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GLOBAL KOLMOGOROV TORI IN THE PLANETARY N-BODY PROBLEM. ANNOUNCEMENT OF RESULT

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ABSTRACT. We improve a result in [9] by proving the existence of a positive measure set of $(3n-2)$-dimensional quasi-periodic motions in the spacial, planetary $(1+n)$-body problem away from co-planar, circular motions. We also prove that such quasi-periodic motions reach with continuity corresponding $(2n-1)$-dimensional ones of the planar problem, once the mutual inclinations go to zero (this is related to a speculation in [2]). The main tool is a full reduction of the SO(3)-symmetry, which retains symmetry by reflections and highlights a quasi-integrable structure, with a small remainder, independently of eccentricities and inclinations.

1. Set up and background

In [2], V. I. Arnold, partly solving but undoubtedly clarifying important mathematical settings of the more than centennial question (going back to the investigations by Sir Isaac Newton, in the 17^{th} century) on the motions of the planetary system, asserted his "Theorem on the stability of planetary motions" as follows.

Theorem 1.1 ("Theorem on the stability of planetary motions" [2, Ch. III, p. 125])**.** *For the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small.* [...] *In particular* [...] *in the many-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities belong to this set, the distances of the bodies from each other will remain perpetually bounded.*

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I am indebted to Jacques Féjoz who, letting me know of the work by Harrington [15], inspired in me the present application of the $\mathcal{P}$-coordinates.

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Arnold proved Theorem 1.1 in the particular case of the planar three-body problem. To extend his statement to the general case, he provided, in his paper, sketchy ideas and ingenious conjectures. Completing his ideas and conjectures was revealed to be absolutely not trivial. The geometric and symplectic structure of the problem, responsible for certain strong degeneracies partly known and partly unknown to him, has been clarified only recently [8]. The complete, general proof of Theorem 1.1 has been reached thanks to contributions by J. Laskar, P. Robutel, M. Herman, J. Féjoz, L. Chierchia, and the author [20, 28, 16, 11, 23, 9].

An important assumption in Arnold's statement, followed by all previously mentioned papers, is the initial one, where he requires the orbits of the planets to be "close to circles lying in a single plane." The mathematical reason that lead Arnold to consider this assumption was the necessity of treating one of the many degeneracies of the problem, the so-called "proper degeneracy," recalled below. From the physical point of view, such constraint may be regarded as almost natural, or at least not too annoying, since, for example, it is observed that most of the planets of the Solar System have indeed small eccentricities and inclinations. However, it is a fact that some trans-Neptunian objects, or a large number of asteroids, does not fulfill this condition, and for them, numerical estimates or extensions allowing to go beyond this constraint seem not to be enough. Arnold himself dedicated a paragraph in his paper [2, Chapter III, §1, no 6] to a discussion on how to eliminate this assumption in the case of the planar three-body problem, using very a particular tool for this case. Also, the cases of the spatial three-body problem and of the planar general problem have been studied, and, for them, similar results have been recently obtained thanks to some special circumstances arising in such problems [24].

The purpose of this paper is to present and illustrate the main ideas of the proof of a new statement (Theorem 2.1 below) of Theorem 1.1 for the general problem, where the first assumption is removed. We shall prove that, for arbitrary values of eccentricities and relatively small inclinations, a positive measure set of quasi-periodic motions with trajectories close to Keplerian ellipses with those eccentricities and inclinations do exists, and *its measure is ruled only by the distances among the planets*, thus (as the main novelty with respect to existing quoted formulations), is independent on eccentricities and inclinations. The proof is based on a different procedure for eliminating the proper degeneracy, with respect to Arnold's. We shall obtain a new, *explicit* normal form for the planetary problem (Proposition 3.3 below). Here, by "explicit," we mean that we are able to compute any relevant quantity for the system, for example, its "torsion." This computational aspect will be achieved thanks to two ingredients. The first ingredient (actually, the main novelty of the paper) is a new set of canonical coordinates for the planetary problem, presented in §3.1 below, which, thanks to its nice parity properties, allows for explicit analytical expressions of an integrable problem close to the problem under investigation. The second ingredient consists of a suitable choice of the planets' distances, which will allow the use of normal form and KAM techniques. The complete proof is publicly available in [25].

Before presenting our results, we remark that at the time of submission of the present manuscript to this journal and the arXive repository (June 2014), J. Féjoz was also working on the N-body problem and announced, in his recent lectures, to have obtained, via an independent proof, similar KAM results. Even so, at the time of revision of the present paper (February 2015), no manuscript of Féjoz's work is publicly available and hence no detailed comparison of strategies is possible. It is, however, a pleasure to mention here a recent private conversation with him, during which he kindly told the author that he used, for his proof, Poincaré's coordinates and, as a common point with our strategy, the planets' distances as a "smallness parameter."

To illustrate our results, we need to recall main difficulties, features and tools of the problem. Therefore, we dedicate this section to this purpose (referring however the reader to the aforementioned literature, or to the review papers [12, 4, 7] or, finally, to the introduction of [24] for more details), and defer to the next §2 the precise statement and technical aspects of our result.

Consider $(1+n)$ masses in the configuration space $E^3=\mathbb{R}^3$ interacting through gravity. Let such masses be denoted as $m_0, \mu m_1, \ldots, \mu m_n$, where m_0 is a leading mass ("sun," of "order one"), while $\mu m_1, \ldots, \mu m_n$ are n smaller masses ("planets," of "order μ," with μ a very small number). This problem, a sub-problem (usually referred to as "planetary" system) of the more general N-body problem, emulates the Solar System; hence, the study of it has a relevant physical meaning. It is very natural to regard this system (which is Hamiltonian[1]) as a small perturbation of the leading dynamical problem consisting of the gravitational interaction of the sun separately with each planet. This corresponds to what follows. After letting the system be free of the invariance by translations (i.e., eliminating the motion of the sun), one can write the $3n$-degrees of freedom Hamiltonian governing the motions of the planets as

$$\begin{aligned} \mathrm{H}_{\mathrm{hel}}(y,x) &= \sum_{i=1}^{n} h_{2\mathrm{B}}^{(i)}(y^{(i)},x^{(i)}) + \mu f_{\mathrm{hel}}(y,x) \\ &= \sum_{i=1}^{n} \Big(\frac{|y^{(i)}|^2}{2\mathfrak{m}_i} - \frac{\mathfrak{m}_i\mathfrak{M}_i}{|x^{(i)}|}\Big) \\ &+ \mu \sum_{1\le i<j\le n} \Big(\frac{y^{(i)}\cdot y^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)}-x^{(j)}|}\Big) \end{aligned} \tag{1}$$

where $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) = q^{(i)} - q^{(0)}$ denote the "heliocentric distances," $y^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)})$ their generalized conjugated momenta and $\mathfrak{m}_i := \frac{m_0 m_i}{m_0+\mu m_i}$ and $\mathfrak{M}_i := m_0 + \mu m_i$ the "reduced masses."

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[1] That is, its motions are described by equations of the form

$$\begin{cases} \dot y_j^{(i)} = -\partial_{x_j^{(i)}} \mathrm{H}_{3+3n}(p,q) \\ x_j^{(i)} = \partial_{y_j^{(i)}} \mathrm{H}_{3+3n}(p,q) \end{cases}$$

where $(p^{(i)}, q^{(i)}) := (p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, q_1^{(i)}, q_2^{(i)}, q_3^{(i)}$ are canonical coordinates of the point-mass i, and H_{3+3n} is a suitable $(3+3n)$-degrees of freedom Hamilton function, depending on $(p,q) = (p^{(0)}, \cdots, p^{(n)}, q^{(0)}, \cdots, q^{(n)})$.

In order to exploit the integrability of the "two-body terms"

$$h_{2\mathrm{B}}^{(i)} := \frac{|y^{(i)}|^2}{2\mathfrak{m}_i} - \frac{\mathfrak{m}_i \mathfrak{M}_i}{|x^{(i)}|},$$

a natural approach is to put the system in Delaunay[2] coordinates. This is a system of canonical action-angle variables ($(\Lambda, \Gamma, \mathrm{H}, \ell, g, \mathrm{h}) \in \mathbb{R}^{3n} \times \mathbb{T}^{3n}$), whose role is the one of transforming (via the Liouville-Arnold Theorem) $h_{2\mathrm{B}}^{(i)}$ into "Kepler form," i.e., a function of actions only. It is well known that, due to the too many integrals of $h_{2\mathrm{B}}^{(i)}$, this integrated form

$$h_{\mathrm{K}}^{(i)} = -\frac{\mathfrak{m}_i^3 \mathfrak{M}_i^2}{2\Lambda_i^2} \tag{2}$$

exhibits a dramatic loss of degrees of freedom: two actions, $\Gamma_i := |x^{(i)} \times y^{(i)}|$ and $\mathrm{H}_i := x_1^{(i)} y_2^{(i)} - x_2^{(i)} y_1^{(i)}$, disappear completely. This circumstance is usually called the "proper degeneracy."

Let us denote by

$$\mathrm{H}_{\mathrm{Del}} = h_{\mathrm{K}}(\Lambda) + \mu f_{\mathrm{Del}}(\Lambda, \Gamma, \mathrm{H}, \ell, g, \mathrm{h}) \tag{3}$$

where

$$h_{\mathrm{K}}(\Lambda_1, \cdots, \Lambda_n) := -\sum_{1 \le i \le n} \frac{\mathfrak{m}_i^3 \mathfrak{M}_i^2}{2\Lambda_i^2} \tag{4}$$

the system (1) expressed in Delaunay coordinates. The purpose is to determine a positive measure set of quasi-periodic motions for this system.

In 1954, A. N. Kolmogorov [19] discovered a breakthrough property of quasi-integrable dynamical systems: for a regular, slightly perturbed system

$$\mathrm{H}(I, \varphi) = \mathrm{h}(I) + \mu \mathrm{f}(I, \varphi) \qquad (I, \varphi) \in A \times \mathbb{T}^{\nu}$$

where $A \subset \mathbb{R}^{\nu}$ is open, a great number of quasi-periodic motions $(I_0, \varphi_0) \to (I_0, \varphi_0 + \partial_I \mathrm{h}(I_0) t)$ of the unperturbed system h may be continued in the dynamics of the perturbed system, provided the Hessian $\partial_I^2 \mathrm{h}(I)$ does not vanish identically in A. Due to the proper degeneracy, for the planetary system expressed in Delaunay variables (3), taking $I := (\Lambda, \Gamma, \mathrm{H})$ and $\varphi := (\ell, g, \mathrm{h})$, Kolmogorov's non-degeneracy assumption is clearly violated. Despite of this fact, the perturbing function has good parity properties: Arnold noticed that such parities help in determining a quasi-integrable structure in *all* the variables for the planetary system, as now we explain.

