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Samuel Häfner

The Author:

Samuel Häfner, Assistant

University of Basel
Faculty of Business and Economics
Peter Merian-Weg 6

CH - 4002 Basel

phone: +41(0)61 267 33 59

E-Mail : samuel.haefner@unibas.ch

http://wwz.unibas.ch/personen/profil/person/haefner/

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Contact:
WWZ Forum | Peter Merian-Weg 6 | CH-4002 Basel | forum-wwz@unibas.ch | wwz.unibas.ch

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Abstract

A multi-stage model on the course of war is presented: Individual battles are modeled as private value all-pay auctions with asymmetric combatants of two opposing teams. These auctions are placed within a multi-stage framework with a tug-of-war structure. Such framing provides a microfounded rationale for the use of the popular logit Tullock contest success function in models of militarized conflicts, yields new theoretical justification for existing empirical findings with respect to war, and provides new hypotheses regarding strategic battlefield behavior.

Keywords: Auction, War, Multi-Stage Contest, Tug-of-War, Tullock Contest Success Function, Microfoundation
JEL-Classifications: D74, F51, H56

*University of Basel, Faculty of Business and Economics, Peter Merian-Weg 6, Basel, Switzerland. Email: samuel.haefner@unibas.ch

1 Introduction

The history of mankind is a history of war. No age has passed without epic battles; and no era without bloodshed and death. Even so, and in spite of its disastrous consequences for combatants and civilians alike, the occurrence and the course of war is still very little understood.

In recent political science literature, the occurrence of war is explained by a breakdown in peaceful bargaining between nations due to either information incompleteness (e.g., Reiter 2003; Maoz & Siverson 2008) or to anticipated changes in the bargaining environment (e.g., Powell 2004*b*, 2006). The course of war is modeled as a multi-stage interaction with player's alternately deciding whether to keep on fighting or to agree on an allocation proposal made by the opponent (cf. Filson & Werner 2001, 2004, Powell 2004*a*, Leventoglu & Slantchev 2007). These approaches have certainly increased our understanding of war and the deliberative aspects that go along with it. However, a blind spot remains: the particular war or battle incident is not modeled explicitly, but is represented by either a simple probability measure or by a function which maps actions into outcome probabilities. It is these probabilities or functions which the participants resort to when deciding over their actions. Probabilities or contest success functions (CSF) themselves, however, remain unexplained by the models.

This paper sets out to explain this lottery character of militarized interaction. In order to do so, a different perspective on war is adopted: if we want to understand war, we need to understand what happens on the battlefield. This leads to the study of individual combatant behavior. It is hardly surprising that this has been done before: going back two centuries in the history of military thought, the work of the Prussian military theorist Carl von Clausewitz reads like a modern, decision-theoretic analysis of battlefield behavior: a contestant going to battle will, as von Clausewitz (1982 [1832], 104) argues, consider possible enemy types as a guideline for his actions. In particular, he argues as follows:

> "If we desire to defeat the enemy, we must proportion our efforts to his powers of resistance."

These "powers of resistance" are two-dimensional. They are given by

> "the product of two factors which cannot be separated, namely, *the sum of available means* and *the strength of [w]ill*."

Handling these two dimensions is not straightforward, as one of the two is prone to uncertainty:

> "The sum of available means may be estimated in a measure, as it depends (although not entirely) upon numbers; but the strength of volition is more

> difficult to determine, and can only be estimated *to a certain extent* by the strength of the motives."

And as if foreseeing Bayesian concepts, von Clausewitz (1982 [1832], 109) continues:

> "From character, the measures, the situation of the adversary, and the relations with which he is surrounded, each side will draw conclusions by the *law of probability* as to the designs of the other, and act accordingly." (emphasis added)

Clausewitz indicates that warfare necessitates handling two type dimensions in an appropriate fashion. One of these – material strength – is common knowledge; the other one – an individual motive – is private information. Acknowledging the fact that both sides choose their actions interdependently, we are in a modern game theoretic framework. More concretely, Clausewitz's writing neatly fits into a description of an auction with asymmetrically effective bid submissions: his "powers of resistance" are interpretable as winning relevant bids which are given by the product of "the strength of will" – that is of actual bids – and of some efficacy parameter that he calls "the sum of available means". The "strength of will" depends on individual "motives". These are interpreted as private combatant valuations. Valuations, in turn, are drawn from commonly known distributions – as introduced by the agents' "strength of the motives". Lastly, we may reasonably assume that bids are paid irrespective of the battle outcome. Following this reasoning, a battle is structurally equivalent to a modified private value all-pay auction.

This very idea of a battle is spelled out in the following. It will be combined with modeling war as a multi-stage contest, with the succession of battles being determined by a tug-of-war structure. This yields a course of war that has a simple Markov structure with a Bayesian Nash equilibrium at every stage.

The set-up allows for a description of war by means of the logit Tullock contest success function – so, the following may be read as a microfoundation of this popular CSF. Microfoundations in other contexts exist and are found for example in Fullerton & McAfee (1999) for research tournaments, or in Lagerlöf (2007) for classical rent-seeking contests.[1] In contrast to these, however, the logit CSF gained here does not follow directly from the stochastic environment chosen but from the strategies employed by the agents. It emerges as a consequence of equilibrium play.

Beyond that, the results will be used to formulate hypotheses on battlefield behavior. Some fit in with recent empirical findings, some are novel: Fighting efforts are fiercest

[1] Further, Baye & Hoppe (2003) find that rent-seeking games, innovation tournaments and patent races are strategically equivalent and expressible via a Tullock contest success function. Axiomatic studies are found in Skaperdas (1996), Kooreman & Schoonbeek (1997), or in Clark & Riis (1998). For a more extensive survey of the literature, see Konrad (2009).

when the advantaged nation is on the edge of defeat and decrease with its closeness to victory. These fluctuations in efforts across war states are the more pronounced the more asymmetric the adversaries are. The materially disadvantaged nation is found to fight more aggressively, and war duration is negatively correlated with the degree of asymmetry between warring parties.

The remainder of the paper is organized as follows: Section 2 introduces the basic ideas and reviews the relevant literature. In Section 3, the model is formalized and the equilibrium characterized. Comparative statics are presented in Section 4. Section 5 relates the results to war empirics and concludes.

2 Explicitly Modeling War

There are two distinct features of the approach that require some introductory remarks before the formal model can be stated: (i) war as a tug-of-war and (ii) battle as a private value all-pay-auction. The following considerations are essential for the model to be set up.

2.1 War as Tug-of-War

A tug-of-war is a sports contest between two teams pulling on opposite ends of a rope. The aim is to pull the adversary over a ground mark initially lying between the teams. The crucial feature of this type of game is that the winner is not declared by means of absolute gains, but in terms of gain differences over the adversary. Applying this idea to the case of militarized disputes, war is assumed to consist of a sequence of battles and a nation is said to be victorious as soon as its battle victories exceed its battle defeats by a certain number. That is, contestants need to top the adversary by a given number of surplus victories - irrespective of absolute victories. Such a design renders a contest potentially infinitely long but this in turn will yield a structure that is elegantly analytically tractable.

In political science, the idea of modeling war as a tug-of-war is found in Smith (1998) and in Smith & Stam (2003, 2004). These models assume that a tug-of-war structure is a suitable analogy for militarized disputes and defend this assertion with the arguments that wars usually exhibit changing positional advantages and that our every-day vocabulary of militarized disputes is replete with notions that link war to this kind of sports contest. Other formulations of multi-stage contests with a tug-of-war structure are found in Konrad & Kovenock (2005) or Agastya & McAfee (2006).

Contrary to the approaches above, however, the following approach will not model war as a succession of interactions between entire teams, but rather as interactions between individual units of two opposing teams. In every such battle, a new unit is sent to field. The aim of the units is to win a battle in order to get their team closer to overall victory. To keep the analysis as simple as possible, the units will be assumed to consist of one single combatant.

