

## Divulgación

# A Model for Calculation of a Productivity Bonus

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RECIBIDO: 8 de febrero de 2017

ACEPTADO: 25 de enero de 2018

### ABSTRACT

A productivity bonus represents a defrayal that some working places extends to their workers, in order to gratificate them due to the accomplishment of some productive goals or objectives. It depends upon any organization the establishment of some method to calculate this payment and to define on which variables the calculation would be based upon. The objective of this document concerns the raising of a mathematical model that could be deployed towards the computation of this quantity. A mathematical analysis is carried out on top of the basic structure of a given fixed set of an enterprise process performance metrics, or key performance indicators (KPIs). The model takes as inputs the goals and control limits (parameter values of the metrics that are commonly found on many organizations) (Duke, 2013), their value accomplishment result, and a free parameter. A typical real life example is exposed, as a case of study, where it is applied this calculation scheme; as the result, a productivity bonus was successfully calculated, so concluding that this result can be a useful device to carry out this task.

### KEYWORDS AND PHRASES

Productivity Bonus Computation.

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2010 *Mathematics Subject Classification*: 26C15 Rational functions.

<sup>1</sup>This work has been developed for *Programa de Investigacion Continua (PIC), Grupo Villacero*.

## 1. INTRODUCTION

In productive enterprises a productivity bonus represents an earning that an employee acquires as a result of the achievements of some process targets or performance aimings, during a fixed period of time. According to the in force laws and/or agreements and guidelines of some countries, this subject can be regarded as a legal requirement, or by the other hand, a merely extra payment. In both cases this item is given in terms of the achievement of some goals, objectives or any other subject of some enterprise.

It is desirable that this outlay could be determined on a more objective, unbiased and fine-grained basis, in order to ensure this amount of money to be given in a further fair way. According to Suff *et al.*(2007), the state of the art of the payment to any wage earner, nowadays lies in the premise that it should be carried out by means of their performance; on this work it is assumed precisely the same for the productivity bonus. There exist some methods that show how could it be determined on a general way (Suff *et al.*, 2007; Martínez Luna, 2001).

Nevertheless, there is not available information that describes a methodology or formula in order to determine on a fine-grained basis this bonus; this is the main motivation in the development of this research. But on the other hand, there exist a great deal of industry standards alike (ISO 9001, 2015) that deals with process performance metrics (Mäquilä, 2014; Duke, 2013). Many working plants cope with those standards; so what if we deploy them to develop an scheme to compute this quantity.

On this paper, a mathematical model aimed towards the determination of this outlay is developed, where mainly those process performance metrics are taken into account as well as other control variables, to end with a framework that makes the calculation possible. The result of this analysis is a formula whose input elements are precisely those variables; the output represents the actual amount of the productivity bonus in some predefined currency.

Throughout this document the insights of how this model was developed are exposed, as well as its deployability. Firstly, a theoretical framework where a mathematical structure is raised is set up; also a scheme that settles the way the metrics and control variables interact with it. Thereupon the formulated structure is fitted to a frame that fairly fulfill with the portrayal of the factual productivity bonus quantification. Lastly a quintessential instance is bestowed, where the usage of this outcome is exerted.

## 2. MATHEMATICAL FRAMEWORK

### 2.1. Development of a mathematical structure

Let  $\mathbb{Q}$  be the field of rational numbers,  $\mathbb{N}_0$  the field of the naturals and  $\mathbb{R}$  the field of the reals; both with coordinate wise operations of addition and scalar multiplication. So we have that  $\mathbb{N}_0 \subset \mathbb{Q} \subset \mathbb{R}$ ; assign  $F = \mathbb{R}$  and consider the index set  $\Lambda = \{1, 2, 3\}$ . Let the set  $\bigcup_{k \in \Lambda} e_k$  be the standard basis for  $F^3$ ; if  $\beta_k = e_k$ , then the euclidean  $\mathbb{R}^3$  vector space can be represented by:

$$V_{\mathbb{R}^3} = \left\{ \bigoplus_{k \in \Lambda} W_k : W_k = \text{span}(\beta_k); \quad \bigcap_{k \in \Lambda} \beta_k = \{\emptyset\}; \quad \bigcap_{k \in \Lambda} W_k = \{0\} \right\},$$

with the coordinate-wise operations of addition and scalar multiplication, whose metric is represented by its inner product on  $\mathbb{R}^3$ , which on this vector space is the Euclidean definition of length (Friedberg *et al.*, 2003), so define  $\langle \cdot, \cdot \rangle$  to be the norm or length of  $x$  by  $\|\cdot\|$  for some  $x \in V_{\mathbb{R}^3}$ , provided that  $V_{\mathbb{R}^3}$  is an inner product space.

Define the linear operator  $T_1: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as the dilatation or contraction by a factor  $k \in F$  by  $T_1(w) = kw$  for  $w \in W_k$  whose standard matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Since a transformation } T: V \rightarrow V \text{ has a scalar eigenvalue } \lambda$$

if there is a non-zero eigenvector  $\vec{\zeta} \in V$  such that  $T_1(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$ ; for some  $I \in I_n$ , we have (Hefferon, 2017)

$$(\lambda_1 I - A_1)\vec{\zeta} = 0,$$

$$\det(\lambda_1 I - A_1) = \begin{vmatrix} \lambda_1 - k & 0 & 0 \\ 0 & \lambda_1 - k & 0 \\ 0 & 0 & \lambda_1 - k \end{vmatrix} = 0.$$

From that the characteristic polynomial  $(\lambda_1 - k)^3 = 0$  is obtained, whose only solution is  $\lambda_1 = k$ ; hence the eigenspace of  $T_1$  corresponding to the eigenvalue  $\lambda_1$  is:

$$E_{\lambda_1} = N(\lambda_1 I - A_1) = \left\{ \vec{\zeta} \in \mathbb{R}^3: \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{\zeta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The result is the general solution to the system:

$$\vec{\zeta} = r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{for } \{r, s, t\} \in \mathbb{R}.$$

