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A Macro Model of Heterogeneous Growth

Akitaka Dohtani

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Faculty of Economics
University of Toyama

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Author: Akitaka Dohtani

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Affiliation: Faculty of Economics, University of Toyama, 3190 Gofuku, Toyama, 930, JAPAN.

e-mail address: doutani@eco.u-toyama.ac.jp

Abstract: Developing the AK model, we construct an endogenous growth model with many industries. Unlike the original AK model, our model generates endogenous growth accompanied by the change of relative price. The growth rate of each industry is determined by such fundamental parameters as the rate of technological progress of the industry, the elasticity of marginal productivity of the industry and the elasticity of marginal utility of the goods produced by the industry. In our model, the persistent change of relative prices admits of consistent growths of heterogeneous industries. Therefore, our model gives a theoretical explanation of the persistent transition of industrial structure accompanied by a change of relative prices. The transition of industrial structure depends on the fundamental parameters. We derive an equation that relates the growth rate of relative price of an industry to the growth rates of capital stock and production of the industry. By using our model, we unifiedly explain many empirical facts that have been known so far. We also give a new theoretical viewpoint about the empirical fact that the relative price of investment is higher in poor countries relative to rich countries. We demonstrate that the empirical fact results from the myopia concerning consumption in the poor countries. Moreover, in the case where the number of consumption-goods industry is one, we incorporate population growth. In the modified model, we derive an equation which relates the growth rate of relative wage to the growth rates of capital stock and relative price.

Keywords: Structural transformation; Heterogeneous growth; Change of relative prices; Change of relative wage.

JEL classification: C61; L16; O40; O41

1. Introduction

The most influential contributions in modern theories of economic growth have been those of Solow (1956) and Swan (1956). The Solow-Swan model is of an exogenous type. To go further, one has to construct endogenous growth models, that is, to construct a model that determines the long-run growth rate within the model. A base line delivering endogenous growth is the AK type of growth models. See Romer (1986)), Jones and Manuelli (1990), Rebelo (1991), and Barro and Sala-i-Martin (1992). The AK models of infinite horizon optimization have its origin in the Ramsey model (Ramsey (1928)). The AK models have been referred very often as the simplest model that makes clear how the absence of diminishing returns can lead to endogenous growth.

In this paper, modifying the AK model of decentralized infinite horizon optimization by Barro and Sala-i-Martin (1992), we try to construct an endogenous growth model that explains the transition of the relative scales of heterogeneous industries with different production functions. Some of our results are similar to those in the AK model. Like the AK model, the growth rate of each industry in our model depends on parameters concerning production and utility functions. On the other hand, there exist some important features distinguished from the AK model. Industries in our model grow through a persistent change of relative prices. In other word, a persistent change of relative prices admits of a consistent growth of heterogeneous industries. Moreover, we see that a persistent change of relative prices also yields a persistent change of the relative scales of heterogeneous industries (therefore, a persistent change of industrial structure). Thus, our model explains the process of structural transformation, which describes the persistent transition of industrial structure accompanied by a change of relative prices.[1]

What are the factors that yield a change of industrial structure? This problem has been investigated in many papers. For the view point of demand side, see Kongsamut, et al. (2001) and Herrendorf, et al. (2013). On the other hand, for the view point of supply side, see Acemoglu and Guerrieri (2008) and Herrendorf, et al. (2015). In a

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[1] Ngai and Pissarides (2007) analyzed the effect of a change of relative prices on a change of industrial structure. Our result shows that there is a close relation between both changes although both changes in our model are endogenously yielded.

different context, this paper investigates the problem from both sides. Like these papers, a change of industrial structure in our model depends on such parameters as not only the elasticity of marginal productivity and the rate of technological progress but also the elasticity of marginal utility.[2] However, unlike these papers, since our model is endogenous, a (persistent) change of industrial structure is yielded from not (persistent) changes but differences among the parameters of industries. Our model derives analytical and clear results on gowth rates of goods and relative prices, some of which have not been known yet. Especially, we derive an interesting equation that relates the rates of relative price change to the grwoth rates of goods.

It has been well known that the relative price of investment-goods in poor countries is higher than that of rich countries. See Hsieh and Klenow (2007). They conclude that the high relative price of investment-goods in poor countries is due to the low price of consumption-goods in those countries. However, in our model, there exist several possible sources of the well-known empirical fact. By arguing such possible source, we see that our endogenous growth model ties with the empirical evidence concerning the relative price of investment.

Moreover, by incorporating population growth, we extend our endogenous growth model. We derive not only the growth rates of goods and relative price but also the growth rate of wage. We derive an interesting equation that relates the growth rate of wage to growth rates of goods and relative price.

2. Background of the Model

We consider a decentralized and closed economy with two sectors; consumption-goods and investment-goods sectors. The models consist of a representative household and a representative investment-goods industry, and more-than-one consumption-goods industries. Let n be the number of consumption-goods industries. For simplicity, we assume that the household owns the initial endowment

[2] For the production function $f(K)$, the elasticity of marginal productivity is defined as $f''(K)K/f'(K)$. On the other hand, for the utility function $U(C)$, the elasticity of marginal utility is also defined in the sama way.

of capital stock which can be used by any industry. The household distributes the endowment to all industries. Capital goods owned by the household are lent to the investment-goods sector. Without loss of generality, we assume that the depreciation rate of capital stock is zero. The consumption-goods industries rent capital goods from the investment-goods industry. The household has a claim on the consumption-goods sector's net cash flow. There is a competitive credit market in which the household can borrow and lend. To rule out Ponzi-game finance, we assume the credit market imposes a constraint on the amount of borrowing. The two forms of assets, capital and loans are assumed to be perfect substitutes as stores of value. Then, they must pay the same real rate of return, and the interest rate on debt must be equal to the rental rate on capital.

The symbols used in this paper are as follows:

$K_0 =$ Initial endowment of capital stock (given),

$C_j =$ Consumption of the goods produced by industry j,

$s =$ Rate of time preference (constant),

$Q_j =$ Quantity produced by industry $j = C_j$,

$K_j =$ Capital stock of industry j,

$\Pi_j =$ Profit of industry j,

$K_I =$ Capital stock of the investment-goods industry,

$r =$ Interest rate $=$ rental rate on capital (constant),

$P_j =$ Price of the goods produced by industry j,

$P_I =$ Rental price of capital stock,

where $j \in \{1, \cdots, n\} \equiv N$. We denote by $\bullet_t$ the value of $\bullet$ at time t. For example, we denote by K_{jt} the value of capital stock of industry j $(j \in N)$ at time t.

3. The Model

Throughout this paper, we assume that any function is continuously differentiable. As stated above, we extend the AK model of Barro and Sala-i-Martin

(1992). In Sections 3 and 4 we assume that population is constant. We assume the following additive utility function of the representative household:

$$U(C_{1t},\cdots,C_{nt}) = \sum_{k\in N} C_{kt}{}^{a_k} / a_k,$$

where $0 < a_j < 1, \quad j \in N$.

We next consider the investment-goods industry. We assume that the production function of the representative firm in the industry is of the AK type. The firm solves the optimization problem:

$$\max \ (P_I A K_{It} - r K_{It}),$$

where A is a positive constant. The optimization problem of the investment-goods industry is essentially the same as that of the firm in Barro and Sala-i-Martin (1982). The condition for profit maximization requires that the marginal product of capital equals r. That is, $P_I = r / A$. Without loss of generality we here assume $P_I = r / A = 1$.

The global absence of diminishing returns to capital in the production function may seem unrealistic. It is, however, plausible if capital, K_{It}, broadly includes human capital, knowledge, and public infrastructure in addition to physical capital. For this point, see for example Barro and Sala-i-Martin (1995, Ch. 4).

We next consider the representative household who solves the following optimization problem:

$$\max \int_{R^1_+} \sum_{k\in N} C_{kt}{}^{a_k} e^{-st} / a_k dt$$

$$\text{subject to} \quad \dot{K}_{It} = \sum_{k\in N} \Pi_{kt} + rK_{It} - \sum_{k\in N} P_{kt} C_{kt},$$

where R^1_+ is the set of non-negative real numbers. We here assume that in maximizing overall utility the representative household considers that the path of the profits is given exogenously.

