Plenarias

Computer algebra systems for the 21st century: new kind of dynamic representations [1]

James J. Kaput

Universidad de Massachusetts - Dartmouth , United States

Resumen. La idea central de la presentación es la extensión de las representaciones, heredadas de los medios dinámicos en el siglo 21, y cómo estas van más allá de lo que anteriormente era posible en términos de construir las grandes ideas de la Matemática del Cambio y la Variación, incluyendo las ideas subyacentes al Cálculo, en formas más accesibles para nuevas poblaciones de estudiantes.

Three Starting-Point Trends and Assumptions

We begin by listing some working assumptions concerning three trends: (1) the evolving importance of the graphical side of quantitative reasoning in tomorrow's society for the great majority of citizens (both technical specialists and the population in general), (2) the changing roles of the technologies that we use in mathematics education, and (3) the changing nature of the technologies that we use in mathematics education. The first trend suggests that we are undergoing a deep evolution in underlying representational infrastructures due to the computational medium. The second involves a widening of the roles of technology from assisting with the manipulation and linking of notations towards supplying links to phenomena that are embodied in the wider world that has existed apart from mathematical notations and from computers to include, or at least tightly link to those phenomena within the cybernetic universe. The third simply reflects the diversification of technologies that is rapidly turning, to include more personal and portable devices that are increasingly networkable. In discussing these trends and our assumptions regarding them, we will offer some concrete examples of more graphically intensive approaches to substantial mathematics intended to illustrate the need for deep review of appropriate content that has historically been approached through extended instruction in formal mathematics with significant application to modeling and making sense of the experienced world postponed or left to others outside mathematics.

If we had space, we would have included an examination of classroom connectivity across diverse hardware systems. This is an area where we are now concentrating our efforts, particularly in integrating desk-top and hand-held technologies described below within a wireless classroom network. We see considerable promise in engaging students in joint construction of and classroom display of mathematical objects, particularly parametrically definable families of objects.

The Evolving Importance of the Graphical Side of Quantitative Reasoning

World-wide, and certainly in the United States, we have heard repeated calls for at least 20 years to integrate algebraic reasoning across all grades, make fuller connections with other mathematical topics, and updating to account for the increasing impacts of computer technologies, including CAS's, and more recently, graphing calculators. The latter recommendations attend to the importance of graphical representation of variable quantities and iterative functions. However, the underlying assumption of these and most recommendations for use of graphical representations is that the primary notation is character-string-based, and that the graphs are either defined by formulas, or approximated by formulas (e.g., linear regression). Mathematical power is assumed to follow from the ability to exploit the syntactical coherence of character string-based notations.

Our assumption is that the ability not only to interpret but to create and manipulate graphical depictions of variable quantities apart from character-string expressions are important skills for most citizens. Consider the Question posed in Figure 1.

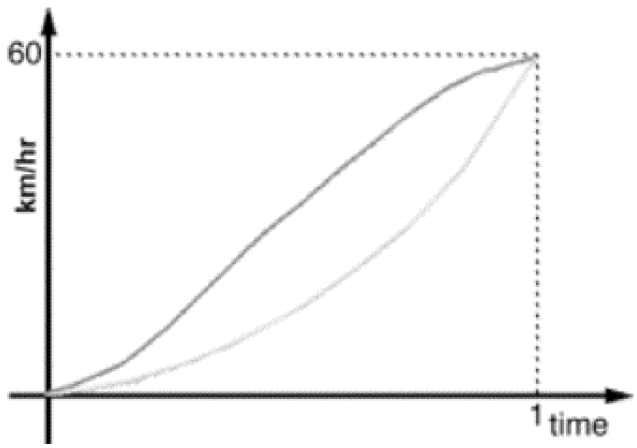

Figure 1. Both the red ca r and the green car reach 60 km/hr after one minute as shown on the two velocity graphs. Are they side by side after one minute or is one ahead?

Now consider this question replaced by one of the form with a similar graph: You are given two job offers, each of which brings you from 68 thousand pesos to 113 thousand pesos annual salary after 4 years, where your monthly salary under the two offers is depicted in the given figure (see Figure 2).

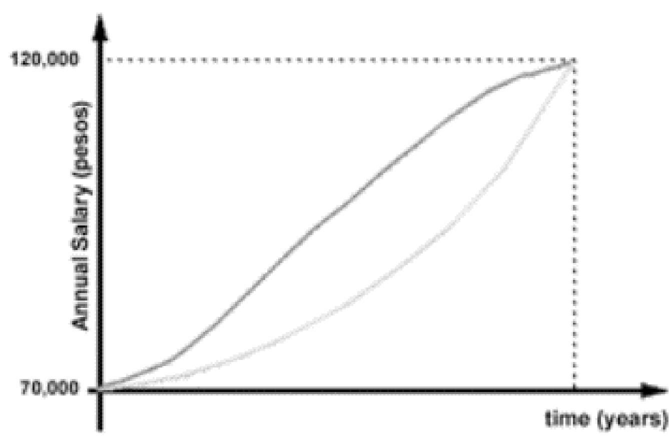

Figure 2. You have two salary offers, each getting you from 70 to 120 thousand pesos in 4 years as shown by the two graphs. Are these equivalent offers or, if not, which is better?

And then consider a follow-up question based on extensions of the given salary graphs so that the two salary graphs extend continuously to the right in such a way that they reverse dominance and so that the area between the two extended pieces is nearly the same as the area between them to the left of the point of intersection: Does the extension of the job offers change the choice of which is the more valuable? (See Figure 3.)

