Moving energies as first integrals of nonholonomic systems with affine constraints

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Abstract

In nonholonomic mechanical systems with constraints that are affine (linear nonhomogeneous) functions of the velocities, the energy is typically not a first integral. It was shown in [15] that, nevertheless, there exist modifications of the energy, called there moving energies, which under suitable conditions are first integrals. The first goal of this paper is to study the properties of these functions and the conditions that lead to their conservation. In particular, we enlarge the class of moving energies considered in [15]. The second goal of the paper is to demonstrate the relevance of moving energies in nonholonomic mechanics. We show that certain first integrals of some well known systems (the affine Veselova and LR systems), which had been detected on a case-by-case way, are instances of moving energies. Moreover, we determine conserved moving energies for a class of affine systems on Lie groups that include the LR systems, for the heavy convex rigid body that rolls without slipping on a uniformly rotating plane, and for an n-dimensional generalization of the Chaplygin sphere on a uniformly rotating hyperplane.

Keywords: Moving energies · Nonholonomic mechanical systems · Affine constraints · Nonhomogeneous constraints · Conservation of energy · LR systems · Rolling rigid bodies · Veselova system

MSC: 70F25, 37J60, 37J15, 70E18

1 Introduction

Conservation of energy in time-independent mechanical systems with nonholonomic constraints is an important feature that has received extended consideration. It is well known that the energy is conserved if the nonholonomic constraints are linear—or more generally homogeneous—functions of the velocities [29, 26] and that this typically does not happen if the constraints are arbitrary nonlinear functions of the velocities (see e.g. [3, 25, 24]). The situation is better understood in the case of systems with nonholonomic constraints that are *affine* (namely, linear non-homogeneous) functions of the velocities—a case which is important in mechanics and is the one that we consider in this paper.

The conditions under which the energy is conserved in nonholonomic systems with affine constraints have been clarified in [14], and are very special. However, it was noticed in [15] that, in such systems, when the energy is not conserved, there may exist modifications of it which are conserved. Such functions were called *moving energies* in [15] because they were there constructed by means

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of time-dependent changes of coordinates that transform the nonholonomic system with affine constraints into a nonholonomic system with linear constraints, for which the energy is conserved. In [15], the existence of conserved moving energies was linked to the presence of symmetries, and in such cases the conserved moving energy is the sum of two non-conserved functions: the energy, and the momentum of an infinitesimal generator of the symmetry group, whose flow gives the required time-dependent change of coordinates. Even though examples of conserved moving energies in systems formed by spheres rolling on rotating surfaces were given in [15], it was there left unclear how general and effective this mechanism can be.

The aim of the present paper is to investigate this question. We will in fact extend in a natural way the notion of moving energy, going beyond the relation to a time-dependent change of coordinates. Specifically, we define here a moving energy as the difference between the energy of the system and the momentum of a vector field defined on the configuration manifold of the system. This will allow a clearer, simpler and more general treatment. We will investigate which vector fields produce a conserved moving energy (Proposition 4) and how this relates to the existence of symmetries of the Lagrangian (Corollary 5).[1] We will investigate various properties of these moving energies, including their 'weak-Noetherianity' (Corollary 6) and their non-uniqueness (Propositions 7 and 8). We will also compare this extended notion of moving energy to the one originally given in ref. [15] (Proposition 9).

Next, we will investigate the relevance of these functions in nonholonomic mechanics. To this end, we will first show that certain known first integrals of some important nonholonomic systems with affine constraints are instances of moving energies. Specifically, we consider a class of affine nonholonomic systems on Lie groups which includes the (affine) Veselova system [33, 32] and the more general (affine) LR systems introduced in [32]. Furthermore, we will determine conserved moving energies for other important nonholonomic systems: a convex body that rolls on a rotating plane and the n-dimensional Chaplygin sphere that rolls on a rotating hyperplane.[2]

The importance of conserved moving energies will be illustrated by indicating some dynamical consequences of their existence (Corollary 14) and remarking their usage for the Hamiltonization of reduced systems (see the Remark at the end of section 4.2).

The resulting picture is that the notion of a moving energy is a unifying concept in the study of nonholonomic systems with affine constraints. Our view is that this class of functions—rather than the energy itself—should be considered the primary 'energy-like' first integrals to be considered in these systems.

In section 2 we recall the setting of mechanical systems with nonholonomic affine constraints and review some of their properties, that are needed in the subsequent study. In particular, given the role played by symmetry in the conservation of moving energies, we give there a 'Noether theorem' for nonholonomic systems with affine constraints that extends previous formulations (Proposition 2). In section 3 we introduce the moving energies and study their existence and properties. Sections 4, 5 and 6 are devoted to the aforementioned examples. In the Appendix, we derive the reduced equations of motion for the system treated in Section 5.

Throughout the work, we assume that all objects (functions, manifolds, distributions, etc.) are smooth and that all vector fields are complete.

2 Nonholonomic systems with affine constraints

2.1 The setting. We start with a Lagrangian system with n-dimensional configuration manifold Q and Lagrangian $L : TQ \to \mathbb{R}$, that describes a holonomic mechanical system. We assume

[1] While the writing of this article was almost completed, we were informed of the existence of the very recent article [23]. This article considers moving energies from the point of view of Noether symmetries for time-dependent systems—calling them Noether integrals—and generalizes to this context some of the results of [15]; in particular, it proves a statement analogous to Corollary 5.

[2] As a bonus, we will also determine the reduced equations of motion of these systems.

that the Lagrangian has the mechanical form

$$L = T + b - V \circ \pi\,, \tag{1}$$

where T is a Riemannian metric on Q, b is a 1-form on Q regarded as a function on TQ, V is a function on Q and $\pi : TQ \to Q$ is the tangent bundle projection. We interpret T as the kinetic energy, V as the potential energy of the positional forces that act on the system, and the 1-form b as the generalized potential of the gyrostatic forces that act on the system.

We add now the nonholonomic constraint that, at each point $q \in Q$, the velocities of the system belong to an affine subspace $\mathcal{M}_q$ of the tangent space T_qQ. Specifically, we assume that there are a nonintegrable distribution $\mathcal{D}$ on Q of constant rank r, with $1 < r < n$, and a vector field Z on Q such that, at each point $q \in Q$,

$$\mathcal{M}_q = Z(q) + \mathcal{D}_q\,. \tag{2}$$

Note that the vector field Z is defined up to a section of $\mathcal{D}$. The affine distribution $\mathcal{M}$ with fibers $\mathcal{M}_q$ may also be regarded as a submanifold $M \subset TQ$ of dimension $n + r$. This submanifold is an affine subbundle of TQ of rank r and will be called the *constraint manifold.* The case of linear constraints is recovered when the vector field Z is a section of the distribution $\mathcal{D}$, since then $\mathcal{M} = \mathcal{D}$.