Arnold's procedure goes as follows. Following Poincaré, one switches from Delaunay coordinates to a new set of canonical coordinates $(\Lambda, \lambda, \eta, \xi, \mathrm{p}, \mathrm{q})$. These are not in action-angle form, but are in mixed action-angle (the couples (Λ, λ)) and rectangular form (the $\mathrm{z} := (\eta, \xi, \mathrm{p}, \mathrm{q})$). The variables (Λ, λ) have roughly the same meaning of the (Λ, ℓ); the z are defined in a neighborhood of $\mathrm{z} = 0 \in \mathbb{R}^{4n}$ and the vanishing of (η_i, ξ_i) or of $(\mathrm{p}_i, \mathrm{q}_i)$ corresponds to the vanishing of the i^{th} eccentricity, inclination, respectively.

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[2] Delaunay and (see below) Poincaré coordinates are widely described in the literature. A definition may be found, e.g., in [8, 13]. Note that $(\mathrm{H}, \mathrm{h}) \in \mathbb{R}^n \times \mathbb{T}^n$ are denoted as (Θ, θ) in [8]. Delaunay-Poincaré coordinates were used by several authors, including Arnold, Nekhorossev, Herman, Laskar, Chenciner, Féjoz, Robutel, etc. We shall see below that, due to the proper degeneracy, there is a certain freedom in choosing canonical coordinates for the planetary system. See the definition of *Kepler map* in §3.2.

Let us denote as

$$\mathrm{H_P} = h_{\mathrm{K}}(\Lambda) + \mu f_{\mathrm{P}}(\Lambda, \lambda, \mathrm{z}) \qquad \mathrm{z} = (\eta, \xi, \mathrm{p}, \mathrm{q}) \tag{5}$$

the system (1) expressed in Poincaré variables.

Since the perturbation f_{hel} in (1) does not change under reflection

$$(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \to (\mathrm{r}_1 y_1^{(i)}, \mathrm{r}_2 y_2^{(i)}, \mathrm{r}_3 y_3^{(i)}, \mathrm{r}_1' x_1^{(i)}, \mathrm{r}_2' x_2^{(i)}, \mathrm{r}_3' x_3^{(i)}) \ \ \mathrm{r}_i, \mathrm{r}_i' = \pm 1 \tag{6}$$

and rotation transformations

$$(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \to \big(\mathrm{R}(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}), \mathrm{R}'(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})\big) \ \ \mathrm{R}, \mathrm{R}' \in \mathrm{SO}(3) \tag{7}$$

and due to the fact that the transformations (respectively, reflections with respect to the coordinate planes and rotation about the k-axis)

$$\begin{array}{lll} \mathcal{R}_1^- : & q'^{(i)} = \big(-x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\big), & p'^{(i)} = \big(y_1^{(i)}, -y_2^{(i)}, -y_3^{(i)}\big) \\ \mathcal{R}_2^- : & q'^{(i)} = \big(x_1^{(i)}, -x_2^{(i)}, x_3^{(i)}\big), & p'^{(i)} = \big(-y_1^{(i)}, y_2^{(i)}, -y_3^{(i)}\big) \\ \mathcal{R}_3^- : & q'^{(i)} = \big(x_1^{(i)}, x_2^{(i)}, -x_3^{(i)}\big), & p'^{(i)} = \big(y_1^{(i)}, y_2^{(i)}, -y_3^{(i)}\big) \\ \mathcal{R}_g : & q'^{(i)} = \big(\mathcal{R}_g^{(3)}(x_1^{(i)}, x_2^{(i)}), x_3^{(i)}\big), & p'^{(i)} = \big(\mathcal{R}_g^{(3)}(y_1^{(i)}, y_2^{(i)}), y_3^{(i)}\big) \end{array} \tag{8}$$

where

$$\mathcal{R}_g^{(3)} := \begin{pmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{pmatrix} \qquad g \in \mathbb{T}$$

have a nice expression in Poincaré variables, respectively,

$$\begin{array}{ll} \mathcal{R}_1^- : & (\Lambda_i', \lambda_i', \eta_i', \xi_i', \mathrm{p}_i', \mathrm{q}_i') = (\Lambda_i, -\lambda_i, \eta_i, -\xi_i, -\mathrm{p}_i, \mathrm{q}_i) \\ \mathcal{R}_2^- : & (\Lambda_i', \lambda_i', \eta_i', \xi_i', \mathrm{p}_i', \mathrm{q}_i') = (\Lambda_i, \pi - \lambda_i, -\eta_i, \xi_i, \mathrm{p}_i, -\mathrm{q}_i) \\ \mathcal{R}_3^- : & (\Lambda_i', \lambda_i', \eta_i', \xi_i', \mathrm{p}_i', \mathrm{q}_i') = (\Lambda_i, \lambda_i, \eta_i, \xi_i, -\mathrm{p}_i, -\mathrm{q}_i) \\ \mathcal{R}_g : & (\Lambda_i', \lambda_i', \eta_i', \xi_i', \mathrm{p}_i', \mathrm{q}_i') = (\Lambda_i, \lambda_i + g, \mathcal{R}_{-g}^{(3)}(\eta_i, \xi_i), \mathcal{R}_{-g}^{(3)}(\mathrm{p}_i, \mathrm{q}_i)) \end{array} \tag{9}$$

one then sees that the averaged ("secular") perturbation

$$f_{\mathrm{P}}^{\mathrm{av}}(\Lambda, \eta, \xi, \mathrm{p}, \mathrm{q}) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\mathrm{P}}(\Lambda, \lambda, \eta, \xi, \mathrm{p}, \mathrm{q}) d\lambda$$

enjoys the following symmetries. If we denote

$$t_j := \frac{\eta_j - \mathrm{i}\xi_j}{\sqrt{2}} \quad t_{j+n} := \frac{\mathrm{p}_j - \mathrm{i}\mathrm{q}_j}{\sqrt{2}} \quad t_j^* := \frac{\eta_j + \mathrm{i}\xi_j}{\sqrt{2}\mathrm{i}} \quad t_{j+n}^* := \frac{\mathrm{p}_j + \mathrm{i}\mathrm{q}_j}{\sqrt{2}\mathrm{i}}$$

and by

$$f_{\mathrm{P}}^{\mathrm{av}}(\Lambda, t, t^*) = \sum_{a, a^* \in \mathbb{N}^n} \mathfrak{F}_{a,a^*}(\Lambda) t^{\alpha} {t^*}^{\alpha^*}$$

the Taylor expansion of $f_{\mathrm{P}}^{\mathrm{av}}$ in powers of t, t^*, we then have the following:

Proposition 1.1 (D'Alembert rules)**.**

$$f_{\mathrm{P}}^{\mathrm{av}}(\Lambda, \eta, \xi, \mathrm{p}, \mathrm{q}) = \left\{ \begin{array}{l} f_{\mathrm{P}}^{\mathrm{av}}(\Lambda, \eta, -\xi, -\mathrm{p}, \mathrm{q}) \\ f_{\mathrm{P}}^{\mathrm{av}}(\Lambda, -\eta, \xi, \mathrm{p}, -\mathrm{q}) \\ f_{\mathrm{P}}^{\mathrm{av}}(\Lambda, \eta, \xi, -\mathrm{p}, -\mathrm{q}) \end{array} \right.$$

$$\mathfrak{F}_{a,a^*}(\Lambda) \neq 0 \iff |a|_1 = |a^*|_1, \tag{10}$$

where $|a|_1 := \sum_{i=1}^n a_i$.

By D'Alembert rules, one has that the expansion of $f_{\rm P}^{\rm av}$ around $\rm z = 0$ contains only even monomials and starts with

$$f_{\rm P}^{\rm av}(\Lambda, \eta, \xi, {\rm p}, {\rm q}) = C_0(\Lambda) + \sum_{1\le i,j\le n} \mathcal{Q}^{(h)}_{ij}(\Lambda)(\eta_i\eta_j + \xi_i\xi_j) + \sum_{1\le i,j\le n} \mathcal{Q}^{(v)}_{ij}(\Lambda)({\rm p}_i{\rm p}_j + {\rm q}_i{\rm q}_j) + {\rm O}({\rm z}^4)$$

where $C_0(\Lambda)$, $\mathcal{Q}^{(h)}_{ij}(\Lambda)$ and $\mathcal{Q}^{(v)}_{ij}(\Lambda)$ are suitable coefficients, expressed in terms of Laplace coefficients, computed in [20, 16, 11]. This expansion shows that the point ${\rm z} = (\eta, \xi, {\rm p}, {\rm q}) = 0$ is an *elliptic equilibrium point* for $f_{\rm P}^{\rm av}(\Lambda, \eta, \xi, {\rm p}, {\rm q})$. A natural question is whether, from here, it is also possible to transform $f_{\rm P}^{\rm av}$ into

$$\breve{\rm H}_{\rm P}(\Lambda, \lambda, {\rm z}) = h_{\rm K}(\Lambda) + \mu \breve{f}_{\rm P}(\Lambda, \lambda, {\rm z})$$

where $\breve{f}_{\rm P}^{\rm av}$ is in "Birkhoff normal form" (hereafter, BNF) of a suitable order (say, of order three). This means

$$\breve{f}_{\rm P}^{\rm av} = C_0(\Lambda) + \sum_{i=1}^n \sigma_i(\Lambda) w_i + \sum_{i=1}^n \varsigma_i(\Lambda) w_{i+n} + \sum_{r=2}^3 \sum_{1\le i_1\cdots i_r\le 2n} \tau_{i_1\cdots i_r}(\Lambda) w_{i_1}\cdots w_{i_k} + {\rm O}({\rm z}^7) \tag{11}$$

where $\sigma_i(\Lambda)$, $\varsigma_i(\Lambda)$ are the eigenvalues of $\mathcal{Q}^{(h)}(\Lambda)$, $\mathcal{Q}^{(v)}(\Lambda)$ and, for $1 \le i \le n$, $w_i := \frac{\eta_i^2+\xi_i^2}{2}$, $w_{i+n} := \frac{{\rm p}_i^2+{\rm q}_i^2}{2}$. Then Arnold aims to solve the problem of the proper degeneracy (and hence to prove Theorem 1.1) by obtaining Kolmogorov full-dimensional tori bifurcating from the elliptic equilibrium ${\rm z} = 0$, via the following abstract result.