2.2 Battle as All-Pay-Auction

Returning to Clausewitz's understanding of battle, combatant behavior is understood as the product of interdependent reasoning over innate and adversarial strength attributes. The attributes considered are material strength as well as motivational strength. Such a two-dimensional description is certainly not too remote from current military research: for example, Biddle (2004) advocates the use of differentiated capability measures in order to explain war success and failure. In military psychology and psychiatry on the other hand, the denominations of troop morale, troop cohesion and *esprit de corps* refer to individual and group states that are seen to be highly important for battlefield success (cf. Manning 1994, Siebold 2006, Newsome 2007). These concepts date back to military classics such as du Picq (2008 [1880]) and Marshall (2000 [1947]), both of which analyze and emphasize troop morale and other psychological states as key ingredients to victory.

The Clausewitzian analysis prompts an understanding of battle as being structurally equivalent to a private value all-pay auction. Two considerations diverge from the canonically symmetric all-pay auction model: Firstly, contestants feature asymmetry in their individual strength levels (differing available means in Clausewitzian terms), which is assumed to affect the actual bid needed in order to win. Secondly, contestants need not be symmetrically motivated; that is, in auction theoretic terms, they feature asymmetries in their value distributions (differing strengths of motives, as Clausewitz says). Each of these asymmetries has been analyzed individually in Amman & Leininger (1996), and in Feess, Muehlheusser & Walzl (2008), respectively – but not yet jointly, and not yet in a multistage setting.

Another noteworthy departure from traditional modeling approaches consists in the role of information. The existing literature on multi-stage auctions generally assumes commonly known valuations at every stage (cf. Agastya & McAfee 2006, and Konrad & Kovenock 2005, 2009, 2010). Such an assumption makes sense, of course, if we think of a succession of auctions with personally identical players at every stage. Framed like this, the players' behavior in the first stage is fully type-revealing, and there is no uncertainty in the subsequent stages. The scenario for war is different. War is unlikely to involve a succession of interactions between personally identical players, and hence commonly known valuations are not assumed.

In the war scenario, uncertainty at every stage is established by two distinct features of the model: firstly, teams are modeled to send their members randomly to field, and secondly, valuations are modeled to be independent both within and across teams. The first feature is motivated by the idea of a residual uncertainty that every team faces regarding the factual valuations of its members. The second feature accompanies the

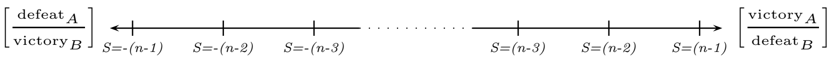

Figure 1: Tug-of-War Structure

assumption that there is no interdependence between individual valuations for victory and some objective value of the goal of war. Both these assumptions have supportive evidence in the aforementioned war psychiatry findings which relate individual combatant motivation more to – notably stochastic – personal and troop circumstances than to specified war objectives (c.f. especially the extended discussion of soldier motivation in Newsome 2007, Chapter 5).

Lastly, note that an all-pay auction constitutes a special case of a contest featuring a discontinuous contest success function. The contestant employing the highest effort wins with a probability of one. Thus, there is no uncertainty incorporated in the success function itself. Of course, this might seem too simple and be at odds with the real terms of war, where imponderabilities besides fighting efforts abound. However, by ignoring these very stochastic effects, analysis can focus on strategy choice that is solely based on the attributes of the two warring nations.

3 The Model

War is regarded as a tug-of-war between two opposing teams $i \in \{A,B\}$, both with a potentially infinite number of members. In every battle, each team sends one member randomly to field. After battle, both members rejoin their respective teams, and again a new fighter is chosen randomly by each team. A team is said to have won the contest, if its number of battle victories exceeds the number of victories by the opposing team by a certain number, n.

If team i leads by z battles, it is said to be in individual state $k_i = z$; if it runs behind z battles, it is said to be in state $k_i = -z$; if both are equal, both are said to be in state $k_i = 0$. Hence, by construction if team A is in state k_A, team B is in state $k_B = -k_A$, and vice versa.

Combining individual state denotations yields what will be called the war state. A war state is described by $S := k_A = -k_B$ referring to team A as being in its individual state k_A and team B being in its individual state $k_B = -k_A$. A graphical intuition for this notation is given in Figure 1. S increases in team A's victories, and decreases in team B's victories respectively. War is over if either team A wins in $S = (n-1)$ or if team B wins in $S = -(n-1)$. The distinction between individual states and war states, respectively, will prove to be helpful in analysis, for it allows us to firstly trace the individual and then the interdependent decision problem.

Members of each team i attach a certain individual valuation v_i to final team victory – that is, battle victories are merely instrumental in nature and do not feature an intrinsic value for agents. Valuations v_i refer to the Clausewitzean notion of motives. Motives v_i are best interpreted as the member's willingness to pay for their team's overall war victory. Since players are utility maximizer, they will never fully pay up to their v_i, but optimize efforts with respect to anticipated adversarial actions.

Individual valuations v_i are assumed to be independently and uniformly distributed on a support $[0, \overline{v}_i]$, $\overline{v}_i \in \mathbb{R}_+$. Valuations are not only independent within but across teams as well. Distributions of possible valuations are denoted by F_i and supposed to be public knowledge. Actual individual valuation draws, however, are known to the holder alone.

The width of the distribution support, $[0, \overline{v}_i]$, parallels the Clausewitzen idea of the strength of motives. The upper bound of the support, $\overline{v}_i$, will further on be denoted as the motivational strength of a team and the notions of valuation and motivation will be used interchangeably.

In battle, contestants exert some level of effort, $b_{i,S}$. This is equivalent to the bid in the classical auction setting: Effort is modeled as a function that has valuation as an argument – $\beta_{i,S} : [0, \overline{v}_i] \to \mathbb{R}$, $v_i \mapsto b_{i,S} = \beta_{i,S}(v_i)$. The notion of effort level captures the Clausewitzian concept of the strength of will which is determined by motives in war. With the terminology used hereafter, $b_{i,S}$ will be denoted as observed battlefield efforts.

Further, members of each respective team are homogeneous in their abilities to fight, but fighting abilities differ between teams. Members of team i are said to have fighting ability $\alpha_i \in \mathbb{R}_+$ which refers to their (common) ability to transform observed battlefield efforts into effective battlefield efforts. This captures the Clausewitzian idea of material strength, which is to be seen as a measure for the effectiveness of observed battle efforts. It will be these effective efforts upon which victory and defeat are decided: A is denoted the winner of the battle if his effective battlefield efforts exceed those of participant B, that is, if $\alpha_A \beta_{A,S}(v_A) > \alpha_B \beta_{B,S}(v_B)$. Translated back into Clausewitzian terms, this means that the player with the higher level of powers of resistance is victorious.

Lastly, agents are assumed to be risk-neutral, and utility functions to be additively separable in the valuation of final victory (if obtained) and the cost of the effort expended. Costs of efforts are assumed to correspond to the effort level chosen, that is, $c(b_{i,S}) = b_{i,S}$.

3.1 The Course of War

To start, the case with teams needing $n = 2$ surplus victories in order to win war is considered. Section 3.2 will then deliver a generalization to any arbitrary $n \geq 2$. Teams begin the contest symmetrically, that is, both teams begin in a state where they fight for the option of imposing decisive defeat upon the opponent in the round to come. For the victorious team, this following round yields the possibility of winning decisively. For the losing team in the preceding round, the second round yields the option of starting anew

– should it win – in a symmetric situation. Hence, in the second round the tug-of-war is either over or starts anew in the initial setting. For analysis then, three individual states $k_i = \{-1, 0, 1\}$ and three war states $S = k_A = \{-1, 0, 1\}$ are of interest.

