Working Paper No. 312

A Macro Model of Heterogeneous Growth II: An Existence Proof of Transitional Dynamics

Akitaka Dohtani

March 2018

UNIVERSITY OF TOYAMA 富山大学

FACULTY OF ECONOMICS
UNIVERSITY OF TOYAMA

Title: A Macro Model of Heterogeneous Growth Ⅱ: An Existence Proof of Transitional Dynamics

Author: Akitaka Dohtani

Date: March 2018

Affiliation: Faculty of Economics, University of Toyama, 3190 Gofuku, Toyama, 930, JAPAN.

e-mail address: doutani@eco.u-toyama.ac.jp

Abstract: In order to makes clear several important sources from which structural transitions occur, Dohtani (2018) constructed an endogenous growth model. However, per capita growth rates of the optimal paths obtained from the model is independent of the initial levels of macroeconomic variables. In other words, the model does not possesses transitional dynamics. This result is inconsistent with the well-known empirical evidence on convergence. See Barro and Sala-i-Martin (Chapters 11 and 12, 1995). In the present paper, by incorporating adjustment costs for investment into the model of Dohtani (2018), we will prove that a modified version yields transitional dynamics.

Keywords: Endogenous Growth; Transitional Dynamics; Investment; Adjustment Costs.

JEL classification: C61, L16; O40; O41

1. Introduction

Dohtani (2018) constructed an endogenous growth model that makes clear several important sources from which structural transitions occur. However, the endogenous growth model of Dohtani (2018) lacks any transitional dynamics. That is, per capita growth rates of the optimal paths obtained from the model is independent of the initial levels of macroeconomic variables. This result is inconsistent with the well-known empirical evidence on convergence. See Barro and Sala-i-Martin (Chapters 11 and 12, 1995). In the present paper, we will prove that a modified version of the growth model of Dohtani (2018) yields transitional dynamics.))

We here briefly explain the method of proving the occurrence of transitional dynamics. Although the AK model lacks transitional dynamics, it has been some extended versions of the AK model possess transitional dynamics. One method is to modify the production function. Another natural method is to incorporate adjustment costs for investment. Investment requires adjustment costs. Eisner and Strotz (1963) tried to derive optimal paths in the growth model with adjustment costs that is given as a monotonously increasing function of investment. The realization of the importance of the adjustment costs in studying economic growth started with this study. It is, however, natural to assume that adjustment costs depend also on capital stock: Adjustment costs $= V(I, K)$. Adjustment costs functions of this type are given by Uzawa (1968) and Barro and Sala-i-Martin (1995). Uzawa considered a model into which Penrose effect is incorporated. The Penrose effect implies that not all of investment is established as capital stock and the rate of establishment increases as investment increases. On the other hand, by incorporating adjustment costs that is required to set up capital stock, Barro and Sala-i-Martin (1995) derived transitional dynamics. By the same approach as of Barro and Sala-i-Martin (1995), we will derive transitional dynamics.

2. Background of the Models

The background of the models is almost the same as Dohtani (2018) and essentially Barro and Sala-i-Martin (1992), So we will briefly explain it.

The models consist of a representative household and a representative investment-goods industry, and a representative consumption-goods industry. Although Dohtani (2018) considered the case where the number of consumption-goods industries is n (≥ 1), for simplicity we here suppose $n=1$. It is not difficult to generalize the model of $n=1$ to that of $n>1$. Such a generalization, however, requires complicated calculations. For simplicity, we assume that the household owns the initial endowment of capital stock which can be used by any industry. The household distributes the endowment to all industries. Capital goods owned by the household are lent to the investment-goods sector. Without loss of generality, we assume that the depreciation rate of capital stock is zero. The consumption-goods industries rent capital goods from the investment-goods industry. The household has a claim on the consumption-goods sector's net cash flow. There is a competitive credit market in which the household can borrow and lend. To rule out Ponzi-game finance, we assume the credit market imposes a constraint on the amount of borrowing. The two forms of assets, capital and loans are assumed to be perfect substitutes as stores of value. Then, they must pay the same real rate of return, and the interest rate on debt must be equal to the rental rate on capital.

The symbols used in this paper are:

$K_0 =$ initial endowment of capital stock (given),
$C =$ consumption of the goods produced by the consumption-goods industry,
$s =$ rate of time preference (constant),
$Q =$ quantity produced by the consumption-goods industry.
$K =$ capital stock of the consumption-goods industry,
$\Pi =$ profit of the consumption-goods industry,
$K_I =$ capital stock of the representative investment-goods industry,
$r =$ interest rate $=$ rental rate on capital (constant),
$P =$ price of the goods produced by the consumption-goods industry.
$P_K =$ the rental price of capital stock (normalized) $= 1$.

We denote by $\bullet_t$ the value of $\bullet$ at time t. For example, we denote by K_t the value of capital stock of the consumption-goods industry at time t.

3. Brief Explanation of the Original Model without Transitional Dynamics

We first briefly explain an endogenous growth model of Dohtani (2018). We first consider the investment-goods sector. The investment-goods sector is assumed to be perfectly competitive. We assume that the production function of the representative industry in the sector is of the AK type. The industry solves the optimization problem:

$$\max\ (\sigma K_I - rK_I),$$

where σ is a positive constant. For the AK model, see Rebelo (1991) and Barro and Sala-i-Martin (Ch.4, 1995). The condition for profit maximization requires that the marginal product of capital equals r. That is, $r = \sigma$.