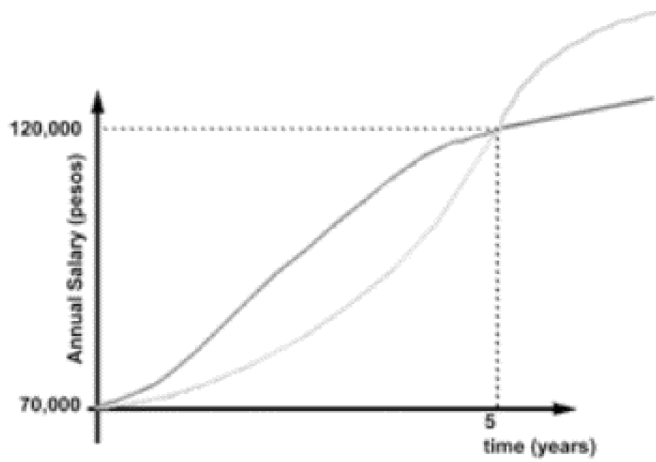

Figure 3. The companies come back with new offers extending for two more years. Are they equivalent or, if not, which one is better—and why?

I wish to draw three inferences from these simple examples:

1. The graphical reasoning—involving areas under or between graphs can be very general, ranging both across contexts and across function-types,

2. The reasoning need not require algebraic notation, and

3. Can link very closely to the kind of knowledge and skill that most citizens should have.

Regarding #1, we normally attribute the generality to the algebraic side of this reasoning, but given the fact that the functions need not have algebraic definitions or origins reveals that the generality may reside elsewhere—in how we approach the mathematics in question and how it is learned. Regarding #2, the reasoning can be purely graphical, and indeed, examination of the reasoning in algebraic contexts shows the algebraic (or numeric) computations of definite integrals typically done in the service of graphical interpretations. Regarding #3, data-defined graphs of functions might also represent competing deficit-reduction proposals, economic data or projections, data regarding toxin or drug concentrations in the human body, accumulation of toxins in the environment, demographic data, competing investment plans, etc. Virtually anything that can be quantified can be graphed and subjected to reasoning about rates and accumulations.

In most countries, including the US, the mathematical instruction needed to learn such skills is postponed till late in the curriculum because it is couched in formalisms requiring lengthy prerequisite study, the net effect of which is to prevent most students from learning them. Newly available representations and instructional approaches exploiting them offer the opportunity to render these skills widely learnable. Note that these skills are not easily described as either algebra or calculus—indeed, I would characterize it as the Mathematics of Change and Variation, the mathematics that formal school algebra and calculus are concerned with. It is also worth remembering that a rather large amount of algebra is taught—and valuable instructional resources are expended—in service of the calculus learning that enables, as a by-product, the kind of reasoning with graphs illustrated in the examples. I suggest that we need to find ways to teach these graphically mediated skills to mainstream students as part of their core curriculum. And secondarily, for students who will work in technical fields requiring more formal methods, some of the same approaches will also serve them well, especially if linked into dynamic Computer Algebra Systems, such as Derive. The next section explores this possibility in more detail in the context of the SimCalc Project.

Illustrations of Graphically-Intensive Approaches to Important and Useful Mathematics

Historical Perspective: Finally, A Dynamic Medium for Graphical Representations

Recall Joseph Lagrange's enthusiasm for coordinate geometry:

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on a rapid pace towards perfection (cited in Kline, 1953, p. 159).

For almost the entire 350 years since coordinate geometry was invented, coordinate graphs of quantitative relationships have been instantiated in static, inert media. Perhaps the real promise of graphical mathematics had to await the availability of dynamic interactive media. Now we can manipulate graphs with the fluency once reserved for character-string manipulation. Moreover, we no longer need to worry about closed-form representations of quantitative relationships in order to analyze them in detail, including analyses that relate functions and their derivatives, or functions and their integrals. In particular, we can define and manipulate functions defined piecewise on intervals. As explored in Kaput (2000), there may be an analogy between the invention of alphabetic phonetic writing, which enabled human writing to tap efficiently into the pre-existing and neurophysiologically well-evolved spoken/oral system of meaning-making and communication—and thereby increase both learnability and expressiveness of written language—and the invention of graphical systems of representing quantitative relationships, which tap efficiently into an even more powerful and neurophysiologically well-evolved system of meaning-making, the *visual* system.

The Base Mathematical Object of the New Century is Parametrized – Generality And Structure Are Alternatively Expressed Via Directly Manipulable Dynamic Objects

Before the instantiating medium became dynamic, *graphical* representations were trapped in the particular. And generality (or abstraction, depending on one's assumptions and definitions) was normally expressed through static inscriptions whose values were presumed to range over some set, usually not given explicitly, where the canonical example is that of algebraically defined functions in closed form, e.g., F(X) = 3X^2 + 2X - 5. (It is important to note that not only do X and F(X) stand for variables over sets of numeric values, but the numerals – and operation signs - in this expression likewise stand for generalizations—generalizations across concrete counting or measuring acts.) But, as illustrated by Dynamic Geometry and in the SimCalc examples below, we can now create and interact with more general mathematical objects, in effect, parametrized objects whose values we can directly manipulate. These extensions are of two types, one "algebraic" where, in most currently available CAS's one can directly manipulate the coefficients – or the graphs—within a *family* of functions, rather than a particular function, as in F(X) = aX^2 + bX + c. The other, non-algebraic type involves direct graphical manipulation of functions not expressed in closed form, as illustrated in the examples below.

