \varphi d\sigma_r = \int_\Omega \varphi d\sigma_r.$$

Thus $\delta_a \succcurlyeq \sigma_r$, but it is clear that δ_a is not greater than σ_r even though $\delta_a(\Omega) = \sigma_r(\Omega) = 1$.

Proposition 2. *If $u, v \in \mathcal{F}_m(\Omega)$ and $u \geq v$, then $H_m(v) \succcurlyeq H_m(u)$.*

Proof. For $\varphi \in \mathcal{E}_{0,m}(\Omega)$, by Theorem 7

$$\begin{aligned}
\int_\Omega -\varphi H_m(u) &= \int_\Omega -u dd^c\varphi \wedge dd^c u^{m-1} \wedge \beta^{n-m} \\
&\leq \int_\Omega -v dd^c\varphi \wedge (dd^c u)^{m-1} \wedge \beta^{n-m} \\
&= \int_\Omega -\varphi dd^c v \wedge (dd^c u)^{m-1} \wedge \beta^{n-m} \\
&\leq \cdots \leq \int_\Omega -\varphi (dd^c v)^m \wedge \beta^{n-m} = \int_\Omega -\varphi H_m(v).
\end{aligned}$$

Thus $H_m(v) \succcurlyeq H_m(u)$. □

The following example shows that the converse implication to the statement given in Proposition 2 is not true.

Example 2. Let Ω is the unit ball $\mathbb{B}$ in $\mathbb{C}^n, n \geq 2$ and define the functions $v(z) = \frac{2}{3}(t^3 - 1), w(z) = t^2 - 1$. Then $v, w \in \mathcal{E}_{0,2}(\mathbb{B}) \cap C^2(\overline{\mathbb{B}})$ and $w \leq v$ on

$\mathbb{B}$, so $H_2(w) \succcurlyeq H_2(v)$ by Proposition 2. For more details, we can compute (see [12])

$$H_2(w)(z) = 4^n n! dV, \quad H_2(v)(z) = 2^{2n-1}(2n+2)(n-1)!|z|^2 dV,$$

where dV is the Lebesgue measure on $\mathbb{C}^n$. By [12] one can compute the solution u to the equation

$$H_2(u) = \frac{H_2(v) + H_2(w)}{2}, \ u \in \mathcal{E}_{0,2}(\mathbb{B}) \cap C^2(\overline{\mathbb{B}}). \tag{1}$$

The solution is given by

$$u(z) = \frac{\sqrt{2}}{3}\left(|z|^2 + 1\right)^{\frac{3}{2}} - \frac{4}{3}.$$

We have $H_2(u) \succcurlyeq H_2(v)$ by (1). Otherwise, $u(0) > -1 = v(0)$, so $v \not\geq u$.

Remark 3. The relation $\succcurlyeq$ defines a partial order on the set of positive Borel measures on Ω. But it is not a total order. To see that consider the Dirac measures δ_z and δ_w, where $z, w \in \Omega$ and $z \neq w$. Choose $\varphi, \psi \in SH_m^-(\Omega)$ such that $\varphi(z) < \varphi(w)$ and $\psi(z) > \psi(w)$. Then $\int_\Omega -\varphi d\delta_z > \int_\Omega -\varphi d\delta_w$ and $\int_\Omega -\psi d\delta_z < \int_\Omega -\psi d\delta_w$, so δ_z and δ_w are not comparable with respect to $\succcurlyeq$.

Definition 7. For a set $E \subset \Omega$, we define the convex hull of E in Ω with respect to the family $SH_m(\Omega) \cap C(\bar{\Omega})$, denoted by $\hat{E}$ as followed

$$\hat{E} = \{z \in \Omega \colon \varphi(z) \leq \sup_E \varphi, \ \forall \varphi \in SH_m(\Omega) \cap C(\overline{\Omega})\}.$$

Remark 4. We have that $\hat{E}$ is closed in Ω. Moreover, if E is relatively compact in Ω, so is $\hat{E}$.

Proposition 3. *Let μ, ν be finite regular measures on Ω such that $\mu(\Omega) = \nu(\Omega)$. If $\nu \succcurlyeq \mu$ then supp $\nu \subset \widehat{\text{supp } \mu}$.*

Proof. Put $K = \text{supp}\mu$. If $\hat{K} = \Omega$ then Proposition 3 is clear. Therefore we assume that $\Omega \backslash \hat{K} \neq \emptyset$. Suppose that $\text{supp}\nu \not\subset \hat{K}$. Since $\hat{K}$ is closed in Ω, it follows that $\nu(\Omega \backslash \hat{K}) > 0$. By the regularity of ν, we can find a compact set $L \in \Omega \backslash \hat{K}$ such that $\nu(L) > 0$. From the definition of $\hat{K}$, for each $z \in L$, there exist a neighborhood $U(z)$ of z and a function $\varphi \in SH_m(\Omega) \cap C(\overline{\Omega})$ such that $\varphi(\xi) > \sup_K \varphi, \ \forall \xi \in U(z)$. We choose $z_1, \ldots, z_k \in L$ such that $L \subset \bigcup_{i=1}^k U(z_i)$. Let $\varphi_1, \ldots, \varphi_k$ be the associated functions and $M_i = \sup_K \varphi_i$, $M = M_1 + \cdots + M_k$. Define

