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Extension Theorems for differential forms on low-dimensional good quotients

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Stefan Heuver

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Gutachter

Prof. Dr. Daniel Greb
Prof. Dr. Stefan Kebekus

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Zusammenfassung

Meine Dissertation zum Thema *Extension Theorems for differential forms on low-dimensional good quotients* beschäftigt sich mit der Frage nach Fortsetzungen von Differentialformen auf Quotienten von Wirkungen reduktiver Gruppen auf glatten Varietäten.

Es sei G eine reduktive Gruppe und V eine glatte G-Varietät über $\mathbb{C}$, dann können wir einen guten Quotienten $\pi : V \to X$ (kurz: $X = V/\!\!/G$) definieren. Wie Boutot im Jahre 1987 bewiesen hat (siehe [Bou87]), hat X rationale Singularitäten.

Es sei σ eine Differentialform auf X die auf dem regulären Ort von X definiert ist und $\eta : \widetilde{X} \to X$ eine Auflösung von X. Wir wollen zeigen, dass der Pullback $\eta^*(\sigma)$ als reguläre Differentialform auf $\widetilde{X}$ fortsetzt. Ist $dim(X) = n$ so folgt der Fortsetzungssatz für 0- und n-Formen aus der Tatsache, dass X normal ist und rationale Singularitäten hat.

Im Allgemeinen ist nicht klar, dass $\eta^*(\sigma)$ eine reguläre Form ist, da ihre Fortsetzung auf $\widetilde{X}$ Pole entlang der exzeptionellen Menge haben kann. Im Jahr 2014 haben Graf und Kovács bewiesen, dass $\eta^*(\sigma)$ auf ganz $\widetilde{X}$ fortsetzt, solange wir logarithmische Pole entlang der exzeptionellen Menge zulassen (siehe [GK14]). In dieser Dissertation verwenden wir zwei Methoden um zu zeigen, dass $\eta^*(\sigma)$ unter speziellen Voraussetzungen keine solchen Pole hat.

In einem ersten Schritt verwenden wir Hodge-theoretische Methoden, die man aus [Nam01] und [SS85] kennt, um zu zeigen, dass der Fortsetzungssatz für 1-und 2-Formen auf X gilt. Zusammen mit der obigen Aussage über 0- und n-Formen erhalten wir so den Fortsetzungssatz für gute Quotienten der Dimensionen 0 bis 3.

In einem zweiten Schritt reduzieren wir uns auf den Fall eines GIT-Quotienten eines Vektorraumes. Dies ermöglicht es uns die partielle Auflösung von Kirwan (siehe [Kir85]) zu verwenden, um $\eta : \widetilde{X} \to X$ in eine Sequenz von Aufblasungen zu zerlegen. Auf diese Sequenz wenden wir eine modifizierte Version der Residuen Sequenz (siehe [EV92]) an, um folgendes Resultat zu erhalten:. *Angenommen der Fortsetzungssatz gilt für alle Formen auf allen guten Quotienten von Dimension kleiner n, dann gilt der Fortsetzungssatz auch für $(n-1)$-Formen auf dem guten n-dimensionalen Quotienten X.* Zusammen mit dem ersten Teil erhalten wir so den Fortsetzungssatz für gute Quotienten der Dimension 4.

Abstract

In my thesis with the title *Extension Theorems for differential forms on low-dimensional good quotients* I am analysing the extension of differential forms on quotients of smooth varieties by the action of reductive groups.

Let G be a reductive group and V a smooth G-variety over $\mathbb{C}$. Then we can define a good quotient $\pi : V \to X$ and write $X = V/\!\!/G$. In 1987, Boutot has proven that X has rational singularities (see [Bou87]).

Let σ be a differential form on X that is defined on the smooth locus on X and $\eta : \widetilde{X} \to X$ a resolution of X. We want to show that the pull-back $\eta^*(\sigma)$ extends as a regular differential form to all of $\widetilde{X}$. If $dim(X) = n$, the Extension Theorem for 0- and n-forms follows from the fact that X is normal with rational singularities.

In general it is not clear that $\eta^*(\sigma)$ is a regular form, since it might have poles along the η-exceptional set in $\widetilde{X}$. In 2014, Graf und Kovács have proven that $\eta^*(\sigma)$ extends to all of $\widetilde{X}$ as long as we allow logarithmic poles along the exceptional set (see [GK14]). In this thesis we will present two methods to show that in certain settings $\eta^*(\sigma)$ has no poles along the exceptional set.

In a first step we use Hodge-theoretic methods, known from [Nam01] and [SS85], to show, that the Extension Theorem is true for 1-and 2-forms on X. In combination with the previous results about 0- and n-forms this yields the Extension Theorem for all good quotients of dimension 0 to 3.

In a second step we reduce the problem to an Extension Theorem for a GIT-quotient of a vector space. In this situation we can use the partial resolution algorithm of Kirwan (see [Kir85]) to split $\eta : \widetilde{X} \to X$ into a sequence of blow-ups. We then apply a modified version of the residue sequence (see [EV92]) to each blow-up to get the following result: *Assume that the Extension Theorem is true for all differential forms on all good quotients of dimension less that n. Then the Extension Theorem is true for $(n-1)$-forms on the n-dimensional good quotient X.* In combination with the results from the previous step, this yields an Extension Theorem for good quotients of dimension 4.

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0. Introduction and main results

0.A. Introduction

In algebraic geometry we analyse algebraic objects to determine the geometric structure of algebraic varieties. In this context differential forms and the sheaves of differential forms are a rich source of information. To give two examples they are used as invariants to classify algebraic varieties and can also provide information on types of singularities.

On a smooth variety traditionally one considers Kähler differential forms. On a normal but singular variety however the sheaf of Kähler differential forms might have torsion and near a singular point a differential form can no longer be evaluated on a vector field. A natural solution to both of the problems is to use differential forms that are defined on the smooth locus instead. From an algebraic point of view these differential forms can be considered as global section of the reflexive hull of the sheaf of Kähler differential forms and are therefore called *reflexive differential forms*.

Let X be a normal variety over $\mathbb{C}$ and σ a reflexive differential form on X. Since certain features of the smooth case (e.g. the Kodaira vanishing and the Serre duality) do not hold for σ on X in general, we would like to consider a resolution $\eta : \widetilde{X} \to X$ of the singularities of X and analyse the pull-back $\eta^*(\sigma)$ on $\widetilde{X}$ instead. This leads to the following problem:

Extension Problem. Let X be an n-dimensional normal variety over $\mathbb{C}$ and $\eta : \widetilde{X} \to X$ any resolution of X. For all $0 \leq p \leq n$, we want to show that given a reflexive p-form $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ on X the pull-back

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big)$$

is a regular p-form on $\widetilde{X}$.

For an arbitrary normal variety X it is not clear, that the pull-back $\eta^*(\sigma)$ is regular on $\widetilde{X}$, since it might have poles along the η-exceptional set E. In [GK14] Graf and Kovács have proven that for a variety X with rational singularities the pull-back $\eta^*(\sigma)$ extends as a p-form to all of $\widetilde{X}$, as long as we allow logarithmic poles (log-poles) along E. In general it is not known that one can avoid these logarithmic poles. However, there are some results for varieties with rational singularities in special cases.

If $p = \dim(X)$ or $p = 0$ the result follows from the definition of a normal variety with rational singularities. For $p = 1$, the result was proven by Graf-Kovács [GK14] for du Bois singularities. For $p = 2$, there is a result by Namikawa [Nam01b], who works on varieties with rational $\mathbb{Q}$-Gorenstein singularities. For arbitrary values of p, the result has been proven by van Straaten-Steenbrink [SS85] for varieties with isolated singularities and by Greb-Kebekus-Kovács-Peternell [GKKP11] for klt-pairs.

Remark. In all these papers the authors show that $\eta^*(\sigma)$ extends to E without poles. This is the reason why we call a result like this *Extension Theorem*.

Notation. If we can prove the assertion of the Extension Problem for a given variety X (and an integer $0 \leq p \leq \dim(X)$) we say that *the Extension Theorem is true for (p-forms on) X*.

Some of the results above work with a different formulation of the Extension Theorem, which uses reflexive sheaves.

Formulation with reflexive sheaves. As [GKKP11] show in Observation 1.3 the Extension Theorem (in our formulation) is true if and only if $\eta_*(\Omega^p_{\widetilde{X}})$ is a reflexive sheaf for all $0 \leq p \leq n$. If $\mathcal{F}$ is a locally free sheaf on $\widetilde{X}$ then $\eta_*\mathcal{F}$ might not necessarily be reflexive. Since $\operatorname{codim}_X(\eta(E)) \geq 2$, the sheaf $\eta_*\mathcal{F}$ is reflexive if and only if any section of $(\eta_*\mathcal{F})_{|X\setminus\eta(E)}$ extends to X. Equivalently, $\eta_*\mathcal{F}$ is reflexive if and only if any section of $\mathcal{F}$ defined on an open set of the form $\eta^{-1}(U)\setminus E$ extends to $\eta^{-1}(U)$.

Remark. Since the reflexivity of a sheaf is a local property, this also shows that proving the Extension Theorem is a local problem.

Based on the results presented above it is natural to assume that an Extension Theorem might be proven for varieties with rational singularities in general. As a matter of fact, to this day no counter example has been presented. Besides klt-pairs, who play an important role in the minimal model program, good quotients are another interesting class of varieties with rational singularities.

Let G be a reductive group and V a smooth irreducible G variety over the complex numbers $\mathbb{C}$. Then, we can define a good quotient $\pi : V \to X := V/\!\!/G$. By Boutot we know that X is a normal variety with rational singularities ([Bou87, Cor.]). Examples for good quotients are Geometric Invariant Theory (GIT)-quotients (see [MFK94]), that are a useful tool in the study of moduli spaces. These varieties come with an extensive amount of extra structures and additional properties. In this thesis we will take advantage of these extra information about the good quotient X to show that for dimension $\dim(X) \leq 4$ one can extend $\eta^*(\sigma)$ as a regular p-form to all of $\widetilde{X}$, regardless of the value of p.

0.B. Main results

In all the theorems of this chapter $X := V/\!\!/G$ is a good quotient, where G is a reductive group and V is a smooth G-variety over $\mathbb{C}$. A precise definition and the main properties of a good quotient can be found in Chapter 3. The main result of the thesis essentially says that a reflexive p-form on a good quotient of dimension less than or equal to 4 lifts to a p-form on any log resolution.

Theorem 0.1 (Extension Theorem for good quotients of dimension 4 or lower)**.** *Let $X := V/\!\!/G$ be a good quotient of dimension $\dim(X) = n \leq 4$. Then for all values $0 \leq p \leq n$ we have the following result: Let $\eta : \widetilde{X} \to X$ be a log resolution and let $\sigma \in H^0\big(X, \Omega^{[p]}_X\big)$ be a reflexive p-form on X. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Remark. The notions of reflexive differential forms and log resolutions will be discussed in Chapter 1. We use a log resolution since it provides additional information about the η-exceptional set. However, as we will see in Chapter 4.A, the result is independent of the choice of resolution.

Remark. The result is only new in dimension 3 and 4. Since the quotient is normal, it is smooth if the dimension $\dim(X) \leq 1$. Then $\widetilde{X} = X$ and the resolution η is just the identity. In dimension 2 the Theorem follows from the fact that X has finite quotient singularities (see [Gur91, Cor. 1]). More information on finite quotient singularities and the Extension Theorem in this case can be found in Chapter 4.C.

We will prove Theorem 0.1 by analysing the cases $p \in \{1, 2\}$ and $p = (n-1)$ separately. It is a corollary of the following two more general results:

Theorem 0.2 (Extension of 1- and 2-forms on good quotients). *Let $X := V/\!\!/G$ be a good quotient of dimension $\dim(X) = n$. Let $\eta : \widetilde{X} \to X$ be a log resolution and $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ a reflexive p-form on X, for $p \in \{1, 2\}$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Theorem 0.2 directly implies Theorem 0.1 in dimension 3. Furthermore, it ensures that in dimension 4 we only have to prove the Extension Theorem for $p = 3$. This case is covered by the following theorem.

Theorem 0.3 (Extension of $(n-1)$-forms on good quotients). *Let $X := V/\!\!/G$ be a good quotient of dimension $n \geq 1$. Let $\eta : \widetilde{X} \to X$ be a log resolution and $\sigma \in H^0\big(X, \Omega_X^{[n-1]}\big)$ a reflexive $(n-1)$-form on X. Assume that the Extension Theorem is true for all reflexive p-forms on any good quotient of dimension less than n. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^{n-1}_{\widetilde{X}}\big).$$

Theorem 0.1 to 0.3 hold for good quotients in general and particularly when X is a GIT-quotient. Since the assertions of the three Extension Theorems are local on X, in Chapter 4.B we will show that for the proofs of these Extension Theorems we can reduce to the case where V is an affine G-variety (or V is a vector space with a linear action of G) and X is the induced affine GIT-quotient. These reductions are crucial for the success of this thesis since they allow us to use properties of certain GIT-quotients such as Luna's Slice Theorem (see Chapter 3.D), the existence of the partial resolution of Kirwan (see Chapter 8) and also provide information about the types of singularities (see Chapter 6.A) of a good quotient. As a consequence, the proofs of the Theorems 0.1 to 0.3 only work for our precise set-up and cannot be generalised for arbitrary varieties with rational singularities. Nevertheless, using this special properties of GIT-quotients also means that the proofs presented in this theses are much more concrete and can provide additional insight.

0.C. Outline of the thesis

There are mainly two approaches to study the extension of reflexive differential forms to a resolution. The first is by studying Hodge-theoretic methods, the other is by using a residue sequence. To arrive at Theorem 0.1 we have to use both methods. This thesis is divided into four parts.

In Part I we introduce the main definitions and present results that will simplify the proofs of the main results. Chapter 1 covers the basic notions of the Extension Problem. After we motivate and define reflexive differential forms we discuss resolutions and rational singularities. Chapter 2 and 3 are designed to given an overview about good quotients and GIT-quotients. After we introduce reductive groups, we define good quotients and (affine or projective) GIT-quotients and discuss their relations and properties. We will see that every good quotient can locally be expressed as an affine GIT-quotient. Using Luna's Slice Theorem (see [Lun73]) we then show that the local structure of an affine GIT-quotient can be analysed by examening a quotient of a vector space by a reductie group insted. Then we explain a method of Luna-Richardon (see [LR79]) that allows us to consider varieties with non-empty staple locus. This reduction later allows us to use the partial resolution algorithm of Kirwan (see Chapter 8). In Chapter 4 we start with the analysis of Extension Theorems. We first present general results such as the independence of the choice of resolutions and the relation between regular differential forms on smooth varieties

and holomorphic forms on manifolds. After this we prove the important reductions of the Extension Theorems for good quotients to affine GIT-quotients of an affine varieties or a vector spaces. The later reduction once again uses Luna's Slice Theorem. We then proof some Extension Theorems for special good quotients that are already known. Finally in Chapter 5 we prepare the proof of Theorem 0.2 by introducing Hodge-theoretic methods and show that cutting down an affine GIT-quotient with hyperplanes results in a variety that is still an affine GIT-quotient.

Part II is devoted to the proof of Theorem 0.2. In Chapter 6 we use the fact that 2-dimensional affine GIT-quotients have finite quotient singularities (see [Gur91, Cor. 1]) and a standard cutting-down technique presented in [GKKP11, Ch. 9.C] to show that good quotients have finite quotient singularities in codimension 2. This implies the Extension Theorem for good quotients in codimension 2 and allows us to use the Hodge-theoretic methods to proof Theorem 0.1 in dimension 3. We then prove Theorem 0.2 in Chapter 7 by modifying a result of Namikawa [Nam01b, Prop. 3]. This proof also relies on the Hodge-theoretic methods presented in Chapter 5.

In Part III we prove Theorem 0.3 by combining the partial resolution algorithm of Kirwan with a residue sequence (see Chapter 9.B). In Chapter 8 we introduce the partial resolution algorithm of Kirwan and show that by Luna-Richardson [LR79] an affine GIT-quotient X of a vector space by a reductive group has a representation $X := V/\!\!/G$ such that the partial resolution algorithm of Kirwan can be applied to X (or V respectively). This allows us to split a resolution into a finite sequence of easier to analyse morphisms. In Chapter 9 we first prove the existence of a residue sequence on certain pairs of good quotients and show that we can apply it to every step in the constructed sequence. We then explain that it is enough to show that the assertion of Theorem 0.3 holds to be true in every step of the partial resolution of Kirwan and reduce the problem to an Extension Theorem in one Kirwan step only. Finally, in Chapter 10 we apply the residue sequence to this single Kirwan step to prove Theorem 0.3.

In Part IV we present additional corollaries that follow from the proof of Theorem 0.3 and explain improvements of certain result in this thesis, that are needed to perfect Theorem 0.1 such that it is true for good quotients in arbitrary dimension.

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Part I.
Preliminaries

The aim of this thesis is to prove Extension Theorems for good quotients of smooth varieties by reductive groups. In general these varieties are singular and the analysis of them requires knowledge about resolutions of singularities and differential forms on normal varieties. This first part is designed to introduce the main definitions and to present useful results that will simplify the proofs of our main theorems.

In this thesis a variety is an integral separated scheme of finite type over $\mathbb{C}$. Varieties are also assumed to be irreducible. A vector space is an affine variety isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$. In particular a vector space is always assumed to be finite dimensional.

1. Differential forms, resolutions and singularities

In this chapter we give a short introduction on differential forms, present useful resolutions and discuss the singularities that can occur on good quotients. The first topic is based on [GKKP11], the last two topics are based on [KM98].

1.A. Kähler differential forms and reflexive differential forms

Differential forms on varieties play an important role in this thesis. Usually by that one means (logarithmic) Kähler differential forms. Let X be a normal variety and D a reduced Weil divisor on X. Then we denote the sheaf of (logarithmic) Kähler 1-forms by Ω^1_X (or $\Omega^1_X\big(\log D\big)$ respectively). For $p \in \mathbb{N}$, we set $\Omega^p_X := \bigwedge^p \Omega^1_X$ and $\Omega^p_X\big(\log D\big) := \bigwedge^p \Omega^1_X\big(\log D\big)$. Then for the sheaf of (logarithmic) 0-forms we have $\Omega^0_X = \Omega^0_X\big(\log D\big) = \mathcal{O}_X$ (see [GKKP11, Ch. 2.A]). More information on logarithmic differential forms can be found in [Iit82, Ch. 11c].

On smooth varieties the Kähler differential forms are a good choice. On singular varieties however working with Kähler differentials is disadvantageous in many cases. The following two facts explain the problems that come with Kähler differentials on singular varieties and also motivate the differential forms we use instead.

Algebraic viewpoint: On smooth varieties the sheaf of Kähler differential forms is locally free. Let X be a singular variety. Then the sheaf Ω^p_X is not locally free and might have both torsion and co-torsion. A natural solution would be to consider the reflexive hull of Ω^p_X instead.

Geometric viewpoint: On smooth varieties a useful property of Kähler differential forms is their pairing with vector fields. Let X be a singular variety. Then near a singular point $x \in X$ a Kähler differential form can no longer be evaluated on a vector field. A natural geometric solution would be to consider differential forms that are defined on the smooth locus of X instead.

These two viewpoints give rise to the following definition of a reflexive differential form on a normal variety.

Definition 1.1 (Reflexive (logarithmic) differential forms [GKKP11, Not. 2.16])**.** *Let X be a normal variety and D a reduced divisor on X. For $0 \leq p \leq \dim(X)$, let Ω^p_X and*

$\Omega^p_X\big(\log D\big)$ *be the sheaves of Kähler p-forms and logarithmic Kähler p-forms on* X*. Then* $\Omega^{[p]}_X := (\Omega^p_X)^{\vee\vee}$ *and* $\Omega^{[p]}_X\big(\log D\big) := \big(\Omega^p_X\big(\log D\big)\big)^{\vee\vee}$*, where* $(\,\cdot\,)^{\vee\vee}$ *is the double dual, are called* sheaf of reflexive p-forms *and* sheaf of reflexive logarithmic p-forms on X*.*

Remark. For more details on reflexive sheaves the reader is referred to [GKKP11, Ch. 2D]. Independently, we would like to make two remarks to show that the reflexive differential forms present a natural solution to both the algebraic and the geometric problem presented above.

1. By definition the sheaves $\Omega^{[p]}_X$ and $\Omega^{[p]}_X\big(\log D\big)$ are reflexive and in particular torsion-free.
2. Consider the pair (X, D) and its regular part $U := (X, D)_{\mathrm{sm}}$ (see [GKKP11, Def. 2.2, 2.4]). If we denote by $i : U \hookrightarrow X$ the embedding of U into X, then $\Omega^{[p]}_X\big(\log D\big) = i_*\big(\Omega^p_U\big(\log D_{|U}\big)\big)$ (see [GKKP11, Not. 2.16]).
3. Let X be a smooth variety. Using the previous remark it is clear that on X the reflexive (logarithmic) differential forms coincide with the (logarithmic) Kähler differential forms.

Definition 1.2 (Reflexive relative differential forms)**.** *Let* $\Psi : X \to T$ *be a morphism from a normal variety* X *to a smooth variety* T*. For* $0 \le p \le dim(X)$*, let* $\Omega^p_{X/T}$ *be the sheaf of relative p-forms on* X *(see [Har77, II. Prop. 8.11]). Then by* $\Omega^{[p]}_{X/T} = (\Omega^p_{X/T})^{\vee\vee}$ *we denote the* sheaf of reflexive relative p-forms on X*.*

Although reflexive differential forms solve some of the problems of Kähler differential forms on normal varieties, their downside is that they do not have a pull-back in general. To be able to explain this problem in greater detail we first like to recall the definition of a resolution.