We assume that the nonholonomic constraint is ideal, namely, that it satisfies d'Alembert principle: when the system is in a configuration $q \in Q$, then the set of reaction forces that the nonholonomic constraint is capable of exerting coincides with the annihilator $\mathcal{D}_q^\circ$ of $\mathcal{D}_q$ (see e.g. [28, 25]). Under this hypothesis there is a unique function $R : M \to \mathcal{D}^\circ$, which is interpreted as associating an ideal reaction force $R(v_q)$ to each constrained kinematic state $v_q \in M$, with the property that the equations of motion of the system are given by the restriction to M of Lagrange equations with reaction forces R; for a detailed proof, see [14]. We will denote $(L, Q, \mathcal{M})$ the nonholonomic system determined by these data.

In bundle coordinates $(q, \dot q)$ on TQ the Lagrangian L has the form

$$L(q, \dot q) = \frac{1}{2}\dot q \cdot A(q)\dot q + b(q)\cdot \dot q - V(q) \tag{3}$$

with $A(q)$ an $n \times n$ symmetric nonsingular matrix and $b(q) \in \mathbb{R}^n$. (In order to keep the notation to a minimum we do not distinguish between global objects and their coordinate representatives). Here, and in all expressions written in coordinates, the dot denotes the standard scalar product in $\mathbb{R}^n$. In bundle coordinates, the fibers of the distribution $\mathcal{D}$ can be described as the null spaces of a q-dependent $k \times n$ matrix $S(q)$ that has everywhere rank k, with $k = n - r$:

$$\mathcal{D}_q = \{\dot q \in T_qQ \,:\, S(q)\dot q = 0\}\,.$$

In turn $\mathcal{M}_q = \{\dot q \in T_qQ \,:\, S(q)(\dot q - Z(q)) = 0\}$ and

$$M = \big\{(q, \dot q) \,:\, S(q)\dot q + s(q) = 0\big\}$$

with

$$s(q) = -S(q)Z(q) \in \mathbb{R}^n\,.$$

In coordinates, the equations of motion of the nonholonomic mechanical system $(L, Q, \mathcal{M})$ are

$$\Big(\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}\Big)\Big|_M \;=\; R|_M \tag{4}$$

with

$$R = S^T(SA^{-1}S^T)^{-1}(SA^{-1}\ell - \sigma) \tag{5}$$

where $\ell \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^k$ have components

$$\ell_i = \sum_{j=1}^{n} \frac{\partial^2 L}{\partial \dot q_i \partial q_j} \dot q_j - \frac{\partial L}{\partial q_i}\,, \qquad \sigma_a = \sum_{i,j=1}^{n} \frac{\partial S_{ai}}{\partial q_j} \dot q_i \dot q_j + \sum_{j=1}^{n} \frac{\partial s_a}{\partial q_j} \dot q_j \tag{6}$$

($i = 1, \dots, n$, $a = 1, \dots, k$). For details see [14]; in the case of linear constraints, these or analogue expressions are given in [1, 12, 2]. We note that the restriction of R to M is independent of the arbitrariness that affects the choices of the vector field Z, of the matrix S and of the vector s, see [14].

2.2 The reaction-annihilator distribution. We need to introduce now the so-called reaction-annihilator distribution $\mathcal{R}^\circ$, from [12, 14]. This object plays a central role in the conservation of energy and of moving energies of nonholonomic systems with affine constraints [14, 15] (as well as in the conservation of momenta in nonholonomic systems with either linear or affine constraints [12, 14]).

The observation underlying the consideration of this object is that, while the condition of ideality assumes that, at each point $q \in Q$, the constraint can—a priori—exert all reaction forces that lie in $\mathcal{D}_q^\circ$, expression (5) shows that, ordinarily, only a subset of these possible reaction forces is actually exerted in the motions of the system. Specifically, in bundle coordinates, $\mathcal{D}_q^\circ$ is the orthogonal complement to $\ker S(q)$, namely the range of $S(q)^T$, but the map

$$S^T (SA^{-1}S^T)^{-1}(SA^{-1}\ell - \sigma)\big|_{M_q} : M_q \to \mathrm{range}\big[S(q)^T\big]$$

need not be surjective. Instead, the reaction forces that the constraint exerts, when the system $(L, Q, \mathcal{M})$ is in a configuration $q \in Q$ with any possible velocity in $\mathcal{M}_q$, are the elements of the set

$$\mathcal{R}_q := \bigcup_{v_q \in \mathcal{M}_q} R(v_q)\,,$$

which is a subset of $\mathcal{D}_q^\circ$—and typically a proper subset of it.

The *reaction-annihilator distribution* $\mathcal{R}^\circ$ of the nonholonomic system $(L, Q, \mathcal{M})$ is the distribution on Q whose fiber $\mathcal{R}_q^\circ$ at $q \in Q$ is the annihilator of $\mathcal{R}_q$. In other words, a vector field Y on Q is a section of $\mathcal{R}^\circ$ if and only if, *in all constrained kinematic states* of the the system, the reaction force does no work on it, namely[3]

$$\langle R(v_q), Y(q)\rangle = 0 \qquad \forall v_q \in T_qM,\ q \in Q\,.$$

This is a system-dependent condition, which is weaker than being a section of $\mathcal{D}$ because

$$\mathcal{D}_q \subseteq \mathcal{R}_q^\circ \qquad \forall\, q \in Q\,.$$

For further details and examples on the reaction-annihilator distribution see [12, 9, 10, 22, 14, 23] and for a discussion of its relation to d'Alembert principle see [13].

