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Monitoring maximal outerplanar graphs

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Abstract

In this paper we define a new concept of monitoring the elements of triangulation graphs by faces. Furthermore, we analyze this, and other monitoring concepts (by vertices and by edges), from a combinatorial point of view, on *maximal outerplanar graphs*.

Keywords: Domination, covering, guarding, triangulation graphs.

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1 Introduction

In Graph Theory, the notion of monitoring vertices, edges of graphs by other vertices or edges has been widely studied. For instance, monitoring vertices by other vertices or edges by other edges leads to well known parameters concerned with *vertex domination* or *edge domination*, respectively. When vertices are to monitor edges we have the well known notions of *vertex covering*. Finally, when edges are to monitor vertices we obtain parameters associated with *edge covering*. A *dominating set* is a set $D \subseteq V$ such that every vertex not in D is adjacent to a vertex in D. A set $D \subseteq E$ is an *edge dominating set* if each edge in E is either in D or is adjacent to an edge in D. A *vertex cover* is a set $C \subseteq V$ if each edge of the graph is incident to at least one vertex of the C. An *edge cover* is a set $C \subseteq E$ which cover every vertex in V. In Computational Geometry, for triangulations or quadrangulations, a different monitoring notion was established – the notion of monitoring bounded faces (*faces*, for short). When the faces are monitored by vertices or edges, we obtain the parameters associated with *vertex guarding* or *edge guarding*, respectively. Being $G = (V, E)$ a triangulation, a *guarding set* is a set $L \subseteq V$ such that every face has a vertex in S. The *guarding number*, $g(G)$, is the number of vertices in a smallest guarding set for G. Concerning edge guarding, an edge $e = (u, v)$ is said to guard a face of F of G if u or v are vertices of F. An *edge guarding set* $L \subseteq E$ is a set which guards every face in G. The *edge guarding number*, $g^e(G)$, is the minimum cardinality of an edge guarding set for G. All the previous described monitoring notions were extended to include its distance versions on plane graphs. For example, domination was extended to *distance domination* and guarding to *distance guarding* [2].

Regarding combinatorial bounds, there are many results about domination and covering for graphs and for triangulations graphs (that is, the graph of a triangulation of a set of points in the plane). In this paper we analyse these monitoring concepts (domination, covering and guarding) from a combinatorial point of view, for a special class of triangulation graphs, the *maximal outerplanar graphs*. A *maximal outerplanar graph* embedded in the plane corresponds to a triangulation of a polygon. Concerning plane graphs, it is natural to extend the notions of monitoring by faces. So, in this paper we also define three new concepts: *face-vertex guarding*, *face guarding* and *face-edge guarding* in triangulation graphs (*triangulations*, for short). Furthermore, we establish tight bounds for the usual and distance versions of monitoring by faces on maximal outerplanar graphs. In the next section we describe some definitions and terminology that will be used throughout this paper.

2 Definitions

As stated above, given a triangulation $T = (V, E)$ its elements (vertices, edges and faces) can be monitored by other vertices, edges or faces (i.e., triangles). First, we start by presenting the terminology that we use when the elements of T are monitored by vertices, at its distance version (see [2], for details). Let $T = (V, E)$ be a triangulation. A *kd-dominating set* for T is a subset $D \subset V$ such that each vertex $u \in V - D$, $dist_T(u, v) \leq k$ for some $v \in D$. Given $n \in \mathbb{N}$, we define $\gamma_{kd}(n) = max\{\gamma_{kd}(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. We say that a triangle T_i of T is *kd-guarded* from by a vertex $v \in V$, if there is a vertex $x \in T_i$ such that $dist_T(x, v) \leq k - 1$. A *kd-guarding set* for T is a subset $L \subseteq V$ such that every triangle of T is kd-guarded by an element of L. Given $n \in \mathbb{N}$, we define $g_{kd}(n) = max\{g_{kd}(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. A *kd-vertex cover* of T, is a subset $C \subseteq V$ such that for each edge $e \in E$ there is a path of length at most k, which contains e and a vertex of C. Given $n \in \mathbb{N}$, we define $\beta_{kd}(n) = max\{\beta_{kd}(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. The kd-domination number $\gamma_{kd}(T)$, the *kd-guarding number* $g_{kd}(T)$ and the *kd-covering number* $\beta_{kd}(T)$ are the number of vertices in a smallest kd-dominating set, kd-guarding set and kd-vertex cover for T, respectively.