1.B. Resolutions of singularities

Definition 1.3 (Resolution [KM98, Not. 04.(9)])**.** *Let* X *be a variety. A* resolution of X *is a surjective birational morphism* $\eta : \widetilde{X} \to X$ *such that* $\widetilde{X}$ *is smooth. By abuse of notation we will sometimes call* $\widetilde{X}$ *the resolution of* X*.*

This is the traditional definition of a resolution. In many cases of practical interest however it is useful to have addition information on the exceptional locus.

Definition 1.4 (Log resolution [GKKP11, Def. 2.12])**.** *Let* X *be a variety. A* log resolution *is a surjective birational morphism* $\eta : \widetilde{X} \to X$ *such that*

(1.4.1) the variety $\widetilde{X}$ *is smooth,*

(1.4.2) the η*-exceptional set* $\mathrm{exc}(\eta)$ *is a divisor with simple normal crossing (snc).*

We call $\eta : \widetilde{X} \to X$ *a* strong log resolution *if the following property holds in addition:*

(1.4.3) The morphism $\eta_{|\eta^{-1}(X_{\mathrm{sm}})} : \eta^{-1}(X_{\mathrm{sm}}) \to X_{\mathrm{sm}}$ *is an isomorphism.*

Remark. By Hironaka's Theorem (c.f. [KM98, pp. 3-4]) (strong) log resolutions exist in our setting. The definition of an snc divisor can be found in [KM98, 0.4.(8)]. Since the η-exceptional set is a divisor, we will sometimes call it *η-exceptional divisor*.

1.C. Pull-back of reflexive differential forms

To illustrate the problem that arises when pulling back a reflexive differential form, we like to consider the following artificial situation:

Let X be a normal variety with only one singular point $x \in X$ and $\eta : \widetilde{X} \to X$ a strong log resolution such that E is the reduced snc divisor with support $\operatorname{supp}(E) = \eta^{-1}(x)$. For $0 \leq p \leq \dim(X)$, a reflexive p-form $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ is a differential form defined outside the singular point $x \in X$. Over $X \backslash \{x\}$ however the strong log resolution η is an isomorphism. Thus, the pull-back $\eta^*(\sigma)_{|\widetilde{X}\backslash E}$ is a regular Kähler differential form on $\widetilde{X}\backslash E$ but the extension $\eta^*(\sigma)$ to all of $\widetilde{X}$ might have poles along E. A concrete example where $\eta^*(\sigma)$ has a pole along E can be found in [GKK10, Ex. 6.3]. In conclusion the pull-back of a reflexive differential form on X might not be a reflexive differential form on $\widetilde{X}$.

In a general set-up it is not clear that one can control the poles along the exceptional set of a pull-back of a reflexive form to a resolution. However, for certain types of singularities of X these poles have already been analysed.

1.D. Rational singularities

Definition 1.5 (Rational singularities [KM98, Def. 5.8])**.** *Let X be a normal variety. We say that X has* rational singularities *if there exists a proper birational map $f : Y \to X$ from a smooth variety Y, such that $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$.*

Example 1.6. The typical example is the singularity of the quadric cone given by the equation $x^2 + y^2 + z^2 = 0$.

Proposition 1.7 ([KM98, Lemma 5.12])**.** *Let X be a variety of dimension n and $f : Y \to X$ a resolution of singularities of X. Then X has rational singularities if and only if X is Cohen-Macaulay (CM) and $f_* \Omega_Y^n = \Omega_X^{[n]}$.*

Proof. See [KM98, Lemma 5.12]. □

If X has rational singularities we get the following useful results.

Corollary 1.8 (Extension of 0 and n-forms)**.** *Let X be a normal n-dimensional variety with rational singularities. Let $\eta : \widetilde{X} \to X$ be any resolution and let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X, for $p \in \{0, n\}$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^p\big).$$

Proof. Since X has rational rational singularities we get the equation $\eta_* \Omega_{\widetilde{X}}^n = \Omega_X^{[n]}$ (see Proposition 1.7). This implies that $\eta_* \Omega_{\widetilde{X}}^n$ is a reflexive sheaf and thus the Extension Theorem for reflexive n-forms on X is true. Since the normality of X implies that $\eta_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, the same is true for reflexive 0-forms on X. □

Proposition 1.9 (Extension with log-poles [GK14, Thm. 4.1])**.** *Let X be a normal n-dimensional variety with rational singularities and let $\eta : \widetilde{X} \to X$ be a log resolution with exceptional divisor $E := \operatorname{exc}(\eta)$. Let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X, for $0 \leq p \leq n$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^p\big(\log E\big)\big).$$

We say that $\eta^*(\sigma)$ has log-poles along E.

Proof. By Theorem S in [Kov99] the variety X is Du Bois. Thus the result follows from [GK14, Thm. 4.1]. □

2. Reductive groups and G-varieties

In this thesis we are analysing Extension Theorems for quotients of smooth varieties by reductive groups. The varieties have to be smooth to allow us to use certain results such as the partial resolution of Kirwan (see Chapter 8) or Luna's Slice Theorem (see Chapter 3.D.1). The group has to be reductive to ensure that we can define the quotients we are interested in.

In this chapter we are going to discuss (linear) algebraic groups and reductive groups and give important examples.

Definition 2.1 ((Affine) algebraic group [Dol03, Ch. 3.3], [Spr81, Def. 2.1.1])**.** *An* (affine) algebraic group G *is an (affine) algebraic variety with a group structure, such that the inversion map* $\iota : G \to G$ *and the group operation* $\mu : G \times G \to G$ *are morphisms of algebraic varieties.*

Example 2.2. Standard examples for affine algebraic groups are $\mathrm{GL}_n := \mathrm{GL}(n, \mathbb{C})$ (see [Spr81, Ex. 2.1.3.(3)] and $(\mathbb{C}^*)^m = (\mathrm{GL}_1)^m$ for n, $m \in \mathbb{N}$.

The following example will be important for the definition of linear algebraic groups.

Example 2.3. Let G be an algebraic group. Let $H \subset G$ be a closed subgroup of G and $i : H \hookrightarrow G$ the inclusion map. Then there exists a algebraic group structure on H such that i is a morphism of varieties (see [Spr81, 2.1.2.(3)]).

Definition 2.4 (Linear algebraic group [Spr81, Ex. 2.1.3.(4)])**.** *A* linear algebraic group *is a closed subgroup of* GL_n*, for some* $n \in \mathbb{N}$.

Remark ([Dol03, p. 37])**.** The affine algebraic groups form a category. One can prove that there exists a closed embedding from any affine algebraic group into GL_n, for some $n \in \mathbb{N}$. Thus, every affine algebraic group is isomorphic to a linear algebraic group.

Example 2.5. The notions of subgroups and group quotients of a linear algebraic group G exist as well.

1. Let $H \subset G$ be a closed subgroup of G and $i : H \hookrightarrow G$ the inclusion map. With the algebraic group structure from Example 2.3, H is an algebraic subgroup and the composition
$$H \hookrightarrow G \to \mathrm{GL}_n$$
makes H into a linear algebraic group.
2. Let $H \subset G$ be a closed normal subgroup of G. Then the group quotient G/H is an affine variety and provided with the usual group structure it becomes a linear algebraic group. The canonical projection $G \to G/H$ is a morphism of varieties (see [Spr81, Prop. 5.2.5]).

Definition 2.6 (G-variety [Dol03, p. 37], [Kra84, Def. 2.1])**.** *Let* G *be a linear algebraic group that acts on a variety* V *via* $\rho : G \times V \to V$ *(with the usual properties of a group action). Then* V *is called* G-variety *if* ρ *is a morphism of varieties. If* $g \in G$ *and* $v \in V$*, we will sometimes use the short notation* $g.v := \rho(g, v)$.

Example 2.7 (Linear representation [Kra84, Ch. 2.3])**.** Let V be a vector space and G a linear algebraic group with linear representation $\varrho : G \to \mathrm{GL}(V)$ on V. Then ϱ induced a G-action $\rho : G \times V \to V$ on V with $\rho(g, v) := \varrho(g) \cdot v$ that makes V into a G-variety. We call such an action *linear action of G on V*.

Remark ([Kra84, Ch. 2.4])**.** Let G be a linear algebraic group and V an affine G-variety. Then the G-action $\rho : G \times V \to V$ on V induces a G-action $\rho' : G \times \mathbb{C}[V] \to \mathbb{C}[V]$ on the affine coordinate ring via

$$\rho'(g, f)(v) = f(\rho(g^{-1}, v), \text{ for } v \in V.$$

A natural way to construct a quotient of an affine variety V by a linear algebraic group G is to consider the sub-algebra $\mathbb{C}[V]^G \subset \mathbb{C}[V]$ of G-invariant polynomials on V. We want the quotient to be the affine variety corresponding to $\mathbb{C}[V]^G$. Thus we have to ensure that $\mathbb{C}[V]^G$ is reduced (which is clear since the algebra is contained in a reduced $\mathbb{C}$-algebra) and finitely generated. The second property is crucial to the definition of the quotient and in general not true. For a non-reductive group G a counter-example was given by Nagata and is explained in [Dol03, Ch. 4.5]. As a consequence we want to restrict ourselves to reductive groups to guarantee that $\mathbb{C}[V]^G$ is finitely generated.

Definition 2.8. *The following definitions can be found in [Dol03, Ch. 3.4].*

(2.8.1) A linear algebraic group T that is isomorphic to $(\mathbb{C}^*)^m$, for some $m \in \mathbb{N}$, is called (algebraic) torus.

(2.8.2) Each linear algebraic group G contains a maximal connected solvable normal subgroup called radical.

(2.8.3) The group G is called semi-simple *if G is connected and its radical is trivial.*

(2.8.4) The group G is called reductive *if its radical is a torus.*

Definition 2.9 (Linear and geometrically reductive group [Dol03, Ch. 3.4])**.** *A linear algebraic group G is called* geometrically reductive *if for any rational representation $\varrho : G \to \mathrm{GL}(V)$ (see [Dol03, p. 37]) and any non-zero G-invariant vector $v \in V$ in a vector space V there exists a homogeneous G-invariant polynomial f on V such that $f(v) \neq 0$. The group G is called* linear reductive *if there exists a linear function f with that property.*

Remark. Since we work over a field of characteristic zero the notions linear reductive, geometrically reductive and reductive are equivalent (see [Dol03, Ch. 3.4]).

Example 2.10. The following examples can be found in [Kra84, Ch. 3.1] and will play an important role in Chapter 4.D. By definition of a reductive group every torus (e.g. $(\mathbb{C}^*)^m$, for $m \in \mathbb{N}$) and every semi-simple group (e.g. SL_n, for $n \in \mathbb{N}$) are reductive. Other important examples of reductive groups are finite groups, products of these three types of groups and GL_n, for $n \in \mathbb{N}$.

3. Good, geometric and GIT-quotients

In this chapter we want to give a short introduction on Mumfords' Geometric Invariant Theory, [MFK94]. The chapter covering the affine case is based on [Kra84]. Our discussion of the projective case is based on [Dol03]. After these introductions we deal with the properties of a GIT-quotient of $\mathbb{C}^n$ in detail and present techniques to reduce to this case. At the end of this chapter we discuss useful properties of affine GIT-quotients and analyse the stable locus of an affine G-variety. Throughout this chapter G will be a reductive group. We start with a general definition of good and geometric quotients.

Definition 3.1 (Good and geometric quotient [Ses72, Def. 1.5, Def. 1.6], [BBŚ97, p. 143]). *Let G be a reductive group and V a G-variety. A morphism $\pi : V \to X$, where X is an algebraic space, is called* good quotient (of V by (the action of) G) *if the following properties are fulfilled:*

(3.1.1) π is G-invariant,

(3.1.2) π is affine,

(3.1.3) $\mathcal{O}_X \cong \pi_*(\mathcal{O}_V^G)$.

It is called geometric quotient *if the following property holds in addition:*

(3.1.4) For each point $x \in X$, the fiber $\pi^{-1}(x)$ is a G-orbit.

By abuse of notation we call X good or geometric quotient and use the notation $X = V/\!\!/G$ or $X = V/G$ respectively.

Remark (Properties of good and geometric quotients [BBŚ97, Ch. 1]). Recall from [BBŚ97, Ch. 1] that a subset $W \subset V$ is called *G-saturated (in V)*, if for all $v \in W$ the closures of the orbit $G(v)$ in V and W coincide. The good quotient X has the following useful properties:

1. Let $W \subset V$ be open and G-saturated in V. Then $\pi(W) \subset X$ is open in X and $\pi_{|W} : W \to \pi(U)$ is a good quotient. Moreover, for any (open) sub-variety $U \subset X$ the preimage $\pi^{-1}(U)$ is a G-saturated (open) subset of V.
2. Let W_1, $W_2 \subset V$ be two disjoint, closed and G-invariant subsets of V. Then the images $\pi(W_1)$ and $\pi(W_2)$ are disjoint in X.

A consequence of the second property is that every fiber of π contains exactly one closed G-orbit. In the special case where X is a geometric quotient, this G-orbit coincides with the fiber of π.

3.A. Affine GIT-quotients

When V is an affine G-variety there exists a intuitively accessible description of an affine GIT-quotient. In most of the proofs we will reduce to this case.

Definition 3.2 (Affine GIT-quotient [Kra84, II.3.2]). *Let G be a reductive group and V an affine G-variety. Then the ring of G-invariant polynomials $\mathbb{C}[V]^G$ is a finitely generated algebra. We call $V/\!\!/G := \operatorname{spec}(\mathbb{C}[V]^G)$ the* affine GIT-quotient of V by G. *The natural projection $\pi : V \to V/\!\!/G$ is called quotient map.*

Remark. The quotient $V/\!\!/G$ might not be a geometric quotient, but is a good quotient (the necessary properties are discussed in [Kra84, II.3.2]). In particular, the quotient map is affine and surjective.

The converse relation between good and affine GIT-quotients is covered by the following lemma.

Lemma 3.3 (Reduction to affine GIT-quotients). *Let G be a reductive group and V a G-variety admitting a good quotient $X := V/\!\!/G$. Then X can locally be described as an affine GIT-quotient.*

Proof. Let $x \in X$ be a point in X and $\pi : V \to X$ the quotient map. Using the definition of a good quotient and the first assertion of the remark to Definition 3.1, one can find an affine open neighbourhood $U \subset X$ of x, such that the preimage $W := \pi^{-1}(U) \subset V$ is an affine open G-saturated subset of V. The induced quotient $W/\!\!/G = U$ is an affine GIT-quotient. □

We will consider the following example many times in this thesis. Although it does not provide an insight into the Extension Theorem of good quotients (the affine GIT-quotient we are going to construct is smooth), it is easy to understand and illustrates most of the constructions and results concerning special good quotients.

Example 3.4. Consider the action of the group $G = \mathbb{C}^*$ of invertible complex numbers on the 2-dimensional complex space $V = \mathbb{C}^2$ via $t.(z_1, z_2) := (t \cdot z_1, t^{-1} \cdot z_2)$. We can write $V = \operatorname{spec}(A)$, where $A := \mathbb{C}[Z_1, Z_2]$ is the polynomial ring of V. The ring of G invariant polynomial is $A^G = \mathbb{C}[Z_1 \cdot Z_2]$. Thus the quotient $X = V/\!\!/G$ is isomorphic to the space of complex numbers $\mathbb{C}$.

3.B. Semi-stable and stable points and GIT-quotients of projective varieties

When V is a projective G-variety, we need to restrict ourselves to the open subsets of (semi-)stable points $V^s \subset V^{ss} \subset V$ to be able to define a GIT-quotient. To give the definition of a (semi-)stable point we first need to recall the definition of a G-linearisation.

Definition 3.5 (G-linearisation of a line bundle [Dol03, Ch. 7.1])**.** *Let V be a G-variety (with a G-action $\rho : G \times V \to V$) and let $\mathcal{L}$ be a line bundle on V. A* G-linearisation of $\mathcal{L}$ *is an G-action $\overline{\rho} : G \times \mathcal{L} \to \mathcal{L}$ such that*

(3.5.1) the diagram

$$\begin{tikzcd} G \times \mathcal{L} \arrow[r, "\overline{\rho}"] \arrow[d] & \mathcal{L} \arrow[d] \\ G \times V \arrow[r, "\rho"] & V \end{tikzcd}$$

commutes and

(3.5.2) the zero section of $\mathcal{L}$ is G-invariant.

By abuse of notation we will refer to $\mathcal{L}$ as a G-linearisation on V.

Remark (The induced action on global sections [Dol03, Ch. 7.3])**.** The G-linearisation $\overline{\rho}$ of $\mathcal{L}$ induces a G-action $\overline{\rho}'$ on the space of global sections $H^0(V, \mathcal{L})$ of $\mathcal{L}$. This action is given by

$$\overline{\rho}'(g, s)(v) := \overline{\rho}(g, s(\rho(g^{-1}, v)))$$

for all $g \in G$, $s \in H^0(V, \mathcal{L})$ and $v \in V$.

Example 3.6. Let V be an affine G-variety and $\mathcal{L} := V \times \mathbb{C}$ the trivial line bundle. Then a G-linearisation $\overline{\rho}$ of $\mathcal{L}$ is given by $\overline{\rho}(g, (v, z)) := (\rho(g, v), z)$, where $g \in G$, $(v, z) \in V \times \mathbb{C}$ and $\rho : G \times V \to V$ is the G-action on V. We will refer to this as the *trivial G-linearisation on V*.

Definition 3.7 ((Semi-)stable point [Dol03, Ch. 8.1])**.** *Let V be a G-variety and $\mathcal{L}$ a G-linearised line bundle on V.*

(3.7.1) A point $v \in V$ is called semi-stable (with respect to $\mathcal{L}$) *if there exists an integer $m \geq 1$ and a G-invariant section $s \in H^0\big(V, \mathcal{L}^{\otimes m}\big)^G$, such that $U_s := \{u \in V \mid s(u) \neq 0\} \subset V$ is affine and $v \in U_s$.*

(3.7.2) A semi-stable point $v \in V^{ss}$ is called stable (with respect to $\mathcal{L}$) *if all orbits of G in U_s are closed and the stabiliser G_v is finite.*

Remark. Note, that the definitions of V^s and V^{ss} do not change, if we replace $\mathcal{L}$ by a positive tensor power of $\mathcal{L}$ (see [Dol03, Rem. 8.1.4]).

Remark. If V is affine, then we can choose $\mathcal{L} := V \times \mathbb{C}$ and the trivial G-linearisation introduced in Example 3.6 on V. Every section $s \in H^0\big(V, \mathcal{L}\big)^G$ corresponds to a G-invariant polynomial on V (because the G-action on the space of global sections corresponds to the G-action on the affine coordinate ring). If for every $v \in V$ we choose s to be the section that corresponds to a G-invariant constant non-zero polynomial, we get $V^{ss} = V$.

Example 3.8. Let V and G be as in Example 3.4. If we consider the trivial linearisation on V we get $V^{ss} = V$. We want to identify those $v \in V$ that are stable.

First let us show that the axes are not in V^s. The origin $0 \notin V^s$, because the stabiliser $G_0 = G$ of $0 \in V$ is not finite. Any other vector $v \neq 0$ on one of the two axes has finite stabiliser $G_v = \{1\} \subset \mathbb{C}^*$ but its orbit $G(v)$ is not closed. In fact $\overline{G(v)} \backslash G(v) = \{0\}$.

Let $v \in V$ be any vector that is not contained in one of the axes. Then the stabiliser G_v is trivial and the orbit $G(V)$ is closed. Thus $v \in V^s$.

Construction of the GIT-quotient. In the setting of the previous definition one can construct the GIT-quotient $V^{ss}/\!\!/G$ in the following way: We first cover V^{ss} by finitely many U_{s_i}, where $s_i \in H^0\big(V, \mathcal{L}^{\otimes m_i}\big)^G$ for $i \in I$ (I is a finite set) and $m_i \geq 1$. Then, gluing the affine GIT-quotients $U_{s_i}/\!\!/G$, we get a quotient $X := V^{ss}/\!\!/G$. More details on this can be found in [Dol03, Thm. 8.1].

Remark. The quotient $X := V^{ss}/\!\!/G$ is a good quotient with quotient map $\pi : V^{ss} \to X$. The set $V^s \subset V^{ss}$ is a saturated open subset and V^s/G is a geometric quotient (see [Dol03, Thm. 8.1]). We are mainly interested in the following two special cases:

1. If V is projective and $\mathcal{L}$ is ample, then the quotient X is projective and given by $X := \operatorname{proj}(R^G)$, where $R := \bigoplus_{n \geq 0} H^0\big(V, \mathcal{L}^n\big)$ (see [Dol03, Prop. 8.1]).
2. If V is affine and $\mathcal{L}$ is the trivial linearisation, then there exists a section $s \in H^0\big(V, \mathcal{L}\big)^G$ that corresponds to a non-zero constant polynomial on V and $U_s = V$. Thus the quotient X is the affine GIT-quotient defined in Definition 3.2.

Definition 3.9 (Projective GIT-quotient)**.** *Let G be a reductive group and V a projective G-variety with a G-linearisation $\mathcal{L}$ such that we can construct a quotient $X := V^{ss}/\!\!/G$. If (some power of) $\mathcal{L}$ corresponds to a projective embedding $V \hookrightarrow \mathbb{P}^n$ for some $n \in \mathbb{N}$ (i.e. it is ample), then we call X* projective GIT-quotient of V by G.