Theorem 1.2 (The Fundamental Theorem, [2])**.** *Let*

$$\mathrm{H} = \mathrm{h}(I) + \mu \mathrm{f}(I, \varphi, u, v) \quad (I, \varphi, u, v) \in A \times \mathbb{T}^\nu \times B \tag{12}$$

where $A \subset \mathbb{R}^\nu$, $B \subset \mathbb{R}^{2\ell}$ are open, $0 \in B$, $(I, \varphi) = (I_1, \ldots, I_\nu, \varphi_1, \ldots, \varphi_\nu)$, and $(u, v) = (u_1, \ldots, u_\ell, v_1, \cdots, v_\ell)$ are real-analytic and

(i) $\det\big(\partial_I^2 \mathrm{h}(I)\big) \not\equiv 0$*;*

(ii)
$$\mathrm{f}^{\rm av} := \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(I, \varphi, u, v)d\varphi = \sum_{r=0}^3 \sum_{1\le i_1\cdots i_r\le m} \beta_{i_1\cdots i_r}(I) w_{i_1}\cdots w_{i_r} + {\rm O}(u, v)^7,$$
where $w_i := \frac{u_i^2+v_i^2}{2}$*;*

(iii) $\det\big(\beta_{ij}(I)\big) \not\equiv 0$.

Then, for any $\kappa > 0$ one can find a number $\varepsilon_0 = \varepsilon_0(\kappa)$ such that, if $0 < \varepsilon < \varepsilon_0$ and $0 < \mu < \varepsilon^8$, the set $\mathrm{F}_\varepsilon := A \times \mathbb{T}^\nu \times B^{2\ell}_\varepsilon(0)$ may be decomposed into a set F^*_ε which is invariant for the motions of H *and a set f_ε the measure of which is smaller than κ. More precisely, F^*_ε foliates into $(\nu+\ell)$-dimensional invariant manifolds $\{\mathcal{T}_\omega\}_\omega$ close to*

$$I_i = I_i^*(\omega) \qquad \varphi_i \in \mathbb{T} \qquad u_j^2 + v_j^2 = \varepsilon^2 I_j^*(\omega),$$

where the motion is analytically conjugated to the linear flow

$$\theta \to \theta + \omega t \qquad \theta \in \mathbb{T}^{\nu+\ell}.$$

Despite of this brilliant strategy, Arnold succeeded in applying Theorem 1.2 to the case of the planar three-body problem only, by explicitly checking assumptions (i)-(iii). For the general case, he was aware of some extra difficulties, about which he gave just some vague indications.

A first problem is represented by the so-called "secular degeneracies." The "first order Birkhoff invariants" $\sigma_1, \dots, \sigma_n, \varsigma_1, \dots, \varsigma_n$ satisfy, identically, two linear combinations with integer coefficients that are often referred to as "rotational" and "Herman" resonance, respectively,

$$\varsigma_n \equiv 0, \qquad \sum_{i=1}^{n} (\sigma_i + \varsigma_i) \equiv 0. \tag{13}$$

Arnold was aware of the former of relations (13), while he did not mention, in his paper, the latter, that seems to be noticed, in its full generality, by M. Herman, in the '90s, from whom it takes its name. He attributed the former to the conservation, along the motion, of the two horizontal components C_1, C_2 of the total angular momentum $\mathrm{C} = (\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3)$, and then managed to eliminate these extra-integrals with a change of coordinates. He proposed two qualitatively different changes, for the case of two, or more than two planets. In the case of two planets, the solution was ready. In the 19^{th} century, Jacobi proved that the order of the differential equations of the motion of the three-body problem may be explicitly reduced from six to four. A reduction by two degrees of freedom is the maximum that one can obtain, since this is the maximum number of commuting, independent integrals that one can form with the three (non-commuting) components of C. Next, Radau found a way to write such equations in Hamiltonian form. Geometrically, the method by Jacobi and Radau consists of referring the system to a rotating reference frame with the z-axis in the direction of the (constant) total angular momentum, and the x-axis in the direction of the (moving) "line of the nodes" – the straight line determined by the intersection of the instantaneous planes of the planets' orbits [18, 27]. The idea works (the system again appears in a adaptable form for the Fundamental Theorem), but Arnold failed its application. Instead of computing explicitly the torsion for this case, for sake of shortness, he invoked certain controversial arguments of continuity with the planar problem (note the number of degrees of freedom of the un-reduced planar problem is the same, four) that in fact were revealed to be affected by the fact that Radau-Jacobi coordinates are not defined for the planar problem. This mistake was pointed out by M. Herman, and repaired by J. Laskar and P. Robutel [20, 28]. Robutel computed the correct torsion and checked its non vanishing.

How to switch to the case of more than two planets was not so clear in the '60s, when no procedure generalizing Jacobi-Radau construction was known. For this more general case, Arnold, surprisingly, suggested a breakthrough procedure, qualitatively very different from the method by Jacobi and Radau. He conjectured that, in order to eliminate the problem of the identically vanishing eigenvalue, just a "partial reduction" (the expression is opposite to the "full reduction," performed by the set of coordinates by Jacobi and Radau mentioned above) should be enough, namely, a reduction of the number of degrees of freedom by just one unit. The idea is genial since he had understood, with the few means at his disposal, that the problem would be solved by a system of coordinates, analogous to Poincaré coordinates, which, including a couple of integrals as a conjugated couple, would render this eigenvalue negligible. He shortly illustrated a formal procedure for computing such coordinates but never completed the proof.

In 2002, Malige, Robutel and Laskar [22] explicitly computed the first orders of Arnold's coordinates for the case of two planets, where, however, the solution via Jacobi-Radau reduction was already available. A global, explicit set of coordinates satisfying Arnold's conjecture and solving the problem appeared in [9], based on the results of the author's PhD dissertation [23]; compare also Theorem 1.3.

In the meantime, in 2004, J. Féjoz, completing investigations by M. Herman, published the proof of Theorem 1.1 for the general case. Herman's ideas were substantially different from Arnold's. Firstly, aiming to avoid too many computations, Herman (extending ideas by H. Russmann [29]) proved an abstract result which relied on the properties of the *first order* coefficients $\beta_1, \ldots, \beta_m$ appearing in the expansion in item (ii) of Theorem 1.2. Instead of assumptions (i) and (iii), Herman's theorem required that the "frequency map"

$$(I_1, \cdots, I_\nu) \in A \subset \mathbb{R}^\nu \to (\partial_{I_1}\mathrm{h}, \cdots, \partial_{I_\nu}\mathrm{h}, \beta_1, \cdots, \beta_m) \in \mathbb{R}^\nu \times \mathbb{R}^m$$

should not be identically contained in any affine hyperplane:

$$\sum_{i=1}^{\nu} c_i \partial_{I_i}\mathrm{h} + \sum_{j=1}^{m} c'_j \beta_j \not\equiv 0 \qquad \forall (c, c') \neq 0. \tag{14}$$

Secondly, since, by (13), condition (14) is evidently violated in the planetary system, Herman (following ideas going back to Poincaré) proposed an indirect solution: to modify the Hamiltonian (5) by adding a commuting integral (a function of the angular momentum), check non-planarity for the modified Hamiltonian so as to prove the existence of quasi-periodic motions for the modified Hamiltonian, and next to recover quasi-periodic motions for the original system by abstract arguments of Lagrangian intersections. J. Féjoz replaced, in the final published version, Herman's ideas with some more subtle arguments of abstract reduction of rotation invariance.

After Herman-Féjoz's proof, some important questions were still unsolved, such as the nature of resonances (13), the existence of a BNF associated to the averaged perturbing function, or the possibility of a direct application of Arnold's and Herman's abstract results (i.e., the verification of condition (iii) in Theorem 1.2, or of (14)). We remark that the possibility of handling a BNF, or some other normal form of different nature, draws perspectives of different applications, for example, to the theory of Nekhorossev, or analysis of instabilities. A positive answer to the question of the BNF has been given in [23, 9]. In the next §3.3 we shall discuss a different normal form for the system.

It might seem a paradoxical fact, but it turns out that the resonances (13) are *not* a true obstacle to the construction of the planetary BNF [22, 9]. Indeed, as a nice effect of the symmetry $\mathcal{R}_g$ in (9), only resonances $\sum_{i=1}^{n}(\sigma_i(\Lambda)k_i + \varsigma_i(\Lambda)k_{i+n}) = 0$ with $\sum_{i=1}^{2n} k_i = 0$ are really important for the construction of BNF, and, evidently, none of the resonances in (13) have this form. Moreover, as proved in [11], (13)'s are the only ones to be *identically satisfied*, a result next improved in [9], where, by direct computation, it has been seen that they are the only ones to be satisfied in an *open set*: compare item (ii)-(c) of Theorem 1.3.

A much more serious problem is the following[3].

Proposition 1.2 (Rotational degeneracy [8]). *For the system (5), BNF can be constructed up to any prefixed[4] order p, but all the coefficient $\tau_{i_1\cdots i_r}(\Lambda)$ of the generic monomial $w_{i_1}\cdots w_{i_r}$ with some of the i_k's equal to $2n$ vanish identically.*

In particular, the "torsion" matrix (the matrix of the second-order coefficients) $\tau = (\tau_{ij})$ has an identically vanishing row and column, hence,

$$\det\tau \equiv 0.$$

This violates assumption (iii) of Theorem 1.2.

However, a such negative result, understood only "a posteriori," is just the counterpart of Theorem 1.3 below.