A combatant going to battle faces the following payoff structure:

$$U_{i,k_i}(v_i, b_{i,k_i}) = \begin{cases} V_{i,k_i+1} - b_{i,k_i} & \text{if } \alpha_i b_{i,k} > \alpha_j \beta_{j,-k_i}(v_j) \\ V_{i,k_i-1} - b_{i,k_i} & \text{else} \end{cases} \tag{1}$$

with V_{i,k_i} denoting the value of the individual state k_i to the combatant of team i returning from battle to that very state.

How do we have to think about V_{i,k_i}? Remember that battles do not feature intrinsic value to combatants. Hence competition in battle is about positions only – that is, agents fight over a more advantageous state for their team in the round to come. The value of the round to come hinges on the expected outcomes of even further battles over which the present combatants have no control. To capture this let V_{i,k_i} with $k_i = \{-1, 0, 1\}, i = \{A, B\}$ be written as follows:

$$V_{i,1} = p_{i,1} v_i + (1 - p_{i,1}) V_{i,0} \tag{2}$$

$$V_{i,0} = p_{i,0} V_{i,1} + (1 - p_{i,0}) V_{i,-1} \tag{3}$$

$$V_{i,-1} = p_{i,-1} V_{i,0} + (1 - p_{i,-1}) 0 \tag{4}$$

The state value is computed as the weighted sum of the value of the next-higher state and the value of the next-lower state which will be reached after the consecutive battle with a battle outcome probability of p_{i,k_i} and $1 - p_{i,k_i}$, respectively. There is no discounting assumed. For example, returning from battle in individual state $k_i = 0$ yields the value of state $k_i = 1$ with probability $p_{i,0}$, and the value of state $k_i = -1$ with probability $(1 - p_{i,0})$. As will be shown shortly, battle outcome probabilities are properly defined in equilibrium and are hence anticipatable.

Rearranging terms leads to:

$$V_{i,1} = \frac{p_{i,1}(1 - p_{i,-1}(1 - p_{i,0}))}{1 - p_{i,-1}(1 - p_{i,0}) - p_{i,0}(1 - p_{i,1})} v_i \tag{5}$$

$$V_{i,0} = \frac{p_{i,0} p_{i,1}}{1 - p_{i,-1}(1 - p_{i,0}) - p_{i,0}(1 - p_{i,1})} v_i \tag{6}$$

$$V_{i,-1} = \frac{p_{i,-1} p_{i,0} p_{i,1}}{1 - p_{i,-1}(1 - p_{i,0}) - p_{i,0}(1 - p_{i,1})} v_i \tag{7}$$

In order to ease notation, let V_{i,k_i} be defined as

$$V_{i,k_i} = \varphi_{i,k_i} v_i \tag{8}$$

where $\varphi_{i,k_i} v_i$ denotes the individual state value for a participant as a fraction φ_{i,k_i} of v_i. In order to complete the description, define further $\varphi_{i,2} := 1$, and $\varphi_{i,-2} := 0$ as the fraction of the valuation gained when the overall war is won, or lost respectively.

Hence, in the case of the winning battle, gains to participant i in state k_i consist in a fraction φ_{i,k_i+1} of the overall valuation for victory, whereas losing yields a (lower) fraction φ_{i,k_i-1}. The game thus yields a payoff of $\varphi_{i,k_i-1}v_i$ with certainty and, in the case of winning, an additional payoff of $\left(\varphi_{i,k_i+1}-\varphi_{i,k_i-1}\right)v_i$.

A closer look at this latter expression is necessary. In order to do so, it is convenient to switch from individual state notation (k_i) to war state notation ($S=k_A=-k_B$). Keeping in mind that $p_{A,k_A}=(1-p_{B,k_B}), \forall k_A=-k_B$, simple algebra is applied to arrive at:

Lemma 1. *Let $\phi_{i,k_i} := \varphi_{i,k_i+1}-\varphi_{i,k_i-1}, i\in\{A,B\}$ denote the fraction of the additional payoff $\phi_{i,k_i}v_i$ in the case of winning a battle in individual state k_i for agents of team i. Then, in every war state $S=k_A=-k_B$, additional payoff fractions are equal for both team's agents and are given by:*

$$\phi_S := \phi_{A,k_A} = \phi_{B,k_B} \tag{9}$$

Lemma 1 states that additional payoffs are equal for both agents for a given battle. This result is important: Expression (9) facilitates analysis, and the optimizing problem for each candidate can finally be stated. Recalling that uniform distribution of valuations is assumed and replacing individual state indices in φ_{i,k_i} by war state indices S, the respective battle-utilities are written as:

$$U_{A,S}(v_A,b_{A,S}) = \frac{\beta^{-1}_{B,S}(\frac{\alpha_A}{\alpha_B}b_{A,S})}{\overline{v}_B}\phi_S v_A + \varphi_{A,S-1}v_A - b_{A,S} \tag{10}$$

$$U_{B,S}(v_B,b_{B,S}) = \frac{\beta^{-1}_{A,S}(\frac{\alpha_B}{\alpha_A}b_{B,S})}{\overline{v}_A}\phi_S v_B + \varphi_{B,-(S-1)}v_B - b_{B,S} \tag{11}$$

Equations (10) and (11) depict respective additional gains from winning the contest multiplied by the probability of winning given a certain effort level $b_{i,S}$ plus the valuation fraction received in the case of defeat minus the cost of the efforts to be expended. The fraction that is gained in the case of defeat is obtained with certainty and will hence not be relevant for the optimizing problem.

In order to find the utility maximizing efforts $b_{A,S}$ and $b_{B,S}$, respectively, FOC's for both candidates with respect to their efforts are taken. Replacing $b_{i,S}$ by the bidding function $\beta_{i,S}(v_i)$ leads to the following system of differential equations:

$$\text{FOC}_{b_{A,S}}: \qquad v_A = \overline{v}_B\frac{\alpha_B}{\alpha_A}\beta'_{B,S}\left(\beta^{-1}_{B,S}\left(\frac{\alpha_A}{\alpha_B}\beta_{A,S}(v_A)\right)\right)\phi_S^{-1} \tag{12}$$

$$\text{FOC}_{b_{B,S}}: \qquad v_B = \overline{v}_A\frac{\alpha_A}{\alpha_B}\beta'_{A,S}\left(\beta^{-1}_{A,S}\left(\frac{\alpha_B}{\alpha_A}\beta_{B,S}(v_B)\right)\right)\phi_S^{-1} \tag{13}$$

Equilibrium analysis will be restricted to bidding functions meeting this differential equation system. The route of looking at differential equation systems of this type is fairly common in the literature on all-pay auctions. For example, Amman & Leininger (1996) thus analyze asymmetric value distributions, and the special case with $\phi_S = 1$, $\overline{v}_A = \overline{v}_B = 1$, and $\alpha_A \neq \alpha_B$ is treated in Feess et al. (2008). Proceeding by "guessing and verifying" leads to:

Proposition 1. *The unique equilibrium effort functions*[2] *complying with (12) and (13) are:*

$$\beta_{A,S}(v_A) = \phi_S \frac{\alpha_B}{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B} \overline{v}_B \overline{v}_A^{\,1-\frac{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B}{\alpha_A \overline{v}_A}} v_A^{\frac{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B}{\alpha_A \overline{v}_A}} \tag{14}$$

$$\beta_{B,S}(v_B) = \phi_S \frac{\alpha_A}{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B} \overline{v}_A \overline{v}_B^{\,1-\frac{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B}{\alpha_B \overline{v}_B}} v_B^{\frac{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B}{\alpha_B \overline{v}_B}} \tag{15}$$

Proof. Since (12) and (13) constitute an ordinary differential equation (ODE) system, we know that, if a continuous and locally differentiable solution trajectory for a given ODE system for a given boundary condition exists, the solution is unique for that boundary condition (cf. Theorem 3.1 in Hale 2009). The only possible boundary condition for the problem is given by $\beta_{A,S}(0) = \beta_{B,S}(0) = 0$: no participant with zero valuation will expend strictly positive efforts, since this necessarily leads to negative utility. Therefore, if there is a solution meeting the boundary condition $\beta_{A,S}(0) = \beta_{B,S}(0) = 0$, the solution is the unique candidate to constitute an equilibrium. Note that (14) and (15) are continuous, locally differentiable and meet the specified boundary condition, and that plugging (14) and (15) into (12) and (13) indeed verifies that the functions presented in Proposition 1 constitute a solution. Hence, (14) and (15) constitute the unique candidate for an equilibrium.

