

MINIMAL CASTELNUOVO-MUMFORD REGULARITY FOR A GIVEN HILBERT POLYNOMIAL

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ABSTRACT. Let K be an algebraically closed field of null characteristic and $p(z)$ a Hilbert polynomial. We look for the minimal Castelnuovo-Mumford regularity $m_{p(z)}$ of closed subschemes of projective spaces over K with Hilbert polynomial $p(z)$. Experimental evidences led us to consider the idea that $m_{p(z)}$ could be achieved by schemes having a suitable *minimal Hilbert function*. We give a constructive proof of this fact. Moreover, we are able to compute the minimal Castelnuovo-Mumford regularity $m_{p(z)}^\varrho$ of schemes with Hilbert polynomial $p(z)$ and given regularity ϱ of the Hilbert function, and also the minimal Castelnuovo-Mumford regularity m_u of schemes with Hilbert function u .

These results find applications in the study of Hilbert schemes. They are obtained by means of *minimal Hilbert functions* and of two new constructive methods which are based on the notion of growth-height-lexicographic Borel set and called *ideal graft* and *extended lifting*.

INTRODUCTION

This paper deals with the Castelnuovo-Mumford regularity $\text{reg}(X)$ of closed subschemes X of projective spaces over an algebraically closed field K of characteristic zero, with a given Hilbert polynomial $p(z)$. Hence, a scheme will be understood to be a closed subscheme of a projective space. We do not establish the dimension of the projective space in which the schemes are embedded, although we will be able to give information also for every given dimension.

As we can read in literature, $\text{reg}(X)$ is “one of the fundamental invariants in Commutative Algebra and Algebraic Geometry” (see [[Brodmann\(2010\)](#)]) and can be also considered as “a measure of the complexity of computing Gröbner bases” [[Bayer and Mumford\(1993\)](#)]. Both these aspects of the Castelnuovo-Mumford regularity are present in this paper. Thus, besides the classical definitions of the Castelnuovo-Mumford regularity in terms of ideal sheaf cohomology and of syzygies, we will also recall its characterization in terms of the generic initial ideal with respect to the degree reverse lexicographic term order (see Definition [2.1](#) and Proposition [4.1](#)).

In his famous paper [[Gotzmann\(1978\)](#)], G. Gotzmann finds a sharp upper bound for the Castelnuovo-Mumford regularity of schemes with a given Hilbert polynomial. This upper bound, called *Gotzmann number*, is fundamental in the study of Hilbert schemes and determines a finite range of possible values for the Castelnuovo-Mumford regularity, given a Hilbert polynomial. Here, we tackle the following question posed by E. Ballico

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in a private conversation: *what is the minimal value $m_{p(z)}$ of the Castelnuovo-Mumford regularity of closed subschemes of projective spaces with given Hilbert polynomial $p(z)$?*

Here, we provide a complete answer to the posed question, because we give a sharp lower bound for the Castelnuovo-Mumford regularity of schemes with a given Hilbert polynomial, thus narrowing the range in which the Castelnuovo-Mumford regularity of a scheme with a given Hilbert polynomial can vary. This additional information can have many significant applications. To offer an example, it can be applied to the study described in [Ballico et al(2011)], from which the question considered in our manuscript arised. In that paper explicit equations and inequalities defining the open locus of the Hilbert scheme of points with bounded regularity are obtained: our result tells when such an open subset is non-empty.

To reach our aim, we study the *minimal functions* and develop two new constructive methods, which we think can have further applications also in more general contexts as, for example, the study of liftings and that of general hypersurface sections.

In some cases, to find the value of $m_{p(z)}$ is almost immediate. For example, for a constant Hilbert polynomial $p(z) = d > 1$, we obtain $m_{p(z)} = 2$. Indeed, having K an infinite cardinality, there exists a scheme $X \subset \mathbb{P}_K^{d-1}$ of d points in generic position, that has Hilbert function $(1, d, d, \dots)$ (for instance, see [Geramita et al(1983), Theorem 2.5]).

However, in general, the posed question has not such a straightforward answer. Some preliminary results highlight the role played in this context by the regularity ϱ_X of the Hilbert function H_X of the scheme X . Indeed, they suggest that, if $\text{reg}(X) = m_{p(z)}$, then the integers ϱ_X and $H_X(\varrho_X - 1)$ are small with respect to the corresponding ones of other schemes with greater Castelnuovo-Mumford regularities (Lemma 2.4 and Proposition 2.5). These facts together with experimental evidences encourage us to look for an answer by focusing on the following two key ideas: to find a suitable notion of minimal function and to consider the minimal Castelnuovo-Mumford regularity $m_{p(z)}^\varrho$ of schemes, by choosing also the regularity ϱ of the Hilbert function. We gained the experimental evidences supporting our ideas by a pencil-and-paper work and by using the applet that is available at http://www.personalweb.unito.it/paolo.lella/HSC/Borel-fixed_ideals.html to compute suitable Borel-fixed ideals. Some of our experiments are reproduced here as examples.

We develop the study of minimal functions with given regularity generalizing a construction that L. G. Roberts introduced in a special case (see [Roberts(1982)]). The minimal functions have very nice properties due to their purely combinatorial structure (see Theorem 3.9) and turn out to be sufficient for a complete description of minimal Castelnuovo-Mumford regularities, in the following sense.

Theorem A. *Let $S(p(z), \varrho)$ be the set of schemes with given Hilbert polynomial $p(z)$ and regularity ϱ of their Hilbert functions. If $S(p(z), \varrho)$ is non-empty, then it contains a scheme X such that both its Hilbert function H_X and its Castelnuovo-Mumford regularity $\text{reg}(X)$ are minimal for schemes in $S(p(z), \varrho)$.*

In [Caviglia and Sbarra(2014), Theorem 3.2], the authors observe that Theorem A holds on fields of any characteristic.

In order to find $m_{p(z)}$, we compare the integers $m_{p(z)}^\varrho$, as ϱ varies among all values for the regularities of Hilbert functions of schemes with the given Hilbert polynomial $p(z)$. Our analysis allows us to prove the following statement, which is even stronger than the one suggested by the preliminary results.

Theorem B. *Let ϱ and ϱ' be the regularities of two Hilbert functions of schemes with Hilbert polynomial $p(z)$. Then,*

- $\varrho < \varrho' \implies m_{p(z)}^\varrho \leq m_{p(z)}^{\varrho'}$;
- if $\varrho < m_{p(z)} - 1$ then $m_{p(z)}^\varrho = m_{p(z)}$; especially, $m_{p(z)} = m_{p(z)}^\varrho$ for the minimal possible ϱ ;
- if $\varrho \geq m_{p(z)} - 1$ then $m_{p(z)}^\varrho = \varrho + 1$ or $m_{p(z)}^\varrho = \varrho + 2$.

The above results are collected in Theorem 7.1 and Corollary 7.2. Their proofs are based on two new constructive methods we conceived by exploiting the features of the growth-height-lexicographic Borel sets, which have been introduced by D. Mall in [Mall(1997), Mall(2000)]. Both these methods combine together properties of two schemes X_1 and X_2 , returning a new scheme with Hilbert function and Castelnuovo-Mumford regularity depending on those of X_1 and X_2 . They differ by the numerical hypotheses they require to be applied.

The first method, called *ideal graft* constructs a scheme with the Hilbert function of X_1 and the Castelnuovo-Mumford regularity depending on that of X_2 . This method has a special role in the comparison of the integers $m_{p(z)}^\varrho$ (Theorem 5.2 and Corollary 5.3).

The second method, called *expanded lifting*, computes a scheme having Hilbert function of X_1 and hyperplane section with the Hilbert function of X_2 . Moreover, the expanded lifting has a total control on the Castelnuovo-Mumford regularity of the constructed scheme (see Theorem 6.4).

Our final result gives a recursive procedure to compute the minimal Castelnuovo-Mumford regularity m_u of a scheme with a Hilbert function u and provides very strict lower and upper bounds for m_u (see Theorem 7.4). This computation allows to determine $m_{p(z)}^\varrho$, $m_{p(z)}$ and also the minimal Castelnuovo-Mumford regularity of schemes with Hilbert polynomial $p(z)$ that are embedded in a given projective space \mathbb{P}_K^n .

The paper is organized in the following way. In section 1, we set some notation about monomial ideals. In section 2, we recall some basic definitions and classical results about Hilbert functions and Castelnuovo-Mumford regularity. We also introduce some new preliminary results (Propositions 2.5 and 2.7). In section 3, we introduce the minimal functions with given regularities, studying their combinatorial properties in relation with the existence of schemes with given Hilbert polynomials and Hilbert function regularity (see Theorem 3.9). Moreover, we describe a closed formula for the minimum ϱ such that $S(p(z), \varrho) \neq \emptyset$ (see Proposition 3.11).

In section 4, we recall the notion of growth-height-lexicographic Borel set, which is crucial for the description of the two constructive methods *ideal graft* and *expanded lifting*, which we propose and describe in sections 5 and 6, respectively. In section 7, we provide the proofs of our main results.

As a natural consequence of the computational point of view that supported this paper, all our results produce some constructive methods. In the appendix A, we describe the main algorithmic procedures to compute the minimal Castelnuovo-Mumford regularities that arise from our exposition.

1. GENERAL SETTING

Let K be an algebraically closed field of characteristic 0, $S := K[x_0, \dots, x_n] = \bigoplus_{t \geq 0} S_t$ be the ring of polynomials over K in $n + 1$ variables with $x_0 < x_1 < \dots < x_n$, where S_t is

the K -vector space of the homogeneous polynomials of degree t , and $\mathbb{P}_K^n = \text{Proj } S$ be the n -dimensional projective space over K .

For a subset $M \subseteq S$ we set $M_t := M \cap S_t$. For a homogeneous ideal I of S , we denote by $I_{\leq t}$ the ideal generated by the polynomials of I of degree $\leq t$.

A *term* of S is a power product $x^\alpha := x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $\alpha_0, \alpha_1, \dots, \alpha_n$ are non negative integers, and $\mathbb{T} := \{x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{n+1}\}$ is the multiplicative monoid of all terms of S . For a monomial ideal J , we denote by $\mathcal{N}(J)$ the *sous-escalier* of J , i.e. the set of all terms outside J .

In our setting, the graded term orders *deglex* and *degrevlex* are defined saying that, given two terms x^α and x^β of \mathbb{T} of the same degree t , x^α is less than x^β with respect to:

- deglex* order, if $\alpha_k < \beta_k$, where $k = \max\{i \in \{0, \dots, n\} : \alpha_i \neq \beta_i\}$;
- degrevlex* order, if $\alpha_h > \beta_h$, where $h = \min\{i \in \{0, \dots, n\} : \alpha_i \neq \beta_i\}$.

Given a degree t , a set $B \subset \mathbb{T}_t$ is a *lex-segment* if it consists of the $|B|$ highest terms of \mathbb{T}_t with respect to the *deglex* order. Given a subset $A \subset \mathbb{T}_t$, a *lex-segment in A* is the intersection of a *lex-segment* of \mathbb{T}_t and A . A monomial ideal J is a *lex-segment ideal* if J_t is a *lex-segment* for every integer t .

2. HILBERT FUNCTION AND CASTELNUOVO-MUMFORD REGULARITY

For a homogeneous ideal $I \subset S$, the Hilbert function of the graded algebra S/I is denoted by $H_{S/I}$ and its Hilbert polynomial is denoted by $p_{S/I}(z) \in \mathbb{Q}[z]$. The *regularity of the Hilbert function of S/I* is the integer $\rho_{S/I} := \min\{t \in \mathbb{N} \mid H_{S/I}(t') = p_{S/I}(t'), \forall t' \geq t\}$.