In the following, we introduce the terminology that we use when the monitoring of the elements of T, at its distance version, is done by edges. A *kd-edge cover* of T, is a subset $C \subseteq E$ such that each vertex, $v \in V$ $dist_T(v, e) \leq k - 1$, for some $e \in C$, where $dist_T(v, e)$ is the minimum distance between the endpoints of e and v (see Fig.1(a)), for a sketch). If the monitored elements are triangles we have the notions of *kd-edge guarding* and $g^e_{kd}(T)$ (see Fig.1(b)). And if they are edges, kd-edge dominating and $\gamma'_{kd}(T)$ (see Fig.1(c)). Given $n \in \mathbb{N}$, the values $\gamma'_{kd}(n)$, $g^e_{kd}(n)$ and $\beta'_{kd}(n)$, are defined similarly to the previous case (monitoring by vertices, in their distance versions).

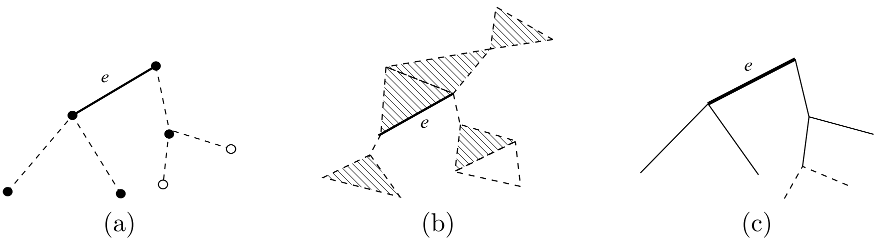


Fig. 1. The edge e: (a) $2d$-edge cover the black vertices; (b) $2d$-edge guard the shadow triangles; (c) $2d$-edge dominate the filled edges.

Next, we will define new monitoring concepts: monitoring the elements of a triangulation by its faces (the usual and the distance versions). Let $T = (V, E)$ be a triangulation. A *face-vertex cover* is a subset C of faces of T such that each vertex $v \in V$ is a vertex of some $T_i \in C$. The *face-vertex covering number* $f^v(T)$ is the number of vertices in a smallest face-vertex cover for T. Given $n \in \mathbb{N}$, we define $f^v(n) = max\{f^v(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. We say that a triangle T_j of T is *guarded* by a triangle T_i of T if they share some vertex. A *face guarding set* is a subset L of triangles of T such that every triangle of T is guarded by an element of L. The *face guarding number*, $g^f(T)$, is the number of triangles in a smallest face guarding set for T. Given $n \in \mathbb{N}$, we define $g^f(n) = max\{g^f(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. Finally, a subset C of triangles of T is *face-edge cover* if each edge $e \in E$ has an endpoint on some $T_i \in C$. The *face-edge covering number* $f^e(T)$ is the minimum cardinality of a face-edge cover for T. Given $n \in \mathbb{N}$, we define $f^e(n) = max\{f^e(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$.

The above defined concepts were extended to its distance versions, "similarly to the monitoring by edges" (see Fig.2). Given $n \in \mathbb{N}$, the values $f^v_{kd}(n), g^f_{kd}(n)$ and $f^e_{kd}(n)$ are defined similarly to $f^v(n), g^f(n)$ and $f^e(n)$, respectively.