In both situations, as in the dynamic geometric counterpart, we see an upward extension of human capability in abstraction and generalization of mathematical objects and operations supported by the notation system. However, it is not by direct extension of individual human psychological capability to abstract or generalize, but is the result of the culturally defined symbol system. Two alternative analogies may help clarify the changes that are underway. One now standard analogy is to treat the symbol system and its instantiation in the computational medium as a mental prosthetic-extension, enabling humans to reach upward in abstraction to more abstract mathematics. Another is to think of the new symbol systems as lowering the abstraction level of the mathematics to within reach of human capabilities by rendering the notations in which the mathematics is embodied more physically concrete and subject to previously developed human capacity to employ hands and eyes in the service of thinking.

We will now examine some specific representational strategies in more detail that enable us to use the visual system in the service of teaching and learning mathematics of the sort illustrated in the examples above. We invite comparisons with Dynamic Geometry as yet another illustration of these ideas.

Summary and Illustrations of SimCalc Representational Strategies

Here we will summarize the core web of four representational innovations employed by the SimCalc Project, all of which require a computational medium for their realization. Cross-platform software, Java MathWorlds for desktop computers, can be viewed and downloaded at http://www.simcalc.umassd.edu and software for hand-helds (TI-83Plus and Palm Pilot) can be downloaded from http://www.simcalc.com. A version incorporating Derive is in development for the TI-92Plus and will be available in Summer, 2002.

1. **Definition and direct manipulation of *graphically defined* functions, especially piecewise-defined functions**, with or without algebraic descriptions. Included is "Snap-to-Grid" control, whereby the allowed values can be constrained as needed—to integers, for example, allowing a new balance between complexity and computational tractability whereby key relationships traditionally requiring difficult computational and conceptual prerequisites can be explored using whole number arithmetic and simple geometry. This allows sufficient variation to model interesting situations (e.g., see the Sack Race Activity Below), avoid the degeneracy of constant rates of change, while postponing (but not ignoring!) the messiness and conceptual challenges of continuous change.

2. **Direct connections between the above representational innovation and simulations**—especially motion simulations—to allow immediate construction and execution of a wide variety of variation phenomena, which puts phenomena at the center of the representation experience, reflecting the purposes for which traditional representations were designed initially, and, most importantly, enabling orders of magnitude tightening of the feedback loop between model and phenomenon.

3. **Direct, hot-linked connections between graphically editable functions and their derivatives or integrals**. Traditionally, connections between descriptions of rates of change (e.g., velocities) and accumulations (positions) are mediated through the algebraic symbol system as sequential procedures employing derivative and integral formulas—but they need not be. In this way, the fundamental idea, expressed in the

Fundamental Theorem of Calculus, is built into the representational infrastructure from the start, in a way analogous to how, for example, the hierarchical structure of the number system is built into the placeholder representational system for numbers—which dramatically democratized numerical calculation.

4. **Importing physical motion-data via MBL/CBL and re-enacting it in simulations, and exporting function-generated data to drive physical phenomena** LBM (Line Becomes Motion), which involves driving physical phenomena, including cars on tracks, using functions defined via the above methods as well as algebraically. Hence there is a two-way connection between physical phenomena and varieties of mathematical notations. Especially through the importing and then re-animating of students' physical motions, this functionality plays an especially important role in SimCalc instructional materials to anchor the visual experience of the simulations in students' kinesthetic experience.

The result of using this array of functionalities, particularly in combination and over an extended period of time, is a qualitative transformation in the mathematical experience of change and variation. However, short term, in less than a minute, using either rate or totals descriptions of the quantities involved, or even a mix of them, a student as young as 11 or 12 years of age can construct and examine a variety of interesting change phenomena that relate to direct experience of daily phenomena. And in more extended investigations, newly intimate connections among physical, linguistic, kinesthetic, cognitive, and symbolic experience become possible.

Determining Mean Values—Exploiting SimCalc Representational Strategies 1–4 Above

We now sample some activities that are possible to help illustrate the ways that common mathematical ideas are approached graphically using the representational strategies outlined above. Figure 4 shows the velocity graphs of two functions, respectively controlling one of the two "elevators" on the left of the figure (graphs on the desktop software are color-coded to match the elevator that they control). The downward-stepping, but positive, velocity function, which controls the left elevator, typically leads to a conflict with expectations, because most students associate it with a downward motion. However, by constructing it and observing the associated motion (often with many deliberate repetitions and variations), the conflicts lead to new and deeper understandings of both graphs and motion. The second, flat, constant-velocity function in Figure 4 that controls the elevator on the right provides constant velocity. It is shown in the midst of being adjusted to satisfy the constraint of "getting to the same floor at exactly the same time." This amounts to constructing the average velocity of the left-hand elevator which has the (step-wise) variable velocity. This in turn reduces to finding a constant velocity segment with the same area under it as does the staircase graph. In this case the total area is 15 and the number of seconds of the "trip" is 5, so the mean value is a whole number, namely, 3. We have "snap-to-grid" turned in this case so that, as dragging occurs, the pointer jumps from point to point in the discrete coordinate system. Note that if we had provided 6 steps for the left elevator instead of 5, the constraint of getting to the same floor at exactly the same time (from the same starting-floor) could not be satisfied with a whole number constant velocity, hence could not be reached with "snap-to-grid" turned on.

