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Dynamic heterogeneous R&D with cross-technologies interactions.*

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Abstract

In many countries, inducing large-scale technological changes has become an important policy objective, as in the context of climate policy or energy transitions. Such large-scale changes require the development of strongly interlinked technologies. But current economic models have little flexibility for describing such linkages. We present a model of induced technological change that covers a fairly large set of cross-technology interactions and that can describe a wide variety of long-run developments. Using this model, we analyse and compare the development induced by optimal firm behaviour and the socially optimal dynamics. We show that the structure of cross-technology interactions is highly important. It shapes the dynamics of technological change in the decentralised and the socially optimal solution, including the prospects of continued productivity growth. It determines whether the decentralised and the socially optimal solution have similar or qualitatively different dynamics. Finally, it is highly important for the question whether simple r&d policies can induce efficient technological change.

Keywords: technological spillovers; social optimality; market inefficiency; optimal control; heterogeneous innovations.

JEL classification: O33, C02, C61, D62.

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1 Introduction

Technological progress is considered to be one of the most important drivers of economic growth and simultaneously seen as the most appealing solution to tackle negative side-effects of growth, such as pollution or resource depletion (Jaffe et al., 2003). Consequently, many countries try to accelerate and steer technological change via dedicated policies. During the past two decades, policies that aim for large-scale technological change have become widespread in industrialised countries; for example, many European countries aim to move towards a "green economy", switching to a "sustainable growth path", or a "turn-around" in their energy systems.

While these aims have a clear motivation in pressing environmental problems, the resulting policies are often either rather costly, as illustrated by the example of the German energy transition, or ineffective, as shown by many pre-2005 national climate policies. One reason is that the process of large-scale technological change is not well understood. In contrast to policies in the 1960s-80s, which usually aimed at supporting one or two individual industries, such large-scale endeavours require an understanding of how the developments of different technologies interact in order to devise efficient policy instruments.

Somewhat surprisingly, economic theory is not well-positioned to support the development of better policy interventions. Albeit the interrelation between the development of different technologies have long been noticed and integrated into economic models, these models typically have a structure that makes them unappealing for modelling large-scale technological change. Many models assume a high amount of symmetry among spillovers between technologies. For example, endogenous growth models typically assume that all developments contribute to public technological knowledge, which in turn influences the development of each technology (see, e.g., Romer (1990), Peretto and Connolly (2007), or Acemoglu and Cao (2015)).[1] Other models allow for asymmetries, but usually only in the context of a few technologies, as, for example, in Fischer et al. (2003).[2]

However, in many applications, we have numerous technologies together with strong asymmetries in cross-technology interactions, where the development of some technologies reduces development costs of other technologies, but not vice versa. Furthermore, some technologies might only be developed as side-effects of other technologies (direct investments being not profitable enough), or there could be disjoint groups of technologies with strong internal but few inter-group spillovers.

[1] An exception is Peretto and Smulders (2002), where a model of group-specific spillovers is introduced. However, even this model assumes full homogeneity of spillovers within groups.

[2] See Jaffe et al. (2003) for an overview over similar approaches.

In this paper, we advance and study a model of cross-technology interactions that is more general than existing models regarding possible interdependencies of technological developments. We first show that the model provides a flexible tool for describing heterogeneous technology development.

Second, we relate the structure of cross-technology interactions, that is, whether they are one- or bi-directional, similar for different technologies or highly diverse, to the dynamics of technological change. We show that the structure of the interactions determines whether unlimited growth of technological quality (productivity) is feasible and whether growth is exponential, linear or a mixture of both. Furthermore, we compare the dynamics induced by individual firm decisions to the socially optimal dynamics and show that, depending on the structure of cross-technology interactions, these might differ not only quantitatively but also qualitatively.

Finally, we show that the problems induced by some types of cross-technology interactions can be solved by a simple subsidising policy, whereas other interaction structures require dynamic or highly differentiated policies.

The main insight of these results is the hitherto mostly disregarded structure of cross-technology interactions has far-reaching consequences both for the dynamics of technological change and for the feasibility of policies that induce efficient technological change. Indeed, it is the structure of these interactions, not their scale, that is of highest relevance for the long-run dynamics and the design of corrective policies. This aspect is missed almost completely by most economic models, which use the idea of a common pool of knowledge (i.e., perfectly homogeneous interactions between technologies) that makes it impossible to study different structures of interactions among technologies.

To highlight the importance of cross-technology interactions, we use a model that is very simple apart from the interaction structure. The central part of our model is thereby close to multi-product innovations models, such as Lambertini (2003); Cellini and Lambertini (2002), but with a fully dynamic approach, following Belyakov et al. (2011) and Bondarev (2012).

We study cross-technology interactions both with a finite number of technologies and a continuum of technologies, as both approaches are widely used in economic models. However, we focus on the dynamics of technological change in the first case, where the dynamics can be investigated in considerable generality, and on the feasibility of corrective policies in the second case, where this is a much more serious problem.

In the next section, we set up the model. In Section 3, we study the dynamics of technological change and show how and why the decentralised solution and a social planner's solution diverge, first for the case of a finite number of technologies and then for a continuum of technologies. In Section 4, we provide a few examples that highlight different aspects of the preceding results. Section 5 concludes.

2 A model of cross-technology interactions

Assume that a set $\mathcal{I}$ of R&D firms (finite or infinite) develops different technologies that are sold in separate markets and that are used for distinct production purposes. Each technology $i \in \mathcal{I}$ is characterised by a quality $q(i,t) \in \mathbb{R}$ that describes the attractiveness of this technology, such as its efficiency in production or its impact on final good quality. This measure $q(i,t)$ can be changed by firm i via R&D efforts $g(i,t)$.

We assume that firm i receives a profit $\pi(i,t,q(i,t)) = \pi_0 + \bar{\pi}(i)\, q(i,t)$ in period t that depends linearly on the quality of its technology.[3] The firm's R&D efforts $g(i,t)$ incur quadratic costs $(1/2)\, g^2(i,t)$. We assume $\bar{\pi}(i) \geq 0 \,\forall\, i \in \mathcal{I}$.

To capture technology interactions, we assume that the quality development of each technology i consists of a part that is controlled by the firm developing this technology and a part that results from the development of the other technologies. Let $q(t)$ be the vector (if $\mathcal{I}$ contains a finite number of technologies) or the function (if $\mathcal{I}$ contains an infinite number of technologies) of qualities of all technologies and $g(t)$ be the corresponding vector/function of r&d efforts. Furthermore, let $\gamma(i) \geq 0$ be the efficiency of the r&d efforts of firm i and γ be the vector or function of these efficiencies for all firms.

Finally, let $\mathcal{F}$ be an operator acting on the space of vectors/functions of qualities q, with a nucleus $F(i,j)$ that describes the influence of the quality of technology j on the development of technology i.

Given these points, we write the development of the quality of all technologies as

$$\dot{q}(t) = \gamma\, g(t) + \mathcal{F}\, q(t). \tag{1}$$

We require $\mathcal{F}$ to be a linear operator on the space of vectors/functions $q(t)$. Thus the term $\mathcal{F}\, q$ in (1) is continuous in $\mathcal{I}$ and the equation is well-defined for each t.

The operator $\mathcal{F}$ is the central part of our model. It can describe a variety of interactions. In case of non-negative values of the nucleus, that is, $\forall i,j \in \mathcal{I}: \;\; F(i,j) \geq 0$,[4] it describes cross-technology spillovers: Gains in understanding one technology also enhance the quality of other technologies, as is often assumed in economic models. However, this description of spillovers is fairly flexible. It captures the often-used idea of a general public knowledge, where each technology contributes in the same way to the development of all other technologies (i.e., $F(i,j) = f > 0$). But it also

[3] As the term π_0 does not influence the subsequent analysis, we assume that it is homogeneous among firms in order to minimise notational effort. We could use $\pi_0(i)$ without any change to results.

[4] This implies that the operator $\mathcal{F}$ is positive with the norm being an operator norm, that is, $||\mathcal{F}|| = \sup_{||q|| \leq 1} ||\mathcal{F}q||$.

captures more interesting settings, such as spillovers only between nearby technologies ($F(i,j) > 0$ only for $i \in [j-\varepsilon, j+\varepsilon]$), distance-dependent spillovers (where $F(i,j)$ declines with $|j-i|$), or sparse, specific spillovers (where $F(i,j) > 0$ only for specific values of i,j). Also, the model can capture asymmetric spillovers (where $F(i,j) \neq F(j,i)$) as well as spillover chains (where technology i enhances technology j, which in turn enhances technology k).

Furthermore, this approach can accommodate cases where the development of technology i has negative effects on some other technologies. Such cases can arise, for example, if different technologies are based on competing sets of basic research and success in one technology shifts the direction of basic research and the education of researchers towards its underlying research lines. As basic research and university training is often funded substantially by public funds, this implies a negative externality among technologies. Of particular interest are cases where the set of technologies can be separated into subsets that are based on similar technological principles, so that there are positive spillovers within these subsets but negative interactions between technologies of different subsets.

