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BOUNDARY FEEDBACK STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS: An Application of Pseudo-Differential Boundary Operators.

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ABSTRACT

The theory of pseudo-differential boundary operators proves to be a fruitful approach to problems arizing in control and stabilization theory of distributed parameter systems. By use of the basic pseudo-differential calculus we can in a direct and simple way obtain existence and stability theorems for boundary feedback semigroups.

I. INTRODUCTION

In this paper we present a brief introduction to the method of pseudo-differential stabilization as developed in [9], and based on the fundamentals from refs. [3] and [4].

Let A be a formally selfadjoint, uniformly strongly elliptic differential operator of order 2m, with smooth coefficients on $\bar{\Omega}$, where Ω is an open, bounded set in R^n, n>1, with smooth boundary Γ. The Dirichlet realization A_γ of A is then the operator acting like A in $L^2(\Omega)$, and with domain

$$D(A_\gamma) = \{u \in H^{2m}(\Omega) \mid \gamma u = 0\} = H^{2m}(\Omega) \cap H_0^m(\Omega). \quad (1)$$

Here γ is the Dirichlet trace operator

$$\gamma u = (u|_\Gamma, (\partial/\partial n)u|_\Gamma, \dots, (\partial/\partial n)^{m-1}u|_\Gamma)^T \quad (2)$$

$(\partial/\partial n)$ is the normal derivative, and $H^{2m}(\Omega)$ is the usual Sobolev space of order 2m, consisting of L^2-functions with L^2-derivatives up to order 2m.

The realization A_γ is associated with the parabolic evolution equation:

$$\begin{aligned} \frac{d}{dt}u(x,t) + Au(x,t) &= 0 \text{ for } x\in\Omega \text{ and } t>0, \\ \gamma u(x,t) &= 0 \text{ for } x\in\Omega \text{ and } t>0, \\ u(x,0) &= u_0(x) \text{ for } x\in\Omega ; \end{aligned} \quad (3)$$

and it is well known that A_γ is the infinitesimal generator of an analytic semigroup, $\exp(-A_\gamma t)$, $t\geq 0$, on $L^2(\Omega)$, giving the solution to (3) as

$$u(x,t) = \exp(- A_\gamma t)u_0(x), \quad (4)$$

for $u_0 \in L^2(\Omega)$, $x\in\Omega$ and $t\geq 0$.

Since A_γ has a compact resolvent, the spectrum of A_γ consists of a sequence of real eigenvalues, converging to infinity. There are only finitely many negative eigenvalues, so we write them as a nondecreasing sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{K-1} \leq 0 < \lambda_K \leq \dots \quad (5)$$

where λ_K is the first positive eigenvalue. Moreover, for simplicity assume that all the negative eigenvalues are simple. Because of the negative eigenvalues of A_γ there are initial data u_0 for which the corresponding solution to (3) blows up (in L^2-norm) as t tends to infinity. This is easily observed from the spectral representation of the solution

$$u(x,t) = \sum_{j\geq 1} \exp(-\lambda_j t)(u_0,\varphi_j)\varphi_j(x), \quad (6)$$

where the φ_j, j = 1,2,... is the set of eigenfunctions and (.,.) is the usual $L^2(\Omega)$-inner product. The boundary stabilization problem is to design a boundary feedback mechanism T'u, such that if the boundary condition $\gamma u=0$ in (3) is replaced by a new boundary condition $\gamma u=T'u$, the resulting boundary feedback system is stable, in the sense that for any initial data $u_0 \in L^2(\Omega)$, the L^2-norm of the corresponding solution goes to zero as t tends to infinity. Moreover, the feedback mechanisms we consider are of the form:

$$T'u = (u,w)g , \quad (7)$$

where $w \in C^\infty(\Omega)$ and $g \in C^\infty(\Gamma)$ are functions to be determined. (For certain choices of Ω or if some of the negative eigenvalues have multiplicities > 1, the feedback must consist of a sum of terms like (7); these technical details are discussed in [5] and [9]).

II. THE FEEDBACK SYSTEM AND THE PSEUDO-DIFFERENTIAL TRANSFORMATION

The boundary feedback stabilization problem can be stated as:

Can we determine functions $w \in C^\infty(\Omega)$, $g\in C^\infty(\Gamma)$, such that the boundary feedback system

$$\begin{aligned} \frac{d}{dt}u(x,t) + Au(x,t) &= 0 \text{ for } x\in\Omega \text{ and } t>0, \\ \gamma u(x,t) &= (u,w)g(x) \text{ for } x\in\Omega \text{ and for } t>0, \\ u(x,0) &= u_0(x) \text{ for } x\in\Omega , \end{aligned} \quad (8)$$

is stable in the sense that the L^2-norm of a solution u(x,t) is exponentially decreasing as t tends to infinity, for any initial data $u_0 \in L^2(\Omega)$?

