Stable determination of an inclusion in a layered medium

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Abstract

We consider the inverse problem of detecting an inclusion in a layered conductor through electrostatic measurements taken on the boundary. We analyze in particular the stability issue showing that the solution depends from the available data with a rate of continuity of logarithmic type.

1 Introduction

In this paper we consider the inverse problem of determining an unknown inclusion contained in a layered medium from electrostatic boundary measurements. In particular we focus our analysis on the stability issue, that is the dependance of the inclusion from the measurements performed. This result follows the line started in [3], where it is considered a domain of constant conductivity inside which a region with different unknown conductivity is located. It is shown that the dependance of the inclusion from the boundary measurements is of logarithmic type. This rate of continuity turns out to be optimal as shown by examples in [10].

The approach to get stability has been later applied to different contexts (more general isotropic conductivity [6], a class of anisotropic conductivities [9], inverse scattering [7], thermal imaging [11, 12], elasticity [4]) and it is based mainly on two arguments:

- quantitative estimates of unique continuation;
- singular solutions.

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Here we will use the same approach. As for the quantitative estimates of unique continuation we will take advantage of the recent paper [13], where it has been derived a three-region inequality for a second order elliptic equation with a jump discontinuous coefficient. A crucial tool to get such inequality is the Carlemann estimate proved in [8]. It is possible indeed to provide a precise evaluation of the propagation of smallness measured on the boundary up to the inclusion.

As for the singular solution method, we will use the asymptotic study contained in [3] to get the stability estimates. Let us mention here that, in the present setting, the inclusion is assumed to be located at a positive distance from the interface of the layer. The major difficulty relays on the fundamental solution argument. In particular it is not clear how to write explicitly such a solution when the boundary of the inclusion and the interface intersect each other.

The three region argument, since the background conductivity is known, does not notice the presence of the interface. Therefore the result, we prove, can be obtained for multi layers media no matter the number of interfaces there are, as long as the inclusion does not touch the interface.

The paper is organized as follows. In the next Section 2, after some notations and definitions, we will state our main result, whose proof is presented in the next Section 3. The proof is based on some auxiliary propositions proved in Section 4.

2 Main Result

Let us first premise some notations and definitions. Let the domain Ω be a bounded open set in $\mathbb{R}^n$ and the layer Σ be a closed hyper-surface contained in Ω. With Σ, the domain Ω is separated into the union of three parts

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-,$$

where $\Omega_\pm$ are open subsets such that $\partial\Omega_- = \partial\Omega \cup \Sigma$ and $\partial\Omega_+ = \Sigma$. We denote by D a subset of Ω such that $D \subset \Omega_+ \subset \Omega$. We consider $\gamma(x)$ the conductivity of Ω of the form

$$\gamma(x) = c_1 + (c_2 - c_1)\chi_{\Omega_+} + (k - c_2)\chi_D,$$

where c_1 and c_2 are given constants and k is an unknown constant.

For points $x \in \mathbb{R}^n$, we will write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$. Moreover, denoted by $\mathrm{dist}(\cdot,\cdot)$ the standard Euclidean distance, we define

$$B_r(x) = \{y \in \mathbb{R}^n | \mathrm{dist}(x, y) \leq r\}, \qquad B'_r(x') = \{y' \in \mathbb{R}^{n-1} | \mathrm{dist}(x', y') \leq r\}$$

as the open balls with radius r centered at x and x' respectively. We write $Q_r(x) = B'_r(x') \times (x_n - r, x_n + r)$ for the cylinder in $\mathbb{R}^n$. For simplicity, we use B_r, B'_r, Q_r instead of $B_r(0), B'_r(0')$ and $Q_r(0)$ respectively. We shall also denote half domain, as well as its associated ball and cylinder

$$\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}; \quad B^+_r = B_r \cap \mathbb{R}^n_+; \quad Q^+_r = Q_r \cap \mathbb{R}^n_+.$$

