ORTHOGONAL POLYNOMIAL INTERPRETATION OF q-TODA AND q-VOLTERRA EQUATIONS

I. AREA, A. BRANQUINHO, A. FOULQUIÉ MORENO, AND E. GODOY

ABSTRACT. The correspondences between dynamics of q-Toda and q-Volterra equations for the coefficients of the Jacobi operator and its resolvent function are established. The main ingredient are orthogonal polynomials which satisfy an Appell condition, with respect to the q-difference operator D_q, and a Lax type theorem for the point spectrum of the Jacobi operator associated with these equations. Examples related with the big q-Legendre, discrete q-Hermite I, and little q-Laguerre orthogonal polynomials and q-Toda and q-Volterra equations are given.

1. INTRODUCTION

The Toda lattice equations describe the oscillations of an infinite system of points joined by spring masses, where the interaction is exponential in the distance between two spring masses [31]. The semi-infinite Toda lattice equations in one time variable are [22, 21]

$$(1)\quad a_{-1}\equiv 0\,,\ a_0\equiv 1\,,\quad \begin{cases} \dfrac{da_n(t)}{dt} = a_n(t)\big(b_{n-1}(t)-b_n(t)\big), \\ \dfrac{db_n(t)}{dt} = a_n(t)-a_{n+1}(t), \end{cases} \qquad n\in\mathbb{N}\,.$$

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Let $t_0 \in \mathbb{R}$ and $\mu(x;t_0)$ be a measure such that all the moments

$$u_n = \int_{\mathbb{R}} x^n d\mu(x;t_0), \qquad n \in \mathbb{N},$$

exist and are finite, and $\{P_n\}_{n\in\mathbb{N}}$ be the sequence of monic orthogonal polynomials with respect to $\mu(x;t_0)$,

$$\int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x;t_0) = h_n^2 \delta_{n,m}\,,$$

where $\delta_{i,j}$ denotes the Kronecker delta. As it is very well-known [4, 11, 30], the sequence $\{P_n(x;t_0) \equiv P_n(x)\}_{n\in\mathbb{N}}$ satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n P_{n-1}(x)\,, \tag{2}$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - b_0$.

The spectral measure of an operator $J(t)$ is the Stieltjes function

$$S(z;t) = e_0^{\top} R_z(t)\, e_0\,,$$

for the resolvent operator, $R_z(t) = [J(t) - z\,\mathcal{I}]^{-1}$, associated with the operator $J(t)$ (cf. [1]). In the case that $J(t)$ is a Jacobi operator

$$J(t) = \big(J_{i,j}(t)\big) = \begin{pmatrix} b_0(t) & 1 & 0 & & & \\ a_1(t) & b_1(t) & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{3}$$

with $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ uniformly bounded, then

$$S(z;t) = \sum_{n=0}^{\infty} \frac{J_{11}^n(t)}{z^{n+1}}\,, \quad |z| > \|J(t)\|, \tag{4}$$

where $\|\cdot\|$ denotes the operator norm. Let $\mathcal{P}$ be the column vector of monic polynomials $P_n(x;t)$ defined by

$$J(t)\,\mathcal{P} = x\,\mathcal{P}\,. \tag{5}$$

By using the Stone-Favard theorem [4], there exists a linear functional $u(t)$ such that $u_n(t) = \langle u(t), x^n \rangle = J_{11}^n(t)$, and so

$$S(z;t) = \sum_{n=0}^{\infty} \frac{u_n(t)}{z^{n+1}} = \Big\langle u(t), \frac{1}{z-x} \Big\rangle,\ |z| > \|J(t)\|. \tag{6}$$

By definition, the diagonal Padé aproximants of index n, Π_n, for $S(z;t)$ is a rational function

$$\Pi_n(z) = Q_n(z)\big/P_n(z), \tag{7}$$

with $\deg P_n \le n$ and $\deg Q_n \le n$, such that $P_n(z)S(z;t) - Q_n(z) = c/z^{n+1} + \cdots$. An important property of the Padé aproximants for the

Stieltjes function (6) of the operator (3) is that their numerators and denominators satisfy a three term recurrence relation as (2) with initial conditions $P_0 = 1$, $P_{-1} = 0$ and $Q_0 = 0$, $Q_{-1} = 1$, where $a_n(t)$ and $b_n(t)$ are the coefficients of the matrix $J(t)$ in (3). The sequence of monic polynomials $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to the linear functional $u(t)$.

We shall consider linear functionals normalized to have their first moment equal to one, i.e.

$$u_0(t) = \langle u(t), 1\rangle = 1. \tag{8}$$

If $b_n(t) \in \mathbb{R}$ and $a_n(t) > 0$, there exists a positive Borel measure $d\mu(x;t)$ supported on $I \subset \mathbb{R}$ such that

$$S(z;t) = \int_I \frac{d\mu(x;t)}{z-x}, \quad |z| > \|J(t)\|,$$

where

$$u_n(t) = \langle u(t), x^n\rangle = \int_I x^n d\mu(x;t) = e_0^\top J^n(t)\, e_0 = J^n_{1,1}(t)\,, \tag{9}$$

with $e_0 = (1, 0, 0, \dots)^\top$.

Let $\{P_n(x;t)\}_{n\in\mathbb{N}}$ be the sequence of orthogonal with respect to $\mu(x;t)$. As it is well known (5) can be written as [4]

$$P_{n+1}(x;t) = \big(x - b_n(t)\big)P_n(x;t) - a_n(t)P_{n-1}(x;t)\,,\ n = 1, 2, \dots\,, \tag{10}$$

with initial conditions $P_0(x;t) = 1$ and $P_1(x;t) = x - b_0(t)$.

The dynamic of the solutions of the Toda equations (1) corresponds to the simple evolution of the measure [22, 21, 23],

$$d\mu(x;t) = \frac{\exp(-xt)d\mu(x,t_0)}{\int \exp(-xt)d\mu(x,t_0)}\,, \tag{11}$$

of the operator $J(t)$.

A difference analogue of a Korteweg-de Vries equation,

$$\frac{d}{dt}\gamma_{n+1}(t) = \gamma_{n+1}(t)\big(\gamma_{n+2}(t) - \gamma_n(t)\big)\,, \qquad n \in \mathbb{N}\,,$$

is called Langmuir lattice or finite difference KDV equation, whose dynamic is given by

$$d\mu(x;t) = \frac{\exp(-x^2t)d\mu(x,t_0)}{\int \exp(-x^2t)d\mu(x,t_0)}\,. \tag{12}$$

In [10, 26] it was studied the construction of a solution of the Toda lattice

$$\begin{cases} \dfrac{da_n(t)}{dt} = a_n(t)\big(b_{n-1}(t) - b_n(t)\big)\,, \\ \dfrac{db_n(t)}{dt} = a_n(t) - a_{n+1}(t)\,, \end{cases} \qquad n \in \mathbb{Z}\,, \tag{13}$$

from another given solution, considering sequences $\{a_n(t)\,,\, b_n(t)\}_{n\in\mathbb{Z}}$, of real functions. Both solutions of (13) were linked to each other by a *Bäcklund* or *Miura transformation*

$$a_n(t) = \gamma_{2n}(t)\gamma_{2n-1}(t)\,,\quad b_n(t) = \gamma_{2n+1}(t) + \gamma_{2n}(t) + C\,,\ n \in \mathbb{Z}\,, \tag{14a}$$

$$\tilde{a}_n(t) = \gamma_{2n+1}(t)\gamma_{2n}(t), \tilde{b}_n(t) = \gamma_{2n+2}(t) + \gamma_{2n+1}(t) + C, n \in \mathbb{Z}\,, \tag{14b}$$

with $C = 0$, where $\{\gamma_n(t)\}$ is a solution of the *Volterra lattice*

$$\dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t)\big(\gamma_{n+2}(t) - \gamma_n(t)\big), \quad n \in \mathbb{Z}\,. \tag{15}$$

In [3], this kind of analysis has been generalized to the full hierarchy of Toda and Volterra lattices studied in [2] and [1] (see also [5, 6]).