So clearly  $\cup_{k \in \Lambda} e_k \subseteq E_{\lambda_1}$ , hence the standard unitary vectors for  $\mathbb{R}^3$  are eigenvectors of  $T_1$ . Since any vector in  $\mathbb{R}^3$  is a linear combination of  $\beta_k = \{e\}_{k \in \Lambda}$ , any vector in  $W_k \in \mathbb{R}^3$  is an eigenvector of  $T_1$ , hence  $\{R(T_1): T_1(w_k); w_k \in W_k\} \subseteq E_{\lambda_1}$ . Consider the subset:

$$\cup_{k \in \Lambda} \hat{w}_k \in \left\{ W_k: W_k \subseteq R(T_1); \quad T_1(e_k) = \sigma_k e_k; \quad \sigma_k = \begin{cases} z \in \mathbb{N}_0, & \text{if } k = 1 \\ \varphi \in \mathbb{Q}, & \text{if } k = 2 \\ \Phi \in \mathbb{R}, & \text{if } k = 3 \end{cases} \right\}. \quad (1)$$

If we represent this subset geometrically, it would look like:

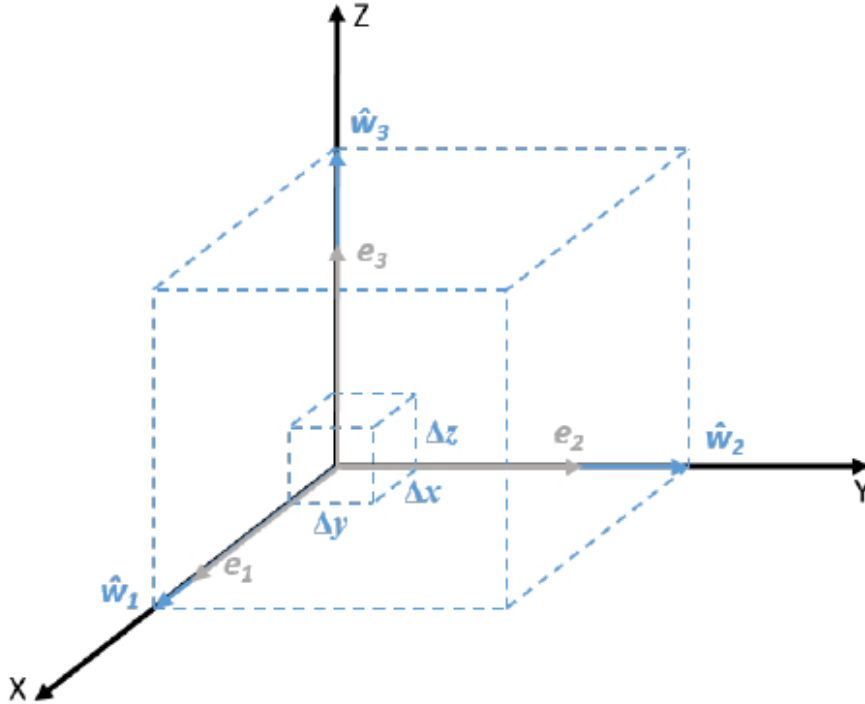


FIGURA 1. Geometrical representation of the subset  $\cup_{k \in \Lambda} \hat{w}_k$ , along with the enclosed volume that it projects.

Now consider from figure 1 the rectangular box  $\widehat{Q}$  that is projected by the subset for every  $1 \leq k \leq 3$  can be seen as a hypersurface in four dimension. Define  $\widehat{Q}$  as a region in space by

$$\widehat{Q} = \{(x, y, z): 0 \leq x \leq \langle \widehat{w}_1 \rangle, 0 \leq y \leq \langle \widehat{w}_2 \rangle, 0 \leq z \leq \langle \widehat{w}_3 \rangle\}.$$

Consider for each  $\{\widehat{Q}\}_{i=1,2,\dots,n}$  the volume  $\Delta V_i$  of  $\widehat{Q}_i$  that could be yielded by  $\Delta x \Delta y \Delta z$ . For some  $\{\widehat{w}_{1_i}, \widehat{w}_{2_i}, \widehat{w}_{3_i}\} \in \widehat{Q}_i$  and any  $f(x, y, z) \in C(\mathbb{R}^3)$  defined on  $\widehat{Q}$ , define the operator  $T_2: C(\mathbb{R}^3) \rightarrow \mathbb{R}$  to be the triple integral of  $f$  over  $\widehat{Q}$  by Smith & Minton (2006).

$$\int \int_{\widehat{Q}} \int f(x, y, z) dV = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f\left(\bigcup_{k \in \Lambda} \widehat{w}_{k_i}\right) \Delta V_i,$$

where the norm  $\|P\|$  represents the longest diagonal of any  $\{\widehat{Q}\}_{i=1,2,\dots,n}$ . What we need is to find the volume of  $\widehat{Q}$  projected on the euclidean space. In order to find it, let  $f(x, y, z) = 1$ , so that  $T_2(f(x, y, z)) = \int \int_{\widehat{Q}} \int 1 dV$ . Now, by the Fubini's theorem we have that

$$\int \int_{\widehat{Q}} \int 1 dV = \int_0^{\langle \widehat{w}_3 \rangle} \int_0^{\langle \widehat{w}_2 \rangle} \int_0^{\langle \widehat{w}_1 \rangle} dx dy dz = \prod_{j \in \Lambda} \langle \widehat{w}_j \rangle. \quad (2)$$

Provided from (1) that

$$\langle \widehat{w}_j \rangle = \begin{cases} \|z \cdot e_1\| = |z|(1) = |z|, & \text{if } j = 1, \\ \|\varphi \cdot e_2\| = |\varphi|(1) = |\varphi|, & \text{if } j = 2, \\ \|\Phi \cdot e_3\| = |\Phi|(1) = |\Phi|, & \text{if } j = 3. \end{cases}$$

Hence the rate of change of the subtended volume of  $\widehat{Q}$  with respect of his components is:

$$\begin{aligned} \left(\frac{\partial V}{\partial x}\right) &= |z| = (1) \cdot |z| \\ \left(\frac{\partial V}{\partial y}\right) &= |\varphi| = (1) \cdot |\varphi| \end{aligned}$$

$$\left(\frac{\partial V}{\partial z}\right) = |\Phi| = (1) \cdot |\Phi|$$

so that  $\widehat{Q}$  expands or contracts by a magnitude equally the scalar product of the norm of the standard basis of each  $\widehat{w}_k$  element component by a scalar defined on (1). The fact that each  $\widehat{w}_k$  is an eigenvector, guarantees that this expansion or contraction on  $\widehat{w}_k$  will be carried out on the same direction of the axis of the unitary vectors that conforms their standard basis, so that  $\widehat{Q}$  will not be distorted by the underlying operations.