Definition 2.1. *Let Ω be the bounded domain in $\mathbb{R}^n$. Given $\alpha \in (0, 1]$, we say a portion S of $\partial\Omega$ is of $C^{1,\alpha}$ class with constants $r, L > 0$ if for any point $p \in S$, there exists a rigid transformation $\varphi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ of coordinates under which we have $p = 0$ and*

$$\Omega \cap B_r = \{(x', x_n) \in B_r | x_n > \varphi(x')\},$$

where $\varphi(\cdot)$ is a $C^{1,\alpha}$ function on B'_r, which satisfies

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$||\varphi||_{C^{1,\alpha}(B'_r)} \leq Lr,$$

where the norm is defined as

$$||\varphi||_{C^{1,\alpha}(B'_r)} := ||\varphi||_{L^\infty(B'_r)} + r||\nabla\varphi||_{L^\infty(B'_r)} + r^{1+\alpha}|\nabla\varphi|_{\alpha,B'_r}$$

$$|\nabla\varphi|_{\alpha,B'_r} := \sup_{\substack{x',y' \in B'_r,\\ x' \neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x' - y'|}.$$

Assumptions and a priori data
For $f \in H^{1/2}(\partial\Omega)$, let u be the solution of the problem

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Our inverse problem is addressed to determine the anomalous region D when the Dirichlet-to-Neumann map Λ_D

$$\begin{array}{cccc} \Lambda_D & : & H^{1/2}(\partial\Omega) & \longrightarrow & H^{-1/2}(\partial\Omega) \\ & & f & \longrightarrow & \frac{\partial u}{\partial\nu}_{|\partial\Omega}, \end{array}$$

is given for any $f \in H^{1/2}(\partial\Omega)$. Here, ν denotes the outer unit normal to $\partial\Omega$, and $\frac{\partial u}{\partial\nu}_{|\partial\Omega}$ corresponds to the current density measured on $\partial\Omega$. Thus, the Dirichlet–to–Neumann map represents the knowledge of infinitely many boundary measurements.

Given constants $r_1, M_1, M_2, \delta_1, \delta_2 > 0$ and $0 < \alpha < 1$, we assume the domain $\Omega \subset \mathbb{R}^n$ is bounded

$$|\Omega| \leq M_2 r_1^n,$$

where $|\cdot|$ denotes the Lebesgue measure.

The interface Σ is C^2 and assumed to stay away from the boundary of the domain, as $\operatorname{dist}(\Sigma, \partial\Omega) \geq \delta_2$, and the inclusion D is assumed to stay away from Σ, as $\operatorname{dist}(D, \Sigma) \geq \delta_1$, and also $\Omega \backslash D$ is connected. Both ∂D and $\partial\Omega$ are of $C^{1,\alpha}$ class with constants r_1, M_1.

We refer to $n, r_1, M_1, M_2, \alpha, \delta_1, \delta_2$ as the **a priori data**. To study the stability, we also denote by D_1 and D_2 two inclusions in Ω, which satisfy the above properties. The associated Dirichlet-to-Neumann map are Λ_{D_1} and Λ_{D_2}.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and we have two known constants c_1, c_2 and one unknown constant k, which are given. Let D_1, D_2 be two inclusions in Ω as above. If for any $\varepsilon > 0$ we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \leq \varepsilon,$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \omega(\varepsilon),$$

where ω is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \leq C|\log t|^{-\eta}, \quad , \forall\, t \in (0,1)$$

and $C > 0$, $0 < \eta \leq 1$ are constants depending on the a priori data only.

3 Proof of the Main Result

The proof of Theorem 2.2 is based on some auxiliary propositions, and their proofs are collected in the next Section 4. In what follows we define layers of our domains. We denote by $\mathcal{G}$ the connected component of $\Omega \backslash (D_1 \cup D_2)$, whose boundary contains $\partial\Omega$. $\Omega_D = \Omega \backslash \overline{\mathcal{G}}$, $S_{2r} := \{x \in \mathbb{R}^n | r \leq dist(x, \Omega) \leq 2r\}$, $S_r := \{x \in \mathcal{C}\Omega | dist(x, \Omega) \leq r\}$ and $\mathcal{G}^h := \{x \in \mathcal{G} | dist(x, \Omega_D) \geq h\}$. We recall that the layer Σ separates the domain into two parts known as Ω_- and Ω_+. We also define $\mathcal{F}^\lambda := \{x \in \Omega_- | dist(x, \Sigma) \geq \lambda\}$, and $\Sigma_\lambda := \{x \in \Omega_- | dist(x, \Sigma) = \lambda\}$

We introduce a variation of the Hausdorff distance called the *modified distance*, which simplifies our proof.