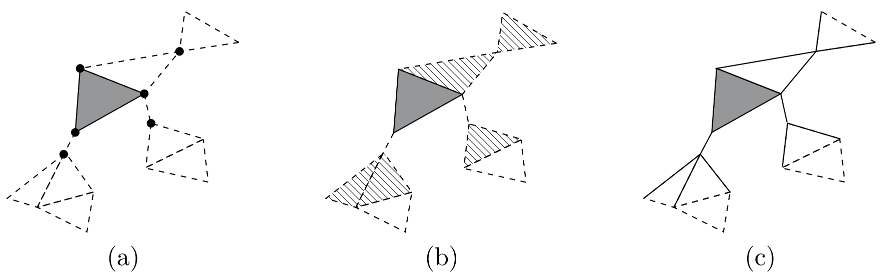


Fig. 2. The gray face:(a) $2d$-face-vertex cover the black vertices; (b) $2d$-face-face guard the shadow triangles; (c) $2d$-face-edge cover the filled edges.

Our main goal is to obtain combinatorial bounds related to the monitoring numbers on triangulation. As stated in the introduction some of these bounds are already known, so, as is evident, we studied the unknown ones. We start by studying a special class of triangulations, namely the maximal outerplanar graphs; and concerning the distance versions we begin with distance 2.

3 Monitoring maximal outerplanar graphs

In this section we establish tight bounds for the minimum number of vertices, edges and faces that monitor the different elements (vertices, edges and faces) of a special class of triangulation graphs – the maximal outerplanar graphs – which correspond, as stated above, to triangulations of polygons. We call the edges on the exterior face *exterior edges*, otherwise they are *interior edges*. In the following tables are summarized our, and related, results concerning the monitoring the different elements of maximal outerplanar graphs. In Table 1 we present the results regarding usual monitoring versions ($k = 1$), and in Table 2 we show the results related to distance monitoring versions ($k = 2$).

		Monitored elements		
		Vertices	Faces	Edges
Monitored by	Vertices	**Dominating** $\gamma(n) = \lfloor \frac{n+n_2}{4} \rfloor$ [1] [1], [7]	**Guarding** $g(n) = \lfloor \frac{n}{3} \rfloor$ [3]	**Covering** $\beta(n) = \lfloor \frac{2n}{3} \rfloor$ (here)
	Edges	**Edge-Covering** $\beta'(n) = \lceil \frac{n}{2} \rceil$ (here)	**Edge-Guarding** $g^e(n) = \lfloor \frac{3n}{10} \rfloor + 1$ [5]	**Edge-Dominating** $\gamma'(n) = \lfloor \frac{n+1}{3} \rfloor$ [4]
	Faces	**Face-vertex Covering** $f^v(n) = \lfloor \frac{n}{2} \rfloor$ (here)	**Face-face Guarding** $g^f(n) = \lfloor \frac{n}{4} \rfloor$ (here)	**Face-edge Covering** $f^e(n) = \lfloor \frac{n}{3} \rfloor$ (here)

Table 1
A summary of new and related results for usual monitoring.

		Monitored elements		
		Vertices	Faces	Edges
Monitored by	Vertices	**Dominating** $\gamma_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ [2]	**Guarding** $g_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ [2]	**Covering** $\beta_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ [2]
	Edges	**Edge-Covering** $\beta'_{2d}(n) = \lfloor \frac{n}{4} \rfloor$ (here)	**Edge-Guarding** $g^e_{2d}(n) = \lfloor \frac{n}{6} \rfloor$ (here)	**Edge-Dominating** $\gamma'_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ (here)
	Faces	**Face-vertex Covering** $f^v_{2d}(n) = \lfloor \frac{n}{4} \rfloor$ (here)	**Face-face Guarding** $g^f_{2d}(n) = \lfloor \frac{n}{6} \rfloor$ (here)	**Face-edge Covering** $f^e_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ (here)

Table 2
A summary of new results for monitoring at distance 2.

