



Dina dos Santos
Tavares

Cálculo das Variações Fracionário
Fractional Calculus of Variations



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Tese de Doutoramento apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática e Aplicações, PDMA, da Universidade de Aveiro e Universidade do Minho, realizada sob a orientação científica do Prof. Doutor Ricardo Miguel Moreira de Almeida, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro, e do Prof. Doutor Delfim Fernando Marado Torres, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro.



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Fractional Calculus of Variations

Ph.D. thesis submitted to the University of Aveiro in fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics, Doctoral Programme in Mathematics and Applications (PDMA), of the University of Aveiro and University of Minho, under the supervision of Professor Ricardo Miguel Moreira de Almeida, Assistant Professor of the Department of Mathematics of University of Aveiro and Professor Delfim Fernando Marado Torres, Full Professor with Habilitation and tenure of the Department of Mathematics of University of Aveiro.

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agradecimentos

Ao longo da realização deste trabalho foram várias as pessoas que, de diversas formas, me incentivaram e me apoiaram de modo a poder concretizá-lo. Desta forma, gostaria de deixar aqui um especial agradecimento aos meus orientadores, Professor Doutor Ricardo Almeida e Professor Doutor Delfim F. M. Torres, pela sugestão do tema, pelo apoio e incentivo constantes, pela partilha de saber e, principalmente, pela orientação agradável e sempre disponível.

Agradeço ainda ao meu marido Nuno, à minha filha Lara e aos meus pais e irmã pelo suporte emocional e pelo carinho e força que sempre me transmitiram.

A todos os meus colegas da Escola Superior de Educação e Ciências Sociais do Instituto Politécnico de Leiria e do PDMA, que me acompanharam e encorajaram ao longo deste percurso, gostaria de expressar o meu sincero agradecimento.

Gostaria também de agradecer à FCT (Fundação para a Ciência e Tecnologia) pelo apoio financeiro que me concedeu através da bolsa de Doutoramento com a referência SFRH/BD/42557/2007.



acknowledgements

During this work, there were several people who, in different ways, encouraged me and supported me in order to be able to concretize it. In this way, I would like to express my special acknowledgement to my advisers, Professor Ricardo Almeida and Professor Delfim F. M. Torres, for the suggestion of the field, for the constant support and encouragement, for sharing knowledge and, especially, for the pleasant and always available guidance.

Thank you also to my husband Nuno, my daughter Lara and my parents and sister for the emotional support and for the attention and strength they have always transmitted me.

To all my colleagues from the School of Education and Social Sciences of the Polytechnic Institute of Leiria and the PDMA, who accompanied me and encouraged me along this journey, I would like to express my sincere thanks.

I would also like to acknowledge the FCT (Portuguese Foundation for Science and Technology) for the financial support, through the Ph.D. fellowship SFRH/BD/42557/2007.



palavras-chave

Cálculo fracionário, cálculo das variações fracionário, condições necessárias de optimalidade, condições de transversalidade, aproximação numérica, estimação de erros.

resumo

O cálculo de ordem não inteira, mais conhecido por cálculo fracionário, consiste numa generalização do cálculo integral e diferencial de ordem inteira. Esta tese é dedicada ao estudo de operadores fracionários com ordem variável e problemas variacionais específicos, envolvendo também operadores de ordem variável. Apresentamos uma nova ferramenta numérica para resolver equações diferenciais envolvendo derivadas de Caputo de ordem fracionária variável. Consideram-se três operadores fracionários do tipo Caputo, e para cada um deles é apresentada uma aproximação dependendo apenas de derivadas de ordem inteira. São ainda apresentadas estimativas para os erros de cada aproximação. Além disso, consideramos alguns problemas variacionais, sujeitos ou não a uma ou mais restrições, onde o funcional depende da derivada combinada de Caputo de ordem fracionária variável. Em particular, obtemos condições de otimalidade necessárias de Euler–Lagrange e sendo o ponto terminal do integral, bem como o seu correspondente valor, livres, foram ainda obtidas as condições de transversalidade para o problema fracionário.

2010 Mathematics

26A33, 33F05, 34A08, 35R11, 49K05, 49K10.

Subject Classification:

keywords

Fractional calculus, fractional calculus of variations, fractional necessary optimality conditions, transversality conditions, numerical approximation, error estimation.

abstract

The calculus of non-integer order, usual known as fractional calculus, consists in a generalization of integral and differential integer-order calculus. This thesis is devoted to the study of fractional operators with variable order and specific variational problems involving also variable order operators. We present a new numerical tool to solve differential equations involving Caputo derivatives of fractional variable order. Three Caputo-type fractional operators are considered, and for each one of them, an approximation formula is obtained in terms of standard (integer-order) derivatives only. Estimations for the error of the approximations are also provided. Furthermore, we consider variational problems subject or not to one or more constraints, where the functional depends on a combined Caputo derivative of variable fractional order. In particular, we establish necessary optimality conditions of Euler-Lagrange. As the terminal point in the cost integral, as well the terminal state, are free, thus transversality conditions are obtained.

2010 Mathematics

26A33, 33F05, 34A08, 35R11, 49K05, 49K10.

Subject Classification:

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Introduction

This thesis intends to deepen the study of the fractional calculus, giving special emphasis to variable order operators.

Fractional calculus is a recent field of mathematical analysis and it is a generalization of integer differential calculus, involving derivatives and integrals of real or complex order [48, 76]. The first note about this idea of differentiation for non-integer numbers, dates back to 1695, with a famous correspondence between Leibniz and L'Hôpital. In a letter, L'Hôpital asked Leibniz about the possibility of the order n in the notation $\frac{d^n y}{dx^n}$, for the n th derivative of the function y , to be a non-integer, $n = 1/2$. Since then, several mathematicians investigated this approach, like Lacroix, Fourier, Liouville, Riemann, Letnikov, Grünwald, Caputo, and contributed to the grown development of this field. Currently, this is one of the most intensively developing areas of mathematical analysis as a result of its numerous applications. The first book devoted to the fractional calculus was published by Oldham and Spanier in 1974, where the authors systematized the main ideas, methods and applications about this field [58].

In the recent years, fractional calculus has attracted the attention of many mathematicians, but also some researchers in other areas like physics, chemistry and engineering. As it is well known, several physical phenomena are often better described by fractional derivatives [44, 67, 116]. This is mainly due to the fact that fractional operators take into consideration the evolution of the system, by taking the global correlation, and not only local characteristics. Moreover, integer-order calculus sometimes contradict the experimental results and therefore derivatives of fractional order may be more suitable [45].

In 1993, Samko and Ross devoted to investigate operators when the order α is not a constant during the process, but variable on time $\alpha(t)$ [89]. With this work, an interesting recent generalization of the theory of fractional calculus is developed to allow the fractional order of the derivative to be non-constant, depending on time [25, 69, 70]. With this

approach of variable order fractional calculus, the non-local properties are more evident and numerous applications have been found in physics, mechanics, control and signal processing [28, 46, 71, 74, 82, 83, 110].

Although there are many variants of fractional derivative definitions, the most commonly used are the Riemann–Liouville, the Caputo, and the Grünwald-Letnikov derivatives. For more about the development of fractional calculus, we suggest Samko, Kilbas and Marichev [88], Samko and Ross [89], Podlubny [76], Kilbas, Srivastava and Trujillo [48] or Mainardi [58].

One difficult issue that usually arises when dealing with such fractional operators, is the extreme difficulty in solving analytically such problems [16, 117]. Thus, in most cases, we do not know the exact solution for the problem and one needs to seek a numerical approximation. Several numerical methods can be found in the literature, typically applying some discretization over time or replacing the fractional operators by a proper decomposition [16, 117].

Recently, new approximation formulas were given for fractional constant order operators, with the advantage that higher-order derivatives are not required to obtain a good accuracy of the method [13, 77, 78]. These decompositions only depend on integer-order derivatives, and by replacing the fractional operators that appear in the problem by them, one leaves the fractional context ending up in the presence of a standard problem, where numerous tools are available to solve them [9].

The first goal of this thesis is to extend such decompositions to Caputo fractional problems of variable order. For three types of Caputo derivatives with variable order, we obtain approximation formulas for this fractional operators and respectively upper bound formulas for the errors.

Then, we focus our attention in a special operator, introduced by Malinowska and Torres, the combined Caputo fractional derivative, that is an extension of the left and the right fractional Caputo derivatives [60]. Considering $\alpha, \beta \in (0, 1)$ and $\gamma \in [0, 1]$, the combined Caputo fractional derivative operator ${}^C D_\gamma^{\alpha, \beta}$ is a convex combination of the left and the right Caputo fractional derivatives, defined by

$${}^C D_\gamma^{\alpha, \beta} = \gamma {}^C D_a^\alpha + (1 - \gamma) {}^C D_b^\beta.$$

Therefore, we consider this fractional operator with variable fractional order, i.e., the

combined Caputo fractional derivative of variable order. This operator is defined by

$${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t) = \gamma_1 {}^C D_a^{\alpha(\cdot)} x(t) + \gamma_2 {}^C D_t^{\beta(\cdot)} x(t)$$

where $\gamma = (\gamma_1, \gamma_2) \in [0, 1]^2$, with γ_1 and γ_2 not both zero.

With this fractional operator, we study different types of fractional calculus of variations problems, where the Lagrangian depends on the referred derivative.

The calculus of variations is a mathematical subject that appeared formally in the XVII century with the solution to the brachistochrone problem, that deals with the extremization (minimization or maximization) of functionals [112]. Usually, functionals are given by an integral that involves one or more functions or/and its derivatives. This branch of mathematics has proved to be relevant because of the numerous applications existing in real situations.

The fractional variational calculus is a recent mathematical field that consists in minimizing or maximizing functionals that depend on fractional operators (integrals or/and derivatives). This subject was introduced by Riewe in 1996, where the author generalizes the classical calculus of variations, using fractional derivatives and allows to obtain conservation laws with nonconservative forces such as friction [84, 85]. Later appeared several works on various aspects of the fractional calculus of variations and involving different fractional operators, like the Riemann–Liouville, the Caputo, the Grünwald–Letnikov, the Weyl, the Marchaud or the Hadamard fractional derivatives [2, 6, 12, 14, 18, 37, 39, 47].

For the state of the art of the fractional calculus of variations, we refer the readers to the books [9, 59, 62].

Specifically, here we study some problems of the calculus of variations with integrands depending on the independent variable t , an arbitrary function x and a fractional derivative ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x$. The endpoint of the cost integral, as well the terminal state are considered to be free. The fractional problem of the calculus of variations consists in finding the maximizers or minimizers to the functional

$$\mathcal{J}(x, T) = \int_a^T L(t, x(t), {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t)) dt + \phi(T, x(T)),$$

where ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t)$ stands for the combined Caputo fractional derivative of variable fractional order, subject to the boundary condition $x(a) = x_a$. For all variational problems presented here, we establish necessary optimality conditions and the transversality optimality conditions.

The work is organized in two parts as follows. In the first part, called Synthesis, we review the basic concepts of fractional calculus (Chapter 1) and of the fractional calculus of variations (Chapter 2). In Chapter 1, we start with a brief overview about fractional calculus and an introduction to the theory of some special functions in fractional calculus. Then, we recall several fractional operators (integrals and derivatives) definitions and some properties of the considered fractional derivatives and integrals are introduced. In the end of this chapter, we review integration by parts formulas for different operators. Chapter 2 presents a short introduction to the classical calculus of variations and review different variational problems, like the isoperimetric problems or problems with variable endpoints. In the end of this chapter, we introduce the theory of the fractional calculus of variations and some fractional variational problems with variable order.

In the second part, called Original Work and divided in two chapters, we systematize the new results that were published during my PhD project in peer reviewed international journals [103–107].

In Chapter 3, considering three types of fractional Caputo derivatives of variable order, we present new approximation formulas for those fractional derivatives and prove upper bound formulas for the errors.

In Chapter 4, we introduce the combined Caputo fractional derivative of variable order and corresponding higher-order operators. Some properties are also given. Then, we prove fractional Euler–Lagrange equations for several types of fractional variational problems of the calculus of variations, with or without constraints.

Part I

Synthesis

Chapter 1

Fractional calculus

In this chapter, a brief introduction to the theory of fractional calculus is presented. We start with a historical perspective of the theory, with a strong connection with the development of classical calculus (Section 1.1). Then, in Section 1.2, we review some definitions and properties about a few special functions that will be needed. We end with a review on fractional integrals and fractional derivatives of noninteger order and with some formulas of integration by parts, involving fractional operators (Section 1.3).

The content of this chapter can be found in some classical books on fractional calculus, for example [9, 48, 62, 76, 88].

1.1 Historical perspective

Fractional Calculus (FC) is considered as a branch of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. Therefore, FC is an extension of the integer-order calculus that considers integrals and derivatives of any real or complex order [48, 88], i.e., unify and generalize the notions of integer-order differentiation and n -fold integration.

FC was born in 1695 with a letter that L'Hôpital wrote to Leibniz, where the derivative of order $1/2$ is suggested [73]. After Leibniz had introduced in his publications the notation for the n th derivative of the function y :

$$\frac{d^n y}{dx^n},$$

L'Hôpital wrote a letter to Leibniz to ask him about the possibility of a derivative of integer

order to be extended in order to have a meaning when the order is a fraction: "What if n be $1/2$?" [86]. In his answer, dated on 30 September 1695, Leibniz replied that "This is an apparent paradox from which, one day, useful consequences will be drawn" and, today, we know it is truth. Then, Leibniz still wrote about derivatives of general order and in 1730, Euler investigated the result of the derivative when the order n is a fraction. But, only in 1819, with Lacroix, appeared the first definition of fractional derivative based on the expression for the n th derivative of the power function. Considering $y = x^m$, with m a positive integer, Lacroix developed the n th derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n,$$

and using the definition of Gamma function, for the generalized factorial, he got

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Lacroix also studied the following example, for $n = \frac{1}{2}$ and $m = 1$:

$$\frac{d^{1/2} x}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{\frac{1}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \quad (1.1)$$

Since then, many mathematicians, like Fourier, Abel, Riemann, Liouville, among others, contributed to the development of this subject. One of the first applications of fractional calculus appear in 1823 by Niels Abel, through the solution of an integral equation of the form

$$\int_0^t (t-\tau)^{-\alpha} x(\tau) d\tau = k,$$

used in the formulation of the Tautochrone problem [1, 86].

Years later, different forms of fractional operators have been introduced along time, like the Riemann–Liouville, the Grünwald-Letnikov, the Weyl, the Caputo, the Marchaud or the Hadamard fractional derivatives [32, 48, 73, 76]. The first approach is the Riemann–Liouville, which is based on iterating the classical integral operator n times and then considering the Cauchy's formula where $n!$ is replaced by the Gamma function and hence the fractional integral of noninteger order is defined. Then, using this operator, some of the fractional derivatives mentioned above are defined.

During three centuries, FC was developed but as a pure theoretical subject of mathematics. In recent times, FC had an increasing of importance due to its applications in

various fields, not only in mathematics, but also in physics, mechanics, engineering, chemistry, biology, finance, and others areas of science [44, 45, 53, 58, 72, 75, 95, 98]. In some of these applications, many real world phenomena are better described by noninteger order derivatives, if we compare with the usual integer-order calculus. In fact, fractional order derivatives have unique characteristics that may model certain dynamics more efficiently: first we can consider any real order for the derivatives, and thus we are not restricted to integer order derivatives only; secondly, they are nonlocal operators, in opposite to the usual derivatives, containing memory. With the memory property, one can take into account the past of the processes. Signal processing, modeling and control are some areas that have been the object of more intensive publishing in the last decades.

In most applications of the FC, the order of the derivative is assumed to be fixed along the process, that is, when determining what is the order $\alpha > 0$ such that the solution of the fractional differential equation $D^\alpha y(t) = f(t, y(t))$ better approaches the experimental data, we consider the order α to be a fixed constant. Of course, this may not be the best option, since trajectories are a dynamic process, and the order may vary. More interesting possibilities arise when one considers the order α of the fractional integrals and derivatives not constant during the process but to be a function $\alpha(t)$, depending on time. Then, we may seek what is the best function $\alpha(\cdot)$ such that the variable order fractional differential equation $D^{\alpha(t)}y(t) = f(t, y(t))$ better describes the model. This approach is very recent. One such fractional calculus of variable order was introduced in 1993 by Samko and Ross [89]. Afterwards, several mathematicians obtained important results about variable order fractional calculus, and some applications appeared, like in mechanics, in the modeling of linear and nonlinear viscoelasticity oscillators and in other phenomena where the order of the derivative varies with time. See, for instance, [11, 15, 27, 70, 82, 87, 94].

The most common fractional operators considered in the literature take into account the past of the process: they are usually called left fractional operators. But in some cases we may be also interested in the future of the process, and the computation of $\alpha(t)$ to be influenced by it. In that case, right fractional derivatives are then considered. Recently, in some works, the main goal is to develop a theory where both fractional operators are taken into account. For that, some combined fractional operators are introduced, like the symmetric fractional derivative, the Riesz fractional integral and derivative, the Riesz–Caputo fractional derivative and the combined Caputo fractional derivative that consists in a linear combination of the left and right fractional operators. For studies with fixed

fractional order, see [49, 61–63].

Due to the growing number of applications of fractional calculus in science and engineering, numerical approaches are being developed to provide tools for solving such classical problems. At present, there are already vast studies on numerical approximate formulas for such fractional calculus, [50, 52]. For example, for numerical modeling of time fractional diffusion equations, we refer the reader to [38].

1.2 Special functions

Before introducing the basic facts on fractional operators, we recall four types of functions that are important in Fractional Calculus: the *Gamma*, *Psi*, *Beta* and *Mittag-Leffler* functions. Some properties of these functions are also recalled.

Definition 1 (Gamma function). *The Euler Gamma function is an extension of the factorial function to real numbers, and it is defined by*

$$\Gamma(t) = \int_0^{\infty} \tau^{t-1} \exp(-\tau) d\tau, \quad t > 0.$$

For example, $\Gamma(1) = 1$, $\Gamma(2) = 1$ and $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. For positive integers n , we get $\Gamma(n) = (n - 1)!$. We mention that other definitions for the Gamma function exist, and it is possible to define it for complex numbers, except for the non-positive integers.

The Gamma function is considered the most important Eulerian function used in fractional calculus, because it appears in almost every fractional integral and derivative definitions. A basic but fundamental property of Γ , that we will use later, is obtained using integration by parts:

$$\Gamma(t + 1) = t\Gamma(t).$$

Definition 2 (Psi function). *The Psi function is the derivative of the logarithm of the Gamma function:*

$$\Psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}.$$

The follow function is used sometimes for convenience to replace a combination of Gamma functions. It is important in FC because it shares a form that is similar to the fractional derivative or integral of many functions, particularly power functions.

Definition 3 (Beta function). *The Beta function B is defined by*

$$B(t, u) = \int_0^1 s^{t-1}(1-s)^{u-1} ds, \quad t, u > 0.$$

This function satisfies an important property:

$$B(t, u) = \frac{\Gamma(t)\Gamma(u)}{\Gamma(t+u)}.$$

With this property, it is obvious that the Beta function is symmetric, i.e.,

$$B(t, u) = B(u, t).$$

The next function is a direct generalization of the exponential series and it was defined by the mathematician Mittag-Leffler in 1903 [76].

Definition 4 (Mittag-Leffler function). *Let $\alpha > 0$. The function E_α defined by*

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad t \in \mathbb{R},$$

is called the one parameter Mittag-Leffler function.

For $\alpha = 1$, this function coincides with the series expansion of e^t , i.e.,

$$E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

While linear ordinary differential equations present in general the exponential function as a solution, the Mittag-Leffler function occurs naturally in the solution of fractional order differential equations [48]. For this reason, in recent times, the Mittag-Leffler function has become an important function in the theory of the fractional calculus and its applications.

It is also common to represent the Mittag-Leffler function in two arguments. This generalization of Mittag-Leffler function was studied by Wiman (1905) [58].

Definition 5 (Generalized Mittag-Leffler function). *The two-parameter function of the Mittag-Leffler type with parameters $\alpha, \beta > 0$ is defined by*

$$E_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad t \in \mathbb{R}.$$

If $\beta = 1$, this function coincides with the classical Mittag-Leffler function, i.e., $E_{\alpha, 1}(t) = E_\alpha(t)$.

1.3 Fractional integrals and derivatives

In this section, we recall some definitions of fractional integral and fractional differential operators, that includes all we use throughout this thesis. In the end, we present some integration by parts formulas because they have a crucial role in deriving Euler–Lagrange equations.

1.3.1 Classical operators

As it was seen in Section 1.1, there are more than one way to generalize integer-order operations to the non-integer case. Here, we present several definitions and properties about fractional operators, omitting some details about the conditions that ensure the existence of such fractional operators.

In general, the fractional derivatives are defined using fractional integrals. We present only two fractional integrals operators, but there are several known forms of the fractional integrals.

Let $x : [a, b] \rightarrow \mathbb{R}$ be an integrable function and $\alpha > 0$ a real number.

Starting with Cauchy’s formula for a n -fold iterated integral, given by

$$\begin{aligned} {}_a I_t^n x(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \cdots \int_a^{\tau_{n-1}} x(\tau_n) d\tau_n \\ &= \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} x(\tau) d\tau, \end{aligned} \tag{1.2}$$

where $n \in \mathbb{N}$, Liouville and Riemann defined fractional integration, generalizing equation (1.2) to noninteger values of n and using the definition of Gamma function Γ . With this, we introduce two important concepts: the left and the right Riemann–Liouville fractional integrals.

Definition 6 (Left and right Riemann–Liouville fractional integrals). *We define the left and right Riemann–Liouville fractional integrals of order α , respectively, by*

$${}_a I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad t > a$$

and

$${}_t I_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau) d\tau, \quad t < b.$$

The constants a and b determine, respectively, the lower and upper boundary of the integral domain. Additionally, if x is a continuous function, as $\alpha \rightarrow 0$, ${}_aI_t^\alpha = {}_tI_b^\alpha = I$, with I the identity operator, i.e., ${}_aI_t^\alpha x(t) = {}_tI_b^\alpha x(t) = x(t)$.

We present the second fractional integral operator, introduced by J. Hadamard in 1892 [48].

Definition 7 (Left and right Hadamard fractional integrals). *We define the left and right Hadamard fractional integrals of order α , respectively, by*

$${}_aJ_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau, \quad t > a$$

and

$${}_tJ_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln \frac{\tau}{t} \right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau, \quad t < b.$$

The three most frequently used definitions for fractional derivatives are: the Grünwald-Letnikov, the Riemann–Liouville and the Caputo fractional derivatives [73, 76]. Other definitions were introduced by others mathematicians, as for instance Weyl, Fourier, Cauchy, Abel.

Let $x \in AC([a, b], \mathbb{R})$, be an absolutely continuous functions on the interval $[a, b]$, and α a positive real number. Using Definition 6 of Riemann–Liouville fractional integrals, we define the left and the right Riemann–Liouville and Caputo derivatives as follows.

Definition 8 (Left and right Riemann–Liouville fractional derivatives). *We define the left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$, respectively, by*

$$\begin{aligned} {}_aD_t^\alpha x(t) &= \left(\frac{d}{dt} \right)^n {}_aI_t^{n-\alpha} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t (t-\tau)^{n-\alpha-1} x(\tau) d\tau, \quad t > a \end{aligned}$$

and

$$\begin{aligned} {}_tD_b^\alpha x(t) &= \left(-\frac{d}{dt} \right)^n {}_tI_b^{n-\alpha} x(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_t^b (\tau-t)^{n-\alpha-1} x(\tau) d\tau, \quad t < b, \end{aligned}$$

where $n = [\alpha] + 1$.

The following definition was introduced by Caputo in 1967 [22]. The Caputo fractional derivatives, in general, are more applicable and interesting in fields like physics and engineering.

Definition 9 (Left and right Caputo fractional derivatives). *We define the left and right Caputo fractional derivatives of order α , respectively, by*

$$\begin{aligned} {}^C D_t^\alpha x(t) &= {}_a I_t^{n-\alpha} \left(\frac{d}{dt} \right)^n x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \quad t > a \end{aligned}$$

and

$$\begin{aligned} {}^C D_b^\alpha x(t) &= {}_t I_b^{n-\alpha} \left(-\frac{d}{dt} \right)^n x(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \quad t < b, \end{aligned}$$

where $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$.

Obviously, the above defined operators are linear. From these definitions, it is clear that the Caputo fractional derivative of a constant C is zero, which it is false when we consider the Riemann–Liouville fractional derivative: if $x(t) = C$, with C a constant, then we get

$${}^C D_t^\alpha x(t) = {}^C D_b^\alpha x(t) = 0$$

and

$${}_a D_t^\alpha x(t) = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad {}_t D_b^\alpha x(t) = \frac{C(b-t)^{-\alpha}}{\Gamma(1-\alpha)}.$$

For this reason, in some applications, Caputo fractional derivatives seem to be more natural than the Riemann–Liouville fractional derivatives.

Remark 1. [62] *If α goes to n^- , with $n \in \mathbb{N}$, then the fractional operators introduced above coincide with the standard derivatives:*

$${}_a D_t^\alpha = {}^C D_t^\alpha = \left(\frac{d}{dt} \right)^n$$

and

$${}_t D_b^\alpha = {}^C D_b^\alpha = \left(-\frac{d}{dt} \right)^n.$$

The Riemann–Liouville fractional integral and differential operators of order $\alpha > 0$ of power functions return power functions, as we can see below.

Lemma 2 (c.f. Property 2.1 [48]). *Let x be the power function $x(t) = (t - a)^\gamma$. Then, we have*

$${}_a I_t^\alpha x(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} (t - a)^{\gamma + \alpha}, \quad \gamma > -1$$

and

$${}_a D_t^\alpha x(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} (t - a)^{\gamma - \alpha}, \quad \gamma > -1.$$

In particular, if we consider $\gamma = 1$, $a = 0$ and $\alpha = 1/2$, then the left Riemann–Liouville fractional derivative of $x(t) = t$ is $\frac{2\sqrt{t}}{\sqrt{\pi}}$, the same result (1.1) as Lacroix obtained in 1819.

Grünwald (1867) and Letnikov (1868) return to the original sources and started the formulation by the fundamental definition of a derivative as a limit:

$$x^{(1)}(t) = \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}$$

and considering the iteration at the n th order derivative formulas:

$$x^{(n)}(t) = \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} x(t - kh), \quad n \in \mathbb{N}, \quad (1.3)$$

where $\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$, with $n, k \in \mathbb{N}$, the usual notation for the binomial coefficients. The Grünwald-Letnikov definition of fractional derivative consists in a generalization of (1.3) to derivatives of arbitrary order α [76].

Definition 10 (Grünwald–Letnikov fractional derivative). *The α th order Grünwald-Letnikov fractional derivative of function x is given by*

$${}_a^{GL} D_t^\alpha x(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} x(t - k + 1),$$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha-k+1)}$ and $nh = t - a$.

Lemma 3. [76] *Let x be the power function $x(t) = (t - a)^\gamma$, where γ is a real number. Then, for $0 < \alpha < 1$ and $\gamma > 0$, we have*

$${}_a^{GL} D_t^\alpha x(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} (t - a)^{\gamma - \alpha}.$$

Definition 11 (Left and right Hadamard fractional derivatives). *We define the left and right Hadamard fractional derivatives of order α , respectively, by*

$${}_a D_t^\alpha x(t) = \left(t \frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau$$

and

$${}_t D_b^\alpha x(t) = \left(-t \frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\ln \frac{\tau}{t}\right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau,$$

for all $t \in (a, b)$, where $n = [\alpha] + 1$.

Observe that, for all types of derivatives operators, if variable t is the time-variable, the left fractional derivative of $x(t)$ is interpreted as a past state of the process, while the right fractional derivative of $x(t)$ is interpreted as a future state of the process.

1.3.2 Some properties of the Caputo derivative

For $\alpha > 0$ and $x \in AC([a, b]; \mathbb{R})$, the Riemann–Liouville and Caputo derivatives are related by the following formulas (see [48]):

$${}_a^C D_t^\alpha x(t) = {}_a D_t^\alpha \left[x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)(t-a)^k}{k!} \right] \quad (1.4)$$

and

$${}_t^C D_b^\alpha x(t) = {}_t D_b^\alpha \left[x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)(b-t)^k}{k!} \right], \quad (1.5)$$

where $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$.

In particular, when $\alpha \in (0, 1)$, the relations (1.4) and (1.5) take the form:

$$\begin{aligned} {}_a^C D_t^\alpha x(t) &= {}_a D_t^\alpha (x(t) - x(a)) \\ &= {}_a D_t^\alpha x(t) - \frac{x(a)}{\Gamma(1-\alpha)} (t-a)^{-\alpha} \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} {}_t^C D_b^\alpha x(t) &= {}_t D_b^\alpha (x(t) - x(b)) \\ &= {}_t D_b^\alpha x(t) - \frac{x(b)}{\Gamma(1-\alpha)} (b-t)^{-\alpha}. \end{aligned} \quad (1.7)$$

It follows from (1.6) and (1.7) that the left Riemann–Liouville equals to the left Caputo fractional derivative in the case $x(a) = 0$ and the analogue holds for the right derivatives under the assumption $x(b) = 0$.

In Theorem 4, we see that the Caputo fractional derivatives provides a left inverse operator to the Riemann–Liouville fractional integration.

Theorem 4 (c.f. Lemma 2.21 [48]). *Let $\alpha > 0$ and let $x \in C([a, b]; \mathbb{R}^n)$. For Caputo fractional operators, the next rules hold*

$${}_a^C D_t^\alpha {}_a I_t^\alpha x(t) = x(t)$$

and

$${}_t^C D_b^\alpha {}_t I_b^\alpha x(t) = x(t).$$

The next statement characterizes the composition of the Riemann–Liouville fractional integration operators with the Caputo fractional differentiation operators.

Theorem 5 (c.f. Lemma 2.22 [48]). *Let $\alpha > 0$. If $x \in AC^n([a, b]; \mathbb{R})$, then*

$${}_a I_t^\alpha {}_a^C D_t^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!} (t-a)^k$$

and

$${}_t I_b^\alpha {}_t^C D_b^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{(-1)^k x^{(k)}(b)}{k!} (b-t)^k,$$

with $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$.

In particular, when $\alpha \in (0, 1)$, then

$${}_a I_t^\alpha {}_a^C D_t^\alpha x(t) = x(t) - x(a) \quad \text{and} \quad {}_t I_b^\alpha {}_t^C D_b^\alpha x(t) = x(t) - x(b).$$

As the Riemann–Liouville fractional derivative, the Caputo fractional derivative of power functions yield power functions of the same form.

Lemma 6 (c.f. Property 2.16 [48]). *Let $\alpha > 0$. Then, the following relations hold:*

$${}_a^C D_t^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}, \quad \gamma > n-1$$

and

$${}_t^C D_b^\alpha (b-t)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (b-t)^{\gamma-\alpha}, \quad \gamma > n-1,$$

with $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$.

1.3.3 Combined Caputo derivative

In this section, we introduce a special operator for our work, the combined Caputo fractional derivative. We extend the notion of the Caputo fractional derivative to the fractional derivative ${}^C D^{\alpha, \beta}$, that involves the left and the right Caputo fractional derivative, i.e., it combines the past and the future of the process into one single operator.

This operator was introduced by Malinowska and Torres in 2010 [60], motivated by the idea of the symmetric fractional derivative introduced in 2001, by Klimek [49]:

Definition 12 (Symmetric fractional derivative). *Considering the left and right Riemann–Liouville derivatives, the symmetric fractional derivative is given by*

$$Dx(t) = \frac{1}{2} [{}_a D_t^\alpha x(t) + {}_t D_b^\alpha x(t)]. \quad (1.8)$$

Other combined operators were studied. For example, we have the Riesz and the Riesz–Caputo operators [62].

Definition 13 (Riesz fractional operators). *Let $x : [a, b] \rightarrow \mathbb{R}$ be a function of class C^1 and $\alpha \in (0, 1)$. For $t \in [a, b]$, the Riesz fractional integral of order α , is defined by*

$$\begin{aligned} {}^R I_b^\alpha x(t) &= \frac{1}{2\Gamma(\alpha)} \int_a^b |t - \tau|^{\alpha-1} x(\tau) d\tau \\ &= \frac{1}{2} [{}_a I_t^\alpha x(t) + {}_t I_b^\alpha x(t)], \end{aligned}$$

and the Riesz fractional derivative of order α , is defined by

$$\begin{aligned} {}^R D_b^\alpha x(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^b |t - \tau|^{-\alpha} x(\tau) d\tau \\ &= \frac{1}{2} [{}_a D_t^\alpha x(t) - {}_t D_b^\alpha x(t)]. \end{aligned}$$

Definition 14 (Riesz–Caputo fractional derivative). *Considering the left and right Caputo derivatives, Riesz–Caputo fractional derivative is given by*

$$\begin{aligned} {}^{RC} D_b^\alpha x(t) &= \frac{1}{\Gamma(1-\alpha)} \int_a^b |t - \tau|^{-\alpha} \frac{d}{d\tau} x(\tau) d\tau \\ &= \frac{1}{2} [{}_a^C D_t^\alpha x(t) - {}_t^C D_b^\alpha x(t)]. \end{aligned} \quad (1.9)$$

Remark 7. [62] *If α goes to 1, then the fractional derivatives introduced above coincide with the standard derivative:*

$${}^R D_b^\alpha = {}^{RC} D_b^\alpha = \frac{d}{dt}.$$

Similarly to the last operator (1.9), the combined Caputo derivative is a convex combination of the left and the right Caputo fractional derivatives. But in this operator, we consider also other coefficients besides $1/2$ for the convex combination and the orders of two sides of Caputo fractional derivatives can be different (α and β). Therefore, the combined Caputo derivative is a convex combination of the left Caputo fractional derivative of order α and the right Caputo fractional derivative of order β .

Definition 15 (Combined Caputo fractional derivative). [60] Let $\alpha, \beta \in (0, 1)$ and $\gamma \in [0, 1]$. The combined Caputo fractional derivative operator ${}^C D_\gamma^{\alpha, \beta}$ is defined by

$${}^C D_\gamma^{\alpha, \beta} = \gamma {}^C D_a^\alpha + (1 - \gamma) {}^C D_b^\beta, \quad (1.10)$$

which acts on $x \in AC([a, b]; \mathbb{R})$ in the following way:

$${}^C D_\gamma^{\alpha, \beta} x(t) = \gamma {}^C D_a^\alpha x(t) + (1 - \gamma) {}^C D_b^\beta x(t).$$

The operator (1.10) is obviously linear. Observe still that

$${}^C D_0^{\alpha, \beta} = {}^C D_b^\beta \quad \text{and} \quad {}^C D_1^{\alpha, \beta} = {}^C D_a^\alpha.$$

The symmetric fractional derivative and the Riesz fractional derivative are useful tools to describe some nonconservative models. But those types of differentiation do not seem suitable for all kinds of variational problems because they are based on the Riemann–Liouville fractional derivatives and therefore the possibility that admissible trajectories x have continuous fractional derivatives imply that $x(a) = x(b) = 0$ [89]. For more details about the combined Caputo fractional derivative, see [61, 63, 64, 68].

1.3.4 Variable order operators

Very useful physical applications have given birth to the variable order fractional calculus, for example in modeling mechanical behaviors [38, 100]. Nowadays, variable order fractional calculus is particularly recognized as a useful and promising approach in the modelling of diffusion processes, in order to characterize time-dependent or concentration-dependent anomalous diffusion, or diffusion processes in inhomogeneous porous media [99].

Now, we present the fundamental notions of the fractional calculus of variable order [59]. We consider the fractional order of the derivative and of the integral to be a continuous

function of two variables, $\alpha(\cdot, \cdot)$ with domain $[a, b]^2$, taking values on the open interval $(0, 1)$. Let $x : [a, b] \rightarrow \mathbb{R}$ be a function.

First, we recall the generalization of fractional integrals for a variable order $\alpha(\cdot, \cdot)$.