3.C. The case $V = \mathbb{C}^n$ and $\mathbb{C}^*$-actions

Let V be an affine variety with affine coordinate ring $A := \mathbb{C}[V]$. Let $\rho : \mathbb{C}^* \times V \to V$ be any action of the group $\mathbb{C}^*$ on V. Then ρ induces an action $\rho' : \mathbb{C}^* \times A \to A$ on the coordinate ring A via $\rho'(t, f)(v) := f(\rho(t^{-1}, v))$ for all $v \in V$. For every integer $d \in \mathbb{Z}$, we consider the corresponding character $\chi_d : G \to \mathbb{C}^*$ of the reductive group $G = \mathbb{C}^*$ (see [CLS11, §1]). Then $A_d := \{f : V \to \mathbb{C} \mid \rho'(t, f) = \chi_d(t) \cdot f\}$ defines a grading $A = \bigoplus_{d \in \mathbb{Z}} A_d$ on A, where $A_k \cdot A_l \subset A_{k+l}$. We call the $\mathbb{C}^*$-action on V *good* if $A_d = 0$ for all $d < 0$ and $A_0 = \mathbb{C}$. In this case the only fix-point $v_0 \in V$ of the $\mathbb{C}^*$-action (corresponding to the maximal ideal $\bigoplus_{d>0} A_d$) is called *vertex* (see [Pin77, Ch. 1]).

Example 3.10. Let $V = \mathbb{C}^n$ and $\rho : \mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ a $\mathbb{C}^*$-action on $\mathbb{C}^n$ given by $\rho(t, (z_1, \ldots, z_n)) := (t^{q_1} \cdot z_1, \ldots, t^{q_n} \cdot z_n)$. Then ρ is a good $\mathbb{C}^*$-action if and only if $q_i > 0$ for all $i \in \{1, \ldots, n\}$ and $\mathrm{g.c.d.}(q_1, \ldots, q_n) = 1$. The vertex of this good $\mathbb{C}^*$-action is $0 \in \mathbb{C}^n$.

The following result will be useful in Chapter 6.A.

Lemma 3.11 (Good $\mathbb{C}^*$-action on GIT-quotients)**.** *Let G be a reductive group acting linearly on $V = \mathbb{C}^n$. We denote the GIT-quotient by $X := V/\!\!/G$. Then the good $\mathbb{C}^*$-action on V, introduced in the previous example (with $q_i = 1$ for all $i \in \{1, \dots, n\}$), induces a good $\mathbb{C}^*$-action on X.*

Proof. Let $G \times V \to V$, $(g, v) \mapsto \varrho(g)v$ be the linear action of G on V, where $\varrho : G \to \mathrm{GL}(V)$ is a linear representation of G. As we have seen in the previous example, there exists a good $\mathbb{C}^*$-action

$$m : \mathbb{C}^* \times V \to V\,,\ (t, v) \mapsto m_t v$$

on V, where $m_t \in \mathrm{GL}(V)$ is the multiplication with $t \in \mathbb{C}^*$. By definition, for each $g \in G$ and each $t \in \mathbb{C}^*$ we have $\varrho(g)m_t = m_t\varrho(g)$ and the two actions commute. Therefore, m induces a $\mathbb{C}^*$-action on X. The coordinate ring of X is finitely generated (since G is reductive) and given by $\mathbb{C}[X] := \mathbb{C}[V]^G = \bigoplus_{d\in\mathbb{N}_0} \mathbb{C}[X]_{(d)}$, where $\mathbb{C}[X]_{(d)} = \mathbb{C}[V]^G_{(d)} = (\mathbb{C}[V]_{(d)})^G$ are the G-invariant polynomials on V of degree d. We may see that $\mathbb{C}[X]_{(0)} = (\mathbb{C}[V]_0)^G = \mathbb{C}$. Thus, $X/\!\!/\mathbb{C}^* = \{pt\}$ and the $\mathbb{C}^*$-action has a unique fixpoint $x_0 \in \pi^{-1}(X/\!\!/\mathbb{C}^*)$ given by the unique closed orbit. This point x_0 corresponds to the maximal ideal $\bigoplus_{d\in\mathbb{N}_{>0}} \mathbb{C}[X]_{(d)}$ and is the vertex of the good $\mathbb{C}^*$-action on X. □

3.D. Properties and reductions

Our main results are formulated for an arbitrary good quotient $X := V/\!\!/G$ of a smooth variety V by a reductive group G and we would like to analyse the properties of these types of quotients. Some of the properties (e.g. singularities) can be analysed locally. We already mentioned that every good quotient can locally be described as an affine GIT-quotient (see Lemma 3.3). In this chapter we will present a result of Gurjar [Gur91], who uses Luna's Slice Theorem [Lun73] to reduce from the affine case to the case $V = \mathbb{C}^n$. We then recall that any good quotient of a smooth variety has rational singularities.

3.D.1. Luna's Slice Theorem and reduction to GIT-quotients of vector spaces

Our formulation of Luna's Slice Theorem and the preparations are based on [Dré04]. Before we can recall Luna's Slice Theorem, we first need the following result by Matsushima.

Lemma 3.12 (Reductive stabiliser)**.** *Let G be a reductive group and V an affine G-variety. Assume that $v \in V$ has a closed orbit $G(v)$. Then the stabiliser G_v is reductive.*

Proof. This result was proven by [Mat60]. □

For the convenience of the reader we like to recall some of the definitions that will be used in our formulation of Luna's Slice Theorem.

Definition 3.13. *Let G be a reductive group and V, V' affine G-varieties.*

(3.13.1) A morphism $\Phi : V \to V'$ is called G-morphism *if $\Phi(g.v) = g.\Phi(v)$ for all $g \in G$ and $v \in V$ (see [Dré04, Def. 2.4]).*

(3.13.2) A G-morphism $\Phi : V \to V'$ is called strongly étale *if the induced morphism $\Phi_G : V/\!\!/G \to V'/\!\!/G$ of affine GIT-quotients is étale and the quotient morphism $\pi_V : V \to V/\!\!/G$ induces a G-isomorphism $V \cong V' \times_{V'/\!\!/G} (V/\!\!/G)$ (see [Dré04, Def. 4.14]).*

Remark. The definition of étale morphisms and useful results about them can be found in [Dré04, Ch. 4.1]. More details on (strongly) étale (G-)morphisms can be found in ([Dré04, Ch. 4.4]).

Theorem 3.14 (Luna's Slice Theorem [Dré04, Thm. 5.3; Thm. 5.4])**.** *Let G be a reductive group and V an affine G-variety. Let $v \in V$ be a point such that the orbit $G(v)$ is closed (and thus G_v reductive (see Lemma 3.12)). Then there exists a locally closed sub-variety $S \subset V$ of V, called* slice, *such that*

(3.14.1) S is affine and $v \in S$,

(3.14.2) S is G_v invariant,

(3.14.3) the image of the G-morphism $G \times_{G_v} S := (G \times S)/\!\!/G_v \to V$ induced by the G-action on V is a saturated open subset $U \subset V$,

(3.14.4) the restriction $G \times_{G_v} S \to U$ is a strongly étale surjective G-morphism.

If in addition $v \in V$ is smooth, we get an étale G_v-invariant morphism $\Phi : S \to T_v S$ to the tangent space to S at v such that $\Phi(v) = 0$, $T\Phi_v = Id$ and such that

(3.14.5) $T_v V = T_v(G_v) \oplus T_v S$,

(3.14.6) the image of Φ is a saturated open subset $U' \subset T_v S$,

(3.14.7) the restriction $S \to U'$ is a strongly étale surjective G_v-morphism.

Proof. A proof and more details on the theorem can be found in [Dré04, Ch. 5] □

Remark. Since the surjective G-morphism in 3.14.4 is strongly étale, it induces a surjective étale morphism

$$G \times_{G_v} S/\!\!/G \cong S/\!\!/G_v \to U/\!\!/G.$$

Similarly, the G_v-morphism in 3.14.7 induces a surjective étale morphism

$$S/\!\!/G_v \to U'/\!\!/G_v.$$

Building on this, Gurjar proves the following result that will help us to analyse the singularities of good quotients in Chapter 6.A.

Corollary 3.15 (c.f. [Gur91, Thm.])**.** *Let G be a reductive group and V a smooth affine G-variety. Let $X := V/\!\!/G$ be the induced quotient and $x \in X$ any point. Then there exists a reductive group H acting linearly on some $\mathbb{C}^n$, such that the analytic local ring of X in x is isomorphic to the analytic local ring of $\mathbb{C}^n/\!\!/H$ at the image of $0 \in \mathbb{C}^n$ in $\mathbb{C}^n/\!\!/H$.*

Proof. The proof can be found in [Gur91, §1]. □

Remark. In the setting of the previous corollary, let $\pi : V \to X$ be the quotient map and $v \in \pi^{-1}(x)$ a point with closed orbit. We know that $v \in V$ is smooth, $H = G_v$ is the stabilizer of v and $\mathbb{C}^n$ is isomorphic to the tangent space $T_v S$ to the slice S at v (see Theorem 3.14).

3.D.2. Good quotients have rational singularities

In this chapter we want to present a famous result by Boutot that ensures, that the good quotients we are considering are normal varieties with rational singularities. Thus, we can use the results from Chapter 1.D in the proofs of the Theorems 0.1 to 0.3.

Lemma 3.16. *Let G be a reductive group and V an affine G-variety with affine GIT-quotient $X := V/\!\!/G$. If V is normal then X is normal.*

Proof. The following proof is based on [Dol03, Prop. 1.3]. Let A be the affine coordinate ring of V. Then A is normal. Since $X = \operatorname{spec}(A^G)$ we have to show that A^G is normal. Let K be the fractions field of A and let L be the fraction field of A^G. By definition $L \subset K$ and every point $x \in L$ is fixed by the G-action. Thus $L \subset K^G$. We have to show that A^G is integrally closed in L. Let $x \in L$ satisfy the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

with coefficients $a_i \in A^G$ for $0 \leq i \leq n-1$ and $n \geq 1$. We know that $x \in K$ and since A is normal, the equation implies that $x \in A$. Since x is fixed by the G-action we get $x \in A^G$. □

Lemma 3.17 (Boutot)**.** *Let G be a reductive group and V an affine G-variety with affine GIT-quotient $X := V/\!\!/G$. If V has rational singularities then X has rational singularities.*

Proof. This Lemma is a famous result by Boutot [Bou87, Cor.]. A proof can also be found in [Kov00]. □

Proposition 3.18. *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$. Then X is normal and has rational singularities.*

Proof. Since being normal and having rational singularities are local properties, it is enough to prove them in the case where V is an affine G-variety and X is an affine GIT-quotient (see Lemma 3.3). The proposition then follows from the previous two lemmas. □

3.E. Principal points and finite stabiliser

Let G be a reductive group and V a smooth affine G-variety with affine GIT-quotient $X = V/\!\!/G$. If we choose the trivial G-linearisation on V, then all points $v \in V$ are semi-stable, $V^{ss} = V$. The stable points are exactly the $v \in V$ with finite stabiliser and closed orbit. In this chapter we want to present a method by Luna-Richardson to find an affine variety F and a reductive group $\mathcal{W}$, such that $F/\!\!/\mathcal{W} \cong X$ and $F^s \neq \emptyset$. More details on this method can be found in [LR79]. In this chapter we will denote by $O(x)$ the unique closed G-orbit in the fiber of $x \in X$.

Definition 3.19 (Principal point [LR79, Def. 3.2])**.** *Let G be a reductive group and V an affine G-variety with affine GIT-quotient $X := V/\!\!/G$. A point $x \in X$ is called* principal point*, if there exists an open neighbourhood $U \subset X$ of x, such that for all $x' \in U$ the following condition is fulfilled: If $O(x)$ is the closed orbit in the fiber of x and $O(x')$ is the closed orbit in the fiber of x'. Then $O(x)$ and $O(x')$ are G-isomorphic homogeneous spaces. We denote by X_{pr} the space of all principal points.*

Remark. If $v \in O(x)$ and $v' \in O(x')$. Then $O(x)$ and $O(x')$ are G-isomorphic homogeneous spaces if and only if the stabilisers G_x and $G_{x'}$ are conjugated subgroups of G (see [LR79, Rem. 3.3]). The space $X_{\mathrm{pr}} \subset X$ is a dense open subspace of X (see [LR79, Lem. 3.4]).

Lemma 3.20. *Let $H \subset G$ be a closed reductive subgroup of G and $X^H := \{x \in X \mid h.x = x \text{ for all } h \in H\}$ the space of all H-invariant points in X. Then the normaliser $N_G(H) := \{g \in G \mid gH = Hg\}$ is a reductive group with an induced action on X^H.*

Proof. The proof of the first assertion can be found in [LR79, Lem. 1.1]. The second assertion follows directly from the definitions. □

Proposition 3.21 ([LR79, Thm. 4.2])**.** *Let G be a reductive group and V an affine G-variety with affine GIT-quotient $X := V/\!\!/G$. Let $x \in X_{\mathrm{pr}}$ be a principal point and $a \in O(x) \subset V$. Then $H := G_a$ is reductive by Lemma 3.12. Set $\mathcal{W} := N_G(H)/H$ and $F := X^H$. Assume that $F/\!\!/\mathcal{W}$ is irreducible. Then $X \cong F/\!\!/\mathcal{W}$.*

Proof. The proof can be found in [LR79, Thm. 4.2]. □

Remark. If V is smooth, the restriction that $F/\!\!/\mathcal{W}$ is irreducible is not necessary. A detailed explanation can be found in [LR79, Rem. 4.6]. If V is a vector space and G acts linearly on V. Then one can easily check that F is a vector space as well and $\mathcal{W}$ acts linearly on F. Since the vector space F is irreducible, $F/\!\!/\mathcal{W}$ is irreducible as well (see [Kra84, Ch. II.4.3.A]).

Corollary 3.22. *Consider the same notation as in the previous proposition. Let $y \in X_{\mathrm{pr}}$ and $b \in O(y) \cap X^H$. Consider the trivial linearisation on V and F. Then $b \in F^s$.*

Proof. We start by proving that $a \in F^s$. First of all we have to show that $\mathcal{W}_a$ is the trivial group. This is true by the definition of $\mathcal{W}$. Secondly, we want to show that $\mathcal{W}(a)$ is closed in F. Assume that $\mathcal{W}(a)$ is not closed in F and that $v \in \overline{\mathcal{W}(a)} \backslash \mathcal{W}(a)$. Since $G(a) = O(x)$ is closed in V, there exists a $g \in G$ such that $g.v = a$. If $g \in N_G(H)$, then there exists a $g' \in \mathcal{W}$ such that $g'.v = a$, which contradicts the assumption. Thus $g \notin N_G(H)$. However, in this case we have

$$a = g.v = g.(h.v) \neq h.(g.v) = h.a = a, \text{ for } h \in H.$$

This is a contradiction, which shows that $v \in F^s$.

By [LR79, Lem. 3.5] we can deduce that $O(y) \cap X^H \neq \emptyset$ and that G_b is conjugated to H. Therefore, it is enough to show that $\mathcal{W}(b)$ is closed in F. Since $G(b)$ is closed in V, this follows from the same argumentation as above. □

Example 3.23. Let $V = \mathbb{C}^2$ and consider the following action of $G = \mathbb{C}^*$ on V:

$$t.(z_1, z_2) = (z_1, t \cdot z_2), \text{ for } t \in \mathbb{C}^* \text{ and } z_1, z_2 \in V.$$

Using the trivial linearisation on V, every point is semi-stable (i.e. $V^{ss} = V$). Unfortunately, no point is stable, because the only closed orbits are $G(v)$ for points contained in the $(1,0)$-axis, $v \in \mathbb{C} \cdot (1,0)$. These points however are all fixed by the group G and thus $G_v = G$ is not finite, for all $v \in \mathbb{C} \cdot (1,0)$.

The quotient $X := V/\!\!/G$ is isomorphic to the space of complex numbers $\mathbb{C}$. For every two points $x, x' \in X$, the unique closed orbits $O(x) = \{v\}$ and $O(x') = \{v'\}$ each consist of a single point contained in the $(1,0)$-axis and the stabiliser $G_v = G = G_{v'}$. Thus, $X_{\mathrm{pr}} = X$. Let $a \in O(0)$. Then $H = G$, $\mathcal{W}$ is the trivial group, $F = V^G = \mathbb{C} \cdot (1,0)$ and $F/\!\!/\mathcal{W} \cong X$. While the variety X stays the same, the representation via $\mathcal{W}$ and F is superior since $F^s = F$ and $F/\mathcal{W}$ is a geometric quotient.

4. Some easy Extension Theorems and useful reductions

In this chapter we will present useful statements that will help us to simplify the proofs of the Extension Theorems in this thesis. We then discuss special situations in which an Extension Theorem is easy to prove. For the convenience of the reader, let us recall the Extension Problem in the case of a normal variety:

Extension Problem. Let X be an n-dimensional normal variety and $\eta : \widetilde{X} \to X$ any resolution of X. For all $0 \leq p \leq n$, we want to show that given a reflexive p-form $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ on X the pull-back

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big)$$

is a regular p-form on $\widetilde{X}$.

Recall that the assertion of the Extension Problem is local. This means that it is enough to show that $\eta^*(\sigma)$ has no pole near a fiber $\eta^{-1}(x)$, for all $x \in X$. The following result shows that for this purpose it is enough to consider an analytic neighbourhood of $x \in X$.

Lemma 4.1. *In the same setting as above, let $U \subset X$ be an analytic open subset of X with $x \in U$ and η^{an} the analytification of η. Assume that the form*

$$(\eta^*(\sigma))^{\mathrm{an}}_{|(\eta^{\mathrm{an}})^{-1}(U)} \quad ,$$

which is the restriction of the analytification of the pull-back of σ to $(\eta^{\mathrm{an}})^{-1}(U)$, is a holomorphic form. Then $\eta^*(\sigma)$ has no pole near a fiber $\eta^{-1}(x)$.

Proof. We want to show that a rational algebraic differential form on a smooth variety is regular if its analytification is a holomorphic form on the corresponding complex manifold. Using local coordinates, this follows from the fact that a rational algebraic function on a smooth variety is regular if its analytification is a holomorphic function on the corresponding complex manifold (see [Sha13, Book 3, p. 177]). □

To avoid complicated notation when switching between the analytic and algebraic category we will not introduce the analytification of the relevant objects. However, we will use Lemma 4.1 to restrict differential forms to analytic neighbourhoods when analysing their poles.

4.A. Independence of the choice of resolution

Lemma 4.2. *Let Z be a normal variety and Y a smooth variety. Consider a surjective morphism $f : Z \to Y$. Let σ be a rational differential form (i.e. a rational section of the sheaf Ω^*_Y of Kähler differential forms) on Y. Then σ is regular if and only if the pull-back $f^*(\sigma)$ has no poles on Z.*

Proof. The proof can be found in [Kem77, Lem. 2]. □

Corollary 4.3 (Independence of the choice of resolution)**.** *Let X be an n-dimensional normal variety and $\eta_1 : \widetilde{X}_1 \to X$ any resolution of X. Let $0 \leq p \leq n$ and assume that for any reflexive p-form $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ the pull-back*

$$\eta_1^*(\sigma) \in H^0\big(\widetilde{X}_1, \Omega^p_{\widetilde{X}_1}\big)$$

is a regular p-form on $\widetilde{X}_1$. Let $\eta_2 : \widetilde{X}_2 \to X$ be another resolution of X. Then

$$\eta_2^*(\sigma) \in H^0\big(\widetilde{X}_2, \Omega^p_{\widetilde{X}_2}\big)$$

is a regular p-form on $\widetilde{X}_2$, too.

Proof. There exists a resolution $\eta : \widetilde{X} \to X$ that dominates $\widetilde{X}_1$ and $\widetilde{X}_2$. In other words we can consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "\eta"] & \widetilde{X}_2 \arrow[d, "\eta_2"] \\ \widetilde{X}_1 \arrow[r, "\eta_1"'] & X. \end{tikzcd}$$

Applying Lemma 4.2 to the maps p_1 and p_2 the result follows. □

4.B. Reductions for Extension Theorems for good quotients

In this chapter we want to analyse Extension Theorems for an arbitrary good quotient $X := V/\!\!/G$ of a smooth variety V by a reductive group G and present some useful reductions.

Lemma 4.4 (Reduction to affine GIT-quotients)**.** *To prove an Extension Theorem for a good quotient $X := V/\!\!/G$ (of a smooth variety V by a reductive group G) it is enough to prove the Extension Theorem in the case where V is an affine G-variety with affine GIT-quotient $X := V/\!\!/G$.*

Proof. Since proving the Extension Theorem is a local problem the result follows from the fact, that every good quotient can locally be described as an affine GIT-quotient (see Lemma 3.3). □

Lemma 4.5. *Let $g : Y \to X$ be a surjective flat morphism between normal varieties and $\mathcal{F}$ a coherent sheaf on X. Then the following assertions are true:*

(4.5.1) For the pull-back of the dual $\mathcal{F}^\vee$ of the sheaf $\mathcal{F}$ we have the isomorphism $g^*(\mathcal{F}^\vee) \cong (g^*\mathcal{F})^\vee$.

(4.5.2) The pull-back $g^*\mathcal{F}$ is reflexive if and only if $\mathcal{F}$ is reflexive.

Proof. A proof of the first assertion can be found in [Har80, 1.8], where he explains that the problem is local and thus proves the following result for modules on rings instead:

Let A be a noetherian ring, M, N be A-modules, with M finitely generated, and let $A \to B$ be a flat surjective ring homomorphism (for the first assertion we do not need g to be surjective). Then the natural map

$$\mathrm{Hom}_A(M, N) \otimes_A B \to \mathrm{Hom}_B\big(M \otimes_A B, N \otimes_A B\big)$$

is an isomorphism (see [Har80, 1.8]).