The standard Mean Value Theorem, of course, asserts that if a function is continuous over an (open) interval, then its mean value will exist and will intersect that function in that interval. But here the step-wise varying function is *not* continuous, and so the Mean value Theorem conclusion fails—as it would if 6 steps were used. However, if we had used imported data from a student's physical motion, then her (continuous) velocity would necessarily equal her average velocity at one or more times in the interval. We have developed activities involving a second student walking in parallel whose responsibility is to walk at an estimated average speed of her partner. Then the differences between same-velocity and same-position begin to become apparent. Additional activities involve the two students in importing their motion data into the computer (or calculator) *serially* (discussed below) and replaying them *simultaneously*, where the velocity-position distinction becomes even more apparent due to the availability of the respective velocity and position graphs juxtaposed with the cybernetically replayed motion.

Note how the dual perspectives of the velocity and position functions, both illustrated in Figure 4, simultaneously show two different views of the average value situation. In the left-hand graph, we see the connection as a matter of equal areas under respective velocity graphs. In the right-hand graph, we see it through position graphs as a matter of getting to the same place at the same time, one with variable velocity and the other with constant velocity. Depending on the activity, of course, one or the other of the graphs might not be viewable or, if viewable, not editable. For example, another version of this activity involves giving the step-wise varying position function on the right and asking the student to construct its velocity-function mean value on the left. This makes the slope the key issue. By reversing the given and requested function types, equalizing areas becomes the key issue. Importantly, by building in the connections between rate (velocity) and totals (position) quantities throughout, the underlying idea of the Fundamental Theorem of Calculus is always at hand.

Another kind of activity that is easy to imagine based on Figure 4 involves approximation of a linearly decreasing velocity from, say 6 floors/sec to 0 floors/sec in 6 seconds (imagine the constant velocity function being dragged into the appropriate downward-sloping linear function ($V(t) = 6- t$). We systematically refine the staircase and, in an extension of normal area-computation activity, examine how the approximation error relates to the differences in distances traveled by the elevator moving according to the approximating staircase vs. that moving according to the linear velocity function. Actually, when using the SimCalc materials with younger students, the students work first with the step-wise varying velocities. In this case the usual approximation is reversed since the linear function is not yet known to the students—and they are asked to use their available step functions to try to match the motion of an object with linear velocity whose function is hidden (so they adjust their staircases using the differences between the two motions as feedback since they cannot see the graph of the function that they are trying to match—indeed, they initially assume it some hidden step function. With help, they conclude that the "unknown motion" must be linear, and that their staircases can never match it. Furthermore, if the position vs. time graphs of the students' step-wise varying velocity functions are available, then they see how the refinements of the staircases are reflected in refinements of the associated "polygonal parabola"—and how the polygon approaches a smooth curve.

Of interest in these brief examples is how the mathematical issues can be deep and central to the learning of the mathematics of variation, but yet not require algebraic notation to achieve substantive engagement.

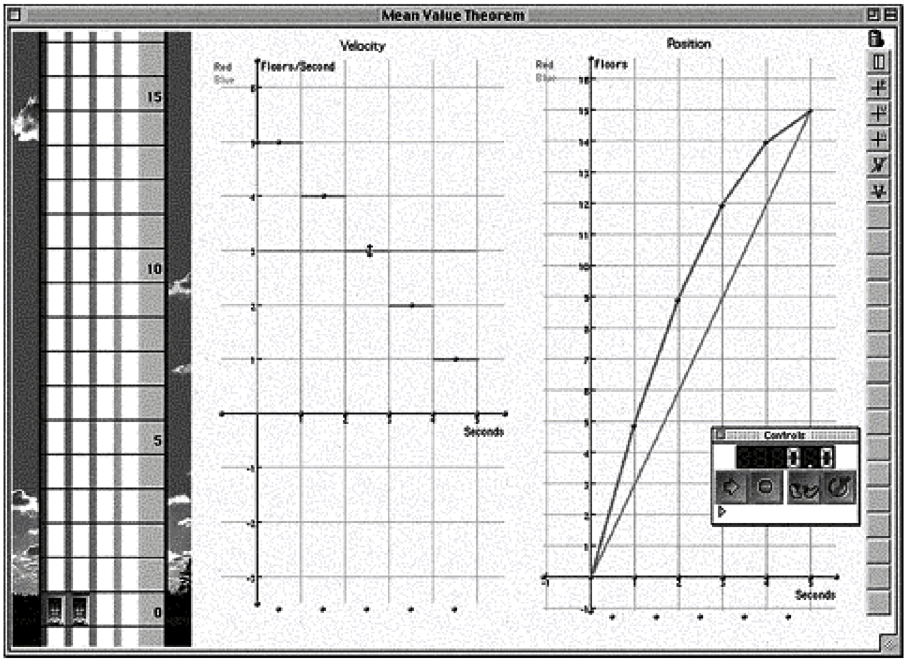

Figure 4. Averages from Both Velocity and Position Perspectives

Basing Mathematical Experience in Physical and Cybernetic Phenomena

Much has been written over the past two decades regarding the educational potential of linking multiple representations of mathematical ideas, especially functions, (including by the author, Kaput, 1986). However, as illustrated so clearly in a detailed study by Schoenfeld and colleagues (Schoenfeld, et al., 1994), multiple, linked representations of functions, even when coupled with carefully designed and supported instruction, can fail to yield stable and robust understanding. The difficulty is rooted in the fact that the representations only refer to each other, and to nothing anchored in the student's wider world of experience. In the words of Anna Sfard (PME-NA, 1995), "The emperor is *only* clothes." Our approach, reflected in Representational Strategies #2 and #4 above, puts phenomena at the center, either physical or cybernetic phenomena (simulations). Put in terms of Figure 5, we link the representations to each other by initially making each refer to common phenomena – they are *about* something other than each other. We put an emperor inside the clothes.