We next consider the represent household. The representative household is assumed to solve the following dynamic optimization problem:

$$\max \int_{R_+^1} U(C)e^{-st}dt \quad \text{subject to} \quad \dot{K}_I = \pi + rK_I - PC,$$

where R_+^1 is the set of non-negative real numbers. In the present paper, we assume that the utility function of the representative household is given as

$$U(C) = C^a / a,$$

where $0 < a < 1$, and U represents utility which is obtained by consuming the goods produced by the consumption-goods industry.

The first order condition of the above optimization problem is given by

(1.1) $$H_C = C^{a-1}e^{-st} - P\eta = 0,$$

(1.2) $$\dot{\eta} = -H_{K_I} = -r\eta,$$

(1.3) $$\dot{K}_I = \pi + rK_I - PC,$$

(1.4) $$\lim_{t\to\infty} K_I(t)\eta(t) = 0.$$

Equation (1.1) yields $P\eta = e^{-st}C^{a-1}$. Equation (1.2) yields $\eta = \eta_0 e^{-rt}$, where η_0 is determined later. Then we have

(2) $$P = \frac{\eta_0 e^{(r-s)t}}{C^{1-a}}.$$

Equation (2) gives a dynamic version of static inverse demand equation. Following Dohtani (2018), Equation (2) is called the dynamic inverse demand equation.

Finally, we consider the consumption-goods sector. Although Dohtani (2018) considered more-than-one consumption-goods industries, for simplicity we consider the case where the number of consumption-goods industry is only one. The production function of the industry is assumed to be

$$Q = \zeta K^m, \quad 0 < m \leq 1, \quad 0 < \zeta.$$

Without loss of generality, we assume $\zeta = 1$. We consider the situation where the consumption-goods market is cleared, so that we have $C = Q = K^m$. Then, substituting this equation into the dynamic inverse demand equation (2) yields

$$\text{(3)} \qquad P = \frac{e^{(r-s)t}}{\eta_0 K^{(1-a)m}} \equiv D(K;t).$$

The consumption-goods industry solves the following static optimization problem under the dynamic inverse demand equation:

$$\text{(4)} \qquad \max(PK^m - K) \quad \text{subject to} \quad P = D(K;t).$$

This model generates endogenous growth. Dohtani (2018) constructed an extended version of the model and derived an optimal growth path of the extended version. The optimal path makes clear several important sources from which structural transitions occur. However, the endogenous growth model of Dohtani (2018) lacks any transitional dynamics. In the next section, by modifying the above-mentioned model, we will prove the occurrence of transitional dynamics.

4. Transitional Dynamics and Adjustment Costs for Investment

Except for the optimization problem of the consumption-goods industry, we consider the same model as that of Section 3. We work under the following assumptions:

Assumption 1: $1 > am$;
Assumption 2: $r > \max\{s, \ (r-s)/(1-am)\}$.

Assumptions 1 and 2 yield

$$G \equiv \frac{r-s}{1-am} > 0\,.$$

We here define $gr(\bullet) =$ the growth rate of $\bullet$.

In the following, we employ the adjustment costs function of Barro and Sala-i-Martin (Ch.3.5, 1995):

$$V(I, K) = I[1 + \phi(I/K)]\,.$$

We here assume the following:

Assumption 3: $\phi(0) = 0$;
Assumption 4: $\phi'(x) > 0$ for any $x > 0$;
Assumption 5: $\phi''(x) \geq 0$ for any $x > 0$.

Using this adjustment function, we rewrite the optimization problem (4) by the following intertemporal optimization problem of the consumption-goods industry as follows:

(5) $$\max \int_{R_+^1} \{PK^m - I(1 + bI/K)\}e^{-rt}dt\,,$$

subject to $\dot{K} = I$, $\dot{P} = P\{r - s - m(1-a)I/K\}$.

The Hamiltonian of the optimization problem (5) is given by

(6) $$H = \{PK^m - I(1 + bI/K)\}e^{-rt} + \lambda I + \mu P\{r - s - m(1-a)I/K\}$$

Since the Hamiltonian is a concave function of the state and the control variables, the sufficient condition for optimization given by

(7.1) $$H_I = -(1 + 2bI/K)e^{-rt} + \lambda - \mu m(1-a)P/K = 0\,,$$
(7.2) $$\dot{\lambda} = -H_K = -(mPK^{m-1} + bI^2/K^2)e^{-rt} + \mu m(1-a)PI/K^2\,,$$
(7.3) $$\dot{\mu} = -H_P = -\mu\{r - s - m(1-a)I/K\} - K^m e^{-rt}\,,$$
(7.4) $$\dot{K} = I\,,$$
(7.5) $$\dot{P} = P\{r - s - m(1-a)I/K\}\,,$$

(7.6) $\lim_{t\to\infty} K(t)\lambda(t) = 0$,

(7.7) $\lim_{t\to\infty} P(t)\mu(t) = 0$.

We now derive optimal paths that satisfy (7). Time derivative of (7.1) yields

(8) $$\dot{\lambda} = 2bI(\dot{I}/I - \dot{K}/K)e^{-rt}/K - r(1+2bI/K)e^{-rt} + m(1-a)\mu P(\dot{\mu}/\mu + \dot{P}/P - \dot{K}/K)/K$$

Substituting (7.3) and (7.4) for (8) implies

(9) $$\dot{\lambda} = (2b\dot{I}/K - 2bI^2/K^2 - r - 2brI/K)e^{-rt} - m(1-a)PK^{m-1}e^{-rt} + m(1-a)\mu PI/K^2$$