Definition 16 (Riemann–Liouville fractional integrals of variable order). *The left and right Riemann–Liouville fractional integrals of order $\alpha(\cdot, \cdot)$ are defined by*

$${}_a I_t^{\alpha(\cdot, \cdot)} x(t) = \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau) - 1} x(\tau) d\tau, \quad t > a$$

and

$${}_t I_b^{\alpha(\cdot, \cdot)} x(t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau, t))} (\tau - t)^{\alpha(\tau, t) - 1} x(\tau) d\tau, \quad t < b,$$

respectively.

We remark that, in contrast to the fixed fractional order case, variable order fractional integrals are not the inverse operation of the variable order fractional derivatives.

For fractional derivatives, we consider two types: the Riemann–Liouville and the Caputo fractional derivatives.

Definition 17 (Riemann–Liouville fractional derivatives of variable order). *The left and right Riemann–Liouville fractional derivatives of order $\alpha(\cdot, \cdot)$ are defined by*

$$\begin{aligned} {}_a D_t^{\alpha(\cdot, \cdot)} x(t) &= \frac{d}{dt} {}_a I_t^{\alpha(\cdot, \cdot)} x(t) \\ &= \frac{d}{dt} \int_a^t \frac{1}{\Gamma(1 - \alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)} x(\tau) d\tau, \quad t > a \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} {}_t D_b^{\alpha(\cdot, \cdot)} x(t) &= - \frac{d}{dt} {}_t I_b^{\alpha(\cdot, \cdot)} x(t) \\ &= \frac{d}{dt} \int_t^b \frac{-1}{\Gamma(1 - \alpha(\tau, t))} (\tau - t)^{-\alpha(\tau, t)} x(\tau) d\tau, \quad t < b, \end{aligned} \tag{1.12}$$

respectively.

Lemma 8 gives a Riemann–Liouville variable order fractional integral and fractional derivative for the power function $(t - a)^\gamma$, where we assume that the fractional order depends only on the first variable: $\alpha(t, \tau) := \bar{\alpha}(t)$, where $\bar{\alpha} : [a, b] \rightarrow (0, 1)$ is a given function.

Lemma 8. [89] Let x be the power function $x(t) = (t - a)^\gamma$. Then, for $\gamma > -1$, we have

$${}_a I_t^{\bar{\alpha}(t)} x(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \bar{\alpha}(t) + 1)} (t - a)^{\gamma + \bar{\alpha}(t)}$$

and

$$\begin{aligned} {}_a D_t^{\bar{\alpha}(t)} x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\bar{\alpha}(t)+1)} (t-a)^{\gamma-\bar{\alpha}(t)} \\ &\quad - \bar{\alpha}^{(1)}(t) \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\bar{\alpha}(t)+2)} (t-a)^{\gamma-\bar{\alpha}(t)+1} \\ &\quad \times [\ln(t-a) - \Psi(\gamma - \bar{\alpha}(t) + 2) + \Psi(1 - \bar{\alpha}(t))]. \end{aligned}$$

Definition 18 (Caputo fractional derivatives of variable order). The left and right Caputo fractional derivatives of order $\alpha(\cdot, \cdot)$ are defined by

$${}_a^C D_t^{\alpha(\cdot, \cdot)} x(t) = \int_a^t \frac{1}{\Gamma(1 - \alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)} x^{(1)}(\tau) d\tau \quad (1.13)$$

and

$${}_t^C D_b^{\alpha(\cdot, \cdot)} x(t) = \int_t^b \frac{-1}{\Gamma(1 - \alpha(\tau, t))} (\tau - t)^{-\alpha(\tau, t)} x^{(1)}(\tau) d\tau, \quad (1.14)$$

respectively.

Of course, the fractional derivatives just defined are linear operators.

1.3.5 Generalized fractional operators

In this section, we present three definitions of one-dimensional generalized fractional operators that depend on a general kernel, studied by Odziejewicz, Malinowska and Torres (see e.g. [59, 67, 72], although Agrawal already, in 2010, had introduced this generalized fractional operators [5]).

Let $\Delta := \{(t, \tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\}$.

Definition 19 (Generalized fractional integrals of Riemann–Liouville type). Let k be a function defined almost everywhere on Δ with values in \mathbb{R} . For all $x : [a, b] \rightarrow \mathbb{R}$, the generalized fractional integral operator K_P is defined by

$$K_P[x](t) = \lambda \int_a^t k(t, \tau) x(\tau) d\tau + \mu \int_t^b k(\tau, t) x(\tau) d\tau, \quad (1.15)$$

with $P = \langle a, t, b, \lambda, \mu \rangle$, where λ and μ are real numbers.

In particular, if we choose special cases for the kernel, we can obtain standard fractional operators or variable order fractional operators.

Remark 9. For special chosen kernels k and parameters P , the operator K_P can be reduced to the classical or variable order Riemann–Liouville fractional integrals:

- Let $K^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha)}(t - \tau)^{\alpha-1}$ and $0 < \alpha < 1$. If $P = \langle a, t, b, 1, 0 \rangle$, then

$$K_P[x](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} x(\tau) d\tau =: {}_a I_t^\alpha[x](t)$$

is the left Riemann–Liouville fractional integral of order α ; if $P = \langle a, t, b, 0, 1 \rangle$, then

$$K_P[x](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} x(\tau) d\tau =: {}_t I_b^\alpha[x](t)$$

is the right Riemann–Liouville fractional integral of order α .

- If $K^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha(t, \tau))}(t - \tau)^{\alpha(t, \tau)-1}$ and $P = \langle a, t, b, 1, 0 \rangle$, then

$$K_P[x](t) = \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} x(\tau) d\tau =: {}_a I_t^{\alpha(\cdot, \cdot)}[x](t)$$

is the left Riemann–Liouville fractional integral of variable order $\alpha(\cdot, \cdot)$; if $P = \langle a, t, b, 0, 1 \rangle$, then

$$K_P[x](t) = \int_t^b \frac{1}{\Gamma(\alpha(t, \tau))} (\tau - t)^{\alpha(t, \tau)-1} x(\tau) d\tau =: {}_t I_b^{\alpha(\cdot, \cdot)}[x](t)$$

is the right Riemann–Liouville fractional integral of variable order $\alpha(\cdot, \cdot)$.

Some other fractional operators can be obtained with the generalized fractional integrals, for example, Hadamard, Riesz, Katugampola fractional operators [5, 59].

The following two new operators, the generalized fractional Riemann–Liouville and Caputo derivatives, are defined as a composition of classical derivatives and generalized fractional integrals.

Definition 20 (Generalized fractional derivative of Riemann–Liouville type). *The generalized fractional derivative of Riemann–Liouville type, denoted by A_P , is defined by*

$$A_P = \frac{d}{dt} \circ K_P.$$

Definition 21 (Generalized fractional derivative of Caputo type). *The generalized fractional derivative of Caputo type, denoted by B_P , is defined by*

$$B_P = K_P \circ \frac{d}{dt}.$$

Considering $K^\alpha(t, \tau) = \frac{1}{\Gamma(1-\alpha)}(t - \tau)^{-\alpha}$ with $0 < \alpha < 1$ and appropriate sets P , this two general kernel operators A_P and B_P can be reduced to the standard Riemann–Liouville and Caputo fractional derivatives, respectively [59].

1.3.6 Integration by parts

In this section, we summarize formulas of integration by parts because they are important results to find necessary optimality conditions when dealing with variational problems.

First, we present the rule of fractional integration by parts for Riemann–Liouville fractional integral.

Theorem 10 (Integration by parts for fractional integrals). *Let $0 < \alpha < 1$, $p \geq 1$, $q \geq 1$ and $1/p + 1/q \leq 1 + \alpha$. If $y \in L_p([a, b]; \mathbb{R})$ and $x \in L_q([a, b]; \mathbb{R})$, then the following formula for integration by parts hold:*

$$\int_a^b y(t) {}_a I_t^\alpha x(t) dt = \int_a^b x(t) {}_t I_b^\alpha y(t) dt.$$

For Caputo fractional derivatives, the integration by parts formulas are presented below [7].

Theorem 11 (Integration by parts for Caputo fractional derivatives). *Let $0 < \alpha < 1$. The following relations hold:*

$$\int_a^b y(t) {}_a^C D_t^\alpha x(t) dt = \int_a^b x(t) {}_t D_b^\alpha y(t) dt + [x(t) {}_t I_b^{1-\alpha} y(t)]_{t=a}^{t=b} \quad (1.16)$$

and

$$\int_a^b y(t) {}_t^C D_b^\alpha x(t) dt = \int_a^b x(t) {}_a D_t^\alpha y(t) dt - [x(t) {}_a I_t^{1-\alpha} y(t)]_{t=a}^{t=b}. \quad (1.17)$$

When $\alpha \rightarrow 1$, we get ${}_a^C D_t^\alpha = {}_a D_t^\alpha = \frac{d}{dt}$, ${}_t^C D_b^\alpha = {}_t D_b^\alpha = -\frac{d}{dt}$, ${}_a I_t^\alpha = {}_t I_b^\alpha = I$, and formulas (1.16) and (1.17) give the classical formulas of integration by parts.

Then, we introduce the integration by parts formulas for variable order fractional integrals.

Theorem 12 (Integration by parts for variable order fractional integrals [71]). *Let $\frac{1}{n} < \alpha(t, \tau) < 1$ for all $t, \tau \in [a, b]$ and a certain $n \in \mathbb{N}$ greater or equal than two, and $x, y \in C([a, b]; \mathbb{R})$. Then the following formula for integration by parts hold:*

$$\int_a^b y(t) {}_a I_t^{\alpha(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_t I_b^{\alpha(\cdot, \cdot)} y(t) dt.$$

In the following theorem, we present the formula involving the Caputo fractional derivative of variable order. It was proved in [71] and is a generalization of the standard fractional formula of integration by parts for a constant α .

Theorem 13 (Integration by parts for variable order Caputo fractional derivatives [71]).

Let $0 < \alpha(t, \tau) < 1 - \frac{1}{n}$ for all $t, \tau \in [a, b]$ and a certain $n \in \mathbb{N}$ greater or equal than two.

If $x, y \in C^1([a, b]; \mathbb{R})$, then the fractional integration by parts formulas

$$\int_a^b y(t) {}_a^C D_t^{\alpha(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_t D_b^{\alpha(\cdot, \cdot)} y(t) dt + \left[x(t) {}_t I_b^{1-\alpha(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}$$

and

$$\int_a^b y(t) {}_t^C D_b^{\alpha(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_a D_t^{\alpha(\cdot, \cdot)} y(t) dt - \left[x(t) {}_a I_t^{1-\alpha(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}$$

hold.

This last theorem have an important role in this work to the proof of the generalized Euler–Lagrange equations.

In the end of this chapter, we present integration by parts formulas for generalized fractional operators [59]. For that, we need the following definition:

Definition 22 (Dual parameter set). Let $P = \langle a, t, b, \lambda, \mu \rangle$. We denote by P^* the parameter set $P^* = \langle a, t, b, \mu, \lambda \rangle$. The parameter P^* is called the dual of P .

Let $1 < p < \infty$ and q be the adjoint of p , that is $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 14 (c.f. Theorem 4.6 of [59]). Let $k \in L_q(\Delta; \mathbb{R})$. Then the operator K_{P^*} is a linear bounded operator from $L_p([a, b]; \mathbb{R})$ to $L_q([a, b]; \mathbb{R})$. Moreover, the following integration by parts formula holds:

$$\int_a^b x(t) \cdot K_P[y](t) dt = \int_a^b y(t) \cdot K_{P^*}[x](t) dt$$

for all $x, y \in L_p([a, b]; \mathbb{R})$.

Chapter 2

The calculus of variations

As part of this thesis is devoted to the fractional calculus of variations, in this chapter we introduce the basic concepts about the classical calculus of variations and the fractional calculus of variations. The study of fractional problems of the calculus of variations and respective Euler–Lagrange type equations is a subject of current strong research.

In Section 2.1, we introduce some concepts and important results from the classical theory. Afterwards, in Section 2.2, we start with a brief historical introduction to the non–integer calculus of variations and then we present recent results on the fractional calculus of variations.

For more information about this subject, we refer the reader to the books [9, 59, 62, 112].

2.1 The classical calculus of variations

The calculus of variations is a field of mathematical analysis that concerns with finding extrema (maxima or minima) for functionals, i.e., concerns with the problem of finding a function for which the value of a certain integral is either the largest or the smallest possible.

In this context, a functional is a mapping from a set of functions to the real numbers, i.e., it receives a function and produces a real number. Let $D \subseteq C^2([a, b]; \mathbb{R})$ be a linear space endowed with a norm $\|\cdot\|$. The cost functional $\mathcal{J} : D \rightarrow \mathbb{R}$ is generally of the form

$$\mathcal{J}(x) = \int_a^b L(t, x(t), x'(t)) dt, \quad (2.1)$$

where $t \in [a, b]$ is the independent variable, usually called time, and $x(t) \in \mathbb{R}$ is a function. The integrand $L : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, that depends on the function x , its derivative x' and the independent variable t , is a real-valued function, called the Lagrangian.

The roots of the calculus of variations appear in works of Greek thinkers, such as Queen Dido or Aristotle in the late of the 1st century BC. During the 17th century, some physicists and mathematicians (Galileo, Fermat, Newton, ...) investigated some variational problems, but in general they did not use variational methods to solve them. The development of the calculus of variations began with a problem posed by Johann Bernoulli in 1696, called the brachistochrone problem: given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in minimal time. The curve that solves the problem is called the *brachistochrone*. This problem caught the attention of some mathematicians including Jakob Bernoulli, Leibniz, L'Hôpital and Newton, which presented also a solution for the brachistochrone problem. Integer variational calculus is still, nowadays, a relevant area of research. It plays a significant role in many areas of science, physics, engineering, economics, and applied mathematics.

The classical variational problem, considered by Leonhard Euler, is stated as follows. Among all functions $x \in C^2([a, b]; \mathbb{R})$ with $a, b \in \mathbb{R}$, find the ones that minimize (or maximize) the functional $\mathcal{J} : D \rightarrow \mathbb{R}$, with

$$\mathcal{J}(x) = \int_a^b L(t, x(t), x'(t)) dt \quad (2.2)$$

subject to the boundary conditions

$$x(a) = x_a, \quad x(b) = x_b, \quad (2.3)$$

with x_a, x_b fixed reals and the Lagrangian L satisfying some smoothness properties. Usually, we say that a function is "sufficiently smooth" for a particular development if all required actions (integration, differentiation, ...) are possible.

Definition 23 (Admissible trajectory). *A trajectory $x \in C^2([a, b]; \mathbb{R})$ is said to be an admissible trajectory if it satisfies all the constraints of the problem along the interval $[a, b]$. The set of admissible trajectories is denoted by D .*

To discuss maxima and minima of functionals, we need to introduce the following definition.

Definition 24 (Local extremizer of a functional). *We say that $x^* \in D$ is a local extremizer to the functional $\mathcal{J} : D \rightarrow \mathbb{R}$ if there exists some real $\epsilon > 0$, such that*

$$\forall x \in D : \quad \|x^* - x\| < \epsilon \Rightarrow \mathcal{J}(x^*) - \mathcal{J}(x) \leq 0 \vee \mathcal{J}(x^*) - \mathcal{J}(x) \geq 0.$$

In this context, as we are dealing with functionals defined on functions, we need to clarify the term of directional derivatives, here called variations. The concept of variation of a functional is central to obtain the solution of variational problems.

Definition 25 (First variation). *Let \mathcal{J} be defined on D . The first variation of a functional \mathcal{J} at $x \in D$ in the direction $h \in D$ is defined by*

$$\delta\mathcal{J}(x, h) = \lim_{\epsilon \rightarrow 0} \left[\frac{\mathcal{J}(x + \epsilon h) - \mathcal{J}(x)}{\epsilon} = \frac{d}{d\epsilon} \mathcal{J}(x + \epsilon h) \right]_{\epsilon=0},$$

where x and h are functions and ϵ is a scalar, whenever the limit exists.

Definition 26 (Admissible variation). *A direction $h \in D$, $h \neq 0$, is said to be an admissible variation for \mathcal{J} at $y \in D$ if*

1. $\delta\mathcal{J}(x, h)$ exists;
2. $x + \epsilon h \in D$ for all sufficiently small ϵ .

With the condition that $\mathcal{J}(x)$ be a local extremum and the definition of variation, we have the following result that offers a necessary optimality condition for problems of calculus of variations [112].

Theorem 15. *Let \mathcal{J} be a functional defined on D . If x^* minimizes (or maximizes) the functional \mathcal{J} over all functions $x : [a, b] \rightarrow \mathbb{R}$ satisfying boundary conditions (2.3), then*

$$\delta\mathcal{J}(x^*, h) = 0$$

for all admissible variations h at x^* .

2.1.1 Euler–Lagrange equations

Although the calculus of variations was born with Johann’s problem, it was with the work of Euler (1742) and Lagrange (1755) that the systematic theory was developed. The common procedure to address such variational problems consists in solving a differential

equation, called the Euler–Lagrange equation, which every minimizer/maximizer of the functional must satisfy.

In Lemma 16, we review an important result to transform the necessary condition of extremum in a differential equation, free of integration with an arbitrary function.

Lemma 16 (Fundamental lemma of the calculus of variations). *Let x be continuous in $[a, b]$ and let h be an arbitrary function on $[a, b]$ such that it is continuous and $h(a) = h(b) = 0$. If*

$$\int_a^b x(t)h(t)dt = 0$$

for all such h , then $x(t) = 0$ for all $t \in [a, b]$.

For the sequel, we denote by $\partial_i z$, $i \in \{1, 2, \dots, M\}$, with $M \in \mathbb{N}$, the partial derivative of a function $z : \mathbb{R}^M \rightarrow \mathbb{R}$ with respect to its i th argument. Now we can formulate the necessary optimality condition for the classical variational problem.

Theorem 17 (The Euler–Lagrange equation for the classical variational problem). *If x is an extremizing of the functional (2.2) on D , subject to (2.3), then x satisfies*

$$\partial_2 L(t, x(t), x'(t)) - \frac{d}{dt} \partial_3 L(t, x(t), x'(t)) = 0 \quad (2.4)$$

for all $t \in [a, b]$.

To solve this second order differential equation, the two given boundary conditions (2.3) provide sufficient information to determine the two arbitrary constants.

Definition 27 (Extremal). *A curve x that is a solution of the Euler–Lagrange differential equation will be called an extremal of \mathcal{J} .*

2.1.2 Problems with variable endpoints

In the basic variational problem considered previously, the functional \mathcal{J} to minimize (or maximize) is subject to given boundary conditions of the form

$$x(a) = x_a, \quad x(b) = x_b,$$

where $x_a, x_b \in \mathbb{R}$ are fixed. It means that the solution of the problem, x , needs to pass through the prescribed points. This variational problem is called a fixed endpoints variational problem. The Euler–Lagrange equation (2.4) is normally a second-order differential

equation containing two arbitrary constants, so with two given boundary conditions provided, they are sufficient to determine the two constants.

However, in some areas, like physics and geometry, the variational problems do not impose the appropriate number of boundary conditions. In these cases, when one or both boundary conditions are missing, that is, when the set of admissible functions may take any value at one or both of the boundaries, then one or two auxiliary conditions, known as the natural boundary conditions or transversality conditions, need to be obtained in order to solve the equation [112]:

$$\left[\frac{\partial L(t, x(t), x'(t))}{\partial x'} \right]_{t=a} = 0 \quad \text{and/or} \quad \left[\frac{\partial L(t, x(t), x'(t))}{\partial x'} \right]_{t=b} = 0 \quad (2.5)$$

There are different types of variational problems with variable endpoints:

- Free terminal point—one boundary condition at the initial point ($x(a) = x_a$). The terminal point is free;
- Free initial point—one boundary condition at the terminal point ($x(b) = x_b$). The initial point is free;
- Free endpoints—both endpoints are free;
- Variable endpoints—the initial point a or/and the endpoint b is variable on a certain set, for example, on a prescribed curve.

In the last situation, one generalization of the variational problem is finding an optimal curve x and the endpoint of the variational integral T , with $T \in [a, b]$. This problem is known in the literature as a free-time problem [26]. An example is the following variable terminal point. Let D denote the subset $C^2([a, b]; \mathbb{R}) \times [a, b]$ endowed with a norm $\|(\cdot, \cdot)\|$. Find the local minimizers of the functional $\mathcal{J} : D \rightarrow \mathbb{R}$, with

$$\mathcal{J}(x, T) = \int_a^T L(t, x(t), x'(t)) dt, \quad (2.6)$$

over all $(x, T) \in D$ satisfying the boundary condition $x(a) = x_a$, with $x_a \in \mathbb{R}$ fixed. The terminal time T and the terminal state $x(T)$ are free.

Definition 28 (Local extremizer of the functional (2.6)). *We say that $(x^*, T^*) \in D$ is a local extremizer (minimizer or maximizer) to the functional $\mathcal{J} : D \rightarrow \mathbb{R}$ if there exists some $\epsilon > 0$ such that*

$$\forall (x, T) \in D : \quad \|(x^*, T^*) - (x, T)\| < \epsilon \Rightarrow J(x^*, T^*) \leq J(x, T) \vee J(x^*, T^*) \geq J(x, T).$$

To develop the necessary optimality condition to problem (2.6) for an extremal (x^*, T^*) , we need to consider an admissible variation of the form:

$$(x^* + \epsilon h, \quad T^* + \epsilon \Delta T),$$

where $h \in C^1([a, b]; \mathbb{R})$ is a perturbing curve that satisfies the condition $h(a) = 0$, ϵ represents a small real number, and ΔT represents an arbitrarily chosen small change in T . Considering the functional $\mathcal{J}(x, T)$ in this admissible variation, we get a function of ϵ , where the upper limit of integration will also vary with ϵ :

$$\mathcal{J}(x^* + \epsilon h, \quad T^* + \epsilon \Delta T) = \int_a^{T^* + \epsilon \Delta T} L(t, (x^* + \epsilon h)(t), (x^* + \epsilon h)'(t)) dt. \quad (2.7)$$

To find the first-order necessary condition, we need to determine the derivative of (2.7) in order to ϵ and setting it equal to zero. With this, we obtain three terms on the equation, where emerges from the first term, the Euler–Lagrange equation and the other two terms, which depend only on the terminal time T , giving transversality conditions.

2.1.3 Constrained variational problems

Variational problems are often subject to one or more constraints (holonomic constraints, integral constraints, dynamic constraints, ...). Isoperimetric problems are a special class of constrained variational problems for which the admissible functions are needed to satisfy an integral constraint.

Here we review the classical isoperimetric variational problem. The classical variational problem, defined above, may be modified by demanding that the class of potential extremizing functions also satisfy a new condition, called an isoperimetric constraint, of the form

$$\int_a^b g(t, x(t), x'(t)) dt = C, \quad (2.8)$$

where g is a given function of t , x and x' , and C is a given real number.

The new problem is called an isoperimetric problem and encompasses an important family of variational problems. In this case, the variational problems are often subject to one or more constraints involving an integral of a given function [37]. Some classical examples of isoperimetric problems appear in geometry. The most famous example consists in finding the curve of a given perimeter that bounds the greatest area and the answer is

the circle. Isoperimetric problems are an important type of variational problems, with applications in different areas, like geometry, astronomy, physics, algebra or analysis.

In the next theorem, we present a necessary condition for a function to be an extremal to a classical isoperimetric problem, obtained via the concept of Lagrange multiplier [112].

Theorem 18 (Euler–Lagrange equation for isoperimetric problems). *Consider the problem of minimizing (or maximizing) the functional \mathcal{J} , defined by (2.2), on D given by those $x \in C^2([a, b]; \mathbb{R})$ satisfying the boundary conditions (2.3) and an integral constraint of the form*

$$\mathcal{G} = \int_a^b g(t, x(t), x'(t)) dt = C,$$

where $g : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a twice continuously differentiable function. Suppose that x gives a local minimum (or maximum) to this problem. Assume that $\delta\mathcal{G}(x, h)$ does not vanish for all $h \in D$. Then, there exists a constant λ such that x satisfies the Euler–Lagrange equation

$$\partial_2 F(t, x(t), x'(t), \lambda) - \frac{d}{dt} \partial_3 F(t, x(t), x'(t), \lambda) = 0, \quad (2.9)$$

where $F(t, x, x', \lambda) = L(t, x, x') - \lambda g(t, x, x')$.

Remark 19. *The constant λ is called a Lagrange multiplier.*

Observe that $\delta\mathcal{G}(x, h)$ does not vanishes for all $h \in D$ if x does not satisfies the Euler–Lagrange equation with respect to the isoperimetric constraint, that is, x is not an extremal for \mathcal{G} .

2.2 Fractional calculus of variations

The first connection between fractional calculus and the calculus of variations appeared in XIX century, with Niels Abel [1]. In 1823, Abel applied fractional calculus in the solution of an integral equation involved in a generalization of the tautochrone problem. But, only in the XX century, both areas were joined in an unique research field: the fractional calculus of variations.

The fractional calculus of variations deals with problems in which the functional, the constraint conditions, or both, depend on some fractional operator [9, 59, 62] and the main goal is to find functions that extremize such a fractional functional. By inserting fractional

operators that are non-local in variational problems, they are suitable for developing some models possessing memory effects.

This is a fast growing subject, and different approaches have been developed by considering different types of Lagrangians, e.g., depending on Riemann–Liouville or Caputo fractional derivatives, fractional integrals, and mixed integer-fractional order operators (see, for example, [10, 12, 14, 18, 19, 29, 80, 102]). In recent years, there has been a growing interest in the area of fractional variational calculus and its applications, which include classical and quantum mechanics, field theory and optimal control.

Although the origin of fractional calculus goes back more than three centuries, the calculus of variations with fractional derivatives has born with the works of F. Riewe only in 1996-1997. In his works, he obtained a version of the Euler–Lagrange equations for problems of the calculus of variations with fractional derivatives, when investigated non-conservative Lagrangian and Hamiltonian mechanics [84, 85]. Agrawal continued the study of the fractional Euler–Lagrange equations [2–4], for some kinds of fractional variational problems, for example, problems with only one dependent variable, for functionals with different orders of fractional derivatives, for several functions, involving both Riemann–Liouville and Caputo derivatives, etc. The most common fractional operators considered in the literature take into account the past of the process: they are usually called left fractional operators. But, in some cases, we may be also interested in the future of the process, and the computation of $\alpha(\cdot)$ to be influenced by it. In that case, right fractional derivatives are then considered.

2.2.1 Fractional Euler–Lagrange equations

Similarly to the classical variational calculus, the common procedure to address such fractional variational problems consists in solving a fractional differential equation, called the fractional Euler–Lagrange equation, which every minimizer/maximizer of the functional must satisfy. With the help of the boundary conditions imposed on the problem at the initial time $t = a$ and at the terminal time $t = b$, one solves, often with the help of some numerical procedure, the fractional differential equation and obtain a possible solution to the problem [21, 56, 79, 97, 114].

Referring again to Riewe’s works [84, 85], friction forces are described with Lagrangians that contain fractional derivatives. Precisely, for r, N and N' natural numbers and assum-

ing $x : [a, b] \rightarrow \mathbb{R}^r$, $\alpha_i, \beta_j \in [0, 1]$ with $i = 1, \dots, N, j = 1, \dots, N'$, the functional defined by Riewe is

$$\mathcal{J}(x) = \int_a^b L \left({}_a D_t^{\alpha_1} [x](t), \dots, {}_a D_t^{\alpha_N} [x](t), {}_t D_b^{\beta_1} [x](t), \dots, {}_t D_b^{\beta_{N'}} [x](t), x(t), t \right) dt. \quad (2.10)$$

He proved that any solution x of the variational problem of extremizing the functional (2.10) satisfies the following necessary condition:

Theorem 20 (Fractional Euler–Lagrange equation for Riewe’s problem). *Considering the variational problem of minimizing (or maximizing) the functional (2.10), the fractional Euler–Lagrange equation is*

$$\sum_{i=1}^N {}_t D_b^{\alpha_i} [\partial_i L] + \sum_{i=1}^{N'} {}_a D_t^{\beta_i} [\partial_{i+N} L] + \partial_{N'+N+1} L = 0.$$

Riewe also illustrated his results considering the classical problem of linear friction [84].

In what follows, we are concerned with problems of the fractional calculus of variations where the functional depends on a combined fractional Caputo derivative with constant orders α and β ([61, Definition15]).

Let D denote the set of all functions $x : [a, b] \rightarrow \mathbb{R}^N$, endowed with a norm $\|\cdot\|$ in \mathbb{R}^N . Consider the following problem: find a function $x \in D$ for which the functional

$$\mathcal{J}(x) = \int_a^b L(t, x(t), {}^C D_\gamma^{\alpha, \beta} x(t)) dt \quad (2.11)$$

subject to given boundary conditions

$$x(a) = x_a, \quad x(b) = x_b$$

archives a minimum, where $t \in [a, b]$, $x_a, x_b \in \mathbb{R}^N$, $\gamma \in [0, 1]$ and the Lagrangian L satisfies some smoothness properties.

Theorem 21 (c.f. Theorem 6 of [60]). *Let $x = (x_1, \dots, x_N)$ be a local minimizer to the above problem with the functional (2.11) subject to two boundary conditions, as defined before. Then, x satisfies the system of N fractional Euler–Lagrange equations*

$$\partial_i L(t, x(t), {}^C D_\gamma^{\alpha, \beta} x(t)) + D_{1-\gamma}^{\beta, \alpha} \partial_{N+i} L(t, x(t), {}^C D_\gamma^{\alpha, \beta} x(t)) = 0, \quad (2.12)$$

$i = 2, \dots, N + 1$, for all $t \in [a, b]$.

Observe that, if the orders α and β go to 1, and if $\gamma = 0$ or $\gamma = 1$, we obtain a corresponding result in the classical context of the calculus of variations. In fact, considering α and β going to 1, the fractional derivatives ${}_a^C D_t^\alpha$ and ${}_a D_t^\alpha$ coincide with the classical derivative $\frac{d}{dt}$; and similarly, ${}_t^C D_b^\beta$ and ${}_t D_b^\beta$ coincide with the classical derivative $-\frac{d}{dt}$.

Variational problems with free endpoints and transversality conditions are also relevant subjects in the fractional calculus of variations. The subject of free boundary points in fractional variational problems was first considered by Agrawal in [3]. In this work, he studied the Euler–Lagrange equation and transversality conditions for the case when both endpoints are given and the optimal function is specified at the lower endpoint but is unspecified at the upper endpoint. After that, some other free-time variational problems involving fractional derivatives or/and fractional integrals are studied [7, 68].

Sometimes, the analytic solution of the fractional Euler–Lagrange equation is very difficult to obtain and, in this case, some numerical methods have been developed to solve the variational problem [9].

2.2.2 Fractional variational problems of variable order

In recent years, motivated by the works of Samko and Ross in 1993, where they investigated integrals and derivatives not of a constant but of variable order [87, 89], some problems of the calculus of variations involving derivatives of variable fractional order have appeared [15, 69].

Considering Definition 19 of the generalized fractional integral of operator K_P , Odziejewicz et al. [59] presented a new variational problem, where the functional was defined by a given kernel. For appropriate choices of the kernel k and the set P , we can obtain a variable order fractional variational problem (see Remark 9).

Let $P = \langle a, t, b, \lambda, \mu \rangle$. Consider the functional \mathcal{J} in $\mathbf{A}(x_a, x_b)$ defined by:

$$\mathcal{J}[x] = \int_a^b L(x(t), K_P[x](t), x'(t), B_P[x](t), t) dt, \quad (2.13)$$

where

$$\mathbf{A}(x_a, x_b) = \{x \in C^1([a, b]; \mathbb{R}) : x(a) = x_a, x(b) = x_b, K_P[x], B_P[x] \in C([a, b]; \mathbb{R})\},$$

and K_P is the generalized fractional integral operator with kernel belonging to $L_q(\Delta; \mathbb{R})$ and B_P the generalized fractional derivative of Caputo type.

The optimality condition for the problem that consists to determine a function that minimize (or maximize) the functional (2.13) is given in the following theorem.

Theorem 22 (c.f. Theorem 4.10 of [59]). *Let $x \in \mathbf{A}(x_a, x_b)$ be a minimizer of functional (2.13). Then, x satisfies the following Euler–Lagrange equation:*

$$\begin{aligned} & \frac{d}{dt} [\partial_3 L(\star_x)(t)] + A_{P^*} [\tau \mapsto \partial_4 L(\star_x)(\tau)](t) \\ & = \partial_1 L(\star_x)(t) + K_{P^*} [\tau \mapsto \partial_2 L(\star_x)(\tau)](t), \end{aligned} \quad (2.14)$$

where $(\star_x)(t) = (x(t), K_P[x](t), x'(t), B_P[x](t), t)$, for $t \in (a, b)$, $P^* = \langle a, t, b, \mu, \lambda \rangle$ and A_P is the generalized fractional derivative of Riemann–Liouville type.

Observe that, if functional (2.13) does not depends on the generalized fractional operators K_P and B_P , this problem coincide with the classical variational problem and Theorem 22 reduces to Theorem 17.

Let $\Delta := \{(t, \tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\}$ and let $1 < p < \infty$ and q be the adjoint of p . A special case of this problem is obtained when we consider $\alpha : \Delta \rightarrow [0, 1 - \delta]$ with $\delta > 1/p$ and the kernel is defined by

$$k^\alpha(t, \tau) = \frac{1}{\Gamma(1 - \alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)}$$

in $L_q(\Delta; \mathbb{R})$.

Theorem 23 (c.f. Corollary 4.18 of [59]). *Consider the problem of minimizing a functional*

$$\mathcal{J}[x] = \int_a^b L \left(x(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[x](t), x'(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[x](t), t \right) dt \quad (2.15)$$

subject to boundary conditions

$$x(a) = x_a, \quad x(b) = x_b, \quad (2.16)$$

where $x', {}_a I_t^{1-\alpha(\cdot, \cdot)}[x], {}_a^C D_t^{\alpha(\cdot, \cdot)}[x] \in C([a, b]; \mathbb{R})$. Then, if $x \in C^1([a, b]; \mathbb{R})$ minimizes (or maximizes) the functional (2.15) subject to (2.16), then it satisfies the following Euler–Lagrange equation:

$$\begin{aligned} & \partial_1 L \left(x(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[x](t), x'(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[x](t), t \right) \\ & - \frac{d}{dt} \partial_3 L \left(x(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[x](t), x'(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[x](t), t \right) \\ & + {}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_2 L \left(x(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[x](\tau), x'(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[x](\tau), \tau \right) \right] (t) \\ & + {}_t D_b^{\alpha(\cdot, \cdot)} \left[\partial_4 L \left(x(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[x](\tau), x'(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[x](\tau), \tau \right) \right] (t) = 0. \end{aligned}$$

In fact, the use of fractional derivatives of constant order in variational problems may not be the best option, since trajectories are a dynamic process, and the order may vary. Therefore, it is important to consider the order to be a function, $\alpha(\cdot)$, depending on time. Then we may seek what is the best function $\alpha(\cdot)$ such that the variable order fractional differential equation $D^{\alpha(\cdot)}x(t) = f(t, x(t))$ better describes the model.

This approach is very recent, and many work has to be done for a complete study of the subject (see, e.g., [17, 28, 89, 94, 111]).

Part II

Original Work

Chapter 3

Expansion formulas for fractional derivatives

In this chapter, we present a new numerical tool to solve differential equations involving three types of Caputo derivatives of fractional variable order. For each one of them, an approximation formula is obtained and it is written in terms of standard (integer order) derivatives only. Estimations for the error of the approximations are also provided. Then, we compare the numerical approximation of some test function with its exact fractional derivative. We present an exemplification of how the presented methods can be used to solve partial fractional differential equations of variable order.