In Eq. (1), $F(i,i)$ can be interpreted as a depreciation of knowledge: If $F(i,i) < 0$, it is necessary to exert some r&d effort to maintain the quality of a technology.[5] To ensure that the problem has reasonable solutions, we assume $F(i,i) < r$ for all $i \in \mathcal{I}$.

Given our assumptions, firm i's optimisation problem over an infinite time horizon and with a discount rate r can therefore be stated as

$$\max_{g(i,t)} \int_0^\infty \mathrm{e}^{-r\,t} \left(\pi_0 + \bar{\pi}(i)\, q(i,t) - \frac{1}{2}\, g^2(i,t) \right) dt, \tag{2}$$

s.t.

$$\dot{q}(i,t) = \gamma(i)\, g(i,t) + \mathcal{F}_i\, q(t), \tag{3}$$

where $\mathcal{F}_i\, q(t)$ denotes the impact of all technologies on technology i implied by the operator $\mathcal{F}$.

As each firm i accounts only for the effects of its decisions on the quality of its technology, the last term in Eq. (3) is, from this firms' perspective, simply a linear term $F(i,i)\, q(i,t)$ plus some function of time $f_i(t)$.

As a benchmark, we also consider the social planner's problem, who jointly maximises the benefit of developing the different technologies. To focus on technology interactions, instead of general externalities, we assume that the social and the private evaluation of technological quality coincide. Thus the social planner uses the private profit $\pi(i, q(i,t))$ to evaluate the quality $q(i,t)$ of technology i at time t. The social planner's decision problem

[5] Also, we could assume $F(i,i) > 0$ to model a case, where a higher quality of technology i reduces the effort required to increase $q(i)$.

is

$$\max_{\tilde{g}(t)} \int_0^\infty \mathrm{e}^{-rt} \int_{\mathcal{I}} \left(\pi_0 + \bar{\pi}(i)\, \tilde{q}(i,t) - \frac{1}{2}\, \tilde{g}^2(i,t) \right) di\, dt, \tag{4}$$

s.t.

$$\dot{\tilde{q}}(t) = \gamma\, \tilde{g}(t) + \mathcal{F}\, \tilde{q}(t) \tag{5}$$

where the tilde is used to differentiate the social planner's decision and state variables from the private ones. Note that the social planner optimises Eqs. (4)–(5) w.r.t. all $g(i,t)$, not just w.r.t. a particular technology.

For many specifications of the cross-technology interactions, problems (2)–(3) and (4)–(5) will have differing solutions, that is, the decentralised solution will not be socially optimal. As the two problems are otherwise identical, the differences result solely from the cross-technology interactions. To investigate the consequences of these interactions, it is thus interesting to study what kind of interventions are required to correct the decentralised solution. In this way, it is possible to differentiate different types of cross-technology interactions with regard to how severe are the problems that they cause.

To this end, we consider a subsidy scheme $s(i,t)$ that increases the marginal profit of firm i w.r.t. to quality $q(i,t)$ at time t to $\bar{\pi}(i)+s(i,t)$.Note that, in general, these subsidies can differ arbitrarily between firms.

There are three ways in which the subsidy scheme can be more or less complex. First, it can have more or less different subsidies. This is a question of the overall complexity of the intervention. Second, it can have a finer or coarser granularity: Subsidies could pick out single technologies or the same subsidy could be used for similar (i.e., close-by) technologies. This is a question as to whether enough information is available to fine-tune subsidies to individual technologies, or whether they can be tuned only to larger sets of technologies. Finally, a subsidy scheme can be constant over time or require adjustments when the technologies develop.

In the most simple case, only a single subsidy that is constant over time and that is paid to all technologies would suffice. In the most difficult case, the social optimum can only be implemented if each technology receives a different subsidy and these subsidies vary strongly even between close-by technologies and over time.

To cover a range of cases, we first define a constrained subsidy scheme, where the number of different subsidies is limited and where it is ensured that subsidies cannot pick out single technologies.

Definition 1 (Constrained subsidy scheme)**.**
Let $M \in \mathbb{N}$. A subsidy scheme $s(i)$ is of the class of constrained subsidies $\mathcal{S}^M$, if it is constant in time and:

1. *there exists a partitioning of $\mathcal{I}$ into M connected sets $\mathcal{J}_1, \ldots, \mathcal{J}_M$, with $s(i)$ being constant w.r.t. i within each set $\mathcal{J}_j$, $j = 1, \ldots, M$;*

2. each of these sets has a strictly positive measure in $\mathcal{I}$.

By this definition, $\mathcal{S}^1$ is the class of constant uniform subsidies, $\mathcal{S}^\infty$ is the class of countably many different constant subsidies, and $\mathcal{S}^M$ with $1 < M < \infty$ is the class of a finite number of different constant subsidies.

Using this definition, we can define different types of policies to correct for the externalities induced by cross-technology interactions.

Definition 2 (Types of corrective policies)**.**

A-1 A policy is simple *, if there exists a finite number $M \in \mathbb{N}$ and a subsidy scheme $s(i) \in \mathcal{S}^M$, so that $s(i)$ implements the social optimum;*

A-2 A simple policy approximates *the socially optimal solution, if there exists a sequence of subsidy schemes $s_k(i)$, $k \in \mathbb{N}$, with $s_k(i) \in \mathcal{S}^k, \forall k \in \mathbb{N}$ and with the property that $\lim_{k\to\infty} s_k(i)$ converges to a subsidy scheme that implements the social optimum.*

Note that there are three possibilities as to why it is impossible to correct cross-technology interactions with a simple policy: an uncountable number of different subsidies might be required, subsidies might have to vary strongly, even between close-by technologies, or subsidies might have to vary over time.

In all cases, it will usually be impossible even to come close to an efficient outcome in applications. Using a very large number of different instruments is typically infeasible in applications, as this requires highly detailed information and impose extreme administrative costs.[6] Using subsidies that vary strongly between close-by technologies is also difficult in practice, as tailoring policies to specific technologies usually requires private information of firms and often runs into legal problems, as apparently similar technologies receive strongly different levels of support. Finally, using a dynamic policy is often impossible due to administrative delays in adjusting policies and a lack of the information that would be necessary to design a dynamic scheme ex-ante.

3 Model solution and results

Having set up the model, we now assess its implications. In particular, we are interested in seeing how strongly these implications differ from the conventional common pool of knowledge setting.

[6]A prominent example is the feed-in tariff scheme that has been used in Germany to support renewables till 2016. The scheme differentiated according to technology, location, and age of an installation, using several hundred levels of subsidies in total. As the scheme was too complex for being adjusted in time to technological changes, it caused excessive costs and large parts of it were recently scrapped in favour of competitive tenders.

In this conventional setting, we would have $F(i,j) \equiv f > 0, \forall i,j \in \mathcal{I}, i \neq j$ and $F(i,i) \equiv \beta < 0 \forall i \in \mathcal{I}$. As we will show below, this setting implies that, both in the decentralised and the social planner's setting, technologies experience either exponential growth (unlimited or up to a given steady state) or exponential decay.

Furthermore, in both models, heterogeneity among technologies can only arise via different efficiencies in research ($\gamma(i)$). If the R&D efficiency is identical for all technologies, the qualities will converge to the same steady state (if a steady state exists) or all technologies will experience unlimited growth. Thus spillovers do not induce heterogeneity.

Finally, the difference between the social planner's and the decentralised solution can be eliminated by a subsidy s that is proportional to f, constant over time and identical for all technologies.

Thus, this setting results in a case that has both simple dynamics and can be solved by a simple policy intervention. We will now analyse the implications of more complex spillovers both for the dynamics of the system and the existence of simple policies which induce an efficient decentralised solution. We will do this successively for the cases of a finite and an infinite number of technologies.

3.1 Finite number of technologies

Consider first the case where $\mathcal{I}$ is finite, that is, we have $N \in \mathbb{N}/\infty$ firms. Then the dynamic problem (2)-(3) is a simple linear-quadratic control problem for every firm i and the operator $\mathcal{F}$ has a finite rank and thus a matrix representation F with entries $F_{i,j}$ describing the influence of technology j on technology i.

Let us first analyse the private r&d dynamics. These are given by a set of N standard optimal control problems:

$$\max_{g_i} \int_0^\infty \mathrm{e}^{-r\,t} \left(\bar{\pi}_i \, q_i(t) - \frac{1}{2} \, g_i^2(t) \right) dt, \tag{6}$$

s.t.

$$\dot{q}_i(t) = \gamma_i \, g_i(t) + \sum_{j=1}^{N} F_{ij} \, q_j(t), . \tag{7}$$

As $\mathcal{F}$ is a linear finite-dimensional operator, the solution to the problem is given by the solutions of the associated canonical system of $2\,N$ equations:

$$\dot{\psi}_i = r \, \psi_i(t) - \bar{\pi}_i - F_{ii} \, \psi_i(t), \tag{8}$$

$$\dot{q}_i(t) = \gamma_i^2 \, \psi_i(t) + \sum_{j=1}^{N} F_{ij} \, q_j(t), \tag{9}$$

$$\lim_{t\to\infty} \mathrm{e}^{-rt} \, \psi_i = 0. \tag{10}$$

As a first step, we consider possible steady states of the system. Often, models of technological change imply rather restrictive steady states, such as a convergence of all qualities to a common state. The following result shows that the above model is highly flexible in this regard.