$$\psi = \max\{\varphi_1, M_1\} + \cdots + \max\{\varphi_k, M_k\}.$$

Then we have $\psi \in SH_m(\Omega) \cap C(\overline{\Omega}), \psi \geq M$ on $\overline{\Omega}$, $\psi = M$ on K and $\psi > M$ on L. Define $\psi_0 = \psi - \max_{\bar{\Omega}} \psi$ and let $M_0 = M - \max_{\overline{\Omega}} \psi$. Then $\psi_0 \in SH_m^-(\Omega) \cap C(\overline{\Omega})$, $\psi_0 \geq M_0$ on Ω, $\psi_0 = M_0$ on K and $\psi_0 > M_0$ on L. Hence,

$$\int_{\Omega} -\psi_0 d\nu < -M_0\nu(\Omega) = -M_0\mu(\Omega) = \int_{\Omega} -\psi_0 d\mu.$$

Proposition 3 is proved by a contradiction. □

4. Maximal Measures and Minimal Functions

We want to study the maximality with respect to the m-subharmonic ordering by using some kind of normalization.

Definition 8. A finite measure μ on Ω is said to be maximal if for any measure ν on Ω such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succcurlyeq \mu$ implies that $\nu = \mu$.

Example 3. For $1 \leq m < n$, we define

$$\varphi_j(z) = \max\left\{-\frac{1}{j}|z|^{2-\frac{2n}{m}}, -1\right\} \in SH_m^-(\mathbb{B})$$

and δ_0 is the Dirac measure defined on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. Then for each measure ν, $\nu(\Omega) = 1$ and $\nu \succcurlyeq \delta_0$ we have

$$\lim_{j\to\infty} \int_{\mathbb{B}} -\varphi_j d\nu = -\nu(\{0\})$$

and

$$-1 \leq \int_{\mathbb{B}} -\varphi_j d\delta_0 \leq \int_{\mathbb{B}} -\varphi_j d\nu \leq 1, \ \forall j.$$

Thus we get $\nu(\{0\}) = 1$, so $\nu = \delta_0$ which implies δ_0 is maximal.

Remark 5. (1) If we can write a maximal measure as the sum $\mu = \mu_1 + \mu_2$ of two finite measures, then these are maximal too. To prove this, assume that μ_1 is not maximal. Then there is a finite measure $\nu \neq \mu_1$ such that $\nu(\Omega) = \mu_1(\Omega)$ and $\nu \succcurlyeq \mu$. We have $(\nu + \mu_2)(\Omega) = \mu(\Omega)$ and $\nu + \mu_2 \succcurlyeq \mu$, but $\nu + \mu_2 \neq \mu$, which is a contradiction.

(2) If μ is maximal measure, so is $c\mu$, for $c > 0$.

(3) We will show that the condition μ_1, μ_2 are maximal does not imply the maximality of $\mu_1 + \mu_2$ (see Example 5). This implies that the set of maximal measures on Ω is not a convex cone.

Definition 9. We say that a set $K \Subset \Omega$ is an interpolation set for $SH_m^-(\Omega)$ if for each $f \in C(K)$, $f < 0$ there exists a function $\varphi \in SH_m^-(\Omega)$ such that $\varphi = f$ on K.

Proposition 4. *If μ is a finite measure on Ω such that $\widehat{supp\mu}$ is contained in some interpolation set K for $SH_m^-(\Omega)$, then μ is maximal.*

Proof. Assume that ν is a measure on Ω such that $\nu(\Omega) = \mu(\Omega)$ and $\nu \succcurlyeq \mu$. By Proposition 3, we have $\operatorname{supp}\nu \subset \widehat{supp\mu} \subset K$. For a given $f \in C(K), f \leq 0$, there exists a function $\varphi \in SH_m^-(\Omega)$ such that $\varphi = f$ on K. We get

$$\int_\Omega -f d\nu = \int_\Omega -\varphi d\nu \leq \int_\Omega -\varphi d\mu = \int_\Omega -f d\mu.$$

This implies that $\int_\Omega f d\mu \geq \int_\Omega f d\nu$ holds for any $f \in C_0(\Omega), f \leq 0$. Hence $\mu \leq \nu$, so $\mu = \nu$. □

Example 4. Let $a_1, \ldots, a_k \in \Omega$. For $1 \leq j \leq k$, we choose M_j such that

$$\psi_j(z) = \sum_{l \neq j}^{k} \ln|z - a_l| + M_j \in SH_m^-(\Omega).$$

For each value $c_j < 0$, we take $d_j > 0$ such that $d_j\psi_j(a_j) = c_j$. Define $\varphi = \max(d_1\psi_1, \ldots, d_k\psi_k)$. Then we have $\varphi \in SH_m^-(\Omega)$ and $\varphi(a_j) = c_j$. Thus the finite set $\{a_1, \ldots, a_k\}$ is an interpolation set for $SH_m^-(\Omega)$. And Proposition 4 implies that the measure $\sum_{j=1}^k b_j\delta_{a_j}$ is maximal, where δ_{a_j} is the Dirac measure at the point a_j and $b_1, \ldots, b_k$ are given nonnegative numbers.

We will show that each finite measure with compacted support is majorized by a maximal measure with the same total mass.