We here consider the demand for the goods produced by the investment-goods industry. In this paper, we assume that the profit of the consumption-goods industry

$k(\in N)$ are given by $\Pi_{jt} = P_{jt}Q_{jt} - P_I K_{jt} = P_{jt}Q_{jt} - K_{jt}$.[3] Therefore, it follows from the budget constraint that

$$\dot{K}_{It} = \sum_{k\in N} \Pi_{kt} + rK_{It} - \sum_{k\in N} P_{kt}Q_{kt} = -\sum_{k\in N} K_{kt} + rK_{It}.$$

It should be noted here that

$$AK_{It} = rK_{It} = \dot{K}_{It} + \sum_{k\in N} K_{kt}.$$

This equation implies that the goods produced by the investment-goods industry are demanded for the accumulation of capital goods by households and the investment of the consumption-goods.

Now we have the following Hamiltonian of the intertemporal optimization problem of the representative household:

$$H = \sum_{k\in N} C_{kt}{}^{a_k} e^{-st} / a_k + \eta_t (\sum_{k\in N} \Pi_{kt} + rK_{It} - \sum_{k\in N} P_{kt}C_{kt}).$$

Since the Hamiltonian is a concave function of the state and the control variables, the sufficient condition for optimization is given by

(1.1) $\partial H / \partial C_{jt} = C_{jt}{}^{a_j - 1} e^{-st} - P_{jt}\eta_t = 0,$

(1.2) $\dot{\eta}_t = -\partial H / \partial K_{It} = -r\eta_t,$

(1.3) $\dot{K}_{It} = \sum_{k\in N} \Pi_{kt} + rK_{It} - \sum_{k\in N} P_{kt}C_{kt},$

(1.4) $\lim_{t\to\infty} K_{It}\eta_t = 0.$

The equation (1.2) yields $\eta_t = \eta_0 e^{-rt}$, where the initial value η_0 is determined in the proof of Theorem 1 in Appendix. Then, from (1.1) we have

[3] The maximization of this profit is discussed a little later.

(2) $$P_{jt} = \frac{e^{(r-s)t}}{\eta_0 C_{jt}{}^{1-a_j}}, \quad j \in N.$$

The equation (2) represents the optimal plan taking into account the growth and the persistent change of relative prices. The equation (2) gives a dynamic version of static inverse demand equation.

Definition 1: We call the equation (2) a dynamic inverse demand equation.■

As we see below, by using the dynamic inverse demand equations, the optimal path of price of the goods produced by consumption-goods industry $j(\in N)$ is also determined.

Finally, we consider the consumption-goods industry. The production function of consumption-goods industry $j(\in N)$ is assumed to be

(3) $$Q_{jt} = D_j(t) K_{jt}{}^{m_j}, \quad 0 < m_j \le 1, \quad 0 < D_j(t).$$

where $D_j(t)$ is an index of the state of technology of the industry *j*. In order to consider the effects of technological progresses on the rate of growths, we assume that $D_j(t)$ grows at the rate $d_j > 0$. Then

$$D_j(t) = D_{j0} e^{d_j t}.$$

For simplicity, we assume $D_{j0} = 1$. We have

$$C_{jt} = Q_{jt} = e^{d_j t} K_{jt}{}^{m_j} \quad (j \in N),$$

because we consider the situation where the consumption-goods market is cleared. Substituting this equation into the dynamic inverse demand equation (2), we have

(4) $$P_{jt} = \frac{e^{(r-s)t}}{\eta_0 e^{(1-a_j)d_j t} K_{jt}{}^{(1-a_j)m_j}} \equiv H_j(K_{jt}, t), \quad j \in N.$$

By assuming the price path P_{jt} $(j \in N)$ is given, the consumption-goods industry

j solves the following optimization problem:

$$\max \Pi_{jt} = \max(P_{jt}Q_{jt} - K_{jt}) = \max(P_{jt}e^{d_j t}K_{jt}{}^{m_j} - K_{jt}), \quad j \in N.$$

Unlike the usual optimization problem, this optimization problem implies that the industry j considers the long-run optimization problem allowing for the growth of demand equation following (4). Therefore, taking into account the change of relative price, the industry j determines both the growth rates of production. By rewriting the usual optimization problem in such a way, we will obtain a theoretical explanation of the optimal growth of heterogeneous industries.

We assumed that the initial values of capital stock of consumption-goods industries K_{j0} and the initial values of capital stock of investment-goods industry K_{I0} are distributed by the representative household and from his/her initial endowment. Thus, the given initial endowment of capital stock

$$K_0 = K_{I0} + \sum_{k \in N} K_{k0}.$$

See Section 2. The initial endowment, K_{j0} $(j \in N)$, is determined in Appendix.

4. Equilibrium Growth Paths

In this section, we derive the equilibrium growth paths of the model of Section 3. Before starting it, we introduce a notion. In this section, we assume

Assumption 1: $1 > a_j m_j$ for any $j \in N$,

Assumption 2: $r > \dfrac{r - s + a_j d_j}{1 - a_j m_j} \equiv G_j > 0$ for any $j \in N$.

Assumption 2 will be used later to guarantee the transversality conditions. See Appendix. We define

$gr(\bullet) =$ growth rate of $\bullet$,

$agr(\bullet) \equiv \lim_{t \to \infty} gr(\bullet)$ (the asymptotic growth rate of $\bullet$).

As is shown in Appendix, the equilibrium growth paths are derived. Consequently, we see that the growth rates of equilibrium growth paths differ from each other. This is a remarkable feature of the model. The growth rate of each path is given as follows.

Theorem 1: Suppose Assumptions 1 and 2 are satisfied. Then there exist equilibrium growth paths which satisfy that for any $j \in N$

$$gr(K_{jt}) = gr(I_{jt}) = G_j, \quad gr(C_{jt}) = gr(Q_{jt}) = m_j G_j + d_j,$$
$$gr(P_{jt}) = (1 - m_j)G_j - d_j, \quad gr(\Pi_{jt}) = G_j,$$
$$agr(K_{It}) = G_{\max} \equiv \max\{G_j : j \in N\}.\blacksquare$$

Proof: See Appendix.■

Thus, we have the following result.

Corollary 1: Suppose Assumptions 1 and 2 are satisfied. Then, we have

$$gr(P_{jt}) = gr(K_{jt}) - gr(C_{jt}) \quad \text{for any} \quad j \in N.$$

We call this equation GRRP (growth rate of relative price) equation.■

Proof: The proof follows directly from Theorem 1. ■

The GRRP equation of Corollary 1 is novel and has not been known yet. The GRRP equation is interesting in the sense that it relates the rates of change of relative prices to the growth rates of goods. Corollary 1 shows that the change of relative prices is inevitable in the growth of heterogeneous industries.

Moreover, we have

Corollary 2: Suppose Assumptions 1 and 2 are satisfied. Then, for the equilibrium growth paths we have

the rate of time preference $s \uparrow$

$$\Rightarrow \; gr(C_{jt})\downarrow,\; gr(P_{jt})\downarrow,\; gr(K_{jt})\downarrow,\; gr(K_{It})\downarrow \text{ for any } j \in N,$$

the elasticity of marginal productivity $(1-m_j)\downarrow$ $(j \in N)$ $(\Leftrightarrow\; m_j \uparrow)$

$$\Rightarrow \; gr(C_{jt})\uparrow,\; gr(P_{jt})\downarrow,\; gr(K_{jt})\uparrow, \text{ and } gr(K_{It})\uparrow.$$

the rate of technological progress $d_j \uparrow$ $(j \in N)$

$$\Rightarrow \; gr(C_{jt})\uparrow,\; gr(P_{jt})\downarrow,\; gr(K_{jt})\uparrow, \text{ and } gr(K_{It})\uparrow.$$

the elasticity of marginal utility $(1-a_j)\downarrow$ $(j \in N)$ $(\Leftrightarrow\; a_j \uparrow)$

$$\Rightarrow \; gr(C_{jt})\uparrow,\; gr(P_{jt})\uparrow,\; gr(K_{jt})\uparrow, \text{ and } gr(K_{It})\uparrow.$$

Moreover, as a_j (resp. m_j) $(j \in N)$ becomes large, the effect of the rate of technological progress d_j on the growth rate of goods becomes large (resp. small). That is

$$\frac{\partial^2 gr(C_{jt})}{\partial a_j \partial d_j} > 0, \quad \frac{\partial^2 gr(K_{jt})}{\partial a_j \partial d_j} > 0, \quad \frac{\partial^2 gr(K_{It})}{\partial a_j \partial d_j} > 0,$$
$$\frac{\partial^2 gr(C_{jt})}{\partial m_j \partial d_j} < 0, \quad \frac{\partial^2 gr(K_{jt})}{\partial m_j \partial d_j} < 0, \text{ and } \frac{\partial^2 gr(K_{It})}{\partial m_j \partial d_j} < 0.$$

As a_j (resp. m_j) $(j \in N)$ becomes large, the effect of the rate of technological progress d_j on the growth rate of P_{jt} becomes large (resp. small). That is

$$\frac{\partial^2 gr(P_{jt})}{\partial a_j \partial d_j} > 0 \text{ and } \frac{\partial^2 gr(P_{jt})}{\partial m_j \partial d_j} < 0. \blacksquare$$

Proof: See Appendix.■

Kongsamut, et al. (2001) and Herrendorf, et al. (2013) investigated the relation between the parameters concerning utility function and the change of industrial structure. On the other hand, Acemoglu and Guerrieri (2008) and Herrendorf, et al. (2015) investigated the relation between the changes of parameters concerning production function and the change of industrial structure. These results give theoretical and endogenous explanations which describes the transition of industrial structure. Theorem 1 and the first result of Corollary 2 provide the information concerning the effects of fundamental parameters on the rate of change of economic variables. These results are almost the same as of those that have already known. On the other hand, the second and the third results of Corollary 2 tell us how fundamental parameters affect the effect of parameters on the rate of change of economic variables.