To prove the second assertion we have to show that the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism if and only if the natural map $g^*\mathcal{F} \to (g^*\mathcal{F})^{\vee\vee}$ is an isomorphism. Using 4.5.1 we have $(g^*\mathcal{F})^{\vee\vee} \cong g^*(\mathcal{F}^{\vee\vee})$. Since proving the equivalence is a local problem, too, it can be deduced from the following claim:

Claim 1. Consider the same notation as above. Then the natural map

$$M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M,N),N)$$

is an isomorphism if and only if the map

$$M \otimes_A B \to \mathrm{Hom}_A(\mathrm{Hom}_A(M,N),N) \otimes_A B$$

is an isomorphism.
Proof (of Claim 1). Since g is flat and surjective, B is a faithfully flat A-module. Thus tensoring a sequence of A-modules with B over A produces an exact sequence if and only if the original sequence was exact. □

Lemma 4.6. *Let $g : Y \to X$ be a surjective étale morphism between normal varieties. Then the Extension Theorem for X is true if and only if it is true for Y.*

Proof. Let $\eta_X : \widetilde{X} \to X$ be a resolution of singularities of X. Consider the following commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{g}"] \arrow[d, "\eta_Y"'] & \widetilde{X} \arrow[d, "\eta_X"] \\ Y \arrow[r, "g"'] & X, \end{tikzcd}$$

where $\widetilde{Y}$ is the fiber product $\widetilde{Y} = Y \times_X \widetilde{X}$. Thus, the morphism $\widetilde{g}$ is étale (see [Dré04, Prop. 4.3.(iii)]) and $\widetilde{Y}$ is smooth (see [Dré04, Prop. 4.3.(vii)]). Since η_X is a resolution, the morphism $\eta_Y : \widetilde{Y} \to Y$ is a resolution of singularities of Y. Let $0 \le p \le n = \dim(X) = \dim(Y)$. Then we want to show that $(\eta_X)_*\Omega^p_{\widetilde{X}}$ is reflexive if and only if the sheaf $(\eta_Y)_*\Omega^p_{\widetilde{Y}}$ is reflexive (this is enough since the Extension Theorem is independent of the choice of resolution (see Corollary 4.3)).

Since for $p = 0$ both sheaves are reflexive (see Corollary 1.8), we may assume that $p > 0$. Since $\widetilde{g}$ is étale the sheaf of relative differential forms $\Omega^p_{\widetilde{Y}/\widetilde{X}} = 0$ (see [Dré04, Prop. 4.2]) and thus $\Omega^p_{\widetilde{Y}} \cong (\widetilde{g})^*\Omega^p_{\widetilde{X}}$. Since g is étale it is flat in particular. Using the cohomology-and-base-change (see [Har77, III: Prop. 9.3]) we get

$$(\eta_Y)_*\Omega^p_{\widetilde{Y}} \cong g^*\big((\eta_X)_*\Omega^p_{\widetilde{X}}\big)$$

and the result follows from Lemma 4.5. □

Lemma 4.7. *Let G be a reductive group and V a smooth affine G-variety with affine GIT-quotient $X := V/\!\!/G$ and quotient map $\pi : V \to X$. Let $x \in X$ be any point in X. Then there exists a vector space W and a reductive subgroup $H \subset G$ acting linearly on W, such that the Extension Theorem for the quotient $W/\!\!/H$ implies the Extension Theorem for X in a neighbourhood of $x \in X$.*

Proof. Let $x \in X$ and $v \in \pi^{-1}(x)$ be a point with closed G-orbit $G(v)$ and therefore reductive stabiliser G_v (see Lemma 3.12). By Theorem 3.14 there exists a locally closed affine G_v-invariant sub-variety $S \subset V$ with $v \in S$ and an open saturated subset $U \subset V$ such that

$$f : S/\!\!/G_v \to U/\!\!/G$$

is a surjective étale morphism. Since $v \in V$ is smooth we also get a linear action of G_v on the tangent space T_vS to the slice S at $v \in S$ and an open saturated subset $U' \subset T_vS$, such that

$$g : S/\!\!/G_v \to U'/\!\!/G_v$$

is a surjective étale morphism. Set $W := T_vS$ and $H := G_v$. Then the result follows by applying Lemma 4.6 to f and g. □

Corollary 4.8 (Reduction to quotients of vector spaces)**.** *To prove an Extension Theorem for an arbitrary good quotient $X := V/\!\!/G$ of a smooth variety by a reductive group it is enough to prove the Extension Theorem in the case where V a vector space with linear G-action and affine GIT-quotient $X := V/\!\!/G$.*

Proof. Since proving the Extension Theorem is a local problem we can use Lemma 4.4 to reduce to the case where V is an affine G-variety with affine GIT-quotient $X := V/\!\!/G$. Using the fact that proving the Extension Theorem for the affine GIT-quotient is a local problem as well, the result follows from Lemma 4.7. □

4.C. Finite quotient singularities

A special type of singularities, that can arise on good quotients of smooth varieties by reductive groups, are finite quotient singularities. In this chapter we will discuss these singularities and prove an Extension Theorem for good quotients with finite quotient singularities. We start by giving a definition of a finite quotient singularity.

Definition 4.9 (Finite quotient singularity)**.** *Let X be a normal n-dimensional variety. Then a point $x \in X$ is called* finite quotient singularity (f.q.s) *if there exists an analytic neighbourhood U of x and a finite group Γ acting linearly on some $\mathbb{C}^n$, such that U is biholomorphic to an open neighbourhood $U' \subset \mathbb{C}^n/\Gamma$ of the image of $0 \in \mathbb{C}^n$ in the quotient $\mathbb{C}^n/\Gamma$.*

Example 4.10. The following two examples will be revisited in Chapter 5.C.

1. If X has klt singularities, then there exists a closed subset $Z \subset X$ with $\operatorname{codim}_X(Z) \geq 3$, such that $X \backslash Z$ has only finite quotient singularities (see [GKKP11, Prop. 9.4]).
2. Let G be a reductive group acting linearly on $\mathbb{C}^n$ such that the quotient $X := \mathbb{C}^n/\!\!/G$ is a surface with an isolated singularity $t \in X$. Then $t \in X$ is a finite quotient singularity (see [Gur91, Cor. 2]).

We will discuss the following example separately, because it plays an important role in Chapter 8.

Example 4.11. Let V be an affine or projective smooth G-variety admitting a GIT-quotient $X := V^{ss}/\!\!/G$ (with respect to some G-linearisation). Let $\pi : V^{ss} \to X$ be the induced quotient map and assume that $V^s = V^{ss}$. Then $\pi : V^{ss} \to X$ is a geometric quotient and X has finite quotient singularities due to the following argument:

Let $x \in X$ be an arbitrary point. By shrinking X to a neighbourhood of $x \in X$, we may assume that V is affine and X is an affine GIT-quotient (see Lemma 3.3). Let $v \in \pi^{-1}(x)$ be a point with closed orbit $G(v)$ and therefore reductive stabiliser $H := G_v$ (see Lemma 3.12). Then (by Corollary 3.15) H acts linearly on some $\mathbb{C}^n$ such that the analytic local ring of X in x is isomorphic to the analytic local ring of $\mathbb{C}^n/\!\!/H$ at the image of $0 \in \mathbb{C}^n$ in $\mathbb{C}^n/\!\!/H$. Since $v \in V^s$, we know that H is finite. Thus $x \in X$ is a finite quotient singularity.

Proposition 4.12. *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$. Assume that X has f.q.s. Let $\eta : \widetilde{X} \to X$ be a resolution and let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X, for $0 \le p \le \dim(X)$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Proof. The result can be deduced from the fact that f.q.s. are klt (see [KM98, Prop. 5.20]) by using [GKKP11, Thm. 1.4]. □

4.D. Good quotients of special reductive groups

As we have seen in Chapter 2 the main examples of reductive groups are semi-simple groups, tori, finite groups and their products. In fact every connected (linear) reductive group is a product of a semi-simple group and a torus (see [Kra84, Ch. 3.5, Satz 4]). Let $X := V/\!\!/G$ be an affine GIT-quotient, where G is a reductive group of one of those three types with a linear action on a vector space V. Then one can prove an Extension Theorem for X in arbitrary dimension. Although, we will not use the Extension Theorems for quotients of finite groups, semi-simple groups and tori, we want to discuss these cases to give a complete picture.

When G is a finite group, we will analyse the Extension Theorem for good quotients of smooth varieties by G. In the other two cases we will restrict ourselves to quotients $X := V/\!\!/G$ of vector spaces.

4.D.1. G is a finite group

Let $X := V/\!\!/G$ be a good quotient of a smooth variety by a finite group. Using Corollary 3.15 it clear that X has finite quotient singularities. Then the Extension Theorem for X follows from Proposition 4.12. A more direct approach uses the fact that the induced projection $\pi : V \to X$ is a finite map.

Lemma 4.13. *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$ with induced quotient map $\pi : V \to X$. Let $\eta : \widetilde{X} \to X$ be a resolution of singularities and σ a reflexive differential form on X. Then, $\eta^*(\sigma)$ has no poles on $\widetilde{X}$ if and only if $\pi^*(\sigma)$ has no poles on V.*

Proof. Without loss of generality, we may assume that V is an affine G-variety and X is an affine GIT-quotient. By [Kempf, Cor. 4] there exists a normal variety Z such that the diagram

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "q"'] & V \arrow[d, "\pi"] \\ \widetilde{X} \arrow[r, "\eta"'] & X \end{tikzcd}$$

commutes and such that p is a birational map and q is the quotient map $Z \to Z/\!\!/G \cong \widetilde{X}$. The variety Z is the normalisation of the component of the fiber product $\widetilde{X} \times_X V$ that is mapped birationally to V. If we apply Lemma 4.2 to the maps p and q the result follows. □

The following is an easy corollary of the previous result and implies the Extension Theorem for X when G is a finite group.

Corollary 4.14. *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$. Assume that the quotient map $\pi : V \to X$ is a finite map. Let $\eta : \widetilde{X} \to X$ be a resolution and $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ a reflexive p-form on X, for $0 \leq p \leq \dim(X)$. Then,*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Proof. Since no divisor $D \subset V$ is mapped into the singular locus of X under π, we see that the pull-back $\pi^*(\sigma)$ has no poles on V. The result now follows from Lemma 4.13. □

4.D.2. G is a torus

Let $G = T$ be a torus acting linearly on a vector space V and $X := V/\!\!/T$ the induced affine GIT-quotient. Then an Extension Theorem for X can be proven by first showing that X is an affine toric variety and then using a result by Danilov ([Dan91]). Let us start by recalling the definition of an affine toric variety.

Definition 4.15 (Affine toric variety [CLS11, Def. 1.1.3])**.** *An* affine toric variety *is a normal variety X containing a torus T_X as a Zariski-open subset such that the action of T_X on itself extends to an action of T_X on X.*

Remark. It will be useful to think of the Definition 4.15 in a slightly different way. Let T be a torus acting linearly on X such that there exists a point $x \in X$ with open T-orbit $T(x)$ such that T_y is trivial for all $y \in T(x)$. Then, the algebraic morphism (induced by the T-action on X)

$$\beta_x : T \to X,$$

with $\beta_x(t) = t.x$, embeds T into X and the image of β is a torus T_X that makes X into a toric variety.

Example 4.16. Obviously every torus is a toric variety. Let $V = \mathbb{C}^n$, for some $n \in \mathbb{N}$. Then V is a toric variety with torus $T_V = (\mathbb{C}^*)^n$.

Before we can show that an affine GIT-quotient $X := V/\!\!/T$ (of a vector space V by a torus T) is a toric variety, we first need to recall the following results about tori and diagonalisable groups.

Definition 4.17 (Diagonalisable group [Bor91, Prop. 8.4.(2)])**.** *We denote by $\mathbb{D}_n \subset \mathrm{GL}_n$, for $n \in \mathbb{N}$, the group of diagonal matrices. A* diagonalisable group (d-group) *is a linear algebraic group that is isomorphic to a closed subgroup of the group $\mathbb{D}_n$, for some $n \in \mathbb{N}$.*

Since every torus is isomorphic to $\mathbb{D}_n$, for some $n \in \mathbb{N}$ (see [Bor91, Ch. 8.5]), it is a special case of a d-group. The following lemmas show the relation between tori and d-groups and address some useful properties.

Lemma 4.18. *Let G be a d-group then the following statements hold.*

(4.18.1) A subgroup $H \subset G$ and the image of a morphism $G \to G'$ to a group G' are d-groups.

(4.18.2) The group G is a torus if and only if it is connected.

Proof. The first assertion is proven in [Bor91, Ch. 8.4, Cor.] and the second assertion is proven in [Bor91, Ch. 8.5, Prop.]. □

Corollary 4.19. *Let T be a torus and let $H \subset T$ be a closed subgroup. Then the following statements hold.*

(4.19.1) H is a normal subgroup.

(4.19.2) If H is connected then H is a torus.

(4.19.3) The quotient group T/H is a torus.

Proof. The first assertion follows from the fact that the torus T is commutative. The second assertion is a direct corollary of the previous lemma. For the quotient group we have to consider the induced surjective morphism $T \to T/H$. Since T is connected, we get that T/H is connected as well. Thus the result follows from the previous lemma. □

Lemma 4.20. *Let T be a torus acting linearly on a vector space V and let $X := V/\!\!/T$ be the induced affine GIT-quotient. Then X is a toric variety.*

Proof. Without loss of generality $V = \mathbb{C}^n$, for some $n \in \mathbb{N}$, and $T_V \subset V$ is the torus that makes V into a toric variety. Since T is acting linearly on V there exists a linear representation $T \to \mathrm{GL}_n$ such that every $t \in T$ corresponds to a diagonal matrix. Thus we may assume that $T \subset T_V$ is a closed connected subgroup of the torus T_V.

Since V is T_V-invariant and the actions of T and T_V commute, we get an action of T_V on X. Let $\pi : V \to X$ be the quotient map. Then a simple topological argument shows that the image $\pi(T_V)$ of the dense T_V-orbit in V is a dense T_V-orbit in X. Let $x \in \pi(T_V)$. Consider the algebraic morphism

$$\beta_x : T_V \to X,$$

with $\beta_x(g) = g.x$, that is induced by the T_V action on X. Then by Chevalley the image $\beta_x(T_V)$ is constructable in X (see [Mum99, I. §8. Cor. 2]) and thus the T_V-orbit $T_V(x) \subset \overline{T_V(x)}$ is open in its closure. Since $T_V(x)$ is dense in X, we have $\overline{T_V(x)} = X$ and the image $\pi(T_V)$ is an open T_V-orbit in X.

Assume that the stabiliser $H := (T_V)_x$ is not trivial. Then, since T_V is commutative, the stabiliser $(T_V)_y = H$ for all $y \in T_V(x)$. Since the orbit $T_V(x)$ is dense in X, we see that H fixes every point in X. Thus the torus T_V/H acts on X and makes X into a toric variety. □

Corollary 4.21 (Extension Theorem for quotients of vector spaces by tori)**.** *Let T be a torus acting linearly on a vector space V and let $X := V/\!\!/T$ be the affine GIT-quotient. Let $\eta : \widetilde{X} \to X$ be a resolution and let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X, for $0 \le p \le \dim(X)$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Proof. Since the Extension Theorem is independent of the choice of resolution (see Corollary 4.3), we may assume that η is a toric resolution. In [Dan91, 1.6] Danilov explains that every toric variety X has a toric resolution $\eta : \widetilde{X} \to X$ such that $\eta_*\Omega^p_{\widetilde{X}} \cong \Omega_X^{[p]}$ for all $0 \le p \le \dim(X)$. Thus $\eta_*\Omega^p_{\widetilde{X}}$ is reflexive for all $0 \le p \le \dim(X)$ and the Extension Theorem is true for X. □

4.D.3. Good quotients that are rational Gorenstein

Before we present the Extension Theorem in the case where G is a semi-simple group, we first would like to discuss the following special case of a good quotient that is Gorenstein. Note that we call a normal variety X Gorenstein if the canonical divisor K_X is Cartier (see [CLS11, Def. 8.2.14]).

Proposition 4.22. *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$. Assume that the canonical divisor K_X is Cartier. Let $\eta : \widetilde{X} \to X$ be a resolution of X and let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X, for $0 \leq p \leq \dim(X)$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Proof. Since X has rational singularities, due to [KM98, Cor. 5.24] K_X Cartier implies that X has canonical singularities (see [KM98, Def. 2.11]) and therefore klt singularities. Thus the assertion follows from [GKKP11, Thm. 4.1]. □

It is important to note, that not every GIT-quotient $X = V/\!\!/G$ (of a smooth variety V by a reductive group G) is Gorenstein. An example can be constructed using the quotient construction of a toric variety.

Example 4.23. Let X be an affine toric variety. Then Cox-Little-Schenck describe a quotient construction of X in [CLS11, § 5.1]. As a consequence, we can think of X as a quotient $V/\!\!/G$ where G is a reductive group and V is a smooth affine G-variety. In order to find a GIT-quotient that is not Gorenstein we thus only have to give an example of a toric variety that is not Gorenstein.

In [CLS11, §1.2] one can find the definition of a cone c and a discussion on how it defines an affine toric variety U_c. In the examples [CLS11, Ex. 1.2.22, Ex. 4.1.4, Ex. 8.2.13] Cox-Little-Schenck discuss the affine toric variety U_c that is induced by the cone $c := \mathrm{cone}(de_1 - e_2, e_2) \subset \mathbb{R}^2$, where e_1, e_2 are the standard basis of $\mathbb{R}^2$ and $d \in \mathbb{N}$ is a positive integer. Using the divisors D_1, D_2 corresponding to the rays of c (see Orbit-Cone Correspondence in [CLS11, Thm. 3.2.6]) one can show that $K_{U_c} = -D_1 - D_2$ ([CLS11, Thm. 8.2.3]). Using [CLS11, Prop. 8.2.12] Cox-Litte-Schenk show that U_c is Gorenstein if and only if $d \leq 2$.

4.D.4. G is a semi-simple group

Let $G = G_{ss}$ be a semi-simple group acting linearly on a vector space V and $X := V/\!\!/G_{ss}$ the affine GIT-quotient. Then we can prove an Extension Theorem for X by analysing the algebraic structure of the ring $\mathbb{C}[X] := \mathbb{C}[V]^{G_{ss}}$.

Lemma 4.24 ([Kra84, Ch. II. 3.3, Satz 2])**.** *Let G_{ss} be a semi-simple group acting linearly on a vector space V and $X := V/\!\!/G_{ss}$ the induced affine GIT-quotient. Then X (or $\mathbb{C}[X]$ respectively) is factorial.*

Proof. The proof can be found in [Kra84, Ch. II. 3.3, Satz 2]. There Kraft also explains that, instead of assuming that V is a vector space, it is enough to consider a factorial variety V. □

Corollary 4.25. *Let G_{ss} be a semi-simple group acting linearly on a vector space V and $X := V/\!\!/G_{ss}$ the induced affine GIT-quotient. Let $\eta : \widetilde{X} \to X$ be a resolution of X and let*

$\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X, for $0 \le p \le \dim(X)$. Then

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

Proof. By Lemma 4.24 X is factorial and affine. By [Har77, II. Prop. 6.2] this implies that the divisor class group is trivial, $\mathscr{C}l(X) = 0$. Thus, every Weil divisor is linear equivalent to 0 and therefore Cartier. In particular, the canonical divisor K_X on X is Cartier. The result now follows from Proposition 4.22. □

5. Preparation for Theorem 0.2

5.A. Hodge-theoretic method

In this chapter we will present the Hodge-theoretic approach to the Extension Theorem. Most of the ideas are based on [SS85] and [Nam01b]. Following the guidelines of these papers we will work in the following setting:

Setting. Let X be a Stein open subset of an algebraic variety with rational singularities and $x \in X$ a point in X. Let $\eta : \widetilde{X} \to X$ be a resolution of X such that $E := (\eta^{-1}(x))_{\mathrm{red}}$ is an snc divisor on $\widetilde{X}$ with support $\eta^{-1}(x)$.

Remark (Working in the analytic category)**.** For the first part of this chapter we do not have to assume that X is an analytic space. In Remark to Lemma 5.2 however it is necessary to consider a small analytic neighbourhood of $x \in X$ and in the proof of Corollary 5.3 we work in this setting.

The main goal of this chapter is to show that the injection

$$\iota : H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E\big)\big)$$

is an isomorphism for all values $0 \le p \le \dim(X)$. To do so we consider the exact sequence (see [Nam01b, p. 8])

$$0 \to \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big) \to \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big) \to \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}} \to 0$$

of sheaves on E. We will refer to this sequence by $(+)$.

Lemma 5.1. *Let $\delta : H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big) \to H^1\big(\widetilde{X}, \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big)$ be the boundary map in the long exact cohomology sequence induced by $(+)$. If δ is injective for some value of $p \le \dim(\widetilde{X})$, then ι is an isomorphism for the same value of p.*

Proof. The following argument goes back to [SS85, Thm. 1.3]. Consider the long exact sequence

$$0 \to H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \xrightarrow{\iota} H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E\big)\big) \to H^0\big(E, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big) \xrightarrow{\gamma} H^1\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to \dots.$$

Then by [SS85, p. 99] δ can be understood as the composition

$$H^0\big(E, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big) \xrightarrow{\gamma} H^1\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to H^1\big(E, \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big).$$

Thus, we get the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & \\
H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) \arrow[d, "\iota"] & \\
H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big(\log E\big)\big) \arrow[d] & \\
H^0\big(E,\, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big) \arrow[d, "\gamma"] \arrow[dr, "\delta"] & \\
H^1\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) \arrow[r] \arrow[d] & H^1\big(E,\, \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big) \\
\dots & .
\end{tikzcd}$$

Since δ is injective the map γ is injective as well. Since the vertical sequence is exact, ι must be surjective. □

Remark. The horizontal map in the last diagram is induced by the exact sequence

$$0 \to \Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big) \to \Omega^p_{\widetilde{X}} \to \hat{\Omega}^p_E \to 0.$$

of sheaves on E. The sheaf $\hat{\Omega}^p_E := \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)$ defined by this sequence is called *sheaf of torsion free p-forms on E*. A prove of its properties can be found in [Keb13, Part I].