In order to constitute an equilibrium, (14) and (15) need to apply with incentive compatibility and a rationality constraint (cf. Myerson 1981). The rationality constraint requires that equilibrium utility $U^*(0) \geq 0$. This condition is met since we have $\beta_{A,S}(0) = \beta_{B,S}(0) = 0$; i.e., no efforts are expended with zero valuation. On the other hand, incentive compatibility implies non-decreasing effort functions (cf. Krishna 2002). Obviously, (14) and (15) are non-decreasing in v_A, and v_B respectively. Hence, (14) and (15) indeed constitute an equilibrium and this equilibrium is unique for the conditions specified in (12) and (13). □

[2]Note that the equilibrium functions obtained correspond to the well-known symmetric equilibrium if $\alpha_A = \alpha_B = 1$ and $\overline{v}_A = \overline{v}_B = 1$ (cf. Krishna 2002).

Hence, for a given set of parameters $(\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$, equilibrium efforts only change in ϕ_S across states. In order to derive ϕ_S, expected battle winning probabilities p_{i,k_i} with randomly chosen combatants are computed.

Lemma 2. *Expected battle outcome probabilities p_i for team i playing against team j, $i,j \in \{A,B\}$, $i \neq j$, are war state independent and given by:*

$$p_i := \int_0^{\overline{v}_i} \frac{\beta_{j,S}^{-1}\left(\frac{\alpha_i}{\alpha_j}\beta_{i,S}(v_i)\right)}{\overline{v}_j} dF_i(v_i) = \frac{\alpha_i \overline{v}_i}{\alpha_i \overline{v}_i + \alpha_j \overline{v}_j}, \forall S \tag{16}$$

Proof. Note that, assuming uniform distribution, the probability of winning for combatant i is given by:

$$P\left(\alpha_i \beta_{i,S}(v_i) > \alpha_j \beta_{j,S}(v_j)\right) = \frac{\beta_{j,S}^{-1}\left(\frac{\alpha_i}{\alpha_j}\beta_{i,S}(v_i)\right)}{\overline{v}_j}. \tag{17}$$

Computing the expected value of this probability along with plugging in equilibrium effort functions (14) and (15) yields the result. □

Agents on the battlefield behave in a way that results in equal outcome probabilities across all war states. Such war-state independent battle outcome probabilities are striking, since incentives change with respect to war states: with Lemma 2, additional payoff fractions, ϕ_S, can now be derived, and it turns out that they are not independent of the war state S.[3]

Proposition 2. *The state variables ϕ_S, $S = \{-1, 0, 1\}$ are given by:*

$$\phi_S = \begin{cases} \dfrac{\alpha_A^2 \overline{v}_A^2}{\alpha_A^2 \overline{v}_A^2 + \alpha_B^2 \overline{v}_B^2} & \text{if } S = -1 \\[2ex] \dfrac{\alpha_A \overline{v}_A \alpha_B \overline{v}_B}{\alpha_A^2 \overline{v}_A^2 + \alpha_B^2 \overline{v}_B^2} & \text{if } S = 0 \\[2ex] \dfrac{\alpha_B^2 \overline{v}_B^2}{\alpha_A^2 \overline{v}_A^2 + \alpha_B^2 \overline{v}_B^2} & \text{if } S = 1 \end{cases} \tag{18}$$

Proof. Substitute (16) into the definition of ϕ_S as given in (9). □

This concludes equilibrium analysis for $n = 2$. By Propositions 1 and 2, the equilibrium is fully characterized.

[3]For an extended discussion on ϕ_S, refer to Section 4.1.2.

3.2 Generalization to $n \geq 2$

The results obtained in Lemmata 1 and 2 as well as those obtained in Proposition 1 carry over to the more general case of $n \geq 2$ surplus battle victories needed in order to win war.

Note that a sufficient condition for the feasibility of such a generalization is that the fraction ϕ_S, $S \in \{-n+1,...,n-1\}$ of the additional payoff $\phi_S v_i$ in the case of winning is equal for agents of both teams for a given S and for all n. Proposition 3 states that this is indeed the case:

Proposition 3. *(Generalization of Lemma 1) Take the case of n surplus battle victories needed in order to win war. Then, the fractions $\phi_{i,S}$, $S \in \{-n+1,...,n-1\}$ of the additional payoff $\phi_{i,S} v_i$ as stated in Lemma 1 are equal for agents of both teams $i = \{A, B\}$ for a given S.*

Proof. See Appendix A. □

As a consequence of Proposition 3, equilibrium bidding functions are as given in Proposition 1, and battle outcome probabilities remain constant across war states and are as given in Lemma 2. Finally, to get an analogon to Proposition 2 for the generalized n-case, it is a matter of simple algebra to compute values of ϕ_S by repeating the steps taken to arrive at Lemma 1 and resorting to battle outcome probabilities p_A. So, in any case of n, ϕ_S is properly defined and this in turn allows for a direct analysis of the general n-case in the following.

3.3 War Winning Probabilities

With state-independent battle outcome probabilities, war boils down to a random walk over war states $S = \{-(n-1),...,n-1\}$ that ends with a victory of the respective leading team in either of the fringe states $S = \{-(n-1), n-1\}$. Assuming war to start in middle state $S = 0$, the problem of overall winning probabilities for war is structurally equivalent to the so-called *Gambler's Ruin Problem* with two gamblers initially equally endowed with n units.[4] Consequently, overall outcome probabilities are to be described as follows:

Proposition 4. *The overall winning probability, π_i^n, of nation i endowed with material strength α_i and motivational support $[0, \overline{v}_i]$ faced with nation j characterized by α_j*

[4] The *Gambler's Ruin Problem* describes a repeated lottery game between two players, A and B, each of whom is endowed with some (finite) capital endowment. At every stage, one unit of capital is transferred from A to B with some probability, or from B to A with the respective complementary probability. The game ends as soon as the first player is bankrupt (cf. Takacs 1969).

and $[0,\overline{v}_j]$ *in a tug-of-war contest with symmetric initial positions and a difference of* n *victories needed in order to win is given by:*

$$\pi_i^n = \frac{p_i^n}{p_i^n + p_j^n} = \frac{\alpha_i^n \overline{v}_i^n}{\alpha_i^n \overline{v}_i^n + \alpha_j^n \overline{v}_j^n} \tag{19}$$

Proof. The probability π_i^n of prevailing in war with symmetric initial positions and with n surplus battles needed in order to win is equal to the complementary absorption probability with equal starting endowment n as given in Stern (1975):

$$\pi_i^n = \frac{1 - \left(\frac{1-p_i}{p_i}\right)^n}{1 - \left(\frac{1-p_i}{p_i}\right)^{2n}} \tag{20}$$

Rearranging leads to Proposition 9. □

Overall outcome probabilities take the shape of the logit Tullock contest success function. This CSF is highly popular in models of militarized conflicts (cf. Garfinkel & Skaperdas (2000); Hirschleifer (2000); Anbarci et al. (2002); Slantchev (2005)). Further, we find it extensively used in rent-seeking analysis (see Nitzan (1994) for an overview). Snyder (1989) applies the idea to political election contests. Usually, forms with $n \leq 2$ are studied. See Baye, Kovenock & de Vries (1994) for a mixed strategy equilibrium with $n > 2$.