However (2) describes the volume of  $\widehat{Q}$  as a whole, it needs to be reformulated on a more fine-grained basis in terms of additional parameters in order to be applicable.

## 2.2. Reformulate $|\varphi|$

Let  $V_{\mathbf{R}^2}$  be the  $\mathbb{R}^2$  euclidean space over  $\mathbb{R}$  with the coordinate-wise operations of addition and scalar multiplication. Define the following subsets on canonical form:

$$S_1 = \left\{ \begin{pmatrix} \phi - 0 \\ \xi - \psi_1 \end{pmatrix} : \phi \in \mathbb{N}; \quad \xi \geq \psi_1; \quad \{\xi, \psi_1\} \in \mathbb{Q}^+ \right\},$$

$$S_2 = \left\{ \begin{pmatrix} \phi - 0 \\ \psi_2 - \psi_1 \end{pmatrix} : \phi \in \mathbb{N}; \quad \psi_2 \geq \psi_1; \quad \{\psi_2, \psi_1\} \in \mathbb{Q}^+ \right\},$$

where  $\psi_1 \leq \xi \leq \psi_2$ , whose entries are restricted to the scalar field indicated in the subsets definition, provided that  $\mathbb{N}_0 \subset \mathbb{Q}^+ \subset \mathbb{R}$ , and  $\{S_1, S_2\} \subseteq V_{\mathbf{R}^2}$ ; those subsets are represented on figure 2. Recall the definition of the  $V_{\mathbf{R}^3}$  vector space on (1), so that the euclidean  $\mathbb{R}^2$  vector space can be expressed by  $V_{\mathbf{R}^2} = W_1 \oplus W_2$ , with  $\beta_2 = \{e_1, e_2\}$  as standard basis.

Define the operator  $T_3: V_{\mathbf{R}^2} \rightarrow V_{\mathbf{R}^2}$  to be the projection on  $W_2$  along  $W_1$  by  $T_2(x) = x_2$ , if for  $x = x_1 + x_2$  with  $x_1 \in W_1$  and  $x_2 \in W_2$ , whose standard matrix is  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .

For nonzero vectors  $\{a, b\} \in \mathbb{R}^n$ , if  $\theta$  is the angle between  $a$  and  $b$  ( $0 \leq \theta \leq \pi$ ), then by Friedberg *et al.* (2003)

$$\|a \times b\| = \|a\| \|b\| \sin \theta$$

and two nonzero vectors  $\{a, b\} \in \mathbb{R}^n$  are parallel if and only if  $a \times b = 0$ . Now,

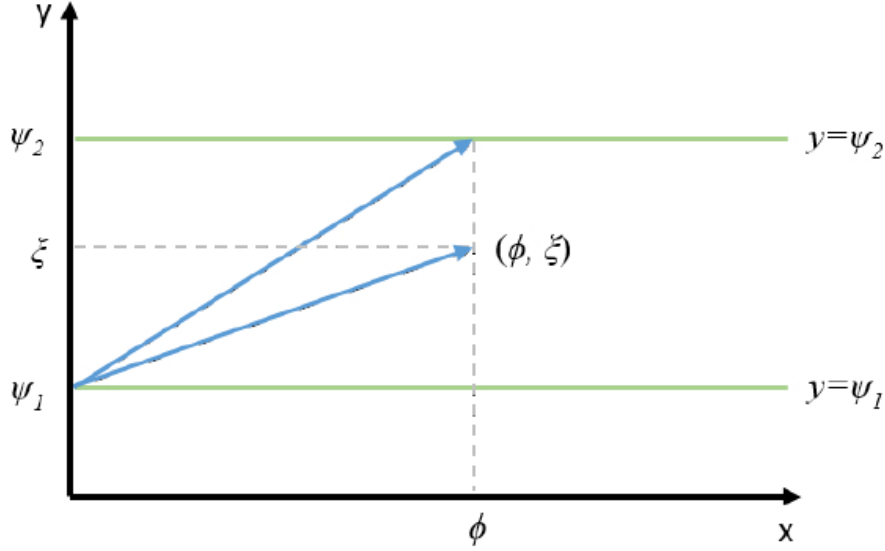


FIGURA 2. Geometrical representation of  $S_1$  and  $S_2$ .

for some  $\{u, v\} \in V_{\mathbf{R}^2}$ , apply  $T_3(u) = \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \hat{u}$  and  $T_3(v) = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \hat{v}$ . Their cross product would be yielded by

$$\begin{aligned} \hat{u} \times \hat{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j}) \times (v_1 \mathbf{i} + v_2 \mathbf{j}) \\ &= u_1 v_1 (\mathbf{i} \times \mathbf{i}) + u_1 v_2 (\mathbf{i} \times \mathbf{j}) \\ &\quad + u_2 v_1 (\mathbf{j} \times \mathbf{i}) + u_2 v_2 (\mathbf{j} \times \mathbf{j}) \\ &= 0. \end{aligned}$$

Clearly  $\hat{u}$  and  $\hat{v}$  are parallel vectors. On a vector space, which is a  $k$ -vector space, by definition for parallel vectors (AlexR, 2013)

$$u \parallel v \Leftrightarrow \exists \lambda \in K: \lambda \cdot u = v,$$

so that if  $\lambda \neq 0$ , one is a scalar multiple of each other. Hence the range  $R(T_3) \subseteq V_{\mathbf{R}^2}$  represent a subset of parallel vectors that are scalar multiple of each other. Aside, since  $V_{\mathbf{R}^2} \ni \{u, v\} \subseteq \text{span}(\beta_2)$ , they are scalar multiple of  $e_2$  also.

