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(Article begins on next page)

ON THE NORMS OF QUATERNIONIC HARMONIC PROJECTION OPERATORS

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Sur les normes des opérateurs de projection harmoniques sur la sphére dans l'espace quaternionique

Abstract. As a consequence of integral bounds for three classes of quaternionic spherical harmonics, we prove some bounds from below for the (L^p, L^2) norm of quaternionic harmonic projectors, for $p \in [1, 2]$.

Résumé. En conséquence d'estimations intégrales pour trois classes d'harmoniques sphériques quaternioniques, nous prouvons quelques minorations pour la (L^p, L^2) norme des projecteurs harmoniques quaternioniques, pour $p \in [1, 2]$.

1. Introduction

In this note, we prove some bounds from below for the (L^p, L^2) norm of the quaternionic harmonic projectors $\pi_{\ell\ell'}$, which are the projection operators mapping the space of square integrable functions defined on the quaternionic unit sphere S^{4n-1} in $\mathbb{H}^n$ onto the subspace $\mathcal{H}^{\ell,\ell'}$, consisting of all quaternionic spherical harmonics of bidegree (ℓ, ℓ'). Here $\ell, \ell' \in \mathbb{N}$, $0 \leqslant \ell' \leqslant \ell$, and $p \in [1, 2]$.

Since the transposed operator $\pi^*_{\ell\ell'} : \mathcal{H}^{\ell\ell'} \to L^q(S^{4n-1})$ is the inclusion operator (here $1/p + 1/q = 1$), we have

$$\|\pi_{\ell\ell'}\|_{(p,2)} \geqslant \frac{\|Y_{\ell\ell'}\|_q}{\|Y_{\ell\ell'}\|_2}, \qquad q \geqslant 2\,,\; Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}\,. \tag{1.1}$$

Thus to prove these inequalities we are led to study the L^q norms of the functions $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$, for $q \geqslant 2$. Our estimates are therefore related to the problem of size concentration of the bigraded spherical harmonics. In the real and complex context, where the analogous question has been largely investigated (see [11] and [4, 5]), it is fully understood that two classes of spherical harmonics with competing behaviours, the highest weight vectors and the zonal functions, play a prominent role in the analysis of the harmonic projectors and also in some related applications (see, e.g., [2, 3, 7]).

The quaternionic framework turns out to be more interesting: indeed, we identify three classes of spherical harmonics with competing behaviours, giving rise, in the light of (1.1), to different bounds from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$ on three subintervals of $p \in [1, 2]$. More precisely, for p close to 1, like in the real and complex framework [11, 4, 5], the

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estimates for $\|\pi_{\ell\ell'}\|_{(p,2)}$ turn out to be sensitive to a high pointwise concentration. Thus we obtain bounds from below by considering the quaternionic zonal functions $\mathbb{Z}_{\ell\ell'}$, which are highly concentrated at the North Pole. When p is close to 2, the estimates are more sensitive to a sparse concentration along the Equator; in this case, we prove our bounds by considering the highest weight spherical harmonics, since these functions spread out in a small neighborhood around the Equator.

Anyway, in a third interval inside $[1,2]$, more precisely when $p \in (4/3, 2(4n-3)/(4n-1))$, the dichotomy between zonal and highest weight harmonics is partially mitigated; we obtain indeed better bounds from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$, by considering a third class of spherical harmonics. We refer to Section 3 for a discussion about these elements of $\mathcal{H}^{\ell\ell'}$, which have no analogous in the real or complex case and are related to representation-theoretic questions on S^{4n-1}.

Finally, in the light of these bounds for the spherical harmonics, in Section 4 we are able to prove $L^p - L^2$ bounds from below for $\pi_{\ell\ell'}$. The proof of the same bounds from above is already under way.