Let us briefly describe the main contents of the chapter. We begin this chapter by formulating the needed definitions (Section 3.1). Namely, we present three types of Caputo derivatives of variable fractional order. First, we consider one independent variable only; then we generalize for several independent variables. The following Section 3.2 is the main core of the chapter, where we prove approximation formulas for the given fractional operators of variable order and respectively upper bound formulas for the errors. To test the efficiency of the proposed method, in Section 3.3 we compare the exact fractional derivative of some test function with the numerical approximations obtained from the decomposition formulas given in Section 3.2. To end, in Section 3.4 we apply our method to approximate two physical problems involving Caputo fractional operators of variable order (a time-fractional diffusion equation and a fractional Burgers' partial differential equation in fluid mechanics) by classical problems that may be solved by well-known standard techniques.

The results of this chapter first appeared in [104].

3.1 Caputo-type fractional operators of variable order

In the literature of fractional calculus, several different definitions of derivatives are found [88]. One of those, introduced by Caputo in 1967 [22] and studied independently by other authors, like Džrbašjan and Nersesjan in 1968 [36] and Rabotnov in 1969 [81], has found many applications and seems to be more suitable to model physic phenomena [31, 34, 57, 65, 96, 101, 115].

3.1.1 Caputo derivatives for functions of one variable

Our goal is to consider fractional derivatives of variable order, with α depending on time. In fact, some phenomena in physics are better described when the order of the fractional operator is not constant, for example, in the diffusion process in an inhomogeneous or heterogeneous medium, or processes where the changes in the environment modify the dynamic of the particle [24, 90, 98]. Motivated by the above considerations, we introduce three types of Caputo fractional derivatives. The order of the derivative is considered as a function $\alpha(t)$ taking values on the open interval $(0, 1)$. To start, we define two different kinds of Riemann–Liouville fractional derivatives.

Definition 29 (Riemann–Liouville fractional derivatives of order $\alpha(t)$ —types I and II). *Given a function $x : [a, b] \rightarrow \mathbb{R}$,*

1. *the type I left Riemann–Liouville fractional derivative of order $\alpha(t)$ is defined by*

$${}_a D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha(t)} x(\tau) d\tau;$$

2. *the type I right Riemann–Liouville fractional derivative of order $\alpha(t)$ is defined by*

$${}_t D_b^{\alpha(t)} x(t) = \frac{-1}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha(t)} x(\tau) d\tau;$$

3. *the type II left Riemann–Liouville fractional derivative of order $\alpha(t)$ is defined by*

$${}_a \mathcal{D}_t^{\alpha(t)} x(t) = \frac{d}{dt} \left(\frac{1}{\Gamma(1 - \alpha(t))} \int_a^t (t - \tau)^{-\alpha(t)} x(\tau) d\tau \right);$$

4. *the type II right Riemann–Liouville fractional derivative of order $\alpha(t)$ is defined by*

$${}_t \mathcal{D}_b^{\alpha(t)} x(t) = \frac{d}{dt} \left(\frac{-1}{\Gamma(1 - \alpha(t))} \int_t^b (\tau - t)^{-\alpha(t)} x(\tau) d\tau \right).$$

The Caputo derivatives are given using the previous Riemann–Liouville fractional derivatives.

Definition 30 (Caputo fractional derivatives of order $\alpha(t)$ —types I, II and III). *Given a function $x : [a, b] \rightarrow \mathbb{R}$,*

1. *the type I left Caputo derivative of order $\alpha(t)$ is defined by*

$${}^C D_t^{\alpha(t)} x(t) = {}_a D_t^{\alpha(t)} (x(t) - x(a)) = \frac{1}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha(t)} [x(\tau) - x(a)] d\tau;$$

2. *the type I right Caputo derivative of order $\alpha(t)$ is defined by*

$${}_t D_b^{\alpha(t)} x(t) = {}_t D_b^{\alpha(t)} (x(t) - x(b)) = \frac{-1}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha(t)} [x(\tau) - x(b)] d\tau;$$

3. *the type II left Caputo derivative of order $\alpha(t)$ is defined by*

$${}^C \mathcal{D}_t^{\alpha(t)} x(t) = {}_a \mathcal{D}_t^{\alpha(t)} (x(t) - x(a)) = \frac{d}{dt} \left(\frac{1}{\Gamma(1 - \alpha(t))} \int_a^t (t - \tau)^{-\alpha(t)} [x(\tau) - x(a)] d\tau \right);$$

4. *the type II right Caputo derivative of order $\alpha(t)$ is defined by*

$${}_t \mathcal{D}_b^{\alpha(t)} x(t) = {}_t \mathcal{D}_b^{\alpha(t)} (x(t) - x(b)) = \frac{d}{dt} \left(\frac{-1}{\Gamma(1 - \alpha(t))} \int_t^b (\tau - t)^{-\alpha(t)} [x(\tau) - x(b)] d\tau \right);$$

5. *the type III left Caputo derivative of order $\alpha(t)$ is defined by*

$${}^C \mathbb{D}_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(1 - \alpha(t))} \int_a^t (t - \tau)^{-\alpha(t)} x'(\tau) d\tau;$$

6. *the type III right Caputo derivative of order $\alpha(t)$ is defined by*

$${}_t \mathbb{D}_b^{\alpha(t)} x(t) = \frac{-1}{\Gamma(1 - \alpha(t))} \int_t^b (\tau - t)^{-\alpha(t)} x'(\tau) d\tau.$$

In contrast with the case when α is a constant, definitions of different types do not coincide.

Theorem 24. *The following relations between the left fractional operators hold:*

$${}^C D_t^{\alpha(t)} x(t) = {}^C \mathbb{D}_t^{\alpha(t)} x(t) + \frac{\alpha'(t)}{\Gamma(2 - \alpha(t))} \int_a^t (t - \tau)^{1 - \alpha(t)} x'(\tau) \left[\frac{1}{1 - \alpha(t)} - \ln(t - \tau) \right] d\tau \quad (3.1)$$

and

$${}^C D_t^{\alpha(t)} x(t) = {}^C \mathcal{D}_t^{\alpha(t)} x(t) - \frac{\alpha'(t) \Psi(1 - \alpha(t))}{\Gamma(1 - \alpha(t))} \int_a^t (t - \tau)^{-\alpha(t)} [x(\tau) - x(a)] d\tau. \quad (3.2)$$

Proof. Integrating by parts, one gets

$$\begin{aligned} {}^C D_t^{\alpha(t)} x(t) &= \frac{1}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha(t)} [x(\tau) - x(a)] d\tau \\ &= \frac{1}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \left[\frac{1}{1 - \alpha(t)} \int_a^t (t - \tau)^{1-\alpha(t)} x'(\tau) d\tau \right]. \end{aligned}$$

Differentiating the integral, it follows that

$$\begin{aligned} {}^C D_t^{\alpha(t)} x(t) &= \frac{1}{\Gamma(1 - \alpha(t))} \left[\frac{\alpha'(t)}{(1 - \alpha(t))^2} \int_a^t (t - \tau)^{1-\alpha(t)} x'(\tau) d\tau \right. \\ &\quad \left. + \frac{1}{1 - \alpha(t)} \int_a^t (t - \tau)^{1-\alpha(t)} x'(\tau) \left[-\alpha'(t) \ln(t - \tau) + \frac{1 - \alpha(t)}{t - \tau} \right] d\tau \right] \\ &= {}^C \mathbb{D}_t^{\alpha(t)} x(t) + \frac{\alpha'(t)}{\Gamma(2 - \alpha(t))} \int_a^t (t - \tau)^{1-\alpha(t)} x'(\tau) \left[\frac{1}{1 - \alpha(t)} - \ln(t - \tau) \right] d\tau. \end{aligned}$$

The second formula follows from direct calculations. \square

Therefore, when the order $\alpha(t) \equiv c$ is a constant, or for constant functions $x(t) \equiv k$, we have

$${}^C D_t^{\alpha(t)} x(t) = {}^C \mathcal{D}_t^{\alpha(t)} x(t) = {}^C \mathbb{D}_t^{\alpha(t)} x(t).$$

Similarly, we obtain the next result.

Theorem 25. *The following relations between the right fractional operators hold:*

$${}^C D_b^{\alpha(t)} x(t) = {}^C \mathbb{D}_b^{\alpha(t)} x(t) + \frac{\alpha'(t)}{\Gamma(2 - \alpha(t))} \int_t^b (\tau - t)^{1-\alpha(t)} x'(\tau) \left[\frac{1}{1 - \alpha(t)} - \ln(\tau - t) \right] d\tau$$

and

$${}^C D_b^{\alpha(t)} x(t) = {}^C \mathcal{D}_b^{\alpha(t)} x(t) + \frac{\alpha'(t) \Psi(1 - \alpha(t))}{\Gamma(1 - \alpha(t))} \int_t^b (\tau - t)^{-\alpha(t)} [x(\tau) - x(b)] d\tau.$$

Theorem 26. *Let $x \in C^1([a, b], \mathbb{R})$. At $t = a$*

$${}^C D_t^{\alpha(t)} x(t) = {}^C \mathcal{D}_t^{\alpha(t)} x(t) = {}^C \mathbb{D}_t^{\alpha(t)} x(t) = 0;$$

at $t = b$

$${}^C D_b^{\alpha(t)} x(t) = {}^C \mathcal{D}_b^{\alpha(t)} x(t) = {}^C \mathbb{D}_b^{\alpha(t)} x(t) = 0.$$

Proof. We start proving the third equality at the initial time $t = a$. We simply note that

$$\left| {}^C \mathbb{D}_t^{\alpha(t)} x(t) \right| \leq \frac{\|x'\|}{\Gamma(1 - \alpha(t))} \int_a^t (t - \tau)^{-\alpha(t)} d\tau = \frac{\|x'\|}{\Gamma(2 - \alpha(t))} (t - a)^{1-\alpha(t)},$$

which is zero at $t = a$. For the first equality at $t = a$, using equation (3.1), and the two next relations

$$\left| \int_a^t (t - \tau)^{1-\alpha(t)} \frac{x'(\tau)}{1 - \alpha(t)} d\tau \right| \leq \frac{\|x'\|}{(1 - \alpha(t))(2 - \alpha(t))} (t - a)^{2-\alpha(t)}$$

and

$$\left| \int_a^t (t - \tau)^{1-\alpha(t)} x'(\tau) \ln(t - \tau) d\tau \right| \leq \frac{\|x'\|}{2 - \alpha(t)} (t - a)^{2-\alpha(t)} \left| \ln(t - a) - \frac{1}{2 - \alpha(t)} \right|,$$

this latter inequality obtained from integration by parts, we prove that ${}_a^C D_t^{\alpha(t)} x(t) = 0$ at $t = a$. Finally, we prove the second equality at $t = a$ by considering equation (3.2): performing an integration by parts, we get

$$\left| \int_a^t (t - \tau)^{-\alpha(t)} [x(\tau) - x(a)] d\tau \right| \leq \frac{\|x'\|}{(1 - \alpha(t))(2 - \alpha(t))} (t - a)^{2-\alpha(t)}$$

and so ${}_a^C \mathcal{D}_t^{\alpha(t)} x(t) = 0$ at $t = a$. The proof that the right fractional operators also vanish at the end point $t = b$ follows by similar arguments. \square

With some computations, a relationship between the Riemann–Liouville and the Caputo fractional derivatives is easily deduced:

$$\begin{aligned} {}_a D_t^{\alpha(t)} x(t) &= {}_a^C D_t^{\alpha(t)} x(t) + \frac{x(a)}{\Gamma(1 - \alpha(t))} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha(t)} d\tau \\ &= {}_a^C D_t^{\alpha(t)} x(t) + \frac{x(a)}{\Gamma(1 - \alpha(t))} (t - a)^{-\alpha(t)} \\ &\quad + \frac{x(a)\alpha'(t)}{\Gamma(2 - \alpha(t))} (t - a)^{1-\alpha(t)} \left[\frac{1}{1 - \alpha(t)} - \ln(t - a) \right] \end{aligned}$$

and

$$\begin{aligned} {}_a \mathcal{D}_t^{\alpha(t)} x(t) &= {}_a^C \mathcal{D}_t^{\alpha(t)} x(t) + x(a) \frac{d}{dt} \left(\frac{1}{\Gamma(1 - \alpha(t))} \int_a^t (t - \tau)^{-\alpha(t)} d\tau \right) \\ &= {}_a^C \mathcal{D}_t^{\alpha(t)} x(t) + \frac{x(a)}{\Gamma(1 - \alpha(t))} (t - a)^{-\alpha(t)} \\ &\quad + \frac{x(a)\alpha'(t)}{\Gamma(2 - \alpha(t))} (t - a)^{1-\alpha(t)} [\Psi(2 - \alpha(t)) - \ln(t - a)]. \end{aligned}$$

For the right fractional operators, we have

$$\begin{aligned} {}_t D_b^{\alpha(t)} x(t) &= {}_t^C D_b^{\alpha(t)} x(t) + \frac{x(b)}{\Gamma(1 - \alpha(t))} (b - t)^{-\alpha(t)} \\ &\quad - \frac{x(b)\alpha'(t)}{\Gamma(2 - \alpha(t))} (b - t)^{1-\alpha(t)} \left[\frac{1}{1 - \alpha(t)} - \ln(b - t) \right] \end{aligned}$$

and

$$\begin{aligned} {}_t\mathcal{D}_b^{\alpha(t)}x(t) &= {}^C\mathcal{D}_b^{\alpha(t)}x(t) + \frac{x(b)}{\Gamma(1-\alpha(t))}(b-t)^{-\alpha(t)} \\ &\quad - \frac{x(b)\alpha'(t)}{\Gamma(2-\alpha(t))}(b-t)^{1-\alpha(t)}[\Psi(2-\alpha(t)) - \ln(b-t)]. \end{aligned}$$

Thus, it is immediate to conclude that if $x(a) = 0$, then

$${}_aD_t^{\alpha(t)}x(t) = {}^C D_t^{\alpha(t)}x(t) \quad \text{and} \quad {}_a\mathcal{D}_t^{\alpha(t)}x(t) = {}^C \mathcal{D}_t^{\alpha(t)}x(t)$$

and if $x(b) = 0$, then

$${}_tD_b^{\alpha(t)}x(t) = {}^C D_b^{\alpha(t)}x(t) \quad \text{and} \quad {}_t\mathcal{D}_b^{\alpha(t)}x(t) = {}^C \mathcal{D}_b^{\alpha(t)}x(t).$$

Next we obtain formulas for the Caputo fractional derivatives of a power function.

Lemma 27. *Let $x(t) = (t-a)^\gamma$ with $\gamma > 0$. Then,*

$$\begin{aligned} {}^C D_t^{\alpha(t)}x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)}(t-a)^{\gamma-\alpha(t)} \\ &\quad - \alpha'(t)\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+2)}(t-a)^{\gamma-\alpha(t)+1} \\ &\quad \times [\ln(t-a) - \Psi(\gamma-\alpha(t)+2) + \Psi(1-\alpha(t))], \\ {}^C \mathcal{D}_t^{\alpha(t)}x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)}(t-a)^{\gamma-\alpha(t)} \\ &\quad - \alpha'(t)\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+2)}(t-a)^{\gamma-\alpha(t)+1} \\ &\quad \times [\ln(t-a) - \Psi(\gamma-\alpha(t)+2)], \\ {}^C \mathbb{D}_t^{\alpha(t)}x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)}(t-a)^{\gamma-\alpha(t)}. \end{aligned}$$

Proof. The formula for ${}^C D_t^{\alpha(t)}x(t)$ follows immediately from [89]. For the second equality, one has

$$\begin{aligned} {}^C \mathcal{D}_t^{\alpha(t)}x(t) &= \frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha(t))} \int_a^t (t-\tau)^{-\alpha(t)} (\tau-a)^\gamma d\tau \right) \\ &= \frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha(t))} \int_a^t (t-a)^{-\alpha(t)} \left(1 - \frac{\tau-a}{t-a} \right)^{-\alpha(t)} (\tau-a)^\gamma d\tau \right). \end{aligned}$$

With the change of variables $\tau - a = s(t - a)$, and with the help of the Beta function $B(\cdot, \cdot)$ (see Definition 3), we prove that

$$\begin{aligned} {}^C\mathcal{D}_t^{\alpha(t)}x(t) &= \frac{d}{dt} \left(\frac{(t-a)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \int_0^1 (1-s)^{-\alpha(t)} s^\gamma (t-a)^{\gamma+1} ds \right) \\ &= \frac{d}{dt} \left(\frac{(t-a)^{\gamma-\alpha(t)+1}}{\Gamma(1-\alpha(t))} B(\gamma+1, 1-\alpha(t)) \right) \\ &= \frac{d}{dt} \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+2)} (t-a)^{\gamma-\alpha(t)+1} \right). \end{aligned}$$

We obtain the desired formula by differentiating this latter expression. The last equality follows in a similar way. \square

Analogous relations to those of Lemma 27, for the right Caputo fractional derivatives of variable order, are easily obtained.

Lemma 28. *Let $x(t) = (b-t)^\gamma$ with $\gamma > 0$. Then,*

$$\begin{aligned} {}^C D_b^{\alpha(t)}x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)} (b-t)^{\gamma-\alpha(t)} \\ &\quad + \alpha'(t) \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+2)} (b-t)^{\gamma-\alpha(t)+1} \\ &\quad \times [\ln(b-t) - \Psi(\gamma-\alpha(t)+2) + \Psi(1-\alpha(t))], \\ {}^C \mathcal{D}_b^{\alpha(t)}x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)} (b-t)^{\gamma-\alpha(t)} \\ &\quad + \alpha'(t) \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+2)} (b-t)^{\gamma-\alpha(t)+1} \\ &\quad \times [\ln(b-t) - \Psi(\gamma-\alpha(t)+2)], \\ {}^C \mathbb{D}_b^{\alpha(t)}x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)} (b-t)^{\gamma-\alpha(t)}. \end{aligned}$$

With Lemma 27 in mind, we immediately see that

$${}^C D_t^{\alpha(t)}x(t) \neq {}^C \mathcal{D}_t^{\alpha(t)}x(t) \neq {}^C \mathbb{D}_t^{\alpha(t)}x(t).$$

Also, at least for the power function, it suggests that ${}^C \mathbb{D}_t^{\alpha(t)}x(t)$ may be a more suitable inverse operation of the fractional integral when the order is variable. For example, consider functions $x(t) = t^2$ and $y(t) = (1-t)^2$, and the fractional order $\alpha(t) = \frac{5t+1}{10}$, $t \in [0, 1]$. Then, $0.1 \leq \alpha(t) \leq 0.6$ for all t . Next we compare the fractional derivatives of x and y

of order $\alpha(t)$ with the fractional derivatives of constant order $\alpha = 0.1$ and $\alpha = 0.6$. By Lemma 27, we know that the left Caputo fractional derivatives of order $\alpha(t)$ of x are given by

$$\begin{aligned} {}_0^C D_t^{\alpha(t)} x(t) &= \frac{2}{\Gamma(3 - \alpha(t))} t^{2-\alpha(t)} - \frac{t^{3-\alpha(t)}}{\Gamma(4 - \alpha(t))} [\ln(t) - \Psi(4 - \alpha(t)) + \Psi(1 - \alpha(t))], \\ {}_0^C \mathcal{D}_t^{\alpha(t)} x(t) &= \frac{2}{\Gamma(3 - \alpha(t))} t^{2-\alpha(t)} - \frac{t^{3-\alpha(t)}}{\Gamma(4 - \alpha(t))} [\ln(t) - \Psi(4 - \alpha(t))], \\ {}_0^C \mathbb{D}_t^{\alpha(t)} x(t) &= \frac{2}{\Gamma(3 - \alpha(t))} t^{2-\alpha(t)}, \end{aligned}$$

while by Lemma 28, the right Caputo fractional derivatives of order $\alpha(t)$ of y are given by

$$\begin{aligned} {}_t^C D_1^{\alpha(t)} y(t) &= \frac{2(1-t)^{2-\alpha(t)}}{\Gamma(3 - \alpha(t))} + \frac{(1-t)^{3-\alpha(t)}}{\Gamma(4 - \alpha(t))} [\ln(1-t) - \Psi(4 - \alpha(t)) + \Psi(1 - \alpha(t))], \\ {}_t^C \mathcal{D}_1^{\alpha(t)} y(t) &= \frac{2(1-t)^{2-\alpha(t)}}{\Gamma(3 - \alpha(t))} + \frac{(1-t)^{3-\alpha(t)}}{\Gamma(4 - \alpha(t))} [\ln(1-t) - \Psi(4 - \alpha(t))], \\ {}_t^C \mathbb{D}_1^{\alpha(t)} y(t) &= \frac{2(1-t)^{2-\alpha(t)}}{\Gamma(3 - \alpha(t))}. \end{aligned}$$

For a constant order α , we have

$${}_0^C D_t^\alpha x(t) = \frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha} \quad \text{and} \quad {}_t^C D_1^\alpha y(t) = \frac{2}{\Gamma(3 - \alpha)} (1-t)^{2-\alpha}.$$

The results can be seen in Figure 3.1.

3.1.2 Caputo derivatives for functions of several variables

Partial fractional derivatives are a natural extension and are defined in a similar way.

Let $m \in \mathbb{N}$, $k \in \{1, \dots, m\}$, and consider a function $x : \prod_{i=1}^m [a_i, b_i] \rightarrow \mathbb{R}$ with m variables.

For simplicity, we define the vectors

$$[\tau]_k = (t_1, \dots, t_{k-1}, \tau, t_{k+1}, \dots, t_m) \in \mathbb{R}^m$$

and

$$(\bar{t}) = (t_1, \dots, t_m) \in \mathbb{R}^m.$$

Definition 31 (Partial Caputo fractional derivatives of variable order—types I, II and III). *Given a function $x : \prod_{i=1}^m [a_i, b_i] \rightarrow \mathbb{R}$ and fractional orders $\alpha_k : [a_k, b_k] \rightarrow (0, 1)$, $k \in \{1, \dots, m\}$,*

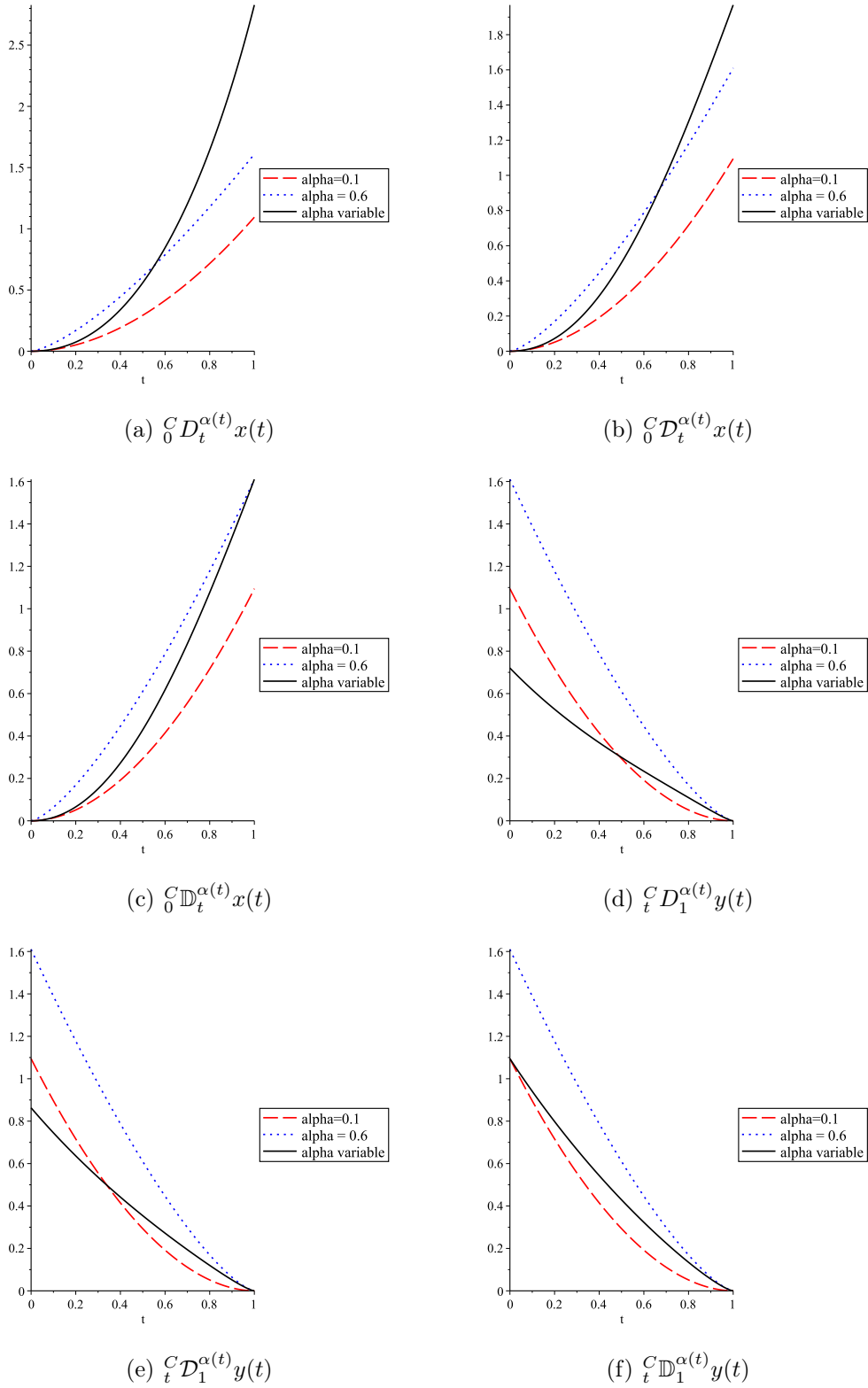


Figure 3.1: Comparison between variable order and constant order fractional derivatives.

1. the type I partial left Caputo derivative of order $\alpha_k(t_k)$ is defined by

$${}^C_{a_k} D_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{1}{\Gamma(1 - \alpha_k(t_k))} \frac{\partial}{\partial t_k} \int_{a_k}^{t_k} (t_k - \tau)^{-\alpha_k(t_k)} (x[\tau]_k - x[a_k]_k) d\tau;$$

2. the type I partial right Caputo derivative of order $\alpha_k(t_k)$ is defined by

$${}^C_{t_k} D_{b_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{-1}{\Gamma(1 - \alpha_k(t_k))} \frac{\partial}{\partial t_k} \int_{t_k}^{b_k} (\tau - t_k)^{-\alpha_k(t_k)} (x[\tau]_k - x[b_k]_k) d\tau;$$

3. the type II partial left Caputo derivative of order $\alpha_k(t_k)$ is defined by

$${}^C_{a_k} \mathcal{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{\partial}{\partial t_k} \left(\frac{1}{\Gamma(1 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{-\alpha_k(t_k)} (x[\tau]_k - x[a_k]_k) d\tau \right);$$

4. the type II partial right Caputo derivative of order $\alpha_k(t_k)$ is defined by

$${}^C_{t_k} \mathcal{D}_{b_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{\partial}{\partial t_k} \left(\frac{-1}{\Gamma(1 - \alpha_k(t_k))} \int_{t_k}^{b_k} (\tau - t_k)^{-\alpha_k(t_k)} (x[\tau]_k - x[b_k]_k) d\tau \right);$$

5. the type III partial left Caputo derivative of order $\alpha_k(t_k)$ is defined by

$${}^C_{a_k} \mathbb{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{1}{\Gamma(1 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{-\alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k d\tau;$$

6. the type III partial right Caputo derivative of order $\alpha_k(t_k)$ is defined by

$${}^C_{t_k} \mathbb{D}_{b_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{-1}{\Gamma(1 - \alpha_k(t_k))} \int_{t_k}^{b_k} (\tau - t_k)^{-\alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k d\tau.$$

Similarly as done before, relations between these definitions can be proven.

Theorem 29. *The following four formulas hold:*

$$\begin{aligned} {}^C_{a_k} D_{t_k}^{\alpha_k(t_k)} x(\bar{t}) &= {}^C_{a_k} \mathbb{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) \\ &+ \frac{\alpha'_k(t_k)}{\Gamma(2 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{1 - \alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k \left[\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - \tau) \right] d\tau, \end{aligned} \quad (3.3)$$

$${}^C_{a_k} D_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = {}^C_{a_k} \mathcal{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) - \frac{\alpha'_k(t_k) \Psi(1 - \alpha_k(t_k))}{\Gamma(1 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{-\alpha_k(t_k)} [x[\tau]_k - x[a_k]_k] d\tau, \quad (3.4)$$

$$\begin{aligned} {}^C D_{b_k}^{\alpha_k(t_k)} x(\bar{t}) &= {}^C \mathbb{D}_{b_k}^{\alpha_k(t_k)} x(\bar{t}) \\ &\quad + \frac{\alpha'_k(t_k)}{\Gamma(2 - \alpha_k(t_k))} \int_{t_k}^{b_k} (\tau - t_k)^{1 - \alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k \left[\frac{1}{1 - \alpha_k(t_k)} - \ln(\tau - t_k) \right] d\tau \end{aligned}$$

and

$${}^C D_{b_k}^{\alpha_k(t_k)} x(\bar{t}) = {}^C \mathcal{D}_{b_k}^{\alpha_k(t_k)} x(\bar{t}) + \frac{\alpha'_k(t_k) \Psi(1 - \alpha_k(t_k))}{\Gamma(1 - \alpha_k(t_k))} \int_{t_k}^{b_k} (\tau - t_k)^{-\alpha_k(t_k)} [x[\tau]_k - x[b_k]_k] d\tau.$$

3.2 Numerical approximations

Let $p \in \mathbb{N}$. We define

$$\begin{aligned} A_p &= \frac{1}{\Gamma(p + 1 - \alpha_k(t_k))} \left[1 + \sum_{l=n-p+1}^N \frac{\Gamma(\alpha_k(t_k) - n + l)}{\Gamma(\alpha_k(t_k) - p)(l - n + p)!} \right], \\ B_p &= \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(1 - \alpha_k(t_k))\Gamma(\alpha_k(t_k))(p - n)!}, \\ V_p(\bar{t}) &= \int_{a_k}^{t_k} (\tau - a_k)^{p-n} \frac{\partial x}{\partial t_k} [\tau]_k d\tau, \\ L_p(\bar{t}) &= \max_{\tau \in [a_k, t_k]} \left| \frac{\partial^p x}{\partial t_k^p} [\tau]_k \right|. \end{aligned}$$

Theorem 30. *Let $x \in C^{n+1}(\prod_{i=1}^m [a_i, b_i], \mathbb{R})$ with $n \in \mathbb{N}$. Then, for all $k \in \{1, \dots, m\}$ and for all $N \in \mathbb{N}$ such that $N \geq n$, we have*

$${}^C \mathbb{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = \sum_{p=1}^n A_p (t_k - a_k)^{p - \alpha_k(t_k)} \frac{\partial^p x}{\partial t_k^p} [t_k]_k + \sum_{p=n}^N B_p (t_k - a_k)^{n - p - \alpha_k(t_k)} V_p(\bar{t}) + E(\bar{t}).$$

The approximation error $E(\bar{t})$ is bounded by

$$E(\bar{t}) \leq L_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n + 1 - \alpha_k(t_k)) N^{n - \alpha_k(t_k)} (n - \alpha_k(t_k))} (t_k - a_k)^{n+1 - \alpha_k(t_k)}.$$

Proof. By definition,

$${}^C \mathbb{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{1}{\Gamma(1 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{-\alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k d\tau$$

and, integrating by parts with $u'(\tau) = (t_k - \tau)^{-\alpha_k(t_k)}$ and $v(\tau) = \frac{\partial x}{\partial t_k} [\tau]_k$, we deduce that

$${}^C \mathbb{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) = \frac{(t_k - a_k)^{1 - \alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \frac{\partial x}{\partial t_k} [a_k]_k + \frac{1}{\Gamma(2 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{1 - \alpha_k(t_k)} \frac{\partial^2 x}{\partial t_k^2} [\tau]_k d\tau.$$

Integrating again by parts, taking $u'(\tau) = (t_k - \tau)^{1-\alpha_k(t_k)}$ and $v(\tau) = \frac{\partial^2 x}{\partial t_k^2}[\tau]_k$, we get

$$\begin{aligned} {}^C \mathbb{D}_{a_k}^{\alpha_k(t_k)} x(\bar{t}) &= \frac{(t_k - a_k)^{1-\alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \frac{\partial x}{\partial t_k} [a_k]_k + \frac{(t_k - a_k)^{2-\alpha_k(t_k)}}{\Gamma(3 - \alpha_k(t_k))} \frac{\partial^2 x}{\partial t_k^2} [a_k]_k \\ &\quad + \frac{1}{\Gamma(3 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{2-\alpha_k(t_k)} \frac{\partial^3 x}{\partial t_k^3} [\tau]_k d\tau. \end{aligned}$$

Repeating the same procedure $n - 2$ more times, we get the expansion formula

$$\begin{aligned} {}^C \mathbb{D}_{a_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^n \frac{(t_k - a_k)^{p-\alpha_k(t_k)}}{\Gamma(p+1 - \alpha_k(t_k))} \frac{\partial^p x}{\partial t_k^p} [a_k]_k \\ &\quad + \frac{1}{\Gamma(n+1 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{n-\alpha_k(t_k)} \frac{\partial^{n+1} x}{\partial t_k^{n+1}} [\tau]_k d\tau. \end{aligned}$$

Using the equalities

$$\begin{aligned} (t_k - \tau)^{n-\alpha_k(t_k)} &= (t_k - a_k)^{n-\alpha_k(t_k)} \left(1 - \frac{\tau - a_k}{t_k - a_k} \right)^{n-\alpha_k(t_k)} \\ &= (t_k - a_k)^{n-\alpha_k(t_k)} \left[\sum_{p=0}^N \binom{n-\alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p} + \bar{E}(\bar{t}) \right] \end{aligned}$$

with

$$\bar{E}(\bar{t}) = \sum_{p=N+1}^{\infty} \binom{n-\alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p},$$

we arrive at

$$\begin{aligned} {}^C \mathbb{D}_{a_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^n \frac{(t_k - a_k)^{p-\alpha_k(t_k)}}{\Gamma(p+1 - \alpha_k(t_k))} \frac{\partial^p x}{\partial t_k^p} [a_k]_k \\ &\quad + \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1 - \alpha_k(t_k))} \int_{a_k}^{t_k} \sum_{p=0}^N \binom{n-\alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p} \frac{\partial^{n+1} x}{\partial t_k^{n+1}} [\tau]_k d\tau + E(\bar{t}) \\ &= \sum_{p=1}^n \frac{(t_k - a_k)^{p-\alpha_k(t_k)}}{\Gamma(p+1 - \alpha_k(t_k))} \frac{\partial^p x}{\partial t_k^p} [a_k]_k \\ &\quad + \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1 - \alpha_k(t_k))} \sum_{p=0}^N \binom{n-\alpha_k(t_k)}{p} \frac{(-1)^p}{(t_k - a_k)^p} \int_{a_k}^{t_k} (\tau - a_k)^p \frac{\partial^{n+1} x}{\partial t_k^{n+1}} [\tau]_k d\tau + E(\bar{t}) \end{aligned}$$

with

$$E(\bar{t}) = \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1 - \alpha_k(t_k))} \int_{a_k}^{t_k} \bar{E}(\bar{t}) \frac{\partial^{n+1} x}{\partial t_k^{n+1}} [\tau]_k d\tau.$$

Now, we split the last sum into $p = 0$ and the remaining terms $p = 1, \dots, N$ and integrate by parts with $u(\tau) = (\tau - a_k)^p$ and $v'(\tau) = \frac{\partial^{n+1}x}{\partial t_k^{n+1}}[\tau]_k$. Observing that

$$\binom{n-\alpha_k(t_k)}{p} (-1)^p = \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(\alpha_k(t_k) - n)p!},$$

we obtain:

$$\begin{aligned} & \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1-\alpha_k(t_k))} \sum_{p=0}^N \binom{n-\alpha_k(t_k)}{p} \frac{(-1)^p}{(t_k - a_k)^p} \int_{a_k}^{t_k} (\tau - a_k)^p \frac{\partial^{n+1}x}{\partial t_k^{n+1}}[\tau]_k d\tau \\ &= \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1-\alpha_k(t_k))} \left[\frac{\partial^n x}{\partial t_k^n}[t_k]_k - \frac{\partial^n x}{\partial t_k^n}[a_k]_k \right] + \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1-\alpha_k(t_k))} \\ & \times \sum_{p=1}^N \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(\alpha_k(t_k) - n)p!(t_k - a_k)^p} \left[(t_k - a_k)^p \frac{\partial^n x}{\partial t_k^n}[t_k]_k - \int_{a_k}^{t_k} p(\tau - a_k)^{p-1} \frac{\partial^n x}{\partial t_k^n}[\tau]_k d\tau \right] \\ &= -\frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1-\alpha_k(t_k))} \frac{\partial^n x}{\partial t_k^n}[a_k]_k + \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1-\alpha_k(t_k))} \frac{\partial^n x}{\partial t_k^n}[t_k]_k \left[1 + \sum_{p=1}^N \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(\alpha_k(t_k) - n)p!} \right] \\ &+ \frac{(t_k - a_k)^{n-\alpha_k(t_k)-1}}{\Gamma(n-\alpha_k(t_k))} \sum_{p=1}^N \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(\alpha_k(t_k) + 1 - n)(p-1)!(t_k - a_k)^{p-1}} \int_{a_k}^{t_k} (\tau - a_k)^{p-1} \frac{\partial^n x}{\partial t_k^n}[\tau]_k d\tau. \end{aligned}$$

Thus, we get

$$\begin{aligned} {}^C_{a_k} \mathbb{D}_{t_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^{n-1} \frac{(t_k - a_k)^{p-\alpha_k(t_k)}}{\Gamma(p+1-\alpha_k(t_k))} \frac{\partial^p x}{\partial t_k^p}[a_k]_k \\ &+ \frac{(t_k - a_k)^{n-\alpha_k(t_k)}}{\Gamma(n+1-\alpha_k(t_k))} \frac{\partial^n x}{\partial t_k^n}[t_k]_k \left[1 + \sum_{p=1}^N \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(\alpha_k(t_k) - n)p!} \right] \\ &+ \frac{(t_k - a_k)^{n-\alpha_k(t_k)-1}}{\Gamma(n-\alpha_k(t_k))} \sum_{p=1}^N \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(\alpha_k(t_k) + 1 - n)(p-1)!(t_k - a_k)^{p-1}} \\ &\times \int_{a_k}^{t_k} (\tau - a_k)^{p-1} \frac{\partial^n x}{\partial t_k^n}[\tau]_k d\tau + E(\bar{t}). \end{aligned}$$

Repeating the process $n - 1$ more times with respect to the last sum, that is, splitting the

first term of the sum and integrating by parts the obtained result, we arrive to

$$\begin{aligned} {}^C \mathbb{D}_{a_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^n \frac{(t_k - a_k)^{p-\alpha_k(t_k)}}{\Gamma(p+1-\alpha_k(t_k))} \frac{\partial^p x}{\partial t_k^p} [t_k]_k \left[1 + \sum_{l=n-p+1}^N \frac{\Gamma(\alpha_k(t_k) - n + l)}{\Gamma(\alpha_k(t_k) - p)(l - n + p)!} \right] \\ &+ \sum_{p=n}^N \frac{\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(1 - \alpha_k(t_k))\Gamma(\alpha_k(t_k))(p - n)!} (t_k - a_k)^{n-p-\alpha_k(t_k)} \\ &\times \int_{a_k}^{t_k} (\tau - a_k)^{p-n} \frac{\partial x}{\partial t_k} [\tau]_k d\tau + E(\bar{t}). \end{aligned}$$

We now seek the upper bound formula for $E(\bar{t})$. Using the two relations

$$\left| \frac{\tau - a_k}{t_k - a_k} \right| \leq 1, \text{ if } \tau \in [a_k, t_k] \quad \text{and} \quad \left| \binom{n-\alpha_k(t_k)}{p} \right| \leq \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{p^{n+1-\alpha_k(t_k)}},$$

we get

$$\begin{aligned} \bar{E}(\bar{t}) &\leq \sum_{p=N+1}^{\infty} \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{p^{n+1-\alpha_k(t_k)}} \\ &\leq \int_N^{\infty} \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{p^{n+1-\alpha_k(t_k)}} dp = \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{N^{n-\alpha_k(t_k)}(n - \alpha_k(t_k))}. \end{aligned}$$

Then,

$$E(\bar{t}) \leq L_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n+1-\alpha_k(t_k))N^{n-\alpha_k(t_k)}(n - \alpha_k(t_k))} (t_k - a_k)^{n+1-\alpha_k(t_k)}.$$

This concludes the proof. \square

Remark 31. In Theorem 30 we have

$$\lim_{N \rightarrow \infty} E(\bar{t}) = 0$$

for all $\bar{t} \in \prod_{i=1}^m [a_i, b_i]$ and $n \in \mathbb{N}$.