Proposition 1. *Let $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_N$ be given and assume that, for at least one technology $i \in \mathcal{I}$, we have $\gamma_i \, \bar{\pi}_i \, \bar{q}_i \neq 0$ and that there is at least one other technology j with $\bar{q}_i \neq 0$. Then, for any given diagonal elements of F (the depreciation of technologies), there is a matrix F so that $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_N$ is the steady-state of the decentralised solution.*

Proof. The ODEs (8) are independent of the states $q_i(t)$ and independent of each other. Furthermore, the optimisation problem has an infinite time horizon with a constant discount rate r. Thus, the co-state for each i is constant over time as a consequence of the transversality condition (10). Under our assumption $F_{i,i} < r$, the system (8) thus has the unique solution:

$$\psi_i = \frac{\bar{\pi}_i}{r - F_{ii}} \rightarrow g_i = \frac{\gamma_i \, \bar{\pi}_i}{r - F_{ii}}. \tag{11}$$

The $F_{i,i}$ (depraciations of the technologies) are given, so that by Eq. (11), the ψ_i are determined. Thus the steady state of the system (9) results from a linear equation system

$$\gamma_i^2 \, \frac{\bar{\pi}_i}{r - F_{ii}} + \sum_{j=1}^{N} F_{ij} \, \bar{q}_j = 0, \; \forall i \in \mathcal{I}. \tag{12}$$

This system is to be solved for the off-diagonal elements $F_{i,j}, \; i, j \in \mathcal{I}, i \neq j$.

To this end, reorder the system, so that the technology with $\gamma_i \, \bar{\pi}_i \, \bar{q}_i \neq 0$ becomes technology 1 and the other technology with $\bar{q}_j \neq 0$ is technology 2. Consider the matrix

$$F = \begin{pmatrix} F_{1,1} & F_{1,2} & 0 & 0 & \ldots & 0 \\ F_{2,1} & F_{2,2} & 0 & 0 & \ldots & 0 \\ F_{3,1} & 0 & F_{3,3} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ F_{N-1,1} & 0 & \ldots & 0 & F_{N-1,N-1} & 0 \\ F_{N,1} & 0 & \ldots & 0 & 0 & F_{N,N} \end{pmatrix}. \tag{13}$$

With this structure, we can solve equation one for $F_{1,2}$ and each other equation i in (12) for $F_{i,1}$ separately. Due to our assumptions, each equation has a solution. □

This proposition shows that the model is highly flexible. Even if firms are homogenous with regard to marginal profits and r&d efficiency, there is no limit to the heterogeneity of the steady-state.

As a second step, we consider the dynamics. To prepare our analysis of policy interventions, we thereby consider not only the decentralised but also the social planner's problem. This problem (Eqs. (4)-(5)) implies the following dynamics

$$\dot{\tilde{\psi}}_i = r\, \tilde{\psi}_i(t) - \bar{\pi}_i - \sum_{j=1}^{N} F_{ji}\, \tilde{\psi}_j(t), \tag{14}$$

$$\dot{\tilde{q}}_i(t) = \gamma_i^2\, \tilde{\psi}_i(t) + \sum_{j=1}^{N} F_{ij}\, \tilde{q}_j(t), \tag{15}$$

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\, \tilde{\psi}_i = 0. \tag{16}$$

The next proposition describes the optimal r&d efforts in both systems.

Proposition 2. *In the decentralised solution, the firm's r&d efforts are constant over time, strictly positive for each firm i with $\gamma_i\, \bar{\pi}_i > 0$, and zero otherwise.*

In the social planner's solution, the r&d efforts are constant over time if and only if the matrix $\tilde{\Psi} := r\, I - F^T$ has the same rank as the augmented matrix $(\tilde{\Psi}|\bar{\pi})$.

Proof. For the decentralised system, Eq. (11) already specifies the optimal r&d efforts and proves that these are constant over time. As g_i is strictly positive if and only if $\bar{\pi}_i g_i > 0$, private r&d efforts for technology i are strictly positive if and only if $\bar{\pi}_i g_i > 0$.

In the social planner's problem, constant r&d efforts result from constant co-states. The dynamic system (14)–(16) admits constant co-state if and only if the system of equations

$$r\, \tilde{\psi}_i - \bar{\pi}_i - \sum_{j}^{N} F_{ji}\, \tilde{\psi}_j = 0, \;\; i \in \mathcal{I}, \tag{17}$$

can be solved. This is the case if and only if the matrix $\tilde{\Psi} := r\, I - F^T$ has the same rank as the augmented matrix $(\tilde{\Psi}|\bar{\pi})$. □

This result provides an important insight. It is to be expected that privately and socially optimal solutions differ regarding the level of r&d efforts. However, Proposition 2 implies that they can also differ in terms of r&d dynamics. In many cases, this will have much more profound effects on the long-term development of technologies. Furthermore, it is much harder to correct, as we will show below.

The result is also surprising, as it stems from a fairly simple model. Typically, a linear-quadratic optimisation problem, such as ours, will have simple solutions, such as, constant optimal controls (r&d efforts in our case).

In fact, this holds in our setting without cross-technology interactions: In the absence of such interactions, F is diagonal and (due to $F_{i,i} < r$), the matrix $\tilde{\Psi}$ has full rank, so that both the decentralised and the social planner's problem imply constant r&d efforts. Thus Proposition 2 is indeed solely a consequence of cross-technology interactions. As we will show in Section 4, situations, where privately and socially optimal dynamics differ, arises easily in plausible models.

As a third step, it is instructive to assess the dynamics in somewhat more detail. In general, a linear-quadratic model can imply three types of dynamics: Exponential growth/decline, (constant/dampened/intensifying) oscillations, and linear growth or decline.

To describe how these patterns are linked to general properties of the matrix describing the cross-technology spillovers, it is helpful to introduce the following measure of the complexity of $\mathcal{F}$: Let $\mu^a_{\mathcal{F}}(\lambda_i), \mu^g_{\mathcal{F}}(\lambda_i)$ be the algebraic and geometric multiplicities of the i-th eigenvalue of $\mathcal{F}$ and define

$$\chi(\mathcal{F}) := \sum_{i}^{K} (\mu^a_{\mathcal{F}}(\lambda_i) - \mu^g_{\mathcal{F}}(\lambda_i)). \tag{18}$$

With this definition, we get the following result.

Proposition 3. *If $\mathcal{F}$ is either symmetric or triangular and in addition semi-simple ($\chi(\mathcal{F}) = 0$) with all eigenvalues being non-zero, then the decentralised dynamics consists solely of exponential growth or decline (including constant qualities).*

If $\mathcal{F}$ is only semi-simple ($\chi(\mathcal{F}) = 0$) and all eigenvalues are non-zero, the decentralised dynamics can include both exponential growth/decline and oscillations.

If $\chi(\mathcal{F}) > 0$, the decentralised dynamics can also include a product of a linear time trend and exponential terms.

If F has eigenvalues that are zero, the decentralised dynamics will also include an additive linear time trend.

Proof. By Prop. 2, r&d efforts $g_i(t)$ are constant and non-negative. Thus in Eq. (7), the dynamics result solely from the matrix F, which is the Jacobian matrix of the system. If F is symmetric or triangular, it has only real eigenvalues. Furthermore, if it is semi-simple, it has no defective eigenvalues (eigenvalues that have a higher algebraic than geometric multiplicity) and, by assumption, no eigenvalue is zero. Thus the dynamics are always exponential growth or decline.

In the second assertion, we have the same situation but without symmetry. Thus there can be complex eigenvalues, which induce oscillations.

Without semi-simplicity, there can be defective eigenvalues (F cannot be diagonalised). In this case, dynamics can include a product of terms that are linear in time and exponential in time.

Finally, if at least one eigenvalue is zero, there will be an additive term that is linear in time in addition to the other dynamics. □

This proposition shows that, even in the decentralised solution, the model encompasses a rich set of dynamics and that rather general properties of the cross-technology interactions are sufficient to decide which dynamics are feasible in a given system.

Most notably, the conventionally used cases of no interactions and perfectly homogeneous interactions (public pool of knowledge) have rather specific implications. With no interactions, F is a diagonal matrix. If none of the diagonal elements (which are the eigenvalues in this case) is zero, we get only exponential dynamics. With perfectly homogeneous interactions, where each technology causes the same spillover f to all other technologies and is depreciated at a homogeneous rate β we get the same result whenever $f \neq \beta$: The matrix F is semi-simple, symmetric and all eigenvalues are non-zero. Thus under common assumptions, only the most simple dynamics occur in our model; more complex cross-technology interactions add interesting dynamic structure.