Theorem 1.3 ([23, 9, 8]). *It is possible to determine a global set of canonical coordinates[5]*

$$RPS = (\Lambda, \lambda, \eta, \xi, p, q) \tag{15}$$

which are related to Poincaré coordinates $(\Lambda, \mathrm{\lambda}, \mathrm{\eta}, \mathrm{\xi}, \mathrm{p}, \mathrm{q})$ by

$$\begin{aligned} &\Lambda = \Lambda, \quad \mathrm{\lambda} = \lambda + \varphi_1(\Lambda, z) \quad \mathrm{\eta}_j + \mathrm{i}\mathrm{\xi}_j = (\eta_j + \mathrm{i}\xi_j)e^{\mathrm{i}\varphi_2(\Lambda,z)} + \mathrm{O}(z^3) \\ &\mathrm{p} = U(\Lambda)p + \mathrm{O}(z^3) \quad \mathrm{q} = U(\Lambda)q + \mathrm{O}(z^3) \qquad (*) \end{aligned}$$

where $U(\Lambda)$ is a $n\times n$ unitary matrix, i.e., verifying $U(\Lambda)U^{\mathrm{t}}(\Lambda) = \mathrm{id}$ and φ_1, φ_2 are suitable functions defined in a global neighborhood of $z = 0$, such that

(i) *(p_n, q_n) are integrals for f_{RPS}, and*

(ii) *D'Alembert rules (9) are preserved and correspond to the reflections and the rotation in (8). In particular, denoting by*

$$\mathrm{H}_{\mathrm{RPS}}(\Lambda, \lambda, \bar z) = h_{\mathrm{K}}(\Lambda) + \mu f_{\mathrm{RPS}}(\Lambda, \lambda, \bar z)$$

the system (1) expressed in the RPS variables, where $\bar z$ denotes z deprived of (p_n, q_n), then

(a) *the point $\bar z = 0 \in \mathbb{R}^{2n-1}$, which corresponds to the vanishing of all eccentricities and mutual inclinations, is an elliptic equilibrium point for $\bar z \to f^{\mathrm{av}}_{\mathrm{RPS}}(\Lambda, \bar z)$;*

(b) *For any fixed $p \in \mathbb{N}, p \geq 2$, it is possible to conjugate $\mathrm{H}_{\mathrm{RPS}}$ to*

$$\breve{\mathrm{H}}_{\mathrm{RPS}}(\Lambda, \breve\lambda, \breve z) = h_{\mathrm{K}}(\Lambda) + \mu \breve f_{\mathrm{RPS}}(\Lambda, \breve\lambda, \breve z)$$

where

$$\begin{aligned} \breve f^{\mathrm{av}}_{\mathrm{RPS}}(\Lambda, \breve\lambda, \breve z) = C_0(\Lambda) &+ \sum_{i=1}^{n}\sigma_i(\Lambda)\breve w_i + \sum_{i=1}^{n-1}\varsigma_i(\Lambda)\breve w_{i+n} \\ &+ \sum_{r=2}^{p}\sum_{1\le i,j\le 2n-1}\tau_{i_1\cdots i_r}(\Lambda)\breve w_{i_1}\cdots\breve w_{i_r} + \mathrm{O}(\breve z^{2p+1}); \end{aligned}$$

[3] Proposition 1.2 answers, in particular, a question raised by M. R. Herman, who, in [16], declared not to know if the planetary torsion might vanish identically. More generally, Proposition 1.2 generalizes Laplace resonance in (13) to any order of BNF.

[4] Namely, with 3 replaced by p and $\mathrm{O}(\mathrm{z}^7)$ by $\mathrm{O}(\mathrm{z}^{2p+1})$ in (11).

[5] RPS stands for Regular, Planetary, and Symplectic.

(c) *for any* $p \in \mathbb{N}, a_-^{(1)} > 0$ *if* $a_+^{(n)} := \infty$, *for any* $1 \le i \le n-1$, *it is possible to choose numbers* $a_+^{(i+1)} > a_-^{(i+1)} \gg a_+^{(i)}$, *such that, if*

$$\mathcal{A} := \big\{\Lambda = (\Lambda_1, \cdots, \Lambda_n) : a_-^{(i)} \le a^{(i)}(\Lambda_i) \le a_+^{(i)}\big\},$$

then $(\sigma(\Lambda), \varsigma(\Lambda)) \cdot k \neq 0$ *for any* $\Lambda \in \mathcal{A}$, $k \in \mathbb{Z}^{2n-1}$, $0 < |k|_1 \le 2p$, $k \neq (1, \cdots, 1)$ *and* $\det \tau(\Lambda) \neq 0$ *for any* $\Lambda \in \mathcal{A}$.

Clearly, Theorem 1.3 above and Theorem 1.2 (with $\nu := n$, $\ell := 2n-1$, $I := \Lambda$, $\varphi := \check{\lambda}$, $(u, v) := \check{z}$) suddenly imply Theorem 1.1, simply replacing[6] "inclinations" with "mutual inclinations" in the statement. That $(*)$ and (i) imply Proposition 1.2 follows by a classical unicity argument in BNF, suitably adapted to the properly-degenerate case; see [8].

We just mention that the variables (15) have been obtained via a suitable "Poincaré regularization" of a set of action-angle variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$, which we may call "planetary" Deprit variables[7], since they are in turn easily related to a set of variables $(\mathrm{R}, \Phi, \Psi, \mathrm{r}, \varphi, \psi)$ studied in the '80s by F. Boigey and, in their full generality, by A. Deprit [3, 10]. The variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ "unfold" and extend to any $n \ge 2$ classical reduction of the nodes by Jacobi and Radau, previously mentioned, [18, 27]. For the relation between the "original" Deprit variables $(\mathrm{R}, \Phi, \Psi, \mathrm{r}, \varphi, \psi)$ and the planetary version $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ or the relation between the latter and Jacobi-Radau reduction of the nodes, see [24].

2. Result

Two joint decisive ingredients lead to the success of Arnold's strategy. These were

- the use of a set of canonical coordinates performing *full or partial reduction* of the SO(3)-symmetry, in order to overcome the rotational degeneracy;
- the *elliptic equilibrium point* of the secular perturbation, in order to overcome the proper degeneracy.

However, the nature of the elliptic equilibrium realized by the Jacobi-Radau ($n = 2$) or RPS ($n \ge 2$) coordinates is very different, and some distinction is to be made.

- The variables (15) realize a *partial reduction* of the SO(3)-invariance: in such variables, the system has $(3n-1)$ degrees of freedom, one over the minimum. As said, this is useful in order to describe with regularity the co-inclined, co-circular configuration and to keep the elliptic equilibrium for $\bar{z} = 0$. On the other hand, the fact of having one more degree of freedom than needed implies that possible $(3n-1)$-dimensional resonant tori corresponding to rotations in the invariable plane of non-resonant $(3n-2)$-dimensional tori are missed, with subsequent under-estimate ($\sim \varepsilon^{4n-2}$ instead of $\sim \varepsilon^{4n-4}$) of the measure of the invariant set F_ε^* mentioned in Theorem 1.2.

---

[6] Substantially, switching from Poincaré to RPS variables corresponds to replacing the n inclinations of the planets with respect to a prefixed frame (i, j, k), with $(n-1)$ mutual inclinations among the planets plus the negligible inclination of the invariable plane with respect to k. Recall that the invariable plane is the plane orthogonal to the total angular momentum C.

[7] The variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$, in such "planetary form," have been rediscovered by the author during her PhD. Note that, apart from a few cases [21, 14] of application to the three-body problem, where they reduce to the variables of Jacobi reduction, Deprit variables seem to have remained un-noticed by most. See also [6] for the proof of the symplecticity of $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ found in [23].

- In [9], a construction is shown that allows one to switch to a "full reduction" to $(3n-2)$ degrees of freedom. Such procedure is a bit involved, but allows one, at the end, to reduce completely the number of degrees of freedom and, simultaneously, to deal with only one singularity. It generalizes the analogous singularity of Jacobi variables for $n = 2$, for which the planar configuration is not allowed. Therefore, one has to discard a positive measure set in order to stay away from it. The measure of the invariant set F^*_ε is therefore estimated as $\sim (\varepsilon^{4n-4} - \varepsilon_0^{4n-4})$ with an arbitrary $0 < \varepsilon_0 < \varepsilon$.
- The completely reduced variables that are obtained via the full reduction of the previous item for the $n = 2$ case are analogues of Jacobi's variables (they are not the same) and lead to the same BNF studied in [28]. Differently from what happens for the above discussed case $n = 2$, for $n \geq 3$, the full reduction studied in [9] loses (besides the $\mathcal{R}_g$-symmetry in (10)) also reflection symmetries and hence the elliptic equilibrium. Such equilibrium needs to be restored via an Implicit Function Theorem procedure that is successful in the range of small eccentricities and inclinations.
- From the two previous items one has that, while a "continuity" (letting the inclinations to zero) between $(3n-1)$-dimensional Lagrangian tori of the partially reduced problem in space (whose existence has been discussed in [11, 9]) and $(2n)$-dimensional Lagrangian tori of the unreduced planar problem follows from [9], instead, an analogous continuity between $(3n-2)$-dimensional Lagrangian tori of the fully reduced problem in space (again discussed in [11, 9]) and $(2n-1)$-dimensional Lagrangian tori of the fully planar problem (discussed in [8]) once inclinations go to zero *is naturally expected but, up to now, remains unproved*. Compare also the arguments in [28, 11] on this issue. As mentioned in the previous section, we recall that a controversial (indeed, erroneous) continuity argument between the planar Delaunay coordinates and the spacial planetary coordinates obtained via Jacobi reduction of the nodes was argued by Arnold [2] in order to infer non-degeneracy of BNF of the spacial three-body problem.
- Recall the definitions of F_ε, F^*_ε in Theorem 1.2. In both the cases discussed above (partial and full reduction), the "density" of F^*_ε inside of F_ε, i.e., the ratio

$$\mathrm{d} := \frac{\operatorname{meas} \mathrm{F}^*_\varepsilon}{\operatorname{meas} \mathrm{F}_\varepsilon}$$

goes to 1 as $\varepsilon \to 0$. That is, *one has to keep more and more close to the co-inclined, co-circular configuration in order to encounter more and more tori*. In [5] it has been proved that one can take

$$\mathrm{d} = 1 - \sqrt{\varepsilon}.$$

Note in fact that the perturbative technique which leads to Theorem 1.2 (or to its improvement discussed in [5]) is developed with respect to ε, rather than with respect to the initial parameter μ appearing in (1). This circumstance is an intrinsic consequence of the fact that the tori obtained via Theorem 1.2 bifurcate from the elliptic equilibrium and that, in general, the Birkhoff series (11) diverges.