Definition 3.1. *The modified distance between D_1 and D_2 is defined as*

$$d_m(D_1, D_2) := \max\left\{ \sup_{x\in\partial\Omega_D\cap\partial D_1} dist(x, \partial D_2), \sup_{x\in\partial\Omega_D\cap\partial D_2} dist(x, \partial D_1)\right\}.$$

With no loss of generality, we can assume that there exists a point $O \in \partial D_1 \cap \partial\Omega_D$ such that the maximum of $d_m = d_m(D_1, D_2) = \mathrm{dist}(O, D_2)$ is attainted. We remark here that d_m is not a metric, and in general, it does not dominate the Hausdorff distance. However, under our *a priori* assumptions on the inclusion, the following lemma holds.

Lemma 3.2. *Under the assumptions of Theorem 2.2, there exists a constant $c_0 \geq 1$ only depending on M_1 and α such that*

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m(D_1, D_2). \tag{3.1}$$

Proof. See [3, Proposition 3.3] □

Another obstacle comes from the fact that the propagation of smallness arguments are based on an iterated application of the three spheres inequality for solutions of the equation over chains of balls contained in $\mathcal{G}$. Therefore, it is crucial to control from below the radii of these balls. In the following Lemma 3.3 we treat the case of points of $\partial\Omega_D$ that are not reachable by such chains of balls. This problem was originally considered by [5] in the context of cracks detection in electrical conductors.

Let us premise some notations. Given $O = (0, \dots, 0)$ the origin, v a unit vector, $H > 0$ and $\vartheta \in \left(0, \frac{\pi}{2}\right)$, we denote

$$C(O, v, \vartheta, H) = \{x \in \mathbb{R}^n : \ |x - (x\cdot v)v| \leq \sin\vartheta |x|, \ 0 \leq x\cdot v \leq H\}$$

the closed truncated cone with vertex at O, axis along the direction v, height H and aperture 2ϑ. Given R, d, $0 < R < d$ and $Q = -de_n$, where $e_n = (0, \dots, 0, 1)$, let us consider the cone $C\left(O, -e_n, \arcsin\frac{R}{d}, \frac{d^2-R^2}{d}\right)$.

From now on, without loss of generality, we assume that

$$d_m(D_1, D_2) = \max_{x\in\partial D_1\cap\partial\Omega_D} \mathrm{dist}(x, \partial D_2)$$

and we write $d_m = d_m(D_1, D_2)$.

We shall make use of paths connecting points in order that appropriate tubular neighborhoods of such paths still remain within $\mathbb{R}^n \setminus \Omega_D$. Let us pick a point $P \in \partial D_1 \cap \partial\Omega_D$, let ν be the outer unit normal to ∂D_1 at P and let $d > 0$ be such that the segment $[(P + d\nu), P]$ is contained in $\mathbb{R}^n \setminus \Omega_D$. Given $P_0 \in \mathbb{R}^n \setminus \Omega_D$, let γ be a path in $\mathbb{R}^n \setminus \Omega_D$ joining P_0 to $P + d\nu$. We consider the following neighborhood of $\gamma \cup [(P+d\nu), P] \setminus \{P\}$ formed by a tubular neighborhood of γ attached to a cone with vertex at P and axis along ν

$$V(\gamma) = \bigcup_{S\in\gamma} B_R(S) \cup C\left(P, \nu, \arcsin\frac{R}{d}, \frac{d^2-R^2}{d}\right). \tag{3.2}$$

Note that two significant parameters are associated to such a set, the radius R of the tubular neighborhood of γ, $\cup_{S\in\gamma} B_R(S)$, and the half-aperture $\arcsin\frac{R}{d}$ of the cone $C\left(P, \nu, \arcsin\frac{R}{d}, \frac{d^2-R^2}{d}\right)$. In other terms, $V(\gamma)$ depends on γ and also on the parameters R and d. At each of the following steps, such two parameters shall be appropriately chosen and shall be accurately specified. For the sake of simplicity we convene to maintain the notation $V(\gamma)$ also when different values of R, d are introduced. Also we warn the reader that it will be convenient at various stages to use a reference frame such that $P = O = (0,\dots,0)$ and $\nu = -e_n$.