Theorem 32. Let $x \in C^{n+1}(\prod_{i=1}^m [a_i, b_i], \mathbb{R})$ with $n \in \mathbb{N}$. Then, for all $k \in \{1, \dots, m\}$ and for all $N \in \mathbb{N}$ such that $N \geq n$, we have

$$\begin{aligned} {}^C D_{a_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^n A_p (t_k - a_k)^{p-\alpha_k(t_k)} \frac{\partial^p x}{\partial t_k^p} [t_k]_k + \sum_{p=n}^N B_p (t_k - a_k)^{n-p-\alpha_k(t_k)} V_p(\bar{t}) \\ &+ \frac{\alpha'_k(t_k)(t_k - a_k)^{1-\alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right) \sum_{p=0}^N \binom{1-\alpha_k(t_k)}{p} \frac{(-1)^p}{(t_k - a_k)^p} V_{n+p}(\bar{t}) \right. \\ &\quad \left. + \sum_{p=0}^N \binom{1-\alpha_k(t_k)}{p} (-1)^p \sum_{r=1}^N \frac{1}{r(t_k - a_k)^{p+r}} V_{n+p+r}(\bar{t}) \right] + E(\bar{t}). \end{aligned}$$

The approximation error $E(\bar{t})$ is bounded by

$$\begin{aligned} E(\bar{t}) &\leq L_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n + 1 - \alpha_k(t_k)) N^{n - \alpha_k(t_k)} (n - \alpha_k(t_k))} (t_k - a_k)^{n+1 - \alpha_k(t_k)} \\ &\quad + |\alpha'_k(t_k)| L_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{\Gamma(2 - \alpha_k(t_k)) N^{1 - \alpha_k(t_k)} (1 - \alpha_k(t_k))} \\ &\quad \times \left[\left| \frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right| + \frac{1}{N} \right] (t_k - a_k)^{2 - \alpha_k(t_k)}. \end{aligned}$$

Proof. Taking into account relation (3.3) and Theorem 30, we only need to expand the term

$$\frac{\alpha'_k(t_k)}{\Gamma(2 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{1 - \alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k \left[\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - \tau) \right] d\tau. \quad (3.5)$$

Splitting the integral, and using the expansion formulas

$$\begin{aligned} (t_k - \tau)^{1 - \alpha_k(t_k)} &= (t_k - a_k)^{1 - \alpha_k(t_k)} \left(1 - \frac{\tau - a_k}{t_k - a_k} \right)^{1 - \alpha_k(t_k)} \\ &= (t_k - a_k)^{1 - \alpha_k(t_k)} \left[\sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p} + \bar{E}_1(\bar{t}) \right] \end{aligned}$$

with

$$\bar{E}_1(\bar{t}) = \sum_{p=N+1}^{\infty} \binom{1 - \alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p}$$

and

$$\begin{aligned} \ln(t_k - \tau) &= \ln(t_k - a_k) + \ln \left(1 - \frac{\tau - a_k}{t_k - a_k} \right) \\ &= \ln(t_k - a_k) - \sum_{r=1}^N \frac{1}{r} \frac{(\tau - a_k)^r}{(t_k - a_k)^r} - \bar{E}_2(\bar{t}) \end{aligned}$$

with

$$\bar{E}_2(\bar{t}) = \sum_{r=N+1}^{\infty} \frac{1}{r} \frac{(\tau - a_k)^r}{(t_k - a_k)^r},$$

we conclude that (3.5) is equivalent to

$$\begin{aligned}
 & \frac{\alpha'_k(t_k)}{\Gamma(2 - \alpha_k(t_k))} \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right) \int_{a_k}^{t_k} (t_k - \tau)^{1 - \alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right. \\
 & \quad \left. - \int_{a_k}^{t_k} (t_k - \tau)^{1 - \alpha_k(t_k)} \ln \left(1 - \frac{\tau - a_k}{t_k - a_k} \right) \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right] \\
 &= \frac{\alpha'_k(t_k)}{\Gamma(2 - \alpha_k(t_k))} \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right) \right. \\
 & \quad \times \int_{a_k}^{t_k} (t_k - a_k)^{1 - \alpha_k(t_k)} \sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p} \frac{\partial x}{\partial t_k} [\tau]_k d\tau \\
 & \quad \left. + \int_{a_k}^{t_k} (t_k - a_k)^{1 - \alpha_k(t_k)} \sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} (-1)^p \frac{(\tau - a_k)^p}{(t_k - a_k)^p} \sum_{r=1}^N \frac{1}{r} \frac{(\tau - a_k)^r}{(t_k - a_k)^r} \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right] \\
 & \quad + \frac{\alpha'_k(t_k)}{\Gamma(2 - \alpha_k(t_k))} \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right) \int_{a_k}^{t_k} (t_k - a_k)^{1 - \alpha_k(t_k)} \bar{E}_1(\bar{t}) \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right. \\
 & \quad \left. + \int_{a_k}^{t_k} (t_k - a_k)^{1 - \alpha_k(t_k)} \bar{E}_1(\bar{t}) \bar{E}_2(\bar{t}) \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right] \\
 &= \frac{\alpha'_k(t_k)(t_k - a_k)^{1 - \alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right) \sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} \frac{(-1)^p}{(t_k - a_k)^p} V_{n+p}(\bar{t}) \right. \\
 & \quad \left. + \sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} (-1)^p \sum_{r=1}^N \frac{1}{r(t_k - a_k)^{p+r}} V_{n+p+r}(\bar{t}) \right] + \frac{\alpha'_k(t_k)(t_k - a_k)^{1 - \alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \\
 & \quad \times \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(t_k - a_k) \right) \int_{a_k}^{t_k} \bar{E}_1(\bar{t}) \frac{\partial x}{\partial t_k} [\tau]_k d\tau + \int_{a_k}^{t_k} \bar{E}_1(\bar{t}) \bar{E}_2(\bar{t}) \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right].
 \end{aligned}$$

For the error analysis, we know from Theorem 30 that

$$\bar{E}_1(\bar{t}) \leq \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{N^{1 - \alpha_k(t_k)}(1 - \alpha_k(t_k))}.$$

Then,

$$\begin{aligned}
 & \left| \int_{a_k}^{t_k} (t_k - a_k)^{1 - \alpha_k(t_k)} \bar{E}_1(\bar{t}) \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right| \\
 & \leq L_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{N^{1 - \alpha_k(t_k)}(1 - \alpha_k(t_k))} (t_k - a_k)^{2 - \alpha_k(t_k)}.
 \end{aligned} \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}
 & \left| \int_{a_k}^{t_k} (t_k - a_k)^{1-\alpha_k(t_k)} \bar{E}_1(\bar{t}) \bar{E}_2(\bar{t}) \frac{\partial x}{\partial t_k} [\tau]_k d\tau \right| \\
 & \leq L_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{N^{1-\alpha_k(t_k)}(1 - \alpha_k(t_k))} (t_k - a_k)^{1-\alpha_k(t_k)} \sum_{r=N+1}^{\infty} \frac{1}{r(t_k - a_k)^r} \int_{a_k}^{t_k} (\tau - a_k)^r d\tau \\
 & = L_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{N^{1-\alpha_k(t_k)}(1 - \alpha_k(t_k))} (t_k - a_k)^{1-\alpha_k(t_k)} \sum_{r=N+1}^{\infty} \frac{t_k - a_k}{r(r+1)} \\
 & \leq L_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{N^{2-\alpha_k(t_k)}(1 - \alpha_k(t_k))} (t_k - a_k)^{2-\alpha_k(t_k)}.
 \end{aligned} \tag{3.7}$$

We get the desired result by combining inequalities (3.6) and (3.7). \square

Theorem 33. *Let $x \in C^{n+1}(\prod_{i=1}^m [a_i, b_i], \mathbb{R})$ with $n \in \mathbb{N}$. Then, for all $k \in \{1, \dots, m\}$ and for all $N \in \mathbb{N}$ such that $N \geq n$, we have*

$$\begin{aligned}
 {}^C \mathcal{D}_{a_k}^{\alpha_k(t_k)} x(\bar{t}) & = \sum_{p=1}^n A_p (t_k - a_k)^{p-\alpha_k(t_k)} \frac{\partial^p x}{\partial t_k^p} [t_k]_k + \sum_{p=n}^N B_p (t_k - a_k)^{n-p-\alpha_k(t_k)} V_p(\bar{t}) \\
 & + \frac{\alpha'_k(t_k) (t_k - a_k)^{1-\alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \left[(\Psi(2 - \alpha_k(t_k)) - \ln(t_k - a_k)) \sum_{p=0}^N \binom{1-\alpha_k(t_k)}{p} \frac{(-1)^p}{(t_k - a_k)^p} V_{n+p}(\bar{t}) \right. \\
 & \left. + \sum_{p=0}^N \binom{1-\alpha_k(t_k)}{p} (-1)^p \sum_{r=1}^N \frac{1}{r(t_k - a_k)^{p+r}} V_{n+p+r}(\bar{t}) \right] + E(\bar{t}).
 \end{aligned}$$

The approximation error $E(\bar{t})$ is bounded by

$$\begin{aligned}
 E(\bar{t}) & \leq L_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n + 1 - \alpha_k(t_k)) N^{n-\alpha_k(t_k)} (n - \alpha_k(t_k))} (t_k - a_k)^{n+1-\alpha_k(t_k)} \\
 & + |\alpha'_k(t_k)| L_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{\Gamma(2 - \alpha_k(t_k)) N^{1-\alpha_k(t_k)} (1 - \alpha_k(t_k))} \\
 & \times \left[|\Psi(2 - \alpha_k(t_k)) - \ln(t_k - a_k)| + \frac{1}{N} \right] (t_k - a_k)^{2-\alpha_k(t_k)}.
 \end{aligned}$$

Proof. Starting with relation (3.4), and integrating by parts the integral, we obtain that

$${}^C \mathcal{D}_{a_k}^{\alpha_k(t_k)} x(\bar{t}) = {}^C D_{a_k}^{\alpha_k(t_k)} x(\bar{t}) + \frac{\alpha'_k(t_k) \Psi(1 - \alpha_k(t_k))}{\Gamma(2 - \alpha_k(t_k))} \int_{a_k}^{t_k} (t_k - \tau)^{1-\alpha_k(t_k)} \frac{\partial x}{\partial t_k} [\tau]_k d\tau.$$

The rest of the proof is similar to the one of Theorem 32. \square

Remark 34. As particular cases of Theorems 30, 32 and 33, we obtain expansion formulas for ${}^C D_t^{\alpha(t)} x(t)$, ${}^C \mathcal{D}_t^{\alpha(t)} x(t)$ and ${}^C \mathbb{D}_t^{\alpha(t)} x(t)$.

With respect to the three right fractional operators of Definition 31, we set, for $p \in \mathbb{N}$,

$$\begin{aligned} C_p &= \frac{(-1)^p}{\Gamma(p+1-\alpha_k(t_k))} \left[1 + \sum_{l=n-p+1}^N \frac{\Gamma(\alpha_k(t_k) - n + l)}{\Gamma(\alpha_k(t_k) - p)(l - n + p)!} \right], \\ D_p &= \frac{-\Gamma(\alpha_k(t_k) - n + p)}{\Gamma(1 - \alpha_k(t_k))\Gamma(\alpha_k(t_k))(p - n)!}, \\ W_p(\bar{t}) &= \int_{t_k}^{b_k} (b_k - \tau)^{p-n} \frac{\partial x}{\partial t_k} [\tau]_k d\tau, \\ M_p(\bar{t}) &= \max_{\tau \in [t_k, b_k]} \left| \frac{\partial^p x}{\partial t_k^p} [\tau]_k \right|. \end{aligned}$$

The expansion formulas are given in Theorems 35, 36 and 37. We omit the proofs since they are similar to the corresponding left ones.

Theorem 35. Let $x \in C^{n+1}(\prod_{i=1}^m [a_i, b_i], \mathbb{R})$ with $n \in \mathbb{N}$. Then, for all $k \in \{1, \dots, m\}$ and for all $N \in \mathbb{N}$ such that $N \geq n$, we have

$${}^C \mathbb{D}_{b_k}^{\alpha_k(t_k)} x(\bar{t}) = \sum_{p=1}^n C_p (b_k - t_k)^{p-\alpha_k(t_k)} \frac{\partial^p x}{\partial t_k^p} [t_k]_k + \sum_{p=n}^N D_p (b_k - t_k)^{n-p-\alpha_k(t_k)} W_p(\bar{t}) + E(\bar{t}).$$

The approximation error $E(\bar{t})$ is bounded by

$$E(\bar{t}) \leq M_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n+1 - \alpha_k(t_k)) N^{n-\alpha_k(t_k)} (n - \alpha_k(t_k))} (b_k - t_k)^{n+1-\alpha_k(t_k)}.$$

Theorem 36. Let $x \in C^{n+1}(\prod_{i=1}^m [a_i, b_i], \mathbb{R})$ with $n \in \mathbb{N}$. Then, for all $k \in \{1, \dots, m\}$ and for all $N \in \mathbb{N}$ such that $N \geq n$, we have

$$\begin{aligned} {}^C D_{b_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^n C_p (b_k - t_k)^{p-\alpha_k(t_k)} \frac{\partial^p x}{\partial t_k^p} [t_k]_k + \sum_{p=n}^N D_p (b_k - t_k)^{n-p-\alpha_k(t_k)} W_p(\bar{t}) \\ &+ \frac{\alpha'_k(t_k)(b_k - t_k)^{1-\alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \left[\left(\frac{1}{1 - \alpha_k(t_k)} - \ln(b_k - t_k) \right) \sum_{p=0}^N \binom{1-\alpha_k(t_k)}{p} \frac{(-1)^p}{(b_k - t_k)^p} W_{n+p}(\bar{t}) \right. \\ &\left. + \sum_{p=0}^N \binom{1-\alpha_k(t_k)}{p} (-1)^p \sum_{r=1}^N \frac{1}{r(b_k - t_k)^{p+r}} W_{n+p+r}(\bar{t}) \right] + E(\bar{t}). \end{aligned}$$

The approximation error $E(\bar{t})$ is bounded by

$$\begin{aligned} E(\bar{t}) &\leq M_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n + 1 - \alpha_k(t_k)) N^{n - \alpha_k(t_k)} (n - \alpha_k(t_k))} (b_k - t_k)^{n+1 - \alpha_k(t_k)} \\ &\quad + |\alpha'_k(t_k)| M_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{\Gamma(2 - \alpha_k(t_k)) N^{1 - \alpha_k(t_k)} (1 - \alpha_k(t_k))} \\ &\quad \times \left[\left| \frac{1}{1 - \alpha_k(t_k)} - \ln(b_k - t_k) \right| + \frac{1}{N} \right] (b_k - t_k)^{2 - \alpha_k(t_k)}. \end{aligned}$$

Theorem 37. Let $x \in C^{n+1}(\prod_{i=1}^m [a_i, b_i], \mathbb{R})$ with $n \in \mathbb{N}$. Then, for all $k \in \{1, \dots, m\}$ and for all $N \in \mathbb{N}$ such that $N \geq n$, we have

$$\begin{aligned} {}^C D_{b_k}^{\alpha_k(t_k)} x(\bar{t}) &= \sum_{p=1}^n C_p (b_k - t_k)^{p - \alpha_k(t_k)} \frac{\partial^p x}{\partial t_k^p} [t_k]_k + \sum_{p=n}^N D_p (b_k - t_k)^{n - p - \alpha_k(t_k)} W_p(\bar{t}) \\ &\quad + \frac{\alpha'_k(t_k) (b_k - t_k)^{1 - \alpha_k(t_k)}}{\Gamma(2 - \alpha_k(t_k))} \left[(\Psi(2 - \alpha_k(t_k)) - \ln(b_k - t_k)) \sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} \frac{(-1)^p}{(b_k - t_k)^p} W_{n+p}(\bar{t}) \right. \\ &\quad \left. + \sum_{p=0}^N \binom{1 - \alpha_k(t_k)}{p} (-1)^p \sum_{r=1}^N \frac{1}{r (b_k - t_k)^{p+r}} W_{n+p+r}(\bar{t}) \right] + E(\bar{t}). \end{aligned}$$

The approximation error $E(\bar{t})$ is bounded by

$$\begin{aligned} E(\bar{t}) &\leq M_{n+1}(\bar{t}) \frac{\exp((n - \alpha_k(t_k))^2 + n - \alpha_k(t_k))}{\Gamma(n + 1 - \alpha_k(t_k)) N^{n - \alpha_k(t_k)} (n - \alpha_k(t_k))} (b_k - t_k)^{n+1 - \alpha_k(t_k)} \\ &\quad + |\alpha'_k(t_k)| M_1(\bar{t}) \frac{\exp((1 - \alpha_k(t_k))^2 + 1 - \alpha_k(t_k))}{\Gamma(2 - \alpha_k(t_k)) N^{1 - \alpha_k(t_k)} (1 - \alpha_k(t_k))} \\ &\quad \times \left[\Psi(2 - \alpha_k(t_k)) - \ln(b_k - t_k) \right] + \frac{1}{N} (b_k - t_k)^{2 - \alpha_k(t_k)}. \end{aligned}$$

3.3 Example

To test the accuracy of the proposed method, we compare the fractional derivative of a concrete given function with some numerical approximations of it. For $t \in [0, 1]$, let $x(t) = t^2$ be the test function. For the order of the fractional derivatives we consider two cases:

$$\alpha(t) = \frac{50t + 49}{100} \quad \text{and} \quad \beta(t) = \frac{t + 5}{10}.$$

We consider the approximations given in Theorems 30, 32 and 33, with a fixed $n = 1$ and $N \in \{2, 4, 6\}$. The error of approximating $f(t)$ by $\tilde{f}(t)$ is measured by $|f(t) - \tilde{f}(t)|$. See Figures 3.2–3.7.

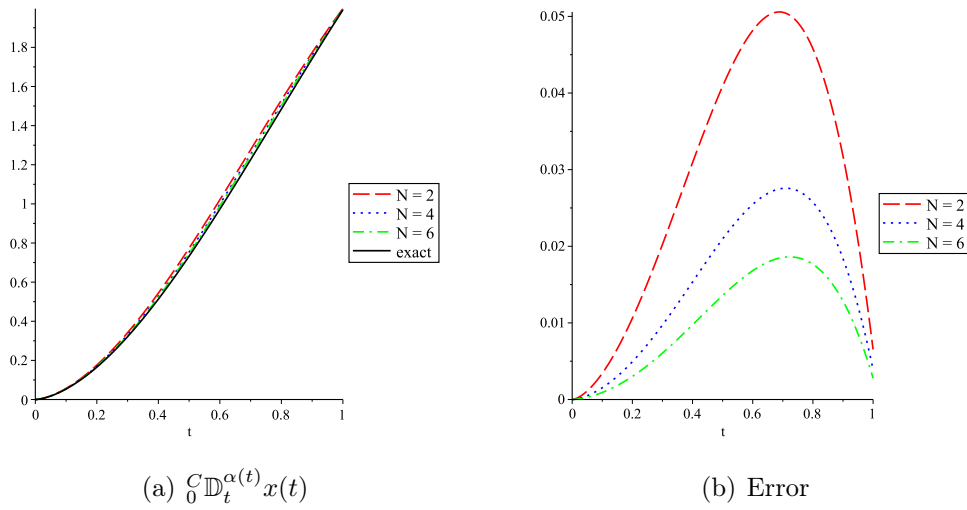


Figure 3.2: Type III left Caputo derivative of order $\alpha(t)$ for the example of Section 3.3—analytic versus numerical approximations obtained from Theorem 30.

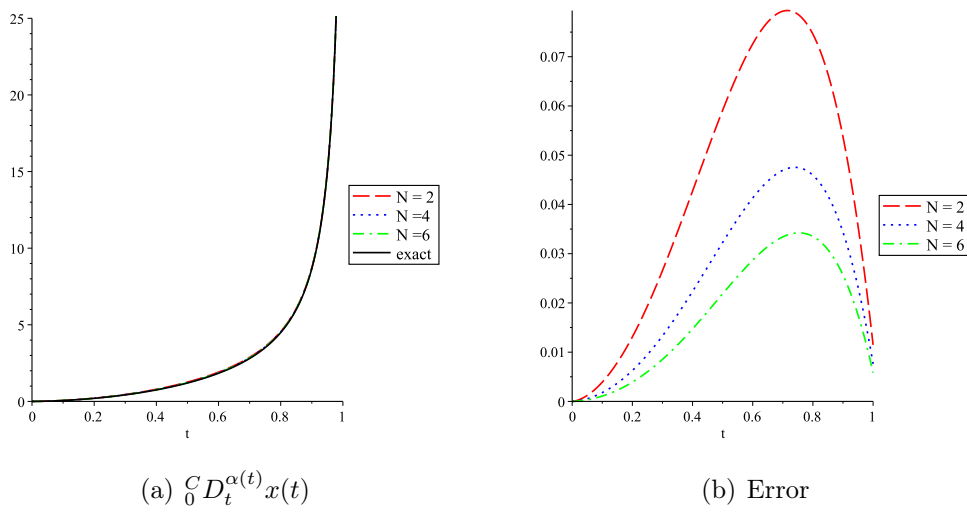


Figure 3.3: Type I left Caputo derivative of order $\alpha(t)$ for the example of Section 3.3—analytic versus numerical approximations obtained from Theorem 32.

3.4 Applications

In this section we apply the proposed technique to some concrete fractional differential equations of physical relevance.

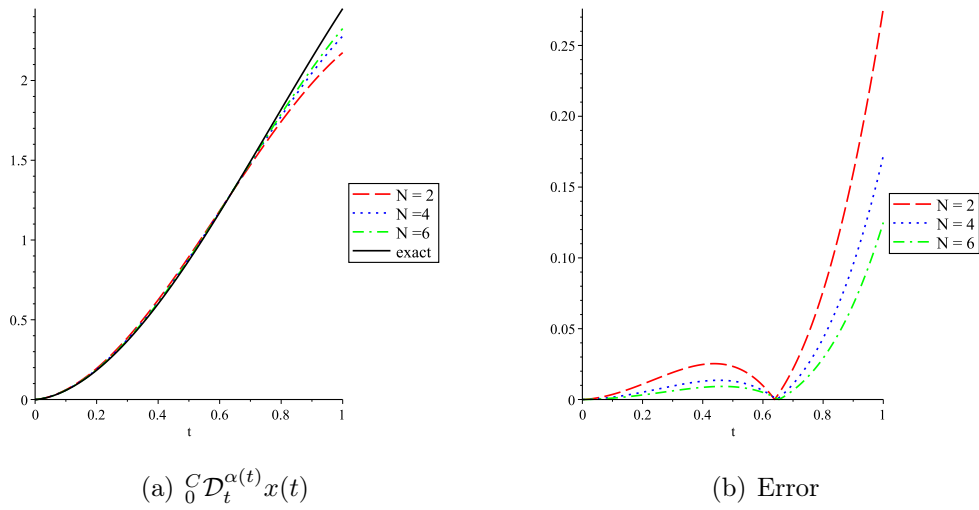


Figure 3.4: Type II left Caputo derivative of order $\alpha(t)$ for the example of Section 3.3—analytic versus numerical approximations obtained from Theorem 33.

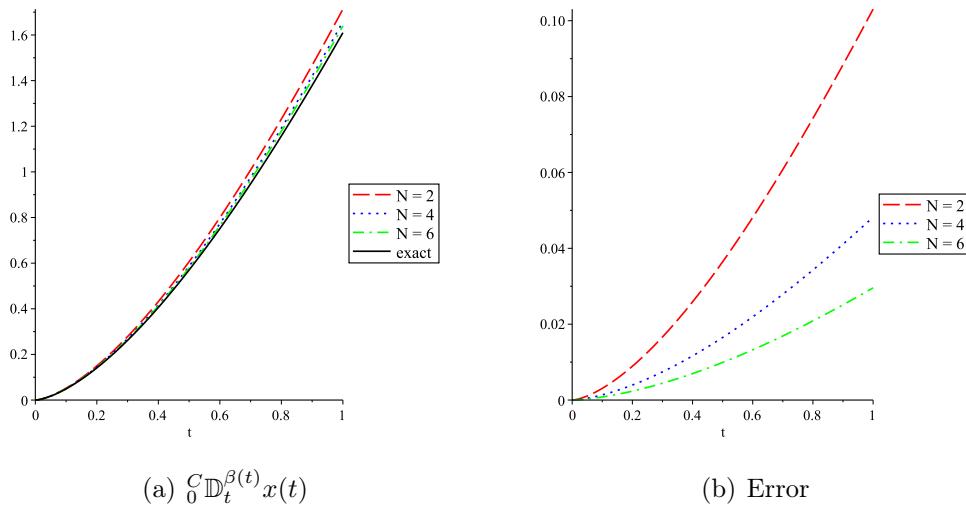


Figure 3.5: Type III left Caputo derivative of order $\beta(t)$ for the example of Section 3.3—analytic versus numerical approximations obtained from Theorem 30.

3.4.1 A time-fractional diffusion equation

We extend the one-dimensional time-fractional diffusion equation [54] to the variable order case. Consider $u = u(x, t)$ with domain $[0, 1]^2$. The partial fractional differential

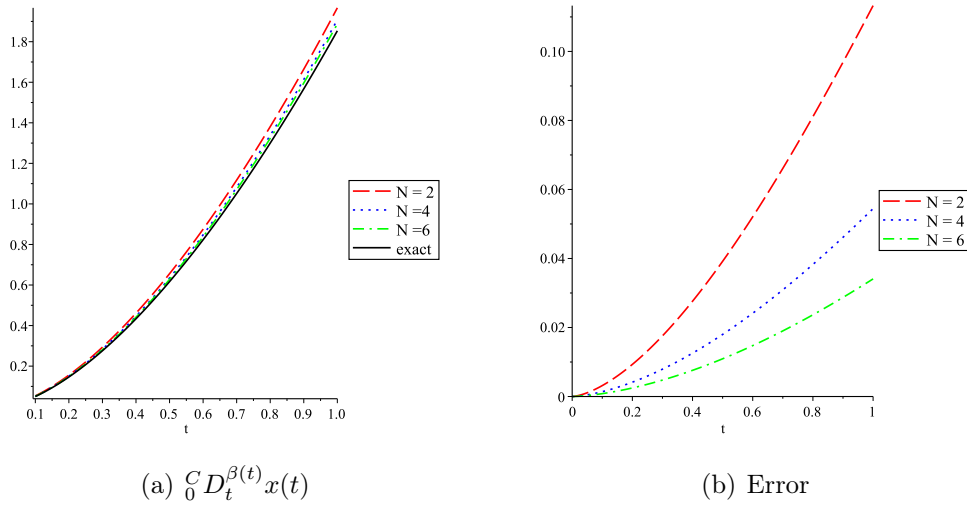


Figure 3.6: Type I left Caputo derivative of order $\beta(t)$ for the example of Section 3.3—analytic versus numerical approximations obtained from Theorem 32.

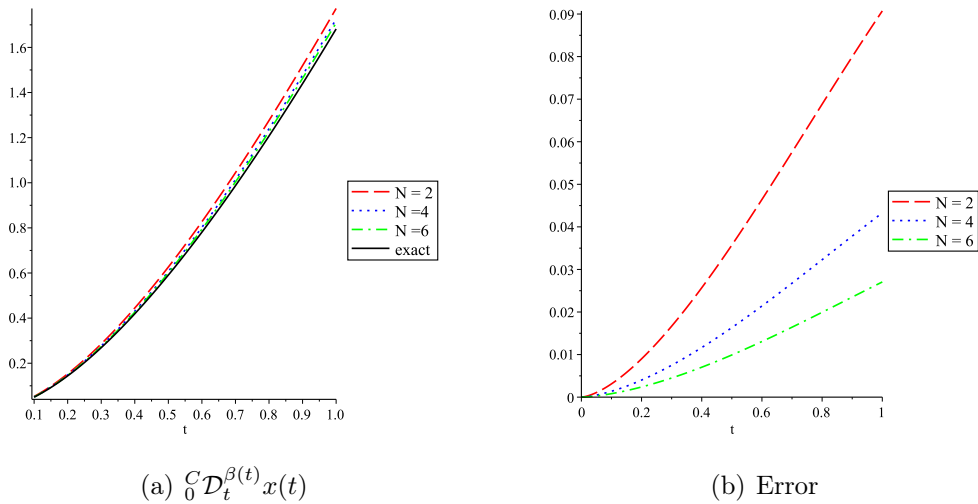


Figure 3.7: Type II left Caputo derivative of order $\beta(t)$ for the example of Section 3.3—analytic versus numerical approximations obtained from Theorem 33.

equation of order $\alpha(t)$ is the following:

$${}^C_0 \mathbb{D}_t^{\alpha(t)} u(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = f(x, t) \quad \text{for } x \in [0, 1], \quad t \in [0, 1], \quad (3.8)$$

subject to the boundary conditions

$$u(x, 0) = g(x), \quad \text{for } x \in (0, 1), \quad (3.9)$$

and

$$u(0, t) = u(1, t) = 0, \quad \text{for } t \in [0, 1]. \quad (3.10)$$

We mention that when $\alpha(t) \equiv 1$, one obtains the classical diffusion equation, and when $\alpha(t) \equiv 0$ one gets the classical Helmholtz elliptic equation. Using Lemma 27, it is easy to check that

$$u(x, t) = t^2 \sin(2\pi x)$$

is a solution to (3.8)–(3.10) with

$$f(x, t) = \left(\frac{2}{\Gamma(3 - \alpha(t))} t^{2-\alpha(t)} + 4\pi^2 t^2 \right) \sin(2\pi x)$$

and

$$g(x) = 0$$

(compare with Example 1 in [54]). The numerical procedure is the following: replace ${}_0^C \mathbb{D}_t^{\alpha(t)} u$ with the approximation given in Theorem 30, taking $n = 1$ and an arbitrary $N \geq 1$, that is,

$${}_0^C \mathbb{D}_t^{\alpha(t)} u(x, t) \approx At^{1-\alpha(t)} \frac{\partial u}{\partial t}(x, t) + \sum_{p=1}^N B_p t^{1-p-\alpha(t)} V_p(x, t)$$

with

$$A = \frac{1}{\Gamma(2 - \alpha(t))} \left[1 + \sum_{l=1}^N \frac{\Gamma(\alpha(t) - 1 + l)}{\Gamma(\alpha(t) - 1) l!} \right],$$

$$B_p = \frac{\Gamma(\alpha(t) - 1 + p)}{\Gamma(1 - \alpha(t)) \Gamma(\alpha(t)) (p - 1)!},$$

$$V_p(x, t) = \int_0^t \tau^{p-1} \frac{\partial u}{\partial t}(x, \tau) d\tau.$$

Then, the initial fractional problem (3.8)–(3.10) is approximated by the following system of second-order partial differential equations:

$$At^{1-\alpha(t)} \frac{\partial u}{\partial t}(x, t) + \sum_{p=1}^N B_p t^{1-p-\alpha(t)} V_p(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = f(x, t)$$

and

$$\frac{\partial V_p}{\partial t}(x, t) = t^{p-1} \frac{\partial u}{\partial t}(x, t), \quad p = 1, \dots, N,$$

for $x \in [0, 1]$ and for $t \in [0, 1]$, subject to the boundary conditions

$$u(x, 0) = 0, \quad \text{for } x \in (0, 1),$$

$$u(0, t) = u(1, t) = 0, \quad \text{for } t \in [0, 1],$$

and

$$V_p(x, 0) = 0, \quad \text{for } x \in [0, 1], \quad p = 1, \dots, N.$$

3.4.2 A fractional partial differential equation in fluid mechanics

We now apply our approximation techniques to the following one-dimensional linear inhomogeneous fractional Burgers' equation of variable order (see [66, Example 5.2]):

$${}_0^C \mathbb{D}_t^{\alpha(t)} u(x, t) + \frac{\partial u}{\partial x}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 2x - 2, \quad \text{for } x \in [0, 1], \quad t \in [0, 1], \quad (3.11)$$

subject to the boundary condition

$$u(x, 0) = x^2, \quad \text{for } x \in (0, 1). \quad (3.12)$$

Here,

$$F(x, t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 2x - 2$$

is the external force field. Burgers' equation is used to model gas dynamics, traffic flow, turbulence, fluid mechanics, etc. The exact solution is

$$u(x, t) = x^2 + t^2.$$

The fractional problem (3.11)–(3.12) can be approximated by

$$At^{1-\alpha(t)} \frac{\partial u}{\partial t}(x, t) + \sum_{p=1}^N B_p t^{1-p-\alpha(t)} V_p(x, t) + \frac{\partial u}{\partial x}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 2x - 2$$

with A , B_p and V_p , $p \in \{1, \dots, N\}$, as in Section 3.4.1. The approximation error can be decreased as much as desired by increasing the value of N .