4 Comparative Statics

Let $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$, $D \in \mathbb{R}_+^4$ denote a description of the exogenous attributes of the two teams in the game and additionally let $p_A|_D$ and $p_B|_D = 1 - p_A|_D$ stand for the battle outcome probabilities generated by description D as in (16). By definition, $p_A|_D \in (0,1)$. Whenever reasonable, reference to D in $p_A|_D$ will be dispensed with and simply p_A be written.

Definition 1. *Team* i *is called advantaged over team* j*, or favorite respectively, if its probability of succeeding in battle exceeds that of its opponent, i.e. if* $p_i > p_j$*. Further, a team* i *is called motivationally or materially advantaged if* $\overline{v}_i > \overline{v}_j$*, or* $\alpha_i > \alpha_j$ *respectively.*

By setting

$$\frac{p_i}{p_j} = \frac{\alpha_i \overline{v}_i}{\alpha_j \overline{v}_j} \tag{21}$$

it is obvious that the question of advantage is driven by material strength and motivational support alike. Being materially disadvantaged does not imply an overall disadvantag as material incapability can be compensated directly by featuring a larger motivational support.

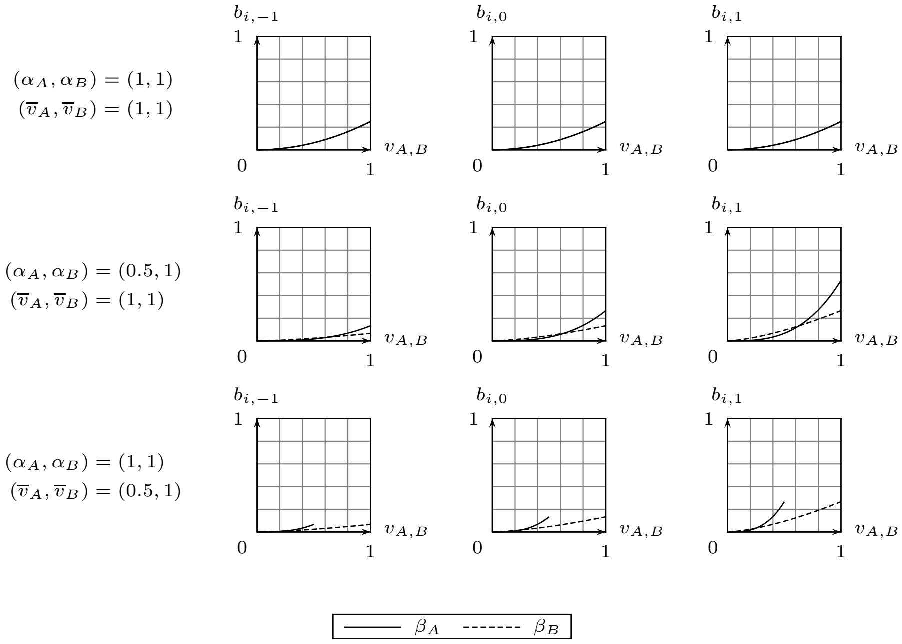

Figure 2: Bidding Behavior with Differing Asymmetries

4.1 Asymmetry and Battlefield Behavior

In the following, a short description of individual battlefield behavior is given. Then, expected individual and aggregate behavior is scrutinized. And beyond that, effects on war duration by shifts in the balance of powers will be analyzed. The term *effort* will always refer to observed battlefield effort.

4.1.1 Individual Efforts

The factors affecting individual efforts are best looked at by distinguishing between effects of material asymmetry on the one hand and effects of motivational asymmetry on the other hand. This will be done by reference to Figure 2. Figure 2 depicts the ($n = 2$)-case with equilibrium efforts $b_{i,S}$ depending on valuation v_i for both candidates i, for war states $S \in \{-1, 0, 1\}$, and for differing degrees of asymmetry.

A graphical intuition for the effect of differing degrees in *material* asymmetry is obtained by comparing the shape of effort functions across equal war states S for differing (α_A, α_B)-values as in the first and second row of Figure 2. With motivational symmetry but material asymmetry, the bidding function of the advantaged party becomes flatter, whereas the underdog with relatively high valuation exerts higher efforts than in the symmetrical case. Taking into account that the underdog needs relatively more effort in order to win, this makes perfect sense: the underdog exerts relatively little effort when valuations are low since his chances of winning are limited by low material strength. An underdog holding a high valuation, however, acts more aggressively as he compensates for the material handicap in order to optimize expected battle utility. So, in terms of effort employed, the aspect of participant type becomes more important with the underdog than with the advantaged party.

With respect to motivational asymmetry, similar effects are observed. The subfigures in the third row of Figure (2) show the effort strategy of the advantaged participant to flatten out and the disadvantaged participant with relatively high valuations to fight more aggressively than in the symmetrical case. The advantaged participant expends less effort, since he can expect an overall lower willingness to pay for victory with members of the adversarial team. On the other hand, underdogs i with valuations close to the upper bound $\overline{v}_i$ play optimally by expending greater efforts than favorite combatants with equal valuations: underdog types with high valuations adopt equilibrium efforts of favorite types j with valuations similarly close to the respective upper bound $\overline{v}_j$ as a guide. This is so in equilibrium, since, by expending less effort, the underdog would lessen his chances of winning by more than he would save on effort expenditure.

4.1.2 Expected Efforts

Since the idea of randomness in choosing agents is a key assumption of the model, the scrutiny of expected behavior is of particular interest. Not only does such analysis yield the sharpest results, but it furthermore yields potentially testable hypotheses against the background of an assumed participant description D.

It will prove to be fruitful to write the expressions of interest with respect to the battle winning probabilities of team A: Let expected individual efforts ($E\beta_{i,S}$) and total expected efforts ($E\beta_{tot,S}$) in war state S be written as

$$E\beta_{A,S} = \phi_S \frac{p_A(1-p_A)}{1+p_A}\overline{v}_A \tag{22}$$

$$E\beta_{B,S} = \phi_S \frac{p_A(1-p_A)}{2-p_A}\overline{v}_B \tag{23}$$

$$E\beta_{tot,S} = \phi_S p_A(1-p_A)\left[\frac{\overline{v}_A}{1+p_A} + \frac{\overline{v}_B}{2-p_A}\right] \tag{24}$$

Absolute Strength Levels A first observation concerns the ratio between respective strength levels for a description $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$; i.e., the ratios between material strength levels on the one hand and motivational strength levels on the other hand.