Corollary 2 shows that as the rate of technological progress (u_j) of a consumption-goods industry is small (resp. large), the rate of relative price change of the industry is large (resp. small). Therefore, compared with the consumption-goods industry, the relative price of the investment-goods industry becomes gradually small (resp. large). In poor (resp. rich) countries, the rate of technological progress (u_j) of a consumption-goods industry is small (resp. large). Therefore, the result of Corollary 2 ties with the empirical evidence of the high relative price of investment in poor countries relative to rich countries. See Hsieh and Klenow (2007). Hsieh and Klenow (2007) concluded that even if investment prices are no higher in poor countries, the relative price of investment is higher in poor countries relative to rich countries. In our model, the price of the investment-goods in our model is constant. Therefore, the above-mentioned empirical fact is derived from the small rates of technological progress (u_j) of consumption-goods industries. Thus, Corollary 2 supports the conclusion of Hsieh and Klenow (2007). However, Corollary 2 also gives other sources of the empirical result. In fact, an increase in m_j or s persistently reduces the relative prices of consumption-goods. Especially, since households are myopic as the parameter s is large, we see that the above empirical fact result concerning poor countries results from the myopia concerning consumption in the poor countries. Unfortunately, many consumers in poor countries appear to be run after by a daily life. Therefore, it appears to be difficult that they avoid the myopia concerning consumption. In this sense, it is natural that the above-mentioned empirical fact is an inevitable result of the myopia.

We here provide a numerical example and describes the transition of industrial structure in our model economy. To stress the effect of the elasticity of marginal utility on the change of industrial structure, we assume $d_j = 0$ $(j \in N)$ and consider the case where $r - s = 0.01$, $a_1 m_1 = 0.23$, $a_2 m_2 = 0.28$, and $a_3 m_3 = 0.32$. If we assume $m_1 = m_2 = m_3 = 0.5$ for simplicity, then we can observe that the transition of the j th industry $(j \in N)$ depends on the parameter a_j of the utility function concerning the consumption-goods produced by the j th industry. See Figure 1. The blue, red, and green lines of Figure 1 describe three utility functions with $a_1 = 0.23/0.5 = 0.46$, $a_2 = 0.28/0.5 = 0.56$, and $a_3 = 0.32/0.5 = 0.64$. Figure 2 describes the relative scale of each industry. The blue line describes the transition of K_{1t}/K_{3t}, the red line describes the transition of K_{2t}/K_{3t}, and the green line describes the transition of $K_{3t}/K_{3t} = 1$. First, the industry 1 leads the model economy (the blue line). After that, the industry 2 leads the model economy (the red line) and finally, the industry 3 leads the model economy (the green line). The industries 1, 2, and 3 correspond to the primary, secondary, and tertiary industries, respectively.

Figures 1 and 2 about here.

The feature of our endogenous growth model is that such a variety of results as Corollaries 1 and 2 are obtained through the model. In the next section, we extend the model in his section by incorporating labor and will consider whether or not we obtain the same as those of the results in this section.

5. Incorporating Population Growth

For simplicity, we have not considered population growth so far. In this section, we show that population growth can be incorporated into the model. Especially, our main interest in this section is whether the GRRP equation is obtained and whether the similar equation concerning the growth rate of wage is obtained.

Moreover, we see that *dynamic optimization of consumption-goods firms* can also be incorporated. To see it, we modify the background of the model. For simplicity, we assume that the number of firms in the consumption-goods sector is one (i.e. $n = 1$). Unlike the model in Section 2, the consumption-goods firm produces goods by using both labor and capital goods and the number of the households is assumed to grow at the constant rate. Moreover, we suppose that the households supply labor to the consumption-goods firm and rent capital goods to the consumption-goods firm. We denote by $K_{00} = K_{I0} + K_0$ the initial endowment owned by the households, where unlike previous sections we denote by K_0 the initial value of capital stock of the consumption-goods firm. The other suppositions are the same as before. We denote by $L_t = L_0 e^{ht}$ the population growing at the rate of h. Define

$$c_t = C_t / L_t,\ \ q_t = Q_t / L_t,\ \ k_{It} = K_{It} / L_t,\ \ k_t = K_t / L_t.$$

In the model in this section, we see from the supposition that the sum of budget constraints of the households is given by

$$\dot{K}_{It} = rK_{It} + \Pi_t + W_t L_t - P_t C_t,$$

where W_t is the relative wage rate. Like in Section 3, we assume $P_I = 1$. Each budget constraint of the households is

$$\dot{k}_{It} = \left(\frac{\dot{K}_{It}}{L_t}\right) = \frac{\dot{K}_{It} L_t - K_{It} \dot{L}_t}{L_t{}^2} = \frac{\dot{K}_{It}}{L_t} - hk_{It} = (r-h)k_{It} + \pi_t + W_t - P_t c_t,$$

where $\pi_t = \Pi_t / L_t$. The intertemporal optimization problem of the households is now given by:

$$\max \int_{\mathrm{R}_+^1} (c_t{}^a / a) e^{-st} dt \qquad \text{subject to} \quad \dot{k}_{It} = (r-h)k_{It} + \pi_t + W_t - P_t c_t,$$

where $1 > a > 0$. Then, the Hamiltonian of the intertemporal optimization problem is given by

$$H = (c_t{}^a / a) e^{-st} + \eta_t \{(r-h)k_{It} + \pi_t + W_t - P_t c_t\}.$$

Therefore, the first condition entails:

(5.1) $\quad \partial H / \partial c_t = c_t{}^{a-1} e^{-st} - P_t \eta_t = 0,$

(5.2) $\quad \dot{\eta}_t = -\partial H / \partial k_{It} = -(r-h)\eta_t,$

(5.3) $\quad \dot{k}_{It} = (r-h)k_{It} + \pi_t + W_t - P_t c_t,$

(5.4) $\quad \lim_{t \to \infty} k_{It} \eta_t = 0.$

Equations (5.1) and (5.2) yield

$$(6) \qquad P_t = e^{(r-h-s)t} c_t{}^{a-1} / \eta_0 = e^{(r-h-s)t} (C_t / L_t)^{a-1} / \eta_0,$$

where η_0 is derived in the proof of Theorem 1 in Appendix. Like Eq. (2), we call Eq. (6) a dynamic inverse demand equation. From the dynamic inverse equation, we have

$$(7) \qquad \frac{\dot{P}_t}{P_t} = r - h - s - (1-a)(\frac{\dot{C}_t}{C_t} - h).$$

We assume that the consumption-goods firm adopts the Cobb-Douglas function of degree one:

$$(8) \qquad C_t = Q_t = \sigma D(t) K_t{}^m L_t{}^{1-m} \Leftrightarrow \ c_t = \sigma D(t) k_t^m, \quad 0 < m \le 1.$$

For simplicity, assume $\sigma = 1.$ Like $D_j(t)$ in Section 3, we assume that $D(t)$ grows at the rate $d > 0$. Then

$$D(t) = D_0 e^{dt}.$$

For simplicity, we assume $D_0 = 1$. Logarithmic differentiation of the production function (8) yields $\dot{C}_t / C_t = d + mI_t / K_t + (1-m)h.$ Substituting this equation into Eq. (7) yields

$$(9) \qquad \frac{\dot{P}_t}{P_t} = r - h - s - (1-a)\{d + m\frac{I_t}{K_t} + (1-m)h - h\}.$$