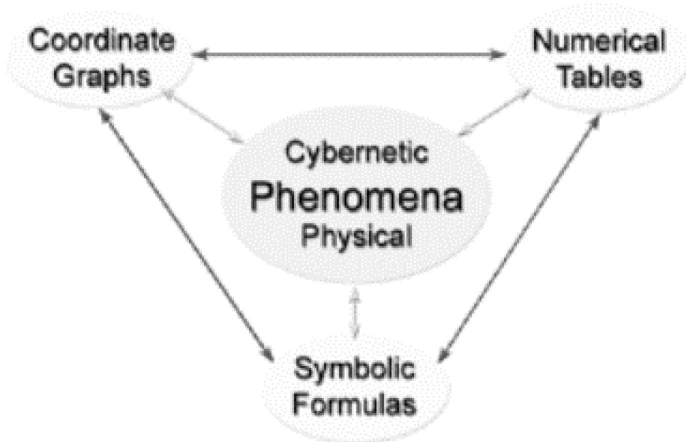

Figure 5. Mathematics Based in Phenomena – Putting an Emperor Inside the Clothes

Distinguishing the Model from That Which Is Modeled

The prior illustrations have focused on motion simulations – cybernetic phenomena. Below we will discuss the use of physical phenomena, emphasized in Representational Strategy #4, but involving parallel software on a different hardware platform. (The matter of parallel software across different platforms will be discussed further in the next section.) By design we normally juxtapose physical and cybernetic data in the same learning experiences rather than treat them separately, in effect combining Representational Strategies #2 and #4. We believe that an important 21st century skill is the ability to understand the connections between simulations and the phenomena that they are alleged to model, and especially the differences between them. As models and simulations become ever more realistic (e.g., embodying virtual reality), it becomes ever more difficult to distinguish model from that which is modeled, and we increasingly are pulled to believe in the models as real – the map is ever more similar to the territory. This hides the assumptions behind the models from view, leading to potentially very dangerous possibilities – as when we come to believe economic or biological models simply because they appear so realistic.

Prior to the computational medium, models and what they were presumed to model occurred in separate realms of human experience, one in the mathematical-semiotic realm and the other in the "external world" realm of whatever was being modeled – physical, social, economic, etc. They were separate in ways that are far less apparent today. Across most fields, models now frequently relate to or are expressed by simulations – in effect, an idealized version of the phenomena being modeled appears inside the synthetic world of the computer, often indistinguishable from the mathematical model itself. Environments which exploit Representational Strategies #2 and #4, such as SimCalc MathWorlds, offer the possibility of making the distinction an explicit object of study in very elementary ways, but also in ways that connect to important mathematical ideas – after all, most of the classic mathematics out of which we build our standard courses in Calculus have their roots in modeling motion and similar phenomena (Kaput, 1994).

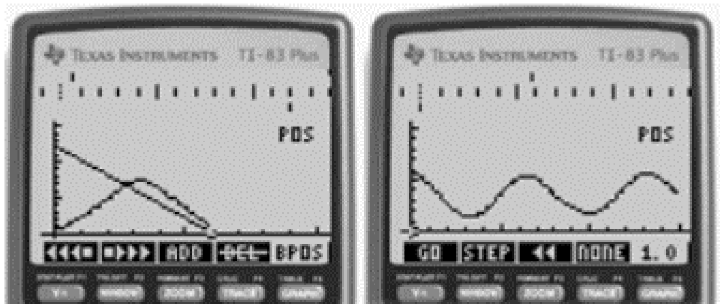

Figure 6

Relating Cybernetic to Physical Phenomena

To illustrate, consider the following three challenges, the latter two of which are reflected in the two parts of Figure 6. These are pictures of screens from a version of SimCalc MathWorlds with the "CBR Animator" running on a TI -83 Plus graphing calculator. The CBR Animator enables the user to import a motion using a data collection device connected to the graphing calculator and then replay that motion perhaps to compare it with a second motion, as in these illustrations.

1. Given the Position vs. time function P(x) = 12-2x which controls the motion of B, walk a motion for A whose position vs. time graph matches that of P(x) as closely as possible, and then explain how the differences between B's graph and A's graph relate to the differences in the motions when you run them side-by-side.

The linearly decreasing function in the left part of Figure 6 is the graph of this function, which controls the motion of B, an object that will move from right to left above the coordinate graphs. The imported motion will control a second object at the top of the screen. Here the activity is relatively straightforward except for the last part, which extends the common matching-motion activity. Issues of continuity, linear vs. non-linear change, starting and ending points in time and position, simultaneous position, and so on, all now have dual status – in the mathematical notion (the coordinate graph) and in the motion simulation, as the imported motion is replayed alongside the algebraically given motion.

2. B has gone to a party and is coming home with a motion controlled by the Position vs. time function P(x) = 12-2x. You stayed home at 0, but now your job is to walk a motion for A starting at 0 that starts at the same time B does, meets B halfway, and then escorts B home. Explain how the differences between B's graph and A's graph relate to the differences in the motions when you run them side-by-side.