Moreover, (7.3) and (7.4) yield

(10) $$2b\dot{I}/K = bI^2/K^2 + 2brI/K + r - maPK^{m-1}$$

Substituting (1.1) for (10) implies

(11) $$2b\dot{I}/K = bI^2/K^2 + 2brI/K + r - mae^{(r-s)t}K^{ma-1}/\eta_0 .$$

For simplicity, we define

$$\Lambda = ma/(2b\eta_0) .$$

Equations (7.4) and (11) yields the following non-autonomous differential equations:

$$\Gamma : \begin{cases} \dot{I} = I^2/2K + rI + rK/(2b) - \Lambda e^{(r-s)t}K^{ma}, \\ \dot{K} = I. \end{cases}$$

We here consider the transformation:

(12) $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{-Gt}I \\ e^{-Gt}K \end{bmatrix}.$$

Then, we have $(x(0), y(0)) = (I(0), K_0)$. By transformation (12), the non-autonomous differential equation Γ is transformed to the autonomous differential equations:

$$\Omega: \begin{cases} \dot{x} = x^2/(2y) + (r-G)x + ry/(2b) - \Lambda y^{ma}, \\ \dot{y} = x - Gy. \end{cases}$$

The equilibrium point of system Ω is given by the following lemma. .

Lemma 1: Denoting the equilibrium point of system Ω by (v, w), we have

$$v = G[\{Gr + r/(2b) - G^2/2\}/\Lambda]^{1/(ma-1)} > 0,$$
$$w = [\{Gr + r/(2b) - G^2/2\}/\Lambda]^{1/(ma-1)} = v/G > 0. ■$$

Proof: The proof is direct.■

Using a solution of system Ω, we derive the optimal path. We start with the phase diagram analysis of system Ω. System Ω is not suitable for the phase diagram analysis. So, we transform system Ω into a more tractable system. We define

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x/y \\ y \end{bmatrix} \equiv \Psi(x, y).$$

It is clear that the Ψ – function is diffeomorphic in $R^2_{++} = \{(x, y) \in R^2 : x > 0, y > 0\}$. Then, we have

$$\dot{\bar{x}} = \frac{\dot{x}y - x\dot{y}}{y^2} = \frac{x^2/2 + (r-G)xy + ry^2/(2b) - \Lambda y^{ma+1}}{y^2}$$
$$= \bar{x}^2/2 + (r-G)\bar{x} + r/(2b) - \Lambda \bar{y}^{ma-1},$$
$$\dot{\bar{y}} = \dot{y} = \bar{x}y - Gy = \overline{xy} - G\bar{y}.$$

Thus, we obtain the following system:

$$\overline{\Omega}: \begin{cases} \dot{\bar{x}} = \bar{x}^2/2 + (r-G)\bar{x} + r/(2b) - \Lambda \bar{y}^{ma-1}, \\ \dot{\bar{y}} = (\bar{x} - G)\bar{y}. \end{cases}$$

In the following, we analyze system $\overline{\Omega}$.

Lemma 2: Denoting the equilibrium point of system $\overline{\Omega}$ by $(\bar{v}, \bar{w})$, we have

$$\bar{v} = v/w = G > 0,$$
$$\bar{w} = w = [\{Gr + r/(2b) - G^2/2\}/\Lambda]^{1/(ma-1)} > 0. \blacksquare$$

Proof: The proof is direct.■

We now start with the phase diagram analysis of system $\overline{\Omega}$.

The $\dot{\bar{x}} = 0$ isocline: $\bar{y} = H\{\bar{x}^2 + 2(r-G)\bar{x} + r/b\}^{-1/(1-am)} \equiv f(\bar{x})$,

The $\dot{\bar{y}} = 0$ isocline: $\bar{y} = 0$ and $\bar{x} = G$,

where $H \equiv (\Lambda/2)^{-1/(1-am)}$. From Assumption 2, we have

(13.1) $\quad f(0) = H(r/b)^{-1/(1-am)} > 0$

(13.2) $\quad f(\bar{x}) = H\{\bar{x}^2 + 2(r-G)\bar{x} + r/b\}^{-1/(1-am)} > 0,$

(13.3) $\quad f'(\bar{x}) = -\{1/(1-am)\}\{2\bar{x} + 2(r-G)\}H\{\bar{x}^2 + 2(r-G)\bar{x} + r/b\}^{-1/(1-am)} < 0.$

Therefore, the phase diagram analysis of system $\overline{\Omega}$ is given as in Figure 1. Thus, we obtain the result on the global existence of stable manifolds:

Figure 1 about here.

Lemma 3: In a neighborhood of w there exists a function $\Theta : R^1 \to R^1$ such that

$$\Theta'(\bar{w}) < 0 \text{ and } \overline{M}^S(\bar{v},\bar{w}) = \{(x,y) \in R^2_{++} : x = \Theta(y)\},$$

where $\overline{M}^S(\bar{v},\bar{w})$ is the stable manifold of $(\bar{v},\bar{w})$. Then, $\Psi(\overline{M}^S(\bar{v},\bar{w}))$ is the stable manifold of (v,w) for system Ω. We define $\Psi(\overline{M}^S(\bar{v},\bar{w})) \equiv M^S(v,w)$.■

Proof: The proof follows directly from (13) and the phase diagram of Figure 1.■

Before getting to the main subject, we prove the following lemma.

Lemma 4: Consider the differential equation:

(14) $\quad \dot{z}(t) = g(t)z(t) - h(t)e^{Bt},$

where

(15.1) $h(t) > 0$ for any $t \geq 0$;

(15.2) $h(t)$ is convergent as $t \to \infty$;

(15.3) $g(t)$ is convergent as $t \to \infty$ and $\lim_{t\to\infty} g(t) > B$.