We set $\Delta^0 H_{S/I}(t) := H_{S/I}(t)$ and, when S/I is not Artinian, we let $\Delta^i H_{S/I}(0) := 1$ and $\Delta^i H_{S/I}(t) := \Delta^{i-1} H_{S/I}(t) - \Delta^{i-1} H_{S/I}(t-1)$, for each $1 \leq i \leq \dim_{\text{Krull}}(S/I)$ and for $t > 0$, calling it *i -th derivative of H* ; we use an analogous notation for Hilbert polynomials.

We define the function $\Sigma H_{S/I} : \mathbb{N} \rightarrow \mathbb{N}$ letting $\Sigma H_{S/I}(0) := 1$ and $\Sigma H_{S/I}(t) := \Sigma H_{S/I}(t-1) + H_{S/I}(t)$ for each $t \geq 1$, and call it *integral of $H_{S/I}(t)$* ; we use an analogous notation for Hilbert polynomials.

Recall that, given two positive integers a and t , the *binomial expansion of a in base t* is the unique writing

$$(1) \quad a = \binom{k(t)}{t} + \binom{k(t-1)}{t-1} + \dots + \binom{k(j)}{j} =: a_t$$

where $k(t) > k(t-1) > \dots > k(j) \geq j \geq 1$. We use the convention that a binomial coefficient $\binom{n}{m}$ is null whenever either $n < m$ or $m < 0$ and $\binom{n}{0} = 1$, for all $n \geq 0$. Referring to [Robbiano(1990)], we let

$$\begin{aligned} (a_t)_+^+ &:= \binom{k(t)+1}{t+1} + \binom{k(t-1)+1}{t} + \dots + \binom{k(j)+1}{j+1}, \text{ and} \\ (a_t)_-^- &:= \binom{k(t)-1}{t-1} + \binom{k(t-1)-1}{t-2} + \dots + \binom{k(j)-1}{j-1}. \end{aligned}$$

For convenience, we also set $((a_t)_-^+)_+^+ := (((a_t)_-^-)_{t-1})_+^+$. By an easy computation one gets (for example, see [Robbiano(1990), Proposition 4.9])

$$(2) \quad ((a_t)_-^-)_+^+ = \begin{cases} a, & \text{if } j > 1 \\ a + k(2) - k(1), & \text{if } j = 1 \end{cases},$$

hence, $((a_t)_-^-)_+^+ \geq a$. Moreover, [Robbiano(1990), Prop. 4.3(a) and 4.6(b)]

$$(3) \quad ((a+1)_t)_+^+ = (a_t)_+^+ + 1 + k(1) \text{ and } ((a+1)_t)_-^- = \begin{cases} (a_t)_-^- + 1 & \text{if } j > 1 \\ (a_t)_-^- & \text{if } j = 1. \end{cases}$$

A numerical function $H : \mathbb{N} \rightarrow \mathbb{N}$ is *admissible* (or an *O-sequence*) if $H(0) = 1$ and $H(t+1) \leq (H(t)_t)_+^\dagger$. In particular, if $H(t) = 0$, then $H(t+h) = 0$ for every $h > 0$. We will say that H is *admissible in \bar{t}* if $H(\bar{t}+1) \leq (H(\bar{t})_{\bar{t}})_+^\dagger$.

Due to [Macaulay(1926)], a numerical function H is admissible if, and only if, it is the Hilbert function of S/I , for a suitable homogeneous ideal I .

Definition 2.1. A homogeneous ideal I is m -regular if the i -th syzygy module of I is generated in degree $\leq m+i$. The *regularity* $\text{reg}(I)$ of I is the smallest integer m for which I is m -regular. The saturation of I is $I^{\text{sat}} := \{f \in S \mid \forall i \in 0, \dots, n, \exists k_i : x_i^{k_i} f \in I\}$ and I is saturated if $I = I^{\text{sat}}$.

With the common notation of the ideal sheaf cohomology, given a scheme $X \subset \mathbb{P}_K^n$ and its (saturated) defining ideal $I = I(X)$, the *Castelnuovo-Mumford regularity* of X is $\text{reg}(X) := \min\{t \in \mathbb{N} \mid H^i(\mathcal{I}_X(t-i)) = 0, \forall t' \geq t, \forall i > 0\}$ and it is equal to $\text{reg}(I)$. Also, we set $H_X(t) := H_{S/I}(t)$, $p_X(z) := p_{S/I}(z)$, $\varrho_X := \varrho_{S/I}$.

If the dimension of a scheme $X \subset \mathbb{P}_K^n$ is $k > 0$, let $h \in S_1$ be a general linear form that is not a zero-divisor on S/I and $J = (I, h)$, where $I := I(X)$. It is well known that the first derivative of the Hilbert function of a projective scheme is admissible, and the converse is true by [Geramita et al(1983), Corollary 3.4]. Indeed, let $Z \subset \mathbb{P}_K^{n-1}$ be the scheme of dimension $k-1$ defined by the saturated ideal $J^{\text{sat}}/(h) = (I, h)^{\text{sat}}/(h)$, i.e. the general hyperplane section of X . Since the linear form h is not a zero-divisor on S/I , we have the short exact sequence $0 \rightarrow (S/I)_{t-1} \xrightarrow{h} (S/I)_t \rightarrow (S/J)_t \rightarrow 0$ that gives $H_{S/J}(t) = \Delta H_{S/I}(t)$, in particular $p_{S/(I,h)} = \Delta p_{S/I}$, and then $\varrho_{S/J} = \varrho_X + 1$, so $\Delta H_X(t) = H_{S/J}(t) \geq H_Z(t)$ for every t and $H_{S/J}(t) = H_Z(t)$ for $t \geq \max\{\varrho_Z, \varrho_{S/J}\}$.

This relation between the first derivative of the Hilbert function of X and the Hilbert function of Z suggests to consider the following partial order.

Definition 2.2 ([Roberts(1982)]). Given two sequences of integers $A = (a_i)_{i \in \mathbb{N}}$ and $B = (b_i)_{i \in \mathbb{N}}$, we let $A \preceq B$, if $a_i \leq b_i$ for every index i .

Remark 2.3. Let f be a Hilbert function with Hilbert polynomial $p(z)$. If g is an other Hilbert function with Hilbert polynomial $\Delta p(z)$ such that $g \preceq \Delta f$, then $\sum g \preceq f$. Moreover, $\sum g$ has Hilbert polynomial $p(z) - c$, where c is a non-negative integer. If \bar{t} is the minimal integer such that $g(\bar{t}) < \Delta f(\bar{t})$, then $\sum g(t) < f(t)$ for every $t \geq \bar{t}$.

By cohomological arguments, one gets $\text{reg}(X) \geq \text{reg}(Z)$ and $\text{reg}(X) \geq \varrho_X + 1$, for every scheme X . In particular, the following result tells how ϱ_X and $\text{reg}(Z)$ determine $\text{reg}(X)$.

Lemma 2.4 ([Cioffi et al(2009), Lemma 3.6]). *With the above notation, we have $\text{reg}(X) = \max\{\text{reg}(Z), \varrho_X + 1\}$.*

For a Cohen-Macaulay scheme $W \subset \mathbb{P}_K^n$ of dimension k and degree d , the Castelnuovo-Mumford regularity is $\text{reg}(W) = \varrho_W + k + 1 \leq d$ (e.g. [Bayer and Stillman(1987)]). Moreover, if $X \subset \mathbb{P}_K^n$ is a scheme of odd dimension k with the same Hilbert function as a Cohen-Macaulay scheme, then $\text{reg}(X) > \varrho_X + 1$ [Cioffi and Di Gennaro(2011), Proposition 2.4], even if the characteristic of K is positive. By exploiting the proof of that result we obtain the following more general statement.

Proposition 2.5. *If $X \subset \mathbb{P}_K^n$ is a scheme with $H_X(\varrho_X - 1) > p_X(\varrho_X - 1)$, then $\text{reg}(X) > \varrho_X + 1$.*

Proof. Let Z be a general hyperplane section of X . By the hypothesis we get $H_Z(\varrho_X) \leq \Delta H_X(\varrho_X) < \Delta p_X(\varrho_X) = p_Z(\varrho_X)$. Hence, we obtain $\varrho_Z > \varrho_X$ and $\text{reg}(X) \geq \text{reg}(Z) \geq \varrho_Z + 1 > \varrho_X + 1$. \square

Polynomials $p(z) \in \mathbb{Q}[z]$ that are Hilbert polynomials of schemes are called *admissible* and are completely characterized in [Hartshorne(1966)].

The *Gotzmann number* r of an admissible polynomial $p(z)$ is the best upper bound for the Castelnuovo-Mumford regularity of a scheme having $p(z)$ as Hilbert polynomial and is computable by using the following unique form of an admissible polynomial:

$$(4) \quad p(z) = \binom{z+k_1}{k_1} + \binom{z+k_2-1}{k_2} + \dots + \binom{z+k_r-(r-1)}{k_r},$$

with $r, k_i \in \mathbb{N}, k_1 \geq k_2 \geq \dots \geq k_r \geq 0$ [Gotzmann(1978)]. We refer to [Green(2010)] for an overview of these arguments. For a constant polynomial $p(z) = d$ we have $r = d$.

A polynomial with Gotzmann number $r = 1$ is of type $p(z) = \binom{h+z}{z}$ and is the Hilbert polynomial of a linear variety X . From now, $p(z)$ is an admissible Hilbert polynomial with Gotzmann number $r > 1$.

Notation 2.6. *We will need the following notation:*

- $S(p(z), \varrho)$ is the set of schemes with Hilbert polynomial $p(z)$ and regularity ϱ of the Hilbert function.
- $F(p(z), \varrho)$ is the set of the Hilbert functions of the schemes in $S(p(z), \varrho)$ and we let $F(p(z)) := \cup_{\varrho} F(p(z), \varrho)$.
- For every $u \in F(p(z), \varrho)$, m_u is the minimal possible Castelnuovo-Mumford regularity of a scheme with Hilbert function u ;
- $M(p(z), \varrho) := \{m_u : u \in F(p(z), \varrho)\}$ and $m_{p(z)}^{\varrho} := \min M(p(z), \varrho)$.
- $\bar{\varrho}_{p(z)}$ is the minimal integer ϱ such that $F(p(z), \varrho) \neq \emptyset$.
- $\varrho_{p(z)}$ is defined in (5) and its meaning is explained by Propositions 2.7 and 3.2.
- \bar{f} and \bar{m} are defined in (8).

Proposition 2.7. *The set $\Pi_{p(z)} := \{1 \leq t \leq r : (p(t+h)_{t+h})_+^{\dagger} \geq p(t+h+1) \geq 1, \forall h \geq 0\}$ is non-empty and the minimal regularity of the admissible functions with Hilbert polynomial $p(z)$ is lower bounded by*

$$(5) \quad \varrho_{p(z)} := \begin{cases} 0, & \text{if } \min \Pi_{p(z)} = 1 \text{ and } p(0) = 1 \\ \min \Pi_{p(z)}, & \text{otherwise.} \end{cases}$$

Moreover,

- (i) $\bar{\varrho}_{p(z)} \geq \max\{\varrho_{p(z)}, \varrho_{\Delta p(z)} - 1\}$
- (ii) if $\varrho_{p(z)} > 0$, then $\varrho_{p(z)+c} \leq \varrho_{p(z)}$ and $\bar{\varrho}_{p(z)+c} \leq \bar{\varrho}_{p(z)}$ for every $c > 0$.