Lemma 3.3. *Under the above notation, there exist positive constants $\overline{d}$, c_1, where $\frac{\overline{d}}{\rho_0}$ only depends on M_1 and α, and c_1 only depends on M_1, α, M_2, and there exists a point $P \in \partial D_1$ satisfying*

$$c_1 d_m \leq dist(P, D_2),$$

and such that, giving any point $P_0 \in S_{2\rho_0}$, there exists a path $\gamma \subset (\overline{\Omega^{\rho_0}} \cup S_{2\rho_0}) \setminus \overline{\Omega_D}$ joining P_0 to $P + \overline{d}\nu$, where ν is the unit outer normal to D_1 at P, such that, choosing a coordinate system with origin O at P and axis $e_n = -\nu$, the set $V(\gamma)$ introduced in (3.2) satisfies

$$V(\gamma) \subset \mathbb{R}^n \setminus \Omega_D,$$

provided $R = \frac{\overline{d}}{\sqrt{1+L_0^2}}$, where L_0, $0 < L_0 \leq M_1$, is a constant only depending on M_1 and α.

Proof. See [4, Lamma 4.2]. ☐

In order to use the information provided by the boundary measurements to evaluate the distance between two inclusions D_1 and D_2, we apply the following identity firstly introduced by Alessandrini in [1]. Let $u_i \in H^1(\partial\Omega)$, $i = 1, 2$, be solutions to (2.1) with conductivities $\gamma_{D_i} = c_1 + (c_2 - c_1)\chi_{\Omega_+} + (k - c_2)\chi_{D_i}$ respectively, we have

$$\int_\Omega (\gamma_{D_1} \nabla u_1 \cdot \nabla u_2) - \int_\Omega (\gamma_{D_2} \nabla u_1 \cdot \nabla u_2) = \int_{\partial\Omega} u_1 [\Lambda_{D_1} - \Lambda_{D_2}] u_2. \tag{3.3}$$

However, when we use the method of fundamental solutions, we will only deal with the interface which is close to Ω_D. This means we are only interested in what is happening inside of Ω_+, where $\chi_{\Omega_+} = 1$ is used for conductivities. Thus, we are only interested in the operator $\mathrm{div}((c_2 + (k - c_2)\chi_{D_i})\nabla\cdot)$ and the associated fundamental solutions Γ_{D_i} for $i = 1, 2$. We apply (3.3) locally to Γ_{D_1} and Γ_{D_2}, obtains

$$\begin{aligned} & \int_\Omega (c_2 + (k - c_2)\chi_{D_1})\, \nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z) \\ & - \int_\Omega (c_2 + (k - c_2)\chi_{D_2})\, \nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z) \\ = & \int_{\partial\Omega} \Gamma_{D_1}(\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(\cdot, z). \end{aligned} \tag{3.4}$$

For $y, z \in \mathcal{G} \cap \mathcal{C}\Omega$, where $\mathcal{C}\Omega$ is the complementary of Ω, we define

$$\begin{aligned} S_{D_1}(y, z) &= (k - c_2) \int_{D_1} \nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z) \\ S_{D_2}(y, z) &= (k - c_2) \int_{D_2} \nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z) \\ f(y, z) &= S_{D_1}(y, z) - S_{D_2}(y, z). \end{aligned}$$

Therefore (3.4) can be written as

$$f(y, z) = \int_{\partial\Omega} \Gamma_{D_1}(\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(\cdot, z), \quad \forall y, z \in \mathcal{C}\overline{\Omega}. \tag{3.5}$$

The following two propositions provide quantitative estimates on $f(y, y)$ and $S_{D_1}(y, y)$, when moving y towards O along $\nu(O)$. Their proof are postponed in the next Section 4.

Proposition 3.4. *Given $\epsilon > 0$, the domain Ω and inclusions D_1, D_2, and a transformation of coordinates defined as $y = h\nu(O)$, if we have*

$$||\Lambda_{D_1} - \Lambda_{D_2}||_{L(H^{1/2}, H^{-1/2})} < \epsilon, \tag{3.6}$$

then for every h where $0 < h < cr, 0 < c < 1$, and c depends on M_1, we have

$$|f(y, y)| \leq C_0 \frac{\epsilon^{Bh^F}}{h^T}. \tag{3.7}$$

Here $0 < T < 1$ and $C_0, B, F > 0$ are constants that depend only on the a priori data.