Define

$$U(t) = \int_{[0,t]} h(u)\exp[\int_{[0,u]} \{B - g(w)\}dw]du.$$

Then, $U(t)$ is convergent as $t \to \infty$. As a solution of equation (14) , we have

$$z(t) = \{z_0 - U(t)\}\exp\{\int_{[0,t]} g(u)du\}, \tag{16}$$

where $z_0 = \lim_{t\to\infty} U(t)$. Then, we have $z(0) = z_0$ and $z(t) > 0$ for ant $t > 0$.■

Proof: In the following, we prove that $U(t)$ is convergent as $t \to \infty$. From condition (15.3), we see that there exists a $T > 0$ such that

$$-V = \sup\{B - g(t) : t > T\} < 0 . \tag{17}$$

Now, define

$$H \equiv \exp[\int_{[0,T]} \{B - g(w)\}dw] .$$

Then, it follows from (17) that for ant $t > T$

$$\begin{aligned} \Omega(t) &\equiv \int_{[T,t]} h(u)\exp[\int_{[0,u]} \{B - g(w)\}dw]du \\ &= \int_{[T,t]} h(u)\exp[\int_{[T,u]} \{B - g(w)\}dw]\exp[\int_{[0,T]} \{B - g(w)\}dw]du \\ &\leq H \int_{[T,t]} h(u)e^{-V(u-T)}du \equiv H\Lambda(t) . \end{aligned} \tag{18}$$

We first prove that $\Lambda(t)$ is convergent as $t \to \infty$. Condition (15.1) shows that $\Lambda(t)$ is monotonously increasing. Noting condition (15.2), we define $h_\infty = \lim_{t\to\infty} h(t)$. Then, for any $\varepsilon > 0$ there exists a $\overline{T} > T$ such that $h(t) < h_\infty + \varepsilon$ for any $t > \overline{T}$. Therefore we have

$$\begin{aligned} \Lambda(t) &= \int_{[0,\overline{T}]} h(u) e^{-V(u-T)} du + \int_{[\overline{T},t]} h(u) e^{-V(u-T)} du \\ &< \int_{[0,\overline{T}]} h(u) e^{-V(u-T)} du + (h_\infty + \varepsilon)\{e^{-V(\overline{T}-T)} - e^{-V(t-T)}\} \\ &\to \int_{[0,\overline{T}]} h(u) e^{-V(u-T)} du + (h_\infty + \varepsilon) e^{-V(\overline{T}-T)} \quad \text{as } t \to \infty . \end{aligned}$$

This implies that $\Lambda(t)$ is bounded from above. Since $\Lambda(t)$ is monotonously increasing, $\Lambda(t)$ is convergent as $t \to \infty$. On the other hand, Condition (15.1) shows that $\Omega(t)$ is monotonously increasing. Therefore, from (18) and convergence of $\Lambda(t)$ we see that $\Omega(t)$ converges as $t \to \infty$, so that as $t \to \infty$

$$\begin{aligned} U(t) &= \int_{[T,t]} h(u) \exp[\int_{[0,u]} \{B - g(w)\} dw] du + \int_{[0,T]} h(u) \exp[\int_{[0,u]} \{B - g(w)\} dw] du \\ &\to \lim_{t\to\infty} \Omega(t) + \int_{[0,T]} h(u) \exp[\int_{[0,u]} \{B - g(w)\} dw] du. \end{aligned}$$

Therefore, we see that $U(t)$ is bounded from above convergent as $t \to \infty$. On the other hand, it follows from condition (15.1) that $U(t)$ is monotonously increasing. Therefore we see that $U(t)$ is convergent as $t \to \infty$. We next prove that $z(t) > 0$ for any $t \geq 0$. Condition (15.1) shows that $U(t)$ is a strictly monotone function. Therefore, from the definition of z_0, we see that $z_0 > U(t)$ for any $t \geq 0$. This shows that $z(t) > 0$ for any $t \geq 0$. Since $U(0) = 0$, it follows directly from equation (16) that $z(0) = z_0$. Finally, we prove that Equation (16) is a solution of Equation (14). We have

$$\begin{aligned} \dot{z}(t) &= \{z_0 - U(t)\} g(u) \exp\{\int_{[0,t]} g(u) du\} - U'(t) \exp\{\int_{[0,t]} g(u) du\} \\ &= \{z_0 - U(t)\} g(u) \exp\{\int_{[0,t]} g(u) du\} - h(t) \exp[\int_{[0,t]} \{B - g(w)\} dw] \exp\{\int_{[0,t]} g(u) du\} \end{aligned}$$

$$= h(t)e^{Bt} + g(u)z(t) .$$

This implies that Equation (16) is a solution of Equation (14). Thus we complete the proof.■

Definition 1: The solution (16) is called a characteristic solution of system (14).■

We next prove the following lemma concerning the stability of the equilibrium:

Lemma 5: The equilibrium point of system Ω, (v, w) is a saddle point.■

Proof: The Jacobean matrix of system Ω is given by

$$J(v,w) = \begin{bmatrix} r & -G^2/2 + r/(2b) - ma\Lambda w^{ma-1} \\ 1 & -G \end{bmatrix}.$$

Assumption 1 yields

(19) $\quad TrJ(v,w) = r - G > 0 .$

Lemma 1 yields $\Lambda w^{ma-1} = Gr + r/(2b) - G^2/2$, so that

$$\begin{aligned} \det J(v,w) &= -rG + G^2/2 - r/(2b) + ma\Lambda w^{ma-1} \\ &= -rG + G^2/2 - r/(2b) + ma(Gr + r/(2b) - G^2/2) \\ &= -(1-ma)\{G(r - G/2) + r/(2b)\} . \end{aligned}$$

On the other hand, $r > G/2$ yields $\det J(v,w) < 0$. From this fact and (19), we see that the eigenvalues of $J(v,w)$ are real and the signs of the eigenvalues are different each others.■

We assume the following.