In the following, due to lack of space, we will present only two proofs of the results shown in the tables. First, we will prove that $g^f(n) = \lfloor \frac{n}{4} \rfloor, \forall n \geq 4$. In order to do this, and following the ideas of O'Rourke [6], we first need to introduce some lemmas.

[1] n_2 is the number of vertices of degree 2.

Lemma 3.1 *Suppose that $f(m)$ triangles are always sufficient to guard any m-vertex outerplanar maximal graph. Let T be a m-vertex outerplanar maximal graph and e an exterior edge. Then with $f(m-1)$ triangles and an additional "collapsed triangle" at the edge e are sufficient to guard T.*

Lemma 3.2 *Let T be an outerplanar maximal graph with $n \geq 2k$ vertices. There is an interior edge e of T that partitions G into two pieces, one of which contains $m = k, k+1, \ldots, 2k-1$ or $2k-2$ exterior edges of T.*

Theorem 3.3 *Every n-vertex maximal outerplanar graph, with $n \geq 4$, can be face-guarded by $\lfloor \frac{n}{4} \rfloor$ triangles. And this bound is tight.*

Proof.

For $4 \leq n \leq 7$, the truth of the theorem can be easily established. Assume that $n \geq 8$ and that the theorem holds for all $n' < n$. Lemma 3.2 guarantees the existence of an interior edge e that divides T into two maximal outerplanar graphs T_1 and T_2, such that T_1 has m exterior edges of T with $5 \leq m \leq 8$. The vertices of T are labeled with $0, 1, \ldots, n-1$ such that e is $(0, m)$. Each value of m, which is minimal is considered separately. Here, we present the cases $m = 5$ and $m = 8$. **(i)** *Case* $m = 5$. T_1 has $m + 1 = 6$ exterior edges, thus it can be face-guarded with one triangle. T_2 has $n - 4$ exterior edges including e, and by induction hypothesis, it can be face-guarded with $\lfloor \frac{n-4}{4} \rfloor = \lfloor \frac{n}{4} \rfloor - 1$ guards. Thus T_1 and T_2 together can be face-guarded by $\lfloor \frac{n}{4} \rfloor$ guards. **(ii)** *Case* $m = 8$. The presence of any of the internal edges (0,7), (0,6), (0,5), (7,1), (7,2) and (7,3) would violate the minimality of m. Thus, the triangle T' in T_1 that is bounded by e is (0,4,8). Consider the maximal outerplanar graph $T^* = T_2 + (0, 4, 5, 6, 7, 8)$ (see Fig.3(a)). T^* has $n - 3$ exterior edges, applying lemma 3.4 it can be face-guarded with $f(n-3)$ triangles, that is $\lfloor \frac{n}{4} \rfloor - 1$ triangles, and an additional "collapsed triangle" at the edge (0,4). This "collapsed triangle" also face-guards the pentagon (0,1,2,3,4) regardless of the way how it is triangulated. Thus, T is face-guarded by $\lfloor \frac{n}{4} \rfloor$ triangles.

To prove that this upper bound is tight we need to construct a n-vertex maximal outerplanar graph T such that $g^f(n) \geq \lfloor \frac{n}{4} \rfloor$. Fig.3(b) shows a maximal outerplanar graph T for which $g^f(n) \geq \lfloor \frac{n}{4} \rfloor$, since two shadowed triangles can only be face-guarded by different triangles.

□

Next, we will show that $\beta'_{2d}(n) = \lfloor \frac{n}{4} \rfloor$, for any n-vertex maximal outerplanar graph, with $n \geq 4$. In order to do this, we first need to introduce the next lemma.