The answer to the above problem is affirmative if we assume that:

The negative eigenvalues are simple (9)
and
the Neumann traces (i.e. the normal boundary derivatives of order $\geq$ m)

$$(\frac{\partial}{\partial n})^k \varphi_j|_\Gamma , \quad k = m,m+1,\dots,2m-1, \quad j = 1,2,\dots,K-1, \quad (10)$$

of the eigenfunctions φ_j, j = 1,2,...,K-1 are linearly independent.

(When the assumptions (9)-(10) do not hold, the situation is more complicated and, in general, more terms in the feedback are required ; for details, see [5],[6] [7] and [9].)

The treatment of the system (8) is complicated by the fact that the associated realization A_1 of the operator A has the domain

$$D(A_1) = \{u \in H^{2m}(\Omega) \mid \gamma u = (u,w)g\} , \qquad (11)$$

which in contrast to the domain for A_γ is given by a variable, non-local boundary condition. Consider now the solution operator K_γ to the stationary Dirichlet problem for A, i.e. K_γ maps φ into the solution u of

$$\begin{aligned} Au &= 0 \quad \text{in} \quad \Omega \\ \gamma u &= \varphi \quad \text{on} \quad \Gamma \end{aligned} \qquad (12)$$

K_γ is a standard type of Poisson Operator, as defined in the pseudo-differential boundary operator calculus, (see [3],[4]). Moreover, the operator T' (7) is a standard type Trace Operator in this theory. However, the most important property with respect to the problem at hand is that the composition $K_\gamma T'$ is also a standard operator of the class called Singular Green Operators, (introduced in [2]). The properties of Singular Green Operators is throughly discussed in refs. [3] and [4]. In the present case we need only the fact that it is possible to choose T' of the form (7), such that the operator $1-K_\gamma T'$ defines a homeomorphism and an isomorphism in $H^{2m}(\Omega)$, such that

$$1-K_\gamma T' : D(A_1) \tilde{\to} D(A_\gamma) . \qquad (13)$$

Then, if $u \in D(A_1)$, $v = (1-K_\gamma T')u$ belongs to $D(A_\gamma)$ and $Au = Av$. This establishes in a precise manner the factorization

$$A_1 = A_\gamma(1-K_\gamma T') \qquad (14)$$

which can now be used in the discussion of (8).

The evolution problem

$$(d/dt)u + Au = 0, \quad u \in D(A_1) \qquad (15)$$

transforms by (13) and (14) into

$$(d/dt)(1-K_\gamma T')^{-1}v + Av = 0, \quad v \in D(A_\gamma) \qquad (16)$$

or alternatively

$$(d/dt)v + (1-K_\gamma T')Av = 0 , \quad v \in D(A_\gamma) . \qquad (17)$$

Since A is a differential operator with smooth coefficients, the operator $G = -K_\gamma T'$ is also a Singular Green Operator (of finite rank), so we observe that our feedback problem (8) (by the transformation (15)-(17)) is in fact nothing but a finite dimensional perturbation:

$$(d/dt)v + Av + Gv = 0 , \quad v \in D(A_\gamma) \qquad (18)$$

of the Dirichlet evolution problem (3):

$$(d/dt)v + Av = 0 , \quad v \in D(A_\gamma) \qquad (19)$$

As shown in refs. [9] , [10] and [11] , the stabilization of the system (18) is straightforward, as finite dimensional pole placement techniques can be employed, (cf.[12]). The result is that under the assumptions (9)-(10), the operator T' (7) can be chosen such that $1-K_\gamma T'$ has the abovementioned properties, and such that the operator A + G with domain $D(A_\gamma) = H^{2m}(\Omega) \cap H_0^m(\Omega)$, is the infinitesimal generator of an analytic semigroup, $\exp(-(A+G)t)$, $t \geq 0$, on L^2, giving the solution to (18) as:

$$v(x,t) = \exp(-(A+G)t)v_0(x) \qquad (20)$$

where $x \in \Omega$, $t \geq 0$, for initial data $v_0 \in L^2(\Omega)$.

Also (what is the key point):

$$||v(.,t)|| \leq M \exp(-(\lambda_K+\varepsilon)t)||v_0|| , \qquad (21)$$

with $M>1, \varepsilon>0$.

As shown in [9], the operator A_1, with domain $D(A_1)$, is then also the infinitesimal generator of an analytic semigroup, $\exp(-A_1 t)$, $t \geq 0$, on $L^2(\Omega)$, which is the transform of the semigroup $\exp(-(A+G)t)$ under $(1-K_\gamma T')$:

$$\exp(-A_1 t) = (1-K_\gamma T')^{-1}\exp(-(A+G)t)(1-K_\gamma T') \quad (22)$$

for which we have the estimate

$$||u(.,t)|| \leq M \exp(-(\lambda_K+\varepsilon)t)||u_0|| , \qquad (23)$$

for the solution u(x,t) of (8).

The formula (22) shows that when we impose a boundary feedback on the originally "free" system (3), we are performing a pseudo-differential"change of coordinates" in the space $H^{2m}(\Omega)$. The pseudo-differential approach allows us to obtain stabilization results on the system (8), together with other perturbations of the free system (3), in a unified setting. Moreover, we can consider hyperbolic problems as well as parabolic problems, as described in ref. [9] .

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