Lemma 1. *Assume that μ and ν are measures on Ω such that $\nu \succcurlyeq \mu$. If $\int_\Omega \varphi d\mu = \int_\Omega \varphi d\nu > -\infty$ for some negative strictly m-subharmonic function φ. Then $\mu = \nu$.*

Proof. For given $f \in C_0^\infty(\Omega)$, choose a constant $c > 0$ so that $(\pm f + c\varphi) \in SH_m^-(\Omega)$. Then we have

$$\begin{aligned}\int_\Omega (\pm f + c\varphi) d\mu &= \int_\Omega \pm f d\mu + c\int_\Omega \varphi d\mu \geq \int_\Omega (\pm f + c\varphi) d\nu \\ &= \int_\Omega \pm f d\nu + c\int_\Omega \varphi d\nu,\end{aligned}$$

which implies that $\int_\Omega \pm f d\mu \geq \int_\Omega \pm f d\nu$. So $\mu = \nu$. □

Theorem 9. *Let μ be a finite measure on Ω with compact support. Then there is a maximal measure μ_0 such that $\mu_0 \succcurlyeq \mu$ and $\mu_0(\Omega) = \mu(\Omega)$.*

Proof. Put $K = \widehat{\operatorname{supp}\mu}$ and

$$\mathcal{M}_\mu = \{\nu \colon \nu \succcurlyeq \mu, \nu(\Omega) = \mu(\Omega)\}.$$

Because $\mu \in \mathcal{M}_\mu$, so $\mathcal{M}_\mu \neq \emptyset$. By Proposition 3, $\operatorname{supp}\nu \subset K$ for each $\nu \in \mathcal{M}_\mu$. Let ρ be the exhaustion function of Ω that is negative, continuous strictly m-subharmonic. We define

$$A = \sup_{\nu \in \mathcal{M}_\mu} \int_\Omega (-\rho) d\nu.$$

Since ρ is bounded on K, it follows that A is finite. Let $\{\nu_j\}_j$ be a sequence in $\mathcal{M}_\mu$ such that $\int_\Omega(-\rho)d\nu_j \to A$, as $j \to \infty$. By Remark 1, we may assume that ν_j tend to some measure μ_0 in the weak*-topology and $\mu_0(\Omega) \leq \mu(\Omega)$. For each $\varphi \in \mathcal{E}_{0,m} \cap C(\overline{\Omega})$,

$$\int_\Omega (-\varphi)d\mu_0 = \lim_{j\to\infty} \int_\Omega (-\varphi)d\nu_j \geq \int_\Omega (-\varphi)d\mu,$$

which implies that $\mu_0 \succcurlyeq \mu$. By Remark 2 and the fact $\mu_0 \leq \mu(\Omega)$, we get $\mu_0(\Omega) = \mu(\Omega)$. Thus $\mu_0 \in \mathcal{M}_\mu$. Take a function $f \in C_0(\Omega)$, $f = 1$ on K. We get

$$\int_\Omega (-\rho)d\mu_0 = \int_\Omega (-\rho)f d\mu_0 = \lim_{j\to\infty} \int_\Omega (-\rho)f d\nu_j = \lim_{j\to\infty} \int_\Omega (-\rho)d\nu_j = A.$$

Suppose that ν be any measure on Ω such that $\nu \geq \mu_0$ and $\nu(\Omega) = \mu(\Omega)$. Then $\nu \in \mathcal{M}_\mu$ and $A \geq \int_\Omega(-\rho)d\nu \geq \int_\Omega(-\rho)d\mu_0 = A$. Hence $\int_\Omega(-\rho)d\nu = \int_\Omega(-\rho)d\mu_0 = A$. Lemma 1 implies that $\nu = \mu_0$, so Theorem 9 is finished. $\square$

Definition 10. A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any function $v \in \mathcal{F}_m(\Omega)$, the conditions $H_m(u)(\Omega) = H_m(v)(\Omega)$ and $v \leq u$ imply $v = u$.

Proposition 5. *Let $u \in \mathcal{F}_m(\Omega)$ be such that $H_m(u)$ is a maximal measure. Then u is minimal.*

To prove this proposition we need the following lemma.

Lemma 2. *If $u, v \in \mathcal{F}_m(\Omega)$, $H_m(u) = H_m(v)$ and $u \leq v$ then $u = v$.*

Proof. We use a method from [7]. Using integration by parts, we have

$$\begin{aligned}
\int_\Omega -(u-v)(dd^c\rho)^m \wedge \beta^{n-m} &= \int_\Omega d(u-v) \wedge d^c\rho \wedge (dd^c\rho)^{m-1} \wedge \beta^{n-m} \\
&\leq \left[\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c\rho)^{m-1} \wedge \beta^{n-m}\right]^{\frac{1}{2}} \\
&\quad \times \left[\int_\Omega d\rho \wedge d^c\rho \wedge (dd^c\rho)^{m-1} \wedge \beta^{n-m}\right]^{\frac{1}{2}},
\end{aligned}$$

where $\rho \in \mathcal{E}_{0,m}(\Omega) \cap C^\infty(\Omega)$ is a strictly m-subharmonic exhaustion function of Ω (see [2]). Hence, to prove $u = v$ it is enough to show that

$$\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c\rho)^{m-1} \wedge \beta^{n-m} = 0. \tag{2}$$