2.3 Conservation of energy. The *energy* of the nonholonomic system $(L, Q, \mathcal{M})$ is the restriction $E_L|_M$ to the constraint manifold M of the energy

$$E_L := \langle p, \cdot\rangle - L$$

[3]Here and in the sequel, $\langle\ ,\ \rangle$ denotes the cotangent-tangent pairing.

of the Lagrangian system (L,Q). Here $p : TQ \to \mathbb{R}$ is the momentum 1-form generated by the Lagrangian L, regarded as a function on TQ. If the Lagrangian is of the form (1), then in coordinates $p = \frac{\partial L}{\partial \dot q} = A(q)\dot q + b(q)$, and

$$E_L = T + V \circ \pi \,.$$

We note that, in Lagrangian mechanics, the function E_L is variously called energy, generalized energy, Jacobi integral, Jacobi-Painlevè integral (see the discussion in section 1.1 of [15]). We simply call it energy.

As we have already mentioned, it is well known that the energy is always conserved if the constraints are linear in the velocities. For affine constraints, with constraint distribution as in (2), the situation is as follows:

Proposition 1. [14] *The energy of $(L,Q,\mathcal{M}=Z+\mathcal{D})$ is conserved if and only if Z is a section of $\mathcal{R}^\circ$.*

Thus, energy conservation is not a universal property of nonholonomic systems with affine constraints. Instead, it is a system-dependent property. In particular, note that $\mathcal{R}^\circ$ depends on the potentials b and V that enter the Lagrangian, see (6). Therefore, changing the (active) forces that act on the system, within the class of gyrostatic and conservative forces, may destroy or restore the conservation of energy. For some examples of this phenomenon, which includes e.g. a sphere that rolls inside a rotating cylinder, see [14]. An extension of Proposition 1 to a time-dependent setting is given in Corollary 4.2 of [23].

2.4 Conservation of momenta of vector fields. We conclude this short panoramic of nonholonomic systems with affine constraints with some results on the conservation of momenta of vector fields and of lifted actions. Even though we will not strictly need these results in the sequel, they will be useful for appreciating certain aspects of the conservation of moving energies. Moreover, we will introduce here some notation which will be used throughout this work.

Given a nonholonomic system $(L,Q,\mathcal{M})$, we define the *momentum* of a vector field Y on Q as the restriction to M of the function

$$J_Y := \langle p, Y\rangle : TQ \to \mathbb{R}$$

(in coordinates, $J_Y(q,\dot q) = \frac{\partial L}{\partial \dot q}(q,\dot q)\cdot Y(q)$). A geometric characterization of the vector fields whose momenta are first integrals of a nonholonomic system $(L,Q,\mathcal{M})$ with affine constraints does not exist. However, just as in the case of systems with linear constraints, see Proposition 2 of [12], we may characterize those among them which have another property as well.

Here and everywhere in the sequel we denote by Y^{TQ} the tangent lift of a vector field Y on a manifold Q, namely the vector field on TQ which, in bundle coordinates, is given by $Y^{TQ} = \sum_i Y_i \partial_{q_i} + \sum_{ij} \dot q_j \frac{\partial Y_i}{\partial q_j}\partial_{\dot q_i}$.

Proposition 2. *Any two of the following three conditions imply the third:*

i. Y *is a section of* $\mathcal{R}^\circ$.

ii. $\hat Y(L)|_M = 0$.

iii. $J_Y|_M$ *is a first integral of* $(L,Q,\mathcal{M})$.

Proof. We may work in coordinates. It is understood that all functions are evaluated in M, and time derivatives are along the flow of the equations of motion (4). Compute

$$\frac{dJ_Y}{dt} = \sum_i \frac{dp_i}{dt}Y_i + \sum_{ij} p_i \dot q_j \frac{\partial Y_i}{\partial q_j} = \sum_i \big(\frac{\partial L}{\partial q_i} + R_i\big)Y_i + \sum_{ij}\frac{\partial L}{\partial \dot q_i}\dot q_j \frac{\partial Y_i}{\partial q_j} = \hat Y(L) + R\cdot Y \,. \qquad (7)$$

From this it follows that, at each point $q \in Q$, the vanishing in all of $\mathcal{M}_q$ of any two among $\frac{dJ_Y}{dt}$, $R \cdot Y$ and $\hat{Y}(L)$ implies the vanishing in all of $\mathcal{M}_q$ of the third. But the vanishing in all of $\mathcal{M}_q$ of $R \cdot Y$ is equivalent to the fact that Y belongs to the fiber at q of $\mathcal{R}^\circ$. □

Let now Ψ be an action of a Lie group G on Q. For each $g \in G$ we write as usual $\Psi_g(q)$ for $\Psi(g,q)$. The infinitesimal generator relative to an element $\xi \in \mathfrak{g}$, the Lie algebra of G, is the vector field

$$Y_\xi := \frac{d}{dt}\Psi_{\exp(t\xi)}\big|_{t=0}$$

on Q. The tangent lift Ψ^{TQ} of Ψ is the action of G on TQ given by $\Psi_g^{TQ}(v_q) = T_q\Psi_g \cdot v_q$ for all $v_q \in TQ$ (in coordinates, $\Psi_g^{TQ}(q,\dot{q}) = \big(\Psi_g(q), \Psi_g'(q)\dot{q}\big)$ with $\Psi_g' = \frac{\partial \Psi_g}{\partial q}$). For any $\xi \in \mathfrak{g}$, the ξ-component of the momentum map of Ψ^{TQ} is the momentum J_{Y_ξ} of Y_ξ.

The following consequence of Proposition 2 extends a result in [14] and is a possible statement of a 'Nonholonomic Noether theorem' for nonholonomic systems with affine constraints:

Corollary 3. *Assume that L is invariant under Ψ^{TQ}, namely $L \circ \Psi_g^{TQ} = L$ for all $g \in G$. Given $\xi \in \mathfrak{g}$, $J_{Y_\xi}|_M$ is a first integral of $(L,Q,\mathcal{M})$ if and only if Y_ξ is a section of $\mathcal{R}^\circ$.*

3 Moving energies

3.1 Definition and conservation. In all of this section, $(L,Q,\mathcal{M})$ is a nonholonomic system with affine constraints and we freely use the notation and the terminology introduced in the previous section.

Definition . *The* moving energy of $(L,Q,\mathcal{M})$ generated by a vector field Y on Q *is the restriction to M of the function*

$$E_{L,Y} := E_L - \langle p, Y\rangle$$

(in coordinates, $E_{L,Y} = E_L - p \cdot Y = p\cdot(\dot{q} - Y) - L$ with $p = \frac{\partial L}{\partial \dot{q}}$).