2. Notation and preliminaries

We denote by $\mathbb{H}$ the skew field of all quaternions $q = x_0 + x_1 i + x_2 j + x_3 k$ over $\mathbb{R}$, where x_0, x_1, x_2, x_3 are real numbers and the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ik = -ki = -j$, $jk = -kj = i$. The conjugate $\overline{q}$ and the modulus $|q|$ are defined by $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and $|q|^2 = q\overline{q} = \sum_{j=0}^{3} x_j^2$, respectively. For $n \geqslant 1$ the symbol $\mathbb{H}^n$ will denote the n-dimensional vector space over $\mathbb{H}$. By abuse of notation, we write q also to denote $(q_1, \ldots, q_n) \in \mathbb{H}^n$. Sometimes we will adopt a complex notation, writing $q = (z_1 + jz_{n+1}, \ldots, z_n + jz_{2n})$, with $z_1, \ldots, z_{2n} \in \mathbb{C}$.

S^{4n-1} is the unit sphere in $\mathbb{H}^n$, that is,

$$S^{4n-1} = \{q = (q_1, \ldots, q_n) \in \mathbb{H}^n \, : \langle q, q\rangle = 1\}\,;$$

here the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$ is defined as $\langle q, q' \rangle = q_1 \overline{q_1'} + \ldots + q_n \overline{q_n'}$, $q, q' \in \mathbb{H}^n$. S^{4n-1} may be identified with K/M, where $K = \mathrm{Sp(n)} \times \mathrm{Sp(1)}$ and $M = \mathrm{Sp(n-1)} \times \mathrm{Sp(1)}$, $\mathrm{Sp(n)}$ denoting the group of $n \times n$ matrices A with quaternionic entries, such that $\overline{A^T} A = A\overline{A^T} = I_n$. We introduce on S^{4n-1} the coordinate system

$$\begin{cases} q_1 = \cos\theta\,(\cos t + \tilde{q}\sin t) \\ q_s = \sigma_s \sin\theta\,, \qquad s = 2, \ldots, n\,, \end{cases} \tag{2.1}$$

where $\theta \in [0, \pi/2]$, $t \in [0, \pi]$, $\sigma_s \in \mathbb{H}$ with $\sum_{s=2}^{n} |\sigma_s|^2 = 1$. Moreover, $\tilde{q} \in \mathbb{H}$ with $|\tilde{q}|^2 = 1$ and $\Re\tilde{q} = 0$; we will write $\tilde{q} = \cos\psi\, i + \sin\psi \cos\varphi\, j + \sin\psi\, \sin\varphi\, k$, with $\psi \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. We remark that $(\sin t\, \sin\psi\, \sin\varphi\,, \sin t\, \sin\psi \cos\varphi\,, \sin t\, \cos\psi\,, \cos t)$ yields a coordinate system for $\mathrm{Sp}(1)$.

The normalized invariant measure $d\sigma = d\sigma_{S^{4n-1}}$ on S^{4n-1} with respect to the spherical coordinates (2.1) is, up to a constant $C = C(n)$,

$$\sin^{4n-5}\theta \cos^3\theta\, d\theta\, \sin^2 t\, dt\, d\sigma_{S^{4n-5}}\, d\sigma(\tilde{q})\,, \tag{2.2}$$

$d\sigma(\tilde{q})$ denoting the measure on the unit sphere in $\mathbb{R}^3$.

By $L^2(S^{4n-1})$ we denote the Hilbert space of square integrable functions on S^{4n-1}, with respect to the inner product

$$(f,g)_{L^2} = \int_{S^{4n-1}} f(q)\,\overline{g(q)}\,d\sigma\,.$$

Johnson and Wallach, starting from some earlier work by Kostant [10], proved in [9] that this space may be decomposed as

$$L^2(S^{4n-1}) = \bigoplus_{\ell\geqslant\ell'\geqslant 0} \mathcal{H}^{\ell\ell'}\,, \tag{2.3}$$

where each subspace $\mathcal{H}^{\ell\ell'}$

(1) is irreducible under K;

(2) is generated under K by the "highest weight vector"

$$\mathrm{P}_{\ell,\ell'}(z,\bar z) = \bar z_{n+1}^{\ell-\ell'}(z_1\bar z_{n+2} - z_2\bar z_{n+1})^{\ell'}\,; \tag{2.4}$$

(3) is finite dimensional.