A solution to this challenge is given in the left screen of Figure 6. The issues introduced in the previous challenge relating the differences in the graphs to the differences in the motions become even more salient here. Furthermore, the motion in this case has a somewhat more realistic context (albeit a "toy" story nonetheless). Other more elaborate stories, including dances and more dramatic situations, link more complex mathematical functions to students real experiences even more strongly. Some of our curricular activities involve students building their own stories as well.

3. (a) How might you walk in a circle to produce the function whose graph is given in the right hand part of Figure 6? OR: (b) If you walk in a circle while pointing your motion-

detector straight at a wall, what kind of position vs. time graph results?

Here the idea is that a student could hold the data collection device and aim it at a wall (in a perpendicular orientation to the wall) while walking in a circle, basically creating a sine function plus a constant, where the constant depends on the distance between the circle and the wall. It is surprising how difficult question (a) is for many students and teachers who presumably know trigonometry. Even more difficult is the (b) variant of this question. Once one graph has been produced it becomes interesting to create variants of it, physically creating different members of the broad family of functions described by

$$P(x) = A + B \, SIN \, (Cx + D).$$

We have done versions of this activity using the connectivity and aggregation capability discussed below, where different students are assigned values that systematically vary one of the literal parameters of the generic formula.

It should be apparent from the above examples that by mixing cybernetic and physical phenomena, including phenomena defined by algebraically defined functions, we can bring the differences between them into stark relief while simultaneously addressing core mathematical ideas and skills.

Diversification of Hardware Platforms and Technologies

Parallel Software and Curricula for Graphing Calculators

In the left two pictures of Figure 7 below are partially analogous software configurations for the TI-83Plus illustrated earlier in Figure 4 above—two elevators controlled by two velocity graphs. Instead of the clicking and drag/drop interface of the desktop software, most user interaction is through the SoftKeys that appear across the bottom of the screen which are controlled by the HardKeys immediately beneath them. The left-most screen depicts the Animation Mode, with two elevators on the left controlled respectively by the staircase and constant velocity functions to their right. The middle screen depicts the Function-Edit Mode, which shows a “HotSpot” on the constant-velocity graph. The user adjusts the height and extent of a graph segment via the four calculator cursor keys (not shown), and can add or delete segments via the SoftKeys. Other features allow the user to scale the graph and animation views, display labels, enter functions in text-input mode, generate time-position output data, and so on—very much in parallel with Computer MathWorlds, but without the benefits of a direct-manipulation interface. The right-most screen shows a horizontal motion world with both position and velocity functions displayed (hot-linkable if needed, as with the computer software). This kind of horizontal motion is the same as appears in Figure 6 above.

We have developed a full, document-oriented Flash ROM software system for the TI-83+ and a core set of activities embodying a common set of curriculum materials that parallels the computer software to the extent possible given the processing and screen constraints (96 by 64 pixels—with only 90 by 54 at best available for coordinate graphs). Considering Figure 4 earlier, the parallelism is evident in the Calculator MathWorlds screens shown. We have also developed a prototype version of MathWorlds for the PalmPilot Operating System.

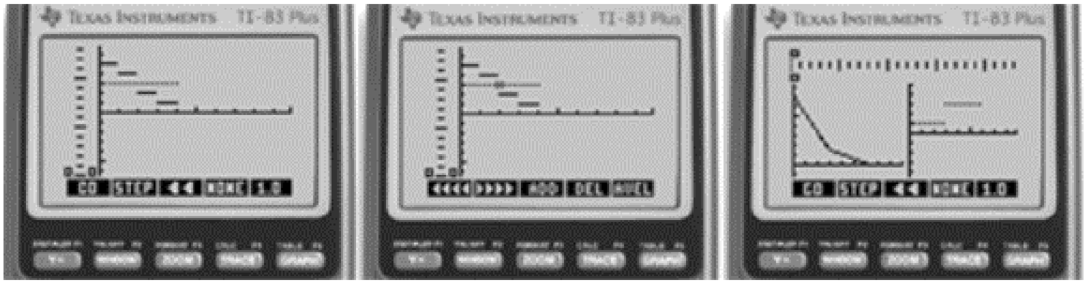

Figure 7. TI-83+ Calculator MathWorlds

We will soon have a version available for the TI-92Plus that includes Derive as a subset, as illustrated in Figures 8 and 9—where screens from a prototype are shown (the final interface will likely be modified). In Figure 8 we see a graphical edit of a piecewise defined function with its piecewise derivative function (two horizontal segments) below on the split screen, and the familiar "elevators" on the left, again, comparable to those above in Figures 4 and 7.

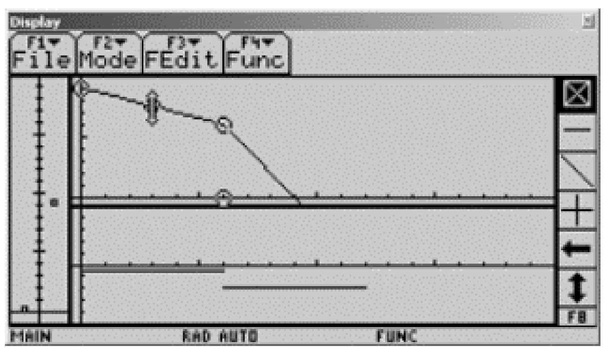

Figure 8. TI-92Plus Calculator MathWorld Graphical Edit Screen and Animation

In Figure 9 below, we illustrate a function edit screen with an algebraically rather complicated function defined using Derive, and its derivative partially viewable below. You will note that this function's domain is specified as the half open interval (0, 3] (and actually, since the left end of the domain is involved, the function is actually defined on the closed interval [0, 3]), and additional functions could be defined for other intervals as needed. Alternatively, the function could be defined globally as usual. Importantly, once a finite domain is specified, the function can be animated in the usual SimCalc MathWorlds way (with a horizontal or vertical motion) and, indeed, it can also control a "meter" – a thermometer-like display to represent non-motion quantities.