As we have mentioned in the Introduction, the notion of moving energy given here is an extension of that originally given in [15], which has a kinematical interpretation. A comparison between the two is done in section 3.4 below.

Obviously, the consideration of moving energies has interest only when the energy is not conserved, namely if Z is not a section of $\mathcal{R}^\circ$. The central question, then, is which vector fields Y produce conserved moving energies for a given nonholonomic system $(L,Q,\mathcal{M})$. The situation is very similar to that of which vector fields produce conserved momenta, see Proposition 2:

Proposition 4. *Any two of the following three conditions imply the third:*

i. $Y - Z$ *is a section of* $\mathcal{R}^\circ$.

ii. $Y^{TQ}(L) = 0$ *in* M.

iii. $E_{L,Y}|_M$ *is a first integral of* $(L,Q,\mathcal{M} = Z + \mathcal{D})$.

Proof. We work in coordinates. All functions are evaluated in M. We have

$$\frac{d}{dt}E_L = \frac{d}{dt}(p\cdot\dot{q} - L) = \Big(\dot{p} - \frac{\partial L}{\partial q}\Big)\cdot\dot{q} + \Big(p - \frac{\partial L}{\partial \dot{q}}\Big)\cdot\ddot{q} = R\cdot\dot{q} = R\cdot Z$$

given that $\dot{q} - Z \in \mathcal{D}$ and R annihilates $\mathcal{D}$. Therefore, by (7),

$$\frac{dE_{L,Y}}{dt} = R\cdot(Y - Z) + Y^{TQ}(L)$$

and the proof goes as that of Proposition 2. □

Proposition 4 does not characterize all vector fields that generate conserved moving energies, but only those which satisfy either one (and hence the other) of the two conditions i. and ii. It has some immediate consequences:

Corollary 5.

i. If $Y^{TQ}(L)|_M = 0$ then $E_{L,Y}|_M$ is a first integral of $(L, Q, \mathcal{M} = Z + \mathcal{D})$ if and only if $Y - Z$ is a section of $\mathcal{R}^\circ$.

ii. If $Z^{TQ}(L)|_M = 0$ then $E_{L,Z}|_M$ is a first integral of $(L, Q, \mathcal{M} = Z + \mathcal{D})$.

iii. Assume that L is invariant under the tangent lift Ψ^{TQ} of an action Ψ on Q, namely $L \circ \Psi_g^{TQ} = L$ for all $g \in G$. Then for any $\xi \in \mathfrak{g}$, $E_{L,Y_\xi}|_M$ is a first integral of $(L, Q, \mathcal{M} = Z + \mathcal{D})$ if and only if $Y_\xi - Z$ is a section of $\mathcal{R}^\circ$.

Statement ii. is a particular case of statement i., but we have made it explicit because—as special as it may appear—it is precisely the case of all the LR systems and of their generalizations considered in section 4. Statement iii. formalizes the idea that, in presence of a symmetry group of the Lagrangian, the natural candidates to generate conserved moving energies are the infinitesimal generators of the group action that are sections of $\mathcal{R}^\circ$. This situation is met in all the examples in this paper and in [15].

Remarks. i. Since the fibers of $\mathcal{D}$ are contained in those of $\mathcal{R}^\circ$, the condition in item i. of Proposition 4 is independent of the arbitrariness in the choice of the component along $\mathcal{D}$ of the vector field Z.

ii. Statement iii. of Corollary 5 generalizes Theorem 2 of [15] in two respects. First, it drops the assumption of the invariance of the distribution $\mathcal{D}$ under the group action. (This hypothesis is present in [15] because it is related to the possibility of interpreting the moving energy as the energy in a different system of coordinates, which is the case considered there). Second, Theorem 2 of ref. [15] required $Z - Y_\xi$ to be a section of $\mathcal{D}$, not of the larger distribution $\mathcal{R}^\circ$.

3.2 Weak Noetherianity of the moving energies. Changing the Lagrangian changes $\mathcal{R}^0$ and—as it happens for the energy—may destroy or restore the conservation of the moving energy generated by a given vector field Y. Since the distribution $\mathcal{D}$ is independent of the Lagrangian, moving energies generated by vector fields Y such that $Y - Z$ is a section of $\mathcal{D}$—and not just of $\mathcal{R}^\circ$—are in this respect special. Specifically, Proposition 4 has the following consequence:

Corollary 6. *Consider an affine distribution $\mathcal{M} = Z + \mathcal{D}$ and a vector field Y on Q. If $Y - Z$ is a section of $\mathcal{D}$, then Y generates a conserved moving energy for any nonholonomic system $(L, Q, \mathcal{M})$ whose Lagrangian satisfies $Y^{TQ}(L)|_M = 0$.*

In particular, if we have an action Ψ of a Lie group G on Q and, for some $\xi \in \mathfrak{g}$, the infinitesimal generator Y_ξ is such that $Y_\xi - Z$ is a section of $\mathcal{D}$, then Y_ξ generates a conserved moving energy for all nonholonomic systems $(L, Q, \mathcal{M})$ with Ψ^{TQ}-invariant Lagrangian L. This may be viewed as a 'weak Noetherian' property (in the sense of [27, 10, 11]) of moving energies whose generators Y are such that $Y - Z$ is a section of $\mathcal{D}$.

3.3 Non-uniqueness of moving energies, and of their generators. A system may have different conserved moving energies and, on the other hand, different vector fields may produce the same moving energy. The following Proposition is a direct consequence of the definitions:

Proposition 7. *Consider a nonholonomic system with affine constraints $(L, Q, \mathcal{M})$ which has a conserved moving energy $E_{L,Y_1}|_M$. Then, for any vector field Y_2 on Q, the moving energy $E_{L,Y_2}|_M$ is conserved if and only if $J_{Y_1 - Y_2}|_M$ is a conserved momentum.*

Typically, this situation is met in presence of symmetries of the Lagrangian.

We analyze the second question only in the special case of a Lagrangian $L = T - V \circ \pi$ without terms that are linear in the velocities; the general case can be easily worked out. We denote here by $\perp$ the orthogonality with respect to the Riemannian metric defined by the kinetic energy T and by $\langle Z \rangle$ the distribution on Q generated by Z.