Proof. Observe that $(p(r+h)_{r+h})_+^{\dagger} = p(r+h+1)$, for every $h \geq 0$, by the well-known fact that the Gotzmann number r of an admissible polynomial $p(z)$ is the regularity of the unique saturated lex-segment ideal with Hilbert polynomial $p(z)$ (for example, see [Green(2010), Proposition 3.7]). Hence, the set $\Pi_{p(z)}$ is non-empty.

Let f be an admissible function with Hilbert polynomial $p(z)$ and regularity ϱ . Then, for every $s \geq \varrho$, we have $(p(s)_s)_+^{\dagger} = (f(s)_s)_+^{\dagger} \geq f(s+1)_{s+1} = p(s+1)_{s+1} \geq 1$ and $s \in \Pi_{p(z)}$. Hence, the regularity of a Hilbert function with Hilbert polynomial $p(z)$ is lower bounded by $\varrho_{p(z)}$.

For the assertion (i), if g is a function of $F(p(z), \bar{\varrho}_{p(z)})$, then $\bar{\varrho}_{p(z)} \geq \varrho_{p(z)}$ and Δg is admissible with regularity $\bar{\varrho}_{p(z)} + 1$, hence $\varrho_{\Delta p(z)} \leq \bar{\varrho}_{p(z)} + 1$. For (ii), we obtain both $\varrho_{p(z)+c} \leq \varrho_{p(z)}$ and $\bar{\varrho}_{p(z)+c} \leq \bar{\varrho}_{p(z)}$ by $\Delta(p(z) + c) = \Delta p(z)$ and by the first formula of (3), because $((p(t) + c)_t)_+^\dagger \geq (p(t)_t)_+^\dagger + c \geq p(t + 1) + c$, for every $t \geq \varrho_{p(z)}$. \square

3. MINIMAL FUNCTIONS

In this section, we will prove there exists the minimum of the Hilbert functions with Hilbert polynomial $p(z)$ and regularity ϱ , for every $\varrho \geq \varrho_{p(z)}$, with respect to the partial order \preceq of Definition 2.2. Each of these minimal functions will be of the following type.

Definition 3.1. For every $\varrho \geq \varrho_{p(z)}$, we let

$$(6) \quad f_{p(z)}^\varrho(t) := \begin{cases} p(t), & \text{if } t \geq \varrho \\ (f_{p(z)}^\varrho(t+1)_{t+1})_-, & \text{otherwise} \end{cases}$$

$$(7) \quad g_{p(z)}^\varrho(t) := \begin{cases} p(t), & \text{if } t \geq \varrho \\ p(t) + 1, & \text{if } t = \varrho - 1 \\ (g_{p(z)}^\varrho(t+1)_{t+1})_-, & \text{otherwise.} \end{cases}$$

Proposition 3.2. For every integer $\varrho_{p(z)} \leq \varrho \leq r - 1$, $f_{p(z)}^\varrho$ is admissible, hence $\varrho_{p(z)}$ is the minimal regularity of Hilbert functions with Hilbert polynomial $p(z)$. Moreover, $f_{p(z)}^\varrho \succeq f_{p(z)}^{\varrho+1}$ and $f_{p(z)}^\varrho$ is the minimal Hilbert function with Hilbert polynomial $p(z)$ and regularity $\leq \varrho$. In particular, $f_{p(z)}^{\varrho_{p(z)}}$ has regularity $\varrho_{p(z)}$ and, if $\varrho > \varrho_{p(z)}$, then $f_{p(z)}^\varrho(\varrho - 1) \leq p(\varrho - 1)$.

Proof. By Proposition 2.7, the regularity of every Hilbert function with Hilbert polynomial $p(z)$ is lower bounded by $\varrho_{p(z)}$.

With the notation introduced in formula (1), for a positive integer a , $(a_t)_-$ is the smallest integer b such that $a \leq (b_{t-1})_+^\dagger$, thanks to formula (2). Hence, by construction the numerical function $f_{p(z)}^\varrho$ is the minimal Hilbert function with Hilbert polynomial $p(z)$. Moreover, $f_{p(z)}^{\varrho_{p(z)}}$ has regularity $\varrho_{p(z)}$, by definition of $\varrho_{p(z)}$. The last assertion holds by construction, also if $\varrho_{p(z)} = 0$ and $\varrho = 1$, because in this case $f_{p(z)}^{\varrho_{p(z)}}(0) = p(0) = 1$. \square

Remark 3.3. In [Roberts(1982)], L.G. Roberts introduces the minimal Hilbert function we denote by $f_{p(z)}^{r-1}$, showing that it is the Hilbert function of the so-called tight fan constructed by Hartshorne and also the Hilbert function of the saturated lex-segment ideal with the given Hilbert polynomial [Roberts(1982), Lemma 5.1 and Theorem 5.3]. This function attains the equality in the Hyperplane Restriction Theorem due to M. Green [Green(2010), Theorem 3.4].

Remark 3.4. (1) For every $\varrho \geq r$, we can set $f_{p(z)}^\varrho := f_{p(z)}^{r-1}$, because $(p(r+h)_{r+h})_+^\dagger = p(r+h+1)$, for every $h \geq 0$, [Green(2010), Proposition 3.7].

(2) If $p(z) = d$ is a constant polynomial, for every $1 \leq \varrho \leq d - 1$, the corresponding minimal function f_d^ϱ has regularity equal to ϱ and every scheme with this Hilbert function has Castelnuovo-Mumford regularity $\varrho + 1$, because it is Cohen-Macaulay.

(3) If the regularity of $f_{p(z)}^\varrho$ is $\varrho' < \varrho$, then $f_{p(z)}^{\varrho'} = f_{p(z)}^\varrho$. For example, if we take the admissible polynomial $p(z) = 5z - 3$ with Gotzmann number $r = 7$, we obtain $\varrho_{p(z)} = 3$ and $f_{p(z)}^3 = f_{p(z)}^4$ because $(p(4)_4)_- = p(3)$, so the regularity of $f_{p(z)}^4$ is $3 < \varrho = 4$. On the

other hand, we have $f_{p(z)}^4 \neq f_{p(z)}^5$ because $p(4) = 17$ and $f_{p(z)}^5(4) = 16$. Anyway, there exists the minimal function with regularity ϱ even when the regularity of $f_{p(z)}^\varrho$ is strictly lower than ϱ .

Proposition 3.5. *For every $\varrho \geq \varrho_{p(z)}$, the function $g_{p(z)}^\varrho$ is admissible and, if the regularity of $f_{p(z)}^\varrho$ is $\varrho' < \varrho$, then $g_{p(z)}^\varrho$ is the minimal admissible function with regularity ϱ and Hilbert polynomial $p(z)$. Moreover, $f_{p(z)}^\varrho \preceq g_{p(z)}^\varrho$.*

Proof. It is enough to argue as in the proof of Proposition 3.2. \square

Now, as we announced at the beginning of this section, for every $\varrho \geq \varrho_{p(z)}$ we have the minimal Hilbert function with regularity ϱ , that is either $f_{p(z)}^\varrho$ or $g_{p(z)}^\varrho$. Anyway, we are interested in Hilbert functions of schemes, that are admissible functions with also admissible first derivative.

Example 3.6. If we consider $p(z) = 6z^2 - 18z + 37$ we get $\varrho_{p(z)} = 1$, $\Delta p(z) = 12z - 24$ and $\varrho_{\Delta p(z)} = 5$, so $f_{p(z)}^1$ is admissible but its first derivative $(1, 24, 12z - 24)$ is not, because of its behavior in the degrees $\varrho_{p(z)} \leq t \leq \varrho_{\Delta p(z)}$. For example, $\Delta f_{p(z)}^1$ is not admissible in 2 because $\Delta f_{p(z)}^1(2) = 0$.

We will show that the construction of a minimal function $f_{p(z)}^\varrho$ guarantees the admissibility of the first derivative of the function in the degrees strictly lower than ϱ . As a consequence, we will prove there exists the minimum of $F(p(z), \varrho)$ with respect to the partial order \preceq of Definition 2.2, as soon as $F(p(z), \varrho)$ is non-empty, and this minimal function is either $f_{p(z)}^\varrho$ or $g_{p(z)}^\varrho$.

The following technical result is used in Lemma 3.8 to detect when the first derivative of a minimal function $f_{p(z)}^\varrho$ is admissible.

Lemma 3.7. *Assume that an integer a with binomial expansion $\binom{k(t)}{t} + \dots + \binom{k(j)}{j}$ in base t can be written also as $a = \binom{h(t)}{t} + \binom{h(t-1)}{t-1} + \dots + \binom{h(j')}{j'}$ with $h(u) > h(u-1)$, for every $j' < u \leq t$, and $k(t) > h(t)$. Then $a = \binom{k(t)}{t}$ and the two writings of a are $\binom{k(t)}{t}$ and $\sum_{i=0}^t \binom{k(t)-1-i}{t-i}$, respectively.*

Proof. Let $b = k(t)$. By the definition of binomial expansion we have $\binom{b}{t} \leq \binom{h(t)}{t} + \binom{h(t-1)}{t-1} + \dots + \binom{h(j')}{j'}$. Replacing $\binom{b}{t}$ by $\sum_{i=0}^t \binom{b-1-i}{t-i}$ we get $\sum_{i=0}^t \binom{b-1-i}{t-i} \leq \sum_{i=0}^{t-j'} \binom{h(t-i)}{t-i} = a$. On the other hand, for every $i = 0, \dots, t-j'$ we have $\binom{b-1-i}{t-i} \geq \binom{h(t-i)}{t-i} \geq \binom{h(t-i)}{t-i}$, hence $j' = 0$ and by the equality between the two members we obtain $a = \binom{b}{t}$. \square

Lemma 3.8. (i) *For every $\varrho \geq \varrho_{p(z)}$ and $1 \leq t < \varrho$, $\Delta f_{p(z)}^\varrho$ is admissible in t , because*

$$\Delta f_{p(z)}^\varrho(t) = (\Delta f_{p(z)}^\varrho(t+1)_{t+1})^-.$$

(ii) *For every $\varrho > \varrho_{p(z)}$ and $t < \varrho$, $\Delta f_{p(z)}^\varrho(t) \leq \Delta f_{p(z)}^{\varrho-1}(t)$.*

(iii) *For every $\varrho \geq \max\{\varrho_{p(z)}, \varrho_{\Delta p(z)} - 1\}$, $\Delta f_{p(z)}^\varrho$ is admissible.*

Proof. (i) For every $1 \leq t \leq \varrho-1$ by construction we have $f_{p(z)}^\varrho(t+1) = \binom{k(t+1)}{t+1} + \dots + \binom{k(j)}{j}$, $f_{p(z)}^\varrho(t) = \binom{k(t+1)-1}{t} + \dots + \binom{k(j)-1}{j-1}$ and $f_{p(z)}^\varrho(t-1) = \binom{k(t+1)-2}{t-1} + \dots + \binom{k(j)-2}{j-2}$ and obtain $\Delta f_{p(z)}^\varrho(t) = (\Delta f_{p(z)}^\varrho(t+1)_{t+1})^-$.