Chapter 4

Fractional calculus of variations

In this chapter, we consider general fractional problems of the calculus of variations, where the Lagrangian depends on a combined Caputo fractional derivative of variable fractional order ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)}$ given as a combination of the left and the right Caputo fractional derivatives of orders, respectively, $\alpha(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$. More specifically, here we study some problems of the calculus of variations with integrands depending on the independent variable t , an arbitrary function x and a fractional derivative ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x$ defined by

$${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t) = \gamma_1 {}^C D_a^{\alpha(\cdot)} x(t) + \gamma_2 {}^C D_b^{\beta(\cdot)} x(t)$$

where $\gamma = (\gamma_1, \gamma_2) \in [0, 1]^2$, with γ_1 and γ_2 not both zero.

Starting from the fundamental variational problem, we investigate different types of variational problems: problems with time delay or with higher-order derivatives, isoperimetric problems, problems with holonomic constraints and problems of Herglotz and those depending on combined Caputo fractional derivatives of variable order. Our variational problems are known as free-time problems because, in general, we impose a natural boundary condition at the initial time $t = a$, but we consider the endpoint b of the integral and the terminal state $x(b)$ to be free (variable). The main results provide necessary optimality conditions of Euler–Lagrange type, described by fractional differential equations of variable order, and different transversality optimality conditions.

Our main contributions are to consider the Lagrangian containing a fractional operator where the order is not a constant, and may depend on time. Moreover, we do not only assume that $x(b)$ is free, but the endpoint b is also free.

In Section 4.1, we introduce the combined Caputo fractional derivative of variable order

and provide the necessary concepts and results needed in the sequel (Section 4.1.1). We deduce two formulas of integration by parts that involve the combined Caputo fractional derivative of higher-order (Section 4.1.2).

The fractional variational problems under our consideration are formulated in terms of the fractional derivative ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)}$. We discuss the fundamental concepts of a variational calculus such as the Euler–Lagrange equations and transversality conditions (Section 4.2), variational problems involving higher-order derivatives (Section 4.3), variational problems with time delay (Section 4.4), isoperimetric problems (Section 4.5), problems with holonomic constraints (Section 4.6) and the last Section 4.7 investigates fractional variational Herglotz problems.

Some illustrative examples are presented for all considered variational problems.

The results of this chapter first appeared in [103, 105–107].

4.1 Introduction

In this section, we recall the fundamental definition of the combined Caputo fractional derivative presented in Section 1.3.3 (see Definition 15), and generalize it for fractional variable order. In the end, we prove an integration by parts formula, involving the higher-order Caputo fractional derivative of variable order.

4.1.1 Combined operators of variable order

Motivated by the combined fractional Caputo derivative, we propose the following definitions about combined variable order fractional calculus.

Let $\alpha, \beta : [a, b]^2 \rightarrow (0, 1)$ be two functions and $\gamma = (\gamma_1, \gamma_2) \in [0, 1]^2$ a vector, with γ_1 and γ_2 not both zero.

Definition 32 (Combined Riemann–Liouville fractional derivative). *The combined Riemann–Liouville fractional derivative of variable order, denoted by $D_\gamma^{\alpha(\cdot), \beta(\cdot)}$, is defined by*

$$D_\gamma^{\alpha(\cdot), \beta(\cdot)} = \gamma_1 {}_a D_t^{\alpha(\cdot)} + \gamma_2 {}_t D_b^{\beta(\cdot)},$$

acting on $x \in C([a, b]; \mathbb{R})$ in the following way:

$$D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t) = \gamma_1 {}_a D_t^{\alpha(\cdot)} x(t) + \gamma_2 {}_t D_b^{\beta(\cdot)} x(t).$$

Definition 33 (Combined Caputo fractional derivative). *The combined Caputo fractional derivative operator of variable order, denoted by ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)}$, is defined by*

$${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} = \gamma_1 {}^C D_a^{\alpha(\cdot)} + \gamma_2 {}^C D_b^{\beta(\cdot)},$$

acting on $x \in C([a, b]; \mathbb{R})$ in the following way:

$${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t) = \gamma_1 {}^C D_a^{\alpha(\cdot)} x(t) + \gamma_2 {}^C D_b^{\beta(\cdot)} x(t).$$

In the sequel, we need the auxiliary notation of the dual fractional derivative, defined by

$$D_{\bar{\gamma}}^{\beta(\cdot), \alpha(\cdot)} = \gamma_2 {}_a D_t^{\beta(\cdot)} + \gamma_1 {}_t D_T^{\alpha(\cdot)}, \quad (4.1)$$

where $\bar{\gamma} = (\gamma_2, \gamma_1)$.

To generalize the concept of the combined fractional derivative to higher- orders, we need to review some definitions of higher-order operators.

Let $n \in \mathbb{N}$ and $x : [a, b] \rightarrow \mathbb{R}$ be a function of class C^n . The fractional order is a continuous function of two variables, $\alpha_n : [a, b]^2 \rightarrow (n - 1, n)$.

Definition 34 (Higher-order Riemann–Liouville fractional integrals). *The left and right Riemann–Liouville fractional integrals of order $\alpha_n(\cdot, \cdot)$ are defined respectively by*

$${}_a I_t^{\alpha_n(\cdot)} x(t) = \int_a^t \frac{1}{\Gamma(\alpha_n(t, \tau))} (t - \tau)^{\alpha_n(t, \tau) - 1} x(\tau) d\tau$$

and

$${}_t I_b^{\alpha_n(\cdot)} x(t) = \int_t^b \frac{1}{\Gamma(\alpha_n(\tau, t))} (\tau - t)^{\alpha_n(\tau, t) - 1} x(\tau) d\tau.$$

With this definition of Riemann–Liouville fractional integrals of variable order, and considering $n = 1$, in Appendix A.1 we implemented, in the open source software package Chebfun [109], two functions `leftFI(x, alpha, a)` and `rightFI(x, alpha, b)` that approximate, respectively, the Riemann–Liouville fractional integrals ${}_a I_t^{\alpha_n(\cdot)} x(t)$ and ${}_t I_b^{\alpha_n(\cdot)} x(t)$. With these two functions we present also, in Appendix A.1, an illustrative example where we determine computational approximations to the fractionals integrals for a specific power function of the form $x(t) = t^\gamma$ (see Example 4 in Appendix A.1). For this numerical computations, we have used MATLAB [55].

Definition 35 (Higher-order Riemann–Liouville fractional derivatives). *The left and right Riemann–Liouville fractional derivatives of order $\alpha_n(\cdot, \cdot)$ are defined by*

$${}_a D_t^{\alpha_n(\cdot, \cdot)} x(t) = \frac{d^n}{dt^n} \int_a^t \frac{1}{\Gamma(n - \alpha_n(t, \tau))} (t - \tau)^{n-1-\alpha_n(t, \tau)} x(\tau) d\tau$$

and

$${}_t D_b^{\alpha_n(\cdot, \cdot)} x(t) = (-1)^n \frac{d^n}{dt^n} \int_t^b \frac{1}{\Gamma(n - \alpha_n(\tau, t))} (\tau - t)^{n-1-\alpha_n(\tau, t)} x(\tau) d\tau,$$

respectively.

Definition 36 (Higher-order Caputo fractional derivatives). *The left and right Caputo fractional derivatives of order $\alpha_n(\cdot, \cdot)$ are defined by*

$${}_a^C D_t^{\alpha_n(\cdot, \cdot)} x(t) = \int_a^t \frac{1}{\Gamma(n - \alpha_n(t, \tau))} (t - \tau)^{n-1-\alpha_n(t, \tau)} x^{(n)}(\tau) d\tau \quad (4.2)$$

and

$${}_t^C D_b^{\alpha_n(\cdot, \cdot)} x(t) = (-1)^n \int_t^b \frac{1}{\Gamma(n - \alpha_n(\tau, t))} (\tau - t)^{n-1-\alpha_n(\tau, t)} x^{(n)}(\tau) d\tau, \quad (4.3)$$

respectively.

Remark 38. *Definitions 35 and 36, for the particular case of order between 0 and 1, can be found in [59]. They seem to be new for the higher-order case.*

Considering Definition 36, in Appendix A.2 we turn to Chebfun again to implement two new functions `leftCaputo(x, alpha, a, n)` and `rightCaputo(x, alpha, b, n)` that approximate, respectively, the higher-order Caputo fractional derivatives ${}_a^C D_t^{\alpha_n(\cdot, \cdot)} x(t)$ and ${}_t^C D_b^{\alpha_n(\cdot, \cdot)} x(t)$. With these two functions, we present, also in Appendix A.2, an illustrative example where we study approximations to the Caputo fractional derivatives for a specific power function of the form $x(t) = t^\gamma$ (see Example 5 in Appendix A.2).

Remark 39. *From Definition 34, it follows that*

$${}_a D_t^{\alpha_n(\cdot, \cdot)} x(t) = \frac{d^n}{dt^n} {}_a I_t^{n-\alpha_n(\cdot, \cdot)} x(t), \quad {}_t D_b^{\alpha_n(\cdot, \cdot)} x(t) = (-1)^n \frac{d^n}{dt^n} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} x(t)$$

and

$${}_a^C D_t^{\alpha_n(\cdot, \cdot)} x(t) = {}_a I_t^{n-\alpha_n(\cdot, \cdot)} \frac{d^n}{dt^n} x(t), \quad {}_t^C D_b^{\alpha_n(\cdot, \cdot)} x(t) = (-1)^n {}_t I_b^{n-\alpha_n(\cdot, \cdot)} \frac{d^n}{dt^n} x(t).$$

In Lemma 40, we obtain higher-order Caputo fractional derivatives of a power function. For that, we assume that the fractional order depends only on the first variable: $\alpha_n(t, \tau) := \bar{\alpha}_n(t)$, where $\bar{\alpha}_n : [a, b] \rightarrow (n - 1, n)$ is a given function.

Lemma 40. *Let $x(t) = (t - a)^\gamma$ with $\gamma > n - 1$. Then,*

$${}^C D_t^{\bar{\alpha}_n(t)} x(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \bar{\alpha}_n(t) + 1)} (t - a)^{\gamma - \bar{\alpha}_n(t)}.$$

Proof. As $x(t) = (t - a)^\gamma$, if we differentiate it n times, we obtain

$$x^{(n)}(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - n + 1)} (t - a)^{\gamma - n}.$$

Using Definition 36 of the left Caputo fractional derivative, we get

$$\begin{aligned} {}^C D_t^{\bar{\alpha}_n(t)} x(t) &= \int_a^t \frac{1}{\Gamma(n - \bar{\alpha}_n(t))} (t - \tau)^{n-1-\bar{\alpha}_n(t)} x^{(n)}(\tau) d\tau \\ &= \int_a^t \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - n + 1)\Gamma(n - \bar{\alpha}_n(t))} (t - \tau)^{n-1-\bar{\alpha}_n(t)} (\tau - a)^{\gamma - n} d\tau. \end{aligned}$$

Now, we proceed with the change of variables $\tau - a = s(t - a)$. Using the Beta function $B(\cdot, \cdot)$, we obtain that

$$\begin{aligned} {}^C D_t^{\bar{\alpha}_n(t)} x(t) &= \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - n + 1)\Gamma(n - \bar{\alpha}_n(t))} \int_0^1 (1 - s)^{n-1-\bar{\alpha}_n(t)} s^{\gamma - n} (t - a)^{\gamma - \bar{\alpha}_n(t)} ds \\ &= \frac{\Gamma(\gamma + 1)(t - a)^{\gamma - \bar{\alpha}_n(t)}}{\Gamma(\gamma - n + 1)\Gamma(n - \bar{\alpha}_n(t))} B(\gamma - n + 1, n - \bar{\alpha}_n(t)). \\ &= \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \bar{\alpha}_n(t) + 1)} (t - a)^{\gamma - \bar{\alpha}_n(t)}. \end{aligned}$$

The proof is complete. □

Considering the higher-order left Caputo fractional derivative's formula of a power function of the form $x(t) = (t - a)^\gamma$, deduced before, in Appendix A.2 we determine the left Caputo fractional derivative of the particular function $x(t) = t^4$ for several values of t and compare them with the approximated values obtained by the `Chebfun` function `leftCaputo(x,alpha,a,n)` (see Example 7 in Appendix A.2).

For our next result, we assume that the fractional order depends only on the second variable: $\alpha_n(\tau, t) := \bar{\alpha}_n(t)$, where $\bar{\alpha}_n : [a, b] \rightarrow (n - 1, n)$ is a given function. The proof is similar to that of Lemma 40, and so we omit it here.

Lemma 41. *Let $x(t) = (b - t)^\gamma$ with $\gamma > n - 1$. Then,*

$${}^C D_b^{\bar{\alpha}_n(t)} x(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \bar{\alpha}_n(t) + 1)} (b - t)^{\gamma - \bar{\alpha}_n(t)}.$$

The next step is to consider a linear combination of the previous fractional derivatives to define the combined fractional operators for higher-order.

Let $\alpha_n, \beta_n : [a, b]^2 \rightarrow (n - 1, n)$ be two variable fractional orders, $\gamma^n = (\gamma_1^n, \gamma_2^n) \in [0, 1]^2$ a vector, with γ_1^n and γ_2^n not both zero, and $x \in C^n([a, b]; \mathbb{R})$ a function.

Definition 37 (Higher-order combined Riemann–Liouville fractional derivative). *The higher-order combined Riemann–Liouville fractional derivative is defined by*

$$D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} = \gamma_1^n {}_a D_t^{\alpha_n(\cdot, \cdot)} + \gamma_2^n {}_t D_b^{\beta_n(\cdot, \cdot)},$$

acting on $x \in C^n([a, b]; \mathbb{R})$ in the following way:

$$D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t) = \gamma_1^n {}_a D_t^{\alpha_n(\cdot, \cdot)} x(t) + \gamma_2^n {}_t D_b^{\beta_n(\cdot, \cdot)} x(t).$$

In our work, we use both Riemann–Liouville and Caputo derivatives definitions. The emphasis is, however, in Caputo fractional derivatives.

Definition 38 (Higher-order combined Caputo fractional derivative). *The higher-order combined Caputo fractional derivative of x at t is defined by*

$${}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t) = \gamma_1^n {}_a^C D_t^{\alpha_n(\cdot, \cdot)} x(t) + \gamma_2^n {}_t^C D_b^{\beta_n(\cdot, \cdot)} x(t).$$

Similarly, in the sequel of this work, we need the auxiliary notation of the dual fractional derivative:

$$D_{\bar{\gamma}^i}^{\beta_i(\cdot, \cdot), \alpha_i(\cdot, \cdot)} = \gamma_2^i {}_a D_t^{\beta_i(\cdot, \cdot)} + \gamma_1^i {}_t D_T^{\alpha_i(\cdot, \cdot)}, \quad (4.4)$$

where $\bar{\gamma}^i = (\gamma_2^i, \gamma_1^i)$.

Some computational aspects about the combined Caputo fractional derivative of variable order, are also discussed in Appendix A.3, using the software package `Chebfun`. For that, we developed the new function `combinedCaputo(x, alpha, beta, gamma1, gamma2, a, b, n)` and obtained approximated values for a particular power function (See Example 8 in Appendix A.3).

4.1.2 Generalized fractional integration by parts

When dealing with variational problems, one key property is integration by parts. Formulas of integration by parts have an important role in the proof of Euler–Lagrange conditions. In the following theorem, such formulas are proved for integrals involving higher-order Caputo fractional derivatives of variable order.

Let $n \in \mathbb{N}$ and $x, y \in C^n([a, b]; \mathbb{R})$ be two functions. The fractional order is a continuous function of two variables, $\alpha_n : [a, b]^2 \rightarrow (n - 1, n)$.

Theorem 42 (Integration by parts). *The higher-order Caputo fractional derivatives of variable order satisfy the integration by parts formulas*

$$\int_a^b y(t) {}^C D_t^{\alpha_n(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_t D_b^{\alpha_n(\cdot, \cdot)} y(t) dt + \left[\sum_{k=0}^{n-1} (-1)^k x^{(n-1-k)}(t) \frac{d^k}{dt^k} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}$$

and

$$\int_a^b y(t) {}^C D_t^{\alpha_n(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_a D_t^{\alpha_n(\cdot, \cdot)} y(t) dt + \left[\sum_{k=0}^{n-1} (-1)^{n+k} x^{(n-1-k)}(t) \frac{d^k}{dt^k} {}_a I_t^{n-\alpha_n(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}.$$

Proof. Considering the Definition 36 of left Caputo fractional derivatives of order $\alpha_n(\cdot, \cdot)$, we obtain

$$\int_a^b y(t) {}^C D_t^{\alpha_n(\cdot, \cdot)} x(t) dt = \int_a^b \int_a^t y(t) \frac{1}{\Gamma(n - \alpha_n(t, \tau))} (t - \tau)^{n-1-\alpha_n(t, \tau)} x^{(n)}(\tau) d\tau dt.$$

Using Dirichelt's Formula, we rewrite it as

$$\begin{aligned} \int_a^b \int_t^b y(\tau) \frac{(\tau - t)^{n-1-\alpha_n(\tau, t)}}{\Gamma(n - \alpha_n(\tau, t))} x^{(n)}(t) d\tau dt &= \int_a^b x^{(n)}(t) \int_t^b \frac{(\tau - t)^{n-1-\alpha_n(\tau, t)}}{\Gamma(n - \alpha_n(\tau, t))} y(\tau) d\tau dt \\ &= \int_a^b x^{(n)}(t) {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) dt. \end{aligned} \quad (4.5)$$

Using the (usual) integrating by parts formula, we get that (4.5) is equal to

$$- \int_a^b x^{(n-1)}(t) \frac{d}{dt} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) dt + \left[x^{(n-1)}(t) {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}.$$

Integrating by parts again, we obtain

$$\int_a^b x^{(n-2)}(t) \frac{d^2}{dt^2} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) dt + \left[x^{(n-1)}(t) {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) - x^{(n-2)}(t) \frac{d}{dt} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}.$$

If we repeat this process $n - 2$ times more, we get

$$\begin{aligned} & \int_a^b x(t) (-1)^n \frac{d^n}{dt^n} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) dt + \left[\sum_{k=0}^{n-1} (-1)^k x^{(n-1-k)}(t) \frac{d^k}{dt^k} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b} \\ &= \int_a^b x(t) {}_t D_b^{\alpha_n(\cdot, \cdot)} y(t) dt + \left[\sum_{k=0}^{n-1} (-1)^k x^{(n-1-k)}(t) \frac{d^k}{dt^k} {}_t I_b^{n-\alpha_n(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}. \end{aligned}$$

The second relation of the theorem for the right Caputo fractional derivative of order $\alpha_n(\cdot, \cdot)$, follows directly from the first by Caputo–Torres duality [23]. \square

Remark 43. If we consider in Theorem 42 the particular case when $n = 1$, then the fractional integration by parts formulas take the well-known forms presented in Theorem 13.

Remark 44. If x is such that $x^{(i)}(a) = x^{(i)}(b) = 0$, for all $i = 0, \dots, n-1$, then the higher-order formulas of fractional integration by parts given by Theorem 42 can be rewritten as

$$\int_a^b y(t) {}_a^C D_t^{\alpha_n(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_t D_b^{\alpha_n(\cdot, \cdot)} y(t) dt$$

and

$$\int_a^b y(t) {}_t^C D_b^{\alpha_n(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_a D_t^{\alpha_n(\cdot, \cdot)} y(t) dt.$$

4.2 Fundamental variational problem

This section is dedicated to establish necessary optimality conditions for variational problems with a Lagrangian depending on a combined Caputo derivative of variable fractional order. The problem is then stated in Section 4.2.1, consisting of the variational functional

$$\mathcal{J}(x, T) = \int_a^T L(t, x(t), {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)) dt + \phi(T, x(T)),$$

where ${}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)$ stands for the combined Caputo fractional derivative of variable fractional order (Definition 33), subject to the boundary condition $x(a) = x_a$.

In this problem, we do not only assume that $x(T)$ is free, but the endpoint T is also variable. Therefore, we are interested in finding an optimal curve $x(\cdot)$ and also the endpoint of the variational integral, denoted in the sequel by T .

We begin by proving in Section 4.2.1 necessary optimality conditions that every extremizer (x, T) must satisfy. The main results of this section provide necessary optimality conditions of Euler–Lagrange type, described by fractional differential equations of variable order, and different transversality optimality conditions (Theorems 45 and 46). Some particular cases of interest are considered in Section 4.2.2. We end with two illustrative examples (Section 4.2.3).

4.2.1 Necessary optimality conditions

Let $\alpha, \beta : [a, b]^2 \rightarrow (0, 1)$ be two functions. Let D denote the set

$$D = \{(x, t) \in C^1([a, b]; \mathbb{R}) \times [a, b] : {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \text{ exists and is continuous on } [a, b]\}, \quad (4.6)$$

endowed with the norm $\|(\cdot, \cdot)\|$ defined on the linear space $C^1([a, b]; \mathbb{R}) \times \mathbb{R}$ by

$$\|(x, t)\| := \max_{a \leq t \leq b} |x(t)| + \max_{a \leq t \leq b} \left| {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \right| + |t|.$$

Definition 39. We say that $(x^*, T^*) \in D$ is a local minimizer to the functional $\mathcal{J} : D \rightarrow \mathbb{R}$ if there exists some $\epsilon > 0$ such that

$$\forall (x, T) \in D : \quad \|(x^*, T^*) - (x, T)\| < \epsilon \Rightarrow \mathcal{J}(x^*, T^*) \leq \mathcal{J}(x, T).$$

Along the work, we denote by $\partial_i z$, $i \in \{1, 2, 3\}$, the partial derivative of a function $z : \mathbb{R}^3 \rightarrow \mathbb{R}$ with respect to its i th argument, and by L a differentiable Lagrangian $L : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

Consider the following problem of the calculus of variations:

Problem 1. Find the local minimizers of the functional $\mathcal{J} : D \rightarrow \mathbb{R}$, with

$$\mathcal{J}(x, T) = \int_a^T L(t, x(t), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)) dt + \phi(T, x(T)), \quad (4.7)$$

over all $(x, T) \in D$ satisfying the boundary condition $x(a) = x_a$, for a fixed $x_a \in \mathbb{R}$. The terminal time T and terminal state $x(T)$ are free.

The terminal cost function $\phi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is at least of class C^1 .

For simplicity of notation, we introduce the operator $[\cdot]_\gamma^{\alpha,\beta}$ defined by

$$[x]_\gamma^{\alpha,\beta}(t) = (t, x(t), {}^C D_\gamma^{\alpha(\cdot),\beta(\cdot)} x(t)). \quad (4.8)$$

With the new notation, one can write (4.7) simply as

$$\mathcal{J}(x, T) = \int_a^T L[x]_\gamma^{\alpha,\beta}(t) dt + \phi(T, x(T)).$$

The next theorem gives fractional necessary optimality conditions to Problem 1.

Theorem 45 (Necessary optimality conditions for Problem 1). *Suppose that (x, T) is a local minimizer to the functional (4.7) on D . Then, (x, T) satisfies the fractional Euler–Lagrange equations*

$$\partial_2 L[x]_\gamma^{\alpha,\beta}(t) + D_\gamma^{\beta(\cdot),\alpha(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) = 0, \quad (4.9)$$

on the interval $[a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - {}_T D_t^{\beta(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right) = 0, \quad (4.10)$$

on the interval $[T, b]$. Moreover, (x, T) satisfies the transversality conditions

$$\begin{cases} L[x]_\gamma^{\alpha,\beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'(T) = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \left[\gamma_2 {}_T I_t^{1-\beta(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - {}_a I_t^{1-\beta(\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=b} = 0. \end{cases} \quad (4.11)$$

Proof. Let (x, T) be a solution to the problem and $(x + \epsilon h, T + \epsilon \Delta T)$ be an admissible variation, where $h \in C^1([a, b]; \mathbb{R})$ is a perturbing curve, $\Delta T \in \mathbb{R}$ represents an arbitrarily chosen small change in T and $\epsilon \in \mathbb{R}$ represents a small number. The constraint $x(a) = x_a$ implies that all admissible variations must fulfill the condition $h(a) = 0$. Define $j(\cdot)$ on a neighborhood of zero by

$$\begin{aligned} j(\epsilon) &= \mathcal{J}(x + \epsilon h, T + \epsilon \Delta T) \\ &= \int_a^{T+\epsilon \Delta T} L[x + \epsilon h]_\gamma^{\alpha,\beta}(t) dt + \phi(T + \epsilon \Delta T, (x + \epsilon h)(T + \epsilon \Delta T)). \end{aligned}$$

The derivative $j'(\epsilon)$ is

$$\begin{aligned} j'(\epsilon) &= \int_a^{T+\epsilon \Delta T} (\partial_2 L[x + \epsilon h]_\gamma^{\alpha,\beta}(t) h(t) + \partial_3 L[x + \epsilon h]_\gamma^{\alpha,\beta}(t) {}^C D_\gamma^{\alpha(\cdot),\beta(\cdot)} h(t)) dt \\ &\quad + L[x + \epsilon h]_\gamma^{\alpha,\beta}(T + \epsilon \Delta T) \Delta T + \partial_1 \phi(T + \epsilon \Delta T, (x + \epsilon h)(T + \epsilon \Delta T)) \Delta T \\ &\quad + \partial_2 \phi(T + \epsilon \Delta T, (x + \epsilon h)(T + \epsilon \Delta T)) [(x + \epsilon h)(T + \epsilon \Delta T)]'. \end{aligned}$$

Considering the differentiability properties of j , a necessary condition for (x, T) to be a local extremizer is given by $j'(\epsilon)|_{\epsilon=0} = 0$, that is,

$$\int_a^T (\partial_2 L[x]_\gamma^{\alpha, \beta}(t) h(t) + \partial_3 L[x]_\gamma^{\alpha, \beta}(t)^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t)) dt + L[x]_\gamma^{\alpha, \beta}(T) \Delta T + \partial_1 \phi(T, x(T)) \Delta T + \partial_2 \phi(T, x(T)) [h(T) + x'(T) \Delta T] = 0. \quad (4.12)$$

The second addend of the integral function (4.12),

$$\int_a^T \partial_3 L[x]_\gamma^{\alpha, \beta}(t)^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t) dt, \quad (4.13)$$

can be written, using the definition of combined Caputo fractional derivative, as

$$\begin{aligned} & \int_a^T \partial_3 L[x]_\gamma^{\alpha, \beta}(t)^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t) dt \\ &= \int_a^T \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \left[\gamma_1 {}^C D_t^{\alpha(\cdot, \cdot)} h(t) + \gamma_2 {}^C D_b^{\beta(\cdot, \cdot)} h(t) \right] dt \\ &= \gamma_1 \int_a^T \partial_3 L[x]_\gamma^{\alpha, \beta}(t) {}^C D_t^{\alpha(\cdot, \cdot)} h(t) dt \\ & \quad + \gamma_2 \left[\int_a^b \partial_3 L[x]_\gamma^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h(t) dt - \int_T^b \partial_3 L[x]_\gamma^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h(t) dt \right]. \end{aligned}$$

Integrating by parts (see Theorem 13), and since $h(a) = 0$, the term (4.13) can be written as

$$\begin{aligned} & \gamma_1 \left[\int_a^T h(t) {}_t D_T^{\alpha(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) dt + \left[h(t) {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} \right] \\ & + \gamma_2 \left[\int_a^b h(t) {}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) dt - \left[h(t) {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} \right. \\ & \quad - \left(\int_T^b h(t) {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) dt - \left[h(t) {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} \right. \\ & \quad \left. \left. + \left[h(t) {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} \right) \right]. \end{aligned}$$

Unfolding these integrals, and considering the fractional operator $D_\gamma^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)}$ with $\bar{\gamma} =$

(γ_2, γ_1) , then (4.13) is equivalent to

$$\begin{aligned}
 & \int_a^T h(t) D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) dt \\
 & + \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & + \left[h(t) \left(\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=T} \\
 & + \left[h(t) \gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

Substituting these relations into equation (4.12), we obtain

$$\begin{aligned}
 0 &= \int_a^T h(t) \left[\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & + \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & + h(T) \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\
 & + \Delta T \left[L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'(t) \right]_{t=T} \\
 & + h(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned} \tag{4.14}$$

As h and ΔT are arbitrary, we can choose $\Delta T = 0$ and $h(t) = 0$, for all $t \in [T, b]$, but h is arbitrary in $t \in [a, T]$. Then, for all $t \in [a, T]$, we obtain the first necessary condition (4.9):

$$\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) = 0.$$

Analogously, considering $\Delta T = 0$, $h(t) = 0$, for all $t \in [a, T] \cup \{b\}$, and h arbitrary on (T, b) , we obtain the second necessary condition (4.10):

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) = 0.$$

As (x, T) is a solution to the necessary conditions (4.9) and (4.10), then equation (4.14) takes the form

$$\begin{aligned}
 0 &= h(T) \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\
 & + \Delta T \left[L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'(t) \right]_{t=T} \\
 & + h(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned} \tag{4.15}$$

Transversality conditions (4.11) are obtained for appropriate choices of variations. \square

In the next theorem, considering the same Problem 1, we rewrite the transversality conditions (4.11) in terms of the increment ΔT on time and on the consequent increment Δx_T on x , given by

$$\Delta x_T = (x + h)(T + \Delta T) - x(T). \quad (4.16)$$

Theorem 46. *Let (x, T) be a local minimizer to the functional (4.7) on D . Then, the fractional Euler–Lagrange equations (4.9) and (4.10) are satisfied together with the following transversality conditions:*

$$\begin{cases} L[x]_\gamma^{\alpha,\beta}(T) + \partial_1\phi(T, x(T)) \\ \quad + x'(T) \left[\gamma_{2T} I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \gamma_{1t} I_T^{1-\alpha(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=T} = 0, \\ \left[\gamma_{1t} I_T^{1-\alpha(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=T} + \partial_2\phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - {}_a I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=b} = 0. \end{cases} \quad (4.17)$$

Proof. The Euler–Lagrange equations are deduced following similar arguments as the ones presented in Theorem 45. We now focus our attention on the proof of the transversality conditions. Using Taylor’s expansion up to first order for a small ΔT , and restricting the set of variations to those for which $h'(T) = 0$, we obtain

$$(x + h)(T + \Delta T) = (x + h)(T) + x'(T)\Delta T + O(\Delta T)^2.$$

Rearranging the relation (4.16) allows us to express $h(T)$ in terms of ΔT and Δx_T :

$$h(T) = \Delta x_T - x'(T)\Delta T + O(\Delta T)^2.$$

Substitution of this expression into (4.15) gives us

$$\begin{aligned} 0 = & \Delta x_T \left[\gamma_{1t} I_T^{1-\alpha(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) + \partial_2\phi(t, x(t)) \right]_{t=T} \\ & + \Delta T \left[L[x]_\gamma^{\alpha,\beta}(t) + \partial_1\phi(t, x(t)) \right. \\ & \quad \left. - x'(t) \left(\gamma_{1t} I_T^{1-\alpha(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right) \right]_{t=T} \\ & + h(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - {}_a I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right) \right]_{t=b} + O(\Delta T)^2. \end{aligned}$$

Transversality conditions (4.17) are obtained using appropriate choices of variations. \square

4.2.2 Particular cases

Now, we specify our results to three particular cases of variable terminal points.

Vertical terminal line

This case involves a fixed upper bound T . Thus, $\Delta T = 0$ and, consequently, the second term in (4.15) drops out. Since Δx_T is arbitrary, we obtain the following transversality conditions: if $T < b$, then

$$\begin{cases} \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} = 0; \end{cases}$$

if $T = b$, then $\Delta x_T = h(b)$ and the transversality conditions reduce to

$$\left[\gamma_1 {}_t I_b^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} + \partial_2 \phi(b, x(b)) = 0.$$

Horizontal terminal line

In this situation, we have $\Delta x_T = 0$ but ΔT is arbitrary. Thus, the transversality conditions are

$$\begin{cases} L[x]_{\gamma}^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) \\ \quad + x'(T) \left[\gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases}$$

Terminal curve

Now the terminal point is described by a given curve $\psi : C^1([a, b]) \rightarrow \mathbb{R}$, in the sense that $x(T) = \psi(T)$. From Taylor's formula, for a small arbitrary ΔT , one has

$$\Delta x(T) = \psi'(T)\Delta T + O(\Delta T)^2.$$

Hence, the transversality conditions are presented in the form

$$\begin{cases} L[x]_{\gamma}^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T))\psi'(T) \\ \quad + (x'(T) - \psi'(T)) \left[\gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases}$$

4.2.3 Examples

In this section, we show two examples to illustrate the new results. Let $\alpha(t, \tau) = \alpha(t)$ and $\beta(t, \tau) = \beta(\tau)$ be two functions depending on a variable t and τ only, respectively. Consider the following fractional variational problem: to minimize the functional

$$\mathcal{J}(x, T) = \int_0^T \left[2\alpha(t) - 1 + \left({}^C D_{\gamma}^{\alpha(\cdot), \beta(\cdot)} x(t) - \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} - \frac{(10-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^2 \right] dt$$

for $t \in [0, 10]$, subject to the initial condition $x(0) = 0$ and where $\gamma = (\gamma_1, \gamma_2) = (1/2, 1/2)$. Simple computations show that for $\bar{x}(t) = t$, with $t \in [0, 10]$, we have

$${}^C D_{\gamma}^{\alpha(\cdot), \beta(\cdot)} \bar{x}(t) = \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} + \frac{(10-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))}.$$

For $\bar{x}(t) = t$ the functional reduces to

$$\mathcal{J}(\bar{x}, T) = \int_0^T (2\alpha(t) - 1) dt.$$

In order to determine the optimal time T , we have to solve the equation $2\alpha(T) = 1$. For example, let $\alpha(t) = t^2/2$. In this case, since $\mathcal{J}(x, T) \geq -2/3$ for all pairs (x, T) and $\mathcal{J}(\bar{x}, 1) = -2/3$, we conclude that the (global) minimum value of the functional is $-2/3$, obtained for \bar{x} and $T = 1$. It is obvious that the two Euler–Lagrange equations (4.9) and (4.10) are satisfied when $x = \bar{x}$, since

$$\partial_3 L[\bar{x}]_{\gamma}^{\alpha, \beta}(t) = 0 \quad \text{for all } t \in [0, 10].$$

Using this relation, together with

$$L[\bar{x}]_{\gamma}^{\alpha, \beta}(1) = 0,$$

the transversality conditions (4.11) are also verified.