Proposition 8. *Assume that the Lagrangian does not contain gyrostatic terms. Let Y_1 and Y_2 be two vector fields on Q. Then $E_{L,Y_1}|_M = E_{L,Y_2}|_M$ if and only if $Y_1 - Y_2$ is a section of $(\mathcal{D} \oplus \langle Z \rangle)^\perp$.*

Proof. The equality $E_{L,Y_1}|_M = E_{L,Y_2}|_M$ is equivalent to the condition $\langle p, Y_1 - Y_2 \rangle|_M = 0$. Since $p = A\dot{q}$, this is in turn equivalent to the condition that, at each point $q \in Q$, $Y_1(q) - Y_2(q)$ is T-orthogonal to the fiber $\mathcal{M}_q$, namely, to all tangent vectors $Z(q) + u$ with $u \in \mathcal{D}_q$. Since $0 \in \mathcal{D}_q$, this is equivalent to the two conditions $Y_1(q) - Y_2(q) \perp Z(q)$ and $Y_1(q) - Y_2(q) \perp \mathcal{D}_q$. It follows that $E_{L,Y_1}|_M = E_{L,Y_2}|_M$ if and only if, for all $q \in Q$, $(Y_1 - Y_2)(q) \perp \mathcal{D}_q \oplus \langle Z \rangle_q$. □

3.4 Kinematically interpretable moving energies. We conclude this analysis of moving energies with a comparison with the class of them which were originally introduced in [15] and have a kinematical interpretation.

The construction of moving energies in [15] is as follows. One looks for a time-dependent change of coordinates $\mathcal{C}$ that transforms the nonholonomic system with affine constraints $(L, Q, \mathcal{M})$ into a nonholonomic system $(\tilde{L}, Q, \tilde{\mathcal{M}})$ with *linear* constraints. If time-independent, the transformed system has a conserved energy $E_{\tilde{L}}|_{\tilde{M}}$. The moving energy of ref. [15] is the pull-back $E^\star_{L,\mathcal{C}}$ of the function $E_{\tilde{L}}|_{\tilde{M}}$ to the original system. $E^\star_{L,\mathcal{C}}$ is always a conserved function for the original system, but the interesting case is when it is time-independent. In that case, $E^\star_{L,\mathcal{C}}$ coincides with a moving energy $E_{L,Y_\mathcal{C}}$ as defined here, with a vector field $Y_\mathcal{C}$ constructed out of the time-dependent change of coordinates $\mathcal{C}$, see below. Clearly, the time-independence of both—the transformed system $(\tilde{L}, Q, \tilde{\mathcal{M}})$ and the function $E^\star_{L,\mathcal{C}}$—is a rather strong condition, and typically requires some symmetry of the system.

We will say that a moving energy $E_{L,Y}$ is *kinematically interpretable* when there is a time-dependent change of coordinates such that $E_{L,Y}|_M = E^\star_{L,\mathcal{C}}$.

The mechanism of ref. [15] has a simple mechanical interpretation for systems that consist of rigid bodies constrained to roll without sliding on moving surfaces. In such cases, the moving energy $E_{L,Y_\mathcal{C}}|_M = E^\star_{L,\mathcal{C}}$ is the energy of the system relative to a moving reference frame in which the surface is at rest (so that the nonholonomic constraint of rolling without sliding is no longer affine but linear), written however in the original coordinates. Clearly, as pointed out in [5], in situations like these, instead of describing the system using the original system of coordinates and consider the (kinematically interpretable) moving energy, one might as well describe the system using the new, time-dependent coordinate system (or, when possible, a moving reference frame) and consider the energy $E_{\tilde{L}}|_{\tilde{M}}$. Ref. [5] illustrates this procedure on a number of examples.[4]

We thus investigate here the conditions under which a moving energy is kinematically interpretable.

Let $\mathcal{C}$ be a time-dependent diffeomorphism of Q onto itself, namely a smooth map $\mathcal{C} : \mathbb{R} \times Q \to Q$ such that, for each $t \in \mathbb{R}$, the map $\mathcal{C}_t := \mathcal{C}(t, \ ) : Q \to Q$ is a diffeomorphism. As shown in [15] (Proposition 1; see also Proposition 4 of [14]), the tangent-bundle lift of $\mathcal{C}$ conjugates a nonholonomic system with affine constraints $(L, Q, \mathcal{M})$ to a nonholonomic system with affine constraints $(\tilde{L}, Q, \tilde{\mathcal{M}})$, which is in general time-dependent. For our purposes, it is sufficient to describe the transformed system in coordinates. The transformed Lagrangian is

$$\tilde{L}(q, \dot{q}, t) = L\big(\mathcal{C}_t(q), \mathcal{C}'_t(q)\dot{q} + \dot{\mathcal{C}}_t(q)\big)$$

[4]Ref. [5] overlooks somewhat the need for symmetry. As a consequence, some of the statements there are not entirely correct (see footnote nr. 2 in [14]). As for the use of the term 'Jacobi integral' instead of 'energy' in ref. [5], see the comment at the beginning of section 2.3 above.

and the transformed, time-dependent, constraint distribution $\tilde{M}$ has fibers

$$\tilde{\mathcal{M}}_{t,q} = \big(\mathcal{C}_t'(q)\big)^{-1}\big[\mathcal{D}_{\mathcal{C}_t(q)} + Z(\mathcal{C}_t(q)) - \dot{\mathcal{C}}_t(q)\big]$$

where $\mathcal{C}_t'$ is the Jacobian matrix of $\mathcal{C}_t$ and $\dot{\mathcal{C}}_t = \frac{\partial \mathcal{C}_t}{\partial t}$. Furthermore, as proven in [15] (Proposition 3), the pull-back under $\mathcal{C}$ of the energy of $(\tilde{L}, Q, \tilde{\mathcal{M}})$—namely, the moving energy of $(L, Q, \mathcal{M})$—is time-independent if and only if the map $\mathcal{C}$ is the flow Φ of a vector field Y on Q. In that case, $\dot{\mathcal{C}}_t = Y \circ \Phi_t = \Phi_t' Y$, where the last equality uses the invariance of a vector field under its own flow, and

$$\tilde{L}(q, \dot{q}, t) \;=\; L\big(\Phi_t(q), \Phi_t'(q)[\dot{q} + Y(q)]\big) \tag{8}$$

$$\mathcal{M}_{t,q} \;=\; \big(\Phi_t'(q)\big)^{-1}\big[\mathcal{D}_{\Phi_t(q)} + (Z - Y)(\Phi_t(q))\big]\,. \tag{9}$$