Lemma 5.2. *The cohomology groups $H^k(\widetilde{X},\, \mathbb{C})$ and $H^k_E(\widetilde{X},\, \mathbb{C})$ carry mixed Hodge structures (MHS) with filtrations F and W, such that the boundary morphism*

$$\delta : H^0\big(E,\, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big) \to H^1\big(E,\, \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big)$$

can be interpreted as the map

$$Gr^p_F H^{p+1}_E(\widetilde{X},\, \mathbb{C}) \to Gr^p_F H^{p+1}(\widetilde{X},\, \mathbb{C})$$

for all $0 \leq p \leq \dim(\widetilde{X})$.

Proof. The proof of this lemma can be found in [Nam01b, p. 7]. □

Remark. We would like to recall the fact that after shrinking X to a small analytic neighbourhood of $x \in X$, we get an isomorphism $H^k(\widetilde{X},\, \mathbb{C}) \cong H^k(E,\, \mathbb{C})$ (see [Nam01b, p. 7]). A good description of the MHS on $H^k(\widetilde{X},\, \mathbb{C})$ and $H^k_E(\widetilde{X},\, \mathbb{C})$ can be found in [Ste83, (1.5), (1.6)].

By combining the previous two lemmas the problem of ι being an isomorphism reduces to the following corollary.

Corollary 5.3. *Let $U := \widetilde{X}\backslash E$ be the complement of E in $\widetilde{X}$. If the morphism of cohomology groups $\alpha : H^p(\widetilde{X}, \mathbb{C}) \to H^p(U, \mathbb{C})$ is a surjection for some value of $p \leq \dim(\widetilde{X})$, then ι is an isomorphism for the same value of p.*

Proof. Consider the exact local cohomology sequence (see [Nam01b, p. 7])

$$\cdots \to H^p(\widetilde{X}, \mathbb{C}) \xrightarrow{\alpha} H^p(U, \mathbb{C}) \to H_E^{p+1}(\widetilde{X}, \mathbb{C}) \xrightarrow{\beta} H^{p+1}(\widetilde{X}, \mathbb{C}) \to \dots.$$

Since α is surjective the exactness implies that β is injective. The map β however is a morphism of mixed Hodge structures. Thus, $Gr_F^p H_E^{p+1}(\widetilde{X}, \mathbb{C}) \to Gr_F^p H^{p+1}(\widetilde{X}, \mathbb{C})$ is injecive. The statement of the corollary now follows by using Lemma 5.2 and Lemma 5.1. □

5.B. Two Extension Theorems

In this chapter we will recall two important Extension Theorems that were proven using the method presented in the previous chapter. In both cases the result follows from Corollary 5.3.

The first result on spaces with isolated singularities was proven by van Straten-Steenbrink in [SS85, Thm. 1.3]. In their paper they work on an arbitrary contractible Stein space with an isolated singularity. Thus, their proof, although still using Corollary 5.3, slightly differs from our proof. The case $p \leq n$ can be found in [SS85, Thm. 1.3], whereas the case $p = n$ follows from [SS85, Cor. 1.4].

Corollary 5.4 (Extension Theorem for isolated rational singularities)**.** *Let X be an affine variety with rational singularities, $\dim(X) := n \geq 2$ and $x \in X$ the only singular point of X. Let $\eta : \widetilde{X} \to X$ be a log-resolution of X such that $E := \operatorname{exc}(\eta)$ is an snc divisor on $\widetilde{X}$. For $0 \leq p \leq n$, let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be a reflexive p-form on X. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^p\big).$$

Proof. We have to consider two cases. If $p = n$ the result follows from Corollary 1.8. Assume that $p < n$. By Proposition 1.9 the pull-back $\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^p\big(\log E\big)\big)$ is a regular p-form on $\widetilde{X}$ with logarithmic poles along E. Let E' be the divisor that contains all components of E that are mapped to $x \in X$ via η. Since E is a reduced snc divisor, E' is a reduced snc divisor as well. Note that X is smooth outside of $x \in X$. Thus $\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^p\big(\log E'\big)\big)$. We only have to show, that $\eta^*(\sigma)$ has no logarithmic poles along E'. Since proving the Extension Theorem is a local problem by Lemma 4.1 we can replace X by a small Stein open subset of an algebraic variety such that we get an isomorphism $H^k(\widetilde{X}, \mathbb{C}) \cong H^k(E', \mathbb{C})$ (see Remark to Lemma 5.2). We want to use Corollary 5.3 and thus have to show that the restriction map

$$\alpha : H^p(\widetilde{X}, \mathbb{C}) \to H^p(U, \mathbb{C})$$

is a surjection for $p < n$, where $U := \widetilde{X}\backslash E'$ is the complement of E' in $\widetilde{X}$. This statement is a corollary of the decomposition theorem in intersection cohomology and goes back to Goreski and MacPherson (see [Ste83, Thm. 1.11]). □

Remark. To get an idea how the surjectivity of α can be deduced from the decomposition theorem in intersection cohomology we recommend the book [Dim04] of Dimca who works on complex algebraic spaces. His version of the decomposition theorem can be found in

[Dim04, Thm. 5.4.10]. By [Dim04, Cor. 5.4.11] $H^p(\widetilde{X}, \mathbb{C})$ contains $IH^p(X)$ as a direct summand. By Proposition 5.4.4 in the same book we know that $IH^p(X) = H^p(X\backslash\{x\})$ for $p < n$. Since $x \in X$ is the only singularity of X and E' has support $\eta^{-1}(x)$ we also get that $H^p(U) = H^p(X\backslash\{x\})$. Since α is the restriction map $H^p(\widetilde{X}, \mathbb{C}) \to H^p(U, \mathbb{C})$ (see [Ste83, Rem. 1.10]), this implies that α is surjective.

The second result is due to Namikawa [Nam01b, Lem. 2], who works in the setting of Chapter 5.A.

Corollary 5.5. *Let X be a Stein open subset of an algebraic variety with rational singularities of dimension $n \geq 3$ and $x \in X$ a point in X. Let $\eta : \widetilde{X} \to X$ be a resolution of X such that $E := (\eta^{-1}(x))_{\mathrm{red}}$ is an snc divisor on $\widetilde{X}$. Then*

$$H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E\big)\big)$$

is an isomorphism for $p \in \{1, 2\}$.

Proof. We once again we want to use Corollary 5.3 and thus have to show that

$$\alpha : H^p(\widetilde{X}, \mathbb{C}) \to H^p(U, \mathbb{C})$$

is a surjection for $p \in \{1, 2\}$, where $U := \widetilde{X}\backslash E$ is the complement of E in $\widetilde{X}$. Namikawa proves this in [Nam01b, Lem. 2] by exploiting the geometric structure of $H^{p-1}(\widetilde{X}, \mathcal{O}^*_X)$. □

The following remark will be useful for the proof of Theorem 0.2 in Chapter 7.

Remark. Consider the exact sequence (+)

$$0 \to \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big) \to \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big) \to \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}} \to 0$$

of sheaves on E we saw in Chapter 5.A. Because α is surjective for $p \in \{1, 2\}$ by Lemma 5.2 and Lemma 5.1 we can deduce that the boundary map

$$\delta : H^0\big(E, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big) \to H^1\big(E, \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big)$$

in the long exact sequence is injective for these values of p. Thus

$$H^0\big(E, \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big) \to H^0\big(E, \Omega^p_{\widetilde{X}}\big(\log E\big)/\Omega^p_{\widetilde{X}}\big(\log E\big)\big(-E\big)\big)$$

is surjective for $p \in \{1, 2\}$.

Corollary 5.6 (Extension Theorem for 1-, 2-forms on (X, x))**.** *Let X be an affine variety with rational singularities of dimension $n \geq 3$ and $x \in X$ a point in X. Let $\eta : \widetilde{X} \to X$ be a resolution of X such that $E := (\eta^{-1}(x))_{\mathrm{red}}$ is an snc divisor on $\widetilde{X}$. For $0 \leq p \leq n$, let $\sigma \in H^0\big(X, \Omega^{[p]}_X\big)$ be a reflexive p-form on X such that the pull-back $\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E\big)\big)$ is a regular p-form on $\widetilde{X}$ with logarithmic poles along E. Then $\eta^*(\sigma)$ has no pole along E, for all values $0 \leq p \leq n$.*

Proof. Since proving the Extension Theorem is a local problem we can restrict ourselves to a Stein open subset of X such that the conditions of Chapter 5.A are fulfilled. By Lemma 4.1 the result now follows directly from the previous corollary. □

5.C. Cutting down affine GIT-quotients

In this chapter we will show that affine GIT-quotients are stable under general hyperplane sections. We later use the results of this chapter to show that affine GIT-quotient have finite quotient singularities in codimension 2 and to prove Theorem 0.2.

Lemma 5.7. *Let G be a reductive group and V a smooth affine G-variety. Assume that the affine GIT-quotient $X := V/\!\!/G$ has dimension $\dim(X) \geq 2$ and let $\pi : V \to X$ be the quotient map. Let $H \in |\mathscr{L}|$ be a general element of an ample basepoint-free linear system corresponding to $\mathscr{L} \in Pic(X)$. Then the following statements hold.*

(5.7.1) The divisor H is irreducible and normal.

(5.7.2) If H is smooth, then X is smooth along H.

(5.7.3) If $\eta : \widetilde{X} \to X$ is a (strong) log-resolution and $\widetilde{H} := \pi^{-1}(H)$, then the restriction $\eta_{|H} : \widetilde{H} \to H$ is a (strong) log-resolution with exceptional set $\operatorname{exc}(\eta_{|H}) = \operatorname{exc}(\eta) \cap H$.

(5.7.4) The preimage $\pi^{-1}(H)$ is a smooth affine G-invariant hyperplane in V and H is the affine GIT-quotient $H = \pi^{-1}(H)/\!\!/G$.

Proof. The first three statements can be found in [GKKP11, Lem. 2.22, Lem. 2.23]. Assertion 5.7.4 can be deduced from the following observations. If $\pi(v) = x$ for some $x \in H$ and $v \in V$, then $\pi(g.v) = x$ for all $g \in G$. Assume that for $1 \leq l \leq n$, $f_1, \dots f_l \in \mathbb{C}[X]$ are the defining polynomials of $H = \{x \in X \mid f_1(x) = \cdots = f_l(x) = 0\}$. Then $\pi^{-1}(H) = \{v \in V \mid f_1 \circ \pi(v) = \cdots = f_l \circ \pi(v) = 0\}$. Thus $\pi^{-1}(H)$ is a G-invariant hyperplane on V. This hyperplane is smooth by Bertini's Theorem (c.f. [Har77, II.8.18]) and affine since it is the fiber product in the pull-back diagram

$$\begin{tikzcd} \pi^{-1}(H) \arrow[r] \arrow[d] & V \arrow[d] \\ H \arrow[r] & X, \end{tikzcd}$$

where H, X and V are affine. □

Part II.
The 3-dimensional case

This part is dedicated to the proof of Theorem 0.2. Recall that it is enough to prove Theorem 0.2 in the case where the good quotient X is an affine GIT-quotient (see Lemma 4.4). We will first show that affine GIT-quotients have finite quotient singularities in codimension 2. Then we prove Theorem 0.2 by adjusting the proof of [Nam01b, Prop. 3] to 1- and 2-forms on affine GIT-quotients.

6. Extension in codimension 2

6.A. Singularities of GIT-quotients in codimension 2

In this chapter we generalise a result by Gurjar. He showed that 2-dimensional affine GIT-quotients have finite quotient singularities. Building on this, we use a standard cutting down technique, presented by Greb-Kebekus-Kovàcs-Peternell in the case of klt-pairs (see [GKKP11, Ch. 9.C]), to show that GIT-quotients have finite quotient singularities in codimension two.

Proposition 6.1 (c.f. [Gur91, Cor. 2])**.** *Let G be a reductive group and V a smooth affine G-variety with affine GIT-quotient $X := V/\!\!/G$. Assume that* $\dim(X) = 2$. *Then X has finite quotient singularities.*

Proof. By Corollary 3.15 we can assume that $X = \mathbb{C}^n/\!\!/H$, where H is a reductive group acting linearly on $\mathbb{C}^n$. Thus, Lemma 3.11 asserts that X has a good $\mathbb{C}^*$-action. Therefore, [Pin77] has shown that X is locally isomorphic to $\mathbb{C}^2/\!\!/\Gamma$ where Γ is a finite group acting linearly on $\mathbb{C}^2$. □

Proposition 6.2 (Affine GIT-quotients have finite quotient singularities in codimension 2)**.** *Let G be a reductive group and V a smooth affine G-variety with affine GIT quotient $X := V/\!\!/G$ and quotient map $\pi : V \to X$. Then there exists a closed subset $Z \subset X$ with* $\operatorname{codim}_X(Z) \geq 3$ *such that $X \backslash Z$ has finite quotient singularities.*

Remark. The proposition of course is true if $\dim(X) = 0$ or $\dim(X) = 1$, because in both cases X is smooth. The case $\dim(X) = 2$ is covered in Proposition 6.1 and was proven by Gurjar. Due to this, we will prove Proposition 6.2 only for $\dim(X) \geq 3$.

Proof. We basically follow the proof presented in [GKKP11, Ch. 9.C] and fill in additional steps where they are needed. We start by reducing the problem to a more simple case:

Step 1: Recall that X is a normal variety and the singular locus $T := X_{\text{sing}}$ has codimension at least 2. We can find a closed subset $T' \subset T$ such that every irreducible component of $T \backslash T'$ has codimension 2 and $\operatorname{codim}_X(T') \geq 3$. The assertion of Proposition 6.2 is local on X. Thus, we may assume that $T \subset X$ is irreducible with $\operatorname{codim}_X(T) = 2$.

By [GKKP11, Prop. 2.25] there exists an open set $X^0 \subset X$ such that $T^0 := T \cap X^0$ is non-empty and a diagram

$$\begin{tikzcd} \mathcal{X}^0 \arrow[r, "\gamma", "finite,\ etale"'] \arrow[d, "\Phi"'] & X^0 \\ S^0 & \end{tikzcd}$$

such that the restriction of Φ to any connected component of $\gamma^{-1}(T^0)$ is an isomorphism. The subspace $X \backslash (X^0 \cup T) \subset X$ is smooth and $\operatorname{codim}_X(T \backslash T^0) \geq 3$. Consequently, it is sufficient to prove Proposition 6.2 for points contained in X^0. Moreover, since the assertion in the Proposition 6.2 is local in the analytic topology, it suffices to prove it for the variety $\mathcal{X}^0$ instead, even after removing all but one component of $\gamma^{-1}(T^0)$. In conclusion we may assume the following:

> There exists a surjective morphism $\Phi : X \to S$ with connected fibers, such that the restriction $\Phi_{|T} : T \to S$ is an isomorphism.

Step 2: We now want to reduce our problem to a case where the fibers of the morphism Φ are surfaces. Let $S^0 \subset S$ be a Zariski-open, dense subset. Then X is smooth at all points of $X \backslash (\Phi^{-1}(S^0) \cup T)$ and $\operatorname{codim}_X(T \backslash \Phi^{-1}(S^0)) \geq 3$. As above, it is sufficient to prove Proposition 6.2 for the open set $\Phi^{-1}(S^0) \subset X$ only. Consider the following diagram

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "\lambda"'] & X \arrow[d, "\Phi"] \\ & S \end{tikzcd} .$$

For each $s \in S$, we denote the fibers by $X_s := \Phi^{-1}(s)$ and $V_s := \pi^{-1}(X_s) = \lambda^{-1}(s)$. Then the Generic Flatness Lemma [FGI+05, Lem. 5.12] and the Lemma 5.7 allow us to assume the following:

> The morphism Φ is flat. Given any point $s \in S$ the preimage $X_s := \Phi^{-1}(s)$ is a normal surface, and moreover an affine GIT-quotient $X_s = V_s /\!\!/ G$, where V_s is a smooth variety. If $t_s \in T$ is the unique point that maps to $s \in S$, then X_s is smooth away from $t_s \in X_s$. Using Proposition 6.1, X_s has only finite quotient singularities. In particular, X_s is klt [KM98; 5.20].

Step 3: We are now in the situation of [GKKP11, 9.8] and can adopt their proof from here on. Proposition 6.2 follows from the argument in [GKKP11, 9.C.2 and 9.C.3]. □

Remark. Since the assertion of the previous proposition is local on X, by Lemma 3.3 it is true for good quotients as well.

6.B. Two Extension Theorems

Recall that in Chapter 4.C we have proven an Extension Theorem for good quotients with finite quotient singularities (see Proposition 4.12). Using Proposition 6.2 we are now in a position to prove the following two Extension Theorems.

Corollary 6.3 (Extension Theorem for GIT-quotients in codimension 2)**.** *Let X and Z be as in Proposition 6.2. For $0 \leq p \leq \dim(X)$, let $\sigma \in H^0\big(X\backslash Z\,,\,\Omega_X^{[p]}\big)$ be a reflexive p-form on $X\backslash Z$ and $\eta : \widetilde{X} \to X$ be a log-resolution. Then $\eta^*(\sigma) \in H^0\big(\eta^{-1}(X\backslash Z)\,,\,\Omega^p_{\widetilde{X}}\big)$.*

Proof. This follows from the Extension Theorem for good quotients with finite quotient singularities (see Proposition 4.12). □

Remark. Similarly to the remark to Proposition 6.2, by Lemma 4.4 this Extension Theorem is true for good quotients as well.

At this point we can already prove Theorem 0.1 for $\dim(X) = 3$ using the previous corollary together with Corollary 5.6.

Corollary 6.4 (Extension Theorem for good quotients in dimension 3)**.** *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$ of dimension $\dim(X) = 3$. Let $\sigma \in H^0\big(X\,,\,\Omega_X^{[p]}\big)$ be a reflexive p-form, $0 \leq p \leq \dim(X)$, and let $\eta : \widetilde{X} \to X$ be a log-resolution. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}\,,\,\Omega^p_{\widetilde{X}}\big).$$

Proof. Without loss of generality V is affine and X is an affine GIT-quotient (see Lemma 4.4). The proposition is clear for $p = 0$ and $p = n$ by the Corollary 1.8. Thus we only have to check the assertion for $1 \leq p \leq 2$.

Let $\sigma \in H^0\big(X\,,\,\Omega_X^{[p]}\big)$ be a reflexive p-form, for $1 \leq p \leq 2$. By Proposition 1.9, $\eta^*(\sigma) \in H^0\big(\widetilde{X}\,,\,\Omega^p_{\widetilde{X}}(\log E)\big)$, where $E := \mathrm{exc}(\eta)$ is the exceptional divisor. Thus, we only have to show that σ extends on divisors contained in E. Let $E' \subset E$ be any reduced component of E. Then either $\eta(E') = \{pt\}$ is a point (*) or $\eta(E')$ has codimension 2 (**).

By Corollary 6.3 there exists a closed subset $Z \subset X$ with $\mathrm{codim}_X(Z) \geq 3$ such that every reflexive p-form σ on $X\backslash Z$ extends to a regular p-form on $\eta^{-1}(X\backslash Z)$, for each $0 \leq p \leq 3$. Thus for each divisor E' of type (**) the following holds: Let $e \in E'$ be a generic point of E', then $\eta^*(\sigma)$ is regular in $e \in E'$. Let $e' \in E'$ be an arbitrary point of E'. Since $\widetilde{X}$ is smooth we can find an open neighbourhood $U \subset \widetilde{X}$ of e' where $\Omega^p_{\widetilde{X}}(\log E)$ is free and $\eta^*(\sigma)$ is regular in a generic point of U. Then by [GW10, Thm. 6.45] $\eta^*(\sigma)$ is regular in $e' \in E'$. Since $e' \in E'$ was an arbitrary point, $\eta^*(\sigma)$ extends to all of E' as a regular form.

It is only left to check that $\eta^*(\sigma)$ extends as a regular form on a divisor E' of type (*). This case however is covered in Corollary 5.6. □

7. Proof of Theorem 0.2

In this chapter we are going to prove Theorem 0.2. First let us recall the statement.

Theorem (Extension of 1- and 2-forms on good quotients)**.** *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$ of dimension $\dim(X) = n$. Let $\eta : \widetilde{X} \to X$ be a log resolution and $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ a reflexive p-form on X, for $p \in \{1, 2\}$. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big).$$

To prove this Theorem we need the following lemma by Namikawa.

Lemma 7.1. *Let X be a normal, affine variety with rational singularities and $x \in X$ a point in X. Let $\eta : \widetilde{X} \to X$ be a resolution of X such that $E := (\eta^{-1}(x))_{\mathrm{red}}$ is an snc divisor on $\widetilde{X}$. Then $H^0\big(E, \hat{\Omega}^p_E\big) = 0$ for all $p > 0$.*

Proof. We outline a proof that uses a Hodge-theoretic method. More details on this argument can be found in [Nam01a, Lemma 1.2]. Recall that the cohomology group $H^i\big(E, \mathbb{C}\big)$ carries a mixed Hodge structure. Assume that $H^0\big(E, \hat{\Omega}^p_E\big) \neq 0$ for some $p > 0$. Then by Hodge symmetry we get that $H^p\big(E, \hat{\Omega}^0_E\big) = H^p\big(E, \mathcal{O}_E\big) \neq 0$. However, since X has rational singularities $H^p\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big) = 0$, from which one can prove that $H^p\big(E, \mathcal{O}_E\big) = 0$. This contradicts the assumption. □

Proof (of Theorem 0.2). The organisation of the following proof follows [Nam01b, pp. 10-12]. Without loss of generality V is affine and X is a an affine GIT-quotient (see Lemma 4.4). By $E := \mathrm{exc}(\eta)$ we denote the exceptional divisor. Recall that E is an snc divisor.