Assumption 6: $(x^{**}(0), y^{**}(0)) = (\Theta(K_0), K_0) \in M^S(v,w)$.

We denote by $(x^{**}(t), y^{**}(t))$ the path of system Ω with the initial value that satisfies Assumption 6. Then, we have

$$(x^{**}(t), y^{**}(t)) \in M^S, \quad \lim_{t\to\infty}(x^{**}(t), y^{**}(t)) = (v, w).$$

Since $(x^{**}(t), y^{**}(t))$ converges to the equilibrium point,

$$\lim_{t\to\infty} \dot{x}^{**}(t) = \lim_{t\to\infty} \dot{y}^{**}(t) = 0$$

is satisfied. Therefore, we have

$$(20) \qquad \begin{aligned} 0 &= \lim_{t\to\infty} \dot{x}^{**}(t) \\ &= \lim_{t\to\infty}\{x^{**}(t)^2/\{2y^{**}(t)\} + (r-G)x^{**}(t) + ry^{**}(t)/(2b) - \Lambda y^{**}(t)^{ma}\} \\ &= v^2/(2w) + (r-G)v + rw/(2b) - \Lambda w^{ma} \end{aligned}$$

From (3) and (20), we see that the following equation must be satisfied.

$$\begin{aligned} \Lambda w^{ma-1} = w^{ma-1}ma/(2b\eta_0) &= v^2/(2w^2) + (r-G)v/w + r/(2b) \\ &= -G^2/2 + rG + r/(2b). \end{aligned}$$

Therefore, we obtain

$$(21) \qquad \eta_0 = \frac{w^{ma-1}ma}{2brG + r - bG^2}.$$

Thus, η_0 is given by (21). Moreover, it follows from (3) that

$$(22) \qquad P_0 = \frac{1}{\eta_0 K(0)^{(1-a)m}} = \frac{w^{ma-1}ma}{(2brG + r - bG^2)K(0)^{(1-a)m}}.$$

We now obtain the following main theorem concerning the existence of the optimal growth path:

Theorem 1: We assume Assumptions 1 to 5. We define

$$(23) \qquad \begin{bmatrix} Q^{**}(t) \\ C^{**}(t) \\ I^{**}(t) \\ K^{**}(t) \\ P^{**}(t) \end{bmatrix} = \begin{bmatrix} y^{**}(t)^m e^{mGt} \\ y^{**}(t)^m e^{mGt} \\ x^{**}(t)e^{Gt} \\ y^{**}(t)e^{Gt} \\ P_0 \exp[(r-s)t - m(1-a)\int_{[0,t]} x^{**}(u)/y^{**}(u)du] \end{bmatrix},$$

where

$$P_0 = \frac{w^{ma-1}ma}{(2brG + r - bG^2)y^{**}(t)^{(1-a)m}}.$$

See equation (22). Then, we have $P^{**}(t) = D(K;t)$ and

(24) $$\dot{K}_I = \pi(t) + rK_I - P^{**}(t)C^{**}(t) = rK_I - I^{**}(t)\{1 + bI^{**}(t)/K^{**}(t)\}$$

possesses a characteristic solution. Denoting it by $K_I^{**}(t)$, the optimal path is given by (23) and $K_I^{**}(t)$. Moreover, we have

$$agr(K^{**}) = G,\ \ agr(C^{**}) = agr(Q^{**}) = mG,\ \ agr(P^{**}) = (1-m)G,$$
$$agr(K_I^{**}) = G.$$

On the other hand, the growth rate of $K_I^{**}(t)$ is given by G. We also obtain that there exists a $T > 0$ such that for any $t > T$.

$$\pi_C^{**}(t) = P^{**}(t)Q^{**}(t) - I^{**}(t)\{1 + bI^{**}(t)/K^{**}(t)\} > 0. \blacksquare$$

Proof: The initial point of η, η_0 is determined in this proof. We have

$$\frac{\dot{P}^{**}(t)}{P^{**}(t)} = r - s - m(1-a)\frac{x^{**}(t)}{y^{**}(t)} = r - s - m(1-a)\frac{\dot{K}^{**}(t)}{K^{**}(t)}$$

This shows that

$$\{\log \dot{P}^{**}(t)\} = r - s - m(1-a)\{\log \dot{K}^{**}(t)\}.$$

Therefore, we have $\log\{P^{**}(t)\} = (r-s)t - \log\{K^{**}(t)^{m(1-a)}\} + A,$ where A is the constant of integration. This implies $P^{**}(t) = e^A e^{(r-s)t}/K^{**}(t)^{(1-a)m}$. From (3), we have $A = -\log \mu_0$ and

$$P^{**}(t) = e^{(r-s)t}/\{\eta_0 K^{**}(t)^{(1-a)m}\} = D(K;t).$$

We next prove that the differential equation (24) possesses the characteristic equation. From a simple calculation, we see that conditions (1.2) satisfied. Defining $g(t) = r$ and $B = G,$ we have

$$\begin{aligned}\dot{K}_I &= rK_I - I^{**}(t)\{1 + bI^{**}(t) / K^{**}(t)\}\\ &= rK_I - y^{**}(t)\{1 + bx^{**}(t) / y^{**}(t)\}e^{Gt}.\end{aligned}$$