Now, the consumption-goods firm is assumed to solve the profit maximization problem under Eq. (9). Like in Section 3, by assuming the price path P_t is given, the consumption-goods industry solves the following optimization problem:

$$\max \int_{R_+^1} (P_t D_t K_t{}^m L_t{}^{1-m} - W_t L_t - I_t) e^{-rt} dt \quad \text{subject to} \quad \dot{K}_t = I_t.$$

We The Hamiltonian is given by

$$H = (P_t D_t K_t{}^m L_t{}^{1-m} - W_t L_t - I_t)e^{-rt} + \lambda_t I_t.$$

The sufficient condition for optimization is given by

(10.1) $\partial H / \partial L_t = \{(1-m)P_t D_t K_t{}^m L_t{}^{-m} - W_t\}e^{-rt} = 0;$

(10.2) $\partial H / \partial I_t = -e^{-rt} + \lambda_t = 0;$

(10.3) $\dot{\lambda}_t = -\partial H / \partial K_t = -mP_t D_t K_t{}^{m-1} L_t{}^{1-m} e^{-rt};$

(10.4) $\dot{K}_t = I_t;$

(10.5) $\lim_{t\to\infty} K_t \lambda_t = 0.$

In this section, we assume the following conditions:

Assumption 3: $1 > am$;

Assumption 4: $r - h > \max\{s, G_h = \dfrac{r-h-s+ad}{1-am}\}.$

Assumptions 3 and 4 yield $G_h > 0$. Assumptions 3 and 4 play the same roles as Assumptions 1 and 2. We now obtain the following result:

Theorem 2: Suppose Assumptions 3 and 4 are satisfied. Then there exist equilibrium growth paths which satisfy

$$gr(k_t) = gr(W_t) = gr(k_{It}) = G_h, \quad gr(c_t) = gr(q_t) = mG_h + d,$$
$$gr(P_t) = (1-m)G_h - d, \quad gr(\pi_t) = G_h. \blacksquare$$

Proof: See Appendix.■

Like Corollary 1, we can derive the GRRP equation concerning per capita. Moreover, we can also derive the equation concerning growth rate of relative wage:

Corollary 3: We have the GRRP equation concerning per capita variables:

$$gr(P_t) = gr(k_t) - gr(c_t).$$

Moreover, for growth rate of relative wage we have the following equality on the equilibrium growth paths:

$$gr(W_t) = gr(k_t) = gr(P_t) + gr(c_t).$$

We call this equation GRRW (growth rate of relative wage) equation.■

Proof: The proof follows directly from Theorem 2. ■

Like the GRRP equation, the GRRW equation is interesting in the sense that it relates the growth rate of relative wage to the growth rate of relative price and the growth rates of goods. The equation is also novel.

Corollary 4: Concerning the per capita variables and the growth rates of P_t and W_t, we obtain the same results as of Corollary 2. Moreover,

$$\begin{aligned}&\text{the growth rate of population } h \uparrow \\ &\quad \Rightarrow\ gr(c_t)\downarrow,\ gr(P_t)\downarrow,\ gr(k_t)\downarrow,\ gr(k_{It})\downarrow,\ gr(W_t)\downarrow \ \text{ for any } j \in N,\end{aligned}$$

Moreover, as a (resp. m) becomes large, the effect of the growth rate of population h on the growth rate of goods becomes large (resp. small). That is

$$\frac{\partial^2 gr(c_t)}{\partial a \partial h} < 0, \quad \frac{\partial^2 gr(k_t)}{\partial a \partial h} < 0, \quad \frac{\partial^2 gr(k_{It})}{\partial a \partial h} < 0,$$
$$\frac{\partial^2 gr(c_t)}{\partial m \partial h} < 0, \quad \frac{\partial^2 gr(k_t)}{\partial m \partial h} < 0, \ \text{ and } \ \frac{\partial^2 gr(k_{It})}{\partial m \partial h} < 0.$$

As a (resp. m) becomes large, the effect of the growth rate of population h on the growth rate of P_t becomes large (resp. small). That is

$$\frac{\partial^2 gr(P_t)}{\partial a \partial h} < 0 \ \text{ and } \ \frac{\partial^2 gr(P_t)}{\partial m \partial h} > 0.$$

As a (resp. m) becomes large, the effect of the growth rate of population h on the growth rate of W_t becomes large (resp. small). That is

$$\frac{\partial^2 gr(W_t)}{\partial a \partial h} < 0 \ \text{ and } \ \frac{\partial^2 gr(W_t)}{\partial m \partial h} < 0. \blacksquare$$

Proof: See Appendix. ■

Thus, we see that almost the same result as before can be obtained even if population growth is incorporated into the endogenous growth model.

6. Conclusions and Final Remark

In this paper, developing the AK model of decentralized infinite horizon optimization, we constructed an endogenous growth model that explains the transition of the relative scales of heterogeneous industries with different production functions. Unlike the AK model, we assumed that households and representative firms of industries plan the optimal schedules allowing for the growth and the persistent change of relative prices. Not only the growth rate of each industry but also the growth rate of relative price depends on the elasticity of marginal productivity of the industry and the elasticity of marginal utility of the utility function of the goods produced by the industry. We proved that as the productivity of an industry gets high or the elasticity of marginal utility gets large, the relative price in the industry decreases and the growth rate of the industry increases. Consequently, the relative scales of industries change. This provides a theoretical explanation of the persistent transition of industrial structure.

Moreover, we derived two important equations. The first one is an equation that connects the growth rate of relative price with the growth rates of goods. By incorporating population growth, we derived an second equation that connects the growth rate of relative wage with the growth rate of relative price and the growth rates of goods. These equations are novel.

We showed that an increase in time preference or the rate of technological progress or a decrease in the elasticity of marginal productivity reduces the growth rate of relative price of consumption-goods. Thus, we argued several possible sources concerning the change of relative price, which may explain the empirical fact that the relative price of investment-goods in poor countries is higher than that of rich countries. Among the possible sources, we stressed that the empirical fact result concerning poor countries results from the myopia of consumers. We demonstrated that the empirical fact result concerning poor countries results from the myopia concerning consumption in the poor countries. Since many consumers in many poor countries appear to be run after by a daily life, our result concerning myopia is natural. Moreover, for the empirical fact concerning poor countries, Hsieh and Klenow (2007) demonstrated that even if investment prices are no higher in poor countries, the relative price of investment is higher in poor countries relative to rich countries. Our results concerning the above-mentioned possible sources also support it.

In our model, the change of relative prices admits of the differences among growth rates of heterogeneous industries. In other words, if relative prices do not change, then consistent optimal growth of heterogenous industries cannot be accomplished. In other words, optimal coexistence of heterogeneous industries can be achieved through such a price mechanism. The persistent transition of industrial structure emerges as an inevitable consequence of the optimal coexistence of heterogeneous industries.

Our results are analytically connected to fundamental economic parameters. This is a remarkable feature of our model. Moreover, though analyses of our model are slightly complicated, our model itself is relatively flexible. Therefore, it is expected that our model has further development potential, although we must leave it as a future research.

We could consider the endogenous growth model with the other types of utility functions:

$$U(C_{1t},\cdots,C_{nt}) = \prod_{k\in N} C_{kt}{}^{a_k}, \quad U(C_{1t},\cdots,C_{nt}) = \sum_{k\in N} \log C_{kt}.$$

where $0 < a_j < 1$ $(j \in N)$. Moreover, we also could consider the endogenous growth model with the CES production under the slightly specialized production function with $m_j = m$:

$$U(C_{1t},\cdots,C_{nt}) = \left(\sum_{k\in N} C_{kt}^{a}\right)^{1/a}, \quad Q_{jt} = D_j(t) K_{jt}{}^{m}.$$

where $0 < a < 1$. Even if we assume these utility functions, we obtain the same results as Theorems 1 and 2 and Corollaries 1 and 3. In order to consider the dependence of growth rates on various fundamental parameters and the difference among the growth rates of industries, we employed the additive utility function provided in Section 3.

7. Appendix

In this Appendix, we prove Theorem 1, Corollary 2, Theorem 2 and Corollary 4.

Proof of Theorem 1: Before determining the initial value of capital stock of each industry, we derive the equilibrium growth paths assuming that the initial values of capital stock of each industry are given. After deriving them, we will calculate the initial values of capital stock. By assuming the relative price path is given, we maximize the profit of industry j $(\in N)$.