Nevertheless, we have to consider also the case the binomial expansion of $f(t)$ in base t is different from the writing obtained by the binomial expansion of $f(t+1)$ in base $t+1$. This is exactly the case considered in Lemma 3.7, for which $f(t) = \binom{k(t+1)-1}{t} + \binom{k(t+1)-2}{t-1} \dots + \binom{k(t+1)-(t+1)}{0} = \binom{k(t+1)}{t}$ and the operation that consists in subtracting a unit to both the integers of the binomial coefficients gives the same result.

(ii) By Proposition 3.2, we have $f_{p(z)}^\varrho(\varrho-1) \leq p(\varrho-1) = f_{p(z)}^{\varrho-1}(\varrho-1)$.

Let c be the non-negative integer such that $f_{p(z)}^\varrho(\varrho-1) = p(\varrho-1) - c$. Then, formula (3) and the definition of a minimal function give:

$$\begin{aligned} \Delta f_{p(z)}^\varrho(\varrho-1) &= f_{p(z)}^\varrho(\varrho-1) - f_{p(z)}^\varrho(\varrho-2) = p(\varrho-1) - c - ((p(\varrho-1) - c)_{\varrho-1})^- \leq \\ &\leq p(\varrho-1) - c - (p(\varrho-1)_{\varrho-1})^- + c = \Delta f_{p(z)}^{\varrho-1}(\varrho-1). \end{aligned}$$

Now, the thesis is proved applying fact (i).

(iii) Due to the definition of $\varrho_{\Delta p(z)}$, to Proposition 3.2 and to fact (i), it is enough to show that the first derivative of $f_{p(z)}^\varrho$ is admissible in ϱ , i.e. $(\Delta f_{p(z)}^\varrho(\varrho)_\varrho)_+^\dagger \geq \Delta f_{p(z)}^\varrho(\varrho+1)$. We know that $\Delta f_{p(z)}^{r-1}$ is admissible because $f_{p(z)}^{r-1}$ is the Hilbert function of the unique saturated lex-segment ideal with Hilbert polynomial $p(z)$. So, the admissibility of $\Delta f_{p(z)}^\varrho$ in ϱ follows from that of $\Delta f_{p(z)}^{\varrho+1}$. Indeed, by fact (ii) we obtain $(\Delta f_{p(z)}^\varrho(\varrho)_\varrho)_+^\dagger \geq (\Delta f_{p(z)}^{\varrho+1}(\varrho)_\varrho)_+^\dagger \geq \Delta f_{p(z)}^{\varrho+1}(\varrho+1) \geq \Delta p(\varrho+1) = \Delta f_{p(z)}^\varrho(\varrho+1)$. \square

Theorem 3.9. (i) $\bar{\varrho}_{p(z)} = \max\{\varrho_{p(z)}, \varrho_{\Delta p(z)} - 1\}$ and $f_{p(z)}^{\bar{\varrho}_{p(z)}} \in F(p(z), \bar{\varrho}_{p(z)})$;
 (ii) for all $\varrho > \bar{\varrho}_{p(z)}$,

$$F(p(z), \varrho) \neq \emptyset \Leftrightarrow \begin{cases} f_{p(z)}^\varrho \in F(p(z), \varrho), & \text{if } f_{p(z)}^\varrho \text{ has regularity } \varrho, \\ g_{p(z)}^\varrho \in F(p(z), \varrho), & \text{otherwise.} \end{cases}$$

Proof. (i) First, recall that $\bar{\varrho}_{p(z)} \geq \max\{\varrho_{p(z)}, \varrho_{\Delta p(z)} - 1\}$ by Proposition 2.7. Then, by Lemma 3.8(iii) we obtain $\bar{\varrho}_{p(z)} = \max\{\varrho_{p(z)}, \varrho_{\Delta p(z)} - 1\}$ and $f_{p(z)}^{\bar{\varrho}_{p(z)}}$ belongs to $F(p(z), \bar{\varrho}_{p(z)})$, because the regularity of $f_{p(z)}^{\bar{\varrho}_{p(z)}}$ is $\bar{\varrho}_{p(z)}$ by the definition of $\bar{\varrho}_{p(z)}$.

(ii) The case involving $f_{p(z)}^\varrho$ follows from Lemma 3.8. Thus, suppose that $f_{p(z)}^\varrho$ has regularity $\varrho' < \varrho$. If $g_{p(z)}^\varrho$ belongs to $F(p(z), \varrho)$, obviously $F(p(z), \varrho)$ is non-empty. Vice versa, let $g := g_{p(z)}^\varrho$ and observe that $(\Delta g(t)_t)_+^\dagger \geq \Delta g(t+1)$, for every $t \geq \varrho+1$, by definition of $\bar{\varrho}_{p(z)}$, and for every $1 \leq t \leq \varrho-2$, by arguments analogous to those of the proof of Lemma 3.8.

It remains to prove $(\Delta g(\varrho)_\varrho)_+^\dagger \geq \Delta g(\varrho+1)$ and $(\Delta g(\varrho-1)_{\varrho-1})_+^\dagger \geq \Delta g(\varrho)$.

For the degree ϱ , if g' belongs to $F(p(z), \varrho)$, then $g(\varrho-1) \leq g'(\varrho-1)$, hence $\Delta g(\varrho) \geq \Delta g'(\varrho)$ and $\Delta g'$ admissible implies Δg admissible in the degree ϱ .

For the degree $\varrho-1$, we obtain $(\Delta f_{p(z)}^\varrho(\varrho-1)_{\varrho-1})_+^\dagger \geq \Delta f_{p(z)}^\varrho(\varrho) = \Delta g(\varrho) + 1$ by Lemma 3.8 and by the definition of g . Moreover, by (3)

$$\Delta g(\varrho-1) = p(\varrho-1) + 1 - ((p(\varrho-1) + 1)_{\varrho-1})^- \geq$$

$$\geq p(\varrho-1) + 1 - (p(\varrho-1)_{\varrho-1})^- - 1 = p(\varrho-1) - (p(\varrho-1)_{\varrho-1})^- = \Delta f_{p(z)}^\varrho(\varrho-1)$$

and $\Delta f_{p(z)}^\varrho$ admissible implies the first derivative of g admissible in the degree $\varrho-1$. \square

Example 3.10. Consider the admissible polynomial $p(z) = 5z - 3$ of Remark 3.4(3) with Gotzmann number $r = 7$ and $\varrho_{p(z)} = 3$. We have that $f_{p(z)}^3 = f_{p(z)}^4 = (1, 4, 8, 12, p(z))$, meanwhile $f_{p(z)}^4 \neq f_{p(z)}^5$. So, we can take the function $g_{p(z)}^4 = (1, 4, 8, 13, p(z))$, that is the minimal admissible function with regularity 4 and Hilbert polynomial $p(z)$. But, its first derivative $\Delta g_{p(z)}^4 = (1, 3, 4, 5, 4, 5)$ is not admissible. Thus, by Theorem 3.9 we can affirm that $F(p(z), 4) = \emptyset$, i.e. there are not schemes with Hilbert function having regularity 4 and Hilbert polynomial $p(z)$.

If the Gotzmann number r of the polynomial $p(z)$ is big, a computation of $\varrho_{p(z)}$ and of $\bar{\varrho}_{p(z)}$ performed according to formula (5) and to Theorem 3.9(i) can be very expensive. Indeed, we have to test $(p(t)_t)_+ \geq p(t+1)$, for every $\varrho_{p(z)} - 1 \leq t \leq r - 2$. For example, for the polynomial $p(z) = 2z^3 - 6z^2 + 29z - 20$ (see Example A.2) we find $r = 218498$ and $\varrho_{p(z)} = \bar{\varrho}_{p(z)} = 2$. Anyway, Proposition 2.7(ii) will help to compute $\varrho_{p(z)}$ and $\bar{\varrho}_{p(z)}$ in an efficient way, by induction on the degree of $p(z)$, as the following proposition shows using the features of minimal functions.

Proposition 3.11. *If $\deg p(z) > 0$, $\bar{\varrho}_{p(z)} = \sigma - 1$, with $\sigma := \min \{t \geq \max\{\varrho_{\Delta p(z)}, 1\} : \sum f_{\Delta p(z)}^t(t-1) \leq p(t-1)\}$.*

Proof. If $\varrho_{\Delta p(z)} = 0$, then we can observe that $f_{\Delta p(z)}^0 = f_{\Delta p(z)}^1$. Thus, in any case the Hilbert function $\sum f_{\Delta p(z)}^\sigma$ belongs to $F(p(z) - c, \sigma - 1)$ for some non-negative integer c , by construction and according to Remark 2.3. Then, by Proposition 2.7(ii), we have $\varrho_{p(z)} \leq \bar{\varrho}_{p(z)} \leq \bar{\varrho}_{p(z)-c} \leq \sigma - 1$.

Now, observe that $\Delta f_{p(z)}^{\bar{\varrho}_{p(z)}}$ is an admissible function of regularity $\bar{\varrho}_{p(z)} + 1$ and Hilbert polynomial $\Delta p(z)$. Thus, we obtain $f_{\Delta p(z)}^{\bar{\varrho}_{p(z)}+1} \leq \Delta f_{p(z)}^{\bar{\varrho}_{p(z)}}$, hence $\sum f_{\Delta p(z)}^{\bar{\varrho}_{p(z)}+1} \leq f_{p(z)}^{\bar{\varrho}_{p(z)}}$ and $\bar{\varrho}_{p(z)} + 1 \geq \sigma$. \square

Remark 3.12. Proposition 3.11 gives a closed formula for $\bar{\varrho}_{p(z)}$ and also the opportunity to compute efficiently $\varrho_{p(z)}$. Indeed, as $\bar{\varrho}_{p(z)} = \max\{\varrho_{p(z)}, \varrho_{\Delta p(z)} - 1\}$, if $\sigma > \varrho_{\Delta p(z)}$ then $\varrho_{p(z)} = \bar{\varrho}_{p(z)} = \sigma - 1$. Otherwise we have to check that $p(z)$ is admissible in the degrees $t \leq \sigma - 2$, instead of $t \leq r - 2$ as in the definition of $\varrho_{p(z)}$ of Proposition 2.7 (see Algorithms 1 and 2).

4. BOREL IDEALS AND GROWTH-HEIGHT-LEXICOGRAPHIC BOREL SETS

The *Borel order* $<_B$ is the partial order on \mathbb{T} for which, given two terms x^α and x^β , we have $x^\alpha <_B x^\beta$ if there is a finite sequence of terms $\tau_1 = x^\alpha, \tau_2, \dots, \tau_k = x^\beta$ such that $\tau_h = \frac{x_i}{x_j} \tau_{h-1}$ with $j < i$, for every $1 < h \leq k$.

A set $B \subset \mathbb{T}_t$ is called a *Borel set* if, for every term x^α in B and x^β in \mathbb{T}_t , $x^\alpha <_B x^\beta$ implies that x^β belongs to B . From the definition it follows immediately that, if $B \subset \mathbb{T}_t$ is a Borel set, then the set $N := \mathbb{T}_t \setminus B$ has the property that, for every $x^\gamma \in N$ and $x^\delta \in \mathbb{T}_t$ with $x^\delta <_B x^\gamma$, x^δ belongs to N .

A monomial ideal J is *strongly stable* if J_t is a Borel set, for each t . Every lex-segment ideal is a strongly stable ideal.