The model predicts quantitatively the same behavior for an armed conflict between materially weak adversaries as it does for materially stronger candidates - given that the balance of powers in terms of material attributes is the same. Such is not the case for motivational attributes. Given a certain ratio of team motivation, expected efforts increase in absolute levels of team motivation:

Proposition 5. *(Absolute strength levels) Take any description $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$.*

(a) *Keep a certain material strength ratio, $\frac{\alpha_A}{\alpha_B}$, fixed. Then, expected individual and expected total efforts do not change ceteris paribus in absolute levels of (α_A, α_B).*

(b) *Keep a certain motivational strength ratio, $\frac{\overline{v}_A}{\overline{v}_B}$, fixed. Then, expected individual and expected total efforts increase ceteris paribus in absolute levels of $(\overline{v}_A, \overline{v}_B)$.*

Proof. Note that by (16), p_A is homogeneous of degree zero in (α_A, α_B) as well as in $(\overline{v}_A, \overline{v}_B)$. By (18), ϕ_i, as well is homogeneous of degree zero in (α_A, α_B) and in $(\overline{v}_A, \overline{v}_B)$. Hence, by (22), (23), and (24), expected efforts only change in p_A and in $\overline{v}_i$, and thus do not change if both α_A and α_B are multiplied by some constant c. This proves (a). On the other hand, if both $\overline{v}_A$ and $\overline{v}_B$ are multiplied by some constant c, p_A does not change, but as expected individual and total efforts are homogeneous of degree 1 in $(\overline{v}_A, \overline{v}_B)$, both measures change by factor c. This proves (b). □

Relative Aggressiveness Let us switch to the analysis of expected behavior under differing advantages in both α_i and $\overline{v}_i$. As it turns out, weakness in material strength leads to aggression, whereas weakness in motivational strength leads to moderation:

Proposition 6. *(Relative Aggressiveness) Take any description $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$. Generally,*

$$\frac{\partial \frac{E\beta_{A,S}}{E\beta_{B,S}}}{\partial \alpha_A} < 0, \quad \frac{\partial \frac{E\beta_{A,S}}{E\beta_{B,S}}}{\partial \overline{v}_A} > 0, \forall S. \tag{25}$$

And especially

(a) *If $\alpha_A > \alpha_B, \overline{v}_A = \overline{v}_B$, then $E\beta_{A,S} < E\beta_{B,S}, \forall S$.*

(b) *If $\alpha_A = \alpha_B, \overline{v}_A > \overline{v}_B$, then $E\beta_{A,S} > E\beta_{B,S}, \forall S$.*

Proof. See B. □

Proposition 6 holds for all war states and for all n, since by (22) and (23), $\frac{E\beta_{A,S}}{E\beta_{B,S}}$ is independent of S. A ceteris paribus increase in innate material strength, α_i, yields a more favorable strength ratio $\frac{\alpha_i}{\alpha_j}$ and leads to decreasing own expected efforts relative to the efforts of the adversary. Taking a description D with equal motivational strength, $\overline{v}_A = \overline{v}_B$, yields the sharpest result in this vein, since then combatants of the materially advantaged team expend relatively less expected effort than their underdog counterparts. The same effect with inverse signs is observed for motivational attributes. Here, the greater the advantage, the more efforts are expended relatively: combatants of a favorite team that is favorite only because of motivational advantages fight more aggressively than their adversaries.

Embattled Favorite With perfectly symmetric teams, we can see by inspection of (22), (23), and (18) that expected efforts are the same for both teams and across all war states. Above, we have seen that with differing balances of power, expected efforts differ between teams within a certain war state. This is not the end of the story: asymmetry between teams affects expected efforts across war states, as well. In particular, greatest expected individual and total efforts are observed when the favorite team is on the edge of defeat.

Proposition 7. *(Embattled Favorite) Take any description $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$ with respective $p_A|_D > p_B|_D$ and n surplus battles needed in order to win war, then*

$$\phi_{-(n-1)} > \phi_{-(n-2)} > \ldots > \phi_{n-2} > \phi_{n-1} \tag{26}$$

and hence

$$E\beta_{i,-(n-1)} > E\beta_{i,-(n-2)} > \ldots > E\beta_{i,n-2} > E\beta_{i,n-1},\ i \in \{A,B\} \tag{27}$$

$$E\beta_{tot,-(n-1)} > E\beta_{tot,-(n-2)} > \ldots > E\beta_{tot,n-2} > E\beta_{tot,n-1} \tag{28}$$

Proof. See C. □

Expected individual and total efforts decrease monotonically in the favorite's closeness to victory. To make sense of this effect, we must take a closer look at the war state variable ϕ_S. ϕ_S refers to the difference in winning and losing fractions of war victory valuations and has been shown to be equal for both participants for a given war state S.

An intuition for the differences ϕ_S goes as follows: Note that it is ϕ_S alone that affects behavior across states for a given description D. Hence, we can interpret ϕ_S as an incentive to expend effort. This incentive looses more force, the closer the advantaged team comes to decisive victory: to the advantaged team, losing while being the front runner bears a comparably small risk of overall defeat, since the chances of returning to the decisive fringe state remain intact. For the disadvantaged, the incentive to expend effort while being on the edge of defeat is small as there is little chance of winning in future battles. Contrarily, if the disadvantaged participant is closer to victory, he then has a strong incentive to strive, since there is little chance that his team will ever return to such a favorable position again. Likewise, the advantaged participant faces high stakes on the edge of defeat.

Effort Volatility Battlefield behavior exhibits greatest effort when the favorite team is on the edge defeat and the least effort when it is close to victory – we shall refer to this fact as fringe state volatility. We will see that fringe state volatility is positively dependent on asymmetries between nations.

In order to compare strength asymmetry and effort volatility, it is convenient to introduce a measure for the degree of dispersion in any two variables (x,y) that is capable of analogously capturing the ideas of strength asymmetry and effort volatility in a reasonable way. A useful measure is the absolute logarithmic difference between x and y:

Definition 2. *Let*

$$\delta(x,y) := \left|\ln\left(\frac{x}{y}\right)\right| \textit{with } x,y \in \mathbb{R}_+ \tag{29}$$

denote a measure for the dispersion between x and y.[5]

Approximately, $\delta(x,y)$ describes the absolute percentage difference between x and y. $\delta(x,y)$ is symmetric in the sense that $\delta(x,y) = \delta(y,x)$, homogeneous of degree zero in (x,y) and increasing in $\max\{x,y\}$ as well as decreasing in $\min\{x,y\}$. As we are dealing with ratio measures, this is very convenient.

With this, the imbalance of powers between two teams are defined by resorting to battle outcome probabilities. Let

$$I(p_A) := \delta(p_A, 1-p_A) = |\ln(p_A) - \ln(1-p_A)| \tag{30}$$

stand for the degree of imbalance between A and B. $I(p_A)$ features a global minimum at $p_A = (1-p_A) = 0.5$ and exhibits positive slopes with $p_A \neq 0.5$.

Analogously, volatility $V(p_A)$ in efforts is understood as dispersion between expected efforts in the two fringe states $S = \{-(n-1), n-1\}$:

$$V(p_A) := \delta(E\beta_{i,-(n-1)}, E\beta_{i,n-1}) = \delta(E\beta_{tot,-(n-1)}, E\beta_{tot,n-1}) \tag{31}$$

$$= \left|\ln\left[\phi_{-(n-1)}(p_A)\right] - \ln\left[\phi_{n-1}(p_A)\right]\right| \tag{32}$$

We can now return to the claim that an increase in the imbalance of powers leads to a higher fringe state volatility in efforts:

Proposition 8. *(Effort Volatility) Take any description $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$ with respective battle outcome probability $p_A|_D$ and n surplus battle victories needed in order to win. Then volatility $V(p_A)$ between expected individual as well as between expected total efforts in fringe war states $S = \{-(n-1), n-1\}$ is increasing in the imbalance of powers $I(p_A)$.*

Proof. See D. □

[5] Note that $\delta(x,y)$ is a reformulation of the Thompson metric for the two-dimensional case, $d(x,y) = max\{\ln(\frac{x}{y}), \ln(\frac{y}{x})\}$.