(A.1) $$\Pi_{jt} = P_{jt} e^{d_j t} K_{jt}{}^{m_j} - K_{jt}$$

The first order condition yields $m_j P_{jt} e^{d_j t} K_{jt}{}^{m_j - 1} = 1$ so that from the dynamic inverse demand equation (2), we have

$$1 = \frac{m_j e^{(r-s)t} e^{d_j t} K_{jt}^{m_j - 1}}{\eta_0 C_{jt}^{1-a_j}} = \frac{m_j e^{(r-s)t} e^{d_j t} K_{jt}^{m_j - 1}}{\eta_0 e^{(1-a_j) d_j t} K_{jt}^{(1-a_j) m_j}} = \frac{m_j e^{(r-s)t} e^{d_j t}}{\eta_0 e^{-a_j d_j t} K_{jt}^{1-a_j m_j}}$$

where η_0 is determined later. Thus, we have

(A.2) $$K_{jt} = K_{j0} e^{G_j t}, \quad K_{j0} = (m_j \eta_0{}^{-1})^{\frac{1}{1-a_j m_j}},$$

for any $j \in N$. It follows from (A.2) that we have the following equilibrium growth paths of $Q_{jt} = C_{jt}$:

(A.3) $$Q_{jt} = C_{jt} = e^{d_j t} K_{jt}^{m_j} = (m_j \eta_0{}^{-1})^{\frac{m_j}{1-a_j m_j}} e^{(m_j G_j + d_j)t}$$

where $j \in N$. We have

(A.4) $$r - s = (1 - a_j m_j) G_j - a_j d_j .$$

Therefore, from (4) and (A.2), the growth rate of the equilibrium growth path of P_{jt}

$(j \in N)$ is given by

(A.5.1) $$gr(P_{jt}) = r - s - (1-a_j)d_j - (1-a_j)m_j G_{jt}$$
$$= \frac{(1-a_j m_j)(r-s) - (1-a_j m_j)(1-a_j)d_j - (m_j - a_j m_j)(r - s + a_j d_j)}{1-a_j m_j}$$
$$= \frac{r - s - a_j m_j (r-s) - (1-a_j m_j)d_j + (1-a_j m_j)a_j d_j}{1-a_j m_j}$$
$$\frac{-(m_j - a_j m_j)(r-s) - (m_j - a_j m_j)a_j d_j}{1-a_j m_j}$$
$$= \frac{r - s - m_j(r-s) + a_j d_j - m_j a_j d_j}{1-a_j m_j} - d_j = (1-m_j)G_j - d_j.$$

Moreover, the initial value of the equilibrium growth path of P_{jt} is given by

(A.5.2) $$P_{j0} = \frac{1}{\eta_0 K_{j0}{}^{(1-a_j)m_j}} = \frac{1}{\eta_0 \{(a_j m_j \eta_0{}^{-1})^{1/(1-a_j m_j)}\}^{(1-a_j)m_j}}$$
$$= (a_j m_j)^{-\frac{(1-a_j)m_j}{1-a_j m_j}} \eta_0^{-1+\frac{(1-a_j)m_j}{1-a_j m_j}}$$
$$= (a_j m_j)^{-\frac{(1-a_j)m_j}{1-a_j m_j}} \eta_0^{-\frac{1-m_j}{1-a_j m_j}}$$

where $j \in N$. Thus, if η_0 is determined, all optimal paths are completely determined.

Now, for a given K_{I0}, we determine η_0. Since we have $\Pi_{jt} = P_{jt}Q_{jt} - K_{jt} = P_{jt}C_{jt} - K_{jt}$, it follows from (A.2) that

(A.6) $$\dot{K}_{It} = \sum_{k \in N} \Pi_{kt} + rK_{It} - \sum_{k \in N} P_{kt}C_{kt}$$
$$= rK_{It} - \sum_{k \in N} K_{kt} = rK_{It} - \sum_{k \in N} K_{k0}e^{G_k t}.$$

To solve the differential equation (A.6), we prepare a sublemma.

Sublemma1: We consider the differential equation $\dot{x}_t = ax_t + f(t)$, where a is a constant real number and $f(t)$ is a continuous function. The solution of the differential equation is given by

$$x_t = x_0 e^{at} + e^{at} \int_{[0,t]} e^{-au} f(v) dv. ■$$

Proof: See Perko (1996, Remark 2 in Section 1.10).■

From Sublemma 1, the solution of (A.6) is given by

$$K_{It} = K_{I0} e^{rt} - \int_{[0,t]} e^{r(t-v)} \sum_{k \in N} K_{k0} e^{G_k v} dv$$
$$= K_{I0} e^{rt} - \sum_{k \in N} \frac{K_{k0}}{G_k - r} e^{G_k t} + e^{rt} \sum_{k \in N} \frac{K_{k0}}{G_k - r}.$$

Define

(A.7) $$K_{It} = \sum_{k \in N} \frac{K_{k0}}{r - G_k} e^{G_k t}.$$

Equation (4.2) yields $\eta_t = \eta_0 e^{-rt}$ and Assumption 2 yields $r - G_j > 0$ for any $j \in N$. Therefore, we see from (A.7) that

$$\lim_{t \to \infty} K_{It} \eta_t = \lim_{t \to \infty} \sum_{k \in N} \frac{K_{k0} \eta_0}{r - G_k} e^{-(r - G_k)t} = 0.$$

Thus, the transversality condition (1.4) is satisfied. Now, we determine η_0. Since the initial endowment of capital stock which the household possesses is given by $K_0 = K_{I0} + \sum_{k \in N} K_{k0}$, we see from the equations (A.2) and (A.7) that

(A.8) $$K_0 = K_{I0} + \sum_{k \in N} K_{k0} = \sum_{k \in N} \frac{1 + r - G_k}{r - G_k} (a_k m_k \eta_0{}^{-1})^{1/(1 - a_k m_k)} \equiv \Theta(\eta_0).$$

Thus, η_0 must satisfy (A.8). We consider the continuous function $\Theta : \mathrm{R}^1_+ \to \mathrm{R}^1_+$, where $\mathrm{R}^1_+ \equiv \{v \in \mathrm{R}^1 : v > 0\}$. The $\Theta -$function is continuously differentiable. Clearly, we have

(A.9) $$\Theta'(v) < 0, \quad \lim_{u \to \infty} \Theta(v) = 0, \text{ and } \lim_{u \to 0} \Theta(v) = \infty.$$

Figure 3 about here.

(A.9) proves that the $\Theta-$function is a strictly monotone decreasing function. See Figure 1. Therefore, the inverse of the $\Theta-$ function exists: $\Theta^{-1}: \mathrm{R}_{+}^{1} \to \mathrm{R}_{+}^{1}$ $(w \to \Theta^{-1}(w))$, For a given K_0, the initial value of η_t is now given by $\eta_0 = \Theta^{-1}(K_0)$. Thus, we see that the optimal paths are given by (A.2) to (A.5) with $\eta_0 = \Theta^{-1}(K_0)$. Therefore, for any $j \in N$, we can see from (A.2) to (A.5) that the profit of each industry on the equilibrium growth paths is given by

$$
\begin{aligned}
\Pi_{jt} &= P_{jt}Q_{jt} - K_{jt} = P_{jt}D_jK_{jt}^{m} - K_{jt} \\
&= \{(a_jm_j)^{-\frac{(1-a_j)m_j}{1-a_jm_j}} \eta_0^{-\frac{1-m_j}{1-a_jm_j}} e^{\{(1-m_j)G_j - d_j\}t} (a_jm_j\eta_0{}^{-1})^{\frac{m_j}{1-a_jm_j}} e^{m_jG_jt} e^{d_jt} \\
&\qquad - (a_jm_j)^{\frac{1}{1-a_jm_j}} \eta_0^{-\frac{1}{1-a_jm_j}} e^{G_jt} \\
&= \{(a_jm_j)^{-\frac{a_jm_j}{1-a_jm_j}} \eta_0^{-\frac{1}{1-a_jm_j}} e^{G_jt} - (a_jm_j)^{\frac{1}{1-a_jm_j}} \eta_0^{-\frac{1}{1-a_jm_j}} e^{G_jt} \\
&= \{(a_jm_j)^{\frac{a_jm_j}{1-a_jm_j}} - (a_jm_j)^{\frac{1}{1-a_jm_j}}\} \eta_0^{\frac{-1}{1-a_jm_j}} e^{G_jt} \\
&= (a_jm_j)^{\frac{a_jm_j}{1-a_jm_j}} (1-a_jm_j) \eta_0^{\frac{-1}{1-a_jm_j}} e^{G_jt}.
\end{aligned}
$$