If $m = 1$ then (2) is clear. For $m \geq 2$ and $j + k = m - 1$, we have

$$\begin{aligned} 0 &\leq \int_\Omega -(u-v)(dd^c u)^j \wedge (dd^c v)^k \wedge dd^c\rho \wedge \beta^{n-m} \\ &= \int_\Omega -\rho dd^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge \beta^{n-m} \\ &\leq \int_\Omega -(u-v) \sum_{a+b=m-1} (dd^c u)^a \wedge (dd^c v)^b \wedge dd^c\rho \wedge \beta^{n-m} \\ &= \int_\Omega -\rho dd^c(u-v) \wedge \sum_{a+b=m-1} (dd^c u)^a \wedge (dd^c v)^b \wedge \beta^{n-m} \\ &= \int_\Omega -\rho(H_m(u) - H_m(v)) = 0. \end{aligned}$$

Thus, for every couple $j, k, j + k = m - 2$ we have

$$\begin{aligned} &\int_\Omega -udd^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c\rho \wedge \beta^{n-m} \\ &= \int_\Omega -\rho dd^c(u-v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge \beta^{n-m} = 0. \end{aligned}$$

Similarly, $\int_\Omega -vdd^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c\rho \wedge \beta^{n-m} = 0$. So

$$\begin{aligned} &\int_\Omega -(u-v)dd^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c\rho \wedge \beta^{n-m} \\ &= \int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c\rho \wedge \beta^{n-m} = 0, \end{aligned} \tag{3}$$

for every couple $j, k, j + k = m - 2$. Assume that

$$\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c\rho)^l \wedge \beta^{n-m} = 0 \tag{4}$$

for $j + k = m - l - 1$. By (3), (4) is true for $l = 1$. For $j + k = m - l - 2$ we have

$$\begin{aligned} &\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \ (dd^c\rho)^{l+1} \wedge \beta^{n-m} \\ &= \int_\Omega -\rho(dd^c(u-v))^2 \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c\rho)^l \wedge \beta^{n-m} \\ &= \int_\Omega d\rho \wedge d^c(u-v) \wedge dd^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c\rho)^l \wedge \beta^{n-m} \\ &\leq \left| \int_\Omega d\rho \wedge d^c(u-v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c\rho)^l \wedge \beta^{n-m} \right| \\ &\quad + \left| \int_\Omega d\rho \wedge d^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c\rho)^l \wedge \beta^{n-m} \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\int_\Omega d\rho \wedge d^c\rho \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m}\right]^{\frac{1}{2}} \\
&\quad \times \left[\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m}\right]^{\frac{1}{2}} \\
&\quad + \left[\int_\Omega d\rho \wedge d^c\rho \wedge (dd^c u)^{j} \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m}\right]^{\frac{1}{2}} \\
&\quad \times \left[\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^{j} \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m}\right]^{\frac{1}{2}} \\
&= 0,
\end{aligned}
$$

by assumption (4). So (2) is true by taking $l = m-1$ in (4). □

Proof of Proposition 5. Assume that $v \in \mathcal{F}_m(\Omega), H_m(v)(\Omega) = H_m(u)(\Omega)$ and $v \leq u$. Since $v \leq u$, Proposition 2 implies that $H_m(v) \succcurlyeq H_m(u)$. From the assumption $H_m(u)$ is maximal, we get $H_m(u) = H_m(v)$. Now Proposition 5 follows from Lemma 1. □

Lemma 3. *Assume that $u, v \in \mathcal{E}_m(\Omega)$ and $u \geq v$. Then $\chi_{\{u=-\infty\}} H_m(u) \leq \chi_{\{v=-\infty\}} H_m(v)$.*

Proof. We use a method from [1]. For $\epsilon > 0$ small enough, set $w_j = \max\{(1-\epsilon)u - j, v\}$. Then we have $w_j = (1-\epsilon)u - j$ on the open set $\{v < -\frac{j}{\epsilon}\}$. Therefore

$$
H_m(w_j) = (1-\epsilon)^m H_m(u) \text{ on } \{v < -\frac{j}{\epsilon}\}.
$$

Hence $H_m(w_j) \geq (1-\epsilon)^m \chi_{\{u=-\infty\}} H_m(u)$. Letting $j \to \infty$, then we get $H_m(v) \geq (1-\epsilon)^m \chi_{\{u=-\infty\}} H_m(u)$. The proof is complete by letting $\epsilon \to 0^+$. □

Lemma 4. *For each $u \in \mathcal{F}_m(\Omega)$, if $H_m(u)$ is carried by an m-polar set, then $H_m(u) = \chi_{\{u=-\infty\}} H_m(u)$.*

Proof. We use the same idea as in [5]. We choose a sequence $\{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega)$, $u_j \downarrow u$. Then $\frac{u_j}{1-u_j} \downarrow \frac{u}{1-u} \in \mathcal{F}_m(\Omega) \cap L^\infty(\Omega)$. For each $v \in C^2(\Omega)$,

$$
\frac{\partial}{\partial z_l \partial \bar{z_k}} \left(\frac{v}{1-v}\right) = \frac{v_{l\bar{k}}}{(1-v)^2} + \frac{2 v_l v_{\bar{k}}}{(1-v)^3}, \forall\, 1 \leq l, k \leq n.
$$