The following Proposition is essentially contained in [15]:

Proposition 9. *Let $(L, Q, \mathcal{M})$ be a nonholonomic system with affine constraints, with $\mathcal{M} = Z + \mathcal{D}$, and let Y be a vector field on Q. Let $(\tilde{L}, Q, \tilde{\mathcal{M}})$ be the nonholonomic system conjugate to $(L, Q, \mathcal{M})$ by the flow Φ of Y.*

i. $\tilde{M}$ is a linear subbundle of TQ if and only if $Y - Z$ is a section of $\mathcal{D}$.

ii. Assume that $Y - Z$ is a section of $\mathcal{D}$, that $Y^{TQ}(L) = 0$ and that the linear distribution $\mathcal{D}$ is invariant under the flow Φ of Y, in the sense that $\mathcal{D}_{\Phi_t(q)} = \Phi_t'(q)\mathcal{D}_q$ for all t and q. Then $\tilde{L}$ is time-independent and $\tilde{\mathcal{M}} = \mathcal{D}$.

Proof. (i) This follows from (9). (ii) The condition $Y^{TQ}(L) = 0$ means $L(\Phi_t(q), \Phi_t'(q)\dot{q}) = L(q, \dot{q})$ for all t, q and $\dot{q}$. Together with (8), this gives $\tilde{L}(q, \dot{q}, t) = L(q, \dot{q} + Y(q))$ and shows that $\tilde{L}$ is time-independent. If $Z - Y$ is a section of $\mathcal{D}$ then equality (9) gives $\tilde{\mathcal{M}}_{t,q} = (\Phi_t'(q))^{-1}\mathcal{D}_{\Phi_t(q)}$ and, by the invariance of $\mathcal{D}$, $\tilde{\mathcal{M}}_{t,q} = \mathcal{D}_q$. □

Statement i. of this Proposition implies that a necessary condition for a moving energy $E_{L,Y}|_M$ to be kinematically interpretable is that it is generated by a vector field Y such that $Y - Z$ is a section of $\mathcal{D}$, not just of $\mathcal{R}^\circ$ as in Proposition 4. This shows that, unless $\mathcal{R}^\circ = \mathcal{D}$, the class of moving energies considered here is broader than that considered in [15]. Furthermore, statement ii. gives sufficient conditions for this interpretation to be feasible.

We remark that all the moving energies produced in the examples of sections 4, 5 and 6 fullfil the hypotheses of Proposition 9 and are thus kinematically interpretable.

4 Moving energies for LR systems

4.1 A moving energy for a class of affine nonholonomic systems on Lie groups. As a first example, we consider here a class of nonholonomic systems $(L, Q, \mathcal{M})$ with affine constraints whose configuration manifold Q is a Lie group G. This class includes the so-called (affine) LR systems, that we will consider in the next subsection.

As usual, we denote by L and R, respectively, the actions of G on itself by left and right translations.[5] A function $f : TG \to \mathbb{R}$ is left-invariant if it is invariant under the lifted action L^{TG} on TG, namely $f \circ L_g^{TG} = f$ for all $g \in G$. A vector field Y on G is right-invariant if $(R_g)_* Y = Y$ for all $g \in G$.

Corollary 5 has the following immediate consequence (which we formulate in a way that takes into account the fact that the vector field Z that determines the inhomogeneous part of the affine constraint distribution is nonunique):

[5] The symbols L and R are also used for other objects, but there will be no risk of confusion from the context.

Proposition 10. *Consider a nonholonomic system with affine constraints $(L, G, \mathcal{M})$, where G is a Lie group. Assume that the Lagrangian L is left-invariant and that the affine distribution $\mathcal{M}$ can be written as $\mathcal{M} = Z + \mathcal{D}$ with a right-invariant vector field Z. Then, the moving energy $E_{L,Z}|_M$ is a first integral of $(L, G, \mathcal{M})$.*

Proof. Since the flow of a right-invariant vector field consists of left-translations, under the stated hypotheses we have $Z^{TG}(L) = 0$ in all of TG. Hence, the conclusion follows either from item ii. of Corollary 5 or from Corollary 6, given that $Z - Z = 0$ is certainly a section of $\mathcal{D}$. □

If the Lagrangian has the form (1), then the condition of left-invariance implies that the kinetic energy T is a left-invariant Riemannian metric on G, that the gyrostatic term b is a left-invariant 1-form on G, and that the potential energy V is a constant. We give here the expression of $E_{L,Z}$ in the case in which the Lagrangian does not contain gyrostatic terms, so that $L = T$.

We employ the left-trivialization of TG, namely the identification $\Lambda : TG \to G \times \mathfrak{g}$ given by the maps $T_g L_{g^{-1}} : T_g G \to T_e G \equiv \mathfrak{g}$. We write Ω for $T_g L_{g^{-1}} \dot{g}$ (the 'body angular velocity'; in a matrix group, $\Omega = g^{-1}\dot{g}$). We denote $\langle\ ,\ \rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}$ the $\mathfrak{g}^*$-$\mathfrak{g}$ pairing. On account of the left-invariance of the Lagrangian $L = T$, its left-trivialization $T \circ \Lambda^{-1}$, which we keep denoting T, is given by

$$T(g, \Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}$$

where $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is the positive definite symmetric tensor determined by the kinetic energy at the group identity e. Furthermore, the left-trivialization of the Legendre transformation is the map $\mathfrak{g} \to \mathfrak{g}^*$ given by $\Omega \mapsto \mathbb{I}\Omega$. Given that the vector field Z is right-invariant, there exists a Lie algebra vector $\zeta \in \mathfrak{g}$ such that

$$Z(g) = T_e R_g \cdot \zeta \quad \forall g \in G\,,$$

and its left-trivialization is $\mathrm{Ad}_{g^{-1}}\zeta$. Hence, the left-trivialization of the momentum J_Z is $\langle \mathbb{I}\Omega, \mathrm{Ad}_{g^{-1}}\zeta\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}$ and that of $E_{T,Z}$ is

$$E_{T,Z}(g, \Omega) \;=\; \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}} - \langle \mathbb{I}\Omega, \mathrm{Ad}_{g^{-1}}\zeta\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}\,. \tag{10}$$

We conclude this subsection with two remarks. First, even though the left-invariance of the Lagrangian requires the potential energy to be a constant, and hence $L = T + b$, on account of Corollary 6 any Z-invariant potential energy V can be included into the Lagrangian without destroying the conservation of the moving energy, which becomes the restriction to M of $T + V - J_Z$. Hence, in the absence of gyrostatic terms,

$$E_{T-V,Z} = E_{T,Z} + V\,. \tag{11}$$

In the special case of the LR systems, a particular class of Z-invariant potentials has been identified in [20] (section 4.6).