Then, we see

(A.10) $\qquad gr(\Pi_{jt}) = G_j,$

for any $j \in N$. Finally, we prove the results on the asymptotic growth of K_{It}. We define $\Psi = \{k \in N : G_k = G_{\max}\}$. The growth rate of K_{It} is given by

$$
\frac{\dot{K}_{It}}{K_{It}} = \frac{\sum_{k \in N} \{G_kK_{k0}/(r - G_k)\} e^{G_kt}}{\sum_{k \in N} \{K_{k0}/(r - G_k)\} e^{G_kt}}
$$

$$= \frac{\sum_{k\in N\backslash\Psi} G_k\{K_{k0}/(r-G_k)\}e^{(G_k-G_{\max})t} + G_{\max}\sum_{k\in\Psi} K_{k0}/(r-G_{\max})}{\sum_{k\in N\backslash\Psi} \{K_{k0}/(r-G_k)\}e^{(G_k-G_{\max})t} + \sum_{k\in\Psi} K_{k0}/(r-G_{\max})},$$

where $A\backslash B$ is the difference of A and B. Therefore, since $G_k < G_{\max}$ for any $k \in N\backslash\Psi$, we see

$$agr(K_{It}) = \lim_{t\to\infty} \dot{K}_{It}/K_{It} = G_{\max}.$$

This completes the proof of Theorem 1.■

Proof of Corollary 2: The growth rates of the optimal growth path are given by

(A.11.1) $\quad gr(K_{jt}) = gr(I_{jt}) = G_j,$

(A.11.2) $\quad gr(C_{jt}) = gr(Q_{jt}) = m_j G_j + d_j,$

(A.11.3) $\quad gr(P_{jt}) = (1-m_j)G_j - d_j,$

for any $j \in N$. On the other hand, we have

(A.12.1) $$\frac{\partial gr(K_{jt})}{\partial a_j} = \frac{\partial gr(I_{jt})}{\partial a_j} = \frac{\partial G_j}{\partial a_j} = \frac{(r-s)m_j + d_j}{(1-a_j m_j)^2} > 0,$$

(A.12.2) $$\frac{\partial gr(K_{jt})}{\partial m_j} = \frac{\partial gr(I_{jt})}{\partial m_j} = \frac{\partial G_j}{\partial m_j} = \frac{a_j}{1-a_j m_j} G_j > 0,$$

(A.12.3) $$\frac{\partial gr(K_{jt})}{\partial d_j} = \frac{\partial gr(I_{jt})}{\partial d_j} = \frac{\partial G_j}{\partial d_j} = \frac{a_j}{1-a_j m_j} > 0,$$

(A.12.4) $$\frac{\partial gr(K_{jt})}{\partial s} = \frac{\partial gr(I_{jt})}{\partial s} = \frac{\partial G_j}{\partial s} = -\frac{1}{1-a_j m_j} < 0,$$

for any $j \in N$. Thus, we see from (A.12) and Assumption 1 that

(A.13.1) $$\frac{\partial gr(C_{jt})}{\partial a_j} = \frac{\partial gr(Q_{jt})}{\partial a_j} = \frac{\partial(m_j G_j + d_j)}{\partial a_j} = \frac{m_j\{(r-s)m_j + d_j\}}{(1-a_j m_j)^2} > 0,$$

(A.13.2) $$\frac{\partial gr(C_{jt})}{\partial m_j} = \frac{\partial gr(Q_{jt})}{\partial m_j} = \frac{\partial(m_j G_j + d_j)}{\partial m_j} = \frac{1}{1-a_j m_j} G_j > 0,$$

(A.13.3) $$\frac{\partial gr(C_{jt})}{\partial d_j} = \frac{\partial gr(Q_{jt})}{\partial d_j} = \frac{\partial(m_j G_j + d_j)}{\partial d_j} = \frac{1}{1-a_j m_j} > 0,$$

(A.13.4) $$\frac{\partial gr(C_{jt})}{\partial s}=\frac{\partial gr(Q_{jt})}{\partial s}=\frac{\partial (m_jG_j+d_j)}{\partial s}=-\frac{m_j}{1-a_jm_j}>0,$$

(A.13.5) $$\frac{\partial gr(P_{jt})}{\partial a_j}=\frac{\partial\{(1-m_j)G_j-d_j\}}{\partial a_j}=\frac{(1-m_j)\{(r-s)m_j+d_j\}}{(1-a_jm_j)^2}>0,$$

(A.13.6) $$\frac{\partial gr(P_{jt})}{\partial m_j}=\frac{\partial\{(1-m_j)G_j-d_j\}}{\partial m_j}=-G_j\{1-\frac{(1-m_j)a_j}{1-a_jm_j}\}$$

$$=-G_j\frac{1-a_j}{1-a_jm_j}<0,$$

(A.13.7) $$\frac{\partial gr(P_{jt})}{\partial d_j}=\frac{\partial\{(1-m_j)G_j-d_j\}}{\partial d_j}=\{\frac{(1-m_j)a_j}{1-a_jm_j}-1\}=-\frac{1-a_j}{1-a_jm_j}<0,$$

(A.13.8) $$\frac{\partial gr(P_{jt})}{\partial s}=\frac{\partial\{(1-m_j)G_j-d_j\}}{\partial s}=-\frac{1-m_j}{1-a_jm_j}<0.$$

for any $j\in N$. On the other hand, since $agr(K_{It})=\max\{G_j : j\in N\}$, the results on $agr(K_{It})$ follows directly from the results of (A.12). The equation (A.11) and the inequalities in (A.13) complete the proof of the first half of Corollary 2. We next prove the latter half.

(A.14.1) $$\frac{\partial^2 gr(K_{jt})}{\partial a_j\partial d_j}=\frac{\partial}{\partial a_j}\left(\frac{a_j}{1-a_jm_j}\right)=\frac{1}{(1-a_jm_j)^2}>0\,,$$

(A.14.2) $$\frac{\partial^2 gr(C_{jt})}{\partial a_j\partial d_j}=\frac{\partial}{\partial a_j}\left(\frac{1}{1-a_jm_j}\right)=\frac{m_j}{(1-a_jm_j)^2}>0\,,$$

(A.14.3) $$\frac{\partial^2 gr(P_{jt})}{\partial a_j\partial d_j}=\frac{\partial}{\partial a_j}\left(-\frac{1-a_j}{1-a_jm_j}\right)=\frac{(1-m_j)}{(1-a_jm_j)^2}>0\,,$$

(A.14.4) $$\frac{\partial^2 gr(K_{jt})}{\partial m_j\partial d_j}=\frac{\partial}{\partial m_j}\left(\frac{a_j}{1-a_jm_j}\right)=\frac{a_j^2}{(1-a_jm_j)^2}>0\,,$$

(A.14.5) $$\frac{\partial^2 gr(C_{jt})}{\partial m_j\partial d_j}=\frac{\partial}{\partial m_j}\left(\frac{1}{1-a_jm_j}\right)=\frac{a_j}{(1-a_jm_j)^2}>0\,,$$

(A.14.6) $$\frac{\partial^2 gr(P_{jt})}{\partial m_j\partial d_j}=\frac{\partial}{\partial m_j}\left(-\frac{1-a_j}{1-a_jm_j}\right)=-\frac{(1-a_j)a_j}{(1-a_jm_j)^2}<0\,.$$

By the same argument as above, the results on $agr(K_{It})$ follow directly from (A.14.1) and (A.14.2). The inequalities of (A.14) now prove the proof of the latter half of Corollary 2. Thus, we complete the proof of Corollary 2.■

Proof of Theorem 2: Eqs. (10.2) and (10.3) yield

$$re^{-rt} = mP_t D_t K_t{}^{m-1} L_t{}^{1-m} e^{-rt}.$$

Therefore, we have

(A.15) $$r = mP_t D_t k_t{}^{m-1}.$$

The dynamic inverse demand equation (6) and the production function (8) yield

(A.16) $$P_t = \frac{e^{(r-h-s)t}}{\eta_0 c_t{}^{1-a}} = \frac{e^{(r-h-s)t}}{\eta_0 D_t^{(1-a)} k_t{}^{(1-a)m}} = \frac{e^{\{r-h-s-(1-a)d\}t}}{\eta_0 k_t{}^{(1-a)m}}.$$