In the following, we shall use the symbols c and C with $0 < c,\ C < \infty$ to denote constants which are not necessarily equal at different occurrences. They depend only on the dimension n and on the Lebesgue indices p or q. The symbol $\simeq$ between two positive expressions means that their ratio is bounded above and below by such constants. For two positive quantities a and b, we write $a \lesssim b$ instead of $a \leqslant Cb$ and $a \gtrsim b$ for $b \lesssim a$. Finally, we will denote by $I_{\mathbb{S}}$ the set of indices $\{(\ell,\ell') \in \mathbb{N}\times\mathbb{N} : 0 \leqslant \ell' \leqslant \ell\}$.

3. The main estimates

In [6] we started studying the $L^p - L^2$ norm of the joint spectral projectors $\pi_{\ell\ell'}$, $(\ell,\ell') \in I_{\mathbb{S}}$, mapping $L^p(S^{4n-1})$ onto $\mathcal{H}^{\ell\ell'}$, $1 \leqslant p \leqslant 2$. We proved sharp bounds for these norms under the additional assumptions $\ell - \ell' \leqslant c_0$ or $\ell' \leqslant c_1$, for some positive constants c_0, c_1. In this note, we prove some crucial estimates from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$ in the general case. As illustrated in the Introduction, we are led to study the L^q norms of the eigenfunctions $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$, for $q \geqslant 2$.

Estimates for zonal functions. We call *zonal function of bidegree* (ℓ,ℓ') with pole $e_1 = (1,0\ldots,0)$ a M-invariant function in $\mathcal{H}^{\ell\ell'}$. An explicit formula for the zonal function $\mathrm{Z}_{\ell\ell'}$ with pole e_1 is given for all $(\ell,\ell') \in I_{\mathbb{S}}$ by

$$\mathrm{Z}_{\ell\ell'}(\theta,t) = \frac{d_{\ell\ell'}}{\omega_{4n-1}}\,\frac{\sin\left((\ell-\ell'+1)t\right)}{(\ell-\ell'+1)\,\sin t}\,(\cos\theta)^{\ell-\ell'}\,\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}\,, \tag{3.1}$$

where $t \in [0,\pi]$, $\theta \in [0,\frac{\pi}{2}]$, ω_{4n-1} denotes the surface area of S^{4n-1}, $P_{\ell'}^{(2n-3,\ell-\ell'+1)}$ is the Jacobi polynomial and $d_{\ell\ell'}$ is the dimension of $\mathcal{H}^{\ell\ell'}$, given by

$$d_{\ell\ell'} = (\ell+\ell'+2n-1)(\ell-\ell'+1)^2\,\frac{(\ell+2n-2)!}{(\ell+1)!(2n-3)!}\,\frac{(\ell'+2n-3)!}{\ell'!(2n-1)!}\,,\quad \ell\geqslant\ell'\geqslant 0. \tag{3.2}$$

We recall the Mehler–Heine formula for the so-called disk polynomials, proved in [1, p. 10]. The symbol J_α denotes the Bessel function of the first kind of order α.

Proposition 3.1. *Fix $n \in \mathbb{N}$. Let $j, k \in \mathbb{N}$, $j \leqslant k$. Then*

$$\lim_{\substack{j\to+\infty \\ k\to+\infty}} \Big(\cos(\frac{\theta}{\sqrt{jk}})\Big)^{k-j} \frac{P_j^{(2n-3,k-j)}\big(\cos(\frac{2\theta}{\sqrt{jk}})\big)}{P_j^{(2n-3,k-j)}(1)} = \Gamma(2n-2)\frac{J_{2n-3}(2\theta)}{\theta^{2n-3}}.$$

This limit holds uniformly in every compact interval.

We also recall (see [1, p. 12]) that for all $j, k \in \mathbb{N}$, $j \leqslant k$,

$$\sup_{\theta\in[0,\pi/2]} \Big|(\cos\theta)^{k-j} \frac{P_j^{(2n-3,k-j)}\big(\cos(2\theta)\big)}{P_j^{(2n-3,k-j)}(1)}\Big| \leqslant 1. \tag{3.3}$$

For $q \geqslant 2$ set

$$\mathcal{I}_q = \Big(\int_0^{\pi/2} \Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos\theta)^{\ell-\ell'}\Big|^q (\sin\theta)^{4n-5}(\cos\theta)^3\, d\theta\Big)^{1/q}. \tag{3.4}$$