P. D. Lax in [15] put the inverse scattering method for solving the KdV equation into a more general framework which subsequently paved the way to generalizations of the technique as a method for solving other partial differential equations. He considered two time-dependent operators $\mathcal{L}$ and $\mathcal{M}$, where $\mathcal{L}$ is the operator of the spectral problem and $\mathcal{M}$ is the operator governing the associated time evolution of the eigenfunctions

$$\mathcal{L}\, v = \lambda\, v\,, \frac{dv(t)}{dt} = \mathcal{M}\, v\,;$$

and hence we get

$$\frac{d\mathcal{L}}{dt} = \mathcal{L}\,\mathcal{M} - \mathcal{M}\,\mathcal{L}$$

if, and only if, $\dfrac{d\lambda(t)}{dt} = 0$. Magnus [18] showed that Toda type integrable systems of the form

$$\frac{\partial}{\partial t}A = \frac{\partial}{\partial x}H + HA - AH$$

is imbedded in the theory of semiclassical orthogonal polynomials (see e.g. [19]). Moreover, from this partial differential equation he derived Painlevé equations for the coefficients of the three term recurrence relation satisfied by these semiclassical orthogonal polynomials. Recently, Ormerod et. al. [24] presented a framework for the study of q-difference equations satisfied by q-semi-classical orthogonal systems, to derive as an example, the q-difference equation satisfied by a deformed version of the little q-Jacobi polynomials as a gauge transformation of a special case of the associated linear problem for q-Painlevé VI.

Naturally arises the problem of characterizing, in terms of their spectral points, the Jacobi operators $\mathcal{L}$ such that

$$\mathrm{D}_q\,\mathcal{L} = \mathcal{L}(qt)\,\mathcal{M} - \mathcal{M}\,\mathcal{L}(t),$$

where the q-difference operator

$$\mathrm{D}_q\,\mathcal{L} = \frac{\mathcal{L}(qt) - \mathcal{L}(t)}{(q-1)t}.$$

The history of q-calculus (and q-hypergeometric functions) dates back to the eighteenth century. In fact it can be taken as far back as Leonhard Euler (1707–1783), who first introduced the q in his Introductio [8] in the tracks of Newton's infinite series.

In q-calculus we are looking for q-analogues of mathematical objects that have the original object as limits when q tends to 1. For instance, there are two types of q-addition [7], the Nalli-Ward-Al-Salam q-additionand the Jackson-Hahn-Cigler q-addition. The first one is commutative and associative, while the second one is neither. This is one of the reasons why sometimes more than one q-analogue exists.

Recently, it has been presented the q-analogue of Toda lattice system of difference equations by discussing the q-discretization in three aspects: differential q-difference, q-difference-q-difference and q-differential-q-difference Toda equation [29]. Moreover, a new integrable equation which is a generalization of q-Toda equation has been constructed in [20], presenting its soliton solutions. But the first studies to construct Lax pairs for the q-Painlevé equations were those of Jimbo and Sakai [12] and Sakai [27, 28] using the Birkhoff theory of linear difference and q-difference equations.

In this work we analyze the correspondence between dynamics of q-Toda and q-Volterra equations for the coefficients of the Jacobi operator. For a given solution of a q-Toda lattice we construct a solution of the q-Volterra lattice, and from the latter by using Bäcklund or Miura transformations we derive another solution of the initial q-Toda equation. Equivalent conditions in terms of q-difference equation for the Jacobi matrix, the linear functional, the moments and the Stieltjes function are proved. The main ingredient are orthogonal polynomials which satisfy an Appell condition with respect to the q-difference operator D_q as well as a Lax type theorem for the point spectrum of the Jacobi operator associated with these equations. By limit as $q \uparrow 1$ we recover known results of the continuous case. An explicit example related with the big q-Laguerre orthogonal polynomials and q-Toda and q-Volterra equations is given.

2. q-Toda equations

Let $\mu \in \mathbb{R}$ be fixed. A set $A \subseteq \mathbb{R}$ is called a μ-geometric set if for $t \in A$, $\mu t \in A$. Unless we say otherwise we shall always assume that $0 < q < 1$. Let f be a function defined on a q-geometric set $A \subseteq \mathbb{R}$. The q-difference operator, acting on the variable t, is defined by

$$\mathrm{D}_q f(t) = \frac{f(qt) - f(t)}{(q-1)\,t}\,, \qquad t \in A \setminus \{0\}\,.$$

If $0 \in A$, we say that f has the q-derivative at zero if the limit

$$\lim_{n\to\infty} \frac{f(tq^n) - f(0)}{tq^n}\,, \qquad t \in A\,,$$

exists and does not depend on t. We then denote this limit by $\mathrm{D}_q f(0)$.

The main aim of this paper is to analyze the following system of difference equations (q-Toda equations):

$$\begin{cases} \mathrm{D}_q a_n(t;q) = \alpha_1^n(t;q)\big(b_{n-1}(t;q) - b_n(qt;q)\big)\,, \\ \mathrm{D}_q b_n(t;q) = \alpha_1^n(t;q) - \alpha_1^{n+1}(t;q)\,, \end{cases} \qquad n \in \mathbb{N}\,, \tag{16}$$

where

$$\alpha_1^n(t;q) = \frac{g_n(t;q)}{1 + (q-1)\,t\,b_0(qt;q)}\,, \tag{17}$$

$$g_n(t;q) = \prod_{k=1}^{n} \frac{a_k(qt;q)}{a_{k-1}(t;q)}\,, \tag{18}$$

assuming that $1 + (q-1)\,t\,b_0(qt;q) \neq 0$ and $a_0(t;q) = 1$.

Next, we present the central theorem of this work.