Applying the transformation for some  $u \in S_1$  and  $v \in S_2$ , we would have

that

$$T_3(u) = \begin{pmatrix} 0 \\ \xi - \psi_1 \end{pmatrix} = \text{proy}_y S_1,$$

$$T_3(v) = \begin{pmatrix} 0 \\ \psi_2 - \psi_1 \end{pmatrix} = \text{proy}_y S_2.$$

In terms of the previously established definition of  $S_1$  and  $S_2$ , the following possibilities may apply:

$$\begin{aligned} (\psi_2 < 1; \quad \xi < \psi_2) &\implies (\|\text{proy}_y S_2\| > \|\text{proy}_y S_1\|) < \|e_2\|, \\ (\psi_2 < 1; \quad \xi = \psi_2) &\implies (\|\text{proy}_y S_2\| = \|\text{proy}_y S_1\|) < \|e_2\|, \\ (\psi_2 = 1; \quad \xi = \psi_2) &\implies (\|\text{proy}_y S_2\| = \|\text{proy}_y S_1\|) = \|e_2\|, \\ (\psi_2 > 1; \quad \xi < \psi_2) &\implies (\|\text{proy}_y S_2\| > \|\text{proy}_y S_1\|) > \|e_2\|, \\ (\psi_2 > 1; \quad \xi < \psi_2) &\implies (\|\text{proy}_y S_2\| > \|e_2\|) > \|\text{proy}_y S_1\|, \end{aligned}$$

provided no restriction on  $\|e_2\|$  with respect to the other eigenvectors (according to the definition of  $S_1$  and  $S_2$ ); figure 3 shows an scheme in about what the above is based upon. Since  $\psi_1 \leq \xi \leq \psi_2$ , we have so far that always  $\|\text{proy}_y S_2\| > \|\text{proy}_y S_1\|$  hold.

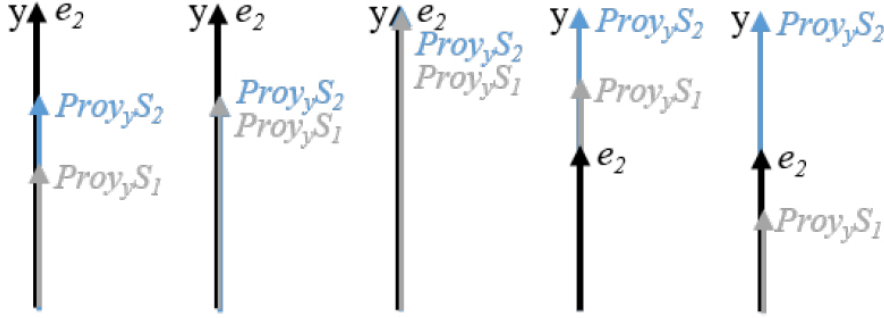


FIGURA 3. Possible arrangements of  $\text{proy}_y S_2$ ,  $\text{proy}_y S_1$  and  $e_2$  against each other.

In each case it can be seen the position of  $\text{proy}_y S_1$  with respect to  $\text{proy}_y S_2$ ; however if we stretched or contracted  $\text{proy}_y S_2$  towards the unit vector of the standard basis  $e_2$ , at the same time it is done also with  $\text{proy}_y S_1$ , then  $\text{proy}_y S_1$  would hold a position with respect both to  $\text{proy}_y S_2$  and to the unitary vector. Then within this transformation,  $\text{proy}_y S_1$  and  $\text{proy}_y S_2$  could be towards the representation of  $|\varphi|$ .



Recall the operator of dilatation or contraction now  $T_4: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , whose standard matrix is  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . It can be demonstrated that its corresponding eigenspace is:

$$E_{\lambda_2} = N(\lambda_2 I - A_4) = \left\{ \vec{\zeta} \in \mathbb{R}^2: \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{\zeta} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\},$$

with general solution:

$$\vec{\zeta} = r \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{for } \{r, s\} \in \mathbb{R}.$$

Now let  $T_4(\text{proy}_y S_2) = \varsigma \cdot \text{proy}_y S_2$  denote the dilatation or contraction of  $\text{proy}_y S_2$  towards the unitary vector. The only possibility is that  $\varsigma = \left[ \frac{1}{\psi_2 - \psi_1} \right]$ ; so in terms of matrix representation, we have:

$$T_4(\text{proy}_y S_2) = \begin{bmatrix} \varsigma & 0 \\ 0 & \varsigma \end{bmatrix} \begin{bmatrix} 0 \\ \psi_2 - \psi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2.$$

Since after the transformation  $\text{proy}_y S_2$  has changed by a scalar multiple of  $\varsigma$ , now  $\text{proy}_y S_1$  must be applied the same transformation in order to hold its position with respect to  $\text{proy}_y S_2$ :

$$T_4(\text{proy}_y S_1) = \begin{bmatrix} \varsigma & 0 \\ 0 & \varsigma \end{bmatrix} \begin{bmatrix} 0 \\ \xi - \psi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\xi - \psi_1}{\psi_2 - \psi_1} \end{bmatrix}.$$

Now after the transformation  $\text{proy}_y S_1$  holds its position against  $\text{proy}_y S_2$ , but this last one is already a unitary vector, hence  $\begin{bmatrix} 0 \\ \frac{\xi - \psi_1}{\psi_2 - \psi_1} \end{bmatrix}$  holds also a position against the unitary vector as required. Cause clearly  $\{R(T_4), e_2\} \subseteq E_{\lambda_1}$ , so they are eigenvectors of  $T_4$ , by that those set of vectors are coplanars, and after the transformation will not change its original direction, as required.