This implies that

$$
\frac{H_m(u_j)}{(1-u_j)^{2m}} \leq H_m\left(\frac{u_j}{1-u_j}\right).
$$

The function $\frac{1}{(1-t)^{2m}}$ is convex on $[-\infty, 0]$, hence by [11, Proposition 2.1], $\frac{1}{(1-u)^{2m}} - 1 \in SH_m^-(\Omega)$. For every fixed k,

$$\begin{aligned}\left(\frac{1}{(1-u_k)^{2m}} - 1\right) H_m(u) &\geq \lim_{j\to\infty} \left(\frac{1}{(1-u_k)^{2m}} - 1\right) H_m(u_j) \\ &\geq \lim_{j\to\infty} \left(\frac{1}{(1-u_j)^{2m}} - 1\right) H_m(u_j) \geq \lim_{j\to\infty} \left(\frac{1}{(1-u)^{2m}} - 1\right) H_m(u_j) \\ &= \left(\frac{1}{(1-u)^{2m}} - 1\right) H_m(u).\end{aligned}$$

Letting $k \to \infty$, we get $\frac{H_m(u_j)}{(1-u_j)^{2m}}$ tends weakly to $\frac{H_m(u)}{(1-u)^{2m}}$. Moreover, $H_m\left(\frac{u_j}{1-u_j}\right)$ tends weakly to $H_m\left(\frac{u}{1-u}\right)$. Hence,

$$\frac{H_m(u)}{(1-u)^{2m}} \leq H_m\left(\frac{u}{1-u}\right). \tag{5}$$

Theorem 8 shows that there exist $\varphi \in \mathcal{E}_{0,m}(\Omega)$ and $f \in L^1(H_m(\varphi))$ such that

$$H_m(u) = fH_m(\varphi) + \nu,$$

where ν is carried by an m-polar set. Moreover, (5) implies that $\frac{H_m(u)}{(1-u)^{2m}}$ has no mass on m-polar sets. Hence, $\frac{\nu}{(1-u)^{2m}} = 0$, so ν is carried by the set $\{u = -\infty\}$. $\square$

Theorem 10. *Let $u \in \mathcal{F}_m(\Omega)$ be such that $H_m(u)$ is carried by an m-polar set. Then u is a minimal function.*

Proof. Assume that $v \in \mathcal{F}_m(\Omega)$, $v \leq u$ and $H_m(v)(\Omega) = H_m(u)(\Omega)$. By Lemmas 3 and 4,

$$\int_\Omega H_m(v) \geq \int_\Omega \chi_{\{v=-\infty\}} H_m(v) \geq \int_\Omega \chi_{\{u=-\infty\}} H_m(u) = \int_\Omega H_m(u).$$

Hence, $H_m(v) = \chi_{\{v=-\infty\}} H_m(v)$. By Lemma 3 again, $H_m(u) \leq H_m(v)$. Combine this with $H_m(u)(\Omega) = H_m(v)(\Omega)$, we get $H_m(u) = H_m(v)$. Lemma 2 implies that $u = v$. $\square$

Proposition 6. *Assume that μ is a finite measure on Ω such that $\widehat{supp\mu}$ is contained in a level set $\{z \in \Omega\colon \psi(z) = c\}$, where $c > -\infty$ and $\psi < 0$ is a strictly m-subharmonic function on Ω. Then μ is maximal.*

Proof. Suppose that $\nu \succcurlyeq \mu$ and $\nu(\Omega) = \mu(\Omega)$. By Proposition 3, $\mathrm{supp}\nu \subset \{z \in \Omega\colon \psi(z) = c\}$. Thus,

$$\int_\Omega -\psi d\nu = \int_\Omega -c d\nu = \int_\Omega -c d\mu = \int_\Omega -\psi d\mu < \infty.$$

Therefore, Lemma 1 implies that $\nu = \mu$, and the proof is complete. $\square$

The following example confirms Remark 5(3).

Example 5 [3, Examples 4.15, 4.16]. We consider the unit disc $\mathbb{D}$ in $\mathbb{C}$. Define the sets $S_1 = \{z = \frac{1}{2}e^{i\theta} : 0 \le \theta \le \pi\}$ and $S_2 = \{z = \frac{1}{2}e^{i\theta} : \pi < \theta < 2\pi\}$. Let σ be the area measure on the circle $\partial\mathbb{D}(0, \frac{1}{2})$ and define $\mu_j = \sigma_{|S_j}$, for $j = 1, 2$. We have $S_j \subset \{\psi = |z|^2 - 1 = -\frac{3}{4}\}$. Let $h_j = h_{1,S_j,\mathbb{D}}$ be the 1-relative extremal function for S_j over $\mathbb{D}$. Then $h_j \in \mathcal{E}_{0,1}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and $h_j = -1$ on S_j. Moreover, h_j is harmonic on the connected set $\mathbb{D}\backslash S_j$, which implies that $h > -1$ on $\mathbb{D}\backslash S_j$. Hence $\hat{S}_j = S_j$ and Proposition 6 deduces that μ_1 and μ_2 are maximal measures. But $\sigma = \mu_1 + \mu_2$ is not maximal (see Example 1).

We will show that each function in $\mathcal{F}_m(\Omega)$ is minorized by a minimal function with the same total Hessian mass.