Theorem 1. *Let us assume that the sequences $\{a_n(t) \equiv a_n(t;q)\}_{n\in\mathbb{N}}$ and $\{b_n(t) \equiv b_n(t;q)\}_{n\in\mathbb{N}}$ are uniformly bounded. The following conditions are equivalent:*

(1) *The Jacobi matrix $J(t)$ defined in (3) satisfies the matrix q-difference equation*

$$\mathrm{D}_q J(t) = A(t)J(t) - J(qt)A(t)\,, \tag{19}$$

where

$$A(t)=\begin{pmatrix} b_0(qt) & 0 & & \\ g_1(t) & b_0(qt) & 0 & \\ 0 & g_2(t) & b_0(qt) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{20}$$

(2) *The moments $u_n(t)$, defined by (9), satisfy*

$$\mathrm{D}_q u_n(t) = -u_{n+1}(qt) + u_1(qt)u_n(t)\,, \qquad n\in\mathbb{N}\,. \tag{21}$$

(3) *The Stieltjes function associated with $J(t)$ satisfies*

$$\mathrm{D}_q S(z;t) = -zS(z;qt) + u_1(qt)S(z,t) + 1. \tag{22}$$

(4) *The linear functional $u(t)$ associated with $J(t)$ satisfies*

$$\mathrm{D}_q u(t) = -xu(qt) + u_1(qt)u(t)\,. \tag{23}$$

(5) *The monic polynomials $\{P_n(x,t;q)\equiv P_n(x;t)\}_{n\in\mathbb{N}}$ defined by the three term recurrence relation*

$$P_{n+1}(x;t) = (x-b_n(t))P_n(x;t) - a_n(t)P_{n-1}(x;t)\,, \quad n=1,2,\dots\,,$$

with $P_{-1}(x;t)=0$ and $P_0(x;t)=1$, satisfy an Appell condition

$$\mathrm{D}_q P_n(x;t) = \alpha_1^n(t)\,P_{n-1}(x;t)\,, \tag{24}$$

where

$$\alpha_1^n(t)\equiv\alpha_1^n(t;q) = \frac{g_n(t)}{1+(q-1)\,t\,u_1(qt)}\,, \quad n=1,2,\dots\,, \tag{25}$$

and $g_n(t)\equiv g_n(t;q)$ is given in (18).

Proof. (1) $\Rightarrow$ (2). By induction it can be proved that

$$\mathrm{D}_q J^n(t) = A(t)J^n(t) - J^n(qt)A(t)\,, \tag{26}$$

where $A(t)$ is defined in (20). By using (9)

$$e_0^\top \mathrm{D}_q J^n(t)e_0 = \mathrm{D}_q\big(e_0^\top J^n(t)e_0\big) = \mathrm{D}_q u_n(t)\,,$$

where $e_0^\top=(1,0,\dots)$. Moreover, from (26) we have

$$\begin{aligned} e_0^\top \mathrm{D}_q J^n(t)e_0 &= u_1(qt)J^n_{1,1}(t) - \big(J^n_{1,1}(qt)u_1(qt) + J^n_{1,2}(qt)a_1(qt)\big)\\ &= u_1(qt)u_n(t) - J^{n+1}_{1,1}(qt)\,, \end{aligned}$$

since $b_0(qt)=u_1(qt)$, which completes the proof.

(2) $\Rightarrow$ (3). From (6), then

$$\begin{aligned} \mathrm{D}_q S(z;t) &= \sum_{n=0}^{\infty}\frac{\mathrm{D}_q u_n(t)}{z^{n+1}} = -\sum_{n=0}^{\infty}\frac{u_{n+1}(qt)}{z^{n+1}} + u_1(qt)\sum_{n=0}^{\infty}\frac{u_n(t)}{z^{n+1}}\\ &= -zS(z;qt) + u_1(qt)S(z;t) + 1\,, \end{aligned}$$

where we have used that $u_0(t) = 1$.

(3) $\Rightarrow$ (4). By using (6) and (8), if we apply the D_q operator, we have that the equation (22) reads as

$$\begin{aligned}\mathrm{D}_q S(z;t) &:= \Big\langle \mathrm{D}_q u(t), \frac{1}{z-x}\Big\rangle = \Big\langle u(qt), \frac{-z}{z-x} + 1\Big\rangle + \Big\langle u(t), \frac{u_1(qt)}{z-x}\Big\rangle \\ &= \Big\langle u(qt), \frac{-x}{z-x}\Big\rangle + \Big\langle u(t), \frac{u_1(qt)}{z-x}\Big\rangle ,\end{aligned}$$

which implies

$$\Big\langle \mathrm{D}_q u(t) + x u(qt) - u_1(qt) u(t), \frac{1}{z-x}\Big\rangle = 0 ,$$

and so, all the moments for the linear functional $\mathrm{D}_q u(t) + x u(qt) - u_1(qt) u(t)$ are zero, and (23) is obtained.

(4) $\Rightarrow$ (5). First of all, let us show that a regular linear functional $u(t)$ satisfying (21), is such that $1 + (q-1)\, t\, u_1(qt) \neq 0$. Let us assume that $u_1(qt) = 1/((1-q)t)$. Then, from (21) we obtain that $u_2(qt) = 1/((1-q)^2\, t^2)$ which yields

$$\det H_1(qt) = \begin{vmatrix} u_0(qt) & u_1(qt) \\ u_1(qt) & u_2(qt) \end{vmatrix} = \begin{vmatrix} 1 & 1/((1-q)t) \\ 1/((1-q)t) & 1/((1-q)^2\, t^2) \end{vmatrix} = 0 ,$$

in contradiction with being $u(t)$ a regular linear functional (cf. for instance [4]).

Let $\{P_n(x;t)\}_{n\in\mathbb{N}}$ be the sequence of monic orthogonal polynomials with respect to the linear functional $u(t)$. Since $\{P_n(x;t)\}_{n\in\mathbb{N}}$ is a basis in the space of polynomials of degree n, we have

$$\mathrm{D}_{q^{-1}}\, P_n(x;qt) = \sum_{k=1}^{n} \alpha_k^n(t)\, P_{n-k}(x;t) . \tag{27}$$

By convention we shall assume that $\alpha_1^0(t) = 0$. We shall prove for $n > 1$ that $\alpha_k^n(t) = 0$ for $k = 2, \ldots, n$ and $\alpha_1^n(t) \neq 0$. By applying the linear functional $u(t)$ to (27) and using the orthogonality of $P_n(x;t)$ it holds

$$\alpha_n^n(t) \big\langle u(t), P_0(x;t)\big\rangle = \big\langle u(t), \mathrm{D}_{q^{-1}}\, P_n(x;qt)\big\rangle = -\big\langle \mathrm{D}_q u(t), P_n(x;qt)\big\rangle ,$$

where we have used that [17]

$$\big\langle \mathrm{D}_q u, p(x;t)\big\rangle = -\frac{1}{q}\big\langle u, \mathrm{D}_{q^{-1}}\, p(x;t)\big\rangle , \quad \text{and} \qquad \mathrm{D}_q\, \mathrm{D}_{q^{-1}}\, f(t) = \frac{1}{q}\, \mathrm{D}_{q^{-1}}\, \mathrm{D}_q f(t) .$$

From (23) we obtain

$$\begin{aligned}\alpha_n^n(t) \big\langle u(t), P_0(x;t)\big\rangle &= \big\langle x u(qt), P_n(x;qt)\big\rangle - u_1(qt) \big\langle u(t), P_n(x;qt)\big\rangle \\ &= -u_1(qt) \big\langle u(t), P_n(x;qt)\big\rangle ,\end{aligned}$$

for $n > 1$, by using the orthogonality. From (27) the above expression can be written as

$$\alpha_n^n(t)\big\langle u(t), P_0(x;t)\big\rangle = -u_1(qt)\big\langle u(t), P_n(x;t) + (q-1)\,t \sum_{k=1}^{n} \alpha_k^n(t)\, P_{n-k}(x;t)\big\rangle$$
$$= -u_1(qt)(q-1)\,t\,\alpha_n^n(t)\langle u(t), P_0(x;t)\rangle\,,$$

for $n > 1$, by using again the orthogonality. Since $\big\langle u(t), P_0(x;t)\big\rangle = 1\,,$

$$(1 + (q-1)\,t\,u_1(qt))\alpha_n^n(t) = 0\,,$$

and we obtain $\alpha_n^n(t) = 0$. Similar arguments recursively can be used to prove that $\alpha_k^n(t) = 0$, for $k = 2, \ldots, n$. Therefore,

$$\mathrm{D}_{q^{-1}}\, P_n(x;qt) = \alpha_1^n(t)\, P_{n-1}(x;t)\,.$$

Finally, we will determine $\alpha_1^n(t)$ by using the orthogonality of $P_n(x;t)$:

$$\begin{aligned} \alpha_1^n\big\langle u(t), x^{n-1}P_{n-1}(x;t)\big\rangle &= \big\langle u(t), \mathrm{D}_{q^{-1}}(x^{n-1}P_n(x;qt))\big\rangle \\ &= -\big\langle \mathrm{D}_q u(t), x^{n-1}P_n(x;qt)\big\rangle\,. \end{aligned}$$