Step 1 (Preparations): Let $n := \dim(X)$ be the dimension of X. We denote by X_{sing} the singular locus of X. By Corollary 6.3 the extension of $\eta^*(\sigma)$ is clear outside a certain locus $T \subset X_{\mathrm{sing}}$ of $codim_X(T) \geq 3$. Outside T the GIT-quotient X has finite quotient singularities only (see Proposition 6.2). Thus it is left to check the extension of $\eta^*(\sigma)$ over T. Let $E^0 \subset E$ be an irreducible component of E with $T^0 := \eta(E^0) \subset T$. We put $k := \dim(T) - \dim(T^0)$ and prove the extension of $\eta^*(\sigma)$ along E^0 by induction on k.

Step 2 (Case $k = 0$): In this case $\dim(T) = \dim(T^0)$. We set $l := \mathrm{codim}_X(T)$. Recall that $l \geq 3$.

Step 2.1 (Cutting down): As in Lemma 5.7 we consider $n - l$ general hyperplanes $H_1, \ldots, H_{n-l}$ and denote by $H := H_1 \cap \cdots \cap H_{n-l}$ a general l-dimensional complete intersection in X. Let $t_0 \in T^0 \cap H$. Using Lemma 5.7 we know the following facts. By replacing X by a small open neighbourhood of t_0 we may assume that $T^0 \cap H = \{t_0\}$. The preimage $\widetilde{H} := \eta^{-1}(H)$ is a resolution of singularities. Since X is an affine GIT-quotient, $H = \pi^{-1}(H)/\!\!/G$ is an affine GIT-quotient as well. Thus H has a unique distinguished singular point t_0 such that $H_{\mathrm{sing}} \backslash \{t_0\}$ contains only finite quotient singularities. Similar to the proof of Proposition 6.2 we can replace X by a smaller neighbourhood of t_0 such that there exists a variety S and a flat surjective morphism $\Phi : X \to S$ with connected fibers, such that the restriction $\Phi_{|T^0} : T^0 \to S$ is an isomorphism and such that there exist a point $s_0 \in S$ with $\Phi^{-1}(s_0) = H$. By choosing X small enough we can also ensure that Ω^p_s is trivial for all $0 \leq p \leq \dim(S)$. Given any point $s \in S$ the preimage $X_s := \Phi^{-1}(s)$ is an l-dimension affine GIT-quotient $X_s = \pi^{-1}(X_s)/\!\!/G$. If $t_s \in T^0 \cap X_s$ is the unique point that maps to $s \in S$, then X_s has finite quotient singularities away from $t_s \in X_s$. The map $\eta : \widetilde{X} \to X$ gives a simultaneous resolution of the fibers X_s for $s \in S$. Let $E' \subset \widetilde{X}$ be the union of all irreducible components of E that map into T^0. Then $E' \to S$ is a proper map and by Lemma 5.7 every fiber E'_s is a simple normal crossing divisor. Note that E'_s has support $\eta^{-1}(t_s)$, for all $s \in S$.

Step 2.2 (Filtrations): We consider the composition $\Psi := \Phi \circ \eta : \widetilde{X} \to S$. After shrinking S we may assume, that Ψ is smooth. By [Keb13, Prop. 3.11] there are filtrations

$$\hat{\Omega}^q_{E'} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^q \supset \mathcal{F}^{q+1} = \{0\}$$

and

$$\Omega^q_{\widetilde{X}}(\log E') = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \cdots \supset \mathcal{G}^q \supset \mathcal{G}^{q+1} = \{0\}$$

for $0 \leq q \leq \dim(\widetilde{X}) - 1$, where $\hat{\Omega}^q_{E'}$ is the sheaf of torsion free q-forms on E'. These filtrations induce exact sequences

$$0 \to \mathcal{F}^{r+1} \to \mathcal{F}^r \to (\Psi_{|E'})^*\Omega^r_S \otimes \hat{\Omega}^{q-r}_{E'/S} \to 0$$

on E' and

$$0 \to \mathcal{G}^{r+1} \to \mathcal{G}^r \to \Psi^*\Omega^r_S \otimes \Omega^{q-r}_{\widetilde{X}/S}(\log E') \to 0$$

on $\widetilde{X}$ for $0 \leq r \leq q$, where $\hat{\Omega}^{q-r}_{E'/S}$ and $\Omega^{q-r}_{\widetilde{X}/S}$ are sheaves of relative torsion free $(q-r)$-forms over S (see [Keb13, Ch. 2.1]). In this proof we only have to consider $q \in \{1, 2\}$. For $q = 1$, we have $\mathcal{F}^1 \cong (\Psi_{|E'})^*\Omega^1_S$ and $\mathcal{G}^1 \cong \Psi^*\Omega^1_S$ and get one exact sequence each:

$$\begin{aligned} &0 \to (\Psi_{|E'})^*\Omega^1_S \to \hat{\Omega}^1_{E'} \to \hat{\Omega}^1_{E'/S} \to 0 \\ &0 \to \Psi^*\Omega^1_S \to \Omega^1_{\widetilde{X}}(\log E') \to \Omega^1_{\widetilde{X}/S}(\log E') \to 0 \end{aligned} \tag{1}$$

For $q = 2$, we have $\mathcal{F}^2 \cong (\Psi_{|E'})^*\Omega^2_S$ and $\mathcal{G}^2 \cong \Psi^*\Omega^2_S$. We set $\mathcal{F} := \mathcal{F}^1$ and $\mathcal{G} := \mathcal{G}^1$ and get two exact sequences each:

$$\begin{aligned} &0 \to \mathcal{F} \to \hat{\Omega}^2_{E'} \to \hat{\Omega}^2_{E'/S} \to 0 \\ &0 \to \mathcal{G} \to \Omega^2_{\widetilde{X}}(\log E') \to \Omega^2_{\widetilde{X}/S}(\log E') \to 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &0 \to (\Psi_{|E'})^*\Omega^2_S \to \mathcal{F} \to (\Psi_{|E'})^*\Omega^1_S \otimes \hat{\Omega}^1_{E'/S} \to 0 \\ &0 \to \Psi^*\Omega^2_S \to \mathcal{G} \to \Psi^*\Omega^1_S \otimes \Omega^1_{\widetilde{X}/S}(\log E') \to 0 \end{aligned} \tag{3}$$

Step 2.3 (The long exact cohomology sequences): Let us now consider the sequence (+), that we have already seen in Chapter 5.A

$$0 \to \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \to \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \to \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}} \to 0.$$

We want to prove the following claim.

Claim 1. The induced maps

$$\gamma_p : H^0\big(E', \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big)\big) \to H^0\big(E', \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big)\big)$$

are surjective for $p \in \{1, 2\}$.

Proof (of Claim 1). Recall from Chapter 5.A that $\Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \cong \hat{\Omega}^p_{E'}$ for all $0 \leq p \leq \dim(\widetilde{X})$. By tensoring the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-E') \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{E'} \to 0$$

with $\Omega^p_{\widetilde{X}}(\log E')$, we see that $\Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \cong \Omega^p_{\widetilde{X}}(\log E') \otimes \mathcal{O}_{E'}$ for all $0 \leq p \leq \dim(\widetilde{X})$. This induces the natural map

$$\hat{\Omega}^p_{E'} \cong \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \to \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \cong \Omega^p_{\widetilde{X}}(\log E') \otimes \mathcal{O}_{E'},$$

for all $0 \leq p \leq \dim(\widetilde{X})$. Similarly we get a natural map between the sheaves

$$\hat{\Omega}^p_{E'/S} \to \Omega^p_{\widetilde{X}/S}(\log E') \otimes \mathcal{O}_{E'}.$$

Thus, if we tensor the second sequence in (1) with $\mathcal{O}_{E'}$ we have a map from each element in the first sequence in (1) to the second sequence, which induces the following commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0\big(S,\, \Omega^1_S\big) \arrow[r] \arrow[d] & H^0\big(S,\, \Omega^1_S\big) \arrow[d] \\
H^0\big(E',\, \hat{\Omega}^1_{E'}\big) \arrow[r, "\gamma_1"] \arrow[d] & H^0\big(E',\, \Omega^1_{\widetilde{X}}(\log E') \otimes \mathcal{O}_{E'}\big) \arrow[d] \\
H^0\big(E',\, \hat{\Omega}^1_{E'/S}\big) \arrow[r, "\mu_1"] \arrow[d] & H^0\big(E',\, \Omega^1_{\widetilde{X}/S}(\log E') \otimes \mathcal{O}_{E'}\big) \arrow[d] \\
\cdots & \cdots.
\end{tikzcd} \tag{I}$$

In this diagram we can identify $H^0\big(E',\, \Psi^*\Omega^1_S \otimes \mathcal{O}_{E'}\big) \cong H^0\big(S,\, \Omega^1_S\big)$, because $\Psi^*_{|E'} : E' \to S$ is a proper map.

Analogous, tensoring the second sequence in each (2) and (3) with $\mathcal{O}_{E'}$ we get the following two commutative diagrams with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0\big(E',\, \mathcal{F}\big) \arrow[r] \arrow[d] & H^0\big(E',\, \mathcal{G} \otimes \mathcal{O}_{E'}\big) \arrow[d] \\
H^0\big(E',\, \hat{\Omega}^2_{E'}\big) \arrow[r, "\gamma_2"] \arrow[d] & H^0\big(E',\, \Omega^2_{\widetilde{X}}(\log E') \otimes \mathcal{O}_{E'}\big) \arrow[d] \\
H^0\big(E',\, \hat{\Omega}^2_{E'/S}\big) \arrow[r, "\mu_2"] \arrow[d] & H^0\big(E',\, \Omega^2_{\widetilde{X}/S}(\log E') \otimes \mathcal{O}_{E'}\big) \arrow[d] \\
\cdots & \cdots.
\end{tikzcd} \tag{II}$$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0\big(S,\, \Omega^2_S\big) \arrow[r] \arrow[d] & H^0\big(S,\, \Omega^2_S\big) \arrow[d] \\
H^0\big(E',\, \mathcal{F}\big) \arrow[r] \arrow[d] & H^0\big(E',\, \mathcal{G}\otimes\mathcal{O}_{E'}\big) \arrow[d] \\
H^0\big(E',\, \hat{\Omega}^1_{E'/S}\big)^{n-l} \arrow[r, "(\mu_1)^{n-l}"] \arrow[d] & H^0\big(E',\, \Omega^1_{\widetilde{X}/S}(\log E')\otimes\mathcal{O}_{E'}\big)^{n-l} \arrow[d] \\
\dots & \dots.
\end{tikzcd} \tag{III}$$

In diagram (III) we were allowed to exchange the map

$$H^0\big(E',\, (\Psi_{|E'})^*\Omega^1_S\otimes\hat{\Omega}^1_{E'/S}\big) \to H^0\big(E',\, (\Psi^*\Omega^1_S\otimes\hat{\Omega}^1_{\widetilde{X}/S}\otimes\mathcal{O}_{E'}\big)$$

by $(\mu_1)^{n-l}$, because Ω^1_S is trivial. Here, $n-l$ is the dimension of S.

We first want to show that $H^0\big(E',\, \hat{\Omega}^p_{E'/S}\big) = H^0\big(E',\, \Omega^p_{\widetilde{X}/S}(\log E')\otimes\mathcal{O}_{E'}\big) = 0$ for $p\in\{1,\,2\}$. Then the assertion of the Claim 1 for $p=1$ follows from (I) and for $p=2$ from (II) and (III).

Claim 2. $H^0\big(E',\, \hat{\Omega}^p_{E'/S}\big) = 0$ for $p\in\{1,\,2\}$.

Proof (of Claim 2). By Lemma 7.1 we know that $H^0\big(E'_s,\, \hat{\Omega}^p_{E'_s}\big) = 0$ for $s\in S$ and $p\in\{1,\,2\}$. We assume that there exists a non-zero section $\tau\in H^0\big(E',\, \hat{\Omega}^p_{E'/S}\big)$ and use the assertion of Lemma 7.1 to get a contradiction. If $s\in S$ is general then the restriction $\tau_{|E'_s}$ does not vanish,

$$\tau_{|E'_s}\in H^0\big(E'_s,\, (\hat{\Omega}^p_{E'/S})_{|E'_s}\big)\backslash\{0\}.$$

By [Keb13, Cor. 3.10] we have the following isomorphism:

$$H^0\big(E'_s,\, (\hat{\Omega}^p_{E'/S})_{|E'_s}\big)\cong H^0\big(E'_s,\, \hat{\Omega}^p_{E'_s}\big).$$

However, by Lemma 7.1 the right-hand side is zero. This contradicts the assumption and ends the proof of Claim 2. Q.E.D.

Claim 3. $H^0\big(E',\, \Omega^p_{\widetilde{X}/S}(\log E')\otimes\mathcal{O}_{E'}\big) = 0$ for all $p\in\{1,\,2\}$.

Proof (of Claim 3). Recall from the remark to Lemma 5.5 that the maps $\mu_p(s) : H^0\big(E'_s,\, \hat{\Omega}^p_{E'_s}\big)\to H^0\big(E'_s,\, \Omega^p_{\widetilde{X}_s}(\log E'_s)\otimes\mathcal{O}_{E'_s}\big)$ are surjective for all $s\in S$ and $p\in\{1,\,2\}$. Thus $H^0\big(E'_s,\, \hat{\Omega}^p_{E'_s}\big) = 0$ implies that $H^0\big(E'_s,\, \Omega^p_{\widetilde{X}_s}(\log E'_s)\otimes\mathcal{O}_{E'_s}\big) = 0$, hence $H^0\big(E',\, \Omega^p_{\widetilde{X}/S}(\log E')\otimes\mathcal{O}_{E'}\big) = 0$ for all $p\in\{1,\,2\}$ by the same argument as in the proof of Claim 2. Q.E.D.

Now we can finish the proof of Claim 1 by analysing the cases $p=1$ and $p=2$ separately.

<u>Case $p=1$</u>: Consider the diagram (I). The first horizontal map $H^0\big(S,\Omega^1_S\big)\to H^0\big(S,\Omega^1_S\big)$ is surjective. Thus Claim 2 and 3 ensure that γ_1 is surjective as well.

Case $p = 2$: First consider diagram (III). Since the top map $H^0\big(S, \Omega^2_S\big) \to H^0\big(S, \Omega^2_S\big)$ is surjective, Claim 2 and 3 ensure that $H^0\big(E', \mathcal{F}\big) \to H^0\big(E', \mathcal{G} \otimes \mathcal{O}_{E'}\big)$ is surjective as well. This map is the first horizontal map in diagram (II). By the same argument as in the case $p = 1$ using Claim 2 and 3 we see that γ_2 is surjective. This finishes the proof of Claim 1. Q.E.D.

Step 2.4 (Proof of case $k = 0$): By taking the cohomology of the exact sequence

$$0 \to \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \to \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big) \to \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}} \to 0$$

and by applying Claim 1 of Step 2.3 we see that the boundary map

$$\delta : H^0\big(E', \Omega^p_{\widetilde{X}}\big(\log E'\big)/\Omega^p_{\widetilde{X}}\big) \to H^1\big(E', \Omega^p_{\widetilde{X}}/\Omega^p_{\widetilde{X}}\big(\log E'\big)\big(-E'\big)\big)$$

in the long exact sequence is an injection for $p \in \{1, 2\}$. Similar to the proof of Lemma 5.1 we get that $\tau : H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E'\big)\big)$ is surjective. This finishes the proof of the case $k = 0$.

Step 3 (Case $k > 0$): We assume that the result holds for all cases where $0 \leq k \leq m$ for some $m \in \mathbb{N}$ and want to show it for $k = m + 1$. We once again set $l := \operatorname{codim}_X(T)$.

Step 3.1 (Cutting down): As in Step 2.1 we consider $n - l - k$ general hyperplanes $H_1, \ldots, H_{n-l-k}$ and denote by $H := H_1 \cap \cdots \cap H_{n-l-k}$ a general $(l + k)$-dimensional complete intersection in X. Let $t_0 \in T^0 \cap H$. As before we know the following facts. By replacing X by a small open neighbourhood of t_0 we may assume that $T^0 \cap H = \{t_0\}$. The preimage $\widetilde{H} := \eta^{-1}(H)$ is a resolution of singularities. Since X is an affine GIT-quotient H is an affine GIT-quotient as well. As in Step 2.1 we can define a flat surjective morphism $\Phi : X \to S$ with connected fibers, such that the restriction $\Phi_{|T^0} : T^0 \to S$ is an isomorphism and such that there exist a point $s_0 \in S$ with $\Phi^{-1}(s_0) = H$. Given any point $s \in S$ the preimage $X_s := \Phi^{-1}(s)$ is a $(l+k)$-dimension affine GIT-quotient X_s. By $t_s \in T^0 \cap X_s$ we denote the unique point that maps to $s \in S$. The map $\eta : \widetilde{X} \to X$ gives a simultaneous resolution of the fibers X_s, for $s \in S$. Let $E' \subset \widetilde{X}$ be the union of all irreducible components of E that map into T^0. Then $E' \to S$ is a proper map and by Lemma 5.7 every fiber E'_s is a simple normal crossing divisor. Note that E'_s has support $\eta^{-1}(t_s)$, for all $s \in S$.

Step 3.2 (Proof of case $k > 0$): By the induction hypothesis $H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E\big)\big) \cong H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E'\big)\big)$ for all $p \in \{1, 2\}$. Thus we only have to prove that $\tau : H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big(\log E'\big)\big)$ is surjective for $p \in \{1, 2\}$. Following the argumentation of Step 2.2 to Step 2.4 we get the result. $\square$

Part III.
The 4-dimensional case

In this part we want to analyse Theorem 0.1 when X is a 4-dimensional good quotient of a smooth variety by a reductive group. Let $\eta : \widetilde{X} \to X$ be a log resolution and $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ a reflexive p-form on X, $0 \leq p \leq 4$. For $p = 0$ and $p = 4$, the extension of $\eta^*(\sigma)$ to all of $\widetilde{X}$ without poles follows from the fact that X has rational singularities (see Corollary 1.8). For $p \in \{1, 2\}$, the result was proven in the last chapter. Thus, it is only left to check the Extension Theorem for 3-forms.

Remark. Since $\dim(X) = n = 4$, in this part we are interested in the special case of an $(n-1)$-form. When (X, D) is a klt-pair this case is handled separately (see [GKK10, Prop. 6.1]). However, those kind of arguments won't work in our situation, because of two reasons. First of all we do not know anything about the discrepancy (see [GKK10, Prop. 5.1]). Secondly we do not have a duality between $(n-1)$-forms and vector fields (see [GKK10, Prop. 6.1]), because in general K_X is not Cartier and thus $\mathcal{O}_X(K_X) \ncong \mathcal{O}_X$.

By Proposition 1.9 we already know that $\eta^*(\sigma)$ extends with log-poles along the exceptional divisor $E := \operatorname{exc}(\eta)$. In this case the residue sequence introduced in [EV92] is a useful tool to show that $\eta^*(\sigma)$ in fact extends without these poles. The details of this idea will be explained in Chapter 9.B. Examples how this sequence is used in the case of a klt-pair (X, D) can be found in [GKKP11, Rem. 11.8, 25.E]. Similar to the proof in the klt case, we would like to split the log resolution $\eta : \widetilde{X} \to X$ into a finite sequence of surjective, birational morphisms

$$\widetilde{X} \to X_k \to \cdots \to X_1 \to X_0 := X, \tag{$\star$}$$

such the following properties hold.

1. The morphism $\widetilde{X} \to X_k$ is a log resolution and X_k has finite quotient singularities.
2. Each morphism $X_{i+1} \to X_i$ contracts exactly one irreducible divisor that is a strict transform of an irreducible component of E that is not contracted by $\widetilde{X} \to X_k$.

Then we can apply a modified version of the residue sequence to each step separately. The desired sequence $(\star)$ is induced by the partial resolution algorithm of Kirwan and will be introduced in the next chapter. A modified version of the residue sequence that can be applied in this case will be introduced in Chapter 9.B.

8. The partial resolution of Kirwan

Let V be a smooth projective variety and suppose that a reductive group G acts linearly on V with respect to a projective embedding $V \subset \mathbb{P}^N$. Then $V^{ss}/\!\!/G$ is a projective GIT-quotient in the sense of Definition 3.9 (i.e. the G-linearisation $\mathcal{L}$ on V, corresponding to the projective embedding, is ample). Throughout this chapter, whenever we consider a G-action on a projective variety V, we think of an action of this type. By abuse of notation we will write $X = V/\!\!/G$ instead of $X = V^{ss}/\!\!/G$ to simplify the diagrams in the following chapters.

Theorem 8.1 (Kirwan's partial Resolution, [Kir85, 6.1, 6.3, 6.9])**.** *Let G be a reductive group and V a smooth projective G-variety with a G-linearisation $\mathcal{L} = \mathcal{L}_0$ on V, such that*

we get a projective GIT-quotient $X := V/\!\!/G$. If $V^s \neq \emptyset$, then there exists a finite sequence of blow-ups

$$\begin{tikzcd} V_k \arrow[r, "F_k"'] \arrow[rrrr, bend left, "F"] & V_{k-1} \arrow[r, "F_{k-1}"'] & \dots \arrow[r, "F_2"'] & V_1 \arrow[r, "F_1"'] & V_0 = V, \end{tikzcd} \qquad \text{for } k \in \mathbb{N}, \tag{K1}$$

along smooth G-invariant subspaces $W_i \subset V_i$, for $0 \leq i \leq k-1$, such that we have a G-action on each V_{i+1} (lifting the action on V_i) and a G-linearisation $\mathcal{L}_{i+1}$ on V_{i+1} with the following properties:

1. *Every semi-stable point in V_k is stable (with respect to $\mathcal{L}_k$).*
2. *For all $0 \leq i \leq k-1$, every lift in V_{i+1} of a stable (respectively not semi-stable) point in V_i (with respect to $\mathcal{L}_i$) is stable (respectively not semi-stable) in V_{i+1} (with respect to $\mathcal{L}_{i+1}$).*

Moreover, this sequence induces a sequence of blow-ups of the induced GIT-quotients

$$\begin{tikzcd} X_k \arrow[r, "f_k"] \arrow[rrrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 = X \end{tikzcd} \tag{K2}$$

along the images $W_i/\!\!/G$ of W_i in $X_i := V_i/\!\!/G$, for $0 \leq i \leq k-1$, such that the diagram

$$\begin{tikzcd} V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \dots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[r, "F_1^{ss}"'] \arrow[d, "\pi_1"'] & V_0^{ss} := V^{ss} \arrow[d, "\pi_0 = \pi"] \\ X_k \arrow[r, "f_k"] \arrow[rrrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 := X \end{tikzcd} \tag{K3}$$

commutes. Here, F_i^{ss} and F^{ss} denote the restrictions of F_i and F to the semi-stable locus. We will refer to this as the partial resolution of Kirwan.