Lemma 3.2. *For all $q \geqslant 2$ and for all $(\ell, \ell') \in I_{\mathbb{S}}$ such that ℓ' is sufficiently great, we have*

$$\frac{\mathcal{I}_q}{\mathcal{I}_2} \gtrsim (\ell')^{(2n-2)(\frac12-\frac1q)-\frac12}\,\ell^{(2n-2)(\frac12-\frac1q)} \Bigg\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\Big(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\Big)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1}\Bigg\|_{L^q([0,1];\,\theta^{4n-5}d\theta)}$$

Proof. Observe that

$$\begin{aligned}
\big(\mathcal{I}_q\big)^q &\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos\theta)^{\ell-\ell'}\Big|^q (\sin\theta)^{4n-5}(\cos\theta)^3\, d\theta \\
&= \int_0^{1/\sqrt{\ell\ell'}} \Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos\theta)^{\ell-\ell'+\frac3q}\Big|^q (\sin\theta)^{4n-5}\, d\theta \\
&\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos\theta)^{\ell-\ell'+1}\Big|^q (\sin\theta)^{4n-5}\, d\theta,
\end{aligned}$$

where the last inequality follows from the fact that $\theta \in (0, 1/\sqrt{\ell\ell'})$. Then, after a change of variables we get

$$\begin{aligned}
\big(\mathcal{I}_q\big)^q &\gtrsim \int_0^1 \Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1}\Big|^q (\sin(\theta/\sqrt{\ell\ell'}))^{4n-5}\frac{d\theta}{\sqrt{\ell\ell'}} \\
&\simeq \int_0^1 \Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1}\Big|^q \big(\theta/\sqrt{\ell\ell'}\big)^{4n-5}\, d\theta/(\sqrt{\ell\ell'}) \\
&\simeq (\ell\ell')^{-(2n-2)} \Bigg\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\Big(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\Big)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1}\Bigg\|^q_{L^q([0,1];\,\theta^{4n-5}d\theta)}.
\end{aligned} \tag{3.5}$$

For $q = 2$ we obtain a more precise estimate. Indeed, from standard properties of zonal harmonics it follows that $||\mathrm{Z}_{\ell\ell'}||_2 \simeq (d_{\ell\ell'})^{1/2}$, that is, by means of (3.1),

$$d_{\ell\ell'} \simeq (d_{\ell\ell'})^2 \int_0^\pi \Big|\frac{\sin\big((\ell-\ell'+1)t\big)}{(\ell-\ell'+1)\,\sin t}\Big|^2 \sin^2 t\, dt$$

$$\times \int_0^{\pi/2} \Big| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'} \Big|^2 (\sin\theta)^{4n-5} (\cos\theta)^3 \, d\theta.$$

Since

$$\int_0^{\pi} \Big| \frac{\sin\left((\ell-\ell'+1)t\right)}{(\ell-\ell'+1)\,\sin t} \Big|^2 \sin^2 t \, dt \simeq (\ell-\ell'+1)^{-2}, \tag{3.6}$$

we have

$$(\mathcal{I}_2)^2 \simeq (\ell-\ell'+1)^2 (d_{\ell\ell'})^{-1}. \tag{3.7}$$

Then, combining (3.5) and (3.7), we get for all $q > 2$

$$\begin{aligned}
\frac{\mathcal{I}_q}{\mathcal{I}_2} &\gtrsim (\ell-\ell'+1)^{-1} (d_{\ell\ell'})^{1/2} (\ell\ell')^{-(2n-2)/q} \Bigg\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\Big(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\Big)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \Bigg\|_{L^q([0,1];\,\theta^{4n-5}d\theta)} \\
&\gtrsim (\ell')^{(2n-3)/2} \ell^{(2n-2)/2} (\ell\ell')^{-(2n-2)/q} \Bigg\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\Big(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\Big)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \Bigg\|_{L^q([0,1];\,\theta^{4n-5}d\theta)} \\
&\gtrsim (\ell')^{(2n-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \ell^{(2n-2)(\frac{1}{2}-\frac{1}{q})} \Bigg\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\Big(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\Big)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \Bigg\|_{L^q([0,1];\,\theta^{4n-5}d\theta)}.
\end{aligned}$$