A strongly stable ideal is always Borel-fixed (Borel, for short), i.e. it is fixed under the action of the Borel subgroup of the upper-triangular invertible matrices. If $\text{ch}(K) = 0$, also the vice versa holds. In $\text{ch}(K) = 0$ Galligo and in any characteristic Bayer and

Stillman guarantee that in generic coordinates the initial ideal of an ideal I , with respect to (w.r.t.) a given term order, is a constant Borel ideal, called the *generic initial ideal* of I . We denote by $\text{gin}(I)$ the generic initial ideal of I w.r.t. the degrevlex order and recall that $\text{reg}(I) = \text{reg}(\text{gin}(I))$ [Bayer and Stillman(1987)].

Proposition 4.1. *Let $J \subset S$ be a Borel ideal.*

- (i) *The saturation J^{sat} of J is the ideal generated by the terms minimal generators of J in which we set the least variable equal to 1 [Green(2010), Proposition 2.9].*
- (ii) *$\text{reg}(J)$ is the maximal degree of its minimal generators [Bayer and Stillman(1987), Proposition 2.9].*

Now, we recall the notion of growth-height-lexicographic Borel set which has been introduced by D. Mall [Mall(1997), Mall(2000)].

Definition 4.2 ([Mall(1997), Definition 2.7], [Mall(2000), Definition 2.7]). Let $B \subset \mathbb{T}_t$ be a Borel set. The set $B^{(i)} = \{\tau \in B : \min(\tau) = i\}$ is the *growth class* of B of *growth* i . The *growth-vector* of B is $gv(B) := (|B^{(0)}|, |B^{(1)}|, \dots, |B^{(n)}|)$.

Definition 4.3 ([Mall(1997), Def. 2.13], [Mall(2000), Def. 2.8]). Let $B \subset \mathbb{T}_t$ be a Borel set. For every $i \in \{0, \dots, t\}$, let $B(i) := \{x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n} \in B : \alpha_0 = i\}$. The *height-vector* of B is the vector $hv(B) := (|B(0)|, |B(1)|, \dots, |B(t)|)$.

Given a Borel set B , the *first expansion* of B is $\{x_0, \dots, x_n\} \cdot B$ (see [Macaulay(1926), Marinari and Ramella(1999), Marinari and Ramella(2006)] for a description of this topic). For a strongly stable ideal J , we call $\mathbb{T}_{j+1} \setminus (\{x_0, \dots, x_n\} \cdot J_j)$ the *first expansion* of $\mathcal{N}(J)_j$.

Remark 4.4. Given a Borel set $B \subset \mathbb{T}_t$, we have: for every $i \neq j$, $B^{(i)} \cap B^{(j)} = \emptyset$ and $B(i) \cap B(j) = \emptyset$; $B(0) = \bigcup_{i \geq 1} B^{(i)}$; $B^{(0)} = \bigcup_{i \geq 1} B(i)$.

Proposition 4.5 ([Mall(1997), Proposition 3.2]). *Given a Borel set $B \subset \mathbb{T}_t$, let $I = (B)^{\text{sat}} \subset S$ be the saturation of the homogeneous ideal generated by B . If f is the Hilbert function of S/I , then:*

- (i) $f(j) = \binom{j+n}{n} - \sum_{i=t-j}^t |B(i)|$, for every $j \leq t$;
- (ii) $f(j) - f(j-1) = \binom{j+n-1}{n-1} - |B(t-j)|$ for every $j \leq t$;
- (iii) $f(t+k) = \binom{t+k+n}{n} - \sum \binom{i+k}{k} |B(i)|$, for every $k \geq 1$.

Definition 4.6 ([Mall(1997), Def. 2.10 and 2.14], [Mall(2000), Def. 4.3]).

(1) The Borel set $B \subset \mathbb{T}_t$ is *growth-lexicographic* if all growth classes of B are lex-segments in the corresponding growth classes $\mathbb{T}_t^{(i)}$ of \mathbb{T}_t .

(2) The Borel set $B \subset \mathbb{T}_t$ is *growth-height-lexicographic* if $B(0)$ is growth-lexicographic and $B(i)$ is a lex-segment in $\mathbb{T}_t(i)$, for all $0 < i \leq t$.

Theorem 4.7. *Let $B \subset \mathbb{T}_t$ be a Borel set. Then there is a unique growth-height-lexicographic Borel set $L_{gh}(B) \subset \mathbb{T}_t$ with $gv(L_{gh}(B)) = gv(B)$ and with $hv(L_{gh}(B)) = hv(B)$. Moreover, the ideals $J := (B)^{\text{sat}}$ and $I := (L_{gh}(B))^{\text{sat}}$ have the same Hilbert function and $\text{reg}(J) \leq \text{reg}(I) \leq t$.*

Proof. For the first part of this result we refer to [Mall(1997), Th. 2.17] and [Mall(2000), Theorem 4.4] to obtain that $L_{gh}(B)$ is the Borel set L that consists of the union of the lex-segments $L^{(i)}$ in $\mathbb{T}_t^{(i)}$ with $|B^{(i)}|$ elements, for every $i = 0, \dots, n$, and of the lex-segments $L(j)$ in $\mathbb{T}_t(j)$ with $|B(j)|$ elements, for every $j = 1, \dots, t$.

For the second part, first we note that $H_{S/J} = H_{S/I}$ follows straightforwardly from Proposition 4.5. Then, we observe that $\text{reg}(I) \leq t$, because I is generated by terms of degree $\leq t$ and is a strongly stable ideal. It remains to show that $\text{reg}(J) \leq \text{reg}(I)$.

Assume there are an integer $m > \text{reg}(I)$ and a minimal generator of J of degree m . Observe that $m \leq t$. Thus, every term of $I \cap K[x_1, \dots, x_n]_m$ is multiple of a term of the set L' of terms in $I \cap K[x_1, \dots, x_n]_{m-1}$, while there are terms of $J \cap K[x_1, \dots, x_n]_m$ that are not multiple of a term in the set B' of the terms in $J \cap K[x_1, \dots, x_n]_{m-1}$. Observe that the set of terms in $I \cap K[x_1, \dots, x_n]_h$ is $L(t-h) \cdot x_0^{-t+h}$ and the set of terms in $J \cap K[x_1, \dots, x_n]_h$ is $B(t-h) \cdot x_0^{-t+h}$, for every $h \leq t$. Moreover, $|L'| = |L(t-m+1)| = |B(t-m+1)| = |B'|$ and L' is a lex-segment. Then, we should have $\dim_K(B' \cdot R)_m < \dim_K(L' \cdot K[x_1, \dots, x_n])_m$, but this is a contradiction with the fact that the first expansion of an ideal generated by a lex-segment is the smallest possible. \square

5. IDEAL GRAFT AND MINIMAL CASTELNUOVO-MUMFORD REGULARITIES

In this section, first we exploit the features of the growth-height-lexicographic Borel sets to construct a scheme X obtained by a so-called *ideal graft* of two given schemes X_1 and X_2 . Indeed, the Hilbert function of X will be the “graft” of the other two, because it will coincide with the Hilbert function of X_1 up to a certain degree and with the Hilbert function of X_2 from this degree on. Then, we apply this new construction to find a scheme X such that the regularity ϱ_X and the value $H_X(\varrho_X - 1)$ are small, according to Lemma 2.4 and Proposition 2.5, respectively. In conclusion, we prove that each $m_{p(z)}^{\varrho}$ is achieved by a scheme with the minimal function having regularity ϱ .

Remark 5.1. If $I \subset K[x_0, \dots, x_{n'}]$ is a strongly stable ideal, then for every integer $n > n'$ the ideal $(I + (x_{n'+1}, \dots, x_n)) \subseteq K[x_0, \dots, x_n]$ is a strongly stable ideal.

Theorem 5.2 (Ideal graft). *Let q, w be Hilbert functions of projective schemes and suppose $m > 1$ is an integer such that $w(m-1) = q(m-1)$ and $w(m-2) \leq q(m-2)$ (so that $\Delta w(m-1) \geq \Delta q(m-1)$). Then, the function*

$$h(j) := \begin{cases} w(j) & \text{for all } j < m \\ q(j) & \text{for all } j \geq m \end{cases}$$

is the Hilbert function of a projective scheme and $m_h \leq \max\{m, m_q\}$.

Proof. Let $s := \max\{m, m_q\}$. By the hypotheses, there are a saturated polynomial ideal I , defining a projective scheme with Hilbert function q and Castelnuovo-Mumford regularity m_q , and a saturated polynomial ideal I' , defining a projective scheme with Hilbert function w . Replacing I and I' by $\text{gin}(I)$ and $\text{gin}(I')$, respectively, we can suppose that they are Borel saturated. Moreover, by Remark 5.1, we can suppose that the ideals I and I' are in the same polynomial ring $S = K[x_0, \dots, x_n]$. Nevertheless, we can suppose $\text{reg}(I') \leq s$, replacing I' possibly by $(I'_{\leq s})$, which gives a Hilbert function equal to w at least until the degree s . Hence, we let $L := L_{gh}(I_s)$ and $L' := L_{gh}(I'_s)$ and we can replace again I and I' by $(L)^{\text{sat}}$ and $(L')^{\text{sat}}$, respectively.

For each saturated Borel ideal $V \subset S$ with $H_{S/V} = v$ and integer k , we get $\Delta v(k) = v(k) - v(k-1) = \binom{k+n-1}{n-1} - |V \cap \mathbb{T}_k(0)|$. Then in our hypotheses, we have $|I' \cap \mathbb{T}_{m-1}(0)| \leq |I \cap \mathbb{T}_{m-1}(0)|$, where by construction $I \cap \mathbb{T}_{m-1}(0) = x_0^{-(s-m+1)}L(s-m+1)$ and $I' \cap \mathbb{T}_{m-1}(0) = x_0^{-(s-m+1)}L'(s-m+1)$. Recall that $L(s-m+1)$ e $L'(s-m+1)$ are both lex-segments in $\mathbb{T}_s(s-m+1)$, hence $I' \cap \mathbb{T}_{m-1}(0) \subseteq I \cap \mathbb{T}_{m-1}(0)$.

Let $G' := I' \cap \mathbb{T}_{\leq m-1}(0)$ and $G := I \cap \mathbb{T}_{\geq m}(0)$ and then consider the ideal J that is generated by the terms of the set $G' \cup G$. First, we observe that:

i) J is Borel and saturated, because it is generated by a suitable union of Borel sets of terms in which the variable x_0 does not occur;

ii) $\text{reg}(J) \leq s$, because J is a Borel ideal generated by $G' \cup G_{\leq s}$, by the hypotheses on I and I' .

It remains to show that $h = H_{S/J}$. By construction, for every $k \leq m-1$ we have $J_k = I_k$, hence $h(k) = w(k)$. By induction on k , we show that $h(k) = q(k)$, for every $k \geq m-1$. For $k = m-1$, we have $h(m-1) = w(m-1) = q(m-1)$, by the hypotheses. Now, let $k \geq m$ and assume the thesis is true for $k-1$. We have $h(k) - h(k-1) = q(k) - q(k-1)$ because J and I contain the same terms of $\mathbb{T}_k(0)$. Then, by the inductive hypothesis we obtain $h(k) = q(k)$. \square

Corollary 5.3. *Let ϱ be any integer such that $F(p(z), \varrho) \neq \emptyset$ and let $w \in F(p(z), \varrho)$. For every $q \in F(p(z))$, we get $m_w \leq \max\{\varrho + 2, m_q + 1\}$. If moreover $q \in F(p(z), \varrho)$, then:*

- (i) $m_q < m_w \iff m_q = \varrho + 1$, $m_w = \varrho + 2$ and $q(\varrho - 1) < w(\varrho - 1)$;
- (ii) either $|M(p(z), \varrho)| = 1$ or $M(p(z), \varrho) = \{\varrho + 1, \varrho + 2\}$;
- (iii) $m_{p(z)}^{\varrho} \geq m_{f_{p(z)}^{\varrho}}$ and $m_{p(z)}^{\varrho} = \begin{cases} m_{f_{p(z)}^{\varrho}}, & \text{if } f_{p(z)}^{\varrho} \text{ has regularity } \varrho \\ m_{g_{p(z)}^{\varrho}}, & \text{otherwise.} \end{cases}$

Proof. For the first assertion, it is enough to apply Theorem 5.2 to the functions w and q with $m = \max\{\varrho_w + 2, m_q + 1\}$, obtaining an ideal J such that $H_{S/J} = w$ and so $m_w \leq \text{reg}(J) \leq \max\{m, m_q\} = \max\{\varrho_w + 2, m_q + 1\}$.