Remark. The following remaks should help the reader to understand the diagram (K3) better and should illustrate the fact that we can construct the sequence ($\star$), presented in the introduction to Part III from (K3).

1. In this thesis we want $k \in \mathbb{Z}$ to be the minimal number such that the assertion of the previous theorem are true. We call k the *number of Kirwan steps*.
2. Let $0 \leq i \leq k-1$ be an integer. The G-invariant and smooth subset $W_i \subset V_i$ contains all semi-stable points in V_i (with respect to $\mathcal{L}_i$) that have a stabiliser isomorphic to a reductive group $R_i \subset G$ of maximal dimension (see [Kir85, Ch. 5]). By [Kir85, Lem. 6.1] we know that every semi-stable point in the F_{i+1}-exceptional locus (with respect to $\mathcal{L}_{i+1}$) is no longer fixed by a conjugate of R_i.
3. Since $V_k^{ss} = V_k^s$ the quotient $\pi_k : V_k^{ss} \to X_k$ is a geometric quotient and X_k has only finite quotient singularities (see Example 4.11). Since $f : X_k \to X$ is surjective we can call f a partial resolution.
4. Let $E_f \subset X_k$ be the f-exceptional divisor in X_k. Then E_f consists of k irreducible components $E_1, \dots, E_k$, where E_i is the strict transform of the f_i-exceptional divisor, for $1 \leq i \leq k$.

Lemma 8.2 (The exceptional locus and fibers of the blow-ups on the level of quotients)**.** *In the setting of Theorem 8.1, let $1 \leq i \leq k$ be a positive number. Consider the i-th Kirwan step with the same notation as in the previous Theorem and the corresponding diagram:*

$$\begin{tikzcd} V_i^{ss} \arrow[r, "F_i^{ss}"] \arrow[d, "\pi_i"'] & V_{i-1}^{ss} \arrow[d, "\pi_{i-1}"] \\ X_i \arrow[r, "f_i"'] & X_{i-1}. \end{tikzcd}$$

(8.2.1) Let T_{i-1} be the image $W_{i-1}/\!\!/G$ of W_{i-1} in X_{i-1}. Then the exceptional locus $f_i^{-1}(T_{i-1})$ of f_i is a quotient $F_i^{-1}(W_{i-1})/\!\!/G$ (with respect to the restriction of the linearisation $\mathcal{L}_i$ to $F_i^{-1}(W_{i-1})$) of a smooth projective variety $F_i^{-1}(W_{i-1})$ by G.

(8.2.2) Let $x_{i-1} \in T_{i-1}$ be a point in the quotient. Then there exists a $m \in \mathbb{N}_+$ such that $f_i^{-1}(x_{i-1})$ is isomorphic to a projective GIT-quotient $\mathbb{P}^m/\!\!/H$ (with respect to some H-linearisation on $\mathbb{P}^m$), where $H \subset G$ is a reductive subgroup of G.

Proof. The first assertion follows from the definition of a blow-up of a smooth variety along a smooth subspace and [Kir85, Lem. 3.11, Rem. 6.8]. Since W_{i-1} is smooth and G-invariant the preimage $F_i^{-1}(W_{i-1})$ is smooth with an induced G-action. By the construction of diagram $(K3)$ there exists a G-linearisation on $F_i^{-1}(W_{i-1})$, such that the GIT-quotient $F_i^{-1}(W_{i-1})/\!\!/G$ coincides with the f_i-exceptional divisor (see [Kir85, Rem. 6.8]).

The proof of the second assertion is based on [Kir85, Rem. 6.4]. Let $x_{i-1} \in T_{i-1}$ be a point in the quotient. Then there exists a $v_{i-1} \in \pi_{i-1}^{-1}(x_{i-1}) \subset W_{i-1}$ with closed G-orbit. Thus, the preimage $F_i^{-1}(v_{i-1})$ is isomorphic to the projectification $\mathbb{P}(N_{v_{i-1}})$ of the normal space $N_{v_{i-1}}$ to W_{i-1} at the point v_{i-1}. The stabiliser $G_{v_{i-1}}$ acts linearly on $\mathbb{P}(N_{v_{i-1}})$. This action is induced by its canonical action of on the normal space $N_{v_{i-1}}$. Since $v_{i-1} \in W_{i-1} \cap V_{i-1}^{ss}$ the group $G_{v_{i-1}}$ is a reductive subgroup of G (see previous remark, assertion 2). As before there exists a $G_{v_{i-1}}$-linearisation on $\mathbb{P}(N_{v_{i-1}})$, such that $\mathbb{P}(N_{v_{i-1}})/\!\!/G_{v_{i-1}}$ coincides with the preimage $f_i^{-1}(x_{i-1})$ (see [Kir85, Rem. 6.8]). □

Remark. As we have seen in the previous lemma, the f_i-exceptional locus $f_i^{-1}(T_{i-1})$ and the fiber of a point x_{i-1} under the blow-up f_i can be expressed as GIT-quotients of smooth varieties. Thus, the exceptional locus and these fibers are normal varieties and the notion of a reflexive differential form on them is well defined.

The partial resolution was formulated and proven only in the case where V is a smooth projective G-variety with $V^s \neq \emptyset$. By Corollary 4.8 we can prove Theorem 0.3 by reducing to the case of an affine GIT-quotient $X := V/\!\!/G$ of a vector space V by a group G with linear action on V. In general this vector space might not fulfil the condition $V^s \neq \emptyset$. Thus, we need to find a partial resolution algorithm for this case as well. Since we are only interested in the GIT-quotient $X := V/\!\!/G$, we can change G and V such that the partial resolution algorithm of Kirwan induces a new algorithm in this case, too. As a preparation we need the following two Lemmas.

Lemma 8.3. *Let Z be a smooth variety with an action of a reductive group G. Let $U \subset Z$ be an open G-invariant subvariety admitting a good quotient $\pi : U \to U/\!\!/G$. If $U/\!\!/G$ is quasi-projective, then there exists a G-linearisation on Z such that U is a G-saturated subset of Z^{ss}.*

Proof. The result follows from [Hau04, Thm. on p.1]. In his paper Hausen works on a normal variety Z instead of a smooth variety and thus uses a Weil divisor D on Z for the G-linearisation instead of a line bundle $\mathcal{L}$. In Definition 1.1 he then explains that in the smooth case, where D is Cartier (and thus corresponds to a line bundle $\mathcal{L}$), his definition of a G-linearisation induced by D coincides with the original definition of a G-linearisation induced by $\mathcal{L}$ in [MFK94]. □

Corollary 8.4. *Let V be a vector space and G a reductive group acting linearly on V. Let $X := V/\!\!/G$ be the induced GIT-quotient. Then there exists a smooth projective G-variety $\overline{V}$, together with a G-linearisation on $\overline{V}$, such that V is a G-saturated subset of $\overline{V}^{ss}$ and the G-action on $\overline{V}$ is linear with respect to some projective embedding.*

Proof. We choose $\overline{V} := \mathbb{P}(V \oplus \mathbb{C})$ with the induced G-action. Then the result follows from Lemma 8.3. The G-action is linear with respect to some projective embedding since any chosen linearisation $\mathcal{L}$ on $\mathbb{P}(V \oplus \mathbb{C})$ is ample. □

Proposition 8.5 (Kirwan's partial resolution for GIT-quotients of vector spaces)**.** *Let G be a reductive group acting linearly on a vector space V and consider the induced affine GIT-quotient $X := V/\!\!/G$. Let $\mathcal{L} = \mathcal{L}_0$ be the trivial G-linearisation on V, such that $V^s \neq \emptyset$. Then there exists a finite sequence of blow-ups*

$$\begin{tikzcd} V_k \arrow[r, "F_k"'] \arrow[rrrr, bend left, "F"] & V_{k-1} \arrow[r, "F_{k-1}"'] & \dots \arrow[r, "F_2"'] & V_1 \arrow[r, "F_1"'] & V_0 = V, \end{tikzcd} \qquad \text{for } k \in \mathbb{N}, \tag{K1v}$$

along smooth G-invariant subspaces $W_i \subset V_i$, for $0 \leq i \leq k-1$, such that we have a G-action on each V_{i+1} (lifting the action on V_i) and a G-linearisation $\mathcal{L}_{i+1}$ on V_{i+1} with the following properties.

1. *Every semi-stable point in V_k is stable (with respect to $\mathcal{L}_k$).*
2. *For all $0 \leq i \leq k-1$, every lift in V_{i+1} of a stable (respectively not semi-stable) point in V_i (with respect to $\mathcal{L}_i$) is stable (respectively not semi-stable) in V_{i+1} (with respect to $\mathcal{L}_{i+1}$).*

Moreover, this sequence induces a sequence of blow-ups of the induced GIT-quotients

$$\begin{tikzcd} X_k \arrow[r, "f_k"] \arrow[rrrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 = X \end{tikzcd} \tag{K2v}$$

along the images $W_i/\!\!/G$ of W_i in $X_i := V_i/\!\!/G$, for $0 \leq i \leq k-1$, such that the diagram

$$\begin{tikzcd} V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \dots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[r, "F_1^{ss}"'] \arrow[d, "\pi_1"'] & V_0^{ss} := V^{ss} \arrow[d, "\pi_0 = \pi"] \\ X_k \arrow[r, "f_k"] \arrow[rrrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 := X \end{tikzcd} \tag{K3v}$$

commutes. Here, F_i^{ss} and F^{ss} denote the restrictions of F_i and F to the semi-stable locus. We will refer to this as the partial resolution of Kirwan for vector spaces.

Proof. By Corollary 8.4 there exists a smooth projective G-variety $\overline{V}$ togehter with some G-linearisation such that V is a G-saturated subset in $\overline{V}^{ss}$ and the G-action on $\overline{V}$ is linear with respect to some embedding. Since $V^s \neq \emptyset$ it follows that $\overline{V}^s \neq \emptyset$. Thus, Kirwan's partial resolution algorithm works for $\overline{V}$ and induces the diagrams $(K1) - (K3)$. If we consider $(K3)$ only over $V \subset \overline{V}^{ss}$ (or $X = V/\!\!/G$ respectively) the restriction defines the diagrams $(K1v) - (K3v)$ with the needed properties. □

Remark. By construction, the partial resolution of Kirwan for vector spaces has the same properties as the original partial resolution of Kirwan. In particular, the remark to Theorem 8.1 and Lemma 8.2 are fulfilled in this setting as well.

Lemma 8.6. *Let G' be a reductive group acting linearly on a vector space V'. Let $X := V'/\!\!/G'$ be the induced GIT-quotient. Then there exists a reductive group G acting linearly on a vector space V such that $X \cong V/\!\!/G$ and such that, using the trivial G-linearisation on V, we have $V^s \neq \emptyset$.*

Proof. The existence of V and G follows from Proposition 3.21 together with the remark to this proposition. By Corollary 3.22 we know that $V^s \neq \emptyset$. □

The previous lemma implies the following result:

Corollary 8.7. *Let X be a good quotient of a smooth variety by a reductive group. Then we can find an affine GIT-quotient $V/\!\!/G$, where G is a reductive group acting linearly on a vector space V with the following properties:*

1. *The Extension Theorem for $V/\!\!/G$ implies the Extension Theorem for X.*
2. *The partial resolution algorithm presented in Proposition 8.5 works for $V/\!\!/G$.*

Proof. The first assertion follows from Lemma 4.7 and Corollary 4.8. The second assertion is a direct consequence of the previous lemma. □

The following example illustrates the partial resolution algorithm of Kirwan.

Example 8.8. Let V, G and X be as in Example 3.4. Since V is an affine G-variety with trivial G-linearisation, we know that $V^{ss} = V$. In Example 3.8 we have seen that the axes of $\mathbb{C}^2$ contain all the vectors $v \in V$ with $v \notin V^s$ and that $0 \in V^{ss}$ is the semi-stable vector that has the largest stabilizer (with respect to the dimension). Since the orbit $G(0)$ is closed the space along which we have to blow up according to the algorithm of Kirwan is $W := \{0\}$.

Consider the blow-up $F : \widetilde{V} \to V$ of V in $0 \in V$. The fiber $F^{-1}(\{0\})$ is isomorphic to $\mathbb{P}^1$. One can easily see that the points v_1, v_2 corresponding to $[1:0]$ and $[0:1] \in \mathbb{P}^1$ are the only non-semi-stable points of $\widetilde{V}$, because polynomials of the form $Z_1 \cdot Z_2$ vanish only in these points. Except for v_1 and v_2 every other point in $v \in \widetilde{V}$ has trivial stabiliser. Since $(\widetilde{V})^{ss} = \widetilde{V} \backslash \{v_1, v_2\}$ the orbit $G(v)$ is closed in $(\widetilde{V})^{ss}$. Thus $(\widetilde{V})^{ss} = (\widetilde{V})^s$ and the algorithm stops. The quotient $\widetilde{V}/\!\!/G$ is isomorphic to $\mathbb{C}$ and the induced blow-up of quotients $f : \widetilde{V}/\!\!/G \to V/\!\!/G$ is an isomorphism.

9. Preparation for Theorem 0.3

In this chapter we want to present some reduction steps and results that will help us proving Theorem 0.3. For the convenience of the reader lets recall the theorem.

Theorem (Extension of $(n-1)$-forms on good quotients)**.** *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$ of dimension $n \geq 1$. Let $\eta : \widetilde{X} \to X$ be a log resolution and $\sigma \in H^0\big(X, \Omega_X^{[n-1]}\big)$ a reflexive $(n-1)$-form on X. Assume that the Extension Theorem is true for all reflexive p-forms on any good quotient of dimension less than n. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^{n-1}\big).$$

9.A. Reduction steps

First we would like to recall that, since proving the Extension Theorem is a local problem, it is enough to prove Theorem 0.3 when V is a vector space, G a reductive group acting linearly on V and X is the induced affine GIT-quotient (see Corollary 4.8). In this chapter we will reduce the theorem above to an extension problem concerning a single blow up step in the partial resolution of Kirwan. Since for $\dim(X) \leq 1$ the result is clear (in this cases X is smooth), from now on we will assume that $n \geq 2$. To be able to use an induction step we will work in the following setting:

Setting 1. Let G be a reductive group and V a smooth G-variety admitting a GIT-quotient $X := V/\!\!/G$ (with respect to some G-linearisation). Assume that for V the partial resolution algorithm of Kirwan works and that we get the following commutative diagram (we know from (K3) or (K3v) respectively):

$$\begin{tikzcd}
V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \dots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[r, "F_1^{ss}"'] \arrow[d, "\pi_1"'] & V_0^{ss} := V^{ss} \arrow[d, "\pi_0"'] \\
X_k \arrow[r, "f_k"] \arrow[rrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 := X.
\end{tikzcd}$$

Lemma 9.1. *In Setting 1 there exists a log resolution $\eta : \widetilde{X} \to X$ that factors through the partial resolution of Kirwan.*

Proof. Consider a strong log resolution of X_k and a strong log resolution of X Then there exists a resolution $\eta : \widetilde{X} \to X$ that dominates both and fulfils the conditions of the lemma. □

Setting 2. Let G, V and $X := V/\!\!/G$ be as in Setting 1 and $\eta : \widetilde{X} \to X$ the resolution introduced in the previous lemma. Then we get the following diagram:

$$\begin{tikzcd}
 & V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \dots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[r, "F_1^{ss}"'] \arrow[d, "\pi_1"'] & V_0^{ss} := V^{ss} \arrow[d, "\pi_0"'] \\
\widetilde{X} \arrow[r, "\eta_k"] \arrow[rrrr, bend right, "\eta"'] & X_k \arrow[r, "f_k"] \arrow[rrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 := X.
\end{tikzcd} \tag{R1}$$

Remark. The composition $\eta_i := \eta_k \circ f_k \circ \cdots \circ f_i : \widetilde{X} \to X_i$ is a log resolution of X_i, for $i \in \{1, \ldots, k\}$.

Reduction 1. Because of the Corollary 8.7 and the invariance of the choice of resolution (see Corollary 4.3), Theorem 0.3 is a direct corollary of the following proposition:

Proposition 9.2. *Let G, V, X and $\eta : \widetilde{X} \to X$ be as in Setting 2 together with the diagram (R1). Assume that $\dim(X) = n \geq 2$ and that the Extension Theorem is true for all reflexive p-forms on any GIT-quotient of dimension less than n. Let $\sigma \in H^0\big(X, \Omega_X^{[n-1]}\big)$ be a reflexive $(n-1)$-form on X. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^{n-1}\big).$$

Reduction 2. Recall that X_k has finite quotient singularities. By Proposition 4.12 the pull-back of any reflexive differential form on X_k by η_k extends as a regular form to all of $\widetilde{X}$. Thus $\eta^*(\sigma) = \eta_k^*(f^*(\sigma)) \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^{n-1}\big)$ exactly if $f^*(\sigma) \in H^0\big(X_k, \Omega_{X_k}^{[n-1]}\big)$. In other words, it is enough to show that the pull-back of σ via f is a reflexive differential form without poles on X_k.

Reduction 3. We want to show that in order to prove Proposition 9.2 it is enough to prove the following lemma:

Lemma 9.3. *Let G be a reductive group and V a smooth G-variety admitting a GIT-quotient $X := V/\!\!/G$ (with respect to some G-linearisation). Assume that for V the partial resolution algorithm of Kirwan works and that we get the following diagram (that we know from Setting 2)*

$$\begin{tikzcd}
 & V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \cdots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[r, "F_1^{ss}"'] \arrow[d, "\pi_1"'] & V_0^{ss} := V^{ss} \arrow[d, "\pi_0"'] \\
\widetilde{X} \arrow[r, "\eta_k"] \arrow[rrrr, bend right, "\eta"'] \arrow[rrrr, bend right=60, "f"'] & X_k \arrow[r, "f_k"] \arrow[rrr, bend right] & X_{k-1} \arrow[r, "f_{k-1}"] & \cdots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 := X,
\end{tikzcd} \tag{R1}$$

where $\eta : \widetilde{X} \to X$ is a log resolution of X, such that the composition $\eta_i := \eta_k \circ f_k \circ \cdots \circ f_i : \widetilde{X} \to X_i$ is a log resolution of X_i, for $i \in \{1, \ldots, k\}$. Assume that $\dim(X) = n \geq 2$ and that the Extension Theorem is true for all reflexive p-forms on any GIT-quotient of dimension less than n. Let $\sigma \in H^0\big(X, \Omega_X^{[n-1]}\big)$ be a reflexive $(n-1)$-form on X. Then σ lifts without poles to all of X_1 under the first Kirwan step f_1. In other words

$$\sigma_1 := f_1^*(\sigma) \in H^0\big(X_1, \Omega_{X_1}^{[n-1]}\big).$$

Assume that the assertion of Lemma 9.3 is true. Then G, V_1 and X_1 fulfil the conditions of Setting 2 and we get a commutative diagram:

$$\begin{tikzcd}
 & V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \dots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[d, "\pi_1 := \pi"] \\
\widetilde{X} \arrow[r, "\eta_k"] \arrow[rrrr, bend right, "\eta"'] & X_k \arrow[r, "f_k"] \arrow[rrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \dots \arrow[r, "f_2"] & X_1.
\end{tikzcd} \tag{R2}$$

By renaming the elements of the diagram we can apply Lemma 9.3 and it follows that

$$f_2^*(\sigma_1) := f_2^*(f_1^*(\sigma)) \in H^0\big(X_2, \Omega_{X_2}^{[n-1]}\big).$$

Continuing with this procedure after finitely many steps we deduce that

$$f_k^* f_{k-1}^* \dots f_1^*(\sigma) := f^*(\sigma) \in H^0\big(X_k, \Omega_{X_k}^{[n-1]}\big).$$

In conclusion, using Reduction 2 we prove Proposition 9.2 by repeatedly applying Lemma 9.3 to the GIT-quotients that appear in the partial resolution of Kirwan.

9.B. The residue sequence for pairs of good quotients

Let Y be a smooth variety and $D \subset Y$ a reduced irreducible snc divisor. We call (Y, D) an *snc pair*. For these pairs [EV92, Prop. 2.3] introduced the following exact sequence

$$0 \to \Omega_Y^p \to \Omega_Y^p(\log D) \xrightarrow{res} \Omega_D^{p-1} \to 0,$$

for all $1 \leq p \leq \dim(Y)$, which is called *residue sequence*. In the case where Y is not smooth the residue sequence was constructed for a dlt pair (Y, D) by [GKKP11, Thm. 11.7]. Assume that $f : Y \to X$ is a resolution of singularities of X where D is the f-exceptional divisor and σ is a reflexive p-form on X. Assume that $f^*(\sigma) \in H^0\big(Y, \Omega_Y^p(\log D)\big)$. Then the residue sequence is a useful tool the examine whether $f^*(\sigma)$ has logarithmic poles along D or not. To this end, consider the induced long exact sequence of cohomology groups

$$0 \to H^0\big(Y, \Omega_Y^p\big) \to H^0\big(Y, \Omega_Y^p(\log D)\big) \xrightarrow{res} H^0\big(D, \Omega_D^{p-1}\big) \to \dots .$$

Then $f^*(\sigma) \in H^0\big(Y, \Omega_Y^p\big)$ if and only if $res(f^*(\sigma)) = 0$ (by abuse of notation we will not differ between the residue map on the level of sheaves or cohomology groups). The following lemma enables us to use a similar approach for GIT-quotients.