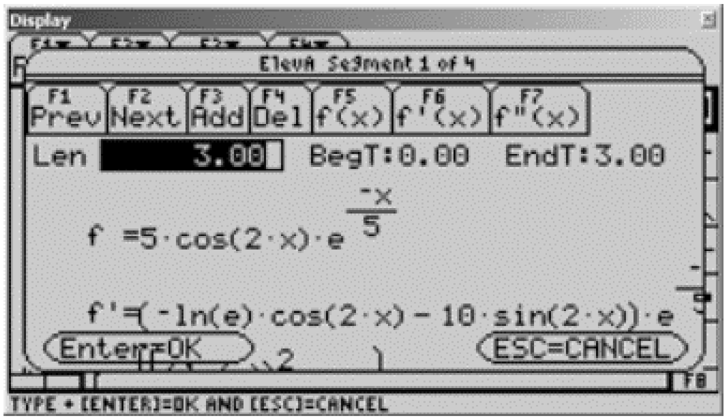

Figure 9. TI-92Plus Calculator MathWorlds Function Edit Screen—Including Derive

In addition, the function's derivatives or integrals can be determined by Derive in Derive's usual ways (the function's first derivative is partially visible on the screen). Functions can also be defined numerically via imported data as with the examples earlier, or internally by the user via a table in the usual ways. Hence we can treat this software either as SimCalc MathWorlds extended by Derive, or as Derive extended by SimCalc MathWorlds. Unlike the more constrained and curriculum-specific versions of MathWorlds for more elementary mathematics, we prefer the latter interpretation for more advanced mathematics, where the educational applications will be largely left open to the users to determine, just as is the case with CAS's in general. In effect *we treat the new software as an extension of a CAS.*

Reflections on the Integration of SimCalc Representational Strategies and a CAS

As suggested earlier, historically, mathematical notations evolved in static, inert media. Hence variation had to be supplied mentally, whether for the function variables or parameters defining function-families. While we cannot concretely illustrate the screen motion of the animations in the static medium of this paper, we trust that the reader can imagine how it provides perceptual support for this mental variation, support that supplants, at a neurophysiological level, the semantics of the needed motion experience as described by, for example, Kosslyn & Koenig in chapter 6 of *Wet Mind: The New Cognitive Neuroscience* (Kosslyn & Koenig, 1992).

It seems that another kind of semiotic support of mental processes and capacities is at work here, broadly analogous to the support that is provided to short-term memory by inert static notations that produce perceptual input that refreshes short-term memory as needed for the task at hand – for example, when we write out sums of several numbers before beginning the summation process, or when we check off items in a list when they have been counted. The difference here is that the motion on the screen (embodied in real time as with any motion) perceptually drives the experience of variation, just as it did *not* need to do for Newton. As Boyer (1959) and Edwards (1979) – reviewed in Kaput (1994) – point out, Newton parametrized most variables with time, so he imagined a particle moving along a line according to whatever conditions defined the motion – conditions that we would now describe in terms of a function.

The computational medium allows us to create dynamic notations that perceptually generate the visual experience of motion, augmentable by physical and kinesthetic experience as discussed above, that in turn can generate the experience of functional variation that we would otherwise need to create through internal mental processes. While these processes can be generated by one who already has the mental structures in place to generate them, they are not necessarily available to one who has no experience with the variation needed. This is the contribution of the computer-generated visual experience that then must be linked by curriculum and pedagogy to the semantics of the more abstract mathematical variation of the calculus of functions, including the two kinds of descriptions of change – rates and totals, "how fast" and "how much" of the quantity in question. Exploiting this power is behind Representational Strategy #2, augmented by #4.

In addition, Representational Strategy #3 that connects rate of change with accumulation – or more traditionally described as the linkage between functions and their derivatives or integrals – can be applied in the context of a CAS, where the accumulated power of the algebraic system as instantiated in the CAS, can be utilized. Given algebraic representation of functions, this linkage has taken the form of serially executable procedures, as embodied in rules for computing derivatives and integrals. However, in the computational medium, we can compute these connections almost instantaneously and hence present functions and their derivatives or integrals side-by-side, as illustrated above. The highly efficient exponential/hierarchical system for organizing quantities (powers of a base, usually ten) and then writing them in extremely compact ways using the combination of Arabic numerals and the placeholder system yielded an extraordinarily efficient, indeed culturally transforming quantitative system for using and computing with numbers based on the representing symbols themselves. Before its appearance, computation was very limited, either to a very small population with larger quantities, or small quantities for a larger population. This dramatically changed when the new number system appeared. To the extent that it was taught by the emerging education system, the new representational system democratized access to numerical computation with large numbers and resulted in an entirely new level of economic activity (Swetz, 1987).