In order to compute the last inner product we shall use

$$P_n(x;qt) = P_n(x;t) + \alpha_1^n(t)(q-1)\,t\,P_{n-1}(x;t)\,,$$

as well as the orthogonality:

$$\begin{aligned} &\big\langle \mathrm{D}_q u(t), x^{n-1}P_n(x;qt)\big\rangle = \big\langle u(qt), x^n P_n(x;qt)\big\rangle - u_1(qt)\big\langle u(t), x^{n-1}P_n(x;qt)\big\rangle \\ &= \big\langle u(qt), x^n P_n(x;qt)\big\rangle - u_1(qt)\big\langle u(t), x^{n-1}\big(P_n(x;t) + \alpha_1^n(t)(q-1)\,t\,P_{n-1}(x;t)\big)\big\rangle \\ &\qquad = \big\langle u(qt), x^n P_n(x;qt)\big\rangle - u_1(qt)\alpha_1^n(q-1)\,t\,\big\langle u(t), x^{n-1}P_{n-1}(x;t)\big\rangle\,, \end{aligned}$$

which gives the value of $\alpha_1^n(t)$ given in (25).

(5) $\Rightarrow$ (1) If we apply $\mathrm{D}_{q^{-1}}$ to the recurrence relation

$$xP_n(x;qt) = P_{n+1}(x;qt) + b_n(qt)P_n(x;qt) + a_n(qt)P_{n-1}(x;qt),$$

we get

$$\begin{aligned} \alpha_1^n(t)xP_{n-1}(x;t) &= \alpha_1^{n+1}(t)P_n(x;t) + b_n(qt)\alpha_1^n(t)P_{n-1}(x;t) \\ &+ \mathrm{D}_q b_n(t)P_n(x;t) + a_n(qt)\alpha_1^{n-1}(t)P_{n-2}(x;t) + \mathrm{D}_q a_n(t)P_{n-1}(x;t). \end{aligned}$$

If we use again the recurrence relation to expand

$$xP_{n-1}(x;t) = P_n(x;t) + b_{n-1}(t)P_{n-1}(x;t) + a_{n-1}(t)P_{n-2}(x;t),$$

by equating the coefficients in $P_n(x,t)$, $P_{n-1}(x;t)$ and $P_{n-2}(x;t)$, we get the q-Toda equations

$$\alpha_1^n(t) = \alpha_1^{n+1}(t) + \mathrm{D}_q b_n(t)\,, \quad \alpha_1^n(t)\big(b_{n-1}(t) - b_n(qt)\big) = \mathrm{D}_q a_n(t)\,, \tag{28}$$

with $\alpha_1^n(t)a_{n-1}(t) = \alpha_1^{n-1}(t)a_n(qt)$, $n \in \mathbb{N}$. $\square$

Remark 1. As a consequence, if we consider the limit as $q \uparrow 1$ in the above Theorem we recover a number of results in relation with the dynamic solutions of the Toda equations (1) that are dispersed in the literature (see [1, 18, 25, 26] and references therein).

The modification of a orthogonality measure by multiplying it by the q-exponential of a polynomial is studied in the next theorem. The notation to be used is that of [9, Chapter 1]. In particular

$$(x;\, q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

Theorem 2. *In the hypothesis of Theorem 1, assume that the normalized functional $u(t)$ verifies*

$$u(t) = \kappa((1-q)xt;\, q)_\infty \, v \,, \tag{29}$$

where κ is the normalizing constant and v is a positive definite linear functional. Then, the coefficients $\{a_n(t)\}_{n\in\mathbb{N}}$, $\{b_n(t)\}_{n\in\mathbb{N}}$ of the Jacobi matrix $J(t)$ associated to $u(t)$ are solution of the q-Toda equations (16).

Proof. Let

$$f(x,t;q) = ((1-q)xt;\, q)_\infty \,, \tag{30}$$

and the moments $\langle v, x^n \rangle = \displaystyle\int x^n d\varrho(x)$, $n = 0, 1, \ldots$. Let $u_n(t)$ be the moments of the linear functional $u(t)$,

$$u_n(t) = \Big(\int f(x,t;q) x^n d\varrho(x) \Big) \Big/ \Big(\int f(x,t;q) d\varrho(x) \Big) \,.$$

Since

$$\mathrm{D}_q(f(t)/g(t)) = \big(\mathrm{D}_q f(t) g(t) - f(t) \mathrm{D}_q g(t) \big) \big/ \big(g(t) g(qt) \big) \,,$$

then

$$\mathrm{D}_q u_n(t) = \frac{\int \mathrm{D}_q f(x,t;q) x^n d\varrho(x)}{\int f(x,qt;q) d\varrho(x)} \\ - \frac{\Big(\int f(x,t;q) x^n d\varrho(x) \Big) \Big(\int \mathrm{D}_q f(x,t;q) d\varrho(x) \Big)}{\Big(\int f(x,t;q) d\varrho(x) \Big) \Big(\int f(x,qt;q) d\varrho(x) \Big)} \,.$$

By using $\mathrm{D}_q f(x,t) = -x f(x,qt)$, we obtain $\mathrm{D}_q u_n(t) = -u_{n+1}(qt) + u_1(qt) u_n(t)$, which completes the proof. $\square$

Remark 2. Notice that [14] from (30)

$$\lim_{q\to 1} f(x,t;q) = \lim_{q\to 1} E_q((q-1)\,x\,t) = \exp(-xt)\,,$$

which is the evolution (11) associated to the continuous case [13, 22], where

$$E_q(z) = (-z;\,q)_\infty = \prod_{k=0}^{\infty}(1+zq^k) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}z^n}{(q;\,q)_n}\,, \tag{31}$$

denotes a q-exponential function [11, p. 306], where the q-shifted factorial is

$$(a;\,q)_n = \prod_{j=0}^{n-1}(1-aq^j)\,.$$

Next, we prove a Lax-type theorem [16, Theorem 3, p. 270].

Theorem 3. *Let $\mathcal{P}$ be the column vector of monic polynomials $P_n(x;t)$ and Let λ be a spectral point of the Jacobi matrix $J(t)$, i.e.*

$$J\,\mathcal{P}(\lambda(t)) = \lambda(t)\,\mathcal{P}(\lambda(t));$$

then, $J(t)$ satisfies (19) with the matrix $A(t)$ defined by (20) if, and only if, $\mathrm{D}_q\lambda(t) = 0$.