Proposition 3.5. *Given $\epsilon > 0$, the domain Ω and inclusions D_1, D_2, and a transformation of coordinates $y = h\nu(O)$ defined as above. Then for every $0 < h < r_0/2$*

$$|S_{D_1}(y,y)| \geq C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3, \tag{3.8}$$

where $r_0 := \frac{r}{2} \min\left[\frac{1}{2}(8M_1)^{-1/\alpha}, \frac{1}{2}\right]$, and C_1, C_2, C_3 are positive constants depending only on the a priori data.

Now, we have all the ingredients to conclude this section with the proof of Theorem 2.2.

Proof of Theorem 2.2. We start from the origin of the coordinate system, point $O \in \partial D_1 \cap \partial \Omega_D$, for which the maximum in Definition 3.1 is attainted

$$d_m := d_m(D_1, D_2) = \operatorname{dist}(O, D_2).$$

Then with a transformation of coordinates $y = h\nu(O)$ where $0 < h < h_1, h_1 := \min\{d_m, cr, r_0/2\}, 0 < c < 1$, where c depends on M_1. By applying [Al-DC] Proposition 3.4 (i); i.e., $|\nabla_x \Gamma_{D_i}(x,y)| \leq c_1 |x-y|^{1-n}$, where $c_1 > 0$ depending only on k, n, α, M_1; we have

$$\begin{aligned} |S_{D_2}(y,y)| &= (k - c_2) \int_{D_2} \nabla \Gamma_{D_1}(\cdot, y) \nabla \Gamma_{D_2}(\cdot, y) \\ &\leq (k - c_2) \int_{D_2} (c_1 |\cdot - y|^{1-n})^2 \leq (k - c_2) c_1^2 \int_{D_2} (|d_m - h|^{1-n})^2 \\ &\leq (k - c_2) c_1^2 |d_m - h|^{2-2n} |D_2| \leq C_4 |d_m - h|^{2-2n}. \end{aligned} \tag{3.9}$$

Here $|D_2|$ is the measure of the inclusion D_2 which is bounded by $|D_2| \leq |\Omega| \leq M_2 r_1^n$. Thus, C_4 depends on $k, n, \alpha, M_1, M_2, r_1$. From (3.7), we already have the upper bound of $f(y,y)$. Moreover, if we apply the triangular inequality, we obtain

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \leq |S_{D_1}(y,y) - S_{D_2}(y,y)| = |f(y,y)| \leq C_0 \frac{\epsilon^{Bh^F}}{h^T}. \tag{3.10}$$

Meanwhile, (3.8) gives us the lower bound of $S_{D_1}(y,y)$. Therefore, together with (3.9) and (3.10), we obtain

$$C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3 \leq C_4 |d_m - h|^{2-2n} + C_0 \frac{\epsilon^{Bh^F}}{h^T}$$

We can rearrange terms, and with a bit modification of the notations of the constants, (in particular, let $C_3 = C_2 d_m^{2-2n}$) we have

$$C_1 h^{2-n} \leq C_4 |d_m - h|^{2-2n} + C_0 \frac{\epsilon^{Bh^F}}{h^T}.$$

We can simplify the above as, by setting $C_5 = C_4/C_0$ and $C_6 = C_1/C_0$

$$C_5|d_m - h|^{2-2n} \geq C_6 h^{2-n} - \frac{\epsilon^{Bh^F}}{h^T} = C_6 h^{2-n}(1 - \epsilon^{Bh^F} h^K),$$

where $0 < K = n - 2 - T$. Now let $h = h(\epsilon) = \min\left\{|\ln \epsilon|^{-\frac{1}{2F}}, d_m\right\}$, for $0 < \epsilon \leq \epsilon_1, \epsilon_1 \in (0,1)$ such that $\exp(-B|\ln \epsilon_1|^{1/2}) = 1/2$. It is easy to see if $d_m \leq |\ln \epsilon|^{-\frac{1}{2F}}$, the main Theorem 2.2 is already proved thanks to Lemma 3.2. Because we can set $\eta = \frac{1}{2F} > 0$, then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 |\ln \epsilon|^{-\eta} = \omega(\epsilon) \tag{3.11}$$