We here define $h(t) = y^{**}(t)\{1 + bx^{**}(t) / y^{**}(t)\}$. Then, $h(t) > 0$ for any $t \geq 0$. Since $y^{**}(t)$ converges as $t \to \infty$, $h(t)$ converges as $t \to \infty$. Thus, we see that conditions (15.1) to (15.3) are satisfied. Therefore, we see from Lemma 3 that defining

$$U_1(t) = \int_{[0,t]} h(u)e^{(G-r)t}du,$$

$U_1(t)$ converges as $t \to \infty$ and the differential equation (24) possesses the characteristic solution:

$$K_I^{**}(t) = \{K_{I0}^{**} - U_1(t)\}e^{rt}, \tag{25}$$

where $K_{I0}^{**} = \lim_{t\to\infty} U_1(t)$. Then, we obtain from (1.2) that

$$\lim_{t\to\infty} K_I^{**}(t)\eta^{**}(t) = \lim_{t\to\infty} \eta_0\{K_{I0}^{**} - U_1(t)\} = 0.$$

This proves equation (1.4). Following (7.3), we here define the following differential equation:

$$\begin{aligned}\dot{\mu} &= \{-r + s + m(1-a)I^{**}(t) / K^{**}(t)\}\mu - K^{**}(t)^m e^{-rt}\\ &= \{-r + s + m(1-a)x^{**}(t) / y^{**}(t)\}\mu - y^{**}(t)^m e^{(mG-r)t}.\end{aligned} \tag{26}$$

We define $g(t) = -r + s + m(1-a)x^{**}(t) / y^{**}(t)$, $h(t) = y^{**}(t)$, $B = mG - r$. Then,

$$\lim_{t\to\infty} g(t) = -r + s + m(1-a)v / w,\quad h(t) > 0 \text{ for any } t \geq 0, \text{ and}$$
$$\lim_{t\to\infty} h(t) = w.$$

This implies that conditions (15.1), (15.2) and the first part of condition (15.3) are satisfied. Moreover, Lemma 1, Assumption 2, and the definition of G yield

$$\begin{aligned}\lim_{t\to\infty} g(t) = -r + s + m(1-a)G &= -r + s + m(1-a)G\\ &= (r-s)\{-1 + m(1-a)/(1-am)\}\\ &= (m-1)G > mG - r = B.\end{aligned}$$

This proves the latter part of condition (15.3). Therefore, Lemma 3 shows that equation (26) also possesses a characteristic solution. We define the characteristic solution by

$\mu^{**}(t)$. Then, Lemma 3 shows that defining

$$U_2(t) = \int_{[0,t]} y^{**}(u)\exp[\int_{[0,u]} \{mG - s - m(1-a)x^{**}(z)/y^{**}(z)\}dz]du,$$

$U_2(t)$ converges as $t \to \infty$ and the characteristic solution of (26) is given by

(27) $$\mu^{**}(t) = \{\mu_0{}^{**} - U_2(t)\}\exp\{(-r+s)t + m(1-a)\int_{[0,t]} x^{**}(u)/y^{**}(u)du\}.$$

It follows from the definition that (27) satisfies (7.3). Moreover, we define

(28) $$\lambda^{**}(t) = \{1 + 2bx^{**}(t)/y^{**}(t)\}e^{-rt} + m(1-a)\mu^{**}(t)P^{**}(t)/K^{**}(t).$$

We prove that (28) satisfies condition (7.2). A simple calculation yields

$$\begin{aligned}\frac{\dot{\mu}^{**}(t)}{\mu^{**}(t)} + \frac{\dot{P}^{**}(t)}{P^{**}(t)} - \frac{\dot{K}^{**}(t)}{K^{**}(t)} &= \{-r + s + m(1-a)\frac{x^{**}(t)}{y^{**}(t)}\} - \frac{y^{**}(t)^m e^{(mG-r)t}}{\mu^{**}(t)} \\ &\quad + \{r - s - m(1-a)\frac{x^{**}(t)}{y^{**}(t)}\} - \frac{\dot{y}^{**}(t)}{y^{**}(t)} + G \\ &= -\frac{y^{**}(t)^m e^{(mG-r)t}}{\mu^{**}(t)} - \frac{x^{**}(t)}{y^{**}(t)} \\ &= -\frac{K^{**}(t)^m e^{-rt}}{\mu^{**}(t)} - \frac{I^{**}(t)}{K^{**}(t)}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\dot{\lambda}^{**}(t) &= 2b[\frac{\dot{x}^{**}(t)y^{**}(t) - \dot{y}^{**}(t)x^{**}(t)}{\{y^{**}(t)\}^2}]e^{-rt} - r\{1 + 2b\frac{x^{**}(t)}{y^{**}(t)}\}e^{-rt} \\ &\quad + m(1-a)\{\frac{\dot{\mu}^{**}(t)P^{**}(t)}{K^{**}(t)} + \frac{\dot{P}^{**}(t)\mu^{**}(t)}{K^{**}(t)} - \frac{P^{**}(t)\mu^{**}(t)}{\{K^{**}(t)\}^2}\} \\ &= 2b[\frac{\{x^{**}(t)\}^2}{2\{y^{**}(t)\}^2} + (r-G)\frac{x^{**}(t)}{y^{**}(t)} + \frac{r}{2b} - Ay^{**ma-1} \\ &\quad - \frac{x^{**}(t)}{y^{**}(t)}\{\frac{x^{**}(t)}{y^{**}(t)} - G\}]e^{-rt} - r\{1 + 2b\frac{x^{**}(t)}{y^{**}(t)}\}e^{-rt}\end{aligned}$$

$$+m(1-a)\frac{\mu^{**}(t)P^{**}(t)}{K^{**}(t)}\{\frac{\dot{\mu}^{**}(t)}{\mu^{**}(t)}+\frac{\dot{P}^{**}(t)}{P^{**}(t)}-\frac{\dot{K}^{**}(t)}{K^{**}(t)}\}$$