Substituting Eq. (A.16) into Eq. (A.15) yields

$$r = m\frac{e^{\{r-h-s-(1-a)d\}t}}{\eta_0 k_t^{(1-a)m}} e^{ut} k_t{}^{m-1} = \frac{me^{\{r-h-s+ad\}t} k_t{}^{am-1}}{\eta_0},$$

so that $k_t = (m/\eta_0 r)^{1/(1-am)} e^{G_h t}$. Now, define

$$\eta_0 = m / rk_0{}^{1-am}.$$

Then we have

$$k_t = k_0 e^{G_h t}, \quad q_t = c_t = k_0{}^m e^{mG_h t}.$$

Moreover, from Eq. (A.16), we have

(A.17.1) $$\begin{aligned} gr(P_t) &= r-h-s-(1-a)d-(1-a)m \bullet gr(k_t) \\ &= r-h-s-(1-a)d-(1-a)m\frac{r-h-s+ad}{1-am} \\ &= \frac{(1-am)\{r-h-s-(1-a)d\}-(1-a)m(r-h-s+ad)}{1-am} \\ &= \frac{(1-am)(r-h-s)-(m-am)(r-h-s)}{1-am} \\ &\qquad \frac{-(1-am)(d-ad)-(m-am)au}{1-am} \end{aligned}$$

$$= \frac{(1-m)(r-h-s)+(1-m)ad-(1-am)d}{1-am} = (1-m)G_h - d,$$

(A.17.2) $$P_0 = \frac{1}{\eta_0 k_0^{(1-a)m}} = \frac{rk_0^{1-am}}{mk_0^{(1-a)m}} = \frac{rk_0^{1-m}}{m}.$$

Therefore we obtain from Eq. (10.1) that

$$gr(W_t) = gr(P_t) + m \bullet gr(k_t) = (1-m)G_h + mG_h = G_h,$$
$$W_0 = (1-m)P_0 k_0^m = \frac{(1-m)rk_0}{m}$$

On the other hand, The dynamic equation concerning the equilibrium path of capital stock of the investment-goods firm becomes

$$\dot{k}_{It} = (r-h)k_{It} + \pi_t + W_t - P_t c_t = (r-h)k_{It} - k_0 e^{G_h t}.$$

The solution of the differential equation is given by

$$k_{It} = k_{I0} e^{(r-h)t} - \int_{[0,t]} e^{(r-h)(t-v)} k_0 e^{G_h v} dv$$
$$= \left(k_{I0} - \frac{k_0}{r-h-G_h} \right) e^{(r-h)t} + \frac{k_0 e^{G_h t}}{r-h-G_h}.$$

Now, we set

(A.18) $$k_{I0} = \frac{k_0}{r-h-G_h}.$$

Then we have

$$k_{It} = \frac{k_0}{r-h-G_h} e^{G_h t}$$

and it follows from Assumption 4 that the transversality condition (10.7) is satisfied. Finally, we determine the initial values of capital stocks of consumption-goods and investment-goods firms. In the same way as before, by using Eq. (A.18) we calculate the initial capital stocks of consumption-goods firm and investment-goods firm.

$$K_{00} = K_{I0} + K_0 = k_{I0} L_0 + k_0 L_0 = \left(\frac{1}{r-h-G_h} + 1 \right) k_0 L_0.$$

Therefore, initial values of capital stocks are given by

$$K_0 = k_0 L_0 = \frac{r-h-G_h}{1+r-h-G_h} K_{00},$$
$$K_{I0} = k_{I0} L_0 = \frac{k_0 L_0}{r-h-G_h} = \frac{1}{1+r-h-G_h} K_{00}.$$

From Assumption 4, we see $K_0 < K_{00}$ and $K_{I0} < K_{00}$. Moreover, we obtain the following results on the profit:

(A.19) $$\pi_t = \Pi_t / L_t = P_t D_t {k_t}^m - W_t - I_t / L_t$$
$$= \frac{r{k_0}^{1-m}}{m} e^{(1-m)G_h t} {k_0}^m e^{mG_h t} - \frac{(1-m)rk_0}{m} e^{G_h t} - \frac{\dot{K}_t}{L_t}$$
$$= \frac{rk_0}{m} e^{G_h t} - \frac{(1-m)rk_0}{m} e^{G_h t} - \frac{\dot{K}_t}{L_t} = rk_0 e^{G_h t} - \frac{\dot{K}_t}{L_t}.$$

On the other hand, we have

$$\dot{k}_t = \left(\dot{\frac{K_t}{L_t}}\right) = \frac{\dot{K}_t L_t - K_t \dot{L}_t}{L_t^2} = \frac{\dot{K}_t}{L_t} - k_t h.$$

Therefore, Eq. (A.18) yields

$$\pi_t = rk_0 e^{G_h t} - \dot{k}_t - k_t h = (r - G_h - h) k_0 e^{G_h t}.$$

Thus, we have $gr(\pi_t) = G_h$. Thus we complete the proof of Theorem 2.■

Proof of Corollary 4: The growth rates of the optimal growth path are given by

(A.20.1) $gr(k_t) = gr(W_t) = G_h,$
(A.20.2) $gr(c_t) = mG_h + d,$
(A.20.3) $gr(P_t) = (1-m)G_h - d,$

On the other hand, we have

(A.21.1) $$\frac{\partial gr(k_t)}{\partial h} = \frac{\partial gr(k_{It})}{\partial h} = \frac{\partial gr(W_t)}{\partial h} = \frac{\partial G_h}{\partial h} = \frac{-1}{1-am} < 0,$$

(A.21.2) $$\frac{\partial gr(c_t)}{\partial h} = \frac{\partial (mG_h + d)}{\partial h} = \frac{-m}{1-am} < 0,$$

(A.21.3) $$\frac{\partial gr(P_t)}{\partial h} = \frac{\partial \{(1-m)G_h - d\}}{\partial h} = \frac{-(1-m)}{1-a_j m_j} < 0,$$

Thus, we see from (A.21) and Assumption 3 that

(A.22.1) $$\frac{\partial^2 gr(k_t)}{\partial a \partial h} = \frac{\partial^2 gr(k_{It})}{\partial a \partial h} = \frac{\partial^2 gr(W_t)}{\partial a \partial h} = \frac{\partial}{\partial a}\left(\frac{-1}{1-am}\right) = \frac{-n}{(1-am)^2} < 0,$$

(A.22.2) $$\frac{\partial^2 gr(c_t)}{\partial a \partial h} = \frac{\partial}{\partial a}\left(\frac{-m}{1-am}\right) = \frac{-m^2}{(1-am)^2} < 0,$$

(A.22.3) $$\frac{\partial^2 gr(P_t)}{\partial a \partial h} = \frac{\partial}{\partial a}\left(-\frac{1-m}{1-am}\right) = -\frac{(1-m)m}{(1-am)^2} < 0,$$

(A.22.4) $$\frac{\partial^2 gr(k_t)}{\partial m \partial h} = \frac{\partial^2 gr(k_{It})}{\partial m \partial h} = \frac{\partial^2 gr(W_t)}{\partial m \partial h} = \frac{\partial}{\partial m}\left(\frac{-1}{1-am}\right) = \frac{-a}{(1-am)^2} < 0,$$

(A.22.5) $$\frac{\partial^2 gr(c_t)}{\partial m \partial h} = \frac{\partial}{\partial m}\left(\frac{-m}{1-am}\right) = \frac{-am}{(1-am)^2} < 0,$$

(A.22.6) $$\frac{\partial^2 gr(P_t)}{\partial m \partial h} = \frac{\partial}{\partial m}\left(-\frac{1-m}{1-am}\right) = \frac{1-a}{(1-am)^2} > 0.$$

Thus, from Eqs. (A.2) and (A.22), we complete the proof of Corollary 4.■

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Figure Captions

Figure 1: The effect of the elasticity of marginal utility on the form of the utility function.

Figure 2: The effect of the elasticity of marginal utility on the relative scale of industry.

Figure 3: The Θ-function.