In another case where $d_m \geq |\ln \epsilon|^{-\frac{1}{2F}}$, it is easy to check

$$(d_m - h)^{2-2n} \geq \frac{C_6}{2C_5} h^{2-n} \quad \Longrightarrow \quad d_m \leq C_7 |\ln \epsilon|^{-\frac{n-2}{4F(n-1)}}.$$

Here we can solve d_m because here $h = h(\epsilon) = |\ln \epsilon|^{-\frac{1}{2F}}$, and C_7 depends only on the *a priori* data. Therefore we conclude the proof by setting $\eta = \frac{n-2}{4F(n-1)}$

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 C_7 |\ln \epsilon|^{-\eta} = \omega(\epsilon) \tag{3.12}$$

and for $\epsilon_1 \leq \epsilon$, we can also include the proof because $d_m \leq |\Omega| \leq M_2 r_1^n$.

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 M_2 r_1^n = \omega(\epsilon). \tag{3.13}$$

We can conclude the proof Theorem 2.2 by (3.11), (3.12) and (3.13)

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq C d_m = \omega(\epsilon),$$

where C only depends on the *a priori* data. □

4 Proof of the Auxiliary Propositions

First, we prove Proposition 3.4 mainly followed by [3, Proposition 3.5], with three sphere inequalities. When it comes to the situation that we need to cross Σ during the iterative process, we apply the three-region inequalities. The proof contains two major steps: (1) we need to define our "*smallness*" outside of the domain Ω, and then we will propagate this smallness until it arrives at any given small ball contained inside $\mathcal{F}^\lambda$; (2) we use three-region inequality to propagate the "*smallness*" by crossing Σ; (3) we continue with three-sphere inequalities until the "*smallness*" arrives arbitrarily closed to $O \in \partial D_1$.

Before proving Proposition 3.4, let us briefly recall this result contained in [13, Theorem 3.1]. Based on some suitable Carleman estimate (see [8, Theorem 2.1]), the following three region inequality in the L^2 norm across the interface $y = 0$ holds.

Theorem 4.1. *There exist C and R, depending on λ_0, M_0, n such that if $0 < R_1, R_2 < R$ then*

$$\int_{U_2} |u|^2 dx \leq \left(e^{\tau_0 R_2} + C R_1^{-4}\right) \left(\int_{U_1} |u|^2 dxdy\right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{U_3} |u|^2 dxdy\right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \tag{4.1}$$

where τ_0 is the constant derived in the Carleman estimate [8, Theorem 2.1],

$$\begin{aligned} U_1 &= \{-4R_2 \leq z, \quad \frac{R_1}{8a} < y < \frac{R_1}{a}\}, \\ U_2 &= \{-R_2 \leq z \leq \frac{R_1}{2a}, \quad y < \frac{R_1}{8a}\}, \\ U_3 &= \{-4R_2 \leq z, \quad y < \frac{R_1}{a}\}, \end{aligned} \tag{4.2}$$

$a = \alpha_+/\delta$ and

$$z = \frac{\alpha_-}{\delta} y + \frac{\beta}{2\delta^2} y^2 - \frac{1}{2\delta}|x|^2.$$

We also compute $H := \frac{\delta}{\beta}[\alpha_- - \sqrt{\alpha_-^2 - 2R_2\beta}]$ to measure "*vertical depth*" of the region U_2 below the x-axis.

Proof of Proposition 3.4. Let us consider $f(y, \cdot)$ with a fixed $y \in S_{2r}$ then

$$\Delta_w f(y, w) = 0 \quad \text{in } \mathcal{C}\overline{\Omega}_D. \tag{4.3}$$

For $x \in S_{2r}$, by (3.5) and (3.6), we have the smallness quantity

$$|f(y, x)| \leq C(r, M_1, M_2)||\Gamma_{D_1} - \Gamma_{D_2}|| = \epsilon. \tag{4.4}$$

Also by [3, Proposition 3.4], the uniform bound of f is given as

$$|f(y, x)| \leq ch^{2-2n}, \qquad \text{in } \mathcal{G}^h \cup \mathcal{F}^\lambda. \tag{4.5}$$