For our last example, consider the functional

$$\mathcal{J}(x, T) = \int_0^T \left[2\alpha(t) - 1 + \left({}^C D_{\gamma}^{\alpha(\cdot), \beta(\cdot)} x(t) - \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} - \frac{(10-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^3 \right] dt,$$

where the remaining assumptions and conditions are as in the previous example. For this case, $\bar{x}(t) = t$ and $T = 1$ still satisfy the necessary optimality conditions. However, we cannot assure that $(\bar{x}, 1)$ is a local minimizer to the problem.

4.3 Higher-order variational problems

In this section, we intent to generalize the results obtained in Section 4.2 by considering higher-order variational problems with a Lagrangian depending on a higher-order combined Caputo derivative of variable fractional order, defined by

$${}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t) = \gamma_1^n {}^C D_t^{\alpha_n(\cdot, \cdot)} x(t) + \gamma_2^n {}^C D_b^{\beta_n(\cdot, \cdot)} x(t),$$

subject to boundary conditions at the initial time $t = a$.

In Section 4.3.1, we obtain higher-order Euler–Lagrange equations and transversality conditions for the generalized variational problem with a Lagrangian depending on a combined Caputo fractional derivative of variable fractional order (Theorems 47 and 49).

One illustrative example is discussed in Section 4.3.2.

4.3.1 Necessary optimality conditions

Let $n \in \mathbb{N}$ and $x \in [a, b] \rightarrow \mathbb{R}$ be a function of class C^n . The fractional order is a continuous function of two variables, $\alpha_n : [a, b]^2 \rightarrow (n - 1, n)$.

Let D denote the linear subspace of $C^n([a, b]; \mathbb{R}) \times [a, b]$ such that ${}^C D_{\gamma^i}^{\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot)} x(t)$ exists and is continuous on the interval $[a, b]$ for all $i \in \{1, \dots, n\}$. We endow D with the norm

$$\|(x, t)\| = \max_{a \leq t \leq b} |x(t)| + \max_{a \leq t \leq b} \sum_{i=1}^n \left| {}^C D_{\gamma^i}^{\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot)} x(t) \right| + |t|.$$

Consider the following higher-order problem of the calculus of variations:

Problem 2. Minimize functional $\mathcal{J} : D \rightarrow \mathbb{R}$, where

$$\mathcal{J}(x, T) = \int_a^T L \left(t, x(t), {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} x(t), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t) \right) dt + \phi(T, x(T)), \quad (4.18)$$

over all $(x, T) \in D$ subject to boundary conditions

$$x(a) = x_a, \quad x^{(i)}(a) = x_a^i, \quad \forall i \in \{1, \dots, n - 1\},$$

for fixed $x_a, x_a^1, \dots, x_a^{n-1} \in \mathbb{R}$. Here the terminal time T and terminal state $x(T)$ are both free. For all $i \in \{1, \dots, n\}$, $\alpha_i, \beta_i : [a, b]^2 \subseteq (i - 1, i)$ and $\gamma^i = (\gamma_1^i, \gamma_2^i)$ is a vector. The terminal cost function $\phi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is at least of class C^1 .

For simplicity of notation, we introduce the operator $[\cdot]_\gamma^{\alpha,\beta}$ defined by

$$[x]_\gamma^{\alpha,\beta}(t) = \left(t, x(t), {}^C D_{\gamma^1}^{\alpha_1(\cdot),\beta_1(\cdot)} x(t), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot),\beta_n(\cdot)} x(t) \right).$$

We assume that the Lagrangian $L : [a, b] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a function of class C^1 . Along the work, we denote by $\partial_i L$, $i \in \{1, \dots, n+2\}$, the partial derivative of the Lagrangian L with respect to its i th argument.

Now, we can rewrite functional (4.18) as

$$\mathcal{J}(x, T) = \int_a^T L[x]_\gamma^{\alpha,\beta}(t) dt + \phi(T, x(T)). \quad (4.19)$$

In the previous section, we obtained fractional necessary optimality conditions that every local minimizer of functional \mathcal{J} , with $n = 1$, must fulfill. Here, we generalize those results to arbitrary values of n , $n \in \mathbb{N}$.

Theorem 47 (Necessary optimality conditions for Problem 2). *Suppose that (x, T) gives a minimum to functional (4.19) on D . Then, (x, T) satisfies the following fractional Euler–Lagrange equations:*

$$\partial_2 L[x]_\gamma^{\alpha,\beta}(t) + \sum_{i=1}^n D_{\gamma^i}^{\beta_i(\cdot),\alpha_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) = 0, \quad (4.20)$$

on the interval $[a, T]$, and

$$\sum_{i=1}^n \gamma_2^i \left({}_a D_t^{\beta_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) - {}_T D_t^{\beta_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right) = 0, \quad (4.21)$$

on the interval $[T, b]$. Moreover, (x, T) satisfies the following transversality conditions:

$$\left\{ \begin{array}{l} L[x]_\gamma^{\alpha,\beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'(T) = 0, \\ \sum_{i=1}^n \left[\gamma_1^i (-1)^{i-1} \frac{d^{i-1}}{dt^{i-1}} t I_T^{i-\alpha_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right. \\ \quad \left. - \gamma_2^i \frac{d^{i-1}}{dt^{i-1}} T I_t^{i-\beta_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \sum_{i=j+1}^n \left[\gamma_1^i (-1)^{i-1-j} \frac{d^{i-1-j}}{dt^{i-1-j}} t I_T^{i-\alpha_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right. \\ \quad \left. + \gamma_2^i (-1)^{j+1} \frac{d^{i-1-j}}{dt^{i-1-j}} T I_t^{i-\beta_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right]_{t=T} = 0, \quad \forall j = 1, \dots, n-1, \\ \sum_{i=j+1}^n \left[\gamma_2^i (-1)^{j+1} \left[\frac{d^{i-1-j}}{dt^{i-1-j}} {}_a I_t^{i-\beta_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right. \right. \\ \quad \left. \left. - \frac{d^{i-1-j}}{dt^{i-1-j}} T I_t^{i-\beta_i(\cdot)} \partial_{i+2} L[x]_\gamma^{\alpha,\beta}(t) \right] \right]_{t=b} = 0, \quad \forall j = 0, \dots, n-1. \end{array} \right. \quad (4.22)$$

Proof. The proof is an extension of the one used in Theorem 45. Let $h \in C^n([a, b]; \mathbb{R})$ be a perturbing curve and $\Delta T \in \mathbb{R}$ an arbitrarily chosen small change in T . For a small number $\epsilon \in \mathbb{R}$, if (x, T) is a solution to the problem, we consider an admissible variation of (x, T) of the form $(x + \epsilon h, T + \epsilon \Delta T)$, and then, by the minimum condition, we have that

$$\mathcal{J}(x, T) \leq \mathcal{J}(x + \epsilon h, T + \epsilon \Delta T).$$

The constraints $x^{(i)}(a) = x_a^{(i)}$ imply that all admissible variations must fulfill the conditions $h^{(i)}(a) = 0$, for all $i = 0, \dots, n - 1$. We define function $j(\cdot)$ on a neighborhood of zero by

$$\begin{aligned} j(\epsilon) &= \mathcal{J}(x + \epsilon h, T + \epsilon \Delta T) \\ &= \int_a^{T + \epsilon \Delta T} L[x + \epsilon h]_{\gamma}^{\alpha, \beta}(t) dt + \phi(T + \epsilon \Delta T, (x + \epsilon h)(T + \epsilon \Delta T)). \end{aligned}$$

The derivative $j'(\epsilon)$ is

$$\begin{aligned} j'(\epsilon) &= \int_a^{T + \epsilon \Delta T} \left(\partial_2 L[x + \epsilon h]_{\gamma}^{\alpha, \beta}(t) h(t) + \sum_{i=1}^n \partial_{i+2} L[x + \epsilon h]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma^i}^{\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot)} h(t) \right) dt \\ &\quad + L[x + \epsilon h]_{\gamma}^{\alpha, \beta}(T + \epsilon \Delta T) \Delta T + \partial_1 \phi(T + \epsilon \Delta T, (x + \epsilon h)(T + \epsilon \Delta T)) \Delta T \\ &\quad + \partial_2 \phi(T + \epsilon \Delta T, (x + \epsilon h)(T + \epsilon \Delta T)) (x + \epsilon h)'(T + \epsilon \Delta T). \end{aligned}$$

Hence, by Fermat's theorem, a necessary condition for (x, T) to be a local minimizer of j is given by $j'(0) = 0$, that is,

$$\begin{aligned} \int_a^T \left(\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) h(t) + \sum_{i=1}^n \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma^i}^{\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot)} h(t) \right) dt + L[x]_{\gamma}^{\alpha, \beta}(T) \Delta T \\ + \partial_1 \phi(T, x(T)) \Delta T + \partial_2 \phi(T, x(T)) [h(t) + x'(T) \Delta T] = 0. \end{aligned} \quad (4.23)$$

Considering the second addend of the integral function (4.23), for $i = 1$, we get

$$\begin{aligned} &\int_a^T \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} h(t) dt \\ &= \int_a^T \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \left[\gamma_1^1 {}^C D_a^{\alpha_1(\cdot, \cdot)} h(t) + \gamma_2^1 {}^C D_b^{\beta_1(\cdot, \cdot)} h(t) \right] dt \\ &= \gamma_1^1 \int_a^T \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_a^{\alpha_1(\cdot, \cdot)} h(t) dt \\ &\quad + \gamma_2^1 \left[\int_a^b \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_b^{\beta_1(\cdot, \cdot)} h(t) dt - \int_T^b \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_b^{\beta_1(\cdot, \cdot)} h(t) dt \right]. \end{aligned}$$

Integrating by parts (see Theorem 42), and since $h(a) = 0$, we obtain that

$$\begin{aligned} & \gamma_1^1 \left[\int_a^T h(t) {}_t D_T^{\alpha_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) dt + \left[h(t) {}_t I_T^{1-\alpha_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \right] \\ & + \gamma_2^1 \left[\int_a^b h(t) {}_a D_t^{\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) dt - \left[h(t) {}_a I_t^{1-\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} \right. \\ & \quad - \left(\int_T^b h(t) {}_T D_t^{\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) dt - \left[h(t) {}_T I_t^{1-\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} \right. \\ & \quad \left. \left. + \left[h(t) {}_T I_t^{1-\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \right) \right]. \end{aligned}$$

Unfolding these integrals, and considering the fractional operator $D_{\gamma^1}^{\beta_1, \alpha_1}$ with $\overline{\gamma^1} = (\gamma_2^1, \gamma_1^1)$, then the previous term is equal to

$$\begin{aligned} & \int_a^T h(t) D_{\gamma^1}^{\beta_1(\cdot, \cdot), \alpha_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) dt \\ & + \int_T^b \gamma_2^1 h(t) \left[{}_a D_t^{\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\ & + h(T) \left[\gamma_1^1 {}_T I_T^{1-\alpha_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2^1 {}_T I_t^{1-\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \\ & - h(b) \gamma_2^1 \left[{}_a I_t^{1-\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T I_t^{1-\beta_1(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b}. \end{aligned}$$

Considering the third addend of the integral function (4.23), for $i = 2$, we get

$$\begin{aligned} & \int_a^T \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma^2}^{\alpha_2(\cdot, \cdot), \beta_2(\cdot, \cdot)} h(t) dt = \gamma_1^2 \int_a^T \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_t^{\alpha_2(\cdot, \cdot)} h(t) dt \\ & + \gamma_2^2 \left[\int_a^b \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}_t D_b^{\beta_2(\cdot, \cdot)} h(t) dt - \int_T^b \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}_t D_b^{\beta_2(\cdot, \cdot)} h(t) dt \right] \\ & = \gamma_1^2 \left[\int_a^T h(t) {}_t D_T^{\alpha_2(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) dt \right. \\ & \quad \left. + \left[h^{(1)}(t) {}_t I_T^{2-\alpha_2(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - h(t) \frac{d}{dt} {}_t I_T^{2-\alpha_2(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \right] \\ & + \gamma_2^2 \left[\int_a^b h(t) {}_a D_t^{\beta_2(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) dt \right. \\ & \quad \left. + \left[h^{(1)}(t) {}_a I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - h(t) \frac{d}{dt} {}_a I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} \right] \end{aligned}$$

$$\begin{aligned}
 & - \int_T^b h(t) {}_T D_t^{\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) dt \\
 & - \left[h^{(1)}(t) {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - h(t) \frac{d}{dt} {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} \\
 & + \left[h^{(1)}(t) {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - h(t) \frac{d}{dt} {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T}.
 \end{aligned}$$

Again, with the auxiliary operator $D_{\frac{\gamma^2}{\gamma^2}}^{\beta_2, \alpha_2}$, with $\bar{\gamma}^2 = (\gamma_2^2, \gamma_1^2)$, we obtain

$$\begin{aligned}
 & \int_a^T h(t) D_{\frac{\gamma^2}{\gamma^2}}^{\beta_2(\cdot, \cdot), \alpha_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) dt \\
 & + \int_T^b \gamma_2^2 h(t) \left[{}_a D_t^{\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 & + \left[h^{(1)}(t) \left(\gamma_1^2 {}_t I_T^{2-\alpha_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \gamma_2^2 {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=T} \\
 & - \left[h(t) \left(\gamma_1^2 \frac{d}{dt} {}_t I_T^{2-\alpha_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \gamma_2^2 \frac{d}{dt} {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=T} \\
 & + \left[h^{(1)}(t) \gamma_2^2 \left({}_a I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b} \\
 & - \left[h(t) \gamma_2^2 \left(\frac{d}{dt} {}_a I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - \frac{d}{dt} {}_T I_t^{2-\beta_2(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

Now, consider the general case

$$\int_a^T \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) {}^C D_{\gamma^i}^{\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot)} h(t) dt,$$

$i = 3, \dots, n$. Then, we obtain

$$\begin{aligned}
 & \gamma_1^i \left[\int_a^T h(t) {}_t D_T^{\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) dt \right. \\
 & \quad \left. + \left[\sum_{k=0}^{i-1} (-1)^k h^{(i-1-k)}(t) \frac{d^k}{dt^k} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} \right] \\
 & + \gamma_2^i \left[\int_a^b h(t) {}_a D_t^{\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) dt \right. \\
 & \quad \left. + \left[\sum_{k=0}^{i-1} (-1)^{i+k} h^{(i-1-k)}(t) \frac{d^k}{dt^k} {}_a I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} \right. \\
 & - \int_T^b h(t) {}_T D_t^{\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) dt \\
 & \quad \left. - \left[\sum_{k=0}^{i-1} (-1)^{i+k} h^{(i-1-k)}(t) \frac{d^k}{dt^k} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} \right].
 \end{aligned}$$

Unfolding these integrals, we obtain

$$\begin{aligned}
 & \int_a^T h(t) D_{\gamma^i}^{\beta_i(\cdot, \cdot), \alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) dt \\
 & + \int_T^b \gamma_2^i h(t) \left[{}_a D_t^{\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & + h^{(i-1)}(T) \left[\gamma_1^i {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) + \gamma_2^i (-1)^i {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \\
 & + h^{(i-1)}(b) \gamma_2^i (-1)^i \left[{}_a I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} \\
 & + h^{(i-2)}(T) \left[\gamma_1^i (-1)^1 \frac{d}{dt} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) + \gamma_2^i (-1)^{i+1} \frac{d}{dt} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \\
 & + h^{(i-2)}(b) \gamma_2^i (-1)^{i+1} \left[\frac{d}{dt} {}_a I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) - \frac{d}{dt} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} \\
 & + \dots \\
 & + h(T) \left[\gamma_1^i (-1)^{i-1} \frac{d^{i-1}}{dt^{i-1}} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right. \\
 & \quad \left. + \gamma_2^i (-1)^{2i-1} \frac{d^{i-1}}{dt^{i-1}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \\
 & + h(b) \gamma_2^i (-1)^{2i-1} \left[\frac{d^{i-1}}{dt^{i-1}} {}_a I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) - \frac{d^{i-1}}{dt^{i-1}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b}.
 \end{aligned}$$

Substituting all the relations into equation (4.23), we obtain that

$$\begin{aligned}
 0 & = \int_a^T h(t) \left(\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) + \sum_{i=1}^n D_{\gamma^i}^{\beta_i(\cdot, \cdot), \alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right) dt \\
 & + \int_T^b h(t) \sum_{i=1}^n \gamma_2^i \left[{}_a D_t^{\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & + \sum_{j=0}^{n-1} h^{(j)}(T) \sum_{i=j+1}^n \left[\gamma_1^i (-1)^{i-1-j} \frac{d^{i-1-j}}{dt^{i-1-j}} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right. \\
 & \quad \left. + \gamma_2^i (-1)^{j+1} \frac{d^{i-1-j}}{dt^{i-1-j}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} \\
 & + \sum_{j=0}^{n-1} h^{(j)}(b) \sum_{i=j+1}^n \gamma_2^i (-1)^{j+1} \left[\frac{d^{i-1-j}}{dt^{i-1-j}} {}_a I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right. \\
 & \quad \left. - \frac{d^{i-1-j}}{dt^{i-1-j}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} \\
 & + h(T) \partial_2 \phi(T, x(T)) + \Delta T \left[L[x]_{\gamma}^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'(T) \right].
 \end{aligned} \tag{4.24}$$

We obtain the fractional Euler–Lagrange equations (4.20)–(4.21) and the transversality conditions (4.22) applying the fundamental lemma of the calculus of variations (see, e.g., [112]) for appropriate choices of variations. \square

Remark 48. When $n = 1$, functional (4.18) takes the form

$$\mathcal{J}(x, T) = \int_a^T L(t, x(t), {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t)) dt + \phi(T, x(T)),$$

and the fractional Euler–Lagrange equations (4.20)–(4.21) coincide with those of Theorem 45.

Considering the increment ΔT on time T , and the consequent increment Δx_T on x , given by

$$\Delta x_T = (x + h)(T + \Delta T) - x(T), \quad (4.25)$$

in the next theorem we rewrite the transversality conditions (4.22) in terms of these increments.

Theorem 49. If (x, T) minimizes functional \mathcal{J} defined by (4.19), then (x, T) satisfies the Euler–Lagrange equations (4.20) and (4.21), and the following transversality conditions:

$$\left\{ \begin{array}{l} L[x]_\gamma^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) - x'(T) \sum_{i=1}^n \left[\gamma_1^i (-1)^{i-1} \frac{d^{i-1}}{dt^{i-1}} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right. \\ \quad \left. - \gamma_2^i \frac{d^{i-1}}{dt^{i-1}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} = 0, \\ \sum_{i=1}^n \left[\gamma_1^i (-1)^{i-1} \frac{d^{i-1}}{dt^{i-1}} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right. \\ \quad \left. - \gamma_2^i \frac{d^{i-1}}{dt^{i-1}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \sum_{i=j+1}^n \left[\gamma_1^i (-1)^{i-1-j} \frac{d^{i-1-j}}{dt^{i-1-j}} {}_t I_T^{i-\alpha_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right. \\ \quad \left. + \gamma_2^i (-1)^{j+1} \frac{d^{i-1-j}}{dt^{i-1-j}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} = 0, \quad \forall j = 2, \dots, n-1, \\ \sum_{i=j+1}^n \left[\gamma_2^i (-1)^{j+1} \left[\frac{d^{i-1-j}}{dt^{i-1-j}} {}_a I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right. \right. \\ \quad \left. \left. - \frac{d^{i-1-j}}{dt^{i-1-j}} {}_T I_t^{i-\beta_i(\cdot, \cdot)} \partial_{i+2} L[x]_\gamma^{\alpha, \beta}(t) \right] \right]_{t=b} = 0, \quad \forall j = 0, \dots, n-1. \end{array} \right.$$

Proof. Using Taylor's expansion up to first order, and restricting the set of variations to those for which $h'(T) = 0$, we obtain that

$$(x + h)(T + \Delta T) = (x + h)(T) + x'(T)\Delta T + O(\Delta T)^2.$$

According to the increment on x given by (4.25), we get

$$h(T) = \Delta x_T - x'(T)\Delta T + O(\Delta T)^2.$$

From substitution of this expression into (4.24), and by using appropriate choices of variations, we obtain the intended transversality conditions. \square

4.3.2 Example

We provide an illustrative example. It is covered by Theorem 47.

Let $p_{n-1}(t)$ be a polynomial of degree $n-1$. If $\alpha, \beta : [0, b]^2 \rightarrow (n-1, n)$ are the fractional orders, then ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} p_{n-1}(t) = 0$ since $p_{n-1}^{(n)}(t) = 0$ for all t . Consider the functional

$$\mathcal{J}(x, T) = \int_0^T \left[({}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t))^2 + (x(t) - p_{n-1}(t))^2 - t - 1 \right] dt + T^2$$

subject to the initial constraints

$$x(0) = p_{n-1}(0) \quad \text{and} \quad x^{(k)}(0) = p_{n-1}^{(k)}(0), \quad k = 1, \dots, n-1.$$

Observe that, for all $t \in [0, b]$,

$$\partial_i L[p_{n-1}]_\gamma^{\alpha, \beta}(t) = 0, \quad i = 2, 3.$$

Thus, function $x \equiv p_{n-1}$ and the final time $T = 1$ satisfy the necessary optimality conditions of Theorem 47. We also remark that, for any curve x , one has

$$\mathcal{J}(x, T) \geq \int_0^T [-t - 1] dt + T^2 = \frac{T^2}{2} - T,$$

which attains a minimum value $-1/2$ at $T = 1$. Since $\mathcal{J}(p_{n-1}, 1) = -1/2$, we conclude that $(p_{n-1}, 1)$ is the (global) minimizer of \mathcal{J} .

4.4 Variational problems with time delay

In this section, we consider fractional variational problems with time delay. As mentioned in [108], “We verify that a fractional derivative requires an infinite number of samples capturing, therefore, all the signal history, contrary to what happens with integer order derivatives that are merely local operators. This fact motivates the evaluation of calculation strategies based on delayed signal samples”. This subject has already been studied for constant fractional order [6, 20, 47]. However, for a variable fractional order, it is, to the authors’ best knowledge, an open question. We also refer to the works [30, 33, 51, 113], where fractional differential equations are considered with a time delay.

In Section 4.4.1, we deduce necessary optimality conditions when the Lagrangian depends on a time delay (Theorem 50). One illustrative example is discussed in Section 4.4.2.

4.4.1 Necessary optimality conditions

For simplicity of presentation, we consider fractional orders $\alpha, \beta : [a, b]^2 \rightarrow (0, 1)$. Using similar arguments, the problem can be easily generalized for higher-order derivatives. Let $\sigma > 0$ and define the vector

$${}_{\sigma}[x]_{\gamma}^{\alpha, \beta}(t) = (t, x(t), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t), x(t - \sigma)).$$

For the domain of the functional, we consider the set

$$D_{\sigma} = \{(x, t) \in C^1([a - \sigma, b]; \mathbb{R}) \times [a, b] : {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \text{ exists and is continuous on } [a, b]\}.$$

Let $\mathcal{J} : D_{\sigma} \rightarrow \mathbb{R}$ be the functional defined by

$$\mathcal{J}(x, T) = \int_a^T L_{\sigma}[x]_{\gamma}^{\alpha, \beta}(t) + \phi(T, x(T)), \quad (4.26)$$

where we assume again that the Lagrangian L and the payoff term ϕ are differentiable.

The optimization problem with time delay is the following:

Problem 3. *Minimize functional (4.26) on D_{σ} subject to the boundary condition*

$$x(t) = \varphi(t)$$

for all $t \in [a - \sigma, a]$, where φ is a given (fixed) function.

We now state and prove the Euler–Lagrange equations for this problem.

Theorem 50 (Necessary optimality conditions for (4.26)). *Suppose that (x, T) gives a local minimum to functional (4.26) on D_σ . If $\sigma \geq T - a$, then (x, T) satisfies*

$$\partial_2 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) = 0, \quad (4.27)$$

for $t \in [a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right) = 0, \quad (4.28)$$

for $t \in [T, b]$. Moreover, (x, T) satisfies

$$\begin{cases} L_\sigma[x]_\gamma^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'(T) = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \left[\gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases} \quad (4.29)$$

If $\sigma < T - a$, then Eq. (4.27) is replaced by the two following ones:

$$\partial_2 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + \partial_4 L_\sigma[x]_\gamma^{\alpha, \beta}(t + \sigma) = 0, \quad (4.30)$$

for $t \in [a, T - \sigma]$, and

$$\partial_2 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) = 0, \quad (4.31)$$

for $t \in [T - \sigma, T]$.

Proof. Consider variations of the solution $(x + \epsilon h, T + \epsilon \Delta T)$, where $h \in C^1([a - \sigma, b]; \mathbb{R})$ is such that $h(t) = 0$ for all $t \in [a - \sigma, a]$, and $\epsilon, \Delta T$ are two reals. If we define $j(\epsilon) = \mathcal{J}(x + \epsilon h, T + \epsilon \Delta T)$, then $j'(0) = 0$, that is,

$$\begin{aligned} & \int_a^T (\partial_2 L_\sigma[x]_\gamma^{\alpha, \beta}(t) h(t) + \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t) + \partial_4 L_\sigma[x]_\gamma^{\alpha, \beta}(t) h(t - \sigma)) dt \\ & + L_\sigma[x]_\gamma^{\alpha, \beta}(T) \Delta T + \partial_1 \phi(T, x(T)) \Delta T + \partial_2 \phi(T, x(T)) [h(T) + x'(T) \Delta T] = 0. \end{aligned} \quad (4.32)$$

First, suppose that $\sigma \geq T - a$. In this case, since

$$\int_a^T \partial_4 L_\sigma[x]_\gamma^{\alpha, \beta}(t) h(t - \sigma) dt = \int_{a-\sigma}^{T-\sigma} \partial_4 L_\sigma[x]_\gamma^{\alpha, \beta}(t + \sigma) h(t) dt$$

and $h \equiv 0$ on $[a - \sigma, a]$, this term vanishes in (4.32) and we obtain Eq. (4.12). The rest of the proof is similar to the one presented in Section 4.2, Theorem 45, and we obtain (4.27)–(4.29). Suppose now that $\sigma < T - a$. In this case, we have that

$$\begin{aligned} \int_a^T \partial_4 L_\sigma[x]_\gamma^{\alpha,\beta}(t) h(t - \sigma) dt &= \int_{a-\sigma}^{T-\sigma} \partial_4 L_\sigma[x]_\gamma^{\alpha,\beta}(t + \sigma) h(t) dt \\ &= \int_a^{T-\sigma} \partial_4 L_\sigma[x]_\gamma^{\alpha,\beta}(t + \sigma) h(t) dt. \end{aligned}$$

Next, we evaluate the integral

$$\begin{aligned} \int_a^T \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} h(t) dt \\ = \int_a^{T-\sigma} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} h(t) dt + \int_{T-\sigma}^T \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} h(t) dt. \end{aligned}$$

For the first integral, integrating by parts, we have

$$\begin{aligned} \int_a^{T-\sigma} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} h(t) dt &= \gamma_1 \int_a^{T-\sigma} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_a^C D_t^{\alpha(\cdot,\cdot)} h(t) dt \\ &+ \gamma_2 \left[\int_a^b \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_t^C D_b^{\beta(\cdot,\cdot)} h(t) dt - \int_{T-\sigma}^b \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_t^C D_b^{\beta(\cdot,\cdot)} h(t) dt \right] \\ &= \int_a^{T-\sigma} h(t) \left[\gamma_{1t} D_{T-\sigma}^{\alpha(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) + \gamma_{2a} D_t^{\beta(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) \right] dt \\ &+ \int_{T-\sigma}^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) - {}_{T-\sigma} D_t^{\beta(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) \right] dt \\ &+ \left[h(t) \left[\gamma_{1t} I_{T-\sigma}^{1-\alpha(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) - \gamma_{2T-\sigma} I_t^{1-\beta(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) \right] \right]_{t=T-\sigma} \\ &+ \left[\gamma_2 h(t) \left[-{}_a I_t^{1-\beta(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) + {}_{T-\sigma} I_t^{1-\beta(\cdot,\cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t) \right] \right]_{t=b}. \end{aligned}$$

For the second integral, in a similar way, we deduce that

$$\begin{aligned} \int_{T-\sigma}^T \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} h(t) dt \\ = \gamma_1 \left[\int_a^T \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_a^C D_t^{\alpha(\cdot,\cdot)} h(t) dt - \int_a^{T-\sigma} \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_a^C D_t^{\alpha(\cdot,\cdot)} h(t) dt \right] \\ + \gamma_2 \left[\int_{T-\sigma}^b \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_t^C D_b^{\beta(\cdot,\cdot)} h(t) dt - \int_T^b \partial_3 L_\sigma[x]_\gamma^{\alpha,\beta}(t)_t^C D_b^{\beta(\cdot,\cdot)} h(t) dt \right] \end{aligned}$$

$$\begin{aligned}
 &= \int_a^{T-\sigma} \gamma_1 h(t) \left[{}_t D_T^{\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - {}_t D_{T-\sigma}^{\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 &\quad + \int_{T-\sigma}^T h(t) \left[\gamma_{1t} D_T^{\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + \gamma_{2T-\sigma} D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 &\quad + \int_T^b \gamma_2 h(t) \left[{}_{T-\sigma} D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 &\quad + \left[h(t) \left[-\gamma_{1t} I_{T-\sigma}^{1-\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + \gamma_{2T-\sigma} I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] \right]_{t=T-\sigma} \\
 &\quad + \left[h(t) \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] \right]_{t=T} \\
 &\quad + \left[\gamma_2 h(t) \left[-{}_{T-\sigma} I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] \right]_{t=b}.
 \end{aligned}$$

Replacing the above equalities into (4.32), we prove that

$$\begin{aligned}
 0 &= \int_a^{T-\sigma} h(t) \left[\partial_2 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + \partial_4 L_\sigma[x]_\gamma^{\alpha, \beta}(t + \sigma) \right] dt \\
 &\quad + \int_{T-\sigma}^T h(t) \left[\partial_2 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 &\quad + \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 &\quad + h(T) \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\
 &\quad + \Delta T \left[L_\sigma[x]_\gamma^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'(t) \right]_{t=T} \\
 &\quad + h(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L_\sigma[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

By the arbitrariness of h in $[a, b]$ and of ΔT , we obtain Eqs. (4.28)–(4.31). \square

4.4.2 Example

Let $\alpha, \beta : [0, b]^2 \rightarrow (0, 1)$, $\sigma = 1$, f be a function of class C^1 , and $\hat{f}(t) = {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} f(t)$. Consider the following problem of the calculus of variations:

$$\begin{aligned}
 \mathcal{J}(x, T) &= \int_0^T \left[\left({}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) - \hat{f}(t) \right)^2 + (x(t) - f(t))^2 \right. \\
 &\quad \left. + (x(t-1) - f(t-1))^2 - t - 2 \right] dt + T^2 \rightarrow \min
 \end{aligned}$$

subject to the condition $x(t) = f(t)$ for all $t \in [-1, 0]$.

In this case, we can easily verify that $(x, T) = (f, 2)$ satisfies all the conditions in Theorem 50 and that it is actually the (global) minimizer of the problem.

4.5 Isoperimetric problems

Isoperimetric problems are optimization problems, that consist in minimizing or maximizing a cost functional subject to an integral constraint. From the variational problem with dependence on a combined Caputo derivative of variable fractional order (see Definition 33) discussed in Section 4.2, here we study two variational problems subject to an additional integral constraint. In each of the problems, the terminal point in the cost integral, as well as the terminal state, are considered to be free, and we obtain corresponding natural boundary conditions.

In Sections 4.5.1 and 4.5.2, we study necessary optimality conditions in order to determine the minimizers for each of the problems. We end this section with an example (Section 4.5.3).

For the two isoperimetric problems considered in the next sections, let D be the set given by (4.6) and $[x]_{\gamma}^{\alpha,\beta}(t)$ the vector (4.8).

Picking up the problem of the fractional calculus of variations with variable order, discussed in Section 4.2, we consider also a differentiable Lagrangian $L : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and the functional $\mathcal{J} : D \rightarrow \mathbb{R}$ of the form

$$\mathcal{J}(x, T) = \int_a^T L[x]_{\gamma}^{\alpha,\beta}(t) dt + \phi(T, x(T)), \quad (4.33)$$

where the terminal cost function $\phi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is of class C^1 .

In the sequel, we need the auxiliary notation of the dual fractional derivative:

$$D_{\bar{\gamma},c}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} = \gamma_2 {}_a D_t^{\beta(\cdot,\cdot)} + \gamma_1 {}_t D_c^{\alpha(\cdot,\cdot)}, \quad \text{where } \bar{\gamma} = (\gamma_2, \gamma_1) \quad \text{and } c \in (a, b]. \quad (4.34)$$

With the functional \mathcal{J} , defined by (4.33), we consider two different isoperimetric problems.

4.5.1 Necessary optimality conditions I

The first fractional isoperimetric problem of the calculus of variations is Problem 4.

Problem 4. *Determine the local minimizers of \mathcal{J} over all $(x, T) \in D$ satisfying a boundary condition*

$$x(a) = x_a \quad (4.35)$$

for a fixed $x_a \in \mathbb{R}$ and an integral constraint of the form

$$\int_a^T g[x]_{\gamma}^{\alpha,\beta}(t) dt = \psi(T), \quad (4.36)$$

where $g : C^1([a, b] \times \mathbb{R}^2) \rightarrow \mathbb{R}$ and $\psi : [a, b] \rightarrow \mathbb{R}$ are two differentiable functions. The terminal time T and terminal state $x(T)$ are free.

In this problem, the condition of the form (4.36) is called an isoperimetric constraint. The next theorem gives fractional necessary optimality conditions to this isoperimetric problem.