(i) Assume $m_q < m_w$ and set $m := \varrho + 2$. Then, $q(k) = w(k) = p(k)$ for $k = m-2$ and $k = m-1$, and we can apply Theorem 5.2 to construct a saturated ideal $J \subset S$ with $H_{S/J} = w$ and $\text{reg}(J) \leq \max\{m, m_q\}$. Hence, $m_w \leq \text{reg}(J) \leq \max\{\varrho + 2, m_q\}$ and we obtain $m_q = \varrho + 1$, $m_w = \varrho + 2$. Moreover, if on the contrary $w(\varrho - 1) \leq q(\varrho - 1)$, then we can apply Theorem 5.2 with $m = m_q = \varrho + 1$ obtaining an ideal $J \subset S$ such that $H_{S/J} = w$ and $m_w \leq \max\{m, m_q\} = m_q$, against the assumption.

Statement (ii) is a straightforward consequence of part (i).

For (iii), by the previous results $m_{p(z)}^{\varrho}$ is attained by the minimal function in $F(p(z), \varrho)$, hence by either $f_{p(z)}^{\varrho}$ or $g_{p(z)}^{\varrho}$. Then, we apply Theorem 3.9. \square

The results of Corollary 5.3 confirm that the Hilbert function of a scheme X with minimal Castelnuovo-Mumford regularity must either have minimal regularity or assume the smallest possible value at the degree $\varrho_X - 1$.

Remark 5.4. If $|M(p(z), \varrho)| = 1$, then the unique element m of $M(p(z), \varrho)$ can be bigger than $\varrho + 2$ (see Example A.2).

Example 5.5. For the admissible polynomial $p(z) = 12z - 25$ we have $\bar{\varrho}_{p(z)} = \varrho_{p(z)} = 6$ and $f_{p(z)}^7 = f_{p(z)}^6 = (1, 4, 9, 16, 25, 36, 12z - 25)$. So, we consider $g_{p(z)}^7 = (1, 4, 9, 16, 25, 36, 48, 12z - 25)$ whose first derivative $\Delta g_{p(z)}^7 = (1, 3, 5, 7, 9, 11, 12, 11, 12)$ is admissible. Thus, $F(p(z), 7) \neq \emptyset$ and there are schemes with Hilbert function having regularity 7 and Hilbert polynomial $p(z)$. Moreover, we obtain $m_{p(z)}^7 = m_{g_{p(z)}^7} = 9$ and $m_{p(z)}^6 = m_{f_{p(z)}^6} = 8$.

6. EXPANDED LIFTING

In this section, we use the notion of growth-height-lexicographic Borel set to obtain a scheme X , given the Hilbert function f and a possible hyperplane section Z , where for a “possible” hyperplane section Z we intend that $H_Z \leq \Delta f$ and $p_Z(z) = \Delta p(z)$.

Our first tool is the following variant of [Cioffi et al(2011), Proposition 4.3].

Proposition 6.1. *Let $J \subset S$ be a saturated strongly stable ideal with Hilbert polynomial $p(z)$ and $\text{reg}(J) = m$. Let $x^\beta x_0^b$, with $x_0 \nmid x^\beta$ and $b > 0$, be a term of J of degree $s \geq m$ which is minimal in J_s w.r.t. $<_B$. Then,*

- (i) *the ideal I generated by $J_s \setminus \{x^\beta x_0^b\}$ is strongly stable and $p_{S/I} = p(z) + 1$; in particular, its saturation I^{sat} is strongly stable with $p_{S/I^{\text{sat}}} = p(z) + 1$;*
- (ii) *$H_{S/I^{\text{sat}}}(t) = H_{S/J}(t)$, for $t < |\beta|$, and $H_{S/I^{\text{sat}}}(t) = H_{S/J}(t) + 1$ otherwise; if $|\beta| = m$, then we get $\text{reg}(I^{\text{sat}}) = m + 1$, otherwise $\text{reg}(I^{\text{sat}}) = m$.*

Proof. For part (i), we refer to the proof of [Cioffi et al(2011), Proposition 4.3], in which we replace the Gotzmann number r of $p(z)$ by the regularity m of the ideal J . For part (ii), note that x^β is a minimal generator of J , because $x^\beta x_0^h$ is a minimal term w.r.t. $<_B$ in $J_{|\beta|+h}$, for every $h \geq 0$. Then, $(I^{\text{sat}})_{|\beta|+h}$ is generated by $(J_{|\beta|+h} \cap \mathbb{T}) \setminus \{x^\beta x_0^h\}$ and also the result about the Castelnuovo-Mumford regularity follows. \square

If J has suitable minimal generators, then Proposition 6.1 can be used to *change* the Hilbert function of S/J into a given other Hilbert function, by constructing a new saturated strongly stable ideal.

Proposition 6.2. *Let $L \subset \mathbb{T}_m$ and $L' \subset \mathbb{T}_{m'}$ be two growth-height-lexicographic Borel sets, with $m \leq m'$, and let $I := (L)^{\text{sat}}, I' := (L')^{\text{sat}} \subset S$. If there is an integer $\bar{t} < m$ such that $\Delta H_{S/I}(t) = \Delta H_{S/I'}(t)$, for every $t < \bar{t}$, and $\Delta H_{S/I}(\bar{t}) < \Delta H_{S/I'}(\bar{t})$, then I has a minimal generator of degree \bar{t} . Hence, L contains a term of type $x^\beta x_0^b$, with $|\beta| = \bar{t}$ and $x_0 \nmid x^\beta$ and $b > 0$, which is minimal in L w.r.t. $<_B$.*

Proof. By the hypotheses and the definition of growth-height-lexicographic Borel set, we have $x_0^{-m+\bar{t}+1}(I_{\bar{t}-1} \cap \mathbb{T}) = x_0^{-m'+\bar{t}+1}(I'_{\bar{t}-1} \cap \mathbb{T})$. So, the first expansions of $x_0^{-m+\bar{t}+1}(I_{\bar{t}-1} \cap \mathbb{T})$ and of $x_0^{-m'+\bar{t}+1}(I'_{\bar{t}-1} \cap \mathbb{T})$ are the same and the difference between the Hilbert functions implies the existence of a minimal generator of degree \bar{t} for I , by Proposition 4.5. The last assertion follows because now we can take the minimal term w.r.t. $<_B$ among the terms of L of type $x^\beta x_0^b$, with $|\beta| = \bar{t}$ and $x_0 \nmid x^\beta$ and $b > 0$. \square

Example 6.3. Consider the polynomial $p(z) = 9z - 7$. Observe that for the saturated strongly stable ideal $J := (x_4^2, x_4x_3, x_3^2, x_4x_2, x_3x_2^3, x_2^5) \subset S = K[x_0, \dots, x_4]$ we get $H_{S/J} = (1, 5, 11, 9z-8) \preceq f_{p(z)}^2 = (1, 5, 9z-7)$, but from J we cannot construct a saturated strongly stable ideal I such that S/I has Hilbert function $(1, 5, 9z-7)$ by Proposition 6.1 because J has not a minimal generator of degree 3. Anyway, taking the Borel set $B := J_5 \cap \mathbb{T}$ with growth vector $gv(B) = (42, 26, 15, 5, 1)$ and height vector $hv(B) = (47, 26, 12, 4)$, we can consider the growth-height-lexicographic Borel set $L := L_{gh}(B)$ that generates an ideal whose saturation $I := (x_4^2, x_3x_4, x_2x_4, x_1x_4, x_3^3, x_2x_3^2, x_1^3x_3^2, x_2^4x_3, x_2^5)$ has at least a generator of degree 3, as we expect by Proposition 6.2.

Theorem 6.4 (Expanded lifting). *Let f be a function of $F(p(z), \varrho)$ and Z be a scheme with Hilbert polynomial $\Delta p(z)$ and a Hilbert function g such that $g \preceq \Delta f$. Then there is a scheme X such that $H_X = f$ and $\text{reg}(X) = \max\{\text{reg}(Z), \varrho + 1\}$.*

Proof. Consider $\text{gin}(I(Z)) \subset K[x_1, \dots, x_n]$ and its lifting $J = \text{gin}(I(Z)) \cdot K[x_0, x_1, \dots, x_n]$, that is a saturated strongly stable ideal with regularity $\text{reg}(J) = \text{reg}(Z)$ and $H_{S/J} = \sum g$.

Take $m := \max\{\text{reg}(Z), \varrho + 1\}$, $L := L_{gh}(J_m)$ and let $I := (L)^{\text{sat}}$. By Theorem 4.7, we have $\text{reg}(Z) = \text{reg}(J) \leq \text{reg}(I) \leq m$. Moreover, by construction and by Proposition 4.5, the Hilbert function of S/I is the same as the Hilbert function of S/J , that is $\sum g \preceq f$, with Hilbert polynomial $p(z) - c$, for a non-negative integer c (see Remark 2.3).

If $\sum g = f$, then it is enough to let $X = \text{Proj } S/I$, because in this case $\varrho + 1 \leq \text{reg}(I) \leq m$ and $\text{reg}(Z) = \text{reg}(J) \leq \text{reg}(I) \leq m$, hence $\text{reg}(I) = m$.

Otherwise, let Y be a scheme with $H_Y = f$ and let $J' = \text{gin}(I(Y))$. Take $m' = \max\{m, \text{reg}(Y)\}$, $L' := L_{gh}(J'_{m'})$ and let $I' := (L')^{\text{sat}}$. As before, by construction and by Proposition 4.5, the Hilbert function of S/I' is the same as the Hilbert function of S/J' , that is f . By Theorem 4.7, we have $\text{reg}(Y) = \text{reg}(J') \leq \text{reg}(I') \leq m' = \max\{\text{reg}(Z), \varrho + 1, \text{reg}(Y)\}$, hence $\text{reg}(Y) \leq \text{reg}(I') \leq m$.

Consider $\bar{t} := \min\{t : \sum g(t) < f(t)\}$ and observe that $\bar{t} = \min\{t : g(t) < \Delta f(t)\} < m$, because $g(m) = \Delta p(m) = \Delta f(m)$ by the definition of m .