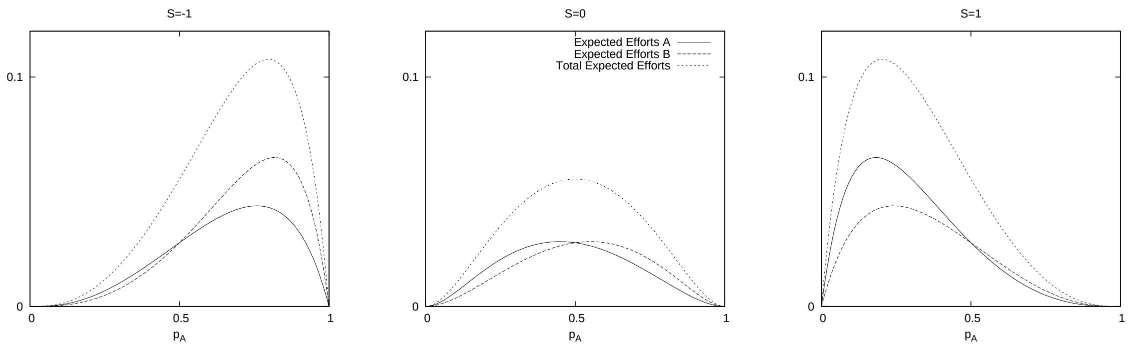

Figure 3: Expected Efforts with $\overline{v}_A = \overline{v}_B = 1$ and changing p_A

We can make sense of this observation with a consideration similar to that above: The higher the asymmetries between contestants are, the higher are the relative battle outcome probabilities of the favorite team. This implies that the stakes of the favorite on the edge of defeat increase in its superiority over its opponent. Analogously, incentives to expend efforts in the decisive state before victory decrease, since chances of returning to that very state increase.

This concludes the comparative statics analysis with respect to efforts. To sum up, changes in the balance of power lead to differing expected battlefield efforts on the one hand and to a certain volatility in observed battlefield efforts across war states on the other hand. In particular, expected individual and total efforts increase with the underdog's closeness to victory.

A graphical intuition for these effects is provided in Figure 4.1.2. The three subfigures depict expected individual as well as total effort levels with respect to p_A in the three war states $S = \{-1, 0, 1\}$ for the $n = 3$ case. Motivational strength for both teams is assumed to be $\overline{v}_A = \overline{v}_B = 1$, and hence the changes in p_A are due to changes in material strength ratio levels only. That is, for $p_A > 0.5$, team A is materially favorite and for $p_A < 0.5$, material advantages lie with team B.

With an increasing imbalance of powers – that is, with p_A moving away from $p_A = 0.5$ in either direction – expected individual efforts of the favorite come to rest below those of the underdog. The underdog behaves more aggressively. Further, the state with the embattled favorite exhibits the highest total and individual efforts of all states. And lastly, comparing effort levels for given p_A's across all three states reveals the volatility effect brought about by the imbalance of powers.

4.2 War Winning Probabilities

With state-independent battle outcome probabilities, war boils down to a random walk over war states $S = \{-(n-1), ..., n-1\}$ that ends with a victory of the respective leading team in either of the fringe states $S = \{-(n-1), n-1\}$. Assuming war to start in middle state $S = 0$, the problem of overall winning probabilities for war is structurally equivalent to the so-called *Gambler's Ruin Problem* with two gamblers initially equally endowed with n units.[6] Consequently, overall outcome probabilities are to be described as follows:

Proposition 9. *The overall winning probability, $P(i|S=0)$, of player i in a tug-of-war contest with symmetric initial positions and a difference of n victories needed in order to win is given by:*

$$P(i|S=0) = \frac{p_i^n}{p_i^n + p_j^n} \tag{33}$$

Proof. The probability π_i^n of prevailing in war with symmetric initial positions and with n surplus battles needed in order to win is equal to the complementary absorption probability with equal starting endowment n as given in Stern (1975):

$$\pi_i^n = \frac{1 - \left(\frac{1-p_i}{p_i}\right)^n}{1 - \left(\frac{1-p_i}{p_i}\right)^{2n}} \tag{34}$$

Rearranging leads to Proposition 9. □

4.3 Asymmetry and War Duration

By taking a closer look at overall winning probabilities, the expected length of war comes into focus. Again, thinking of war as analogous to the Gambler's Ruin Problem proves to be fruitful. Expected war duration can be expressed in terms of the number n of surplus battles needed and battle winning probabilities p_A:

Proposition 10. *(War duration) Take any description $D = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$, and respective battle outcome probability $p_A|_D$. Then:*

(a) *The expected length $L(n, p_A)$ of war with symmetric initial positions and with n surplus battles needed in order to win is given by*

$$L(n,p_A) = \begin{cases} n\left(\frac{1}{1-2p_A}\right)\left(\frac{(1-p_A)^n - p_A^n}{(1-p_A)^n + p_A^n}\right) & \text{if } p_A \neq \frac{1}{2} \\ n^2 & \text{if } p_A = \frac{1}{2} \end{cases} \tag{35}$$

[6] The *Gambler's Ruin Problem* describes a repeated lottery game between two players, A and B, each of whom is endowed with some (finite) capital endowment. At every stage, one unit of capital is transferred from A to B with some probability, or from B to A with the respective complementary probability. The game ends as soon as the first player is bankrupt Takacs (1969).

(b) *Expected war duration $L(n, p_A)$ decreases in the imbalance of powers $I(p_A)$.*

Proof. See E. $\square$

An intuition for this observation goes as follows: the more asymmetric teams are in terms of advantages, the higher are the probabilities that war moves in one rather than in the other direction. The advantaged team is expected to win battles more frequently than the disadvantaged team, and, as a consequence, a decisive fringe war state is reached earlier.

5 Conclusion

A multi-stage private value all-pay auction has been presented and solved for subgame perfect equilibria. The aim of this analysis has been twofold: firstly, endogenizing the lottery character of war; and secondly, providing empirically testable hypotheses on battlefield behavior as well as on the length of war. Thereby, the combination of modeling militarized dispute with a tug-of-war structure and framing individual battles as a modified private value all-pay auction in the spirit of Clausewitz has proven to be fruitful.

The setup chosen has allowed some light to be shed on the relationship between the logit Tullock CSF and the all-pay auction - an endeavor that is certainly interesting for general contest theory, as well. What this model essentially boils down to is an understanding of the Tullock CSF as the product of a discrete random walk between two absorption states. Transition probabilities are described by the strength parameters of the two teams and come about from equilibrium play in repeated private value all-pay auctions that are linked by a tug-of-war structure.

With this result, possibilities for future research in the context of war open up: a nation going to war might previously decide on how to optimally choose its parameters given specified costs. Higher order decision problems regarding the choice of material and motivational strength attributes are certainly natural extensions to the analysis done in this paper.

Further, the model presented is in line with several existing empirical findings: capabilities are often used in order to explain war outcomes – an elaborate and highly differentiated view on the explanatory power of material strength is found in Biddle (2004), for example. Material strength enters the model with the parameter α_i, which is positively correlated with chances of winning both in battle as well as in war. Motivation, on the other hand, is seen as an additional ingredient to victory – treatments thereof are found for example in Manning (1994), Siebold (2006), or Newsome (2007). This relationship, as well, is depicted in the expressions of battle and overall war outcome probabilities. And thirdly, the empirical observation that the duration of war is negatively correlated with the imbalance of powers between nations (cf. **?**) has been successfully replicated.

Last but not least, new hypotheses on battlefield behavior have been presented. These hypotheses refer to expected behavior with respect to war states and with respect to the strength ratios between adversaries. Propositions 5 to 8 may be directly taken to empirical research: The model predicts aggressiveness of the material underdog, increasing fighting intensity when the favorite is losing ground and high fighting volatility accompanying a high imbalance of powers.

Acknoledgments

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A Proof of Proposition 3

Proof. As shown in Proposition 1 and Lemma 2, equality in ϕ_S for both agents in battle leads to equilibrium effort functions that are linear in ϕ_S and to battle outcome probabilities p_i, $i \in \{A,B\}$ that are independent of war state S. In the following, it is shown that the implication works in the opposite direction, too: Assuming state independent outcome probabilities implies equality in ϕ_S for both teams' agents; hence, stating team independent ϕ_S is equivalent to stating war state independent battle outcome probabilities $p_A = 1 - p_B$.