Lemma 9.4 (Residue sequence for good quotients)**.** *Let G be a reductive group and V a smooth G-variety admitting a good quotient $X := V/\!\!/G$. Let $E \subset X$ be a reduced irreducible smooth divisor on X. Then there exists a residue sequence (which is exact)*

$$0 \to \Omega_X^{[p]} \to \Omega_X^{[p]}(\log E) \xrightarrow{res} \Omega_E^{p-1}$$

for all $1 \leq p \leq \dim(X)$, such that the restriction to the sheaves defined on the smooth locus X_{sm} provides the exact residue sequence of the snc pair $(X_{\mathrm{sm}}, E \cap X_{\mathrm{sm}})$:

$$0 \to \Omega_{X_{\mathrm{sm}}}^p \to \Omega_{X_{\mathrm{sm}}}^p(\log(E \cap X_{\mathrm{sm}})) \xrightarrow{res_{|X_{\mathrm{sm}}}} \Omega_{E \cap X_{\mathrm{sm}}}^{p-1} \to 0.$$

Proof. Let $i : X_{\mathrm{sm}} \to X$ be the natural embedding of the smooth locus X_{sm} into X. Consider the exact sequence

$$0 \to \Omega^p_{X_{\mathrm{sm}}} \to \Omega^p_{X_{\mathrm{sm}}}(\log(E \cap X_{\mathrm{sm}})) \xrightarrow{res_{|X_{\mathrm{sm}}}} \Omega^{p-1}_{E \cap X_{\mathrm{sm}}} \to 0.$$

We now can use the fact that push-forward is a left-exact functor to obtain the following exact sequence:

$$0 \to i_*\Omega^p_{X_{\mathrm{sm}}} \to i_*\Omega^p_{X_{\mathrm{sm}}}(\log(E \cap X_{\mathrm{sm}})) \xrightarrow{res_{|X_{\mathrm{sm}}}} i_*\Omega^{p-1}_{E \cap X_{\mathrm{sm}}}$$

By definition of the sheaf of reflexive differential forms, this is the sequence

$$0 \to \Omega^{[p]}_X \to \Omega^{[p]}_X(\log E) \xrightarrow{res} \Omega^{p-1}_E$$

with the claimed property. □

9.C. Log-poles along the exceptional divisor in the first Kirwan step

Our aim is to prove Lemma 9.3. To this end we analyse the properties of the f_1-exceptional divisor $E_1 \subset X_1$ that arises in the first Kirwan step.

Lemma 9.5. *Let X be a normal variety and $f : Y \to X$ a resolution of singularities. Let $E \subset X$ be a reduced divisor on X and $D \subset Y$ the strict transform of E in Y. Consider a reflexive differential form σ on X. Then σ has no pole (a log-pole, a pole of degree 1 or bigger) along E if and only if the pull-back $f^*(\sigma)$ has no pole (a log-pole, a pole of degree 1 or bigger) along D.*

Proof. Since X is normal there exists a point $p \in E \cap X_{\mathrm{sm}}$ and a neighbourhood $U \subset X_{\mathrm{sm}}$ of p such that f is an isomorphism over U. The result of the lemma is true for $\sigma_{|U}$ on $E \cap U$ and thus follows for σ on E. □

Corollary 9.6. *In the setting of Lemma 9.3, let $E_1 \subset X_1$ be the f_1-exceptional divisor. Then*

$$\sigma_1 := f_1^*(\sigma) \in H^0\big(X_1, \Omega^{[n-1]}_{X_1}(\log E_1)\big).$$

Proof. Let $\widetilde{E} = \mathrm{exc}(\eta) \subset \widetilde{X}$ be the exceptional divisor on $\widetilde{X}$. By Proposition 1.9 $\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^{n-1}_{\widetilde{X}}(\log \widetilde{E})\big)$. Recall that $\widetilde{E} = E_{\eta_k} \cup (\bar{E}_1 \cup \cdots \cup \bar{E}_k)$ where E_{η_k} gets contracted by η_k and $\bar{E}_i$ is the irreducible component of $\widetilde{E}$, that is the strict transform of the f_i-exceptional divisor $E_i \subset X_i$, for $i \in \{1, \ldots, k\}$. Consider the log resolution $\eta_i : \widetilde{X} \to X_i$ for $i \in \{1, \ldots, k\}$. Since $\eta^*(\sigma) = \eta_1^*(\sigma_1)$ has at most a log-pole along $\bar{E}_1$ it follows that

$$\sigma_1 := f_1^*(\sigma) \in H^0\big(X_1, \Omega^{[n-1]}_{X_1}(\log E_1)\big).$$

□

10. The proof

In this chapter we want to apply the residue sequence constructed in Lemma 9.4 to the single Kirwan step (that we consider in Lemma 9.3) to prove Lemma 9.3 and thus conclude Theorem 0.3. As a last preparation we need to introduce the notion of an unirational variety.

Definition 10.1 ([Sha77, p. 208]). *A variety X is called* unirational *if there exists a dominant rational morphism $\Phi : \mathbb{P}^m \dashrightarrow X$, for some $m \in \mathbb{N}$.*

Example 10.2. Let G be a reductive group acting on $V = \mathbb{P}^m$. Consider a G-linearisation on V such that we get a projective GIT-quotient $X := V/\!\!/G$. Since the space of semi-stable points V^{ss} is dense in V, the quotient X is a unirational variety.

Lemma 10.3. *Let X be a normal unirational variety and let $0 < p \leq \dim(X)$ be an integer. Assume that for any reflexive p-form $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ on X and any resolution $f : Y \to X$ the pull-back $f^*(\sigma)$ extends to all of Y without poles. Then $H^0\big(X, \Omega_X^{[p]}\big) = 0$.*

Proof. Let $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ be any reflexive p-form on X, for $0 \leq p \leq dim(X)$, and let $f : Y \to X$ be a resolution. Then $f^*(\sigma) \in H^0\big(Y, \Omega_Y^p\big)$. Since X is unirational, Y is unirational as well. By [Sha77, p. 208] we know that $H^0\big(Y, \Omega_Y^p\big) = 0$ for all $0 < p \leq \dim(Y)$. Thus $f^*(\sigma) = 0$ and hence $\sigma = 0$. □

By Reduction 3 in Chapter 9.A we know that Theorem 0.3 can be deduced from Lemma 9.3. Our goal is to prove this lemma. For the convenience of the reader we would like to sum up the essentials of setting we use in the proof:

Setting 3. Let G, V and $X := V/\!\!/G$, with $\dim(X) =: n \geq 2$, be as in Lemma 9.3. Then we get the following diagram

$$\begin{tikzcd}
 & V_k^{ss} \arrow[r, "F_k^{ss}"'] \arrow[d, "\pi_k"'] \arrow[rrr, bend left, "F^{ss}"] & V_{k-1}^{ss} \arrow[r, "F_{k-1}^{ss}"'] \arrow[d, "\pi_{k-1}"'] & \cdots \arrow[r, "F_2^{ss}"'] & V_1^{ss} \arrow[r, "F_1^{ss}"'] \arrow[d, "\pi_1"'] & V_0^{ss} := V^{ss} \arrow[d, "\pi_0"'] \\
\widetilde{X} \arrow[r, "\eta_k"] \arrow[rrrr, bend right, "\eta"'] & X_k \arrow[r, "f_k"] \arrow[rrr, bend right, "f"'] & X_{k-1} \arrow[r, "f_{k-1}"] & \cdots \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & X_0 := X,
\end{tikzcd} \tag{R1}$$

where $\eta : \widetilde{X} \to X$ is a log-resolution. Let $E_1 \subset X_1$ be the f_1-exceptional divisor in X_1 and let $\sigma \in H^0\big(X, \Omega_X^{[n-1]}\big)$ be a reflexive $(n-1)$-form on X. Then we want to show that

$$\sigma_1 := f_1^*(\sigma) \in H^0\big(X_1, \Omega_{X_1}^{[n-1]}\big).$$

By Corollary 9.6 we know that $\sigma_1 \in H^0\big(X_1, \Omega_{X_1}^{[n-1]}(\log E_1)\big)$. Thus we only have to show that σ_1 has no log-pole along E_1.

Reduction 4. Recall that there exists a closed subset $Z \subset X$ with $\operatorname{codim}_X(Z) \geq 3$, such that $X \backslash Z$ has finite quotient singularities (see Proposition 6.2 and remark). Set $T := X_{\text{sing}} \cap Z \subset X$ to be the closed subset of the singular locus that contains all other points. We have to analyse two cases. If $f_1^{-1}(T) \cap E_1 \subset X_1$ is not a divisor in X_1 then the

image of a generic point of E_1 lies in $X \backslash T$. In this case we can show that σ_1 has no pole along E_1, using Corollary 6.3 and Lemma 9.5.

Hence let us assume for the remainder of the proof that $f_1^{-1}(T) \cap E_1 \subset X_1$ is a divisor in X_1. Then it is enough to show that σ_1 has no pole in a generic point of the divisor $f_1^{-1}(T) \cap E_1$ (see proof of Corollary 6.4). Thus, after shrinking T, we may assume that T is irreducible of $\operatorname{codim}_X(T) =: l \geq 3$, $E_1 = f_1^{-1}(T)$ and that $f_{1|E_1} : E_1 \to T$ is a surjective morphism. Since the extension of differential forms is a local problem we may shrink T even further, such that T is smooth and such that the sheaf Ω_T^q of q-forms on T $(0 \leq q \leq \dim(T))$ is trivial for all $0 \leq q \leq \dim(T)$.

We now can prove Lemma 9.3 by analysing different dimensions of T separately. Recall that we work under the following assumption.

Assumption (A1). The Extension Theorem is true for all reflexive p-forms on any GIT-quotient of dimension less than n.

Claim 1. There exists a closed subspace $Z \subset X_1$ with $\operatorname{codim}_{X_1}(Z) \geq 3$ such that $X_1^0 := X_1 \backslash Z$ has finite quotient singularities and such that $E_1^0 := E_1 \cap X_1$ is a smooth irreducible divisor on X_1.

Proof (of Claim 1). Since X_1 and E_1 are GIT-quotients of smooth varieties with respect to some G-linearisation (see Lemma 8.2), they are normal varieties, that are smooth outside codimension 2 and have only f.q.s. outside codimenion 3 by Proposition 6.2. Q.E.D.

Case 1 ($\dim(T) = 0$)**:** Let $T = \{x\}$. Then $E_1 = f_1^{-1}(x)$ is the fiber of x in X_1. By Lemma 8.2 there exists a $m \in \mathbb{N}_{\geq 1}$ and a reductive subgroup $H \subset G$, such that E_1 is the projective GIT-quotient $\mathbb{P}^m /\!\!/ H$ with respect to some H-linearisation on $\mathbb{P}^m$. Thus E_1 is unirational by Example 10.2.

Since E_1 is unirational and by Lemma 8.2 a projective GIT-quotient, assumption (A1) ensures that we can use Lemma 10.3 to get the following result:

$$H^0\big(E_1, \Omega_{E_1}^{[p]}\big) = 0 \text{ for } 0 < p \leq \dim(E_1) = n-1. \tag{+}$$

Let Z, X_1^0, and E_1^0 be as in Claim 1. Recall that $\sigma_1 \in H^0\big(X_1, \Omega_{X_1}^{[n-1]}(\log E_1)\big)$. Thus the restriction $\sigma_1^0 := \sigma_{1|X_1^0} \in H^0\big(X_1^0, \Omega_{X_1^0}^{[n-1]}(\log E_1^0)\big)$ might have a log pole along E_1^0. We can use the residue sequence constructed in Lemma 9.4 to define a $(n-1)$-form $\alpha := res(\sigma_1^0)$ on E_1^0. Since $E_1^0 = E_1 \backslash Z'$, for a closed subset $Z' = Z \cap E_1 \subset E_1$ of $\operatorname{codim}_{E_1}(Z') \geq 2$ and E_1 is normal, we see that (+) implies that $H^0\big(E_1^0, \Omega_{E_1^0}^{n-2}\big) = 0$. Thus $\alpha = 0$ and by Lemma 9.4 hence $\sigma_1^0 \in H^0\big(X_1^0, \Omega_{X_1^0}^{[n-1]}\big)$. Since $X_1^0 = X_1 \backslash Z$ with $\operatorname{codim}_{X_1}(Z) \geq 3$ and X_1 is normal, it follows that $\sigma_1 \in H^0\big(X_1, \Omega_{X_1}^{[n-1]}\big)$. This ends the proof of Case 1.

Case 2 ($\dim(T) > 0$)**:** Let $t \in T$ be a general point of T and the fiber $f_1^{-1}(t) =: E_{1,t} \subset E_1$. By Lemma 8.2 there exists a $m_t \in \mathbb{N}_{\geq 1}$ and a reductive subgroup $H_t \subset G$, such that $E_{1,t}$ is the projective GIT-quotient $\mathbb{P}^{m_t} /\!\!/ H_t$ (with respect to some H_t-linearisation on $\mathbb{P}^{m_t}$). Thus $E_{1,t}$ is unirational.

Since for all $t \in T$ the fiber $E_{1,t}$ is unirational and by Lemma 8.2 a projective GIT-quotient (and therefore normal), assumption (A1) ensures that we can use Lemma 10.3 to

see that $H^0\big(E_{1,t},\, \Omega^{[p]}_{E_{1,t}}\big) = 0$ for $0 < p \leq \dim(E_{1,t})$ and for all $t \in T$. Thus, by the same argument as in [GKKP11, Lem. 25.11], we get the following result:

$$H^0\big(E_1,\, \Omega^{[p]}_{E_1/T}\big) = 0 \text{ for } 0 < p \leq dim(E_1). \tag{++}$$

Let Z, X_1^0 and E_1^0 be as in Claim 1. As before $E_1^0 = E_1 \backslash Z'$, for a closed subset $Z' = Z \cap E_1 \subset E_1$ of $\operatorname{codim}_{E_1}(Z') \geq 2$. Then $H^0\big(E_1,\, \Omega^{[p]}_{E_1/T}\big) = 0$ implies that $H^0\big(E_1^0,\, \Omega^{p}_{E_1^0/T}\big) = 0$.

Assume that $\sigma_1 \in H^0\big(X_1,\, \Omega^{[n-1]}_{X_1}(\log E_1)\big)$ has a true log pole along E_1. Then the restriction $\sigma_1^0 := \sigma_{1|X_1^0} \in H^0\big(X_1^0,\, \Omega^{[n-1]}_{X_1^0}(\log E_1^0)\big)$ has a log pole along E_1^0. We can use the residue sequence constructed in Lemma 9.4 to define a non-vanishing $(n-2)$-form $\alpha := res(\sigma_1^0) \neq 0$ on E_1^0. We will now show that such a form cannot exists.

Consider the induced morphism $\Psi : E_1^0 \to T$ of smooth varieties. After shrinking T, we may assume that Ψ is a smooth morphism. Thus, we get the following filtration (see [Har77, II. Ex. 5.16])

$$\Omega^{n-2}_{E_1^0} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^{n-2} \supset \mathcal{F}^{n-1} = \{0\}.$$

These filtrations induce exact sequences

$$0 \to \mathcal{F}^{r+1} \to \mathcal{F}^r \to \Psi^*\Omega^r_T \otimes \Omega^{n-2-r}_{E_1^0/T} \to 0, \tag{S_r}$$

for all $0 \leq r \leq n-2$, where, by construction, $\mathcal{F}^{n-2} \cong \Psi^*\Omega^{n-2}_T$. If we consider the long exact sequence induced by the sequence above, the following is true for $0 \leq r \leq n-2$: Assume that β is a non-vanishing section of $\mathcal{F}^r$. Then either it induces a non-vanishing section of $\Psi^*\Omega^r_T \otimes \Omega^{n-2-r}_{E_1^0/T}$, or it comes from a non-vanishing section on $\mathcal{F}^{r+1}$. Chasing the $(n-2)$-form α through these sequences we get the following contradiction:

Recall that $\Psi^*\Omega^q_T$ is a trivial bundle for all $0 \leq q \leq \dim(T)$. Thus, by (++) we see, that $H^0\big(E_1^0,\, \Psi^*\Omega^r_T \otimes \Omega^{n-2-r}_{E_1^0/T}\big) = 0$ for all $0 < n-2-r \leq \dim(E_1)$. As a conclusion the last sheaf in the sequence (S_r) has no non-vanishing section for $0 \leq r < n-2$. Therefore, the non-vanishing $(n-2)$-form α on E_1^0 induces a non-vanishing $(n-2)$-form $\alpha' \in H^0\big(E_1^0,\, \Psi^*\Omega^{n-2}_T\big)$. However, $\dim(T) \leq \dim(X) - 3 = n-3$ and such a form cannot exist. As a consequence σ_1 has no log pole along E_1 and $\sigma_1 \in H^0\big(X_1,\, \Omega^{[n-1]}_{X_1}\big)$. $\square$

Part IV.
Further results and open questions

11. Theorem 0.3 for forms of degree $p < n - 1$

In this thesis we discussed an Extension Theorem for reflexive differential forms on an arbitrary n-dimensional good quotient X of a smooth variety by a reductive group G. Let us review what we learned from the thesis.

For 0-forms and n-forms the Extension Theorem follows from the fact that X is normal with rational singularities (see Corollary 1.8). Thus, Theorem 0.2 not only ensures an Extension Theorem for good quotients of dimension 0 to 3, but for reflexive 0-, 1-, 2- and n-forms on a good quotient of arbitrary dimension n as well. Building on this, Theorem 0.3 starts the following short induction process:

Let X be a 4-dimensional good quotient. Then assumption $(A1)$ on page 44 is fulfilled and Theorem 0.3 yields the Extension Theorem in dimension 4. Next we consider a 5-dimensional good quotient X. In this case, the assumption $(A1)$ is fulfilled as well and Theorems 0.2 and 0.3 yield the Extension Theorem for reflexive differential forms of degree 0, 1, 2, 4 and 5 on X. If we now consider a 6-dimensional good quotient, we realise that the assumption $(A1)$ is not necessarily fulfilled and the induction stops.

Assume, that we could prove Theorem 0.3 not only for reflexive $(n-1)$-forms on X, but for reflexive p-forms on X, for all values $2 \leq p \leq n$, instead. Then the induction described above would not stop and would yield an Extension Theorem for good quotients of arbitrary dimension n.

In the following chapters we describe special situations in which the proof in Chapter 10 yields an improved version of Theorem 0.3. Then we explain necessary steps that would allow us to optimise the proof in Chapter 10 such that we would get an Extension Theorem for good quotients of arbitrary dimension.

11.A. A Corollary of the proof in Chapter 10

Going through the proof in Chapter 10 we see that almost every step works for any value $2 < p \leq n$. Only at the end of the proof of Case 2 ($\dim(T) > 0$) one has to realise that the extension of the reflexive p-form σ to X_1 without poles depends on the dimension of T. Recall from the proof, that if the reflexive p-forms σ has a log-pole along E_1, then we can produce a non-zero $(p-1)$-form α' on T (see contradiction on page 45). If $\dim(T) < p-1$ such a form cannot exist and σ has no logarithmic pole along E_1. This argument yields the following corollary.

Corollary 11.1. *Let G be a reductive group and V a smooth G-variety admitting an n-dimensional GIT-quotient $X := V/\!\!/G$ (with respect to some G-linearisation). Assume that the partial resolution algorithm of Kirwan works and let*

$$\widetilde{X} \to X_k \xrightarrow{f_k} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 := X,$$

be the sequence induced by $(R1)$ *on page 39, such that the composition* $\eta : \widetilde{X} \to X$ *is a log resolution. Let* $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$ *be a reflexive p-form on* X*, for* $2 \le p \le n$*. Assume that the Extension Theorem is true for all reflexive q-forms on any GIT-quotient of dimension less than* n*. For all* $0 \le i \le k-1$*, we want* f_i *to be the blow-up along* $T_i \subset X_i$*. Assume that in every step* $0 \le i \le k-1$ *we have* $\dim(T_i) \le p-2$*. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big)$$

for $2 \le p \le n$*.*

11.B. The case $\dim(T_i) = 0$

In the setting of the previous corollary the case $\dim(T_i) = 0$ for all $0 \le i \le k-1$ needs to be considered separately. If $\dim(T_i) = 0$, then in the proof presented in Chapter 10 we only have to consider Case 1. Since the contradiction that uses the dimension of T_i is not needed in this case, it works for a reflexive differential form σ of arbitrary degree $2 \le p \le n$. Together with Theorem 0.2 we get the following result.

Corollary 11.2. *Consider the setting of Corollary 11.1. Assume that in every step* $0 \le i \le k-1$ *we have* $\dim(T_i) = 0$*. Then*

$$\eta^*(\sigma) \in H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big)$$

for $0 \le p \le n$*.*

11.C. How to optimise the proof in Chapter 10

As explained above we would like to optimise the proof in Chapter 10 such that it works for all values $2 \le p \le n$. In this chapter we want to present one way this could be accomplished.

The first step is to show that the left exact sequence in Lemma 9.4 is right exact as well. One idea is to improve a result by [Bri98] to compare the reflexive differential forms on X with G-invariant horizontal forms on V. Then one might be able to deduce an exact residue sequence for reflexive differential forms on X from an exact residue sequence for G-invariant horizontal forms on V. Unfortunately we could not prove the existence of such a residue sequence on V.

The second step is to construct filtrations (similar to the filtrations in the proof of Theorem 0.2) for GIT-quotients with finite quotient singularities.

Assume that the assertions of both steps can be proven. Then one could interchange the arguments in Case 2 of the proof in Chapter 10 with [GKKP11, 25.F.2 - 25.F.4] to proof Theorem 0.3 for all values $2 \le p \le n$.

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