The following proposition gives a similar insight for the social planner's problem.

Proposition 4. *If $\tilde{\Psi} := r\,I - F^T$ has the same rank as $(\tilde{\Psi}|\bar{\pi})$, Proposition 3 also holds for the social planner's solution.*

If the rank of $\tilde{\Psi}$ is strictly smaller than the rank of $(\tilde{\Psi}|\bar{\pi})$ and no eigenvalue of F equals r, the social planner's solution always includes additive linear dynamics for at least one technology.

If the rank of $\tilde{\Psi}$ is strictly smaller than the rank of $(\tilde{\Psi}|\bar{\pi})$ and one eigenvalue of F equals r, the social planner's solution either adds linear dynamics to a technology that does not have these in the decentralised solution or the dynamics include a term that is quadratic in t for at least one technology.

Proof. By Prop. 2, $rank(\tilde{\Psi}) = rank(\tilde{\Psi}|\bar{\pi})$ implies constant socially optimal r&d efforts. Thus the structure of the dynamic system (9) is the same as that of the system analysed in Prop. 3.

If $rank(\tilde{\Psi}) < rank(\tilde{\Psi}|\bar{\pi})$, the system (14) implies that at least one co-state is not constant but changes linearly with time, which implies the same for socially optimal r&d efforts (which, following from Eqs. (4)-(5), are proportional to the co-states). Thus the dynamic system (9) is non-autonomous for at least one $i \in \mathcal{I}$; as the first term on the r.h.s. being a linear function of time for at least one equation.

If no eigenvalue of F is equal to r, then $F + r\,I$ does not have full rank but F has. Consequently, the dynamics implied by the system (9) have a linear part for at least one technology. This can be easily seen by solving the system via first solving the time-autonomous part and then using the variation-of-constant method.

If one eigenvalue of F is equal to r, then both $F + r\,I$ and F do not have full rank. In this case, the decentralised system already has linear dynamics for at least one technology. Let j be a technology that has non-constant r&d efforts in the social planner's solution and no linear dynamics in the decentralised solution. Let k be a technology that has linear dynamics in the decentralised solution. If these technologies are fully separated, that is, if for every $l \in \mathcal{I}$ with $F_{l,k} \neq 0$, we have $F_{j,l} = 0$, the non-constant r&d efforts add linear dynamics to (at least) the technology j. If the technologies are not fully separated or if j already has linear dynamics in the decentralised solution, the non-constant r&d efforts and the existing linear dynamics interact and a term that is quadratic in time results. □

This proposition shows that the condition of Prop. 2 is also crucial for the quality dynamics: If $rank(\tilde{\Psi}) = rank(\tilde{\Psi}|\bar{\pi})$, the socially optimal solution and the decentralised solution have the same type of dynamics. If this condition is not met, the socially optimal solution has always more complex dynamics. Interestingly, this can even result in dynamics that do not exist in the decentralised system.

Another implication of Prop. 3 and 4 relates to long-run growth of technological quality. To have unconstrained growth, it is necessary that the dynamic system does not have a stable steady state. Our above results shed some light on this question. First, as shown in the proofs above, the matrix F is indeed the Jacobian of the quality dynamics (9) or (15). Thus the eigenvalues of F govern the stability of the system, if it is autonomous. Second, it is obvious that once the system becomes non-autonomous, it will not have a steady state, as the non-autonomous term is additive and independent of the states of the system.

Together, these points imply the following result.

Corollary 1. *If $rank(\tilde{\Psi}) = rank(\tilde{\Psi}|\bar{\pi})$ and F has no eigenvalue that is zero, the steady states of the decentralised and of the social planner's solution have the same stability properties (but the steady states themselves can differ).*

If $rank(\tilde{\Psi}) < rank(\tilde{\Psi}|\bar{\pi})$, the social planner's solution has no stable steady state.

Proof. The first assertion is a consequence of Prop. 4 and the fact that the Jacobian of the quality dynamics in both systems is F and thus identical. The second assertion follows from Prop. 4, as the social planner's solution always has an additive non-autonomous part that is independent of $q(t)$. □

Thus, the social planner's solution will have no stable steady state in more cases than the decentralised solution. In other words, if unconstrained growth occurs in the decentralised solution, it is always socially optimal. But there are cases where unconstrained growth would be socially optimal, but does not realise in the private solution.

Finally, it is possible to provide some information on the prospects of unlimited growth in the decentralised solution.

Corollary 2. *Assume that $F_{i,i} < 0 \,\forall\, i \in \mathcal{I}$ and $F_{i,j} \geq 0 \,\forall\, i,j \in \mathcal{I}, i \neq j$. Then:*

1. *Without cross-technology interactions, the system has an asymptotically stable steady state.*

2. *With one-way cross-technology interactions (i.e., either $F_{i,j} = 0 \forall i < j$ or $F_{i,j} = 0 \,\forall\, i > j$), the system has an asymptotically stable steady state.*

3. *With homogeneous interactions (i.e., $F_{i,i} = -\beta \,\forall\, i \in \mathcal{I}$, $F_{i,j} = f > 0 \forall i,j \in \mathcal{I}, i \neq j$), there is a stable steady state if and only if $f < g/N$, where N is the number of technologies. Otherwise, there is unlimited growth.*

4. *With two-way cross-technology interactions (i.e., there exist $F_{i,j} > 0$ with $i < j$ and $F_{i,j} > 0$ with $i > j$), there is no stable steady state and thus there can be unlimited growth.*

Proof. In Case 1 and Case 2, the eigenvalues of the system are the diagonal elements of F, which are all strictly negative by assumption. Thus there is an asymptotically stable steady state.

In Case 3, the eigenvalues of the F are $-f-g$, which is strictly negative, and $N\,f - g$, which is also strictly negative if and only if $f < g/N$. As we only have positive spillovers, the non-existence of an asymptotically stable steady state implies unlimited growth.

In Case 4, it is easy to find examples with strictly positive eigenvalues. Consider, for example, a matrix F that has $-g$ on the diagonal, $F_{i,j} = f$ whenever $|i-j| = 1$, and where all other entries are zero. Such a system always has an eigenvalue of the type $(\alpha(N)f - g)$ with $\alpha(N) > 0$ being a constant that depends on the number of technologies. Thus, if f is sufficiently larger than g. Again, this implies unlimited growth. □

This corollary shows that cross-technology interactions can induce unlimited growth in situations, where such growth would not happen without them. This is a well-known result in endogenous growth theory (see, for example, Smulders (1995)): Positive spillovers in non-rival knowledge creation can induce ongoing economy growth. What our result adds is some structure: Such growth is bound to specific properties of the interactions. It cannot happen in case of one-way interactions (chains of technological knowledge) but requires two-way interactions. In case of a common pool of knowledge, either the number of technologies or the spillovers have to be sufficiently strong to induce ongoing growth.

So far, our results have shown that cross-technology interactions can induce fairly complex dynamics in an otherwise simple model and lead to potentially interesting policy problems. As a final step of our analysis, we investigate whether and how policies can bridge the gap between the decentralised and the social planner's solution.

Comparing the decentralised solution (8)–(10) and the social planner's solution (14)–(16) shows that both become identical, if firms in the decentralised system get a subsidy that increases their profit to $\pi_0+(\bar{\pi}_i+s_i(t))q_i(t)$ with

$$s_i(t) = \sum_{j\neq i}^{N} F_{ji}\,\tilde{\psi}_j(t). \tag{19}$$

Thus the social optimum is, in theory, implementable in the decentralised system. But it is instructive to consider the necessary complexity of such a scheme. The following proposition provides a first result in this regard.

Proposition 5. *The subsidy scheme* (19) *is a simple policy in the sense of Definition 2 if and only if* $rank(\tilde{\Psi}) = rank(\tilde{\Psi}|\bar{\pi})$.

Proof. It is obvious that with I containing a finite number of technologies, the only relevant constraint of Definitions 1 and 2 is that the subsidies cannot vary over time. If $rank(\tilde{\Psi}) = rank(\tilde{\Psi}|\bar{\pi})$, then all $\tilde{\psi}_i$ are constant in the social planner's solution. Thus by Eq. (19), the subsidies are constant. In contrast, if $rank(\tilde{\Psi}) < rank(\tilde{\Psi}|\bar{\pi})$, then Prop. 2 implies that r&d efforts in the social planner's solution vary over time but are constant in the decentralised solution. Thus, in this case, a set of constant subsidies cannot correct incentives at all points of time. □

This result highlights again that cross-technology interactions can have profound effects on the dynamics of technological change and that rather complex policies might be required to implement an efficient solution.

An interesting question left open by Proposition 5 is how complex a "simple" policy needs to be. Our definition of a simple policy states only that the policy can only use a finite number of different subsidies. This will be restrictive in the setting with an uncountable number of technologies analysed in the next section but is a fairly generous requirement in the present setup, as each technology can be treated with a different subsidy. Models involving the assumption of a common pool of knowledge typically show that a single subsidy is sufficient to correct for r&d inefficiencies.