Proposition 7. *Let $\{u_j\}$ be a decreasing sequence in $\mathcal{F}_m(\Omega)$ such that $u_j \downarrow u$ and $H_m(u_j)(\Omega) = H_m(u_{j+1})(\Omega)$ for all j. Then $u \in \mathcal{F}_m(\Omega)$ and $H_m(u)(\Omega) = H_m(u_j)(\Omega)$.*

Proof. We have $u \in SH_m^-(\Omega)$, and by Theorem 4, there exists a sequence $\{w_j\} \subset \mathcal{E}_{0,m}(\Omega) \cap C(\bar{\Omega})$ such that $w_j \downarrow u$ as $j \to \infty$. Set $v_j = \max(w_j, u_j)$. Then $v_j \ge u_j$, $v_j \in \mathcal{E}_{0,m}(\Omega)$ and $v_j \downarrow u$ as $j \to \infty$. Theorem [10, Theorem 3.22] implies that

$$\sup_j \int_\Omega H_m(v_j)(\Omega) \le \sup_j H_m(u_j) = H_m(u_1) < \infty,$$

Thus, $u \in \mathcal{F}_m(\Omega)$. Since the sequence of measures $H_m(v_j)$ converges to the measure $H_m(u)$ in the weak*-topology, we get

$$\liminf_{j\to\infty} H_m(v_j)(\Omega) \ge H_m(u)(\Omega).$$

Moreover, by [10, Theorem 3.22] again, we obtain $H_m(u)(\Omega) \ge H_m(u_j)$ since $u, u_j \in \mathcal{F}_m(\Omega), u \le u_j$. □

Theorem 11. *For each $u \in \mathcal{F}_m(\Omega)$, there exists a minimal function $u_0 \in \mathcal{F}_m(\Omega)$ such that $u_0 \le u$ and $H_m(u_0)(\Omega) = H_m(u)(\Omega)$.*

Proof. Define $S = \{v \in \mathcal{F}_m(\Omega) : v \le u, H_m(v)(\Omega) = H_m(u)(\Omega)\}$. Let T be the totally ordered subset of S and let $t(z) = \inf_{v\in T} v(z)$. We shall prove that $t \in S$. It is obvious that $t \le u$. Let $\{K_i\}$ be a compact exhaustion sets of Ω and let $\{t_j\}$ be a sequence of continuous functions such that $t_j \ge t$ and $t_j \downarrow t$ as $j \to \infty$. For each $z \in K_i$, choose $v_z \in T$ such that $v_z(z) < t_j(z)$ and define the open set $U_z = \{w \in \Omega : v_z(w) < t_j(w)\}$. Take $z_1, \ldots, z_N \in K_i$ such that $\cup_{k=1}^N U_{z_k} \supset K_i$. Since T is totally ordered, we may choose v_i^j to be the smallest of the functions $v_{z_1}, \ldots, v_{z_N}$, which implies that $v_i^j < t_j$ on K_i. Now let $u_1 = v_1^1$ and u_j be the smallest of the functions $\{u_1, \ldots, u_{j-1}, v_j^j\}$ if $j \ge 2$, since T is totally ordered. Then $\{u_j\}$ is a decreasing sequence of functions in T such that $u_j \le v_j^j < t_j$ on K_j. Therefore $u_j \in \mathcal{F}_m(\Omega), H_m(u_j)(\Omega) = H_m(u)(\Omega)$ and $u_j \downarrow t$, as $j \to \infty$. Proposition 7 implies $t \in \mathcal{F}_m(\Omega)$ and $H_m(t)(\Omega) = H_m(u)(\Omega)$.

Hence $t \in S$. Since T is arbitrary, Zorn's lemma deduces that there is a minimal element u_0 of S, so the proof is complete. $\square$

5. Convergence in the Weak*-Topology

We will use the m-subharmonic ordering to obtain some results on weak*-convergence of measures. If Ω is a bounded domain in $\mathbb{C}^n$ and $\{u_j\}$ is a sequence of locally bounded m-subharmonic functions on Ω which is decreasing to a function $u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$, then $H_m(u_j)$ converges to $H_m(u)$ in the weak*-topology (see [4]). The same conclusion holds if $SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ is replaced by the class $\mathcal{E}_m(\Omega)$, where Ω is a bounded m hyperconvex domain (see [9]).

The following example shows that Hessian operator is discontinuous with respec to the convergence in L^1_{loc}. This example follows the idea in [8].

Example 6. For $n \geq 2$, we define

$$u_j(z_1, \ldots, z_n) = \left| \sum_{k=1}^{n} z_k^{2j} \right|^{\frac{1}{2j}}$$

We can compute

$$\frac{\partial^2 u}{\partial z_p \partial \bar{z}_q} = \frac{1}{4} \left| \sum_{k=1}^{n} z_k^{2j} \right|^{\frac{1}{2j}-2} z_p^{2j-1} \bar{z}_q^{2j-1}, \ \forall 1 \leq p, q \leq n.$$

Thus, $H_m(u_j) = 0$, for all j. We have $0 \leq u_j \leq n^{\frac{1}{2j}} u$, where $u(z_1, \ldots, z_n) = \max\{|z_1|, \ldots, |z_n|\}$. Hence, we get $u_j \to u$ in $L^1_{loc}(\mathbb{C}^n)$ as $j \to \infty$. We can show that $H_m(u) \neq 0$. Assume the contrary. Then $H_m(u) = 0$ on the polydisc $\Delta_n(r) = \mathbb{D}(0, r) \times \cdots \times \mathbb{D}(0, r)$, i.e., u is m-maximal function on $\Delta_n(r)$. Note that $u \geq r_1$ outside the compact subset $\overline{\Delta_n}(r_1)$, where $r_1 < r$ but we do not have $u \geq r_1$ on $\Delta_n(r)$.