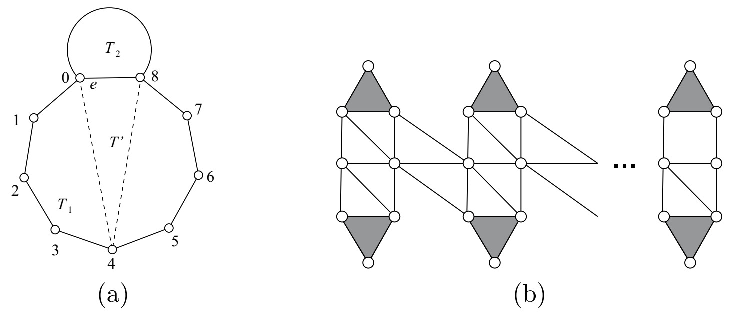


Fig. 3. (a) The triangle T' is (0,4,8); (b) a maximal outerplanar graph T for which $g^f(n) \geq \lfloor \frac{n}{4} \rfloor$.

Lemma 3.4 *Suppose that $f(m)$ edges are always sufficient to $2d$-edge-cover any outerplanar maximal graph with m vertices. Let T be a m-vertex outerplanar maximal graph and $e = (u, v)$ an exterior edge. Then with $f(m-1)$ edges and an additional "collapsed edge" at the vertex u or v are sufficient to $2d$-edge-cover T.*

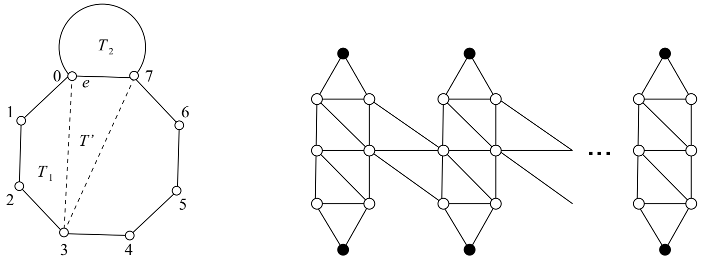
Theorem 3.5 *Every n-vertex maximal outerplanar graph, with $n \geq 4$, can be $2d$-edge covered by $\lfloor \frac{n}{4} \rfloor$ edges. And this bound is tight.*

Proof.

The proof is done by induction on n. For $4 \leq n \leq 9$, the truth of the theorem can be easily established. Assume that $n \geq 10$, and that the theorem holds for $n' < n$. Let T be a maximal outerplanar graph with n vertices. The vertices of T are labeled with $0, 1, 2, \ldots, n-1$. Lemma 3.2 guarantees the existence of an interior edge e (which can be labeled $(0, m)$) that divides T into maximal outerplanar graphs T_1 and T_2, such that T_1 has m exterior edges of T with $5, 6, 7$ or 8. Each value of m, which is minimal, is considered separately. Here, we only present the case $m = 7$. So, if $m = 7$, by the minimality of m the triangle T' supported by the internal edge (0,7) is (0,3,7) or (0,4,7). Since these are equivalent cases, we suppose that T' is (0,3,7) as shown in Fig.4(a). Consider $T^* = T_2 + (0, 1, 2, 3, 7)$. T^* has $n-3$ exterior edges, applying lemma 3.4 it can be $2d$-edge covered with $\lfloor \frac{n}{4} \rfloor - 1$ edges, and an additional "collapsed edge" at the vertex 3 or 7. This "collapsed edge" also $2d$-edge cover the pentagon (3,4,5,6,7). Thus, T can be $2d$-edge covered by $\lfloor \frac{n}{4} \rfloor$ edges.

Now, we will prove that this upper bound is tight. No two black vertices of the maximal outerplanar graph illustrated in Fig.4(b) can be $2d$-edge covered by the same edge, and therefore $\beta'_{2d}(n) \geq \lfloor \frac{n}{4} \rfloor$.

$\square$



(a) (b)

Fig. 4. (a) The triangle T' is (0,3,7); (b) a maximal outerplanar graph T for which $\beta'_{2d}(n) \geq \lfloor \frac{n}{4} \rfloor$.

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