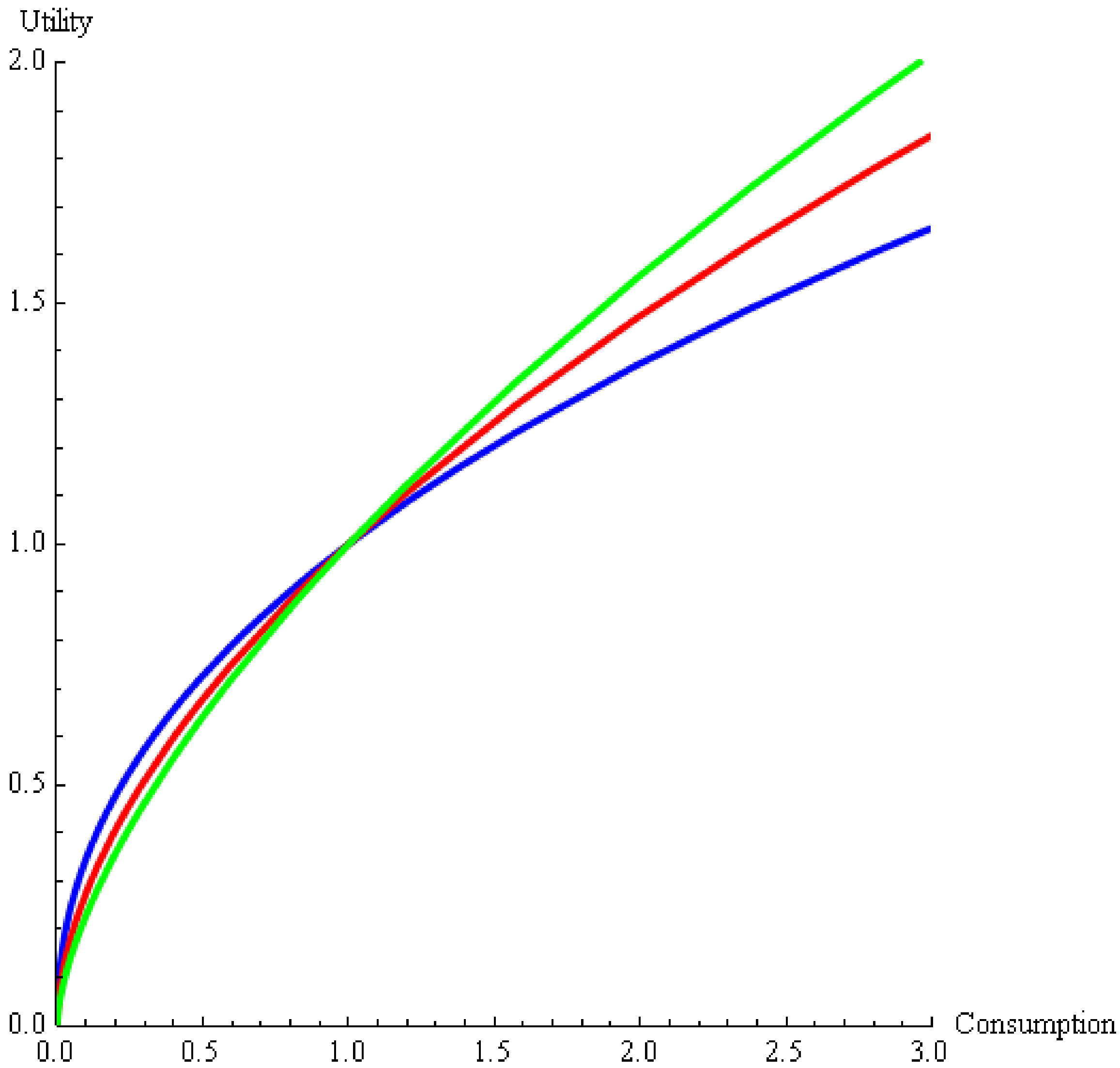

Figure 1

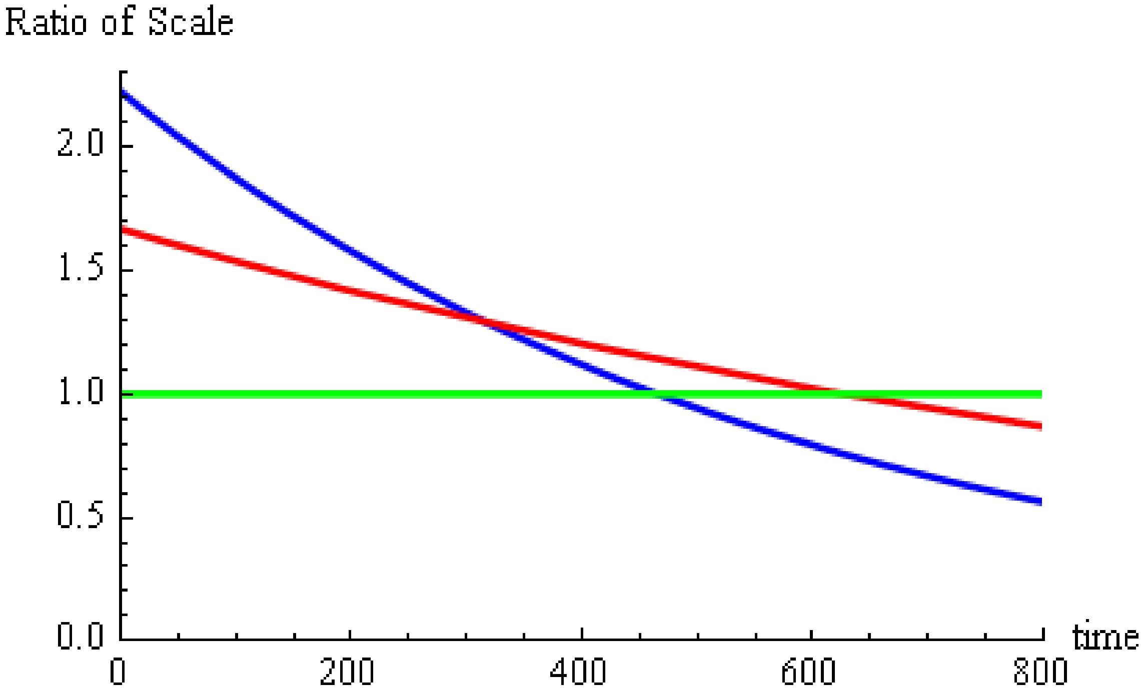

Figure 2

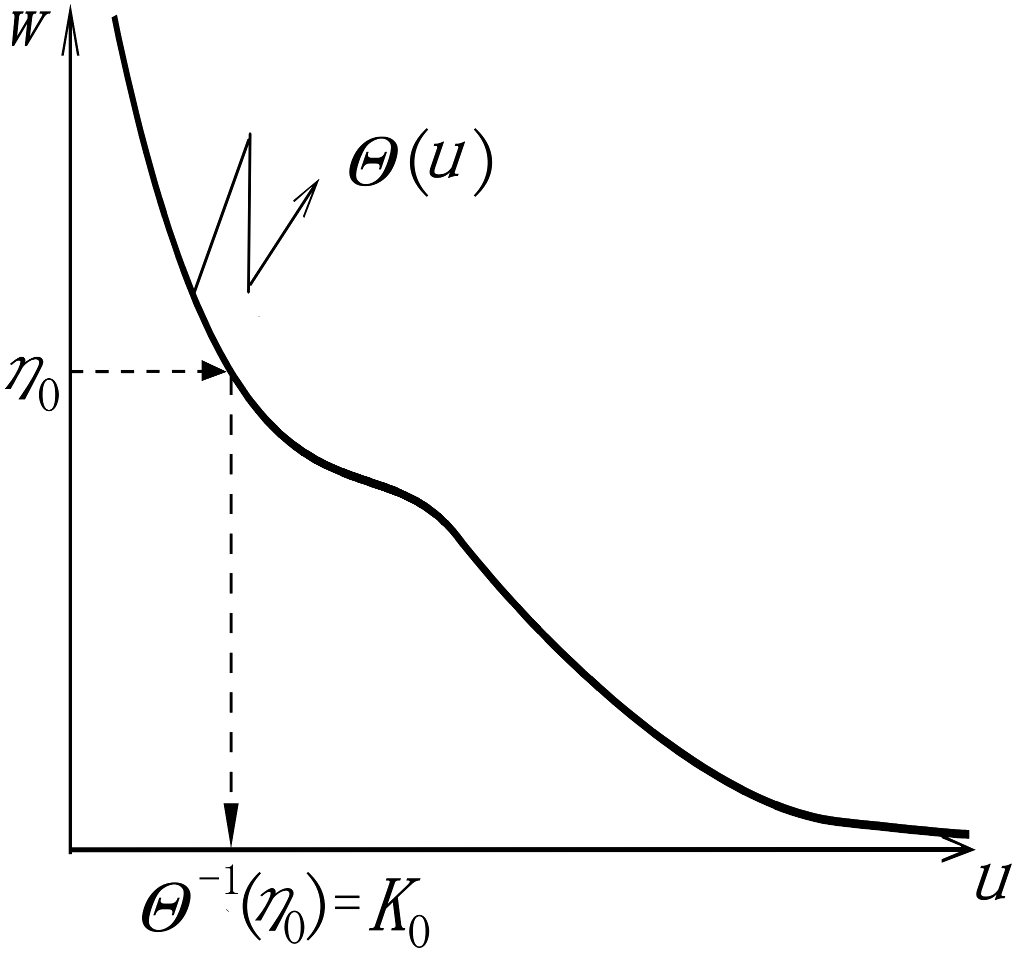

Figure 3