Since  $E_{\lambda_2} \subseteq V_{\mathbf{R}^2}$ , the metric of this eigenvector is represented by its inner product on  $V_{\mathbf{R}^2}$ , which is also the euclidean definition of length (Friedberg

et al., 2003). Let

$$S_3 = \left\{ \begin{pmatrix} 0 \\ \frac{\xi - \psi_1}{\psi_2 - \psi_1} \end{pmatrix} : \psi_1 \leq \xi \leq \psi_2; \quad \{\xi, \psi_k\} \in \mathbb{Q} \right\} \subseteq V_{\mathbf{R}^2},$$

and  $\langle \cdot, \cdot \rangle$  be the norm or length of  $x$  by  $\|\cdot\|$  for some  $x \in V_{\mathbf{R}^2}$ , provided that  $V_{\mathbf{R}^2}$  is also an inner product space. For some  $s_3 \in S_3$ , we have:

$$\|s_3\| = \left\| \begin{pmatrix} 0 \\ \frac{\xi - \psi_1}{\psi_2 - \psi_1} \end{pmatrix} \right\| = \left\langle \begin{pmatrix} 0 \\ \frac{\xi - \psi_1}{\psi_2 - \psi_1} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{\xi - \psi_1}{\psi_2 - \psi_1} \end{pmatrix} \right\rangle^{1/2} = \left| \frac{\xi - \psi_1}{\psi_2 - \psi_1} \right|.$$

Now let

$$|\varphi| := \left| \frac{\xi - \psi_1}{\psi_2 - \psi_2} \right|. \quad (3)$$

### 2.3. Reformulate $|\Phi|, |z|$

Like what was done before, it could be useful to have this vector norm represented by means of another richer structure. From (1) we have that  $|\Phi|$  is an element of the reals; let break it down in terms of other elements from that field, define:

$$|\Phi| := |\widehat{\Phi}| = \begin{cases} B_1, & \text{if } C_1 \geq U_1 \\ B_2, & \text{if } U_1 \leq C_2 \leq U_2 \\ \cdot & \dots \dots \\ B_n, & \text{if } U_{n-1} \leq C_n \leq U_n \end{cases} \quad (4)$$

for some fixed  $\{B_k, C_k, U_k\} \in \mathbb{R}$

Next,  $|z|$  can be acoted to a more reduced set of elements of the scalar field, in order to represent a condition like a go-not go scheme. So, define:

$$|z| := |\widehat{z}| = \{\{0, 1\}: \{0, 1\} \in \mathbb{N}_0\}. \quad (5)$$

The subtended volume on the cube  $\widehat{Q}$  can be represented as a product of the norms of the vector  $\varphi$  previously redefined and by those last two, also redefined.

### 3. PRODUCTIVITY BONUS CALCULATION

#### 3.1. Productivity bonus function definition

Recall the definition from (1) and consider some  $j$  subset  $\cup_{k \in \Lambda} \hat{w}|_k^j$ ; then define the operator  $T_5: \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $\cup_{\substack{k \in \Lambda \\ j \in n}} \{\hat{w}|_k^j\}_{\{j\}} \ni (\mu) \mapsto T_5(\mu)$  taking the result previously yielded by (2), and for  $1 \leq j \leq n$ , by

$$T_5(\mu) = \prod_{k \in \Lambda} \hat{w}|_k^j$$

so that the set of all images of every  $\mu$  of the subset, under  $T_5$  would be

$$R(T_5) = \left\{ T_5(\mu) : \mu \in \bigcup_{\substack{k \in \Lambda \\ j \in n}} \{\hat{w}|_k^j\}_{\{j\}} \subseteq \mathbb{R}^3 \right\}.$$

Let  $A_f: \cup_{n=1}^{\infty} I^n \rightarrow I$  be the quasi-arithmetic mean, or  $f$ -mean function, defined in Hardy *et al.* (1934 or 1952) by the following equation:

$$A_f(x_1, \dots, x_n) := f^{-1} \left( \frac{f(x_1) + \dots + f(x_n)}{n} \right),$$

( $n \in \mathbb{N}$ ;  $\{x_1, \dots, x_n\} \in I$ ;  $I \subseteq \mathbb{R}$ ),

where the mean  $A_f$  is the quasi-arithmetic mean generated by  $f$ . Let  $I = \mathbb{R}$  and  $f(x) = x$ , then  $A_f$  corresponds to the arithmetic mean:

$$A_f(x_1, \dots, x_n) = \left( \frac{1}{n} \right) \sum_{j=1}^n x_j. \tag{6}$$

Since there are at most  $j = n$  3-tuples in  $\cup_{\substack{k \in \Lambda \\ j \in n}} \{\hat{w}|_k^j\}_{\{j\}}$  to be mapped by the operator  $T_5: \mathbb{R}^3 \rightarrow \mathbb{R}$ , clearly  $\dim(R(T_5)) = n$ . So now consider the set  $\hat{\alpha} = \{\alpha_1 \cup \dots \cup \alpha_n : \alpha \in R(T_5)\}$ , applying (6) to it, for all the  $n$  elements, it yields:

$$A_f(\hat{\alpha}) = A_f \left( \prod_{k \in \Lambda} \{\hat{w}|_k^1\}_{\{1\}}, \dots, \prod_{k \in \Lambda} \{\hat{w}|_k^n\}_{\{n\}} \right) = \left( \frac{1}{n} \right) \sum_{j=1}^n \{|z| \cdot |\varphi| \cdot |\Phi|\}_j. \tag{7}$$

From the context of what each term described above represent, subsequently take its positive value:  $|z| = +z$ ,  $|\varphi| = +\varphi$ ,  $|\Phi| = +\Phi$ .