□

Then, for $q \geqslant 2$ set

$$\mathcal{J}_q = \Big(\int_0^{\pi} \Big| \frac{\sin\left((\ell-\ell'+1)t\right)}{(\ell-\ell'+1)\sin t} \Big|^q \sin^2 t \, dt \Big)^{1/q}. \tag{3.8}$$

Lemma 3.3. *For all $q \geqslant 2$ and for all $(\ell,\ell') \in I_{\mathbb{S}}$ such that $\ell-\ell'$ is sufficiently great, we have*

$$\frac{\mathcal{J}_q}{\mathcal{J}_2} \simeq \begin{cases} (\ell-\ell'+1)^{1-3/q} & \text{for all } q > 3 \\ (\log(\ell-\ell'))^{1/3} & \text{for all } q = 3 \\ 1 & \text{for all } q < 3. \end{cases}$$

Proof. We start recalling that

$$\frac{\sin\left((\ell-\ell'+1)t\right)}{\sin t} = O((\ell-\ell'+1)^{1/2}) P_{\ell-\ell'}^{(\frac{1}{2},\frac{1}{2})}(\cos t),$$

[13, p.60]. Thus, using some asymptotic integral estimates in [13, p.391], we see that

$$(\mathcal{J}_q)^q \simeq \int_0^{\pi/2} \Big| \frac{\sin\left((\ell-\ell'+1)t\right)}{(\ell-\ell'+1)\sin t} \Big|^q \sin^2 t \, dt \simeq (\ell-\ell'+1)^{-3}, \tag{3.9}$$

for $q > 3$ and $\ell-\ell'$ sufficiently great. Combining (3.6) and (3.9) we get the expected estimate for $\mathcal{J}_q/\mathcal{J}_2$ for all $q > 3$. The other two cases analogously follow from [13, p.391], and (3.6). □

Combining Lemma 3.2 and Lemma 3.3 gives a bound from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$, with $1 \leqslant p \leqslant 2$.

Proposition 3.4. *Fix $n \geqslant 2$. For all $(\ell,\ell') \in I_{\mathbb{S}}$ such that ℓ' and $\ell-\ell'$ are sufficiently great, and for all $q \geqslant 2$ we have*

$$\frac{||\mathrm{Z}_{\ell\ell'}||_q}{||\mathrm{Z}_{\ell\ell'}||_2} \gtrsim \begin{cases} (\ell-\ell'+1)^{1-3/q}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} & \text{for all } q>3 \\ (\log(\ell-\ell'))^{1/3}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} & \text{for } q=3 \\ (\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} & \text{for all } q<3. \end{cases} \tag{3.10}$$

Proof. As a consequence of Lemma 3.2 for $q>3$ we have

$$\begin{aligned} \frac{||\mathrm{Z}_{\ell\ell'}||_q}{||\mathrm{Z}_{\ell\ell'}||_2} &\gtrsim (\ell-\ell'+1)^{1-3/q}\,\mathcal{I}_q/\mathcal{I}_2 \\ &\simeq (\ell-\ell'+1)^{1-3/q}\,(\ell\ell')^{(2n-2)(1/2-1/q)}(\ell')^{-1/2} \\ &\qquad \times \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos\left(2\theta/\sqrt{\ell\ell'}\right))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1}\right\|_{L^q(\theta^{4n-5}d\theta,[0,1])}. \end{aligned}$$

Then the first inequality in (3.10) follows from a slight variation of Proposition 3.1, (3.3) and some trivial asymptotics for the Bessel function. The proof of the other two inqualities is similar. $\square$

Estimates for the highest weight sherical harmonics. We will estimate the norm of the highest weight spherical harmonics $\mathrm{P}_{\ell,\ell'}$ in $\mathcal{H}^{\ell\ell'}$, defined in (2.4).