For any $0 < \overline{r} < r$ and for every $\overline{\omega} \in \mathcal{F}^\lambda$, we can have smallness on any arbitrarily small ball $B_{\overline{r}/2}(\overline{\omega}) \subset \mathcal{F}^\lambda \subset \mathcal{G}^h$ by iteratively applying three-sphere inequalities on a simple arc $\gamma \in \overline{\Omega_-} \cup \overline{S_r} \cup \overline{S_{2r}}$ which connects $\overline{\omega}$ and x. By [3] (4.21), we can reach $\overline{\omega}$ from x with a finite number s of balls. Thus we obtain

$$||f(y, \cdot)||_{L^\infty(B_{\overline{r}/2}(\overline{\omega}))} \leq C(h^{1-n})^{1-\tau^s} \epsilon^{\tau^s}, \tag{4.6}$$

where C depends on the *a prior* data.

Now let us deal with the situation when smallness is crossing the interface Σ. We define a coordinator system locally in a small neighbor near Σ. For any point $O \in \Sigma$, the outer norm ν from O onto Ω_- with respect to Σ is defined as the y-axis; and its tangential direction is defined as the x-axis. We choose $\frac{R_1}{8a} = 2\lambda$, and $\lambda \leq \frac{\overline{r}}{15}$ for $\mathcal{F}^\lambda$. Now by (4.2), there are three regions U_1, U_2 and U_3 located near the Σ, where $U_1 \subset \mathcal{F}^\lambda$ and $U_2 \cap \Omega_+ \neq \emptyset$.

Notice with the choice of λ, we can use a finite amount of balls to cover the region U_1. We pick up $\overline{\omega}_j \in \Sigma_{\overline{r}/2+\lambda}$, where $\Sigma_{\overline{r}/2+\lambda} := \{x \in \Omega_- | dist(x, \Sigma) = \overline{r}/2 + \lambda\}$. Then there exists $J < \infty$ such that $\cup_{j=1}^J B_{\overline{r}/2}(\overline{\omega}_j) \supset U_1$. By standard bound for L^2 and L^∞ norms, and (4.6), we obtain

$$\begin{aligned} ||f(y,\cdot)||_{L^2(U_2)} &\leq C||f(y,\cdot)||^A_{L^2(U_1)} \cdot ||f(y,\cdot)||^{1-A}_{L^2(U_3)} \\ &\leq C||f(y,\cdot)||^A_{L^\infty(\cup_{j=1}^J B_{\overline{r}/2}(\overline{\omega}_j))} \cdot (h^{2-2n})^{2(1-A)} \\ &\leq C(h^{1-n})^{(1-\tau^s)A} \epsilon^{A\tau^s} \cdot (h^{2-2n})^{2(1-A)} \\ &\leq C(h^{1-n})^{\mathcal{A}} \epsilon^{\mathcal{B}}, \end{aligned} \tag{4.7}$$

where $\mathcal{A} = 4 - 3A - A\tau^s$ and $\mathcal{B} = A\tau^s$, C depends on $\tau_0, R_1, R_2, \lambda_0, M_0, n, J$ and *a prior* data. Then we pick up a small ball inside of $U_2 \cap \Omega_+$ as the start for the rest of the propagation. For the above coordinator system x-O-y, we choose $x_0 = (0, -H/2)$ and $r_0 < H$, so that $B_{r_0/2}(x_0) \subset U_2 \cap \Omega_+$. By (4.7),

$$||f(y,\cdot)||_{L^\infty(B_{r_0/2}(x_0))} \leq ||f(y,\cdot)||_{L^2(U_2)} \leq C(h^{1-n})^{\mathcal{A}} \epsilon^{\mathcal{B}}. \tag{4.8}$$

The rest of the propagation is similar as (4.6). If we choose any $0 < \overline{r_0} < r_0$ and any $\overline{\omega}_0 \in \Omega_+$, by connecting x_0 and $\overline{\omega}_0$ with a simple arc, we obtain

$$||f(y,\cdot)||_{L^\infty(B_{\overline{r}_0/2}(\overline{\omega}_0))} \leq C(h^{1-n})^{\overline{\mathcal{A}}} \epsilon^{\overline{\mathcal{B}}}. \tag{4.9}$$