Finally define the production bonus function  $P$  to be the mean of all the elements in  $\hat{\alpha}$ , as yielded by (7). Substituting each element  $z$ ,  $\varphi$ ,  $\Phi$  by their definition from (3), (4) and (5), it would be:

$$P := \left(\frac{1}{n}\right) \sum_{j=1}^n \hat{z}_j \left[ \frac{\xi_j - LCX_j}{M_j - LCX_j} \right] \hat{\Phi}_j, \quad (8)$$

where a possible representation of each term could be as follows:

$M$ : Goal or target to reach or look for.

$\xi$ : Some indicator about the level with respect to the accomplishment for  $M$ .

$LCX$ : Any control limit established as reference or as a base with respect to  $M$ :  $LCI$  = (lower bound control limit),  $LCS$  = (upper bound control limit).

$\hat{\Phi}$ : Maximum amount of money that could be delivered given in some fixed unit.

$\hat{z}$ : Indicator that enables or disables any summand upon some fixed criterion.

$n$ : Total number of the variables described above, individually established.

### 3.2. Application example

It is presented a possible situation as an example of application of this framework. Certain scenario will be laid down below where what will be needed in terms of mathematics, is to determine the position value of certain quantity (in this situation a metric result obtained from the process  $\xi$ ) with respect to a set of two references: a zero or ground value ( $LCX$  in this case) and a top (but not maximum) value ( $M$  or Goal in this situation); this is, to express  $\xi$  in a scale from  $LCX$  to  $M$ . Thus, the product of this position value just described, by an amount of money would yield what it will be needed: the productivity bonus; this in accordance to what was stated in section 1. Here is where it comes to the aid the formalism developed on sections 2 and 3, and we will see that its usage will come up in a natural way; however, beforehand we know that the model gives room to more than one set of metrics ( $LXC$ ,  $M$ ,  $\xi$ ) to consider. Now the example follows.

At some enterprise, in august the production departments of piping, finishing and varnishing, had recorded accomplishments versus production program about 97%, 92% and 88% respectively. The department of shipments had recorded an accomplishment of 80% versus the shipment program, that

yields 5 000 ton of product shipments for that month. The department of control production, points out that the respectively efficiency according to what was measured for those departments was: 88%, 86% and 90%. In that month it was reported from one client a claim for 10 ton of 1” ASTM A53A pipe (astm, 2012), due to poor workmanship on its varnished surface.

**Process performance measurement framework**

Metric name	Graphic representation	* Formula	Goal	LCI	LCS
Production goal	%Accomplishment prod. planification	Manufacturing output / Planned production	98.0%	90.0%	-
Customer claiming	%Claiming	Faulty workmanship claims / Invoiced material	0.0%	-	1.2%
Prod. efficiency	%Efficiency	Prod. Effective time / Available time	90.0%	70.0%	-

\* Variables are referenced as amount.

FIGURA 4. Definition of deployed process metrics.

Based upon key performance indicators (KPIs), for each department there were defined a set of performance metrics as well as a set of control parameters in order to assign each one the amount of items, and also to point out which one they belong to; those are here represented on a spreadsheet in figures 4 and 5, respectively.

Indicator	i	Department							
		Parameter	Manu- facturing	Shipment	Mech. Maint.	Elec. Maint.	Quality	Other services	
		n ->	z	z	z	z	z	z	
Production goal	1	z <sub>1</sub>   ξ <sub>1</sub>	4	3	2	2	4	3	
Customer claiming	2	z <sub>2</sub>   ξ <sub>2</sub>	1	1	0	0	1	0	
Shipment goal	3	z <sub>3</sub>   ξ <sub>3</sub>	0	1	0	0	1	0	
Prod. efficiency	4	z <sub>4</sub>   ξ <sub>4</sub>	1	0	0	0	0	1	
Mech. Maint. effic.	5	z <sub>5</sub>   ξ <sub>5</sub>	0	0	0	1	0	0	
Elec. Maint. effic.	6	z <sub>6</sub>   ξ <sub>6</sub>	0	0	1	0	0	0	
Work attendancy	7	z <sub>7</sub>   ξ <sub>7</sub>	1	1	1	1	1	1	

FIGURA 5. List of parameters and metrics that belong to each department.

Figures 6 to 8 show the corresponding graphics that display the value for each metric defined on figure 4; this is, the performance of the processes measured and calculated within their definitions just quoted above throughout the year. On figures 6 to 8 it can be seen two horizontal lines

across the process graphics, a dashed red one and a dashed blue one, those represents  $M$  and  $LCX$ , respectively, while the square dots from the green line represents the real values obtained from the process measurement  $\xi$  for each month. Those values will serve as our inputs for this model.