In [6, Lemma 5.3] we proved that for all $\zeta_1 \in \mathbb{R}$, $\zeta_1>0$, and for all $\zeta_2 \in \mathbb{N}$ one has

$$\int_{S^{4n-1}} |\bar z_{n+1}|^{2\zeta_1}|z_1\bar z_{n+2}-z_2\bar z_{n+1}|^{2\zeta_2}d\sigma = \frac{c_n\,\Gamma(\zeta_1+\zeta_2+2)\Gamma(\zeta_2+1)}{\Gamma(\zeta_1+2\zeta_2+2n)\,(\zeta_1+1)}. \tag{3.11}$$

We also proved that as a consequence of (3.11) the following bound holds

$$\|\mathrm{P}_{\ell,\ell'}\|_2 \simeq \left(\frac{(\ell'+1)^{\frac12}}{(\ell+\ell')^{2n-2}\,(\ell-\ell'+1)}\right)^{\frac12}. \tag{3.12}$$

Proposition 3.5. *Let $\mathrm{P}_{\ell\ell'}$ be the highest weight vector defined by (2.4). For all $q \geqslant 2$ we have*

$$\limsup_{\ell'\to+\infty}\left(\frac{(\ell'+1)^{\frac12}}{(\ell+\ell')^{2n-2}\,(\ell-\ell'+1)}\right)^{\frac12-\frac1q}\frac{\|\mathrm{P}_{\ell,\ell'}\|_q}{\|\mathrm{P}_{\ell,\ell'}\|_2} > 0. \tag{3.13}$$

Proof. Fix any $q \geqslant 2$ and let $(\ell,\ell') \in I_{\mathbb{S}}$. First of all, we choose $2\zeta_1 = (\ell-\ell')q$. Then, if $\ell' q \in 2\mathbb{N}$, (3.11) applied to $\mathrm{P}_{\ell\ell'}$ with $2\zeta_2 = \ell' q$ yields

$$\|\mathrm{P}_{\ell,\ell'}\|_q^q = \frac{c_n\,\Gamma(\frac q2\ell+2)\,\Gamma(\frac q2\ell'+1)}{\Gamma(\frac q2(\ell+\ell')+2n)\,\,(\frac q2(\ell-\ell')+1)}.$$

Then a standard application of Stirling's estimate leads to

$$\|\mathrm{P}_{\ell,\ell'}\|_q \simeq \frac{(\frac q2\ell+1)^{\frac12\ell+(1+\frac12)/q}(\frac q2\ell'+1)^{\frac12\ell'+1/(2q)}}{(\frac q2(\ell+\ell')+2n-1)^{\frac12(\ell+\ell')+(2n-1+\frac12)/q}\,\,(\frac q2(\ell-\ell')+1)^{1/q}},$$

which, combined with (3.12), yields

$$\frac{\|\mathrm{P}_{\ell,\ell'}\|_q}{\|\mathrm{P}_{\ell,\ell'}\|_2} \simeq \left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\,(\ell-\ell'+1)}\right)^{\frac{1}{q}-\frac{1}{2}} . \tag{3.14}$$

This proves the assertion under the assumption $\ell' q \in 2\mathbb{N}$.

If $q = \frac{m_0}{n_0}$, for some $m_0,\ n_0 \in \mathbb{N}^*$, it suffices to replace ℓ' with $2n_0\ell'$ and then choose $\zeta_2 = m_0\ell'$. By considering $(\ell,\ell') \in I_{\mathbb{S}}$ such that $\ell \geqslant 2n_0\ell'$, we get an estimate analogous to (3.14) for $\|\mathrm{P}_{\ell,2n_0\ell'}\|_q$, yielding (3.13).

Finally, if q is not rational, the desired estimate follows from the continuity of the L^q norms and the previous arguments for rational values of q. □

Estimates for mixed spherical harmonics. We consider the function $\mathrm{Q}_{\ell\ell'}$, given by

$$\mathrm{Q}_{\ell\ell'}(\theta,\varphi,t) = \left(\sin t\,\sin\psi\, e^{i\varphi}\right)^{\ell-\ell'} (\cos\theta)^{\ell-\ell'} \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}\,, \tag{3.15}$$

for all $(\ell,\ell') \in I_{\mathbb{S}}$, with $t,\psi \in [0,\pi]$, $\varphi \in [0,2\pi]$, $\theta \in [0,\frac{\pi}{2}]$. Observe that $\mathrm{Q}_{\ell\ell'}$ is obtained replacing the factor $\sin\left((\ell-\ell'+1)t\right)/\left((\ell-\ell'+1)\,\sin t\right)^{-1}$ in (3.1) with the highest weight spherical harmonic of degree $\ell-\ell'$ in Σ^3, the unit sphere in $\mathbb{R}^4$. For a discussion about the role of Σ^3 (or, equivalently, of Sp(1)) in our analysis we refer to [6, Remark 2.3].