Theorem 51 (Necessary optimality conditions for Problem 4). *Suppose that (x, T) gives a local minimum for functional (4.33) on D subject to the boundary condition (4.35) and the isoperimetric constraint (4.36). If (x, T) does not satisfies the Euler–Lagrange equations with respect to the isoperimetric constraint, that is, if one of the two following conditions are not verified,*

$$\partial_2 g[x]_{\gamma}^{\alpha,\beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha,\beta}(t) = 0, \quad t \in [a, T], \quad (4.37)$$

or

$$\gamma_2 \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha,\beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha,\beta}(t) \right] = 0, \quad t \in [T, b], \quad (4.38)$$

then there exists a constant λ such that, if we define the function $F : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F = L - \lambda g$, (x, T) satisfies the fractional Euler–Lagrange equations

$$\partial_2 F[x]_{\gamma}^{\alpha,\beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) = 0 \quad (4.39)$$

on the interval $[a, T]$ and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) \right) = 0 \quad (4.40)$$

on the interval $[T, b]$. Moreover, (x, T) satisfies the transversality conditions

$$\begin{cases} F[x]_{\gamma}^{\alpha,\beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'(T) + \lambda \psi'(T) = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha,\beta}(t) \right]_{t=b} = 0. \end{cases} \quad (4.41)$$

Proof. Consider variations of the optimal solution (x, T) of the type

$$(x^*, T^*) = (x + \epsilon_1 h_1 + \epsilon_2 h_2, T + \epsilon_1 \Delta T), \quad (4.42)$$

where, for each $i \in \{1, 2\}$, $\epsilon_i \in \mathbb{R}$ is a small parameter, $h_i \in C^1([a, b]; \mathbb{R})$ satisfies $h_i(a) = 0$, and $\Delta T \in \mathbb{R}$. The additional term $\epsilon_2 h_2$ must be selected so that the admissible variations (x^*, T^*) satisfy the isoperimetric constraint (4.36). For a fixed choice of h_i , let

$$i(\epsilon_1, \epsilon_2) = \int_a^{T+\epsilon_1 \Delta T} g[x^*]_\gamma^{\alpha, \beta}(t) dt - \psi(T + \epsilon_1 \Delta T).$$

For $\epsilon_1 = \epsilon_2 = 0$, we obtain that

$$i(0, 0) = \int_a^T g[x^*]_\gamma^{\alpha, \beta}(t) dt - \psi(T) = \psi(T) - \psi(T) = 0.$$

The derivative $\frac{\partial i}{\partial \epsilon_2}$ is given by

$$\frac{\partial i}{\partial \epsilon_2} = \int_a^{T+\epsilon_1 \Delta T} (\partial_2 g[x^*]_\gamma^{\alpha, \beta}(t) h_2(t) + \partial_3 g[x^*]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_2(t)) dt.$$

For $\epsilon_1 = \epsilon_2 = 0$ one has

$$\left. \frac{\partial i}{\partial \epsilon_2} \right|_{(0,0)} = \int_a^T (\partial_2 g[x]_\gamma^{\alpha, \beta}(t) h_2(t) + \partial_3 g[x]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_2(t)) dt. \quad (4.43)$$

The second term in (4.43) can be written as

$$\begin{aligned} & \int_a^T \partial_3 g[x]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_2(t) dt \\ &= \int_a^T \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \left[\gamma_1 {}^C D_a^{\alpha(\cdot, \cdot)} h_2(t) + \gamma_2 {}^C D_b^{\beta(\cdot, \cdot)} h_2(t) \right] dt \\ &= \gamma_1 \int_a^T \partial_3 g[x]_\gamma^{\alpha, \beta}(t) {}^C D_a^{\alpha(\cdot, \cdot)} h_2(t) dt \\ & \quad + \gamma_2 \left[\int_a^b \partial_3 g[x]_\gamma^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h_2(t) dt - \int_T^b \partial_3 g[x]_\gamma^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h_2(t) dt \right]. \end{aligned} \quad (4.44)$$

Using the fractional integrating by parts formula, (4.44) is equal to

$$\begin{aligned}
 & \int_a^T h_2(t) \left[\gamma_{1t} D_T^{\alpha(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) + \gamma_{2a} D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 & + \int_T^b \gamma_2 h_2(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 & + \left[h_2(t) \left(\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=T} \\
 & + \left[\gamma_2 h_2(t) \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

Substituting these relations into (4.43), and considering the fractional operator $D_{\gamma, c}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)}$ as defined in (4.34), we obtain that

$$\begin{aligned}
 \left. \frac{\partial i}{\partial \epsilon_2} \right|_{(0,0)} &= \int_a^T h_2(t) \left[\partial_2 g[x]_\gamma^{\alpha, \beta}(t) + D_{\gamma, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 & + \int_T^b \gamma_2 h_2(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right] dt \\
 & + \left[h_2(t) \left(\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=T} \\
 & + \left[\gamma_2 h_2(t) \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 g[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

Since (4.37) or (4.38) fails, there exists a function h_2 such that

$$\left. \frac{\partial i}{\partial \epsilon_2} \right|_{(0,0)} \neq 0.$$

In fact, if not, from the arbitrariness of the function h_2 and the fundamental lemma of the calculus of the variations, (4.37) and (4.38) would be verified. Thus, we may apply the implicit function theorem, that ensures the existence of a function $\epsilon_2(\cdot)$, defined in a neighborhood of zero, such that $i(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. In conclusion, there exists a subfamily of variations of the form (4.42) that verifies the integral constraint (4.36). We now seek to prove the main result. For that purpose, consider the auxiliary function $j(\epsilon_1, \epsilon_2) = \mathcal{J}(x^*, T^*)$.

By hypothesis, function j attains a local minimum at $(0, 0)$ when subject to the constraint $i(\cdot, \cdot) = 0$, and we proved before that $\nabla i(0, 0) \neq 0$. Applying the Lagrange multiplier

rule, we ensure the existence of a number λ such that

$$\nabla (j(0, 0) - \lambda i(0, 0)) = 0.$$

In particular,

$$\frac{\partial (j - \lambda i)}{\partial \epsilon_1}(0, 0) = 0. \quad (4.45)$$

Let $F = L - \lambda g$. The relation (4.45) can be written as

$$\begin{aligned} 0 = & \int_a^T h_1(t) \left[\partial_2 F[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) \right] dt \\ & + \int_T^b \gamma_2 h_1(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) \right] dt \\ & + h_1(T) \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\ & + \Delta T \left[F[x]_\gamma^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'(t) + \lambda \psi'(t) \right]_{t=T} \\ & + h_1(b) \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b}. \end{aligned} \quad (4.46)$$

As h_1 and ΔT are arbitrary, we can choose $\Delta T = 0$ and $h_1(t) = 0$ for all $t \in [T, b]$. But h_1 is arbitrary in $t \in [a, T]$. Then, we obtain the first necessary condition (4.39):

$$\partial_2 F[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) = 0 \quad \forall t \in [a, T].$$

Analogously, considering $\Delta T = 0$ and $h_1(t) = 0$ for all $t \in [a, T] \cup \{b\}$, and h_1 arbitrary on (T, b) , we obtain the second necessary condition (4.40):

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) \right) = 0 \quad \forall t \in [T, b].$$

As (x, T) is a solution to the necessary conditions (4.39) and (4.40), then equation (4.46) takes the form

$$\begin{aligned} 0 = & h_1(T) \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\ & + \Delta T \left[F[x]_\gamma^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'(t) + \lambda \psi'(t) \right]_{t=T} \\ & + h_1(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}. \end{aligned} \quad (4.47)$$

Transversality conditions (4.41) are obtained for appropriate choices of variations. \square

In the next theorem, considering the same Problem 4, we rewrite the transversality conditions (4.41) in terms of the increment on time ΔT and on the increment of space Δx_T given by

$$\Delta x_T = (x + h_1)(T + \Delta T) - x(T). \quad (4.48)$$

Theorem 52. *Let (x, T) be a local minimizer to the functional (4.33) on D subject to the boundary condition (4.35) and the isoperimetric constraint (4.36). Then (x, T) satisfies the transversality conditions*

$$\begin{cases} F[x]_{\gamma}^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) + \lambda \psi'(T) \\ \quad + x'(T) \left[\gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} = 0, \\ \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases} \quad (4.49)$$

Proof. Suppose (x^*, T^*) is an admissible variation of the form (4.42) with $\epsilon_1 = 1$ and $\epsilon_2 = 0$. Using Taylor's expansion up to first order for a small ΔT , and restricting the set of variations to those for which $h'_1(T) = 0$, we obtain the increment Δx_T on x :

$$(x + h_1)(T + \Delta T) = (x + h_1)(T) + x'(T)\Delta T + O(\Delta T)^2.$$

Relation (4.48) allows us to express $h_1(T)$ in terms of ΔT and Δx_T :

$$h_1(T) = \Delta x_T - x'(T)\Delta T + O(\Delta T)^2.$$

Substitution this expression into (4.47), and using appropriate choices of variations, we obtain the new transversality conditions (4.49). \square

Theorem 53. *Suppose that (x, T) gives a local minimum for functional (4.33) on D subject to the boundary condition (4.35) and the isoperimetric constraint (4.36). Then, there exists $(\lambda_0, \lambda) \neq (0, 0)$ such that, if we define the function $F : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F = \lambda_0 L - \lambda g$, (x, T) satisfies the following fractional Euler–Lagrange equations:*

$$\partial_2 F[x]_{\gamma}^{\alpha, \beta}(t) + D_{\gamma, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) = 0$$

on the interval $[a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) \right) = 0$$

on the interval $[T, b]$.

Proof. If (x, T) does not verifies (4.37) or (4.38), then the hypothesis of Theorem 51 is satisfied and we prove Theorem 53 considering $\lambda_0 = 1$. If (x, T) verifies (4.37) and (4.38), then we prove the result by considering $\lambda = 1$ and $\lambda_0 = 0$. \square

4.5.2 Necessary optimality conditions II

We now consider a new isoperimetric type problem with the isoperimetric constraint of form

$$\int_a^b g[x]_\gamma^{\alpha,\beta}(t)dt = C, \quad (4.50)$$

where C is a given real number.

Problem 5. Determine the local minimizers of \mathcal{J} (4.33), over all $(x, T) \in D$ satisfying a boundary condition

$$x(a) = x_a \quad (4.51)$$

for a fixed $x_a \in \mathbb{R}$ and an integral constraint of the form (4.50).

In the follow theorem, we give fractional necessary optimality conditions to this problem.

Theorem 54 (Necessary optimality conditions for Problem 5). *Suppose that (x, T) gives a local minimum for functional (4.33) on D subject to the boundary condition (4.51) and the isoperimetric constraint (4.50). If (x, T) does not satisfies the Euler–Lagrange equation with respect to the isoperimetric constraint, that is, the condition*

$$\partial_2 g[x]_\gamma^{\alpha,\beta}(t) + D_{\bar{\gamma},b}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \partial_3 g[x]_\gamma^{\alpha,\beta}(t) = 0, \quad t \in [a, b],$$

is not satisfied, then there exists $\lambda \neq 0$ such that, if we define the function $F : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F = L - \lambda g$, (x, T) satisfies the fractional Euler–Lagrange equations

$$\partial_2 F[x]_\gamma^{\alpha,\beta}(t) + D_{\bar{\gamma},T}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \lambda D_{\bar{\gamma},b}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \partial_3 g[x]_\gamma^{\alpha,\beta}(t) \quad (4.52)$$

on the interval $[a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot,\cdot)} \partial_3 F[x]_\gamma^{\alpha,\beta}(t) - {}_T D_t^{\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) \right) - \lambda \left(\partial_2 g[x]_\gamma^{\alpha,\beta}(t) + \gamma_{1t} D_b^{\alpha(\cdot,\cdot)} \partial_3 g[x]_\gamma^{\alpha,\beta}(t) \right) \quad (4.53)$$

on the interval $[T, b]$. Moreover, (x, T) satisfies the transversality conditions

$$\begin{cases} L[x]_\gamma^{\alpha,\beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'(T) = 0, \\ \left[\gamma_{1t} I_T^{1-\alpha(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} = 0 \\ \left[-\lambda \gamma_{1t} I_b^{1-\alpha(\cdot,\cdot)} \partial_3 g[x]_\gamma^{\alpha,\beta}(t) + \gamma_2 \left({}_T I_t^{1-\beta(\cdot,\cdot)} \partial_3 L[x]_\gamma^{\alpha,\beta}(t) - {}_a I_t^{1-\beta(\cdot,\cdot)} \partial_3 F[x]_\gamma^{\alpha,\beta}(t) \right) \right]_{t=b} = 0. \end{cases} \quad (4.54)$$

Proof. Similarly as done to prove Theorem 51, let

$$(x^*, T^*) = (x + \epsilon_1 h_1 + \epsilon_2 h_2, T + \epsilon_1 \Delta T)$$

be a variation of the solution, and define

$$i(\epsilon_1, \epsilon_2) = \int_a^b g[x^*]_{\gamma}^{\alpha, \beta}(t) dt - C.$$

The derivative $\frac{\partial i}{\partial \epsilon_2}$, when $\epsilon_1 = \epsilon_2 = 0$, is

$$\left. \frac{\partial i}{\partial \epsilon_2} \right|_{(0,0)} = \int_a^b (\partial_2 g[x]_{\gamma}^{\alpha, \beta}(t) h_2(t) + \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t)^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_2(t)) dt.$$

Integrating by parts and choosing variations such that $h_2(b) = 0$, we have

$$\left. \frac{\partial i}{\partial \epsilon_2} \right|_{(0,0)} = \int_a^b h_2(t) \left[\partial_2 g[x]_{\gamma}^{\alpha, \beta}(t) + D_{\bar{\gamma}, b}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t) \right] dt.$$

Thus, there exists a function h_2 such that

$$\left. \frac{\partial i}{\partial \epsilon_2} \right|_{(0,0)} \neq 0.$$

We may apply the implicit function theorem to conclude that there exists a subfamily of variations satisfying the integral constraint. Consider the new function $j(\epsilon_1, \epsilon_2) = \mathcal{J}(x^*, T^*)$. Since j has a local minimum at $(0, 0)$ when subject to the constraint $i(\cdot, \cdot) = 0$ and $\nabla i(0, 0) \neq 0$, there exists a number λ such that

$$\frac{\partial}{\partial \epsilon_1} (j - \lambda i)(0, 0) = 0. \quad (4.55)$$

Let $F = L - \lambda g$. Relation (4.55) can be written as

$$\begin{aligned} 0 &= \int_a^T h_1(t) \left[\partial_2 F[x]_{\gamma}^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \lambda D_{\bar{\gamma}, b}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\ &+ \int_T^b h_1(t) \left[\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right. \\ &\quad \left. - \lambda \left(\partial_2 g[x]_{\gamma}^{\alpha, \beta}(t) + \gamma_{1t} D_b^{\alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t) \right) \right] dt \\ &+ h_1(T) \left[\gamma_{1T} I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_{2T} I_T^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\ &+ \Delta T \left[L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'(t) \right]_{t=T} \\ &+ h_1(b) \left[-\lambda \gamma_{1t} I_b^{1-\alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t) + \gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=b}. \end{aligned}$$

Considering appropriate choices of variations, we obtain the first (4.52) and the second (4.53) necessary optimality conditions, and also the transversality conditions (4.54). \square

Similarly to Theorem 53, the following result holds.

Theorem 55. *Suppose that (x, T) gives a local minimum for functional (4.33) on D subject to the boundary condition (4.51) and the isoperimetric constraint (4.50). Then there exists $(\lambda_0, \lambda) \neq (0, 0)$ such that, if we define the function $F : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F = \lambda_0 L - \lambda g$, (x, T) satisfies the fractional Euler–Lagrange equations*

$$\partial_2 F[x]_{\gamma}^{\alpha, \beta}(t) + D_{\gamma, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - \lambda D_{\gamma, b}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t) = 0$$

on the interval $[a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 F[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) - \lambda \left(\partial_2 g[x]_{\gamma}^{\alpha, \beta}(t) + \gamma_{1t} D_b^{\alpha(\cdot, \cdot)} \partial_3 g[x]_{\gamma}^{\alpha, \beta}(t) \right) = 0$$

on the interval $[T, b]$.

4.5.3 Example

Let $\alpha(t, \tau) = \alpha(t)$ and $\beta(t, \tau) = \beta(\tau)$. Define the function

$$\psi(T) = \int_0^T \left(\frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} + \frac{(b-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^2 dt$$

on the interval $[0, b]$ with $b > 0$. Consider the functional J defined by

$$J(x, t) = \int_0^T \left[\alpha(t) + ({}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t))^2 + \left(\frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} + \frac{(b-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^2 \right] dt$$

for $t \in [0, b]$ and $\gamma = (1/2, 1/2)$, subject to the initial condition

$$x(0) = 0$$

and the isoperimetric constraint

$$\int_0^T {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \left(\frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} + \frac{(b-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^2 dt = \psi(T).$$

Define $F = L - \lambda g$ with $\lambda = 2$, that is,

$$F = \alpha(t) + \left({}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) - \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} - \frac{(b-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^2.$$

Consider the function $\bar{x}(t) = t$ with $t \in [0, b]$. Because

$${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} \bar{x}(t) = \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} + \frac{(b-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))},$$

we have that \bar{x} satisfies conditions (4.39), (4.40) and the two last of (4.41). Using the first condition of (4.41), that is,

$$\alpha(t) + 2 \left(\frac{T^{1-\alpha(T)}}{2\Gamma(2-\alpha(T))} + \frac{(b-T)^{1-\beta(T)}}{2\Gamma(2-\beta(T))} \right)^2 = 0,$$

we obtain the optimal time T .

4.6 Variational problems with holonomic constraints

In this section, we present a new variational problem subject to a new type of constraints. A holonomic constraint is a condition of the form

$$g(t, x) = 0,$$

where $x = (x_1, x_2, \dots, x_n)$, $n \geq 2$, and g is a given function (see, e.g., [112]).

Consider the space

$$U = \{(x_1, x_2, T) \in C^1([a, b]; \mathbb{R}) \times C^1([a, b]; \mathbb{R}) \times [a, b] : x_1(a) = x_{1a} \wedge x_2(a) = x_{2a}\} \quad (4.56)$$

for fixed reals $x_{1a}, x_{2a} \in \mathbb{R}$. In this section, we consider the following variational problem:

Problem 6. Find functions x_1 and x_2 that maximize or minimize the functional \mathcal{J} defined in U by

$$\begin{aligned} \mathcal{J}(x_1, x_2, T) &= \int_a^T L(t, x_1(t), x_2(t), {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x_1(t), {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x_2(t)) dt \\ &+ \phi(T, x_1(T), x_2(T)), \end{aligned} \quad (4.57)$$

where the admissible functions satisfy the constraint

$$g(t, x_1(t), x_2(t)) = 0, \quad t \in [a, b], \quad (4.58)$$

called a holonomic constraint, where $g : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function and continuously differentiable with respect to second and third arguments.

The terminal time T and terminal states $x_1(T)$ and $x_2(T)$ are free and the Lagrangian $L : [a, b] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is a continuous function and continuously differentiable with respect to its i th argument, $i \in \{2, 3, 4, 5\}$. The terminal cost function $\phi : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class C^1 .

4.6.1 Necessary optimality conditions

The next theorem gives fractional necessary optimality conditions to the variational problem with a holonomic constraint. To simplify the notation, we denote by x the vector (x_1, x_2) ; by ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x$ the vector $({}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x_1, {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x_2)$; and we use the operator

$$[x]_\gamma^{\alpha, \beta}(t) := (t, x(t), {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t)).$$

Theorem 56 (Necessary optimality conditions for problem 6). *Suppose that (x, T) gives a local minimum to functional \mathcal{J} as in (4.57), under the constraint (4.58) and the boundary conditions defined in (4.56). If*

$$\partial_3 g(t, x(t)) \neq 0 \quad \forall t \in [a, b],$$

then there exists a piecewise continuous function $\lambda : [a, b] \rightarrow \mathbb{R}$ such that (x, T) satisfies the following fractional Euler–Lagrange equations:

$$\partial_2 L[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot), \alpha(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \lambda(t) \partial_2 g(t, x(t)) = 0 \quad (4.59)$$

and

$$\partial_3 L[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot), \alpha(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) + \lambda(t) \partial_3 g(t, x(t)) = 0 \quad (4.60)$$

on the interval $[a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \lambda(t) \partial_2 g(t, x(t)) \right) = 0 \quad (4.61)$$

and

$${}_a D_t^{\beta(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) + \lambda(t) \partial_3 g(t, x(t)) = 0 \quad (4.62)$$

on the interval $[T, b]$. Moreover, (x, T) satisfies the transversality conditions

$$\begin{cases} L[x]_\gamma^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T)) x'_1(T) + \partial_3 \phi(T, x(T)) x'_2(T) = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} + \partial_3 \phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} = 0 \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases} \quad (4.63)$$

Proof. Consider admissible variations of the optimal solution (x, T) of the type

$$(x^*, T^*) = (x + \epsilon h, T + \epsilon \Delta T),$$

where $\epsilon \in \mathbb{R}$ is a small parameter, $h = (h_1, h_2) \in C^1([a, b]; \mathbb{R}) \times C^1([a, b]; \mathbb{R})$ satisfies $h_i(a) = 0$, $i = 1, 2$, and $\Delta T \in \mathbb{R}$. Because

$$\partial_3 g(t, x(t)) \neq 0 \quad \forall t \in [a, b],$$

by the implicit function theorem there exists a subfamily of variations of (x, T) that satisfy (4.58), that is, there exists a unique function $h_2(\epsilon, h_1)$ such that the admissible variation (x^*, T^*) satisfies the holonomic constraint (4.58):

$$g(t, x_1(t) + \epsilon h_1(t), x_2(t) + \epsilon h_2) = 0 \quad \forall t \in [a, b].$$

Differentiating this condition with respect to ϵ and considering $\epsilon = 0$, we obtain that

$$\partial_2 g(t, x(t)) h_1(t) + \partial_3 g(t, x(t)) h_2(t) = 0,$$

which is equivalent to

$$\frac{\partial_2 g(t, x(t)) h_1(t)}{\partial_3 g(t, x(t))} = -h_2(t). \quad (4.64)$$

Define j on a neighbourhood of zero by

$$j(\epsilon) = \int_a^{T+\epsilon\Delta T} L[x^*]_{\gamma}^{\alpha, \beta}(t) dt + \phi(T + \epsilon \Delta T, x^*(T + \epsilon \Delta T)).$$

The derivative $\frac{\partial j}{\partial \epsilon}$ for $\epsilon = 0$ is

$$\begin{aligned} \left. \frac{\partial j}{\partial \epsilon} \right|_{\epsilon=0} &= \int_a^T (\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) h_1(t) + \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) h_2(t) \\ &\quad + \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t)^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_1(t) + \partial_5 L[x]_{\gamma}^{\alpha, \beta}(t)^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_2(t)) dt \\ &\quad + L[x]_{\gamma}^{\alpha, \beta}(T) \Delta T + \partial_1 \phi(T, x(T)) \Delta T + \partial_2 \phi(T, x(T)) [h_1(T) + x'_1(T) \Delta T] \\ &\quad + \partial_3 \phi(T, x(T)) [h_2(T) + x'_2(T) \Delta T]. \end{aligned} \quad (4.65)$$

The third term in (4.65) can be written as

$$\begin{aligned}
 & \int_a^T \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h_1(t) dt \\
 &= \int_a^T \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \left[\gamma_1 {}^C D_a^{\alpha(\cdot, \cdot)} h_1(t) + \gamma_2 {}^C D_b^{\beta(\cdot, \cdot)} h_1(t) \right] dt \\
 &= \gamma_1 \int_a^T \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_a^{\alpha(\cdot, \cdot)} h_1(t) dt \\
 & \quad + \gamma_2 \left[\int_a^b \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h_1(t) dt - \int_T^b \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h_1(t) dt \right].
 \end{aligned} \tag{4.66}$$

Integrating by parts, (4.66) can be written as

$$\begin{aligned}
 & \int_a^T h_1(t) \left[\gamma_1 {}_T D_T^{\alpha(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) + \gamma_2 {}_a D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & \quad + \int_T^b \gamma_2 h_1(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right] dt \\
 & \quad + \left[h_1(t) \left(\gamma_1 {}_T I_T^{1-\alpha(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=T} \\
 & \quad + \left[\gamma_2 h_1(t) \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

By proceeding similarly to the 4th term in (4.65), we obtain an equivalent expression. Substituting these relations into (4.65) and considering the fractional operator $D_{\gamma, c}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)}$ as defined in (4.34), we obtain that

$$\begin{aligned}
 0 &= \int_a^T \left[h_1(t) \left[\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) + D_{\gamma, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right] \right. \\
 & \quad \left. + h_2(t) \left[\partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) + D_{\gamma, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_5 L[x]_{\gamma}^{\alpha, \beta}(t) \right] \right] dt \\
 & \quad + \gamma_2 \int_T^b \left[h_1(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) \right] \right. \\
 & \quad \left. + h_2(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_5 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_5 L[x]_{\gamma}^{\alpha, \beta}(t) \right] \right] dt \\
 & \quad + h_1(T) \left[\gamma_1 {}_T I_T^{1-\alpha(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_{\gamma}^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T}
 \end{aligned} \tag{4.67}$$

$$\begin{aligned}
 & +h_2(T) \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) + \partial_3 \phi(t, x(t)) \right]_{t=T} \\
 & + \Delta T \left[L[x]_\gamma^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'_1(t) + \partial_3 \phi(t, x(t)) x'_2(t) \right]_{t=T} \\
 & + h_1(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b} \\
 & + h_2(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

Define the piecewise continuous function λ by

$$\lambda(t) = \begin{cases} -\frac{\partial_3 L[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t)}{\partial_3 g(t, x(t))}, & t \in [a, T] \\ -\frac{{}_a D_t^{\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t)}{\partial_3 g(t, x(t))}, & t \in [T, b]. \end{cases} \quad (4.68)$$

Using equations (4.64) and (4.68), we obtain that

$$\lambda(t) \partial_2 g(t, x(t)) h_1(t) = \begin{cases} (\partial_3 L[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t)) h_2(t), & t \in [a, T] \\ ({}_a D_t^{\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t)) h_2(t), & t \in [T, b]. \end{cases}$$

Substituting in (4.67), we have

$$\begin{aligned}
 0 & = \int_a^T h_1(t) \left[\partial_2 L[x]_\gamma^{\alpha, \beta}(t) + D_{\bar{\gamma}, T}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \lambda(t) \partial_2 g(t, x(t)) \right] dt \\
 & + \gamma_2 \int_T^b h_1(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \lambda(t) \partial_2 g(t, x(t)) \right] dt \\
 & + h_1(T) \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) + \partial_2 \phi(t, x(t)) \right]_{t=T} \\
 & + h_2(T) \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) + \partial_3 \phi(t, x(t)) \right]_{t=T} \\
 & + \Delta T \left[L[x]_\gamma^{\alpha, \beta}(t) + \partial_1 \phi(t, x(t)) + \partial_2 \phi(t, x(t)) x'_1(t) + \partial_3 \phi(t, x(t)) x'_2(t) \right]_{t=T} \\
 & + h_1(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_4 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b} \\
 & + h_2(b) \left[\gamma_2 \left({}_T I_t^{1-\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_5 L[x]_\gamma^{\alpha, \beta}(t) \right) \right]_{t=b}.
 \end{aligned}$$

Considering appropriate choices of variations, we obtained the first (4.59) and the third (4.61) necessary conditions, and also the transversality conditions (4.63). The remaining conditions (4.60) and (4.62) follow directly from (4.68). \square

4.6.2 Example

We end this section with a simple illustrative example. Consider the following problem:

$$J(x, t) = \int_0^T \left[\alpha(t) + \left({}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x_1(t) - \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} - \frac{(b-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))} \right)^2 + \left({}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x_2(t) \right)^2 \right] dt \longrightarrow \min,$$

$$x_1(t) + x_2(t) = t + 1,$$

$$x_1(0) = 0, \quad x_2(0) = 1.$$

It is a simple exercise to check that $x_1(t) = t$, $x_2(t) \equiv 1$ and $\lambda(t) \equiv 0$ satisfy our Theorem 56.

4.7 Fractional variational Herglotz problem

In this section, we study fractional variational problems of Herglotz type, depending on the combined Caputo fractional derivatives ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)}$. Two different cases are considered. The variational problem of Herglotz is a generalization of the classical variational problem. It allows us to describe nonconservative processes, even in case the Lagrange function is autonomous (that is, when the Lagrangian does not depend explicitly on time). In opposite to calculus of variations, where the cost functional is given by an integral depending only on time, space and on the dynamics, in the Herglotz variational problem the model is given by a differential equation involving the derivative of the objective function z , and the Lagrange function depends on time, trajectories x and z , and on the derivative of x . The problem of Herglotz was posed by Herglotz in 1930 [43], but only in 1996, with the works [41, 42], it has gained the attention of the mathematical community. Since then, several papers were devoted to this subject. For example, see references [8, 39, 40, 91–93].

In Section 4.7.1, we obtain fractional Euler–Lagrange conditions for the fractional variational problem of Herglotz, with one variable, and the general case, for several independent variables is discussed in Section 4.7.2. Finally, three illustrative examples are presented in detail (Section 4.7.3).

4.7.1 Fundamental problem of Herglotz

Let $\alpha, \beta : [a, b]^2 \rightarrow (0, 1)$ be two functions. The fractional Herglotz variational problem that we study is as follows.

Problem 7. *Determine the trajectories $x \in C^1([a, b]; \mathbb{R})$ satisfying a given boundary condition $x(a) = x_a$, for a fixed $x_a \in \mathbb{R}$, and a real $T \in (a, b]$, that extremize the value of $z(T)$, where z satisfies the differential equation*

$$\dot{z}(x, t) = L(t, x(t), {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t), z(t)), \quad t \in [a, b], \quad (4.69)$$

with dependence on a combined Caputo fractional derivative operator, subject to the initial condition

$$z(a) = z_a, \quad (4.70)$$

where z_a is a fixed real number.

In the sequel, we use the auxiliary notation

$$[x, z]_\gamma^{\alpha, \beta}(t) = (t, x(t), {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t), z(t)).$$

The Lagrangian L is assumed to satisfy the following hypothesis:

1. $L : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is differentiable,
2. $t \rightarrow \lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)$ is such that ${}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t))$, ${}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t))$, and $D_\gamma^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t))$ exist and are continuous on $[a, b]$, where

$$\lambda(t) = \exp \left(- \int_a^t \partial_4 L[x, z]_\gamma^{\alpha, \beta}(\tau) d\tau \right).$$

The following result gives necessary conditions of Euler–Lagrange type for a solution of Problem 7.

Theorem 57 (Necessary optimality conditions for Problem 7). *Let $x \in C^1([a, b]; \mathbb{R})$ be such that z defined by Eq. (4.69), subject to the initial condition (4.70), has an extremum at $T \in (a, b]$. Then, (x, z) satisfies the fractional differential equations*

$$\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) \lambda(t) + D_\gamma^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) = 0, \quad (4.71)$$

on $[a, T]$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right) = 0, \quad (4.72)$$

on $[T, b]$. Moreover, the following transversality conditions are satisfied:

$$\begin{cases} \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right]_{t=T} = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - {}_a I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right]_{t=b} = 0. \end{cases} \quad (4.73)$$

If $T < b$, then $L[x, z]_{\gamma}^{\alpha, \beta}(T) = 0$.

Proof. Let x be a solution of the problem. Consider an admissible variation of x , $\bar{x} = x + \epsilon h$, where $h \in C^1([a, b]; \mathbb{R})$ is an arbitrary perturbation curve and $\epsilon \in \mathbb{R}$ represents a small number. The constraint $x(a) = x_a$ implies that all admissible variations must fulfill the condition $h(a) = 0$. On the other hand, consider an admissible variation of z , $\bar{z} = z + \epsilon \theta$, where θ is a perturbation curve (not arbitrary) such that

1. $\theta(a) = 0$, so that $z(a) = z_a$,
2. $\theta(T) = 0$ because $z(T)$ is a maximum (or a minimum),
i.e., $\bar{z}(T) - z(T) \leq 0$ ($\bar{z}(T) - z(T) \geq 0$),
3. $\theta(t) = \left. \frac{d}{d\epsilon} z(\bar{x}, t) \right|_{\epsilon=0}$, so that the variation satisfies the differential equation (4.69).

Differentiating θ with respect to t , we obtain that

$$\begin{aligned} \frac{d}{dt} \theta(t) &= \left. \frac{d}{dt} \frac{d}{d\epsilon} z(\bar{x}, t) \right|_{\epsilon=0} \\ &= \left. \frac{d}{d\epsilon} \frac{d}{dt} z(\bar{x}, t) \right|_{\epsilon=0} \\ &= \left. \frac{d}{d\epsilon} L(t, x(t) + \epsilon h(t), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) + \epsilon {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t), \bar{z}(t)) \right|_{\epsilon=0}, \end{aligned}$$

and rewriting this relation, we obtain the following differential equation for θ :

$$\dot{\theta}(t) - \partial_4 L[x, z]_{\gamma}^{\alpha, \beta}(t) \theta(t) = \partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) h(t) + \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t).$$

Considering $\lambda(t) = \exp\left(-\int_a^t \partial_4 L[x, z]_{\gamma}^{\alpha, \beta}(\tau) d\tau\right)$, we obtain the solution for the last differential equation

$$\theta(T) \lambda(T) - \theta(a) = \int_a^T (\partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) h(t) + \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t)) \lambda(t) dt.$$

By hypothesis, $\theta(a) = 0$. If x is such that $z(x, t)$ defined by (4.69) attains an extremum at $t = T$, then $\theta(T)$ is identically zero. Hence, we get

$$\int_a^T (\partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) h(t) + \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t)) \lambda(t) dt = 0. \quad (4.74)$$

Considering only the second term in Eq. (4.74) and the definition of combined Caputo derivative operator, we obtain that

$$\begin{aligned} & \int_a^T \lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) \left(\gamma_1^C D_t^{\alpha(\cdot, \cdot)} h(t) + \gamma_2^C D_b^{\beta(\cdot, \cdot)} h(t) \right) dt \\ &= \gamma_1 \int_a^T \lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)^C D_t^{\alpha(\cdot, \cdot)} h(t) dt + \\ & \quad + \gamma_2 \left[\int_a^b \lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)^C D_b^{\beta(\cdot, \cdot)} h(t) dt - \int_T^b \lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)^C D_b^{\beta(\cdot, \cdot)} h(t) dt \right] \\ &= \star. \end{aligned}$$

Using the fractional integration by parts formula and considering $\bar{\gamma} = (\gamma_2, \gamma_1)$, we get

$$\begin{aligned} \star &= \int_a^T h(t) D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) dt \\ & \quad + \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right] dt \\ & \quad + h(T) \left[\gamma_{1T} I_T^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - \gamma_{2T} I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right]_{t=T} \\ & \quad + h(b) \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - {}_a I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right]_{t=b}. \end{aligned}$$

Substituting this relation into expression (4.74), we obtain

$$\begin{aligned} & \int_a^T h(t) \left[\partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) \lambda(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right] dt \\ & \quad + \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right] dt \\ & \quad + h(T) \left[\gamma_{1T} I_T^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - \gamma_{2T} I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right]_{t=T} \\ & \quad + h(b) \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) - {}_a I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)) \right]_{t=b} = 0. \end{aligned}$$

With appropriate choices for the variations $h(\cdot)$, we get the Euler–Lagrange equations (4.71)–(4.72) and the transversality conditions (4.73). \square

Remark 58. If $\alpha(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ tend to 1, and if the Lagrangian L is of class C^2 , then the first Euler–Lagrange equation (4.71) becomes

$$\partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) \lambda(t) + (\gamma_2 - \gamma_1) \frac{d}{dt} [\lambda(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t)] = 0.$$

Differentiating and considering the derivative of the lambda function, we obtain

$$\lambda(t) \left[\partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) + (\gamma_2 - \gamma_1) \left[-\partial_4 L[x, z]_{\gamma}^{\alpha, \beta}(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) + \frac{d}{dt} \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) \right] \right] = 0.$$

As $\lambda(t) > 0$, for all t , we deduce that

$$\partial_2 L[x, z]_{\gamma}^{\alpha, \beta}(t) + (\gamma_2 - \gamma_1) \left[\frac{d}{dt} \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) - \partial_4 L[x, z]_{\gamma}^{\alpha, \beta}(t) \partial_3 L[x, z]_{\gamma}^{\alpha, \beta}(t) \right] = 0.$$

4.7.2 Several independent variables

Consider the following generalization of the problem of Herglotz involving $n + 1$ independent variables. Define $\Omega = \prod_{i=1}^n [a_i, b_i]$, with $n \in \mathbb{N}$, $P = [a, b] \times \Omega$ and consider the vector $s = (s_1, s_2, \dots, s_n) \in \Omega$. The new problem consists in determining the trajectories $x \in C^1(P)$ that give an extremum to $z[x, T]$, where the functional z satisfies the differential equation

$$\begin{aligned} \dot{z}(t) = \int_{\Omega} L(t, s, x(t, s), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t, s), \\ {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} x(t, s), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t, s), z(t)) d^n s \end{aligned} \quad (4.75)$$

subject to the constraint

$$x(t, s) = g(t, s), \quad \text{for all } (t, s) \in \partial P, \quad (4.76)$$

where ∂P is the boundary of P and g is a given function $g : \partial P \rightarrow \mathbb{R}$. We assume that

1. $\alpha, \alpha_i, \beta, \beta_i : [a, b]^2 \rightarrow (0, 1)$ with $i = 1, \dots, n$,
2. $\gamma, \gamma^1, \dots, \gamma^n \in [0, 1]^2$,
3. $d^n s = ds_1 \dots ds_n$,
4. ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t, s), {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} x(t, s), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t, s)$ exist and are continuous functions,

5. the Lagrangian $L : P \times \mathbb{R}^{n+3} \rightarrow \mathbb{R}$ is of class C^1 .

Remark 59. By ${}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t, s)$ we mean the Caputo fractional derivative with respect to the independent variable t , and by ${}^C D_{\gamma^i}^{\alpha_i(\cdot), \beta_i(\cdot)} x(t, s)$ we mean the Caputo fractional derivative with respect to the independent variable s_i , for $i = 1, \dots, n$.