So how many different subsidies are required for different specifications? To investigate this question, it is helpful to use some additional definitions. Let F_0 be a matrix that equals F with regard to all off-diagonal elements but has only zeros on the diagonal, that is, $F_{0_{i,j}} = F_{i,j}$ if $i \neq j, i, j \in \mathcal{I}$ and $F_{0_{i,i}} = 0 \forall\, i \in \mathcal{I}$. Furthermore, let β be the vector of the $F_{i,i}, i \in \mathcal{I}$. Finally, we use the function $\Delta_R(X)$ to denote the number of different[7] rows of the matrix X.

Using these definitions, we can provide an upper boundary to the number of subsidies that are required to implement the socially optimal solution in the decentralised system.

Corollary 3. *Assume that $rank(\tilde{\Psi}) = rank(\tilde{\Psi}|\bar{\pi})$ and let n_S be the number of different, non-zero subsidies in Eq.* (19)*. We then have*

$$n_S \leq \max\{\Delta_R(F_0), \Delta_R(\bar{\pi})\, \Delta_R\left((F_0|\beta)^T\right)\}. \tag{20}$$

Proof. Note first that, with our definitions and by Prop. 5, Eq. (19) can be written as $s = F_0\tilde{\psi}$, where s is the vector of subsidies s_i. Thus the number of subsidies can be at most equal to the number of different elements of $\tilde{\psi}$ and at most equal to the number of different rows of F_0. By Eq. (17), the number of different elements of $\tilde{\psi}$ is itself at most equal to the maximum of product of the number of different elements of $\bar{\pi}$ and the number of different rows of the matrix $(F_0|\beta)^T$ and the number of technologies N. As $\Delta_R(F_0) \leq N$, we thus get Eq. (20). □

To illustrate this result, let us consider two examples. Assume first that we have homogenous interactions, that is, $F(i,j) \equiv f, \forall i,j \in \mathcal{I}, i \neq j$ and $F(i,i) = \beta_i \forall i \in \mathcal{I}$. In this case, the matrix F_0 has always N (number of technologies) different rows (the zero on the diagonal appears in a different column in each row). Furthermore, without constraints on differences between the $\bar{\pi}_i$ and β_i, the second term in the maximum in Eq. (20) can always exceed N, so that the right-hand side of (20) equals N. We might need a different subsidy for each technology. In fact, it is easy to construct examples where this is the case.

However, if we assume that all $\bar{\pi}_i$ and all β_i are identical among firms, the second term in the maximum in Eq. (20) equals 1. Indeed, a single subsidy paid to all technologies suffices in this case.

As a second example, consider the following interaction matrix

$$F = \begin{pmatrix} F_{1,1} & 0 & f_1 & f_2 \\ 0 & F_{2,2} & f_1 & f_2 \\ f_3 & f_4 & F_{3,3} & 0 \\ f_3 & f_4 & 0 & F_{4,4} \end{pmatrix}. \tag{21}$$

The corresponding matrix F_0 (where the diagonal elements are set to zero) has only two distinct rows $(0, 0, f_1, f_2$ and $(f_3, f_4, 0, 0))$. Thus regardless of the assumptions on $\bar{\pi}_i$ and the $F_{i,i}$, at most two subsidies are required to correct for this structure of cross-technology interactions, even though the

[7]Observe that this is the number of different rows not the number of linearly independent rows.

spillovers can take on four different values (f_1 to f_4). Note that if we switch just two values in the matrix (21), for example, $F_{1,2}$ and $F_{4,1}$, we can end up with a need for twice as many different subsidies. This neatly highlights that it is the *structure* of the interactions that is important to determine how complex a policy needs to be.

Overall, the analysis so far has shown that even in a fairly simple model, cross-technology interactions have important consequences for the dynamics of technological change and for the complexity of the policies required to correct for inefficiencies. Thus accounting for these interactions and, in particular, for the structure of these interactions, is important.

This holds even in a simple setting with a finite number of technologies. In the next section, we consider a continuum of technologies.

3.2 Continuum of technologies

Many models of technological change use a continuum of technologies (see Chu et al. (2012), Acemoglu and Cao (2015) for recent examples). To complete our analysis, we now also cover this case.

Thus, we set $\mathcal{I} = [0, 1] \subseteq \mathbb{R}_+$. As we now work in an infinite-dimensional space, we have to be more specific regarding the cross-technology operator $\mathcal{F}$. We assume that $\mathcal{F}$ has an integral representation and is linear and compact. Furthermore, in this setting we take the depreciation of a technology out of the operator $\mathcal{F}$ and integrate it directly into the state equations. Thus we have

$$\dot{q}(i,t) = \gamma(i)\, g(i,t) - \beta(i)\, q(i,t) + \int_{\mathcal{I}} F(i,j)\, q(j,t)\, dj, \tag{22}$$

with

$$\int_{\mathcal{I}} \int_{\mathcal{I}} |F(i,j)^2| di dj < \infty, \tag{23}$$

and $F(i,i) = 0 \forall\, i \in \mathcal{I}$. We do not require the nucleus to be continuous over (i,j), but only to be measurable, defined everywhere and to be a bounded function.

Again, the firms' optimisation problems are separated; each firm i considers the influence of other technologies as a function of time $f(i,t)$. The optimality and transversality conditions for firm i are thus

$$g(i,t) = \gamma(i)\, \psi(i,t), \tag{24}$$

$$\dot{\psi}(i,t) = r\, \psi(i,t) - \bar{\pi}(i) + \beta(i)\, \psi(i,t), \tag{25}$$

$$\dot{q}(i,t) = \gamma(i)\, g(i,t) - \beta(i)\, q(i,t) + f(i,t), \tag{26}$$

$$0 = \lim_{t\to\infty} e^{-r\,t}\, \psi(i,t). \tag{27}$$

Again, we first consider possible steady states and get a result that closely resembles Prop. 1.

Proposition 6. *Let $\bar{q}(i)$ be a continuous and bounded function of $i \in \mathcal{I}$ with $\bar{q}(j) > 0$ on some set $\mathcal{J} \subseteq \mathcal{I}$ whose measure in $\mathcal{I}$ is strictly greater than zero and where we also have $\gamma(j)\,\bar{\pi}(j) > 0$ for all $j \in \mathcal{J}$.*

Then, there is a continuous and bounded operator $\mathcal{F}$, so that $\bar{q}(i)$ is the steady state of the technological development of the decentralised system.

Proof. The conditions (24)–(27) imply

$$\psi(i,t) = \frac{\bar{\pi}(i)}{r + \beta(i)}, \tag{28}$$

$$g(i,t) = \frac{\gamma(i)\,\bar{\pi}(i)}{r + \beta(i)}. \tag{29}$$

Thus the steady state of the system is characterised by

$$\gamma\,\bar{g} - \beta\,q + \mathcal{F}\,q = 0. \tag{30}$$

Let $\mathcal{J}$ be the set of technologies j, for which $\gamma(j)\,\bar{\pi}(j) > 0$ and for which $\bar{q}(j) > 0$. By assumption, this set has a non-zero measure in $\mathcal{I}$.

For each $i \in \mathcal{I}$, we set $F(i,j) = \epsilon(i)$ for all $j \in \mathcal{J}$, and zero otherwise. The value of $\epsilon(i)$ is calculated from $\epsilon(i) = \frac{\beta(i)\,\bar{q}(i) - \gamma(i)\,\bar{g}(i)}{\int_{\mathcal{J}} \bar{q}(j)\,dj}$. As the set $\mathcal{J}$ has a non-zero measure and as $\bar{q}(j) > 0$ for all $j \in \mathcal{J}$, the conditions yields finite values for all $\epsilon(j)$, so that $\mathcal{F}$ is bounded. Furthermore, the conditions ensure that the function $\bar{q}(i)$ solves Eq. (30) with the constructed operator $\mathcal{F}$. □

Thus again, the model is highly flexible in terms of its ability to describe long-run heterogeneity in technological quality. Note that the heterogeneity can even occur if firms are homogeneous with regard to all parts of the model except for the cross-technology interactions.