The following theorem give us a sufficient condition for weak*-convergence for the class $\mathcal{F}_m(\Omega)$.

Theorem 12. *If $u_j \to u$ in $L^1_{loc}(\Omega)$ and there is a strictly m-subharmonic function $v \in \mathcal{E}_{0,m}(\Omega)$ such that*

$$\int_\Omega v H_m(u_j) \to \int_\Omega v H_m(u) \ \textit{as } j \to \infty,$$

then $H_m(u_j)$ tends to $H_m(u)$ in the weak-topology.*

Proof. We use the idea from [6]. For $w \in \mathcal{E}_{0,m}(\Omega)$, using integration by parts (Theorem 7) we have

$$\int_\Omega wH_m(u_j) \le \int_\Omega wH_m\left[(\sup_{s\ge j} u_s)^*\right]\downarrow \int_\Omega wH_m(u) \text{ as } j\to\infty.$$

Hence,

$$\limsup_{j\to\infty}\int_\Omega wH_m(u_j) \le \int_\Omega wH_m(u). \tag{6}$$

Theorem 4 implies that (6) is true for $w \in SH_m^-(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$ be given. By assumption v is strictly m-subharmonic we can choose $A > 0$ large enough such that $(\pm\varphi + Av) \in \mathcal{E}_{0,m}(\Omega)$. By (6) we have

$$\limsup_{j\to\infty}\int_\Omega(\pm\varphi + Av)H_m(u_j) \le \int_\Omega(\pm\varphi + Av)H_m(u).$$

Combining this with assumption $\lim_{j\to\infty}\int_\Omega vH_m(u_j) = \int_\Omega vH_m(u)$ we obtain

$$\limsup_{j\to\infty}\int_\Omega \pm\varphi H_m(u_j) \le \int_\Omega \pm\varphi H_m(u),$$

which implies the desired result. $\square$

Definition 11. If $\{\mu_j\}$ is a sequence of measures such that $\mu_{j+1} \succcurlyeq \mu_j$ for all j, then we say that $\{\mu_j\}$ is m-subharmonically increasing.

Theorem 13. *Let $\{\mu_j\}$ be an m-subharmonically increasing sequence of measures on Ω such that $\sup_j \mu_j(\Omega) < \infty$. Then μ_j converges to a measure μ in the weak*-topology. Moreover, $\int_\Omega(-\varphi)d\mu_j \uparrow \int_\Omega(-\varphi)d\mu$ for each $\varphi \in SH_m^-(\Omega)$.*

Proof. Let $\varphi \in SH_m^-(\Omega)\cap L^\infty(\Omega)$. Then

$$0 \le \int_\Omega(-\varphi)d\mu_1 \le \int_\Omega(-\varphi)d\mu_2 \le \cdots \le \sup_\Omega(-\varphi)\sup_j \mu_j(\Omega) < \infty.$$

so $\lim_{j\to\infty}\int_\Omega(-\varphi)d\mu_j < \infty$. Thus the limit exists for each $\varphi \in C_0(\Omega)$. It follows that this defines a measure μ on Ω that μ_j converges to μ in the weak*-topology. Moreover, we know that $\lim_{j\to\infty}\int_\Omega(-\varphi)d\mu_j = \int_\Omega(-\varphi)d\mu$ for each $\varphi \in \mathcal{E}_{0,m}(\Omega)\cap C(\bar\Omega)$. Now, let $\varphi \in SH_m^-(\Omega)$. As above $\{\int_\Omega(-\varphi)d\mu_j\}$ is an increasing sequence. We always have

$$\lim_{j\to\infty}\int_\Omega(-\varphi)d\mu_j \ge \int_\Omega(-\varphi)d\mu. \tag{7}$$

To show the equality in (7), we assume the contrary, i.e.,

$$\lim_{j\to\infty}\int_\Omega(-\varphi)d\mu_j > \int_\Omega(-\varphi)d\mu.$$

Choose j_0 enough large such that $\int_\Omega(-\varphi)d\mu_{j_0} > \int_\Omega(-\varphi)d\mu$, and a sequence $\{\varphi_k\} \in \mathcal{E}_{0,m}\cap C(\bar\Omega)$ such that $\varphi_k \downarrow \varphi$. Then we might choose k_0 such that $\int_\Omega(-\varphi_{k_0})d\mu_{j_0} > \int_\Omega(-\varphi)d\mu$. It follows that

$$\int_{\Omega}(-\varphi_{k_0})d\mu = \lim_{j\to\infty}\int_{\Omega}(-\varphi_{k_0})d\mu_j \geq \int_{\Omega}(-\varphi_{k_0})d\mu_{j_0}$$
$$> \int_{\Omega}(-\varphi)d\mu \geq \int_{\Omega}(-\varphi_{k_0})d\mu,$$

which is a contradiction. □

If $\{u_j\} \subset \mathcal{F}_m(\Omega)$ converges to $u \in \mathcal{F}_m(\Omega)$ in $L^1_{loc}(\Omega)$, then we can relate the limit measure of sequence $\{H_m(u_j)\}$ in Theorem 13 to $H_m(u)$ as follows.