By Proposition 6.2 applied to L and L' , there is a minimal term $x^\beta x_0^{m-\beta}$ w.r.t. $<_B$ belonging to $L = I_m$, with $|\beta| = \bar{t}$ and $x_0 \nmid x^\beta$ and $b > 0$. As in Proposition 6.1, we consider $I_m \setminus \{x^\beta x_0^{m-\beta}\} = L \setminus \{x^\beta x_0^{m-\beta}\}$, which generates an ideal whose saturation \bar{I} has $H_{S/\bar{I}}(t) = H_{S/I}(t) = \sum g(t) = f(t)$, for $t < \bar{t}$, and $H_{S/\bar{I}}(t) = H_{S/I}(t) + 1 = \sum g(t) + 1 \leq f(t)$ otherwise (see also Remark 2.3). In particular, $\Delta H_{S/\bar{I}}(t) = g(t)$, for $t \neq \bar{t}$, and $\Delta H_{S/\bar{I}}(\bar{t}) = g(\bar{t}) + 1$. Moreover, $\text{reg}(Z) \leq \text{reg}(\bar{I}) \leq m$ because $\text{reg}(\bar{I}) = \text{reg}(I)$.

Thus, if $H_{S/\bar{I}} = f$, then it is enough to let $X = \text{Proj } S/\bar{I}$, because in this case we also have $\varrho + 1 \leq \text{reg}(\bar{I}) \leq m$, hence $\text{reg}(\bar{I}) = m$. Otherwise, we can repeat the above arguments on \bar{I} , redefining g as $g := \Delta H_{S/\bar{I}}$ and noticing that we have again $\Delta p(m) = g(m) = \Delta f(m)$. \square

Example 6.5. For the polynomial $p(z) = 15z - 24$ we have $\bar{\varrho}_{p(z)} = \varrho_{p(z)} = 3$. Consider the functions $f = (1, 5, 11, 15z - 24) \in F(p(z), 3)$ and $g = (1, 3, 6, 10, 15, \dots) \preceq \Delta f = (1, 4, 6, 10, 15, 15, \dots)$. The saturated strongly stable ideal $J' = (x_3^5, x_3^4 x_2, x_2^2 x_3^3, x_2^3 x_3^2, x_3 x_2^4, x_2^5, x_4) \subset K[x_1, \dots, x_4]$ defines a scheme $Z \subset \mathbb{P}_K^3$ with Castelnuovo-Mumford regularity $\text{reg}(Z) = 5$. If $J := J' \cdot K[x_0, \dots, x_4]$ is a lifting of J , then the Hilbert function of $K[x_0, \dots, x_4]/J$ is $\sum g = (1, 4, 10, 20, 15z - 25) \preceq f$. In this case we can apply Proposition 6.1 with $\bar{t} = 1$, taking the saturation $\bar{J} = (x_4^2, x_4 x_3, x_4 x_2, x_4 x_1, x_3^5, x_3^4 x_2, x_3^3 x_2^2, x_3^2 x_2^3, x_3 x_2^4, x_2^5)$ of the ideal generated by $(J)_5 \setminus \{x_0^4 x_4\}$, that defines a curve in \mathbb{P}_K^4 with $\text{reg}(\bar{I}) = 5 = \varrho_{p(z)} + 2$ and $H_{S/\bar{I}} = f$.

7. MINIMAL CASTELNUOVO-MUMFORD REGULARITY

As announced in Notation 2.6, here we set:

$$(8) \quad \bar{f} := f_{p(z)}^{\bar{\varrho}_{p(z)}} \quad \text{and} \quad \bar{m} := m_{\bar{f}}.$$

In this section, we apply the constructive result of Theorem 6.4 to show that $m_{p(z)} = \bar{m}$ and to compute m_u , for every $\varrho \geq \bar{\varrho}_{p(z)}$ and $u \in F(p(z), \varrho)$. We also compute the minimal possible Castelnuovo-Mumford regularity of any scheme with Hilbert polynomial $p(z)$ embedded in a given projective space \mathbb{P}_K^n .

Theorem 7.1. *Let $u \in F(p(z), \varrho)$ and $v \in F(p(z), \varrho')$. Then,*

$$\varrho < \varrho' \implies m_u \leq m_v \quad (\text{in particular } m_{f_{p(z)}^\varrho} \leq m_{f_{p(z)}^{\varrho'}})$$

Hence $m_{p(z)} = \bar{m}$.

Proof. First, we prove that $m_{f_{p(z)}^{\varrho-1}} \leq m_{f_{p(z)}^\varrho}$, for every $\varrho > \bar{\varrho}_{p(z)}$. Suppose $f_{p(z)}^\varrho \neq f_{p(z)}^{\varrho-1}$, hence $f_{p(z)}^\varrho$ has regularity ϱ . By Lemma 3.8(ii), we have $\Delta f_{p(z)}^\varrho(t) \leq \Delta f_{p(z)}^{\varrho-1}(t)$ for every $t < \varrho$. Anyway, we have also $\Delta f_{p(z)}^\varrho(\varrho) > \Delta f_{p(z)}^{\varrho-1}(\varrho) = \Delta p(\varrho)$, because $f_{p(z)}^\varrho(\varrho-1) < p(\varrho-1)$ since $\varrho > \varrho_{p(z)}$, by Proposition 3.2.

Let X be a scheme with $H_X = f_{p(z)}^\varrho$ and $\text{reg}(X) = m_{f_{p(z)}^\varrho}$, and let $g \in F(\Delta p(z), \tilde{\varrho})$ be the Hilbert function of its general hyperplane section. Thus, we obtain $g \preceq \Delta f_{p(z)}^\varrho$ and $m_g \leq m_{f_{p(z)}^\varrho}$, by Lemma 2.4. There are two possible cases: either $g \preceq \Delta f_{p(z)}^{\varrho-1}$ or $g \not\preceq \Delta f_{p(z)}^{\varrho-1}$. If $g \preceq \Delta f_{p(z)}^{\varrho-1}$, then by Theorem 6.4, we have $m_{f_{p(z)}^{\varrho-1}} \leq \max\{m_g, \varrho\} \leq m_{f_{p(z)}^\varrho}$. If $g \not\preceq \Delta f_{p(z)}^{\varrho-1}$, by the described behavior of the first derivatives of our minimal functions, we obtain $\Delta f_{p(z)}^\varrho(\varrho) \geq g(\varrho) > \Delta f_{p(z)}^{\varrho-1}(\varrho) = \Delta p(\varrho)$. Hence, the regularity of g is $\geq \varrho + 1$ and $m_{f_{p(z)}^\varrho} \geq m_g \geq \varrho + 2$. This fact implies the thesis. Indeed, if on the contrary $m_{f_{p(z)}^{\varrho-1}} > m_{f_{p(z)}^\varrho}$, then we could apply Theorem 5.2 to $w := f_{p(z)}^{\varrho-1}$, $q = f_{p(z)}^\varrho$ and $m := \varrho + 2$ getting $m_{f_{p(z)}^{\varrho-1}} = \varrho + 2$ and $m_{f_{p(z)}^\varrho} = \varrho + 1$, meanwhile $m_{f_{p(z)}^\varrho} \geq \varrho + 2$.

Now, if $m_u > m_v$ then, applying Theorem 5.2 to $w := u$, $q := v$ and $m := \varrho' + 2$, we get $\varrho' + 1 \leq m_v < m_u = \varrho' + 2$, hence $m_v = \varrho' + 1$ and $m_u = \varrho' + 2$. We have also $\varrho + 1 \leq m_{p(z)}^\varrho \leq m_{p(z)}^{\varrho'} \leq m_v = \varrho' + 1$, hence $M(p(z), \varrho) = \{m_{p(z)}^\varrho, m_u\}$ with $m_{p(z)}^\varrho \leq \varrho' + 1$ and $m_u = \varrho' + 2 > \varrho + 2$, against Corollary 5.3(i). \square

Corollary 7.2. *Let ϱ be any integer such that $F(p(z), \varrho) \neq \emptyset$. Then,*

$$(9) \quad m_{p(z)}^\varrho = \begin{cases} \bar{m}, & \text{if } \bar{\varrho}_{p(z)} \leq \varrho \leq \bar{m} - 2, \\ m_{f_{p(z)}^\varrho} = \varrho + 1, & \text{if } \bar{m} - 1 \leq \varrho \text{ and } f_{p(z)}^\varrho \text{ has regularity } \varrho, \\ m_{g_{p(z)}^\varrho} = \varrho + 2, & \text{otherwise.} \end{cases}$$

Proof. Consider $\varrho > \bar{\varrho}_{p(z)}$, being the case $\varrho = \bar{\varrho}_{p(z)}$ obvious, and let u be the minimal function in $F(p(z), \varrho)$, so $m_{p(z)}^\varrho = m_u$ by Corollary 5.3(iii). By Theorem 7.1, we have $m_u \geq \bar{m}$. Recall that $\bar{m} > 1$, because we are not considering linear varieties.

By Corollary 5.3(iii), there are two possibilities for u . If $u = f_{p(z)}^\varrho$, we can apply Theorem 5.2 to $w := u$, $q := \bar{f}$ and $m := \max\{\varrho + 1, \bar{m}\}$ by the last part of Proposition 3.2, obtaining $h = w = u$ and $m_u \leq m$, hence $m_u = \varrho + 1$. If $u = g_{p(z)}^\varrho$, then $m_u \geq \varrho + 2$ by Proposition 2.5, because $g_{p(z)}^\varrho(\varrho - 1) > p(\varrho - 1)$ by construction. We apply Theorem 5.2 to $w := u$, $q := \bar{f}$ and $m := \max\{\varrho + 2, \bar{m}\}$, obtaining $h = w = u$ and $m_u \leq m$, hence $m_u = \varrho + 2$. \square

If $p(z)$ is an admissible polynomial for subschemes of \mathbb{P}_K^n , then we can pose the same questions about the Castelnuovo Mumford regularity for a given dimension n of the projective space, and we provide the following answer.

Corollary 7.3. *The minimal possible Castelnuovo-Mumford regularity of a scheme $X \subset \mathbb{P}_K^n$, with $p_X(z) = p(z)$, is $m_{f_{p(z)}^{\varrho_n}}$ with $\varrho_n := \min\{\varrho \geq \bar{\varrho}_{p(z)} : f_{p(z)}^\varrho(1) \leq n + 1\}$.*

Proof. Observe that $n \geq f_{p(z)}^{r-1}(1) - 1$, because $f_{p(z)}^{r-1}$ is the minimum of the Hilbert functions with Hilbert polynomial $p(z)$. Then, apply Theorem 7.1. \square

Finally, we have a recursive procedure to compute m_u , for every $u \in F(p(z), \varrho)$, and we get also lower and upper bounds in terms of $\bar{\varrho}_{\Delta^i p(z)}$.

Theorem 7.4. *Let k be the degree of the polynomial $p(z)$, $u \in F(p(z), \varrho)$ and*

$$M := \max_{1 \leq i \leq k} \{\varrho, \bar{\varrho}_{\Delta^i p(z)}\}, \quad \tilde{\varrho}_u := \min\{t \geq \bar{\varrho}_{\Delta p(z)} : f_{\Delta p(z)}^t \preceq \Delta u\}, \quad \tilde{m} := m_{f_{\Delta p(z)}^{\tilde{\varrho}_u}}.$$

If $k = 0$, then $m_u = \varrho + 1$. If $k > 0$, then $m_u = \max\{\tilde{m}, \varrho + 1\}$. Moreover, we obtain $M + 1 \leq m_u \leq M + 2$.

Proof. The case $k = 0$ holds by Remark 3.4(2). For $k > 0$, observe that \tilde{m} is the lowest Castelnuovo-Mumford regularity of a general hyperplane section of a scheme with Hilbert function u , by Lemma 2.4 and Theorem 7.1, hence $m_u \geq \tilde{m}$. Now, by Theorem 6.4, we can construct a scheme X with $H_X = u$ and $\text{reg}(X) = \max\{\tilde{m}, \varrho + 1\} = m_u$.