- In [2], Arnold realized that, in the case of the planar three-body problem, the series (11) is instead convergent (in this case $f_{\rm P}^{\rm av}$ is integrable). This allows him to prove
$$\mathrm{d} = 1 - \chi(\mu)$$
where $\chi(\mu) \to 0$ as $\mu \to 0$. For this particular case, the tori do not bifurcate from the elliptic equilibrium, but a different quasi-integrable structure is exploited in [2] (besides also a different perturbative technique in place of Theorem 1.2). In [24], a slightly weaker result has been proved for the case of the spacial three-body problem and the planar general problem:
$$\mathrm{d} = 1 - \chi(\mu, \alpha)$$
where α denotes the maximum semi-axes ration and $\chi(\mu, \alpha)$ goes to 0 as $(\mu, \alpha) \to 0$. Note that for such cases $f_{\rm P}^{\rm av}$ is not integrable.
- From the astronomical point of view, the investigation mentioned in the two last items is motivated by the fact that, for example, asteroids or trans-Neptunian planets exhibit relatively large inclinations or eccentricities. From the theoretical point of view, the question is to understand whether it is possible to find different quasi-integrable structures in the planetary N-body problem besides the one determined by the elliptic equilibrium.

We prove the following result.

Theorem 2.1. *Assume that the semi-major axes of the planets are suitably spaced; let α denote the maximum of such ratios. If α is small enough and the mass ratio μ is small with respect to some power of α, one can find a number ε_0 and a positive measure set $\mathrm{F}^*_{\alpha,\mu} \subset \mathrm{F} := \mathrm{F}_{\varepsilon_0}$ of Lagrangian, $(3n-2)$-dimensional, Diophantine tori, the density of which in F goes to one as $(\alpha, \mu) \to (0, 0)$. Letting the maximum of the mutual inclinations go to zero, such $(3n-2)$-dimensional tori are closer and closer to Lagrangian, $(2n-1)$-dimensional, Diophantine tori of the corresponding planar problem.*

In the next sections, we provide the main ideas behind the proof of Theorem 2.1, without entering into the (technical) details of the estimate of the density of $\mathrm{F}^*_{\alpha,\mu}$ for length reasons.

3. Tools and sketch of proof

The proof of Theorem 2.1 relies upon four tools.

3.1. **A symmetric reduction of the SO(3)-symmetry.** The first tool is a new set of canonical action-angle coordinates which perform a reduction of the total angular momentum in the $(1+n)$-body problem, and, simultaneously, keep symmetry by reflection and are regular for planar motions. Their definition is as follows.

Let $a^{(i)} \in \mathbb{R}_+$, $P^{(i)} \in \mathbb{R}^3$, with $|P^{(i)}| = 1$, and $e^{(i)}$, denote, respectively, the *semi-major axis*, the *direction of the perihelion* and the eccentricity of the $i^{\rm th}$ instantaneous ellipse $\mathfrak{E}_i$ through $(x^{(i)}, y^{(i)})$; let $\mathcal{A}^{(i)}$, with $0 \le \mathcal{A}^{(i)} \le \mathcal{A}^{(i)}_{\rm tot} = \pi (a^{(i)})^2 \sqrt{1 - (e^{(i)})^2}$, be the area spanned by $x^{(i)}$ on $\mathfrak{E}_i$ with respect to $P^{(i)}$ and $\mathrm{C}^{(i)} = x^{(i)} \times y^{(i)}$ the $i^{\rm th}$ angular momentum. Define the following partial sums
$$\mathrm{S}^{(j)} := \sum_{k=j}^{n} \mathrm{C}^{(k)} \qquad 1 \le j \le n \tag{16}$$

so that $\mathrm{S}^{(1)} := \mathrm{C}$ is the *total angular momentum*, while $\mathrm{S}^{(n)} = \mathrm{C}^{(n)}$. Define, finally, the following n couples of P-nodes, $(\widetilde{\nu}_j, \widetilde{\mathrm{n}}_j)_{1\le j\le n}$

$$\widetilde{\nu}_1 := k^{(3)} \times \mathrm{C}, \quad \widetilde{\mathrm{n}}_j := \mathrm{S}^{(j)} \times P^{(j)}, \quad \widetilde{\nu}_{j+1} := P^{(j)} \times \mathrm{S}^{(j+1)}, \quad \widetilde{\mathrm{n}}_n := P^{(n)} \tag{17}$$

with $1 \le j \le n-1$. Then define the coordinates

$$\mathcal{P} = (\Lambda, \chi, \Theta, \ell, \kappa, \vartheta) \tag{18}$$

where

$$\begin{aligned} &\Lambda = (\Lambda_1, \cdots, \Lambda_n) \in \mathbb{R}^n && \ell = (\ell_1, \cdots, \ell_n) \in \mathbb{T}^n \\ &\chi = (\chi_0, \bar\chi) \in \mathbb{R} \times \mathbb{R}^{n-1} && \kappa = (\kappa_0, \bar\kappa) \in \mathbb{T} \times \mathbb{T}^{n-1} \\ &\Theta = (\Theta_0, \bar\Theta) \in \mathbb{R} \times \mathbb{R}^{n-1} && \vartheta = (\vartheta_0, \bar\vartheta) \in \mathbb{T} \times \mathbb{T}^{n-1} \end{aligned}$$

with

$$\begin{aligned} &\bar\chi = (\chi_1, \cdots, \chi_{n-1}) && \bar\kappa = (\kappa_1, \cdots, \kappa_{n-1}) \\ &\bar\Theta = (\Theta_1, \cdots, \Theta_{n-1}) && \bar\vartheta = (\vartheta_1, \cdots, \vartheta_{n-1}), \end{aligned}$$

via the following formulas.

$$\begin{aligned} &\Theta_{j-1} = \begin{cases} \mathrm{C}_3 := \mathrm{C} \cdot k^{(3)} \\ \mathrm{S}^{(j)} \cdot P^{(j-1)} \end{cases} && \vartheta_{j-1} = \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, \widetilde{\nu}_1) & j=1 \\ \alpha_{P^{(j-1)}}(\widetilde{\mathrm{n}}_{j-1}, \widetilde{\nu}_j) & 2 \le j \le n \end{cases} \\ &\chi_{j-1} := \begin{cases} \mathrm{G} = |\mathrm{S}^{(1)}| \\ |\mathrm{S}^{(j)}| \end{cases} && \kappa_{j-1} := \begin{cases} \mathfrak{g} := \alpha_{\mathrm{S}^{(1)}}(\widetilde{\nu}_1, \widetilde{\mathrm{n}}_1) & j=1 \\ \alpha_{\mathrm{S}^{(j)}}(\widetilde{\nu}_j, \widetilde{\mathrm{n}}_j) & 2 \le j \le n \end{cases} \\ &\Lambda_i := \mathfrak{M}_i \sqrt{\mathfrak{m}_i a^{(i)}} && \ell_i := 2\pi \frac{\mathcal{A}^{(i)}}{\mathcal{A}^{(i)}_{\rm tot}} := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i \end{aligned} \tag{19}$$

Note the following.

- The variables (18) are very different from the planetary Deprit variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ mentioned in the previous section. For example, they do not provide the Jacobi reduction of the nodes when $n = 2$. Indeed, the definition of (18) is based on $2n$ nodes (17), the nodes between the mutual planes orthogonal to $\mathrm{S}^{(j)}$ and $P^{(j)}$ and $P^{(j)}$ and $\mathrm{S}^{(j+1)}$. Deprit's reduction is instead based on n nodes, the nodes among the planes orthogonal to the $\mathrm{S}^{(j)}$'s. Let us incidentally mention that, for the three-body case ($n = 2$), the variables (19) are trickily related to certain canonical variables introduced in §2.2 of [24]. This relation will be explained elsewhere.
- While, in the case of the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$, inclinations among the $\mathrm{S}^{(j)}$'s cannot be zero, it is not so for the variables (19), where the planar configuration can be reached with regularity. And in fact, in the planar case, the change between planar Delaunay variables $(\Lambda, \Gamma, \ell, g)$ and the planar version $(\Lambda, \chi, \ell, \kappa)$ of (19) reduces to

  $$\begin{cases} \Lambda = \Lambda \\ \ell = \ell \end{cases} \qquad \begin{cases} \chi_{i-1} = \sum_{j=i}^n \Gamma_n \\ \kappa_{i-1} = g_i - g_{i-1} \end{cases} \qquad 1 \le i \le n$$

  with $g_0 \equiv 0$. Note incidentally that the variables (19) are instead singular in correspondence to the vanishing of the inclinations about $P^{(j)}$ and $\mathrm{S}^{(j)}$ or $\mathrm{S}^{(j+1)}$ and $P^{(j)}$; configurations with no physical meaning.
- The variables (18) have in common with the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ and the Delaunay variables $(\Lambda, \Gamma, \mathrm{H}, \ell, g, \mathrm{h})$ the fact of being singular for zero eccentricities (since in this case the perihelia are not defined). We however

give up any attempt to regularize such vanishing eccentricities. The reason is that the Euclidean lengths of the $\mathrm{C}^{(j)}$'s are not[8] actions (apart from $\chi_{n-1} = |\mathrm{C}^{(n)}|$) and hence the regularization does not seem[9] to be (if existing) easy. Note that, since we are interested in high eccentricities motions, we shall have to stay away from these singularities.

- Another remarkable property of the variables (18), besides the one of being regular for zero inclinations, is that they retain the symmetry by reflections, as explained in Proposition 3.1 below. This does not happen for the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$. As we shall explain better in the next section, such symmetry property plays a role in order to highlight a global[10] quasi-integrable structure of H_{χ_0} in (20) below and, especially, to have an explicit expression of it.

Proposition 3.1. *The action-angle coordinates* (18) *are canonical. Moreover, letting* H_{χ_0} *the system* (1) *in these variables,* $(\Theta_0, \vartheta_0, \chi_0)$ *are integrals of motion for* H_{χ_0}*, which so takes the form*

$$\mathrm{H}_{\chi_0} = h_{\mathrm{K}}(\Lambda) + \mu f_{\chi_0}(\Lambda, \bar{\chi}, \bar{\Theta}, \ell, \bar{\kappa}, \bar{\vartheta}). \tag{20}$$

Finally, in such variables, the reflection[11] *transformation*

$$(y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \to (y_1^{(i)}, -y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, -x_2^{(i)}, x_3^{(i)}) \tag{21}$$

is

$$(\Lambda, \chi, \Theta, \ell, \kappa, \vartheta) \to (\Lambda, \chi, -\Theta, \ell, \kappa, 2\pi\mathbb{Z}^n - \vartheta).$$

Therefore, any of the points

$$(\Theta, \vartheta) = (0, \pi k) \quad k \in \{0,1\}^n \qquad \vartheta \mod 2\pi\mathbb{Z}^n \tag{22}$$

which represent a[12] *planar configuration, is an equilibrium point for the function* $(\bar{\Theta}, \bar{\vartheta}) \to f_{\chi_0}(\Lambda, \bar{\chi}, \bar{\Theta}, \ell, \bar{\chi}, \bar{\vartheta})$.