Then rest of the proof is followed by [3] (4.22) to (4.25): we define a truncated cone $C(O, \nu(O), \theta, r)$, in which $O \in D_1$ is the point where the maximum of Definition 3.1 is attained. Then we consider $f(y, w)$ as a function of y to obtain similar results. The last step is to choose $y = w = h\nu(O)$, where $\nu(O)$ is the exterior unit normal to $\partial\Omega_D$ in O, we can obtain

$$|f(y,y)| \leq Ch^T (\epsilon^{\tilde{B}A^{k(h)-1}})^{\gamma A^{k(h)-1}}. \tag{4.10}$$

We observe that for $0 < h < cr$, where $0 < c < 1$ depends on M_1, $k(h) \leq c|\log A| = -c\log h$, we can rewrite

$$A^{k(h)} = \exp\{-c\log h \log A\} = h^{-c\log A} = h^{c|\log A|} = h^Q,$$

$$(A^{k(h)})^2 = (h^Q)^2 = h^F$$

with $F = 2Q = 2c|\log A|$. Therefore

$$\begin{aligned}|f(y,y)| &\leq Ch^{-T}\epsilon^{B(A^{k(h)})^2}\\ &\leq \exp\{-T\log h\}\exp\{B(A^{k(h)})^2\log\epsilon\}\\ &\leq \exp\{-T\log h + Bh^F\log\epsilon\}\\ &= \frac{\epsilon^{Bh^F}}{h^T},\end{aligned}$$

where $B = \frac{\gamma\tilde{B}}{A^2}$ □

The proof of Proposition 3.5 is based on the asymptotic behaviour of the fundamental solutions locally in the neighbour of $O \in D_1$, which is contained complete inside of Ω_+ because $\mathrm{dist}(D,\Sigma) \geq \delta_1$

$$\mathrm{div}[(c_2 + (k - c_2)\chi_D)\nabla\Gamma_D(\cdot - y)] = -\delta(\cdot - y)$$

$$\mathrm{div}[(c_2 + (k - c_2)\chi_+)\nabla\Gamma_+(\cdot - y)] = -\delta(\cdot - y),$$

where χ_+ is the characteristics function if the half-space $\{x_n > 0\}$. If Γ is the standard fundamental solution of the Laplace operator, and $y^* = (y', -y_n)$ is the reflecting point, we have the following relationship

$$\Gamma_+(x,y) = \begin{cases} \frac{1}{k}\Gamma(x,y) + \frac{k-c_2}{k(k+c_2)}\Gamma(x,y^*) & \text{for } \ x_n > 0, y_n > 0\\ \frac{2}{k+c_2}\Gamma(x,y) & \text{for } \ x_n y_n < 0\\ \frac{1}{c_2}\Gamma(x,y) - \frac{k-c_2}{c_2(k+c_2)}\Gamma(x,y^*) & \text{for } \ x_n < 0, y_n < 0.\end{cases}$$

We have the following theorem.

Theorem 4.2. *Let $D \in \mathbb{R}^n$ be an open set with $C^{1,\alpha}$ boundary subjected to constants M_1, r. We have Γ_D and Γ_+ the fundamental solutions defined above, respectively. The following asymptotic estimate holds for any $x, y \in \mathbb{R}^n$*

$$|\nabla\Gamma_D(x,y)| \leq C|x - y|^{1-n}.$$

As for $x \in D \cap B_\rho(O)$ and every $y = h\nu(O)$, with $0 < \rho < r_0$ and $0 < h < r_0$ where $r_0 = \frac{r}{2}\min\{\frac{1}{2}(8M_1)^{-1/2}, \frac{1}{2}\}$ we have

$$|\Gamma_D(x,y) - \Gamma_+(x,y)| \leq \frac{C}{r^\alpha}|x - y|^{\alpha-n+2}$$

$$|\nabla\Gamma_D(x,y) - \nabla\Gamma_+(x,y)| \leq \frac{C}{r^{\alpha^2}}|x - y|^{\alpha^2-n+1},$$

where $C > 0$ only depends on the a priori data.

For both proofs of Theorem 4.2 and Proposition 3.5, we refer to the proofs of Propositions 3.4 and 3.6 in [3] with a slightly modification on the constant coefficients. In fact, during the integration steps, we use $k-c_2$ as the constant instead of $k-1$. This won't affect the proofs since both $k-c_2$ and $k-1$ can be absorbed into a constant C in the final step, where C depends on k. We mention that in our paper, Γ_+ is also represented as a linear combination of standard Laplace Γ with constants coefficients.

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