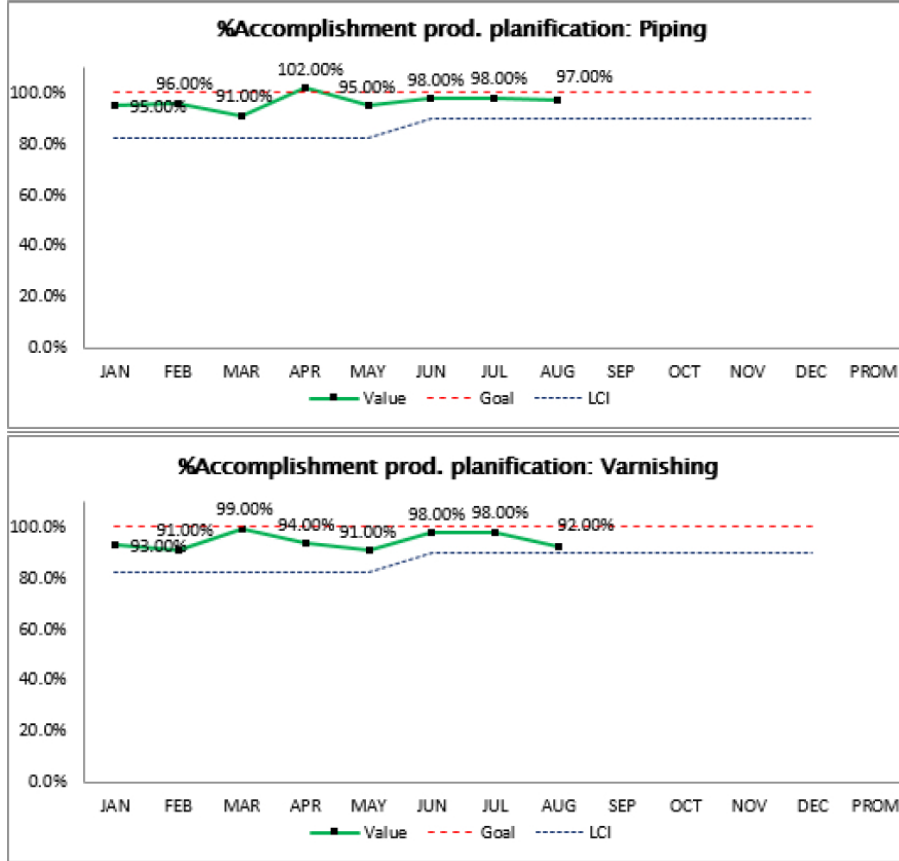


FIGURA 6. Accomplishment of production planification metric for piping and varnishing departments, respectively.

For all departments, for each  $1 \leq i \leq n$ , it was defined  $\hat{\Phi}_i = \hat{\Phi}$  to be the maximum amount of money that could be delivered for every worker, as follows:

$$\hat{\Phi} = \begin{cases} \$150, & \text{for } C > 6\,000 \text{ ton,} \\ \$100, & \text{for } 6\,000 \text{ ton} \leq C \leq 4\,500 \text{ ton,} \\ \$80, & \text{for } C < 4\,500 \text{ ton,} \end{cases}$$

where  $C$  represents the amount of shipments in any month.

According with the information aforementioned and figure 5, calculate the productivity bonus for every people within the situations shown below:

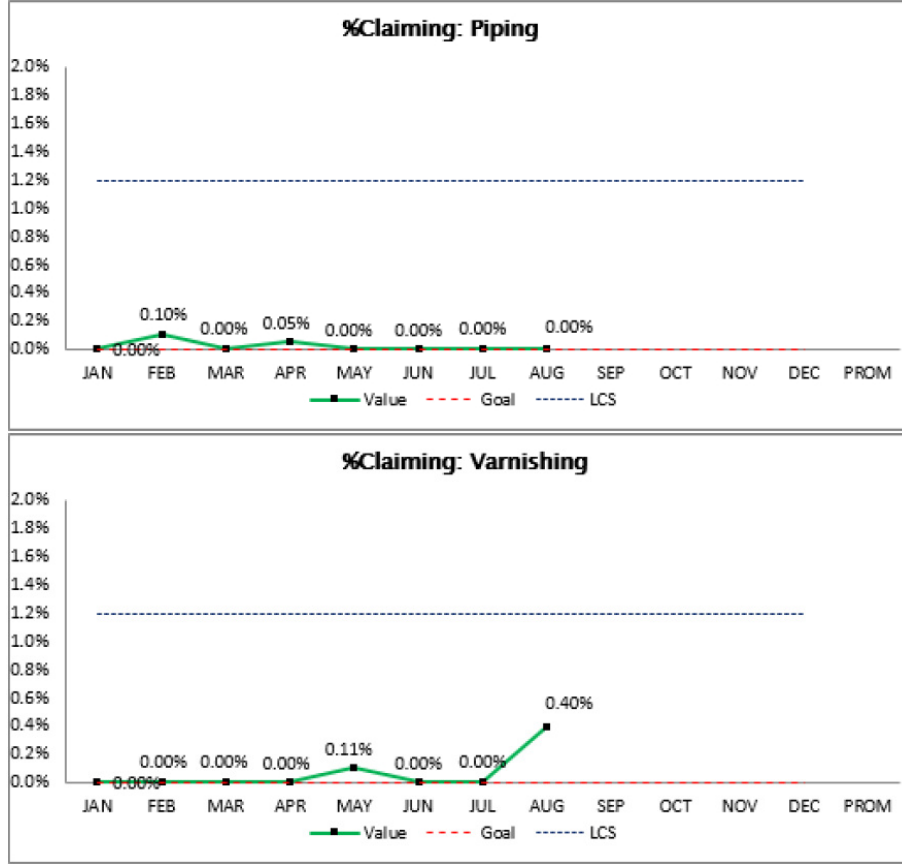


FIGURA 7. Efficiency metric for piping and varnishing departments, respectively.

1. Personnel from process piping department who attended to work every-day on that month.

In this case we have from figure 5:  $z_1 = 1, z_2 = 1, z_3 = 0, z_4 = 1, z_5 = 0, z_6 = 0, z_7 = 1$ , so that  $n = (z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7) = 4$ . Next, from the records proclaimed on august, we have:  $\xi_1 = 0.97, \xi_2 = 0, \xi_3 = 0.80, \xi_4 = 0.88, \xi_5 = 0, \xi_6 = 0, \xi_7 = 0$ . Also, from figures 6 to 8 we have that:  $M_1 = 0.98, LCI_1 = 0.90; M_2 = 0, LCS_2 = 0.012; M_3 = ND, LCI_3 = 0.50; M_4 = 0.90, LC_4 = 0.70; M_5 = ND, LCI_5 = ND; M_6 = ND, LCI_6 = ND$ .

About what work attendancy respects, there is no any specific indicator previously stated, however, by the fact that there is expected to have no absenteeism, we let  $M_7 = 0$ ; on the other hand it is known that for most jobs only 3 days absences are endured, so we let  $LCI_7 = 3$ .