We only recall here that $\mathcal{H}^{\ell\ell'}$ is a joint eigenspace for the spherical Laplacian $\Delta_{S^{4n-1}}$ and for an operator Γ, which essentially concides with the Casimir operator on Sp(1) and in our coordinates reads as

$$\Gamma = \frac{1}{\sin^2 t}\frac{\partial}{\partial t}\sin^2 t\frac{\partial}{\partial t} + \frac{1}{\sin^2 t\,\sin\psi}\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi} + \frac{1}{\sin^2 t}\frac{1}{\sin^2\psi}\frac{\partial^2}{\partial^2\varphi}.$$

We refer to [9] and [8, p. 696] for a discussion about the role of this operator. Then it is easily seen that $\mathrm{Q}_{\ell\ell'}$ belongs to $\mathcal{H}^{\ell\ell'}$, since it is an eigenvector both for $\Delta_{S^{4n-1}}$ and for Γ.

Proposition 3.6. *Fix $n \geqslant 2$. For all $(\ell,\ell') \in I_{\mathbb{S}}$, such that ℓ' and $\ell-\ell'$ are sufficiently great, and for all $q > 2$ we have*

$$\frac{||\mathrm{Q}_{\ell\ell'}||_q}{||\mathrm{Q}_{\ell\ell'}||_2} \gtrsim (\ell-\ell'+1)^{1/2-1/q}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} .$$

Proof. It follows from Lemma 3.2, Proposition 3.1 and some basic estimates for the spherical harmonics in Σ^3 (see [11, Theorem 4.1]). □

4. Bounding the harmonic projections

A comparison between Proposition 3.4, Proposition 3.5 and Proposition 3.6 leads to the following estimate.

Proposition 4.1. *Let $n \geqslant 2$, $1 \leqslant p \leqslant 2$. Set $p_n = 2(4n-3)/(4n-1)$. Then there exists some constant C, only depending on n and p, such that the following estimate holds*

$$||\pi_{\ell\ell'} f||_2 \geqslant C(n,p)\,(1+\ell)^{\alpha(\frac{1}{p},n)}(1+\ell')^{\beta(\frac{1}{p},n)}(\ell-\ell'+1)^{\gamma(\frac{1}{p},n)}\,||f||_p\,, \tag{4.1}$$

where

$$\alpha(\frac{1}{p}, n) := 2(n-1)\big(\frac{1}{p} - \frac{1}{2}\big) \quad \textit{for all } 1 \leqslant p \leqslant 2\,,$$

$$\beta(\frac{1}{p}, n) := \begin{cases} 2(n-1)\big(\frac{1}{p} - \frac{1}{2}\big) - \frac{1}{2} & \textit{if } 1 \leqslant p \leqslant p_n \\ \frac{1}{2}\big(\frac{1}{2} - \frac{1}{p}\big) & \textit{if } p_n \leqslant p \leqslant 2\,, \end{cases}$$

and

$$\gamma(\frac{1}{p}, n) := \begin{cases} 3\big(\frac{1}{p} - \frac{1}{2}\big) - \frac{1}{2} & \textit{if } 1 \leqslant p \leqslant \frac{4}{3} \\ \frac{1}{p} - \frac{1}{2} & \textit{if } \frac{4}{3} \leqslant p \leqslant 2, \end{cases}$$

for all $(\ell, \ell') \in I_{\mathbb{S}}$*, such that* $\ell - \ell'$ *and* ℓ' *are sufficiently great.*

The proof of (4.1) from above, which involves both real and analytic interpolation arguments, multiplier theorems for $\Delta_{S^{4n-1}}$, Γ and for $\mathcal{L}$, and a very detailed analysis of the Jacobi polynomials, is quite long and tangled. This work is already under way.

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