3.2. **An integrability property.** The second tool is an integrability property of the planetary system. To describe it, we generalize the situation some, introducing the concept of a *Kepler map*.

Given $2n$ positive "mass parameters" $\mathfrak{m}_1, \ldots, \mathfrak{m}_n, \mathfrak{M}_1, \ldots, \mathfrak{M}_n$, a set $\mathfrak{X} \subset \mathbb{R}^{5n}$ and a bijection

$$\begin{aligned} \tau : \mathfrak{X} &\to \big\{(\mathfrak{E}_1, \cdots, \mathfrak{E}_n) \in (E^3)^n, \mathfrak{E}_i : \text{ellipse}\big\} \\ \mathrm{X} \in \mathfrak{X} &\to \big(\mathfrak{E}_1(\mathrm{X}), \cdots, \mathfrak{E}_n(\mathrm{X})\big) \end{aligned}$$

which assigns to any $\mathrm{X} \in \mathfrak{X}$ a n-plet of ellipses $(\mathfrak{E}_1, \cdots, \mathfrak{E}_n)$ in the Euclidean space E^3 with strictly positive eccentricities and having a common focus S, we shall say that an injective map

$$\phi : (\mathrm{X}, \ell) \in \mathcal{D}^{6n} := \mathfrak{X} \times \mathbb{T}^n \to (y_\phi(\mathrm{X}, \ell), x_\phi(\mathrm{X}, \ell)) \in (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n$$

---

[8] Indeed, for $1 \le j \le n-1$, $|\mathrm{C}^{(j)}|^2 = \chi_{j-1}^2 + \chi_j^2 - 2\Theta_j^2 + 2\sqrt{(\chi_j^2 - \Theta_j^2)(\chi_{j-1}^2 - \Theta_j^2)}\cos\vartheta_j$.

[9] Recall that $e^{(j)} = 0$ corresponds to $|\mathrm{C}^{(j)}| = \Lambda_j$.

[10] With a remainder independent of eccentricities and inclinations; compare Proposition 3.3.

[11] Note that the reflection in (21) is slightly different from $\mathcal{R}_2^-$ in (8). This is not important, since indeed in (6) the signs s_i and r_i may be chosen independently.

[12] Depending on the signs of the cosines of the mutual inclinations, there are 2^{n-1} planar configurations. The one with all the $\mathrm{C}^{(i)}$ parallel and in the same verse corresponds, in the variables (19), to $(\Theta, \vartheta) = \big((0, \cdots, 0), (\pi, \cdots, \pi)\big)$.

is a *Kepler map* if ϕ associates to $(\mathrm{X},\ell)\in\mathfrak{X}\times\mathbb{T}^n$, with $\ell=(\ell_1,\cdots,\ell_n)$ (*mean anomalies*) an element

$$(y_\phi(\mathrm{X},\ell),x_\phi(\mathrm{X},\ell))=(y_\phi^{(1)}(\mathrm{X},\ell_1),\cdots,y_\phi^{(n)}(\mathrm{X},\ell_n),x_\phi^{(1)}(\mathrm{X},\ell_1),\cdots,x_\phi^{(n)}(\mathrm{X},\ell_n))$$

in the following way. Letting, respectively, $P_\phi^{(i)}(\mathrm{X})$, $a_\phi^{(i)}(\mathrm{X})$, $e_\phi^{(i)}(\mathrm{X})$, and $N_\phi^{(i)}(\mathrm{X})$ be the direction from S to the perihelion, the semi-major axis, the eccentricity and a prefixed direction of the plane of $\mathfrak{E}_i(\mathrm{X})$, $x_\phi^{(i)}(\mathrm{X},\ell_i)$ are the coordinates with respect to a prefixed orthonormal frame (i,j,k) centered in S of the point of $\mathfrak{E}_i(\mathrm{X})$ such that $\frac12 a_\phi^{(i)}\sqrt{1-(e_\phi^{(i)})^2}\ell_i$ (mod $\pi a_\phi^{(i)}\sqrt{1-(e_\phi^{(i)})^2}$) is the area spanned from $P_\phi^{(i)}(\mathrm{X})$ to $x_\phi^{(i)}(\mathrm{X},\ell_i)$ relatively to the positive (counterclockwise) orientation determined by $N_\phi^{(i)}(\mathrm{X})$ and

$$y_\phi^{(i)}(\mathrm{X},\ell_i)=\mathfrak{m}_i\sqrt{\frac{\mathfrak{M}_i}{(a^{(i)})^3}}\partial_{\ell_i}x_\phi^{(i)}(\mathrm{X},\ell_i). \tag{23}$$

A Kepler map will be called *canonical* if any $\mathrm{X}\in\mathfrak{X}$ has the form $\mathrm{X}=(\mathrm{P},\mathrm{Q},\Lambda)$ where $\Lambda=(\Lambda_1,\cdots,\Lambda_n)=(\mathfrak{m}_1\sqrt{\mathfrak{M}_1 a_\phi^{(1)}},\cdots,\mathfrak{m}_n\sqrt{\mathfrak{M}_n a_\phi^{(n)}})$, $\mathrm{P}=(\mathrm{P}_1,\cdots,\mathrm{P}_{2n})$, $\mathrm{Q}=(\mathrm{Q}_1,\cdots,\mathrm{Q}_{2n})$ and the map

$$(\Lambda,\ell,\mathrm{P},\mathrm{Q})\to(y,x)=(y^{(1)},\cdots,y^{(n)},x^{(1)},\cdots,x^{(n)})$$

preserves the standard 2-form:

$$\sum_{i=1}^n d\Lambda_i\wedge d\ell_i+\sum_{i=1}^{2n}d\mathrm{P}_i\wedge d\mathrm{Q}_i=\sum_{i=1}^n\sum_{j=1}^3 dy_j^{(i)}\wedge dx_j^{(i)}.$$

Examples of canonical Kepler maps include

(a) The map ϕ_{Del} which defines the Delaunay variables $(\Lambda,\Gamma,\mathrm{H},\ell,g,\mathrm{h})$;
(b) The map ϕ_{Dep} which defines the planetary Deprit variables $(\Lambda,\Gamma,\Psi,\ell,\gamma,\psi)$;
(c) The map $\phi_{\mathcal{P}}$ which defines the variables $\mathcal{P}=(\Lambda,\chi,\Theta,\ell,\kappa,\vartheta)$ in (19).

The following classical relations then hold for (not necessarily canonical) Kepler maps.

$$\frac{1}{2\pi}\int_{\mathbb{T}}\frac{d\ell_i}{|x_\phi^{(i)}|}=\frac{1}{a_\phi^{(i)}},\qquad \frac{1}{2\pi}\int_{\mathbb{T}}y_\phi^{(i)}d\ell_i=0,\quad \frac{1}{2\pi}\int_{\mathbb{T}}\frac{x_\phi^{(i)}}{|x_\phi^{(i)}|^3}d\ell_i=0. \tag{24}$$

Given a canonical Kepler map ϕ, put $\mathrm{H}_\phi:=\mathrm{H}_{\mathrm{hel}}\circ\phi$, where $\mathrm{H}_{\mathrm{hel}}$ is as in (1). Then

$$\mathrm{H}_\phi=h_{\mathrm{K}}(\Lambda_1,\cdots,\Lambda_n)+\mu\, f_\phi(\mathrm{X},\ell_1,\cdots,\ell_n)$$

where h_{K} is as in (4) and

$$f_\phi(\mathrm{X},\ell_1,\cdots,\ell_n):=\sum_{1\le i<j\le n}\Big(\frac{y_\phi^{(i)}(\mathrm{X},\ell_i)\cdot y_\phi^{(j)}(\mathrm{X},\ell_j)}{m_0}-\frac{m_im_j}{|x_\phi^{(i)}(\mathrm{X},\ell_i)-x_\phi^{(j)}(\mathrm{X},\ell_j)|}\Big)$$

is the perturbing function (1) expressed in the variables $(\Lambda,\ell,\mathrm{P},\mathrm{Q})$. Imposing a suitable restriction of the the domain so as to exclude *orbit collision*, one has that the *secular ϕ-perturbing function*, i.e., the average

$$(f_\phi)_{\mathrm{av}}(\mathrm{X}):=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}f_\phi(\mathrm{X},\ell_1,\cdots,\ell_n)d\ell_1\cdots d\ell_n$$

is well defined. Due to (23), the "indirect" part of the perturbing function, i.e., the term $\sum_{1\le i<j\le n} y_\phi^{(i)}(\mathrm{X},\ell_i)\cdot y_\phi^{(j)}(\mathrm{X},\ell_j)/m_0$ has zero average and hence $(f_\phi)_{\mathrm{av}}$ is just the average of the Newtonian (or "direct") part:

$$(f_\phi)_{\mathrm{av}}(\mathrm{X}) = \sum_{1\le i<j\le n} (f_\phi^{(ij)})_{\mathrm{av}}$$

with

$$(f_\phi^{(ij)})_{\mathrm{av}} := -\frac{m_i m_j}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{d\ell_i d\ell_j}{|x_\phi^{(i)}(\mathrm{X},\ell_i)-x_\phi^{(j)}(\mathrm{X},\ell_j)|} \quad i<j.$$

If we consider the expansion

$$(f_\phi^{(ij)})_{\mathrm{av}} = (f_\phi^{(ij)})_{\mathrm{av}}^{(0)} + (f_\phi^{(ij)})_{\mathrm{av}}^{(1)} + (f_\phi^{(ij)})_{\mathrm{av}}^{(2)} + \cdots$$

where

$$(f_\phi^{(ij)})_{\mathrm{av}}^{(k)}(\mathrm{X}) := -\frac{m_i m_j}{(2\pi)^2}\frac{1}{k!}\frac{d^k}{d\varepsilon^k}\int_{\mathbb{T}^2}\frac{d\ell_i d\ell_j}{|\varepsilon\, x_\phi^{(i)}(\mathrm{X},\ell_i)-x_\phi^{(j)}(\mathrm{X},\ell_j)|}\Big|_{\varepsilon=0},$$

we have that, in this expansion, the two first terms depend only on Λ_j. More precisely, due to (24),

$$(f_\phi^{(ij)})_{\mathrm{av}}^{(0)} = -\frac{m_i m_j}{a^{(j)}}, \quad (f_\phi^{(ij)})_{\mathrm{av}}^{(1)} = 0.$$

Therefore, the term $(f_\phi^{(ij)})_{\mathrm{av}}^{(2)}$ carries the first non-trivial information. In the case of the map $\phi=\phi_{\mathcal{P}}$, we have the following.