Second, in all these systems, the moving energy $E_{L,Z}|_M$ is kinematically interpretable, namely, it can be interpreted as the energy of the system written in suitable time-dependent coordinates, see section 3.4.

4.2 The case of LR systems. LR systems are the subclass of the class of systems on Lie groups considered in the previous subsection in which the affine constraint distribution $\mathcal{M} = Z + \mathcal{D}$ is right-invariant, namely, not only Z but also $\mathcal{D}$ is right-invariant. The right-invariance of $\mathcal{D}$ means that $\mathcal{D}_g = T_e R_g \cdot \mathcal{D}_e$ for all $g \in G$, or equivalently, that $\mathcal{D}$ is the null space of a set of right-invariant 1-forms on G.

LR systems were introduced by Veselov and Veselova in [32], who focussed mostly on the case of linear constraints, with $\mathcal{M} = \mathcal{D}$, and of purely kinetic Lagrangian, namely $L = T$. The prototype of these systems is the renowned Veselova system [33, 31], which describes the motion by inertia of a rigid body with a fixed point under the constraint that the angular velocity remains orthogonal to a direction fixed in space (linear case) or, more generally, that the component of the angular velocity in a direction fixed in space is constant (affine case).

As proven in [32], LR systems have remarkable properties: the existence of an invariant measure and the conservation of the (restriction to the constraint manifold of the) momentum covector. A difference between the linear and the affine cases concerns of course the energy, which is conserved in the former case but not in the latter case. However, it was shown in [16, 17, 18, 21] that the affine Veselova system, and an n-dimensional generalization of it, possess a first integral which was there regarded as an "analog of the Jacobi-Painlevé integral" or as a "modified Hamiltonian", and which turns out to be a moving energy. In fact, in full generality, Proposition 10 implies the following

Corollary 11. *Any affine LR system $(L, G, \mathcal{M} = Z+\mathcal{D})$ has the conserved moving energy $E_{L,Z}|_M$.*

In order to compare the moving energy $E_{L,Z}|_M$ of Corollary 11 with the first integral found in [16, 17, 21], we give the expression of $E_{L,Z}$ in the special case of a Lie group G for which there is an Ad-invariant inner product in $\mathfrak{g}$, which includes the case of $\mathrm{SO}(n)$. Since [16, 17, 21] did not consider gyrostatic terms, on account of (11) we limit ourselves to $L = T$. In general, the right-invariant affine distribution $\mathcal{M} = Z + \mathcal{D}$ can be specified, via right-translations, by a set of independent covectors $a^1, \dots, a^k \in \mathfrak{g}^*$ that span the annihilator $\mathcal{D}_e^\circ$ and by the vector $\zeta \in \mathfrak{g}$ that specifies the vector field Z. Hence, the constraint can be written

$$\langle a^j, \omega - \zeta\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}} = 0, \qquad j = 1, \dots, k \tag{12}$$

where $\omega = \mathrm{Ad}_g\Omega$ is the right-trivialization of the velocity vector $\dot g \in T_gG$. By means of an Ad-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{g}}$, the covectors $a^1, \dots, a^k$ are interpreted as elements of $\mathfrak{g}$ and can be chosen to be orthonormal. Also, Z can be chosen so that ζ belongs to the span of $a^1, \dots, a^k$. Hence $\zeta = \sum_{j=1}^k \langle a^j, \zeta\rangle_{\mathfrak{g}} a^j$ and the left-trivialization of the momentum J_Z takes the form $\sum_{j=1}^k \langle a^j, \zeta\rangle_{\mathfrak{g}} \langle \mathbb{I}\Omega, \gamma^j\rangle_{\mathfrak{g}}$, where $\gamma^j = \mathrm{Ad}_g^{-1} a^j$ are the so-called Poisson vectors. In conclusion,

$$E_{T,V}(g, \Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle_{\mathfrak{g}} - \sum_{j=1}^k \langle a^j, \zeta\rangle_{\mathfrak{g}} \langle \mathbb{I}\Omega, \gamma^j\rangle_{\mathfrak{g}}\,.$$

When $G = \mathrm{SO}(n)$, this coincides with the first integral of the n-dimensional affine Veselova system given in [16, 17, 21], except that the constants $\langle a^j, \zeta\rangle_{\mathfrak{g}}$ are there written, using the constraint (12), as $\langle \gamma^j, \Omega\rangle_{\mathfrak{g}}$.

The right-invariance of the distribution $\mathcal{D}$ makes the affine LR systems very special among those considered in the previous subsection. In particular, as mentioned above, the restriction to the constraint manifold of the momentum covector is conserved [32]. This fact is accounted for, without any computation, by Proposition 2:

Proposition 12. [32] *Consider an affine LR system $(L, G, \mathcal{M} = Z + \mathcal{D})$. Denote Y_ξ the infinitesimal generator of the left-action of G on itself by left-translations associated to $\xi \in \mathfrak{g}$. Then, for any $\xi \in \mathcal{D}_e \subset \mathfrak{g}$, $J_{Y_\xi}|_M = \langle p, Y_\xi\rangle|_M$ is a first integral of the system.*

Proof. $Y_\xi(g) = T_eR_g \cdot \xi$. By left-invariance of L, $Y_\xi^{TG}(L) = 0$. By right-invariance of $\mathcal{D}$, Y_ξ is a section of $\mathcal{D}$. The statement now follows from Proposition 2. □

Note that if $\mathcal{D}$ is not right-invariant then the infinitesimal generators Y_ξ might be not sections of $\mathcal{D}$: this is the reason why, notwithstanding the fact that the Lagrangian has the appropriate invariance property, the momenta $J_{Y_\xi}|_M$ are in general not conserved for the systems considered in the previous subsection.