For the inequalities $M + 1 \leq m_u \leq M + 2$ we argue by induction on k . The first inequality follows again from Lemma 2.4 and from the definition of $\bar{\varrho}_{\Delta^i p(z)}$. For the second one, observe that if $m_u \geq \varrho + 2$, then $m_u = \tilde{m}$, by the previous results. Moreover, by Corollary 5.3(iii), $\tilde{m} = m_{\Delta p(z)}^{\tilde{\varrho}_u}$ because $f_{\Delta p(z)}^{\tilde{\varrho}_u}$ has regularity $\tilde{\varrho}_u$. By induction, we obtain $m_{\Delta p(z)} \leq \max_{1 \leq i \leq k} \{\bar{\varrho}_{\Delta^i p(z)}\} + 2$, because, by Theorem 7.1, $m_{\Delta p(z)}$ is the minimum regularity of a scheme with Hilbert function $f_{\Delta p(z)}^{\bar{\varrho}_{\Delta p(z)}}$, that has regularity $\bar{\varrho}_{\Delta p(z)}$. Moreover, by Theorem 7.1 and Corollary 7.2, we have $\tilde{m} \leq \max\{m_{\Delta p(z)}, \tilde{\varrho}_u + 1\}$. Thus, we complete the proof because $\tilde{\varrho}_u \leq \max\{\bar{\varrho}_{\Delta p(z)}, \varrho + 1\}$, by definition. \square

APPENDIX A. ALGORITHMS AND EXAMPLES

In this appendix, we collect the main algorithms that arise from the results we have described in our exposition. A trial version of these algorithms is available at the web page http://www.personalweb.unito.it/paolo.lella/HSC/Minimal_Hilbert_Functions_and_CM_regularity.html.

Algorithm 1 RHOBAR computes the minimal regularity of a Hilbert function of a scheme with Hilbert polynomial $p(z)$ (see Proposition 3.11).

1: RHOBAR($p(z)$)

Require: $p(z)$ a Hilbert polynomial.

Ensure: $\bar{\varrho}_{p(z)}$, the minimal regularity of a Hilbert function f of a scheme with Hilbert polynomial $p(z)$.

2: **if** $\deg p(z) = 0$ **then**

3: **return** 1;

4: **else**

5: $\varrho_{\Delta p(z)} \leftarrow \text{RHOMIN}(\Delta p(z));$

6: **return** $\min \left\{ t \geq \max\{\varrho_{\Delta p(z)}, 1\} \mid \sum f_{\Delta p(z)}^t(t-1) \leq p(t-1) \right\} - 1;$

7: **end if**

Algorithm 2 RHOMIN computes the minimal regularity of a Hilbert function with Hilbert polynomial $p(z)$ (see Remark 3.12).

```

1: RHOMIN( $p(z)$ )
Require:  $p(z)$  a Hilbert polynomial.
Ensure:  $\varrho_{p(z)}$ , the minimal regularity of a Hilbert function  $f$  with  $f(t) = p(t)$ ,  $t \gg 0$ .
2: if  $\deg p(z) = 0$  then
3:   return 1;
4: else
5:    $\varrho_{\Delta p(z)} \leftarrow \text{RHOMIN}(\Delta p(z));$ 
6:    $\varrho \leftarrow \min \left\{ t \geq \max\{\varrho_{\Delta p(z)}, 1\} \mid \sum f_{\Delta p(z)}^t(t-1) \leq p(t-1) \right\} - 1;$ 
7:   if  $\varrho > \varrho_{\Delta p(z)} - 1$  then
8:     return  $\varrho$ ;
9:   else
10:    while  $\varrho > 1$  and  $(p(\varrho - 1)_{\varrho-1})_+^\dagger \geq p(\varrho)$  do
11:       $\varrho \leftarrow \varrho - 1;$ 
12:    end while
13:    if  $\varrho = 1$  and  $p(0) = 1$  then
14:       $\varrho \leftarrow 0;$ 
15:    end if
16:    return  $\varrho$ ;
17:  end if
18: end if

```

Algorithm 3 This function computes the minimal Castelnuovo-Mumford regularity of a scheme, given the Hilbert polynomial and the regularity of the Hilbert function (see Theorem 7.4).

```

1: MINIMALCMREGULARITY( $p(z), \varrho$ )
Require:  $p(z)$ , a Hilbert polynomial.
Require:  $\varrho$ , an integer such that  $F(p(z), \varrho) \neq \emptyset$ .
Ensure: the minimal Castelnuovo-Mumford regularity  $m_{p(z)}^\varrho$  of a scheme with Hilbert
    function of regularity  $\varrho$  and Hilbert polynomial  $p(z)$ .
2: if  $\deg p(z) = 0$  then
3:   return  $\varrho + 1;$ 
4: else
5:    $\bar{\varrho}_{\Delta p(z)} \leftarrow \text{RHOBAR}(\Delta p(z));$ 
6:    $b \leftarrow \min \{ t \geq \bar{\varrho}_{\Delta p(z)} \mid f_{\Delta p(z)}^t \leq \Delta f_{p(z)}^\varrho \};$ 
7:   return  $\max \{ \varrho + 1, \text{MINIMALCMREGULARITY}(\Delta p(z), b) \};$ 
8: end if

```

Algorithm 4 This function computes the minimal Castelnuovo-Mumford regularity of a scheme, given the Hilbert polynomial (see Theorem 7.4).

1: MINIMALCMREGULARITY($p(z)$)

Require: $p(z)$, a Hilbert polynomial.

Ensure: the minimal Castelnuovo-Mumford regularity $m_{p(z)}$ of a scheme with Hilbert polynomial $p(z)$.

2: **return** MINIMALCMREGULARITY($p(z)$, RHOBAR($p(z)$));

Algorithm 5 This function computes the minimal Castelnuovo-Mumford regularity m_u of a scheme, given the Hilbert function u with Hilbert polynomial $p(z)$ (see Theorem 7.4).

1: MINIMALCMREGULARITY($p(z)$, u)

Require: $p(z)$, a Hilbert polynomial.

Require: u , Hilbert function of a scheme with Hilbert polynomial $p(z)$.

Ensure: the minimal Castelnuovo-Mumford regularity m_u of a scheme with Hilbert function u .

2: $\varrho \leftarrow$ regularity of u ;

3: $b \leftarrow \min \left\{ t \geq \bar{\varrho}_{p(z)} \mid f_{\Delta p(z)}^t \preceq \Delta u \right\}$

4: **return** $\max \{ \varrho + 1, \text{MINIMALCMREGULARITY}(\Delta p(z), b) \}$;

Algorithm 6 This function computes the minimal Castelnuovo-Mumford regularity of a subscheme of \mathbb{P}^n with Hilbert polynomial $p(z)$ (see Corollary 7.3).

1: MINIMALCMREGULARITY($p(z)$, \mathbb{P}^n)

Require: $p(z)$, a Hilbert polynomial.

Require: \mathbb{P}^n , a projective space with $n \geq \deg p(z) + 1$.

Ensure: the minimal Castelnuovo-Mumford regularity of a subscheme of \mathbb{P}^n with Hilbert polynomial $p(z)$.

2: $\bar{\varrho}_{p(z)} \leftarrow \text{RHOBAR}(p(z))$;

3: $\varrho \leftarrow \min \left\{ t \geq \bar{\varrho}_{p(z)} \mid f_{p(z)}^t(1) \leq n + 1 \right\}$

4: **return** MINIMALCMREGULARITY($p(z)$, ϱ);

Example A.1. We compute $m_{p(z)}$ for the Hilbert polynomial $p(z) = \frac{1}{3}z^3 + 2z^2 + \frac{14}{3}z - 4$, applying MINIMALCMREGULARITY($p(z)$). We get $\Delta p(z) = z^2 + 3z + 3$, $\Delta^2 p(z) = 2z + 2$ and $\Delta^3 p(z) = 2$ (see Table 1). Moreover, we have $\varrho_{p(z)} = 3$ and $\varrho_{\Delta p(z)} = 1$, hence $\bar{\varrho}_{p(z)} = \varrho_{p(z)} = 3$. For $u := f_{p(z)}^3 = (1, 6, 17, p(z))$, we compute $\Delta f_{p(z)}^3 = (1, 5, 11, 20, \Delta p(z))$ and have $\tilde{\varrho}_u = 4$. For $u := f_{\Delta p(z)}^4 = (1, 4, 10, 19, \Delta p(z))$, we compute $\Delta f_{\Delta p(z)}^4 = (1, 3, 6, 9, 12, 12, \Delta^2 p(z))$ and get $\tilde{\varrho}_u = 2$, with $f_{\Delta^2 p(z)}^2 = (1, 3, \Delta^2 p(z))$. Since $m_{f_{\Delta^2 p(z)}^2} = 3$, we get $m_{f_{\Delta p(z)}^4} = 5$, hence $m_{p(z)} = 5$.

Example A.2. For the admissible polynomial $p(z) = 6z^2 - 18z + 37$ we already considered in Example 3.6, we get $\Delta p(z) = 12z - 24$, $\varrho_{p(z)} = 1$ and $\varrho_{\Delta p(z)} = 5$, hence $\bar{\varrho}_{p(z)} = 4$ (see

TABLE 1. Example A.1

$p(z)$	r	$\varrho_{p(z)}$	$\bar{\varrho}_{p(z)}$	ϱ	$\tilde{\varrho}$	$m_{p(z)}^{\varrho}$
$\frac{1}{3}z^3 + 2z^2 + \frac{14}{3}z - 4$	10	3	3	3	4	5
$z^2 + 3z + 3$	6	1	1	4	2	5
$2z + 2$	3	1	1	2	1	3
2	2	1	1	1		2

TABLE 2. Example A.2

$p(z)$	r	$\varrho_{p(z)}$	$\bar{\varrho}_{p(z)}$	ϱ	$\tilde{\varrho}$	$m_{p(z)}^{\varrho}$
$2z^3 - 6z^2 + 29z - 20$	218498	2	2	2	4	7
$6z^2 - 18z + 37$	678	1	4	4	5	7
$12z - 24$	42	5	5	5	6	7
12	12	1	1	6		7

Table 2). For $u := f_{p(z)}^4 = (1, 5, 15, 33, p(z))$, we compute $\Delta f_{p(z)}^4 = (1, 4, 10, 18, 28, \Delta p(z))$ and have $\tilde{\varrho}_u = 5$. Then, for $u := f_{\Delta p(z)}^5 = (1, 4, 9, 16, 25, \Delta p(z))$ we compute $\Delta f_{\Delta p(z)}^5 = (1, 3, 5, 7, 9, 11, \Delta^2 p(z))$ and get $\tilde{\varrho}_u = 6$, hence $m_{p(z)} = 7$.

For the admissible polynomial $q(z) = 2z^3 - 6z^2 + 29z - 20$, we have $\Delta q(z) = p(z)$ and $\varrho_{q(z)} = 2$. So, we take $f_{q(z)}^2 = (1, 8, q(z))$ and $\Delta f_{q(z)}^2 = (1, 7, 22, \Delta q(z))$. By $\bar{\varrho}_{\Delta q(z)} = 4$, we get $\tilde{\varrho}_u = 4$ for $u := f_{p(z)}^4 = f_{\Delta q(z)}^4 \preceq f_{\Delta q(z)}^3 \preceq \Delta f_{q(z)}^2$. Using the already computed $m_{f_{p(z)}^4} = m_{p(z)}$, we have also $m_{q(z)} = 7$.

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