Proposition 3.2.

(i) *The functions* $(f_{\phi_{\mathcal{P}}}^{(ij)})_{\mathrm{av}}^{(2)}$ *(more generally,* $(f_{\phi_{\mathcal{P}}}^{(ij)})_{\mathrm{av}}$*) depend only on* Λ_i, Λ_j, $\Theta_i,\ldots,\Theta_{j\wedge(n-1)}$, $\chi_{i-1},\ldots,\chi_{j\wedge(n-1)}$, $\kappa_i,\ldots,\kappa_{j-1}$, $\vartheta_i,\ldots,\vartheta_{j\wedge(n-1)}$ *where* $a\wedge b$ *denotes the minimum of* a *and* b.

(ii) *In particular, for any* $1\le i\le n-1$, *the nearest-neighbor terms* $(f_{\phi_{\mathcal{P}}}^{(i,i+1)})_{\mathrm{av}}^{(2)}$ *(more generally,* $(f_{\phi_{\mathcal{P}}}^{(i,i+1)})_{\mathrm{av}}$*) depend only on the following.* Λ_i, Λ_{i+1}, χ_{i-1}, χ_i, $\chi_{(i+1)\wedge(n-1)}$, Θ_i, $\Theta_{(i+1)\wedge(n-1)}$, κ_i, ϑ_i, $\vartheta_{i+1\wedge(n-1)}$.

(iii) *The function* $(f_{\phi_{\mathcal{P}}}^{(n-1,n)})_{\mathrm{av}}^{(2)}$ *depends only on* Λ_{n-1}, Λ_n, χ_{n-2}, χ_{n-1}, Θ_{n-1}, *and* ϑ_{n-1}, *while it does not depend on* κ_{n-1}. *Then it is integrable.*

(iv) $(f_{\phi_{\mathcal{P}}}^{(n-1,n)})_{\mathrm{av}}^{(2)}$ *is integrable in the Arnold-Liouville sense: there exists a suitable global neighborhood* B^2 *of* $0\in\mathbb{R}^2$ *(where* 0 *corresponds to* $\mathrm{C}^{(\nu-1)}\parallel\mathrm{C}^{(n)}$*), a set* $A\subset\mathbb{R}^4$ *and a real-analytic, canonical change of coordinates*

$$\phi_1:\Big((\Lambda_{n-1},\Lambda_n,\chi_{n-2},\chi_{n-1}),(\tilde\ell_{n-1},\tilde\ell_n,\tilde\kappa_{n-2},\tilde\kappa_{n-1}),(p_{n-1},q_{n-1})\Big)$$
$$\to\Big((\Lambda_{n-1},\Lambda_n,\chi_{n-2},\chi_{n-1}),(\ell_{n-1},\ell_n,\kappa_{n-2},\kappa_{n-1}),(\Theta_{n-1},\vartheta_{n-1})\Big)$$

defined on $A\times\mathbb{T}^4\times B^2$ *which transforms* $(f_{\phi_{\mathcal{P}}}^{(n-1,n)})_{\mathrm{av}}^{(2)}$ *into a function* $h_{\chi_0}^{(2n+1)}$ *depending only on* Λ_{n-1}, Λ_n, χ_{n-2}, χ_{n-1}, $\frac{p_{n-1}^2+q_{n-1}^2}{2}$.

Note the following.

- The main point of Proposition 3.2 is that the action $\chi_{n-1} = |\mathrm{C}^{(n)}|$ is an integral for $(f_{\phi_{\mathcal{P}}}^{(n-1,n)})_{\rm av}^{(2)}$. Clearly, this is general: whatever ϕ is, $|\mathrm{C}^{(j)}|$ is an integral for $(f_{\phi}^{(ij)})_{\rm av}^{(2)}$. This fact has been observed firstly, for the case of the three-body problem, in [15], using Jacobi reduction of the nodes. In that, case Harrington observed that $(f_{\phi_{\rm Jac}}^{(12)})_{\rm av}^{(2)}$ depends only on $(\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, \mathrm{G}, \gamma_1)$ and that the integrability is exhibited via the couple (Γ_1, γ_1). As we already observed, in such case the planetary Deprit variables and the variables obtained by Jacobi reduction of the nodes are the same.
- An important issue that is used in the proof of Theorem 2.1 (precisely, in order to check certain non-degeneracy assumptions involved in Theorem 3.1 below) is the *effective integration* of $(f_{\phi_{\mathcal{P}}}^{(n-1,n)})_{\rm av}^{(2)}$. Clearly, in principle, this could be achieved using any of the sets of variables mentioned in the two previous items: planetary Deprit variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ or the variables (18). However, the integration using planetary Deprit variables carries considerable analytic difficulties and has been performed only qualitatively [21, 14]. Using the variables (18), such integration can be achieved by a suitable convergent Birkhoff series, exploiting the equilibrium points in (22). Compare also Proposition 3.3 and the comments below.

3.3. **Global quasi-integrability of the planetary system.** The third tool is the following.

Proposition 3.3. *There exist natural numbers m, $\nu_1, \ldots, \nu_m$ with $\nu_1 + \cdots + \nu_m = 3n-2 > m$ and a positive real number s such that, if the semi-major axes of the planets are suitably spaced and the maximum semi-axes ratio α is sufficiently small, for any positive number $\bar K$ sufficiently small with respect to some positive power of α^{-1} and any μ small with respect to some power of α, one can find a number $\rho(\alpha, \bar K)$ which goes to 0 as a power law with respect to α and $\frac{1}{\bar K}$, positive numbers $\gamma_1, \ldots, \gamma_m$ depending only on α and μ, a domain $\mathrm{D} \subset \mathbb{R}^{3n-2}$, a global neighborhood $B^{2(n-1)}$ of $0 \in \mathbb{R}^{2(n-1)}$, and, if $C \subset \mathbb{R}^{2n-1}$ is as in Theorem 3.1 with $\nu = 3n-2$ and $\ell = n-1$, a real-analytic and symplectic transformation*

$$\big((\hat\Lambda, \hat\chi), (\hat\ell, \hat\kappa), (\hat p, \hat q)\big) \in C_\rho \times \mathbb{T}_s^{2n-1} \times B_{\sqrt{2\rho}}^{2(n-1)} \to \big((\Lambda, \bar\chi), (\ell, \bar\kappa), (\bar\Theta, \bar\theta)\big),$$

which conjugates the Hamiltonian in (20) to

$$\hat{\mathrm{H}}_{\chi_0} = \hat h_{\chi_0}(\hat\Lambda, \hat\chi, \hat p, \hat q) + \mu \hat f_{\chi_0}(\hat\Lambda, \hat\chi, \hat\ell, \hat\kappa, \hat p, \hat q) \tag{25}$$

where $\hat h_{\chi_0}(\hat\Lambda, \hat\chi, \hat p, \hat q)$ depends on $(\hat p_i, \hat q_i)_{1 \le i \le n-1}$ only via

$$\hat J(\hat p, \hat q) := \Big(\tfrac{\hat p_1^2 + \hat q_1^2}{2}, \ldots, \tfrac{\hat p_{n-1}^2 + \hat q_{n-1}^2}{2}\Big)$$

and letting ω be the gradient of $\hat h_{\chi_0}$ with respect to $(\hat\Lambda, \hat\chi, \hat J)$, then

$$\mathrm{D} \supseteq \omega^{-1}(\mathcal{D}_{\gamma_1, \cdots, \gamma_m, \tau}^{\bar K, 3n-2}) \supset \emptyset.$$

Finally, the following holds. If L, E, $\hat\rho$ are as in Theorem 3.1, then one can take $\hat\rho = \rho$, $L = L_0(\alpha)/\mu$, $E = \mu E_0(\alpha) e^{-\bar K s}$, where $L_0(\alpha)$, $E_0(\alpha)$ do not exceed some power of α^{-1}.

Here are some comments of the proof of Proposition 3.3.

- The function $\hat h_{\chi_0}$ is a sum
$$\hat h_{\chi_0} = \sum_{i=1}^{2n-1} \hat h_{\chi_0}^{(i)} \tag{26}$$
where
$$\hat h_{\chi_0}^{(1)}, \cdots, \hat h_{\chi_0}^{(n)}$$
are close to the respective Keplerian terms
$$h_{\rm K}^{(1)}, \cdots, h_{\rm K}^{(n)}$$
in (2), while
$$\hat h_{\chi_0}^{(n+1)}, \cdots, \hat h_{\chi_0}^{(2n+1)}$$
are as follows. $\hat h_{\chi_0}^{(2n+1)}$ is close to the function $\mu h_{\chi_0}^{(2n-1)}$, where $h_{\chi_0}^{(2n-1)}$ is defined in the last item of Proposition 3.3. For $n \geq 3$ and $2n-2 \geq i \geq n+1$, inductively, $\hat h_{\chi_0}^{(i)}$ is as follows. Consider the "projection[13] over normal modes" of $(f_{\phi_{\mathcal P}}^{(i-n,i-n+1)})^{(2)}_{\rm av} \circ \phi_1 \circ \cdots \circ \phi_{2n-1-i}$ with respect to the variables (p_j, q_j) with $j \geq i-n+1$ and $(\chi_i, \tilde\kappa_i)$ with $i \geq i-n$. This is a function of
$$\Lambda_{i-n}, \cdots, \Lambda_n, \chi_{i-n-1}, \cdots, \chi_{n-1}, \Theta_{i-n}, \vartheta_{i-n}, \frac{p_{i-n+1}^2 + q_{i-n+1}^2}{2}, \cdots, \frac{p_{n-1}^2+q_{n-1}^2}{2}$$
and is integrable in the sense of Liouville-Arnold: there exists ϕ_{2n-i} which lets this projection into a function $h_{\chi_0}^{(i)}$ of
$$\Lambda_{i-n}, \cdots, \Lambda_n, \chi_{i-n-1}, \cdots, \chi_{n-1}, \frac{p_{i-n}^2+q_{i-n}^2}{2}, \cdots, \frac{p_{n-1}^2+q_{n-1}^2}{2}.$$
Then $\hat h_{\chi_0}^{(i)}$ is close to $\mu h_{\chi_0}^{(i)}$.
- The exponential decay of E with respect to $\bar K$ follows from a suitable averaging technique derived from [26], carefully adapted to our case.
- The functions in (26) are of different strengths, with respect to the mass parameter μ and the semi-mjor axes ratios $\alpha_i := \frac{a^{(i)}}{a^{(i+1)}}$. The first n ones, which are, as said, close to be Keplerian, are of order
$$\sim \frac{1}{a^{(1)}}, \cdots, \frac{1}{a^{(n)}}.$$
The remaining $(n-1)$ ones are much smaller:
$$\sim \mu\frac{(a^{(1)})^2}{(a^{(2)})^3}, \cdots, \mu\frac{(a^{(n-1)})^2}{(a^{(n)})^3}$$
(they have the strength of $\mu(f_{\phi_{\mathcal P}}^{(1,2)})^{(2)}_{\rm av}, \ldots, \mu(f_{\phi_{\mathcal P}}^{(n-1,n)})^{(2)}_{\rm av}$, which are so). Therefore, in order to apply a KAM scheme to the Hamiltonian (25), we need a formulation suitably adapted to this case. This is given in the following section.