Again, the private r&d efforts are constant over time, as each firm considers only its own technology. In contrast, the social planner's problem is a joint optimisation of a set of interpendent differential equation. Using the transformation of the system into the infinite-dimensional set of ODEs (following Belyakov et al. (2011), Skritek et al. (2014)), we can write the optimality conditions as:

$$\tilde{g}(i,t) = \gamma(i)\,\tilde{\psi}(i,t), \tag{31}$$

$$\dot{\tilde{\psi}}(i,t) = r\,\tilde{\psi}(i,t) - \bar{\pi}(i) + \beta(i)\,\tilde{\psi}(i,t) - \int_{\mathcal{I}} \tilde{\psi}(j,t)\,F(j,i)\,dj, \tag{32}$$

$$\dot{\tilde{q}}(i,t) = \gamma(i)\,\tilde{g}(i,t) - \beta(i)\,\tilde{q}(i,t) + \int_{\mathcal{I}} F(i,j)\,\tilde{q}(j,t)\,dj. \tag{33}$$

with the following transversality conditions

$$\forall i \in \mathcal{I}: \lim_{t\to\infty} e^{r\,t}\,\tilde{\psi}(i,t) = 0. \tag{34}$$

Eq. (32) admits a constant solution $\tilde{\psi}(i,t)$, if and only if the following integral equation (a Fredholm equation of second kind) has a solution:

$$\tilde{\psi}(i) = \frac{\bar{\pi}(i)}{r+\beta(i)} + \frac{1}{r+\beta(i)} \int_{\mathcal{I}} F(j,i)\,\tilde{\psi}(j)\,dj. \tag{35}$$

Together with our analysis above, this implies the following complement to Prop. 2.

Proposition 7. *In the decentralised solution, r&d efforts are always constant over time. In the social planner's solution r&d efforts are constant over time if and only if Eq.* (35) *has a solution.*

We thus recover the main insight of the finite dimensional case: Cross-technology interactions can lead to a situation where the optimal solution has structurally different dynamics than the decentralised outcome.

However, in the infinite dimensional case it is neither possible to give necessary and sufficient conditions for the operator $\mathcal{F}$ so that Eq. (35) has a solution nor is it possible to characterise the dynamics in detail, apart from cases that are basically direct extensions of the finite dimensional case.

However, it is possible to provide some insights into how complex a policy has to be in order to implement the socially optimal solution. In the infinite-dimensional case, this is of much more interest than in the preceding section, because it is obviously infeasible to have a different subsidy for each technology. Thus the interesting question arises whether it is possible to implement the socially optimal solution with a finite number of subsidies or, at least, to approximate this solution.

Again, comparing the firms' problem (Eqs. (24)–(27)) and the social planner's problem (Eqs. (31)–(34)) shows that the following subsidy scheme will implement the socially optimal solution in the decentralised system:

$$s(i,t) = \int_{\mathcal{I}} F(j,i)\,\tilde{\psi}(j,t)\,dj, \tag{36}$$

where $\tilde{\psi}(j,t)$ denotes the co-states in the socially optimal solution.

Without any constraints on the subsidy scheme, it is hardly surprising that the social optimum can be decentralised. We simply provide an additional incentive to each firm, so that the firm exerts socially optimal R&D efforts. The more interesting question is, whether a feasible subsidy scheme is up to the task. The following proposition yields an answer.

Proposition 8. *Let $K \in \mathbb{N}, K < \infty$ and let $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_K \subseteq [0,1]$ be disjoint connected sets that each have a measure in $\mathcal{I}$ that is strictly greater than zero and whose union equals $\mathcal{I}$. Define*

$$F_K(i,j) = \sum_{k=1}^{K} P_k(j)\,Q_k(i), \tag{37}$$

where

$$P_k(j) = \begin{cases} 1 & \textit{if } j \in \mathcal{J}_k, \\ 0 & \textit{otherwise,} \end{cases} \tag{38}$$

If and only if cross-technology interactions $F(i,j)$ have the structure (37)–(38) *and Eq.* (35) *has a solution, the subsidy scheme* (36) *is a simple policy in the sense of Def. 2.*

Proof. Assume that $F(i,j)$ has the structure (37)–(38) and Eq. (35) has a solution. Then the optimal subsidy (36) is constant and equals

$$s(i) = \int_{\mathcal{I}} \tilde{\psi}(j) \sum_{k=1}^{K} P_k(i)\, Q_k(j)\, dj, \tag{39}$$

$$= \sum_{k=1}^{K} \left(P_k(i) \int_{\mathcal{I}} \tilde{\psi}(j)\, Q_k(j)\, dj \right). \tag{40}$$

By construction, for each $k \in \{1, \ldots, K\}$, the integral is only a function of t and $P_k(i)$ is either 0 or 1. Thus each element of the sum is a constant. For each $t \geq 0$, $s(i,t)$ is therefore a piecewise constant function that takes on at most K different values on the sets $\mathcal{J}_1, \ldots, \mathcal{J}_K$. Each of these sets meets the requirements of Def. 1. Thus a simple solution $s(i,t) \in \mathcal{S}^K$ exists. This proves that the structure (37)–(38) is sufficient for the existence of a simple solution.

Now, consider necessity. As a simple solution exists by assumption, Eq. (35) has a solution, as a dynamic subsidy would be required otherwise. Furthermore, there is a finite natural number K, so that $s(i,t) \in \mathcal{S}^K$. By Definition 1, this implies that the smallest of the sets $\mathcal{J}_1, \ldots, \mathcal{J}_K$ has a minimal size $\epsilon > 0$ and is an interval (a connected subset of $[0,1]$).

By Eq. (36), the optimal subsidy is given by

$$s(i,t) = \int_{\mathcal{I}} \tilde{\psi}(j,t)\, F(j,i)\, dj. \tag{41}$$

Consider the interval $\mathcal{J}_1$, which has a size of at least $\epsilon > 0$. On this interval, the subsidy $s(i,t)$ is constant w.r.t. i (by assumption). As in Eq. (41) only $F(j,i)$ depends on i, this implies that (j,i) is constant w.r.t. i over this interval. Accordingly, on the interval $\mathcal{J}_1$, $F(i,j)$ is a function $Q_1(i)$. Thus, we have $F(i,j) = P_1(j)\, Q_1(i) + F^R(i,j)$ with $P_1(j)$ as defined in Eq. (38) and $F^R(i,j)$ being some function that is zero on $\mathcal{J}_1$.

Repeating this procedure for all other intervals $\mathcal{J}_2, \ldots, \mathcal{J}_K$ leads to $F(i,j) = \sum_{k=1}^{K} P_k(j)\, Q_k(i) + F^R(i,j)$ with the $P_k(j)$ being given by Eq. (38). As $\cup_{k=1}^{K} \mathcal{J}_k = \mathcal{I}$, the function $F^R(i,j)$ needs to be zero everywhere on $\mathcal{I}$. Thus, we get the characterisation of $F(i,j)$ given in Eqs. (37)–(38). The properties of the sets stated in Proposition 8 are directly inherited from Definition 1. □

Proposition 8 shows that highly specific assumptions are necessary to ensure that cross-technology interactions cause problems that can be solved by a simple subsidy scheme. It is necessary that technologies can be grouped into a finite number of sets, where all technologies in a set $\mathcal{J}_i$ cause the same spillover.[8] In Section 4, we will show that this is indeed a rather restrictive assumption; even if we form blocks of technologies that cause constant spillovers to other technologies (which is one of the simplest possible structures apart from fully homogeneous interactions), these blocks have to have a specific form.

Thus the ubiquitous model of a common pool of knowledge describes a rather non-generic case. In most more general cases, we have to expect that technology interactions will cause problems that cannot be solved with a simple policy. Thus, in most applications, we have to expect that the process of technological change will not lead to a socially optimal outcome, with or without state interventions. Note further that this is not a transitory but rather a persistent problem: The long-run qualities arising from privately optimal investments do not converge to the socially optimal long-run quality levels.

This result highlights an important point: If a regulatory authority either does not have the necessary information to use policy interventions that are tailored to specific technologies or is not able to yield a high number of different instruments, R&D subsidies are not sufficient to cope with complex situations of cross-technology interactions. As discussed in the introduction, we face increasingly complex problems of technological change. Thus it appears likely that efficient technology transitions will be more a theoretical benchmark than an achievable outcome.

However, even if an efficient solution cannot be attained, it might be possible to come close to it. In our setting, this corresponds to the question whether a simple approximate solution exists. To gain insight into this question, we use the following definition.

Definition 3. *The cross-technology interactions F have at least a* minimal regularity*, if there exists an $\epsilon > 0$ so that the technology space $\mathcal{I}$ can be partitioned into a number of disjoint, connected sets of minimal size ϵ with $F(i,j)$ being a.e. continuous on each of these sets.*

This condition effectively demands that, except for a countable number of technologies, each technology has some close-by technology that causes similar effects on the development of all other technologies. In other words, there is some minimal regularity as to how interactions change across the set of technologies.

[8]This spillover can vary w.r.t. the receiving technology, but the way in which it varies has to be identical for all technologies in $\mathcal{J}_i$.

As the following proposition shows, this definition is a good indicator for the question, under which conditions, we can find simple approximate solutions.

Proposition 9. *A simple policy can approximate the socially optimal solution if and only if $\mathcal{F}$ has minimal regularity in the sense of Def. 3 and Eq. (35) has a solution.*

Proof. Assume that F has minimal regularity and Eq. (35) has a solution. Then, for every $i \in \mathcal{I}$, there is at most a countable number of points at which $F(i, j)$ is not continuous. Consequently, $F(i, j)$ is a.e. continuous. In this case, Eq. (35) implies that $\bar{\psi}^S$ is also a.e. continuous, so that the optimal subsidy (36) has the same property.