Proof. If we apply the D_q operator to

$$J(t)\mathcal{P}(\lambda(t)) = \lambda(t)\mathcal{P}(\lambda(t))$$

we obtain

$$\mathrm{D}_q J(t)\mathcal{P}(\lambda(t)) + J(qt)\mathrm{D}_q\mathcal{P}(\lambda(t)) = \mathrm{D}_q\lambda(t)\mathcal{P}(\lambda) + \lambda(qt)\mathrm{D}_q\mathcal{P}(\lambda(t))\,.$$

Then,

$$\begin{aligned} A(t)\lambda(t)\mathcal{P}(\lambda(t)) - J(qt)A(t)\mathcal{P}(\lambda(t)) + \big(J(qt) - \lambda(qt)\,\mathcal{I}\big)\mathrm{D}_q\mathcal{P}(\lambda(t)) \\ = \big(\mathrm{D}_q\lambda(t)\big)\mathcal{P}(\lambda(t))\,. \end{aligned}$$

and so

$$\begin{aligned} (J(qt) - \lambda(qt)\,\mathcal{I})\big(\mathrm{D}_q\mathcal{P}(\lambda(t)) - A(t)\mathcal{P}(\lambda(t))\big) \\ = \big(\mathrm{D}_q\lambda(t)\,\mathcal{I} + \big(\lambda(qt) - \lambda(t)\big)A(t)\big)\mathcal{P}(\lambda(t))\,. \end{aligned}$$

Taking into account that the condition

$$(J(qt) - \lambda(qt)\,\mathcal{I})\big(\mathrm{D}_q\mathcal{P}(\lambda(t)) - A(t)\mathcal{P}(\lambda(t))\big) = 0\,,$$

implies the existence of a real parameter s (we can show that $s = b_0(qt)$) such that

$$\mathrm{D}_q\mathcal{P}(\lambda(t)) - A(t)\mathcal{P}(\lambda(t)) = s\,\mathcal{P}(\lambda(qt))\,,$$

and that this equation is the vector representation of (24) (which is equivalent to (19) by Theorem 1), we get that, as $1+(q-1)\,t\,b_0(qt) \neq 0$, $\mathrm{D}_q\lambda(t) = 0$, and this completes the proof. □

3. q-Volterra lattices

Let us now consider the q-Volterra lattice (or q-Langmuir lattice):

$$\mathrm{D}_q\Gamma(t) = B(t)\Gamma(t) - \Gamma(qt)B(t)\,, \tag{32}$$

where

$$\Gamma(t) = (\Gamma_{i,j}(t)) = \begin{pmatrix} 0 & 1 & 0 & & & \\ \gamma_1(t) & 0 & 1 & 0 & & \\ 0 & \gamma_2(t) & 0 & 1 & 0 & \\ & & \ddots & \ddots & \ddots & \end{pmatrix}\,, \tag{33}$$

$$B(t) = \begin{pmatrix} \gamma_1(qt) & 0 & 0 & & \\ 0 & \gamma_1(qt) & 0 & & \\ \eta_1(t) & 0 & \gamma_1(qt) & \ddots & \\ 0 & \eta_2(t) & 0 & \gamma_1(qt) & \\ & \ddots & \ddots & \ddots & \end{pmatrix}\,, \tag{34}$$

$$\eta_n(t) = \frac{\gamma_1(qt)\cdots\gamma_{n+1}(qt)}{\gamma_1(t)\cdots\gamma_{n-1}(t)}\,,\ n = 2,3,\dots\,,\ \eta_1(t) = \gamma_1(qt)\gamma_2(qt)\,. \tag{35}$$

The following equations constitute another formulation of q-Volterra lattice equivalent to (32):

$$\begin{cases} \mathrm{D}_q\gamma_1(t) = -\dfrac{\gamma_1(qt)\gamma_2(qt)}{1+(q-1)\,t\,\gamma_1(qt)}\,, \\ \mathrm{D}_q\gamma_n(t) = \dfrac{\gamma_n(qt)\cdots\gamma_1(qt)(\gamma_{n-1}(t)-\gamma_{n+1}(qt))}{(1+(q-1)\,t\,\gamma_1(qt))\gamma_{n-1}(t)\cdots\gamma_1(t)}\,,\quad n\geq 2, \end{cases} \tag{36}$$

assuming that $1+(q-1)t\gamma_1(qt) \neq 0$ and $\gamma_n(t) \neq 0$.

In a similar way as Theorem 1 for the q-Toda lattices, it can be proved the following result.

Theorem 4. *Let us assume that the sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded. The following conditions are equivalent:*

(1) *The Jacobi matrix $\Gamma(t)$ defined in (33) satisfies the matrix difference equation (32).*
(2) *The moments $u_n(t)$ associated to a symmetric functional $u(t)$, defined by (9), satisfy*

$$\mathrm{D}_q u_n(t) = -u_{n+2}(qt) + u_2(qt)u_n(t)\,, \qquad \textit{when } n \textit{ is even,} \tag{37}$$

since $u_{2n+1}(t) = 0$.

(3) *The Stieltjes function associated with* $\Gamma(t)$ *satisfies*

$$\mathrm{D}_q S(z;t) = -z^2 S(z;qt) + u_2(qt) S(z,t) + z\,. \tag{38}$$

(4) *The linear functional* $u(t)$ *associated with* $\Gamma(t)$ *satisfies*

$$\mathrm{D}_q u(t) = -x^2 u(qt) + u_2(qt) u(t)\,. \tag{39}$$

(5) *The monic symmetric polynomials* $\{R_n(x;t)\}_{n\in\mathbb{N}}$,

$$R_n(-x;t) = (-1)^n R_n(x;t),$$

defined by the three term recurrence relation

$$R_{n+1}(x;t) = x R_n(x;t) - \gamma_n(t) R_{n-1}(x;t)\,, \tag{40}$$

with $R_{-1}(x;t) = 0$ *and* $R_0(x;t) = 1$, *satisfy an Appell condition*

$$\mathrm{D}_q R_n(x;t) = \sigma_2^n(t)\, R_{n-2}(x;t)\,, \tag{41}$$

where for $n = 2, 3, \ldots$

$$\begin{aligned} \sigma_2^n(t) &= \frac{\langle u(qt), x^n R_n(x;qt)\rangle}{(1+(q-1)tu_2(qt))\langle u(t), x^{n-2} R_{n-2}(x;t)\rangle} \\ &= \frac{\eta_{n-1}(t)}{1+(q-1)\,t\,\gamma_1(qt)}\,, \end{aligned} \tag{42}$$

and $\eta_n(t)$ *is defined in (35).*

Proof. (1) $\Rightarrow$ (2) Since the moments u_n can be expressed as $u_n = e_0^\top \Gamma^n(t) e_0$, from (32) we obtain

$$\begin{aligned} e_0^\top \mathrm{D}_q \Gamma^n(t) e_0 &= \gamma_1(qt)\Gamma_{1,1}^n(t) - \left(\Gamma_{1,1}^n(qt)\gamma_1(qt) + \Gamma_{1,3}^n(qt)\eta_1(t)\right) \\ &= u_2(qt)u_n(t) - \Gamma_{1,1}^{n+2}(qt) = -u_{n+2}(qt) + u_2(qt)u_n(t)\,. \end{aligned}$$