In the sequel, we use the auxiliary notation $[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)$ to represent the following vector

$$\left(t, s, x(t, s), {}^C D_\gamma^{\alpha(\cdot), \beta(\cdot)} x(t, s), {}^C D_{\gamma^1}^{\alpha_1(\cdot), \beta_1(\cdot)} x(t, s), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot), \beta_n(\cdot)} x(t, s), z(t) \right).$$

Consider the function

$$\lambda(t) = \exp \left(- \int_a^t \int_\Omega \partial_{2n+4} [x, z]_{n, \gamma}^{\alpha, \beta}(\tau, s) d^n s d\tau \right).$$

Theorem 60. If (x, z, T) is an extremizer of the functional defined by Eq. (4.75), then (x, z) satisfies the fractional differential equations

$$\begin{aligned} \partial_{n+2} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s) \lambda(t) + D_{\bar{\gamma}}^{\beta(\cdot), \alpha(\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \\ + \sum_{i=1}^n D_{\bar{\gamma}^i}^{\beta_i(\cdot), \alpha_i(\cdot)} (\lambda(t) \partial_{n+3+i} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) = 0 \end{aligned} \quad (4.77)$$

on $[a, T] \times \Omega$, and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) - {}_T D_t^{\beta(\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \right) = 0 \quad (4.78)$$

on $[T, b] \times \Omega$.

Moreover, (x, z) satisfies the transversality condition

$$\begin{aligned} \left[\gamma_{1t} I_T^{1-\alpha(\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \right. \\ \left. - \gamma_{2T} I_t^{1-\beta(\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \right]_{t=T} = 0, \quad s \in \Omega. \end{aligned} \quad (4.79)$$

If $T < b$, then $\int_\Omega L[x, z]_{n, \gamma}^{\alpha, \beta}(T, s) d^n s = 0$.

Proof. Let x be a solution of the problem. Consider an admissible variation of x , $\bar{x}(t, s) = x(t, s) + \epsilon h(t, s)$, where $h \in C^1(P)$ is an arbitrary perturbing curve and $\epsilon \in \mathbb{R}$ is such that $|\epsilon| \ll 1$. Consequently, $h(t, s) = 0$ for all $(t, s) \in \partial P$ by the boundary condition (4.76).

On the other hand, consider an admissible variation of z , $\bar{z} = z + \epsilon\theta$, where θ is a perturbing curve such that $\theta(a) = 0$ and

$$\theta(t) = \left. \frac{d}{d\epsilon} z(\bar{x}, t) \right|_{\epsilon=0}.$$

Differentiating $\theta(t)$ with respect to t , we obtain that

$$\begin{aligned} \frac{d}{dt}\theta(t) &= \left. \frac{d}{dt} \frac{d}{d\epsilon} z(\bar{x}, t) \right|_{\epsilon=0} \\ &= \left. \frac{d}{d\epsilon} \frac{d}{dt} z(\bar{x}, t) \right|_{\epsilon=0} \\ &= \left. \frac{d}{d\epsilon} \int_{\Omega} L[\bar{x}, z]_{n,\gamma}^{\alpha,\beta}(t, s) d^n s \right|_{\epsilon=0}. \end{aligned}$$

We conclude that

$$\begin{aligned} \dot{\theta}(t) &= \int_{\Omega} \left(\partial_{n+2} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) h(t, s) + \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}^C D_{\gamma}^{\alpha(\cdot,\cdot), \beta(\cdot,\cdot)} h(t, s) \right. \\ &\quad \left. + \sum_{i=1}^n \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}^C D_{\gamma^i}^{\alpha_i(\cdot,\cdot), \beta_i(\cdot,\cdot)} h(t, s) + \partial_{2n+4} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \theta(t) \right) d^n s. \end{aligned}$$

To simplify the notation, we define

$$B(t) = \int_{\Omega} \partial_{2n+4} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) d^n s$$

and

$$\begin{aligned} A(t) &= \int_{\Omega} \left(\partial_{n+2} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) h(t, s) + \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}^C D_{\gamma}^{\alpha(\cdot,\cdot), \beta(\cdot,\cdot)} h(t, s) \right. \\ &\quad \left. + \sum_{i=1}^n \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}^C D_{\gamma^i}^{\alpha_i(\cdot,\cdot), \beta_i(\cdot,\cdot)} h(t, s) \right) d^n s. \end{aligned}$$

Then, we obtain the linear differential equation

$$\dot{\theta}(t) - B(t)\theta(t) = A(t),$$

whose solution is

$$\theta(T)\lambda(T) - \theta(a) = \int_a^T A(t)\lambda(t) dt.$$

Since $\theta(a) = \theta(T) = 0$, we get

$$\int_a^T A(t)\lambda(t) dt = 0. \tag{4.80}$$

Considering only the second term in (4.80), we can write

$$\begin{aligned}
 & \int_a^T \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \left(\gamma_1 {}_a^C D_t^{\alpha(\cdot,\cdot)} h(t, s) + \gamma_2 {}_t^C D_b^{\beta(\cdot,\cdot)} h(t, s) \right) d^n s dt \\
 &= \gamma_1 \int_a^T \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}_a^C D_t^{\alpha(\cdot,\cdot)} h(t, s) d^n s dt \\
 &+ \gamma_2 \left[\int_a^b \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}_t^C D_b^{\beta(\cdot,\cdot)} h(t, s) d^n s dt \right. \\
 &\quad \left. - \int_T^b \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}_t^C D_b^{\beta(\cdot,\cdot)} h(t, s) d^n s dt \right].
 \end{aligned}$$

Let $\bar{\gamma} = (\gamma_2, \gamma_1)$. Integrating by parts (c.f. Theorem 13), and since $h(a, s) = h(b, s) = 0$ for all $s \in \Omega$, we obtain the following expression:

$$\begin{aligned}
 & \int_a^T \int_{\Omega} h(t, s) D_{\bar{\gamma}}^{\beta(\cdot,\cdot), \alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) d^n s dt \\
 &+ \gamma_2 \int_T^b \int_{\Omega} h(t, s) \left[{}_a D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\
 &\quad \left. - {}_T D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right] d^n s dt \\
 &+ \int_{\Omega} h(T, s) \left[\gamma_1 I_T^{1-\alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\
 &\quad \left. - \gamma_2 {}_T I_t^{1-\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right]_{t=T} d^n s.
 \end{aligned}$$

Doing similarly for the $(i+2)$ th term in (4.80), with $i = 1, \dots, n$, letting $\bar{\gamma}^i = (\gamma_2^i, \gamma_1^i)$, and since $h(t, a_i) = h(t, b_i) = 0$ for all $t \in [a, b]$, we obtain

$$\begin{aligned}
 & \int_a^T \int_{\Omega} \lambda(t) \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \left(\gamma_1 {}_{a_i}^C D_{s_i}^{\alpha_i(\cdot,\cdot)} h(t, s) + \gamma_2 {}_{s_i}^C D_{b_i}^{\beta_i(\cdot,\cdot)} h(t, s) \right) d^n s dt \\
 &= \int_a^T \int_{\Omega} h(t, s) D_{\bar{\gamma}^i}^{\beta_i(\cdot,\cdot), \alpha_i(\cdot,\cdot)} (\lambda(t) \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) d^n s dt.
 \end{aligned}$$

Substituting these relations into (4.80), we deduce that

$$\begin{aligned}
 & \int_a^T \int_{\Omega} h(t, s) \left[\partial_{n+2} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \lambda(t) + D_{\bar{\gamma}}^{\beta(\cdot,\cdot), \alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\
 &+ \sum_{i=1}^n D_{\bar{\gamma}^i}^{\beta_i(\cdot,\cdot), \alpha_i(\cdot,\cdot)} (\lambda(t) \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \left. \right] d^n s dt \\
 &+ \gamma_2 \int_T^b \int_{\Omega} h(t, s) \left[{}_a D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right.
 \end{aligned}$$

$$\begin{aligned}
 & -_T D_t^{\beta(\cdot, \cdot)} \left(\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s) \right) \Big] d^n s dt \\
 & + \int_{\Omega} h(T, s) \left[\gamma_{1t} I_T^{1-\alpha(\cdot, \cdot)} \left(\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s) \right) \right. \\
 & \quad \left. - \gamma_{2T} I_t^{1-\beta(\cdot, \cdot)} \left(\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s) \right) d^n s \right]_{t=T}.
 \end{aligned}$$

For appropriate choices with respect to h , we get the Euler–Lagrange equations (4.77)–(4.78) and the transversality condition (4.79). \square

4.7.3 Examples

We present three examples, with and without the dependence on z .

Example 1. Consider

$$\begin{aligned}
 \dot{z}(t) &= \left({}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \right)^2 + z(t) + t^2 - 1, \quad t \in [0, 3], \\
 x(0) &= 1, \quad z(0) = 0.
 \end{aligned} \tag{4.81}$$

In this case, $\lambda(t) = \exp(-t)$. The necessary optimality conditions (4.71)–(4.72) of Theorem 57 hold for $\bar{x}(t) \equiv 1$. If we replace x by \bar{x} in (4.81), we obtain

$$\begin{aligned}
 \dot{z}(t) - z(t) &= t^2 - 1, \quad t \in [0, 3], \\
 z(0) &= 0,
 \end{aligned}$$

whose solution is

$$z(t) = \exp(t) - (t + 1)^2. \tag{4.82}$$

The last transversality condition of Theorem 57 asserts that

$$L[\bar{x}, z]_{\gamma}^{\alpha, \beta}(T) = 0 \Leftrightarrow \exp(T) - 2T - 2 = 0,$$

whose solution is approximately

$$T \approx 1.67835.$$

We remark that z (4.82) actually attains a minimum value at this point (see Figure 4.1, (a)):

$$z(1.67835) \approx -1.81685.$$

Example 2. Consider now

$$\begin{aligned}
 \dot{z}(t) &= (t - 1) (x^2(t) + z^2(t) + 1), \quad t \in [0, 3], \\
 x(0) &= 0, \quad z(0) = 0.
 \end{aligned} \tag{4.83}$$

Since the first Euler–Lagrange equation (4.71) reads

$$(t - 1)x(t) = 0, \quad \forall t \in [0, T],$$

we see that $\bar{x}(t) \equiv 0$ is a solution of this equation. The second transversality condition of (4.73) asserts that, at $t = T$, we must have

$$L[\bar{x}, z]_{\gamma}^{\alpha, \beta}(t) = 0,$$

that is,

$$(t - 1)(z^2(t) + 1) = 0,$$

and so $T = 1$ is a solution for this equation. Substituting x by \bar{x} in (4.83), we get

$$\begin{aligned} \dot{z}(t) &= (t - 1)(z^2(t) + 1), \quad t \in [0, 3], \\ z(0) &= 0. \end{aligned}$$

The solution to this Cauchy problem is the function

$$z(t) = \tan\left(\frac{t^2}{2} - t\right),$$

(see Figure 4.1, (b)) and the minimum value is

$$z(1) = \tan\left(-\frac{1}{2}\right).$$

Example 3. For our last example, consider

$$\begin{aligned} \dot{z}(t) &= \left({}^C D_{\gamma}^{\alpha(\cdot), \beta(\cdot)} x(t) - f(t)\right)^2 + t^2 - 1, \quad t \in [0, 3], \\ x(0) &= 0, \quad z(0) = 0, \end{aligned} \tag{4.84}$$

where

$$f(t) := \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} - \frac{(3-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))}.$$

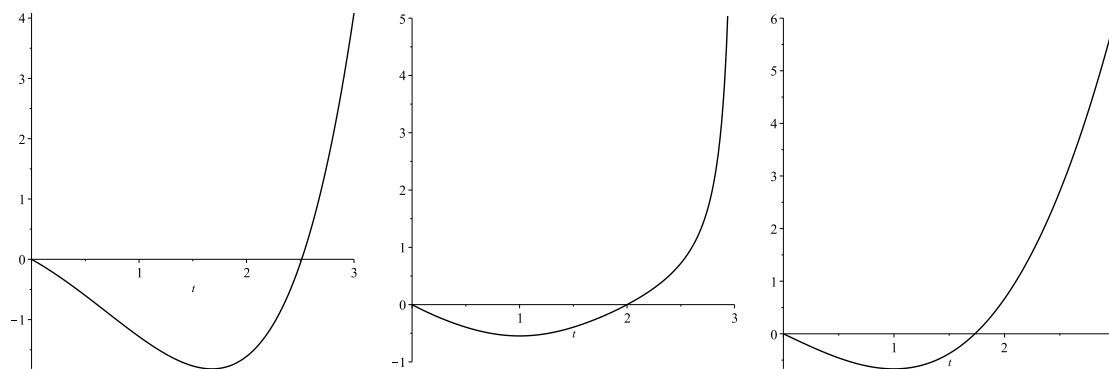
In this case, $\lambda(t) \equiv 1$. We intend to find a pair (x, z) , satisfying all the conditions in (4.84), for which $z(T)$ attains a minimum value. It is easy to verify that $\bar{x}(t) = t$ and $T = 1$ satisfy the necessary conditions given by Theorem 57. Replacing x by \bar{x} in (4.84), we get a Cauchy problem of form

$$\begin{aligned} \dot{z}(t) &= t^2 - 1, \quad t \in [0, 3], \\ z(0) &= 0, \end{aligned}$$

whose solution is

$$z(t) = \frac{t^3}{3} - t.$$

Observe that this function attains a minimum value at $T = 1$, which is $z(1) = -2/3$ (Figure 4.1, (c)).



(a) Extremal z of Example 1. (b) Extremal z of Example 2. (c) Extremal z of Example 3.

Figure 4.1: Graphics of function $z(\bar{x}, t)$.

Appendix

In this appendix, we use a specific MATLAB software, the package `Chebfun`, to obtain a few computational approximations for the main fractional operators in this PhD Thesis.

`Chebfun` is an open source software package that “aims to provide numerical computing with functions” in MATLAB [55]. `Chebfun` overloads MATLAB’s discrete operations for matrices to analogous continuous operations for functions and operators [109]. For the mathematical underpinnings of `Chebfun`, we refer the reader to [109]. For the algorithmic backstory of `Chebfun`, we refer to [35].

In what follows, we study some computational approximations of Riemann–Liouville fractional integrals, of Caputo fractional derivatives and consequently of the combined Caputo fractional derivative, all of them with variable order. We provide, also, the necessary `Chebfun` code for the variable order fractional calculus.

To implement these operators, we need two auxiliary functions: the gamma function Γ (Definition 1) and the beta function B (Definition 3). Both functions are available in MATLAB through the commands `gamma(t)` and `beta(t,u)`, respectively.

A.1 Higher-order Riemann–Liouville fractional integrals

In this section, we discuss computational aspects to the higher-order Riemann–Liouville fractional integrals of variable order ${}_a I_t^{\alpha_n(\cdot)} x(t)$ and ${}_t I_b^{\alpha_n(\cdot)} x(t)$.

Considering the Definition 34 of higher-order Riemann–Liouville fractional integrals, we implemented in `Chebfun` two functions `leftFI(x,alpha,a)` and `rightFI(x,alpha,b)` that approximate, respectively, the Riemann–Liouville fractional integrals ${}_a I_t^{\alpha_n(\cdot)} x(t)$ and ${}_t I_b^{\alpha_n(\cdot)} x(t)$, through the following `Chebfun`/MATLAB code.

```
function r = leftFI(x,alpha,a)
    g = @(t,tau) x(tau)./(gamma(alpha(t,tau)).*(t-tau).^(1-alpha(t,tau)));
```

```

r = @(t) sum(chebfun(@(tau) g(t,tau),[a t], 'splitting', 'on'), [a t]);
end

```

and

```

function r = rightFI(x,alpha,b)
g = @(t,tau) x(tau)./(gamma(alpha(tau,t)).*(tau-t).^(1-alpha(tau,t)));
r = @(t) sum(chebfun(@(tau) g(t,tau),[t b], 'splitting', 'on'), [t b]);
end

```

With these two functions, we illustrate their use in the following example, where we determine computational approximations for Riemann–Liouville fractional integrals of a special power function.

Example 4. Let $\alpha(t, \tau) = \frac{t^2 + \tau^2}{4}$ and $x(t) = t^2$ with $t \in [0, 1]$. In this case, $a = 0$, $b = 1$ and $n = 1$. We have ${}_a I_{0.6}^{\alpha(\cdot, \cdot)} x(0.6) \approx 0.2661$ and ${}_{0.6} I_b^{\alpha(\cdot, \cdot)} x(0.6) \approx 0.4619$, obtained in *Matlab* with our *Chebfun* functions as follows:

```

a = 0; b = 1; n = 1;
alpha = @(t,tau) (t.^2+tau.^2)/4;
x = chebfun(@(t) t.^2, [0,1]);
LFI = leftFI(x,alpha,a);
RFI = rightFI(x,alpha,b);
LFI(0.6)
ans = 0.2661
RFI(0.6)
ans = 0.4619

```

Other values for ${}_a I_t^{\alpha(\cdot, \cdot)} x(t)$ and ${}_t I_b^{\alpha(\cdot, \cdot)} x(t)$ are plotted in Figure 4.2.

A.2 Higher-order Caputo fractional derivatives

In this section, considering the Definition 36, we implement in *Chebfun* two new functions `leftCaputo(x,alpha,a,n)` and `rightCaputo(x,alpha,b,n)` that approximate, respectively, the higher-order Caputo fractional derivatives of variable order ${}_a^C D_t^{\alpha_n(\cdot, \cdot)} x(t)$ and ${}_t^C D_b^{\alpha_n(\cdot, \cdot)} x(t)$.

The following code implements the left operator (4.2):

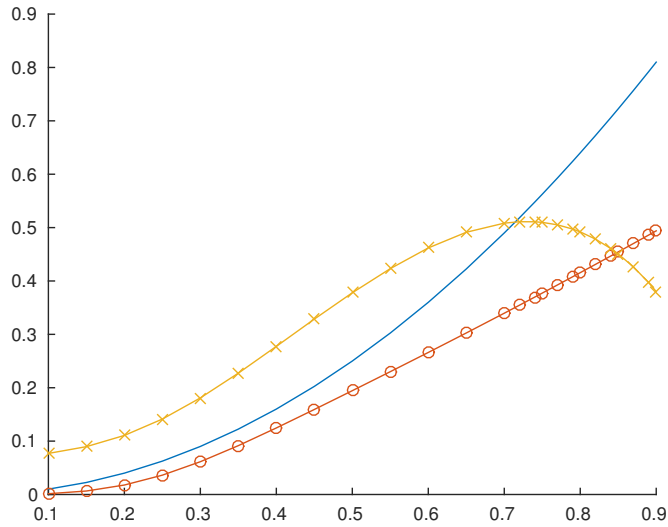


Figure 4.2: Riemann–Liouville fractional integrals of Example 4: $x(t) = t^2$ in continuous line, left integral ${}_a I_t^{\alpha(\cdot,\cdot)} x(t)$ with “o–” style, and right integral ${}_t I_b^{\alpha(\cdot,\cdot)} x(t)$ with “×–” style.

```
function r = leftCaputo(x,alpha,a,n)
    dx = diff(x,n);
    g = @(t,tau) dx(tau)./(gamma(n-alpha(t,tau)).*(t-tau).^(1+alpha(t,tau)-n));
    r = @(t) sum(chebfun(@(tau) g(t,tau),[a t] , 'splitting', 'on'),[a t]);
end
```

Similarly, we define the right operator (4.3) with Chebfun in MATLAB as follows:

```
function r = rightCaputo(x,alpha,b,n)
    dx = diff(x,n);
    g = @(t,tau) dx(tau)./(gamma(n-alpha(tau,t)).*(tau-t).^(1+alpha(tau,t)-n));
    r = @(t) (-1).^n.* sum(chebfun(@(tau) g(t,tau),[t b] , 'splitting', 'on'),[t b]);
end
```

We use the two functions `leftCaputo` and `rightCaputo` to determine approximations for the Caputo fractional derivatives of a power function of the form $x(t) = t^\gamma$.

Example 5. Let $\alpha(t, \tau) = \frac{t^2}{2}$ and $x(t) = t^4$ with $t \in [0, 1]$. In this case, $a = 0$, $b = 1$ and $n = 1$. We have ${}_a^C D_{0.6}^{\alpha(\cdot,\cdot)} x(0.6) \approx 0.1857$ and ${}_b^C D_b^{\alpha(\cdot,\cdot)} x(0.6) \approx -1.0385$, obtained in Matlab with our Chebfun functions as follows:

```

a = 0; b = 1; n = 1;
alpha = @(t,tau) t.^2/2;
x = chebfun(@(t) t.^4, [a b]);
LC = leftCaputo(x,alpha,a,n);
RC = rightCaputo(x,alpha,b,n);
LC(0.6)
ans = 0.1857
RC(0.6)
ans = -1.0385

```

See Figure 4.3 for a plot with other values of ${}^C D_t^{\alpha(\cdot,\cdot)} x(t)$ and ${}^C D_b^{\alpha(\cdot,\cdot)} x(t)$.

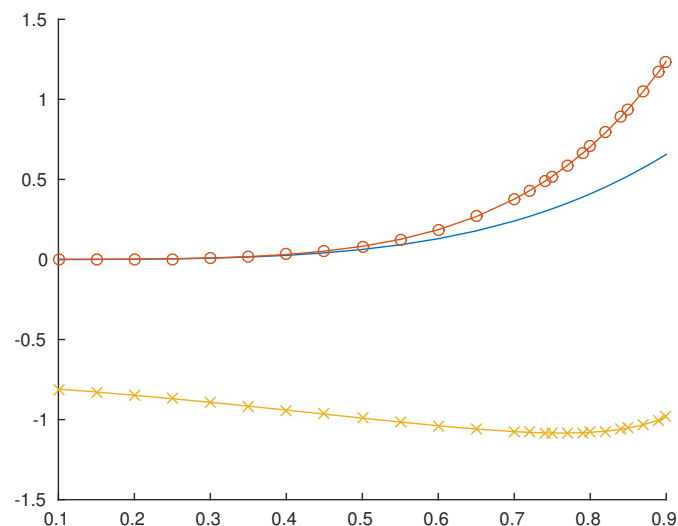


Figure 4.3: Caputo fractional derivatives of Example 5: $x(t) = t^4$ in continuous line, left derivative ${}^C D_t^{\alpha(\cdot,\cdot)} x(t)$ with “o—” style, and right derivative ${}^C D_b^{\alpha(\cdot,\cdot)} x(t)$ with “x—” style.

Example 6. In Example 5, we have used the polynomial $x(t) = t^4$. It is worth mentioning that our *Chebfun* implementation works well for functions that are not a polynomial. For example, let $x(t) = e^t$. In this case, we just need to change

```
x = chebfun(@(t) t.^4, [a b]);
```

in Example 5 by

```
x = chebfun(@(t) exp(t), [a b]);
```

to obtain

```
LC(0.6)
ans = 0.9917
RC(0.6)
ans = -1.1398
```

See Figure 4.4 for a plot with other values of ${}^C D_t^{\alpha(\cdot,\cdot)} x(t)$ and ${}^C D_b^{\alpha(\cdot,\cdot)} x(t)$.

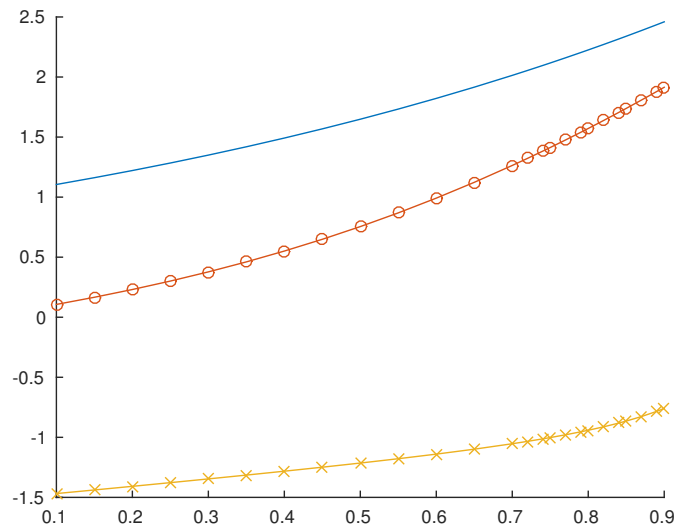


Figure 4.4: Caputo fractional derivatives of Example 6: $x(t) = e^t$ in continuous line, left derivative ${}^C D_t^{\alpha(\cdot,\cdot)} x(t)$ with “o—” style, and right derivative ${}^C D_b^{\alpha(\cdot,\cdot)} x(t)$ with “x—” style.

With Lemma 40 in Section 4.1 we can obtain, analytically, higher-order left Caputo fractional derivative of a power function of the form $x(t) = (t - a)^\gamma$. This allows us to show the effectiveness of our computational approach, that is, the usefulness of polynomial interpolation in Chebyshev points in fractional calculus of variable order. In Lemma 40, we assume that the fractional order depends only on the first variable: $\alpha_n(t, \tau) := \bar{\alpha}_n(t)$, where $\bar{\alpha}_n : [a, b] \rightarrow (n - 1, n)$ is a given function.

Example 7. Let us revisit Example 5 by choosing $\alpha(t, \tau) = \frac{t^2}{2}$ and $x(t) = t^4$ with $t \in [0, 1]$. Table 4.1 shows the approximated values obtained by our *Chebfun* function `leftCaputo(x, alpha, a, n)` and the exact values computed with the formula given by Lemma 40. Table 4.1 was obtained using the following *MATLAB* code:

t	Exact Value	Approximation	Error
0.1	1.019223177296953e-04	1.019223177296974e-04	-2.046431600566390e-18
0.2	0.001702793965464	0.001702793965464	-2.168404344971009e-18
0.3	0.009148530806348	0.009148530806348	3.469446951953614e-18
0.4	0.031052290994593	0.031052290994592	9.089951014118469e-16
0.5	0.082132144921157	0.082132144921157	6.522560269672795e-16
0.6	0.185651036003120	0.185651036003112	7.938094626069869e-15
0.7	0.376408251363662	0.376408251357416	6.246059225389899e-12
0.8	0.704111480975332	0.704111480816562	1.587694420379648e-10
0.9	1.236753486749357	1.236753486514274	2.350835082154390e-10

Table 4.1: Exact values obtained by Lemma 40 for functions of Example 7 versus computational approximations obtained using the `Chebfun` code.

```

format long
a = 0; b = 1; n = 1;
alpha = @(t,tau) t.^2/2;
x = chebfun(@(t) t.^4, [a b]);
exact = @(t) (gamma(5)./gamma(5-alpha(t))).*t.^(4-alpha(t));
approximation = leftCaputo(x,alpha,a,n);
for i = 1:9
    t = 0.1*i;
    E = exact(t);
    A = approximation(t);
    error = E - A;
    [t E A error]
end

```

Computational experiments similar to those of Example 7, obtained by substituting Lemma 40 by Lemma 41 and our `leftCaputo` routine by the `rightCaputo` one, reinforce the validity of our computational methods. In this case, we assume that the fractional order depends only on the second variable: $\alpha_n(\tau, t) := \bar{\alpha}_n(t)$, where $\bar{\alpha}_n : [a, b] \rightarrow (n - 1, n)$ is a given function.

A.3 Higher-order combined fractional Caputo derivative

The higher-order combined Caputo fractional derivative combines both left and right Caputo fractional derivatives, that is, we make use of functions `leftCaputo(x,alpha,a,n)` and `rightCaputo(x,alpha,b,n)` provided in Section A.1 to define Chebfun computational code for the higher-order combined fractional Caputo derivative of variable order:

```
function r = combinedCaputo(x,alpha,beta,gamma1,gamma2,a,b,n)
    lc = leftCaputo(x,alpha,a,n);
    rc = rightCaputo(x,beta,b,n);
    r = @(t) gamma1 .* lc(t) + gamma2 .* rc(t);
end
```

Then we illustrate the behavior of the combined Caputo fractional derivative of variable order for different values of $t \in (0, 1)$, using MATLAB.

Example 8. Let $\alpha(t, \tau) = \frac{t^2 + \tau^2}{4}$, $\beta(t, \tau) = \frac{t + \tau}{3}$ and $x(t) = t$, $t \in [0, 1]$. We have $a = 0$, $b = 1$ and $n = 1$. For $\gamma = (\gamma_1, \gamma_2) = (0.8, 0.2)$, we have ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(0.4) \approx 0.7144$:

```
a = 0; b = 1; n = 1;
alpha = @(t,tau) (t.^2 + tau.^2)/.4;
beta = @(t,tau) (t + tau)/3;
x = chebfun(@(t) t, [0 1]);
gamma1 = 0.8;
gamma2 = 0.2;
CC = combinedCaputo(x,alpha,beta,gamma1,gamma2,a,b,n);
CC(0.4)
ans = 0.7144
```

For other values of ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)$, for different values of $t \in (0, 1)$ and $\gamma = (\gamma_1, \gamma_2)$, see Figure 4.5 and Table 4.2.

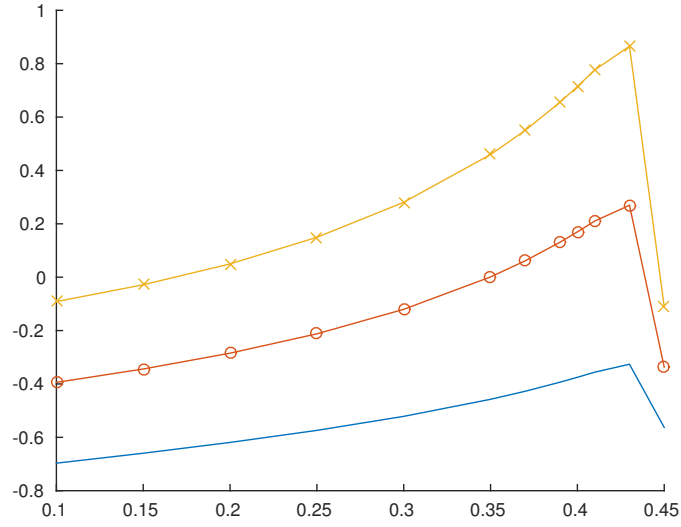


Figure 4.5: Combined Caputo fractional derivative ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)$ for $\alpha(\cdot, \cdot)$, $\beta(\cdot, \cdot)$ and $x(t)$ of Example 8: continuous line for $\gamma = (\gamma_1, \gamma_2) = (0.2, 0.8)$, “o-” style for $\gamma = (\gamma_1, \gamma_2) = (0.5, 0.5)$, and “x-” style for $\gamma = (\gamma_1, \gamma_2) = (0.8, 0.2)$.

t	Case 1	Case 2	Case 3
0.4500	-0.5630	-0.3371	-0.1112
0.5000	-790.4972	-1.9752e+03	-3.1599e+03
0.5500	-3.5738e+06	-8.9345e+06	-1.4295e+07
0.6000	-2.0081e+10	-5.0201e+10	-8.0322e+10
0.6500	2.8464e+14	7.1160e+14	1.1386e+15
0.7000	4.8494e+19	1.2124e+20	1.9398e+20
0.7500	3.8006e+24	9.5015e+24	1.5202e+25
0.8000	-1.3648e+30	-3.4119e+30	-5.4591e+30
0.8500	-1.6912e+36	-4.2280e+36	-6.7648e+36
0.9000	5.5578e+41	1.3895e+42	2.2231e+42
0.9500	1.5258e+49	3.8145e+49	6.1033e+49
0.9900	1.8158e+54	4.5394e+54	7.2631e+54

Table 4.2: Combined Caputo fractional derivative ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)$ for $\alpha(\cdot, \cdot)$, $\beta(\cdot, \cdot)$ and $x(t)$ of Example 8. Case 1: $\gamma = (\gamma_1, \gamma_2) = (0.2, 0.8)$; Case 2: $\gamma = (\gamma_1, \gamma_2) = (0.5, 0.5)$; Case 3: $\gamma = (\gamma_1, \gamma_2) = (0.8, 0.2)$.

Conclusion and future work

A high number of real world phenomena are better described by non-integer order derivatives. Fractional derivatives have unique characteristics, namely, contain memory, that may model certain dynamics more robust and efficiently. When we consider fractional operators, where the fractional order is not a constant and may depend on time, numerous applications have been found.

In Chapter 3, we obtained new approximation formulas for three different fractional variable order operators, namely Caputo fractional derivatives. These decompositions only depend on integer order derivatives and for all of them we determine the upper bound formulas for the errors. Motivated by the possibility of implementing this theory in real problems, we compare the numerical approximation of some test function with its exact fractional derivative and show an exemplification of how the presented methods can be used to solve partial fractional differential equations of variable order.

The results of this part of the thesis can be found in the paper:

- D. Tavares, R. Almeida and D. F. M. Torres, Caputo derivatives of fractional variable order: numerical approximations, *Commun. Nonlinear Sci. Numer. Simul.* 35 (2016), 69–87 [104];

and were presented by the author at the conferences:

- Research Day 2014, University of Aveiro, in 3rd of June of 2014. Poster title: Numerical approximations for the Caputo fractional derivatives;
- The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Madrid, Spain, from 07th to 11th of July of 2014. Talk: Variational problems with dependence on Caputo fractional derivatives of variable order.

In the recent years, the fractional calculus of variations with a Lagrangian depending on fractional derivatives has emerged as an elegant alternative to study non-conservative systems. In Chapter 4, we presented fractional variational problems with variable order where the Lagrangian depends on a combined Caputo derivative of variable order. For all of them, we obtained necessary optimality conditions that the extremizers must satisfy. The terminal point and the terminal state are supposed to be free, and, consequently, transversality conditions are derived. This approach with free endpoint is more general than if we consider fixed boundary problems. We believe that it is reasonable that the order of the fractional operators is influenced also by the acting dynamics, which varies with time, and we trust that the results now obtained can be more effective to provide a mathematical framework to some complex dynamical problems.

Several variational problem are studied, like the classical variational problem with the combined Caputo variable order and its generalization to higher-order fractional derivative and with a state time delay, two isoperimetric problems, with a holonomic constraint and the variational problem of Herglotz, involving the combined Caputo fractional derivative.

The results of Chapter 4 can be found in the following papers:

- Optimality conditions for fractional variational problems with dependence on a combined Caputo derivative of variable order, *Optimization* **64** (2015), no. 6, 1381–1391 [103];
- D. Tavares, R. Almeida and D. F. M. Torres, Constrained fractional variational problems of variable order. *IEEE/CAA Journal Automatica Sinica*, **4** (2017), no. 1, 80–88. [105];
- D. Tavares, R. Almeida and D. F. M. Torres, Fractional Herglotz variational problem of variable order, *Discrete Contin. Dyn. Syst. Ser. S.*, in press. *arXiv:1703.09104* [106];
- D. Tavares, R. Almeida and D. F. M. Torres, Combined fractional variational problems of variable order and some computational aspects, *J. Comput. Appl. Math.*, in press. DOI: 10.1016/j.cam.2017.04.042 [107];

and were presented by the author at the following conferences:

-
- EURO mini conference Optimization in Natural Sciences, University of Aveiro, from 05th to 09th of February of 2014. Talk: Necessary conditions for variational problems with dependence on a combined Caputo derivative;
 - Conference on Differential and Difference Equations and Applications, from 18th to 22nd of May of 2015, Military Academy in Amadora, Portugal. Talk: Isoperimetric problems with dependence on Fractional Derivatives;
 - Research Day 2015, University of Aveiro, Portugal, in 20th of May of 2015. Poster title: An isoperimetric problem with. dependence on Fractional Derivatives;
 - International Meeting on Applied Mathematics in Errachidia, Errachidia, Morocco, from 09th to 12th of May of 2016. Talk: Fractional variational problem of Herglotz;
 - The Cape Verde International Days on Mathematics 2017, Cape Verde, from 08th to 11th of May of 2017. Talk: Fractional variational problems with combined variable order derivatives.

Our work can be extended in several ways. In order to complete the study done here, we can consider new variational problems, such as the generalization of the Herglotz problem and the isoperimetric problems to higher-order derivatives and a state time delay.

It would be interesting to address the question of existence of solutions, and to study direct methods to minimize the proposed functionals. We can also explore some numerical methods for fractional variational problems involving fractional variable order derivatives.

Another interesting idea consists to combine the results of this PhD thesis in the framework of fractional optimal control of epidemic models.

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