As the optimal subsidy is a.e. continuous w.r.t. i, it can be approximated to an arbitrary precision by the step-wise function that constitutes a constrained subsidy according to Definition 1:

$$\int_{\mathcal{I}} \left(s(i) - s^M(i)\right)\, di = \int_{\mathcal{I}} s(i)\, di - \sum_{d=1}^{M} A_d(i_d - i_{d-1}) \overset{M\to\infty}{\to} = 0, \tag{42}$$

where

$$A_d = \frac{\int_{i_{d-1}}^{i_d} s(i)\, di}{i_d - i_{d-1}} \tag{43}$$

is the average value of $s(i)$ over the subset $\mathcal{J}_d$ from Definition 1.

If F does not have minimal regularity, then there is an uncountable number of points at which F is not continuous and, by the above argument, the same holds for the optimal subsidy $s(i)$. The function $s(i)$ thus has an uncountable number of discontinuities, so that no countable number of subsidies can approximate the optimal $s(i)$ in the sense of Def. 2.

If Eq. (35) does not have a solution, a dynamic subsidy is required to approximate the optimal solution. □

This result provides a far more optimistic picture regarding the prospects of interventions then Proposition 8. It shows that, as long as technology interactions originating from close-by technology have similar effects in almost all cases, an efficient outcome can at least be approximated by a feasible policy. By increasing the number of subsidies, we can get arbitrarily close to the socially optimal solution. Note that this result is not trivial, as we have an uncountable number of technologies but restrict the policy to a countable number of instruments.

However, it is obvious that there is a substantial class of problems in which even a simple approximate solution is not achievable. We will provide an example in Section 4. In general terms, this holds when many (an uncountable number) of isolated technologies cause the spillovers. An example are situations, where out of groups of different technologies (such groups could, e.g., be wind power, PV, geothermal energy), only single technologies (e.g., a particular type of PV cell) cause a spillover. In such cases, a feasible policy (that addresses a group of technologies, such as, PV) will not be able to induce an efficient outcome.

Overall, the analysis in this section has shown two main points. First, a generalisation of the commonly used model of cross-technology interactions appears to be useful. The generalised model has a far greater scope to describe eventual outcomes of technological change (in terms of the steady-state distribution of technological qualities). Furthermore, the situation depicted by the model of a common pool of knowledge is non-generic in that it is much too optimistic regarding the prospects of solving the problems caused by cross-technology interactions. Simple solutions can only be found in very special cases and it is not even possible to approximate the social optimum with feasible policy in all cases.

4 Examples

To highlight the most important results of the preceding sections and to show the range of applicability of the model, we give some examples.

4.1 Example 1: One-way interactions with a finite number of technologies

Let us start with one-way technological interactions in a finite-dimensional setting: The technologies can be ordered in a way, so that each technology receives only a spillover from preceding technologies and never from succeeding ones. In this case, the interactions can be described by a triangular matrix, so that we have

$$F = \begin{pmatrix} F_{1,1} & 0 & 0 & \dots & 0 \\ F_{2,1} & F_{2,2} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ F_{N-1,1} & \dots & F_{N-1,N-2} & F_{N-1,N-1} & 0 \\ F_{N,1} & \dots & F_{N,N-2} & F_{N,N-1} & F_{N,N} \end{pmatrix}. \tag{44}$$

The eigenvalues of such a matrix are given by the diagonal elements, which we assume all to be strictly negative. Assume further that each technology (apart from technology 1) receives at least one spillover, that is, for all $i > 1$, there is at least one $j < i$, so that $F_{i,j} \neq 0$. In this case, the geometric multiplicity of each eigenvalue is always one. We assume that all spillovers are non-negative.

Thus, if the $F_{i,i}$ all differ from each other, the matrix is semi-simple (all eigenvalues having an algebraic and geometric multiplicity of one). By

Prop. 3, we thus get exponential dynamics in the decentralised system. Furthermore, by Cor. 2, the decentralised system has an asymptotically stable steady state, so that there is no option for continued growth.

If at least two $F_{i,i}$ coincide (technologies having the same rate of depreciation in quality), the matrix is no longer semi-simple and the system can thus include a term that is a mixture of linear and exponential terms. However, as the exponential terms always dominate in the long run, the system still has an asymptotically stable steady state.

In all these cases, F has full rank. Furthermore, owing to the assumption $F_{i,i} < 0$, the matrix $F - r\, I$ also has full rank. Thus the social planner's solution always has the same type of dynamics as the decentralised solution and a simple solution (a finite number of constant subsidies) will implement the socially optimal outcome in the decentralised system.

As we mainly assumed that the cross-technology interactions have a specific structure (being one-way interactions) without severely limiting constraints on the relative magnitude of the interactions, these conclusions have a considerable range of application.

Figure 1 provides a numerical example for three technologies. Here, each technology causes a positive spillover to the next technology. The first two technologies are easily developed but not very profitable. The third technology is more profitable, but direct r&d has a small efficiency. In this setting, the social planner would maintain the quality of the first technology, strongly develop the second one and thereby induce development of the third technology. This does not happen in the decentralised solution, where all technologies stabilise on a much lower quality level. To correct for this, three constant but different subsidies are required. In the numerical example, we get $s_1 \approx 0.2, s_2 \approx 0.55, s_3 \approx -20.8$; the social planner thus actively discourages the direct development of technology 3, because indirect development via spillovers from technology 2 is much less costly.

4.2 Example 2: A continuum of technologies with homogenous interactions between groups of technologies

As a second example, consider a simple setting with a continuum of technologies. In such a setting, it is usually impossible to obtain closed-form solutions. Thus, we only characterise special settings, where simple policies can (approximately) correct the r&d externalities. To this end, we consider interactions that are contained to groups of technologies and are homogeneous within these groups. Again, this is a setting with many potential applications.

Assume that there is a finite number K of closed, connected sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_K \mathcal{I} \times \mathcal{I}$, so that interactions between technologies i and j take on a constant value f_k, if there is some $k \in \{1, \ldots, K\}$ so that $(i,j) \in \mathcal{S}_k$ or zero, otherwise. If

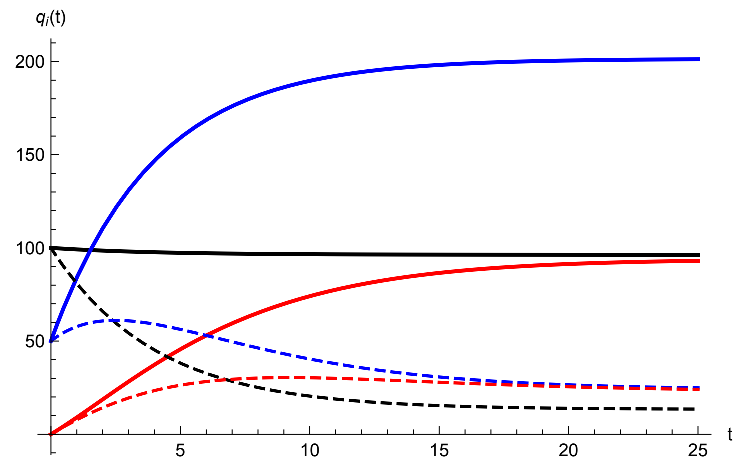


Figure 1: : Example with three technologies: $r = 5\%$, $F_{1,1} = F_{2,2} = F_{3,3} = -25\%$, $(1/2)\ F_{2,1} = F_{3,2} = 10\%$, all other $F_{i,j} = 0$, $\bar{\pi}_1 = \bar{\pi}_2 = 1$, $\bar{\pi}_3 = 25$, $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = 1/25$, $q_1(0) = 100$, $q_2(0) = 50$, $q_3(0) = 0$. Bold lines depict the social planner's solution, dashed lines the decentralised ones.

each of these sets is placed along the diagonal,[9] we have the above special case of interactions within groups of technologies. Otherwise, we have a more general form of interaction.

Define the boundary of the set $\mathcal{S}_k$ as a function within the technology space:

$$\Delta_k : i_k = f_k(j). \tag{45}$$

We characterise our setting in terms of these boundaries.

Corollary 4. *Assume interactions are uniform within closed connected sets $\mathcal{S}_k$ with $k \in \{1, \ldots, K\}$, $K < \infty$ and zero across sets. In addition assume Δ_k is a closed curve for all k. Then:*

1. *If Δ_k for all $k \in K$ is a piecewise-constant function of j over intervals of size $\epsilon > 0$, the nuclei of $\mathcal{F}, \mathcal{F}^*$ are finitely generated and a simple policy can implement the socially optimal solution in the decentralised system.*

2. *If there exists some $\hat{k} \in K$ such that $\Delta_{\hat{k}}$ is convex and nowhere piecewise-constant,[10] a simple policy can approximate the socially optimal solution in the decentralised system.*

[9] That is, if $(i, j) \in \mathcal{S}_k$, then $(i, i) \in \mathcal{S}_k$ and $(j, j) \in \mathcal{S}_k$.