Let $V_{i,S}$ be values of war state S to a randomly chosen agent of team i with valuation v_i and let us assume that outcome probabilities are war state independent and given by $p_A = 1 - p_B$. Then $V_{A,S}$ is given by

$$V_{A,S} = \begin{cases} v_A & \text{if } S = n \\ p_A V_{A,n+1} + (1-p_A) V_{A,n-1} & \text{if } S \in \{-n+1, ..., n-1\} \\ 0 & \text{if } S = -n \end{cases} \tag{36}$$

And analogously, $V_{B,S}$ can be written as

$$V_{A,S} = \begin{cases} v_B & \text{if } S = -n \\ (1-p_A) V_{B,n+1} + p_A V_{B,n-1} & \text{if } S \in \{-n+1, ..., n-1\} \\ 0 & \text{if } S = n \end{cases} \tag{37}$$

The additional payoff in case of winning in war state $S \in \{-n+1,\ldots,n-1\}$ for an agent of team $i \in \{A,B\}$ can be expressed by differences in state values, $\phi_{A,S}v_A = V_{A,S+1} - V_{A,S-1}$, and $\phi_{B,S}v_B = V_{B,S-1} - V_{B,S+1}$, respectively.

Taking these differences in state values as defined in (36) and (37) yields two $(2n-1)$-equation systems for $\{A,B\}$-agents with $2n-1$ unknown $\phi_{i,S}$:

$$\phi_{A,S} = \begin{cases} 1 - p_A \sum_{z=-n+1}^{n-2} \phi_{A,z} & \text{if } S = n-1 \\ p_A \phi_{A,S+1} + (1-p_A)\phi_{A,S-1} & \text{if } S \in \{-n+2,\ldots,n-2\} \\ \dfrac{p_A}{1-p_A}\phi_{A,-n+2} & \text{if } S = -n+1 \end{cases} \tag{38}$$

$$\phi_{B,S} = \begin{cases} 1 - p_A \sum_{z=-n+1}^{n-2} \phi_{B,z} & \text{if } S = n-1 \\ p_A \phi_{B,S+1} + (1-p_A)\phi_{B,S-1} & \text{if } S \in \{-n+2,\ldots,n-2\} \\ \dfrac{p_A}{1-p_A}\phi_{B,-n+2} & \text{if } S = -n+1 \end{cases} \tag{39}$$

From this, it is obvious that $\phi_{A,S} = \phi_{B,S} = \phi_S$ must hold. Hence, surplus payoffs are equal. $\square$

B Proof of Proposition 6

Proof. Take (22),(23) and the definition of p_A from Lemma 2 to write

$$\frac{E\beta_{A,S}}{E\beta_{B,S}} = \frac{\overline{v}_A}{\overline{v}_B}\frac{\alpha_A\overline{v}_A + 2\alpha_B\overline{v}_B}{2\alpha_A\overline{v}_A + \alpha_B\overline{v}_B} \tag{40}$$

Note that the RHS is independent of S. Hence for $\forall S, 0 < D < \infty$:

$$\frac{\partial \frac{E\beta_{A,S}}{E\beta_{B,S}}}{\partial \alpha_A} = -\frac{3\alpha_B\overline{v}_A^2}{(\alpha_B\overline{v}_B + 2\alpha_A\overline{v}_A)^2} < 0 \tag{41}$$

$$\frac{\partial \frac{E\beta_{A,S}}{E\beta_{B,S}}}{\partial \overline{v}_A} = \frac{2\left(\alpha_B^2\overline{v}_B^2 + \alpha_A\alpha_B\overline{v}_A\overline{v}_B + \alpha_A^2\overline{v}_A^2\right)}{\overline{v}_B(\alpha_B\overline{v}_B + 2\alpha_A\overline{v}_A)^2} > 0 \tag{42}$$

Setting $\overline{v}_A = \overline{v}_B$, we have

$$\frac{E\beta_{A,S}}{E\beta_{B,S}} = \frac{\alpha_A + 2\alpha_B}{2\alpha_A + \alpha_B} \tag{43}$$

Now obviously,

$$\alpha_A > \alpha_B \Rightarrow \frac{E\beta_{A,S}}{E\beta_{B,S}} < 1 \Leftrightarrow E\beta_{A,S} < E\beta_{B,S} \tag{44}$$

This proves (a). The proof for (b) repeats the last two steps analogously. $\square$

C Proof of Proposition 7

Proof. By the definitions given in (22), (23), and (24), for a given description D and respective $p_A|_D$, individual as well as total expected efforts vary linearly across states in ϕ_S alone. So, only war state variables ϕ_S have to be looked at. Rewriting (38) and setting $\phi_{A,S} = \phi_S$ yields for $S \in \{-n+1,...,n-2\}$

$$\phi_S = \frac{p_A}{1-p_A}\phi_{S'} \text{ with } S' = S+1 \tag{45}$$

Hence, it must hold for a favorite team A, i.e. $p_A > 0.5$, that

$$\phi_S > \phi_{S'} \text{ with } S' = S+1 \tag{46}$$

That is, ϕ_S (and consequently expected efforts) increases the closer the favorite team A is to defeat. □

D Proof of Proposition 8

Proof. Using (45) with Definition 2 we have

$$\begin{aligned}
V(p_A) &= \delta(E\beta_{i,-n(-1)}, E\beta_{i,n-1})| = \delta(E\beta_{tot,-(n-1)}, E\beta_{tot,n-1})| && (47)\\
&= \left|\ln\frac{\phi_{-(n-1)}}{\phi_{n-1}}\right| && (48)\\
&= \left|\ln\left(\left(\frac{p_A}{1-p_A}\right)^{2n-2}\right)\right| && (49)\\
&= (2n-2)\left|\ln(p_A) - \ln(1-p_A)\right| && (50)
\end{aligned}$$

For $0 < p_A < 1$, $V(p_A)$ exhibits a minimum at $p_A = 0.5$ and positive slopes for $p_A \neq 0.5$. Hence $V(p_A)$ behaves analogously to $I(p_A)$ in the sense that the signs of partial derivatives w.r.t p_A are equal for a given $p_A \neq 0.5$, and $V(p_A) = I(p_A) = 0$ for $p_A = 0.5$. So, an increase in $I(p_A)$ is equivalent to an increase in $V(p_A)$, and vice versa. □

E Proof of Proposition 10

Proof. The expected length $L(n, p_A)$ of war with symmetric initial positions and with n surplus battles needed in order to win is equal to the expected length of the *Gambler's Ruin* lottery game with equal starting endowment n as given in Stern (1975):

$$L(n,p_A) = \begin{cases} n\left(\frac{2p_A}{1-2p_A}\right)\left(1-2\left(\frac{1-\left(\frac{1-p_A}{p_A}\right)^n}{1-\left(\frac{1-p_A}{p_A}\right)^{2n}}\right)\right) & \text{if } p_A \neq \frac{1}{2} \\ n^2 & \text{if } p_A = \frac{1}{2} \end{cases} \tag{51}$$

Rearranging:

$$= \begin{cases} n\left(\frac{1}{1-2p_A}\right)\left(\frac{(1-p_A)^n - p_A^n}{(1-p_A)^n + p_A^n}\right) & \text{if } p_A \neq \frac{1}{2} \\ n^2 & \text{if } p_A = \frac{1}{2} \end{cases} \tag{52}$$

This shows (a). Further, it is a simple exercise in algebra to show that

$$\frac{dL(n,p_A)}{dp_A} = \begin{cases} > 0 & \text{if } p_A < \frac{1}{2} \\ < 0 & \text{if } p_A > \frac{1}{2} \end{cases} \tag{53}$$

Hence, $L(n,p_A)$ behaves analogously to $I(p_A)$. This proves (b). □

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