Just as the base ten placeholder system for numbers semiotically and culturally embodied an extraordinary intellectual achievement that became widely available, the idea that the two kinds of descriptions of situations involving variable quantities based on rates of change and accumulation of variable quantities – dual descriptions in terms of how fast and how much – were equivalent, was itself an extraordinary intellectual achievement, glimpsed by the predecessors of Newton and Leibniz, but recognized for its universality only by the two masters. Newton's predecessor, Isaac Barrow, had demonstrated the equivalence in a few important cases, but did not grasp their universality (Boyer, 1959). And it was Leibniz who developed a notation that expressed that universality in algebraic ways – as a "calculus," that is, as a means of computing based on the symbols themselves – that could be communicated to those who understood and could use algebraic notations. But the results of this computing can now be embodied in what amounts to a new visually explicit notation system that includes the derivative and integral simultaneously, just as the base ten placeholder system embodies in a single character string the result of an extremely sophisticated organization of quantities into an exponentially structured hierarchy – the extraordinary intellectual achievement is crystallized into a single semiotic entity available to a suitably educated individual as an object with reference to something outside itself (Moreno, in preparation).

So, in the 21st century, we have a new symbol system for the mathematics of change and variation that extends the existing base CAS and that, by the evidence accumulated to date, is learnable by mainstream students.

A Larger Perspective for This Change in the Nature of CAS's

This change in representational infrastructure is part of a larger evolution due to the computational medium and the fact that it provides a new medium in which to build new systems and re-instantiate old ones. I see three profound levels of consequences of this change:

Level 1: The knowledge produced in static, inert media can become knowable and learnable in new ways by changing the medium in which the traditional notation systems in which it is carried are instantiated—for example, creating hot-links among dynamically changeable graphs equations and tables in mathematics. Most traditional uses of technology in mathematics education, especially graphing calculators and computers using 20th century Computer Algebra Systems, are of Level 1.

Level 2: New representational infrastructures become possible that enable the reconstitution of previously constructed knowledge through, for example, the new types of visually editable graphs and immediate connections between functions and simulations and/or physical data of the type described above. This is the place of the extended CAS.

Level 3: The construction of new systems of knowledge employing new representational infrastructures—for example, dynamical systems modeling or multi-agent modeling of Complex Systems with emergent behavior, each of which has multiple forms of notations and relationships with phenomena. This is a shift in the nature of mathematics and science towards the use of computationally intensive iterative and visual methods that enable entirely new forms of dynamical modeling of nonlinear and complex systems previously beyond the reach of classical analytic methods—a dramatic enlargement of the MCV that will continue through this new century (Kaput & Roschelle, 1998; Stewart, 1990).

Thus Level 1 change is that associated with traditional CAS's. We have been focusing on a Level 2 change related to extensions of CAS's. Level 3 change is beyond the scope of this paper, but the literature on this topic is wide and growing (Cohen & Stewart, 1994; Haken, 1981; Hall, 1992; Holland, 1995; Kauffman, 1993, 1995; Prigogine & Stengers, 1984). In general, however, as we move from Level 2 to Level 3 change, instead of focusing on the teaching of a very small set of representation systems of the sort we have inherited and extended, we will need to focus on the teaching of *how to learn and use* new representation systems, and how to coordinate among them. Not only will students need to learn multiple ways of representing and reasoning with quantitative relationships, but they will need to *learn how to learn* new systems as they emerge.

References

Boyer, C. (1959). *The history of calculus and its historical development*. New York, NY: Dover Publications.

Cohen, J., & Stewart, I. (1994). *The collapse of chaos: Discovering simplicity in a complex world*. New York: Viking Books.

Edwards, C. (1979). *The historical development of the calculus*. New York, NY: Springer-Verlag, Inc.

Haken, H. (Ed.). (1981a). *Chaos and order in nature*. Berlin: Springer-Verlag.

Hall, N. (1992). *Exploring Chaos: A Guide to the New Science of Disorder*. New York:

Norton.

Holland, J. H. (1995). *Hidden order: How adaptation builds complexity.* New York: Addison-Wesley.

Kaput, J. (1986). Information technology and mathematics: Opening new representational windows. *The Journal of Mathematics Behavior, 5*(2), 187–207.

Kaput, J. (1994). Democratizing access to calculus: New routes using old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*, (pp. 77-156). Hillsdale, NJ: Erlbaum.

Kaput, J. (2000). Implications of the shift from isolated, expensive technology to connected, inexpensive, ubiquitous, and diverse technologies. In M. O. Thomas (Ed.), *TIME 2000: An international conference in Mathematics Education* , (pp 1–25). University of Auckland, New Zealand.

Kaput, J., & Roschelle, J. (1998). The mathematics of change and variation from a millennial perspective: New content, new context. In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Mathematics for a new millennium* (pp. 155–170). London: Springer-Verlag.

Kauffman, S. (1993). *The origins of order: Self organization and selection in evolution*. New York, NY: Oxford Unive rsity Press.

Kauffman, S. (1995). *At home in the universe: The search for the laws of self-organization and complexity.* New York: Oxford.

Kline, M. (1953). *Mathematics in western culture.* New York: Oxford University Press.

Kosslyn, S., & Koenig, O. (1992). *Wet mind: The new cognitive neuroscience*. New York: The Free Press.

Prigogine, I., & Stengers, I. (1984). *Order out of chaos* . New York: Bantam.

Schoenfeld, A., Smith, J., & Arcavi, A. (1994). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology*, (Vol. Vol. 4,). Hillsdale, NJ: Lawrence Erlbaum.

Stewart, I. (1990). Change. In L. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 183–219). Washington, DC: National Academy Press.

Swetz, F. (1987). *Capitalism and arithmetic: The new math of the 15th century.* La Salle: Open Court.

---

[1] N.E. El título en español es *Sistemas algebraicos computacionales para el siglo 21: nuevas clases de representaciones dinámicas*