(2) $\Rightarrow$ (3) It follows from the definition of the Stieltjes function $S(z)$ in terms of moments (6).
(3) $\Rightarrow$ (4) It is a consequence of (6) and (8).
(4) $\Rightarrow$ (5) Let us prove that if $u(t)$ is a regular linear functional satisfying (37) then $1+(q-1)tu_2(qt) \neq 0$. If $u_2(t) = \frac{q}{(1-q)t}$ then from (37) we obtain $u_4(t) = \frac{q^2}{(q-1)^2 t^2}$ and therefore

$$\begin{vmatrix} u_0(qt) & u_1(qt) & u_2(qt) \\ u_1(qt) & u_2(qt) & u_3(qt) \\ u_2(qt) & u_3(qt) & u_4(qt) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \frac{1}{t(1-q)} \\ 0 & \frac{1}{t(1-q)} & 0 \\ \frac{1}{t(1-q)} & 0 & \frac{1}{(1-q)^2t^2} \end{vmatrix} = 0,$$

in contradiction with being $u(t)$ a regular linear functional [4]. The proof follows by using similar arguments as in Theorem 1.
(5) $\Rightarrow$ (1) It follows from (40) by applying the q-difference operator and using (41). $\square$

Next, we state the evolution theorem for q-Volterra lattices.

Theorem 5. *Assume that the normalized symmetric functional $u(t)$ verifies*

$$u(t) = \kappa((1-q)x^2t;\, q)_\infty v\,, \tag{43}$$

where κ is the normalizing constant and v is a positive definite linear functional. Then, the coefficients $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ of the Jacobi matrix $\Gamma(t)$ associated to $u(t)$ are solution of the q-Volterra lattice (36).

Proof. It follows from the fact that the moments of the linear functional $u(t)$ satisfy (37). □

Remark 3. Introducing $g(x,t;q) = ((1-q)x^2t;\, q)_\infty$, we have that [14],

$$\lim_{q\to 1} g(x,t;q) = \lim_{q\to 1} E_q((q-1)\,x^2\,t) = \exp(-x^2t)\,,$$

which is the evolution (12) associated to the continuous case [13, 22], and the q-exponential function E_q is defined in (31).

Next, we state a Lax-type theorem [16, Theorem 3, p. 270] for q-Volterra lattices.

Theorem 6. *Let $\mathcal{R}$ be the column vector of monic polynomials $R_n(x;t)$ and let λ be a spectral point of the Jacobi matrix $\Gamma(t)$, i.e.*

$$\Gamma(t)\,\mathcal{R}(\lambda(t)) = \lambda(t)\,\mathcal{R}(\lambda(t));$$

then, $\Gamma(t)$ satisfies (32) with the matrix $B(t)$ defined by (33) if, and only if, $\mathrm{D}_q\lambda(t) = 0$.

Proof. Following the same ideas of the proof of Theorem 3 we deduce the equivalence between the existence of a real parameter s (we can show that $s = \gamma_1(qt)$) such that

$$\mathrm{D}_q\mathcal{R}(\lambda(t)) - B(t)\mathcal{R}(\lambda(t)) = s\,\mathcal{R}(\lambda(qt))\,, \tag{44}$$

and, as $1 + (q-1)\,t\,\gamma_1(qt) \neq 0$, that $\mathrm{D}_q\lambda(t) = 0$, which completes the proof, since (44) coincides (41) in vector notation. □

4. Bäcklund or Miura transformations and sequences of orthogonal polynomials

Given a family of tridiagonal matrices $\{J(t), t \in \mathbb{R}\}$, as in (3), we consider the sequence of polynomials $\{P_n(x;t)\}_{n\in\mathbb{N}}$ defined in (10). It is well-known [4] that, if $a_n(t) \neq 0$ for $n = 1, 2, \ldots$, then the sequence $\{P_n(x;t)\}_{n\in\mathbb{N}}$ is orthogonal with respect to some quasi-definite moment functional.

Lemma 1. *Let $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ be a solution of the q-Volterra lattice (36). Then $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ defined by $a_0(t)=1$ and*

$$a_n(t)=\gamma_{2n}(t)\gamma_{2n-1}(t)\,, b_n(t)=\gamma_{2n+1}(t)+\gamma_{2n}(t)+C\,, \tag{45}$$

$n=1,\dots,$ are solution of the q-Toda lattice (16). Moreover, the sequences $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$ defined by $\tilde{a}_0(t)=1$ and

$$\tilde{a}_n(t)=\gamma_{2n+1}(t)\gamma_{2n}(t)\,, \tilde{b}_n(t)=\gamma_{2n+2}(t)+\gamma_{2n+1}(t)+C\,, \tag{46}$$

$n=1,\dots,$ are also solution of the q-Toda lattice (16), assuming that $\gamma_0(t)=1$.

Proof. If we apply the D_q operator to the first equation of (45) we obtain

$$\mathrm{D}_q a_n(t)=\mathrm{D}_q\gamma_{2n}(t)\gamma_{2n-1}(t)+\gamma_{2n}(qt)\mathrm{D}_q\gamma_{2n-1}(t)\,.$$

From (36) it yields

$$\begin{aligned}\mathrm{D}_q a_n(t) &= \frac{\gamma_{2n}(qt)\cdots\gamma_1(qt)\big(\gamma_{2n-1}(t)-\gamma_{2n+1}(qt)\big)\gamma_{2n-1}(t)}{(1+(q-1)\,t\,\gamma_1(qt))\gamma_{2n-1}(t)\cdots\gamma_1(t)} \\ &\quad+\gamma_{2n}(qt)\frac{\gamma_{2n-1}(qt)\cdots\gamma_1(qt)\big(\gamma_{2n-2}(t)-\gamma_{2n}(qt)\big)}{(1+(q-1)\,t\,\gamma_1(qt))\gamma_{2n-2}(t)\cdots\gamma_1(t)} \\ &= \frac{\gamma_{2n}(qt)\cdots\gamma_1(qt)\big(\gamma_{2n-1}(t)+\gamma_{2n-2}(t)-\gamma_{2n+1}(qt)-\gamma_{2n}(qt)\big)}{(1+(q-1)\,t\,\gamma_1(qt))\gamma_{2n-2}(t)\cdots\gamma_1(t)}\end{aligned}$$

where by using (45) we finally obtain

$$\mathrm{D}_q a_n(t)=\frac{a_n(qt)\cdots a_1(qt)}{(1+(q-1)\,t\,\gamma_1(qt))a_{n-1}(t)\cdots a_1(t)}\big(b_{n-1}(t)-b_n(qt)\big)\,.$$

Moreover, if we apply the D_q operator to the second equation of (45) we obtain

$$\mathrm{D}_q b_n(t)=\mathrm{D}_q\gamma_{2n+1}(t)+\mathrm{D}_q\gamma_{2n}(t)\,,$$

where by using (45) the result follows.
The results for $\{\tilde{a}_n(t)\}$ and $\{\tilde{b}_n(t)\}$ follow in a similar way. □

Lemma 2. *Let $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ be solution of the q-Toda lattice (16), and $\{P_n(x;t)\}_{n\in\mathbb{N}}$ be the sequence of orthogonal polynomials with Jacobi matrix (3). Let $c\in\mathbb{R}$ such that $P_n(c;t)\neq 0$, for each $n\in\mathbb{N}$ and for all $t\in\mathbb{R}$. Then the sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ defined in (45) are solution of the q-Volterra lattice (36), assuming that $\gamma_0(t)=1$.*

Proof. From [4, Exercise 9.6, page 49] we have that the coefficients $\gamma_n(t)$ have the following representation

$$\gamma_{2n+1}(t) = -\frac{P_{n+1}(c;t)}{P_n(c;t)}\,,\ \gamma_{2n+2}(t) = -a_{n+1}(t)\frac{P_n(c;t)}{P_{n+1}(c;t)}\,,\ n \in \mathbb{N}\,, \tag{47}$$

for the odd and even cases.