$$=2b[-\frac{\{x^{**}(t)\}^2}{2\{y^{**}(t)\}^2}-\Lambda y**^{ma-1}]e^{-rt}$$

$$+m(1-a)\frac{\mu^{**}(t)P^{**}(t)}{K^{**}(t)}\{-\frac{K^{**}(t)^m e^{-rt}}{\mu^{**}(t)}-\frac{I^{**}(t)}{K^{**}(t)}\}$$

$$=[-b\frac{\{x^{**}(t)\}^2}{\{y^{**}(t)\}^2}-2b\Lambda y**^{ma-1}]e^{-rt}$$

$$-m(1-a)P^{**}(t)K^{**}(t)^{m-1}e^{-rt}-\mu^{**}(t)m(1-a)\frac{P^{**}(t)I^{**}(t)}{\{K^{**}(t)\}^2}$$

$$=-[b\frac{\{x^{**}(t)\}^2}{\{y^{**}(t)\}^2}+\{2b\Lambda y**^{ma-1}+m(1-a)P^{**}(t)K^{**}(t)^{m-1}\}]e^{-rt}$$

$$-\mu^{**}(t)m(1-a)\frac{P^{**}(t)I^{**}(t)}{\{K^{**}(t)\}^2}$$

From (3), we here see that

$$\begin{aligned}P^{**}(t)K^{**}(t)^{m-1}&=e^{(r-s)t}K^{**}(t)^{am-1}/\eta_0\\&=e^{(r-s)t}y^{**}(t)^{am-1}e^{(am-1)G}/\eta_0=y^{**}(t)^{ma-1}/\eta_0,\end{aligned}$$

$$2b\eta_0\Lambda+m(1-a)=m.$$

Therefore, we have

$$\begin{aligned}\dot{\lambda}^{**}(t)&=-[b\frac{\{x^{**}(t)\}^2}{\{y^{**}(t)\}^2}+mP^{**}(t)K^{**}(t)^{m-1}]e^{-rt}-\mu^{**}(t)m(1-a)\frac{P^{**}(t)I^{**}(t)}{\{K^{**}(t)\}^2}\\&=-[b\frac{\{I^{**}(t)\}^2}{\{K^{**}(t)\}^2}+mP^{**}(t)K^{**}(t)^{m-1}]e^{-rt}-\mu^{**}(t)m(1-a)\frac{P^{**}(t)I^{**}(t)}{\{K^{**}(t)\}^2}\end{aligned}$$

This implies that condition (7.2) is satisfied. We next prove the transversality condition. We have

$$\text{(29)}\qquad P^{**}(t)\mu^{**}(t)=\{\mu_0{}^{**}-U_2(t)\}\exp\{(-r+s)t+m(1-a)\int_{[0,t]}x^{**}(u)/y^{**}(u)du\}$$

$$\bullet P_0\exp[(r-s)t-m(1-a)\int_{[0,t]}x^{**}(u)/y^{**}(u)du]$$

$$=P_0\{\mu_0{}^{**}-U_2(t)\}.$$

Since we have $\mu_0^{**} = \lim_{t\to\infty} U_2(t)$, we see that $\lim_{t\to\infty} P^{**}(t)\mu^{**}(t) = 0$. This proves the transversality condition (7.6). Moreover, we have

$$\begin{aligned} K^{**}(t)\lambda^{**}(t) &= y^{**}(t)e^{Gt} \bullet \{1+2bx^{**}(t)/y^{**}(t)\}e^{-rt} + ma\mu^{**}(t)P^{**}(t)/K^{**}(t) \\ &= y^{**}(t)\{1+2bx^{**}(t)/y^{**}(t)\}e^{(G-r)t} + ma\mu^{**}(t)P^{**}(t). \end{aligned}$$

Therefore, since $G < r$, we see from (29) that $\lim_{t\to\infty} K^{**}(t)\lambda^{**}(t) = 0$. This proves the transversality condition (7.7). Thus, the optimal path is given by (23) and $K_I^{**}(t)$.

We next derive the growth rate. Using Lemma 2, we calculate the growth rate. In the following, we derive only the growth rates of $P^{**}(t)$ and $K_I^{**}(t)$. The growth rate of the other optimal path can be calculated by the same method. We have

$$\dot{P}^{**}(t)/P^{**}(t) = d\log P^{**}(t)/dt = m(a-1)\dot{y}^{**}(t)/y^{**}(t) + (1-m)G.$$

Therefore, Lemma 2 yields

$$agr(P^{**}) = \lim_{t\to\infty} gr(P^{**}) = (1-m)G.$$

On the other hand, we see from equation (24) that

(30) $$\log K_I^{**}(t) = rt + \log\{K_{I0}^{**} - U_1(t)\}.$$

Moreover, we have

(31) $$\lim_{t\to\infty} \dot{y}(t) = \lim_{t\to\infty}\{x(t) - Gy(t)\} = \lim_{t\to\infty}(v - Gw) = 0.$$

Using L'Hopital's rule, we obtain from Equations (30) and (31) that

$$\begin{aligned} \lim_{t\to\infty} \dot{K}_I^{**}(t)/K_I^{**}(t) &= r - \lim_{t\to\infty} \dot{U}_1(t)/\{K_{I0}^{**} - U_1(t)\} \\ &= r - \lim_{t\to\infty} \frac{\sigma e^E y^{**}(t)^{ma} e^{(G-r)t}}{K_{I0}^{**} - \int_{[0,t]} \sigma e^E y^{**}(u)^{ma} e^{(G-r)u} du} \\ &= r + \lim_{t\to\infty} \frac{\sigma e^E \dot{y}^{**}(t)^{ma} e^{(G-r)t} + (G-r)\sigma e^E y^{**}(t)^{ma} e^{(G-r)t}}{\sigma e^E y^{**}(t)^{ma} e^{(G-r)t}} \\ &= G. \end{aligned}$$