We have to apply (8); throwing away the zero-product terms, calculation

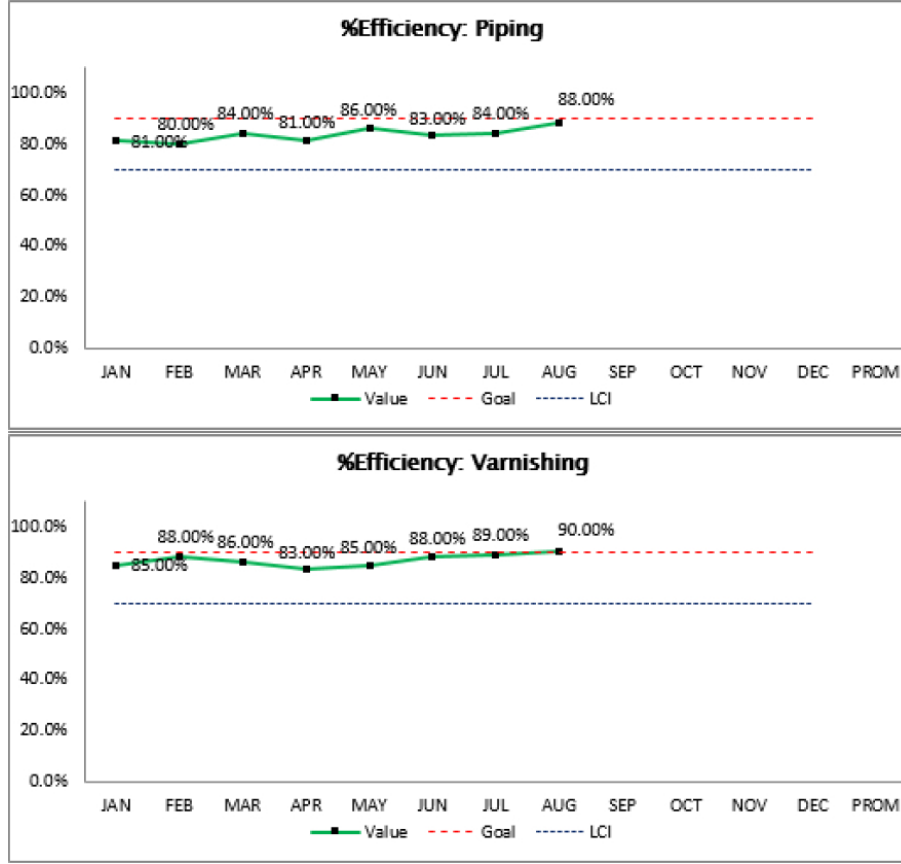


FIGURA 8. Claiming metric for piping and varnishing departments, respectively.

would be as follows:

$$P = \left(\frac{1}{4}\right) \left[ \hat{z}_1 \left( \frac{0.97 - LCI_1}{M_1 - LCI_1} \right) + \hat{z}_2 \left( \frac{0 - LCI_2}{M_2 - LCI_2} \right) + \hat{z}_4 \left( \frac{0.88 - LCI_4}{M_4 - LCI_4} \right) + \hat{z}_7 \left( \frac{0 - LCI_7}{M_7 - LCI_7} \right) \right] \cdot \$100 = \$94.38.$$

**2.** Personnel from process varnishing department who attended to work everyday on that month.

From figure 5 we have:  $z_1 = 1, z_2 = 1, z_3 = 0, z_4 = 1, z_5 = 0, z_6 = 0, z_7 = 1$ , so that  $n = (z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7) = 4$ . From the records proclaimed on august:  $\xi_1 = 0.92, \xi_2 = 0.008, \xi_3 = 0, \xi_4 = 0.90, \xi_5 = 0, \xi_6 = 0, \xi_7 = 0$ . Also from figures 6 to 8:  $M_1 = 0.98, LCI_1 = 0.90; M_2 = 0,$



$LCS_2 = 0.012$ ;  $M_3 = 0.80$ ,  $LCI_3 = 0.50$ ;  $M_4 = 0.90$ ,  $LCI_4 = 0.70$ ;  $M_5 = 0$ ,  $LCI_5 = 0.90$ ;  $M_6 = 0$ ,  $LCI_6 = 0.90$ ;  $M_7 = 0$ ,  $LCI_7 = 3$ , conjointly the same work attendancy criteria.

Applying (8), calculation would be as follows:

$$P = \left(\frac{1}{4}\right) \left[ \hat{z}_1 \left( \frac{0.92 - LCI_1}{M_1 - LCI_1} \right) + \hat{z}_2 \left( \frac{0.008 - LCI_2}{M_2 - LCI_2} \right) + \hat{z}_4 \left( \frac{0.90 - LCI_4}{M_4 - LCI_4} \right) + \hat{z}_7 \left( \frac{0 - LCI_7}{M_7 - LCI_7} \right) \right] \cdot \$100 = \$52.08.$$

Along these examples there could be raised many other situations that could be addressed using this model.

#### 4. CONCLUSION

The mathematical model here envisaged from the ground up, was successfully applied to a real case situation, were a typical scenario on a productive plant was presented. It was proved the fact that it is possible the use of process indicator metrics to calculate the productivity bonus, by the use of this model. As it has been exposed the model requires, first of all, to define a set of specific metric indicators; from those, the variables:  $LCX$  (limit control) and  $M$  (goal), are taken as input parameters. The actual achievement value results of the indicators  $\xi$  came from the actual measure of the process performance during a period of time (a month in this example). As is was shown, it is required to set a free parameter  $\hat{\Phi}$  (amount of money), that represents a fixed quantity that has to be established by any organization. This model furthermore could be set to take as inputs another kind of metrics from different items such as work attendancy as stated here; however there could be some other entries such as sales, safety, etcetera.

Although the theoretical framework here developed was issued to deal with the affair of the bonus calculation, it could as well be applied to some other approaches, such as statistical process control (SPC) and quality engineering, alike Bersimis *et al.* (2005) and Evans (1991) where there raises the need of certain calculations in terms of a ground or zero value ( $LCX$ ) and top or maximum value ( $M$ ); however, this fact is not treated in this work, but it is left as a possible extent.

## 5. ACKNOWLEDGEMENT

The author widely render thanks to the referees who reviewed and gave advice towards the improvement of this document.

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