In Section 2 we have proved that a necessary and sufficient condition for $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ be a solution of a q-Toda lattice is that $\{P_n(x;t)\}_{n\in\mathbb{N}}$ satisfy the Appell condition (24). The result follows by applying the D_q operator to both equations. □

As a consequence, if $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ are solution of the q-Toda lattice defined in (16), then from Lemma 2 we construct a solution of the q-Volterra lattice (36) denoted by $\{\gamma_n(t)\}_{n\in\mathbb{N}}$. Now, from Lemma 1 and these coefficients $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ we construct another solution $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$ of the q-Toda lattice defined in (16).

Let us denote by $J_n(t)$ the finite submatrix formed by the first n rows and columns of $J(t)$. We may summarize these result as follows, which is a q-analogue of [3, Theorem 1.3], where the full Toda and Volterra hierarchy has been considered.

Theorem 7. *Let us consider the family $\{J(t)\}$, $t \in \mathbb{R}$, of tridiagonal infinite matrices defined in (33) and let $c \in \mathbb{C}$ be such that $\det\big(J_n(t) - c\mathcal{I}_n\big) \neq 0$, for each $n \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Then there exists a sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$, $t \in \mathbb{R}$, solution of (36) and there exists a pair of two sequences $\{a_n(t)\}_{n\in\mathbb{N}}, \{b_n(t)\}_{n\in\mathbb{N}}$, and $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}, \{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$, $t \in \mathbb{R}$, solutions of (16) such that (45) and (46) hold.*
Moreover, for each c in the above conditions, the sequences $\{\gamma_n(t)\}_{n\in\mathbb{N}}$, $\{a_n(t)\}_{n\in\mathbb{N}}$, $\{b_n(t)\}_{n\in\mathbb{N}}$, and $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$, $\{\tilde{b}_n(t)\}$ are the unique sequences verifying (45) and (46).

Notice that the condition $\det\big(J_n(t) - c\mathcal{I}_n\big) \neq 0$, is equivalent to $P_n(c;t) \neq 0$ for the monic polynomials $\{P_n(x;t)\}_{n\in\mathbb{N}}$ defined by (10) [4, 23, 30].

5. Examples

We shall show how orthogonal polynomials can be used to provide an explicit solution of q-Volterra equations (16) and q-Toda equations (36).

Let $v^{(c)}$ be the normalized linear functional corresponding to the big q-Legendre polynomials [14, p. 443] defined in terms of a Jackson's

q-integral [7, 9, 14]

$$\langle v^{(c)}, p(x)\rangle = \frac{1}{q(1-c)} \int_{cq}^{q} p(x)\, \mathrm{D}_q x\,, \tag{48}$$

where $c < 0$.

We have the moments,

$$(v^{(c)})_0 = \langle v^{(c)}, 1\rangle = 1\,, \qquad (v^{(c)})_1 = \langle v^{(c)}, x\rangle = -\frac{(c^2-1)q^2}{q+1}\,. \tag{49}$$

5.1. **q-Toda lattices.** Let us consider a q-exponential modification of the linear functional $v^{(c)}$ —see (29)—

$$u(t) = \kappa((1-q)xt;\, q)_\infty\, v^{(c)}, \tag{50}$$

where κ is a normalizing constant for $u(t)$. This linear functional $u(t)$ is a particular case of the big q-Laguerre linear functional [14, p. 479] defined by

$$\langle u^{(a,b)}, p(x)\rangle = \int_{bq}^{aq} \frac{(a^{-1}x, b^{-1}x;\, q)_\infty}{(x;\, q)_\infty} p(x)\, \mathrm{D}_q x\,,$$

which is positive-definite for $0 < aq < 1$ and $b < 0$, for the specific values $a = 1$ and $b^{-1} = (1-q)t$.

We have that the sequence of monic polynomials $\{P_n(x,t;q)\}_{n\in\mathbb{N}}$ orthogonal with respect to $u(t)$ defined in (50) satisfy a three term recurrence relation with coefficients [14, Eq. (14.11.4)]

$$\begin{cases} b_n(t) = \dfrac{q^{n+1}\big((q+1)q^n + qt - t - 2\big)}{(q-1)\,t}\,, \\[2ex] a_n(t) = \dfrac{q^{n+1}\big(q^n-1\big)^2\big(q^n + (q-1)\,t\big)}{(q-1)^2t^2} \end{cases}.$$

From (18), we have $g_n(t;q) = q\big(q^n-1\big)^2((q-1)\,t+1)/((q-1)^2t^2)\,,$ and therefore, using (17), we get $\alpha_1^n(t;q) = (q^n-1)^2/((q-1)^2t^2)\,.$ Thus, we have that $\{b_n(t)\}_{n\in\mathbb{N}}$, $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{\alpha_1^n(t;q)\}_{n\in\mathbb{N}}$ are solution of the q-Toda equations (16), assuming that $1+(q-1)tb_0(qt) = q+(q-1)qt \neq 0$.

Moreover, the limit as $q\uparrow 1$ of $b_n(t;q)$ and $a_n(t;q)$ are

$$\lim_{q\uparrow 1} b_n(t;q) = \frac{2n+t+1}{t} = b_n(t)\,, \qquad \lim_{q\uparrow 1} a_n(t;q) = \frac{n^2}{t^2} = a_n(t)\,,$$

which are explicit solutions of the Toda equations (1). Thus, we have the following limit relation

$$\lim_{q\uparrow 1} P_n(x,t;q) = t^{-n} L_n^{(0)}\big(t(x-1)\big)\,,$$

in terms of monic Laguerre polynomials $L_n^{(\alpha)}(x)$, which satisfy a three-term recurrence relation as (10) with coefficients $a_n(t)$ and $b_n(t)$. Notice that $\{L_n^{(0)}\big(t(x-1)\big)\}_{n\in\mathbb{N}}$ is the sequence of polynomials orthogonal with respect to $\exp(-xt)$ (see [14]).

5.2. **q-Volterra lattices.** Let us now consider the following modification of the big q-Legendre linear functional $v^{(c)}$ already defined in (48),

$$u(t) = \kappa((1-q)x^2t;\, q)_\infty \, v^{(c)}, \tag{51}$$

where κ is a normalizing constant for $u(t)$.