[10] Note that strict convexity is stronger than this requirement: we allow for straight lines, but not for those implying $i = const$ or $j = const$.

Proof. Under the conditions of Assertion 1, all sets $\mathcal{S}_k$ can be partitioned into a finite collection of disjoint, rectangular sets with a size not smaller than $\epsilon \times \epsilon$. By (36), $s(i)$ can take on only a finite number of values. This implies $F(i,j)$ is degenerate and $\mathcal{F}, \mathcal{F}^*$ are of finite rank, so that Prop. 8 applies.

In the context of Assertion 2, sets are convex. Thus $s(i)$ cannot be piecewise-constant, even for uniform interactions over a finite number of sets. But it is a.e. continuous, because interactions are constant within each set, the number of sets is finite, and, due to convexity, each set intersects only twice with a line j and thus can cause at most two discontinuities. In this case, Proposition 9 applies. □

This result shows that, for the case of constant interactions between sets of technologies, the boundaries of these sets play a decisive role. If these boundaries facilitate a decomposition of the set into rectangular subsets, the problem is of the simplest possible class: A simple solution can be found. The reason is that, in this case, not only the size of individual spillovers but also the number of technologies getting the spillover is locally constant.[11] Thus close-by technologies usually cause the same spillovers to the same set of other technologies. Thus the externality can be solved successfully with a finite number of different subsidies.

If the boundaries are more complex (i.e., we have convex sets without vertical or horizontal boundaries), the number of technologies that receive a spillover will usually vary, even between close-by technologies. Thus the subsidy has to vary as well. However, the subsidy will vary continuously (almost everywhere) and can thus be at least approximated by a simple subsidy scheme.

For both cases, it is possible to calculate the subsidy explicitly. Assume that there are K disjoint compact rectangular subsets $\mathcal{S}_k$, $k \in \{1, \ldots, K\}$, of $[0,1]$, where we have, for each subset $\mathcal{S}_k$,

$$F(i,j) = \begin{cases} \bar{f}_k, & \text{if } i,j \in \mathcal{S}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

This resembles Case 1 of Corollary 4.[12] The optimal subsidy scheme consists of a finite number of different subsidies related to the sets $\mathcal{S}_k$ with $k \in \{1, \ldots, K\}$

$$s_k(t) = \bar{f}_k \int_{\underline{j}_k}^{\bar{j}_k} \bar{\psi}^S(j)\, dj, \tag{47}$$

[11] It varies only rarely, that is, a countable number of times.

[12] By combining the disjoint, rectangular set to larger sets, we get the sets described in Case 1.

where $\underline{j}_k$ and $\bar{j}_k$ denote the lower and upper boundary of set k, respectively.

The subsidy for technology i at time t is simply the sum of all the subsidies for all sets $k \in \{1, \ldots, K\}$ that contain technology i:

$$s(i) = \sum_{k=1}^{K} \begin{cases} s_k(t) & \text{if } \exists j \in \mathcal{I} \text{ so that } (j,i) \in \mathcal{S}_k, \\ 0, \text{ otherwise.} \end{cases} \tag{48}$$

This subsidy takes on only a finite number of different values, so that it is a simple policy according to Definition 1.

For Case 2 of Corollary 4, the subsidy is calculated in the same way, only we get instead of (47)

$$s_k(i) = \bar{f}_k \int_{\underline{j}_k(i)}^{\bar{j}_k(i)} \bar{\psi}^S(j)\, dj, \tag{49}$$

where the upper and lower boundaries are now functions of i, so that the set-specific subsidy $s_k(i)$ is also a function of i. As the upper and lower boundaries are continuous functions of i and as there are only finitely many sets K, the subsidy is a.e. continuous. It can thus be approximated by a step-wise function in the sense of Definition 2.

4.3 Example 3: Special cases with specific dynamics

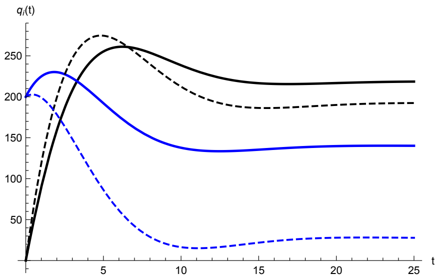
So far, our examples have covered only cases with fairly simple dynamics. As a final example, we consider settings that generate more interesting dynamics. To keep the exposition brief, we constrain ourselves to two technologies, which implies that the examples have to be rather specific to generate complex dynamics.

Consider first a setting with $F_{1,1}, F_{2,2} < 0$ and $F_{1,2} > 0, F_{2,1} < 0$. Thus technology 2 causes a positive spillover on technology 1, which, in turn, has a negative effect on technology 2. As Plot (a) in Figure 2 shows, this setting can cause oscillatory dynamics. However, the decentralised and the social planner's solution still have similar dynamics (albeit the oscillations differ both with regard to amplitude and phase).

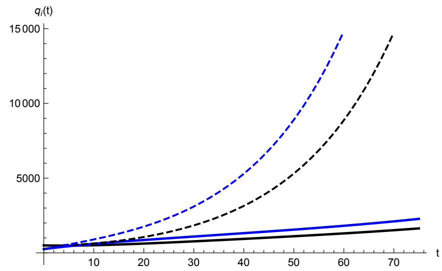
As a final example, we thus consider a case where the matrix $rI - F^T$ does not have full rank. Thus, the social planner's dynamics have a qualitatively different structure than the decentralised dynamics. An example is given in Plot (b) in Figure 2. There, the dynamics in the decentralised system consist of an exponential part, whereas the social planner's dynamics include a linear and an exponential part.

To correct for these differences, a dynamic subsidy is necessary. Note that, in this example, there is no stable steady state, both technologies will have unlimited quality growth.[13] However, the social planner's solution will

[13] This is solely due to the spillovers: Without them the system would converge to a steady state at $q_1(\infty) = 120, q_2(\infty) = 5000/3$.



(a) $F_{1,1} = -1/3$, $F_{2,2} = -1/6$, $F_{1,2} = 13/30$, $F_{2,1} = -13/60$, $\gamma_1 = \gamma_2 = 1$, $\bar{\pi}_1 = 20$, $\bar{\pi}_2 = 10$, $r = 5\%$, $q_1(0) = 0$, $q_2(0) = 200$.



(b) $F_{1,1} = -1/20$, $F_{2,2} = -1/100$, $F_{1,2} = 3/50$, $F_{2,1} = 1/10$, $\gamma_1 = 3/10$, $\gamma_2 = 1$, $\bar{\pi}_1 = 2$, $\bar{\pi}_2 = 1$, $r = 5\%$, $q_1(0) = 500$, $q_2(0) = 250$.

Figure 2: Numerical calculations for example 3. Bold lines depict the social planner's solution, dashed lines the decentralised outcome.

rely on costless spillovers to a much larger extent than the firms, so that growth in the social optimum is slower at the beginning.

5 Conclusions

In this paper we have analysed the effect of cross-technology interactions in r&d on the development of technologies. We have advanced a model that describes interactions among technologies that is considerably richer than existing models. We have shown that the model can cover a wide range of long-run outcomes but is nevertheless structured enough to gain some insights into the effects of cross-technology interactions. In particular, we have provided a number of results that show how the structure of cross-technology interactions influences the dynamics of technological change and characterised settings in which the decentralised dynamics differ qualitatively from the socially optimal ones.

In addition, we have analysed whether a sufficiently simple, and thus potentially implementable, subsidising policy can correct r&d incentives. Again, our results have shown that it is the structure of interactions, not their scale, that determines whether efficient technological change can be achieved by simple means or not.

Our results cover both the case of a finite set and of a continuum of technologies albeit with different foci, as these cases have a strongly differing tractability and different consequences for the feasibility of policies.

The main contribution of this paper is to show that the structure of cross-technology interactions is relevant, if the dynamics of technological change are to be understood and r&d policies are to be designed. The widely used assumption of a common pool of knowledge is too simple and thus misses important aspects of the problems induced by r&d spillovers.

For example, if technologies form a "chain of knowledge", where each technology has only effects on subsequent technologies, different dynamics and policy problems arise than in cases of bi-directional interactions. Similarly, a setting where interactions are homogenous within subgroups of technologies has drastically different policy implications than a setting with more heterogenous interactions. To the best of our knowledge, the structure of cross-technology interactions has so far not been related to the dynamics of technological change or the complexity required for efficient r&d policies.

The trend towards policies aiming to induce broader technology shifts, such as climate and energy policies that aim to replace fossil fuels by renewables, digitisation of whole sectors, or large-scale changes to the mobility system, emphasise that understanding connected technological changes is highly important.

The paper provides a first step in this direction. As the model has deliberately been kept simple to focus the analysis on the effects of cross-technology interactions, it is far from being directly applicable. Rather, it should be seen as an argument that cross-technology interactions have important implications that warrant further study.

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