Corollary 1. *Assume that* $\{u_j\} \subset \mathcal{F}_m(\Omega)$ *such that*

(1) u_j *converges to* $u \in \mathcal{F}_m(\Omega)$ *in* $L^1_{loc}(\Omega)$,
(2) $\{H_m(u_j)\}$ *is* m*-subharmonically increasing,*
(3) $\sup_j H_m(u_j) < \infty$.

Then $H_m(u_j)$ *converges to a measure* μ *in the weak*-topology such that* $\mu \succcurlyeq H_m(u)$. *Moreover,* $\int_\Omega(-\varphi)H_m(u_j) \uparrow \int_\Omega(-\varphi)d\mu$ *for each* $\varphi \in SH_m^-(\Omega)$.

Proof. By Theorem 13 it remains to show that $\mu \succcurlyeq H_m(u)$. By the proof of Theorem 12, assumption (1) implies that $\liminf_{j\to\infty}\int_\Omega(-\varphi)H_m(u_j) \geq \int_\Omega(-\varphi)H_m(u)$ for each $\varphi \in SH_m^-(\Omega)$. □

The following theorem gives us a bridge between convergence in weak*-topology and the concept of maximal measures defined in Sect. 4.

Theorem 14. *Let* $\{u_j\} \subset \mathcal{F}_m(\Omega)$ *such that*

(1) u_j *converges to* $u \in \mathcal{F}_m(\Omega)$ *in* $L^1_{loc}(\Omega)$,
(2) $H_m(u)$ *is a maximal measure,*
(3) $\lim_{j\to\infty} H_m(u_j)(\Omega) = H_m(u)(\Omega)$.

Then $H_m(u_j)$ *converges to* $H_m(u)$ *in the weak*-topology.*

Proof. Assumption (3) implies that there is a subsequence $\{H_m(u_{j_k})\} \subset \{H_m(u_j)\}$ which converging to a measure μ in the weak*-topology. Let $\varphi \in \mathcal{E}_{0,m}(\Omega)\cap C(\bar{\Omega})$ be given. As in the proof of Corollary 1, assumption (a) implies that $\mu \succcurlyeq H_m(u)$. Moreover, by (3) we have $\mu(\Omega) \leq \liminf_{j\to\infty} H_m(u_{j_k})(\Omega) \leq H_m(u)(\Omega)$. Thus, $\mu(\Omega) = H_m(u)(\Omega)$. By assumption (2) we can conclude that $\mu = H_m(u)$. □

Open Question

One might ask if there is a converse of Proposition 5. The answer is affirmative if $n = m = 1$ (see [3, Proposition 4.11]). In higher dimension, the answer is unknown.

Acknowledgements

The author is supported by the Ph.D. programme in the National Science Centre Poland under grant DEC-2013/08/A/ST1/00312 "Hessian type equation in complex geometry". I wish to thank Professor Sławomir Kołodziej for his help to accomplish this work. I am indebted to my advisor, Dr. Rafał Czyż for many stimulating discussions. I am grateful to Dr. Sławomir Dinew for many fruitful comments. I would like to thank the referee for carefully reading my manuscript and for giving such constructive comments which helped improving the paper.

References

[1] Åhag, P., Cegrell, U., Czyż, R., Hiep, P.H.: Monge–Ampère measures on pluripolar sets. J. Math. Pures Appl. **92**, 613–627 (2009)

[2] Åhag, P., Czyż, R., Hed, L.: The geometry of m-hyperconvex domains. J. Geom. Anal. https://doi.org/10.1007/s12220-017-9957-2

[3] Bengtson, B.: An ordering of measures induced by plurisubharmonic functions. Ann. Pol. Math. **119**, 221–237 (2017)

[4] Błocki, Z.: Weak solutions to the complex Hessian equation. Ann. Inst. Fourier (Grenoble) **55**(5), 1735–1756 (2005)

[5] Cegrell, U.: The general definition of the complex Monge–Ampère operator. Ann. Inst. Fourier (Grenoble) **54**, 159–179 (2004)

[6] Cegrell, U.: Weak*-convergence of Monge–Ampère measures. Math. Z. **254**, 505–508 (2006)

[7] Cegrell, U.: A general Dirichlet problem for the complex Monge–Ampère operator. Ann. Pol. Math. **94**, 131–147 (2008)

[8] Fornæss, J.E., Sibony, N.: Oka's inequality for currents and applications. Math. Ann. **301**(3), 399–419 (1995)

[9] Lu, H.C.: Complex Hessian equations, Doctoral thesis, University of Toulouse III Paul Sabatier (2012)

[10] Lu, H.C.: A variational approach to complex Hessian equations in $\mathbb{C}^n$. J. Math. Anal. Appl. **431**(1), 228–259 (2015)

[11] Nguyen, N.C.: Subsolution theorem for the complex Hessian equation. Univ. Iagel. Acta Math. **50**, 69–88 (2013)

[12] Nguyen, V.T.: The convexity of radially symmetric m-subharmonic functions. Complex Var. Elliptic Equ. https://doi.org/10.1080/17476933.2017.1373347

Van Thien Nguyen
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland
e-mail: Thien.Van.Nguyen@im.uj.edu.pl

Received: May 22, 2017.
Accepted: January 12, 2018.