Let us recall the linear functional u_H of discrete q-Hermite I polynomials defined in terms of a Jackson's q-integral [7, 9]

$$\big\langle u_H, p(x)\big\rangle = \int_{-1}^{1} (qx, -qx;\, q)_\infty \, p(x)\, \mathrm{D}_q x\,,$$

and the discrete q-Hermite I polynomials [14, (14.28.1)]

$$h_n(x;q) = q^{n(n-1)/2}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \,\middle|\, q; -qx \right) .$$

In this case, we can identify the linear functional $u(t)$ and the sequence of symmetric orthogonal polynomials $\{R_n(x;t)\}_{n\in\mathbb{N}}$ by doing the substitutions $q \to \sqrt{q}$ and $x \to q^{-1/2}\, x\, \sqrt{(1-q)t}$ in the definition of u_H and $h_n(x;q)$, respectively, i.e. $R_n(x;t) = h_n\big(q^{-1/2}\, x\, \sqrt{(1-q)t}; \sqrt{q}\big)$. Therefore, the sequence of symmetric orthogonal polynomials $\{R_n(x;t)\}_{n\in\mathbb{N}}$ with respect to $u(t)$ satisfy a three term recurrence relation as (40) with

$$\gamma_n(t) = \frac{q^{(n+1)/2}[n/2]_q}{t}\,, \qquad [z]_q := \frac{q^z-1}{q-1}\,. \tag{52}$$

Observe that the difference equation (41) can be written as

$$\mathrm{D}_q R_n(x;t) = \sigma_2^n(t;q)\, R_{n-2}(x;t)\,,\ \ \sigma_2^n(t;q) = \frac{[n/2]_q\,[(n-1)/2]_q}{t^2},$$

$n = 2, 3, \ldots$. It is easy to check that the coefficients $\gamma_n(t)$ defined in (52) are solution of the q-Volterra equations (36). Moreover, if we consider the limit as $q \to 1$ of $\gamma_n(t)$ we obtain

$$\lim_{q\uparrow 1} \gamma_n(t) = \lim_{q\uparrow 1} \frac{q^{(n+1)/2}[n/2]_q}{t} = \frac{n}{2t}\,,$$

which are exactly the coefficients of the three term recurrence relation satisfied by the sequence of monic polynomials $\{\big(2\sqrt{t}\big)^{-n} H_n\big(x\sqrt{t}\big)\}_{n\in\mathbb{N}}$, associated to $\exp(-x^2 t)$ which is the limit of the weight function. Therefore, for the sequence of monic polynomials $\{R_n(x,t)\}_{n\in\mathbb{N}}$, we have

$$\lim_{q\uparrow 1} R_n(x,t) = \big(2\sqrt{t}\big)^{-n} H_n\big(x\sqrt{t}\big)\,,$$

where $H_n(x)$ are the Hermite polynomials.

Using the Miura transformations (45) and relations (52) we obtain explicitly the sequences

$$\begin{cases} a_n(t) = \dfrac{q^{2n+1/2}\,[n]_q\,[n-1/2]_q}{t^2}\,, \\ b_n(t) = \dfrac{q^{n+\frac{1}{2}}\big((q+1)q^n - \sqrt{q} - 1\big)}{(q-1)\,t}\,, \end{cases} \tag{53}$$

which are solutions of the q-Toda equations (16) and different as compared with the solution given in the previous subsection. Therefore, we can identify the sequence of monic polynomials $\{P_n(x,t)\}_{n\in\mathbb{N}}$ generated from the recurrence relation (10) in terms of the sequence of monic little q-Laguerre polynomials $\{P_n(x;a|q)\}_{n\in\mathbb{N}}$ [14, (14.20.1)] as

$$P_n(x,t) = \Big(\Big(\frac{1}{q}-1\Big)t\Big)^{-n} P_n\Big(\Big(\frac{1}{q}-1\Big)tx;\frac{1}{\sqrt{q}}\Big|q\Big)\,.$$

Moreover, the limit of the coefficients given in (53) as $q\uparrow 1$ are

$$\lim_{q\uparrow 1} a_n(t) = \frac{n(2n-1)}{2t^2} = a_n^H(t)\,, \qquad \lim_{q\uparrow 1} b_n(t) = \frac{4n+1}{2t} = b_n^H(t)\,.$$

Notice that the latter coefficients are solution of the Toda equations (1)

$$\frac{d}{dt}a_n^H(t) = a_n^H(t)\big(b_{n-1}^H(t) - b_n^H(t)\big)\,, \qquad \frac{d}{dt}b_n^H(t) = a_n^H(t) - a_{n+1}^H(t)\,,$$

and generate monic Laguerre polynomials $t^{-n}L_n^{(-1/2)}(xt)$. Moreover,

$$\lim_{q\uparrow 1} P_n(x,t) = t^{-n}L_n^{(-1/2)}(xt)\,.$$

Furthermore, from (52) the coefficients $\tilde{a}_n(t)$ and $\tilde{b}_n(t)$ defined in (46) are given by

$$\begin{cases} \tilde{a}_n(t) = \dfrac{q^{2n+3/2}\,[n]_q\,[n+1/2]_q}{t^2}\,, \\ \tilde{b}_n(t) = \dfrac{q^{n+1}\big(q^{n+\frac{1}{2}} + q^{n+\frac{3}{2}} - \sqrt{q} - 1\big)}{(q-1)\,t}\,, \end{cases} \tag{54}$$

and they are new solutions of the q-Toda lattice (16) as compared with the previous subsection. We can identify again the sequence of monic polynomials $\{\tilde{P}_n(x,t)\}_{n\in\mathbb{N}}$ generated from the recurrence relation (10) in terms of the sequence of monic little q-Laguerre polynomials $\{P_n(x;a|q)\}_{n\in\mathbb{N}}$ [14, (14.20.1)] as

$$\tilde{P}_n(x,t) = \left(\left(\frac{1}{q}-1\right)t\right)^{-n} P_n\left(\left(\frac{1}{q}-1\right)tx;\sqrt{q}\,\middle|\,q\right).$$

Moreover, the limit of the coefficients given in (54) as $q\uparrow 1$ are

$$\lim_{q\uparrow 1}\tilde{a}_n(t) = \frac{n(2n+1)}{2t^2} = \tilde{a}_n^H(t)\,, \qquad \lim_{q\uparrow 1}\tilde{b}_n(t) = \frac{4n+3}{2t} = \tilde{b}_n^H(t)\,.$$

Notice that the latter coefficients are solution of the Toda equations

$$\frac{d}{dt}\tilde{a}_n^H(t) = \tilde{a}_n^H(t)\big(\tilde{b}_{n-1}^H(t) - \tilde{b}_n^H(t)\big)\,, \qquad \frac{d}{dt}\tilde{b}_n^H(t) = \tilde{a}_n^H(t) - \tilde{a}_{n+1}^H(t)\,,$$

and generate monic Laguerre polynomials $t^{-n}L_n^{(1/2)}(xt)$. As a consequence,

$$\lim_{q\uparrow 1}\tilde{P}_n(x,t) = t^{-n}L_n^{(1/2)}(xt)\,.$$

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(Area) Departamento de Matemática Aplicada II, E.E. Telecomunicación, Universidade de Vigo, 36310-Vigo, Spain.
E-mail address, Area: area@uvigo.es

(Branquinho) CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 COIMBRA, Portugal.
E-mail address, Branquinho: ajplb@mat.uc.pt

(Foulquié) CIDMA, Department of Mathematics, University of Aveiro, Campus de Santiago, 3810–193 Aveiro, Portugal.
E-mail address, Foulquié: foulquie@ua.pt

(Godoy) Departamento de Matemática Aplicada II, E.E. Industrial, Universidade de Vigo, Campus Lagoas-Marcosende, 36310 Vigo, Spain.
E-mail address, Godoy: egodoy@dma.uvigo.es