[13] By "projection over normal modes" of a given function
$$f(I,\varphi,p,q) = \sum_{(a,b)\in\mathbb N^m\times\mathbb N^m\, \kappa\in\mathbb Z^n} f_k(I) e^{{\rm k}k\cdot\varphi} \prod_{i=1}^m \left(\frac{u_i - {\rm i}v_i}{\sqrt2}\right)^{a_i} \left(\frac{u_i+{\rm i}v_i}{{\rm i}\sqrt2}\right)^{b_i},$$
we mean the function $\sum_{a\in\mathbb N^m} f_0(I)\prod_{i=1}^m(\frac{u_i^2+v_i^2}{2{\rm i}})^{a_i}$.

3.4. **Multi-scale KAM theory.** The fourth tool is a multi-scale KAM Theorem. To quote it, let us fix the following notations.

Given m, $\nu_1, \ldots, \nu_m \in \mathbb{N}$, $\nu := \nu_1 + \cdots + \nu_m$, let us decompose

$$\mathbb{Z}^\nu \setminus \{0\} = \bigcup_{i=1}^m \mathfrak{L}_i \setminus \mathfrak{L}_{i-1}$$

where

$$\mathbb{Z}^\nu =: \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_m = \{0\}$$

is a decreasing sequence of sub-lattices defined by

$$\mathfrak{L}_i := \big\{k = (k_1, \cdots, k_m) \in \mathbb{Z}^\nu = \mathbb{Z}^{\nu_1} \times \cdots \times \mathbb{Z}^{\nu_m} \ : \ k_1 = \cdots = k_i = 0\big\}.$$

Next, given $\gamma, \gamma_1, \ldots, \gamma_m$, $\tau \in \mathbb{R}_+$, define the "multi-scale Diophantine" number sets

$$\begin{aligned}
\mathcal{D}^{\nu,K,i}_{\gamma;\tau} &:= \Big\{\omega \in \mathbb{R}^\nu : |\omega \cdot k| \geq \frac{\gamma}{|k|^\tau} \quad \forall k \in \mathfrak{L}_{i-1} \setminus \mathfrak{L}_i, |k|_1 \leq K\Big\}\\
\mathcal{D}^{\nu,K}_{\gamma_1\cdots\gamma_m;\tau} &:= \bigcap_{i=1}^m \mathcal{D}^{\nu,K,i}_{\gamma_i;\tau}\\
\mathcal{D}^{\nu}_{\gamma_1\cdots\gamma_m;\tau} &:= \bigcap_{K\in\mathbb{N}} \mathcal{D}^{\nu,K}_{\gamma_1\cdots\gamma_m;\tau}.
\end{aligned}$$

Explicitly, a number $\omega = (\omega_1, \cdots, \omega_m) \in \mathbb{R}^\nu = \mathbb{R}^{\nu_1} \times \cdots \times \mathbb{R}^{\nu_m}$ belongs to $\mathcal{D}^\nu_{\gamma_1\cdots\gamma_m;\tau}$ if, for any $k = (k_1, \cdots, k_m) \in \mathbb{Z}^{\nu_1} \times \cdots \times \mathbb{Z}^{\nu_m} \setminus \{0\}$,

$$\Big|\sum_{j=1}^m \omega_j \cdot k_j\Big| \geq \begin{cases} \dfrac{\gamma_1}{|k|^\tau} & \text{if} \quad k_1 \neq 0;\\ \dfrac{\gamma_2}{|k|^\tau} & \text{if} \quad k_1 = 0, \quad k_2 \neq 0;\\ \cdots & \\ \dfrac{\gamma_m}{|k_m|^\tau} & \text{if} \quad k_1 = \cdots = k_{m-1} = 0, \quad k_m \neq 0.\end{cases}$$

Note that the choice $m = 1$ gives the usual Diophantine set $\mathcal{D}^\nu_{\gamma_1,\tau}$. The $m = 2$ case, with $\gamma_1 = \mathrm{O}(1)$ and $\gamma_2 = \mathrm{O}(\mu)$, has been considered in [2] (and [5]) for the proof of Theorem 1.2.

Theorem 3.1 (Multi-scale KAM Theorem)**.** *Let m, ℓ, $\nu_1, \ldots, \nu_m \in \mathbb{N}$, with $\nu := \nu_1 + \cdots + \nu_m \geq \ell$, $\tau_* > \nu$, $\gamma_1 \geq \cdots \geq \gamma_m > 0$, $0 < 4s \leq \bar{s} < 1$, $\rho > 0$, $D \subset \mathbb{R}^{\nu-\ell} \times \mathbb{R}^\ell$, $A := D_\rho$, $B^{2\ell}$ a neighborhood (with possibly different radii) of $0 \in \mathbb{R}^{2\ell}$ such that, if $\bar{I}(u,v) := (\frac{u_1^2+v_1^2}{2}, \cdots, \frac{u_\ell^2+v_\ell^2}{2})$, then $\Pi_{\mathbb{R}^\ell}\mathrm{D} = \bar{I}(B^{2\ell})$, $C := \Pi_{\mathbb{R}^{\nu-\ell}} D$ and let*

$$\mathrm{H}(I, \varphi, u, v) = \mathrm{h}(I, u, v) + \mathrm{f}(I, \varphi, u, v)$$

be real-analytic on $C_\rho \times \mathbb{T}^{\nu-\ell}_{\bar{s}+s} \times B^{2\ell}_{\sqrt{2\rho}}$, where h depends on (u, v) only via $\bar{I}(u, v)$. Assume that $\omega_0 := \partial_{(I,\bar{I})}\mathrm{h}$ is a diffeomorphism of A with non singular Hessian matrix $U := \partial^2_{(I,\bar{I})}\mathrm{h}$ and let U_k denote the $(\nu_k + \cdots + \nu_m) \times \nu$ submatrix of U, i.e., the matrix with entries $(U_k)_{ij} = U_{ij}$, for $\nu_1 + \cdots + \nu_{k-1} + 1 \leq i \leq \nu$, $1 \leq j \leq \nu$, *where* $2 \leq k \leq m$. *Let*

$$\mathrm{M} \geq \sup_A \|U\|, \quad \mathrm{M}_k \geq \sup_A \|U_k\|, \quad \bar{\mathrm{M}} \geq \sup_A \|U^{-1}\|, \quad E \geq \|\mathrm{f}\|_{\rho,\bar{\mathfrak{s}}+s}$$

$$\bar{\mathrm{M}}_k \geq \sup_A \|T_k\| \quad \text{if} \quad U^{-1} = \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix} \qquad 1 \leq k \leq m.$$

Define

$$K := \frac{6}{s} \log_+ \left(\frac{E\mathrm{M}_1^2 L}{\gamma_1^2} \right)^{-1} \quad \textit{where} \quad \log_+ a := \max\{1, \log a\}$$

$$\hat{\rho}_k := \frac{\gamma_k}{3\mathrm{M}_k K^{\tau_*+1}}, \quad \hat{\rho} := \min\{\hat{\rho}_1, \cdots, \hat{\rho}_m, \rho\}$$

$$L := \max\left\{\bar{\mathrm{M}}, \mathrm{M}_1^{-1}, \cdots, \mathrm{M}_m^{-1}\right\}$$

$$\hat{E} := \frac{EL}{\hat{\rho}^2}$$

Then one can find two numbers $\hat{c}_\nu > c_\nu$ *depending only on* ν *such that, if the perturbation* f *is so small that the following "KAM condition" holds*

$$\hat{c}_\nu \hat{E} < 1,$$

then, for any $\omega \in \Omega_* := \omega_0(D) \cap \mathcal{D}^\nu_{\gamma_1,\cdots,\gamma_m,\tau_*}$, *one can find a unique real-analytic embedding*

$$\phi_\omega : \vartheta = (\hat{\vartheta}, \bar{\vartheta}) \in \mathbb{T}^\nu \to (\hat{v}(\vartheta;\omega), \hat{\vartheta} + \hat{u}(\vartheta;\omega), \mathcal{R}_{\bar{\vartheta}+\bar{u}(\vartheta;\omega)} w_1, \cdots, \mathcal{R}_{\bar{\vartheta}+\bar{u}(\vartheta;\omega)} w_\ell)$$
$$\in \operatorname{Re} C_r \times \mathbb{T}^{\nu-\ell} \times \operatorname{Re} B^{2\ell}_{\sqrt{2r}}$$

where $r := c_\nu \hat{E} \hat{\rho}$ *such that* $\mathrm{T}_\omega := \phi_\omega(\mathbb{T}^\nu)$ *is a real-analytic and* ν*-dimensional* H*-invariant torus, on which the* H*-flow is analytically conjugated to* $\vartheta \to \vartheta + \omega t$.

Theorem 3.1 is essentially Proposition 3 of [5] suitably adapted to our case. Applying Theorem 3.1 to the Hamiltonian (25) (with $I := (\hat{\Lambda}, \hat{\chi})$, $\varphi := (\hat{\ell}, \hat{\kappa})$, $(u, v) := (\hat{p}, \hat{q})$, $\nu = 3n-2$, $\ell = n-1$, $m, \nu_1, \ldots, \nu_m$ as in Proposition 3.3) gives the proof of Theorem 2.1. More details will be published elsewhere, [25]. $\square$

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