Noing $\lim_{t\to\infty}(x^{**}(t), y^{**}(t)) = (v, w)$, we see that

$$\lim_{t\to\infty} \pi_C^{**}(t)/e^{(G-r)t} = \lim_{t\to\infty}[P^{**}(t)Q^{**}(t) - I^{**}(t)\{1+bI^{**}(t)/K^{**}(t)\}]/e^{(G-r)t}$$

$$= w\left\{\frac{2bGr + r - bG^2}{ma} - G(1+bG)\right\} > 0.$$

Therefore, we obtain that there exists a $T > 0$ such that $\pi_C^{**}(t) > 0$ for any $t > T$. Thus, we complete the proof.■

References

Barro, R. J., and Sala-i-Martin, X. (1995): *Economic Growth*. New York: McGraw-Hill.

Dohtani, A. (2018): "A Macro Model of Heterogeneous Growth," Mimeo, Faculty of Economics, University of Toyama.

Uzawa, H. (1969): "Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth," *Journal of Political Economy*, 77, 628-652.

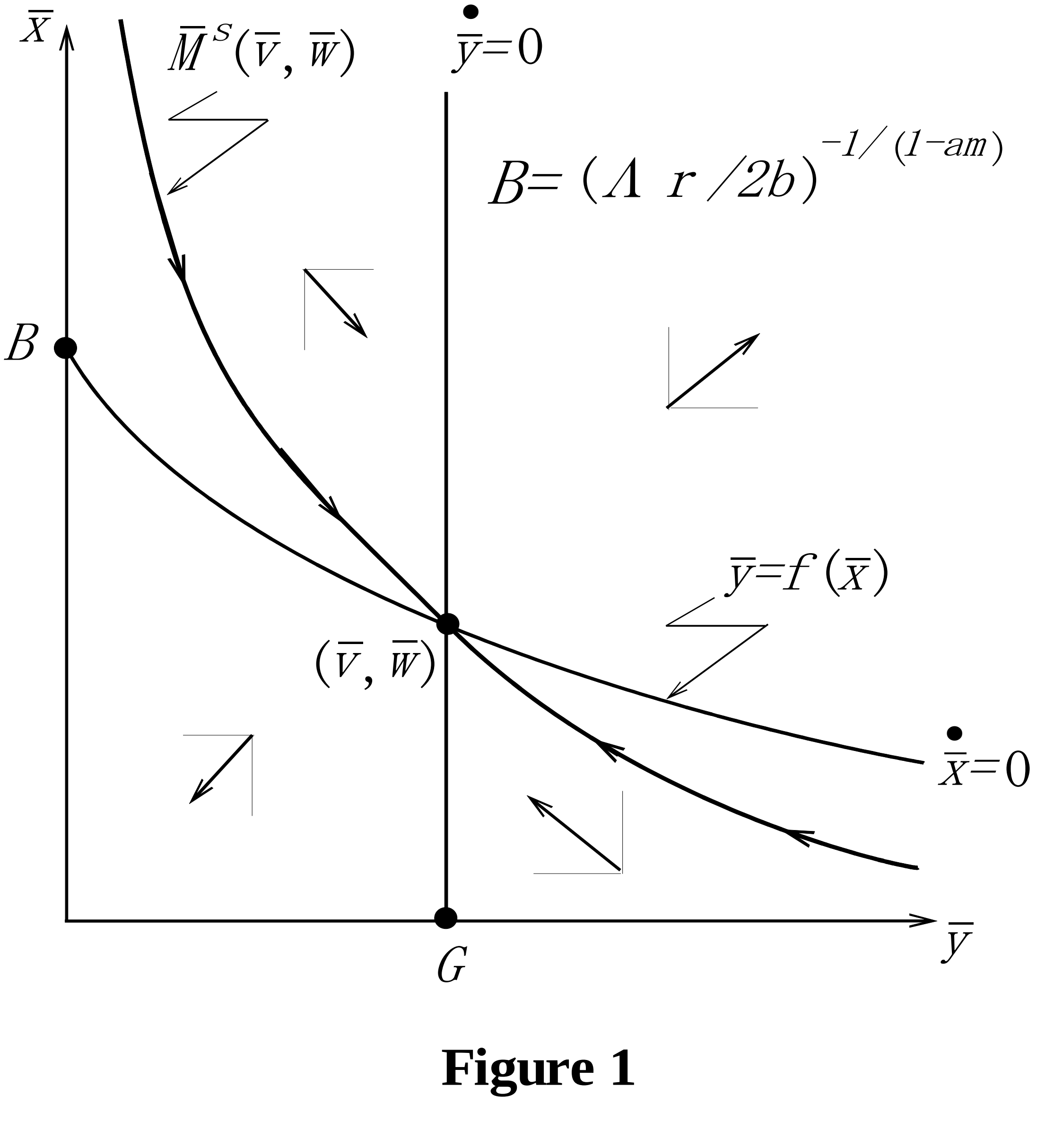
$\bar{x}$
$\bar{M}^s(\bar{v}, \bar{w})$
$\dot{\bar{y}}=0$
$B=(\Lambda\ r/2b)^{-1/(1-am)}$
B
$\bar{y}=f(\bar{x})$
$(\bar{v}, \bar{w})$
$\dot{\bar{x}}=0$
G
$\bar{y}$

Figure 1