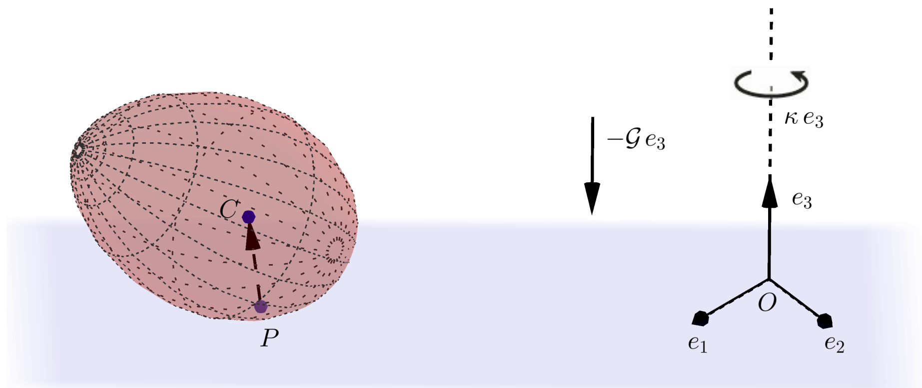

Figure 1: The heavy convex rigid body with smooth surface that rolls without slipping on a plane that rotates with constant angular velocity κe_3.

Proposition 12 implies that the affine LR systems have a multitude of conserved moving energies: for any $\xi \in \mathcal{D}_e$, $E_{L,Z-Y_\xi}|_M = E_{L,Z}|_M - J_{Y_\xi}|_M$ is a conserved moving energy. Note that, on account of statement ii. of Proposition 9, all these moving energies are associated to time-dependent changes of coordinates.

Remark. The 3-dimensional affine Veselova system allows a Hamiltonization after reduction, in terms of a rank-four Poisson structure, with the above moving energy playing the role of the Hamiltonian [21].

5 A convex body that rolls on a steadily rotating plane

5.1 The system. We consider now the system formed by a heavy convex rigid body constrained to roll without slipping on a horizontal plane Π, which rotates uniformly around a vertical axis, see Figure 1. The case in which the plane is at rest is classical and was studied for specific geometries of the body already by Routh [30] and Chaplygin [6] (see [4, 7] for recent treatments) while, to our knowledge, the case in which the plane Π is rotating has been studied only in two particular cases—that of a homogeneous sphere [8, 29, 26] and that of a disk [19] (which however describes the disk in frame co-rotating with the plane, in which the nonholonomic constraint is linear). Here, we exclude the latter case because we assume that the body has a smooth (i.e. C^∞) surface.

We describe the system relatively to a spatial inertial frame $\Sigma_s = \{O; e_1, e_2, e_3\}$. We assume that the plane Π rotates around the axis e_3 of this frame, with constant angular velocity κe_3, $\kappa \in \mathbb{R}$, and that it is superposed to the subspace spanned by e_1 and e_2.

Let $x = (x_1, x_2, x_3)$ be the coordinates in Σ_s of the center of mass C of the body. The system is subject to the holonomic constraint that the body has a point in contact with the plane. The configuration manifold is thus $Q = \mathbb{R}^2 \times \mathrm{SO}(3) \ni (q, g)$, where $q = (x_1, x_2) \in \mathbb{R}^2$ and $g \in \mathrm{SO}(3)$ is the attitude matrix that relates the inertial frame Σ_s to a frame $\Sigma_b = \{C; E_1, E_2, E_3\}$ attached to the body, and with the origin in C. We assume that g is chosen so that the representatives u^s and u^b of a same vector in the two frames Σ_s and Σ_b are related by $u^s = gu^b$.

We denote by ω the angular velocity of the body relative to the inertial frame Σ_s, and by Ω its representative ω^b in the body frame Σ_b.

As in the previous section we will left-trivialize the factor $T\mathrm{SO}(3)$ of TQ, but now we identify the Lie algebra so(3) with $\mathbb{R}^3$ via the hat map $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$, with $\hat{a} = a\times$ for any $a \in \mathbb{R}^3$. Thus, $TQ \equiv \mathbb{R}^2 \times \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^3 \ni (q, \dot{q}, g, \Omega)$.

The holonomic constraint that the body touches the plane is $e_3 \cdot OC = e_3 \cdot PC$, where P is the point of the body in contact with the plane, or $x_3 = e_3 \cdot PC$. Hence, $\dot{x}_3 = e_3 \cdot (\omega \times PC)$. If we denote by $\rho(g)$ the representative in Σ_b of the vector PC and by $\gamma(g)$ the representative of e_3 in Σ_b, which is the so-called Poisson vector and equals $g^{-1}e_3^s = g^{-1}(0,0,1)^T$, then the holonomic constraint is $x_3 = \gamma(g) \cdot \rho(g)$ and

$$\dot{x}_3 = \gamma(g) \cdot [\Omega \times \rho(g)] \,. \tag{13}$$

The potential energy of the weight force is $V(g) = m\mathcal{G}e_3 \cdot OC = m\mathcal{G}\gamma(g) \cdot \rho(g)$, with the obvious meaning of the constants m and $\mathcal{G}$. Here, γ and ρ are known functions of the attitude $g \in \mathrm{SO}(3)$, but they are related by the Gauss map $G : \mathcal{S} \to S^2$ of the surface $\mathcal{S}$ of the body. Specifically, given that γ is the inward unit normal vector to $\mathcal{S}$ at the point of $\mathcal{S}$ of coordinates $-\rho$, $\gamma(g) = -G(-\rho(g))$. Since $\mathcal{S}$ is assumed to be smooth and convex, G is a diffeomorphism and we may also write $\rho = F \circ \gamma$ with $F(\gamma) = -G^{-1}(-\gamma)$. Hence $V = v \circ \gamma$ with $v(\gamma) = m\mathcal{G}\gamma \cdot F(\gamma)$. In the sequel, we shall routinely write γ for $\gamma(g)$ and ρ for $F(\gamma(g))$. With these conventions, the left-trivialized Lagrangian of the system is

$$L(q, g, \dot{q}, \Omega) = \frac{1}{2}\mathbb{I}\Omega \cdot \Omega + \frac{m}{2}\Big(\dot{q}_1^2 + \dot{q}_2^2 + \big(\gamma \cdot [\Omega \times \rho]\big)^2\Big) - v(\gamma) \tag{14}$$

and depends on the attitude g only through γ. Here, $\mathbb{I}$ is the inertia tensor of the body relative to its center of mass.

The condition of no-slipping of the body on the plane is obtained by equating the velocities (relative to Σ_s) of the point P of the body, which is $\frac{d}{dt}(OC) + \omega \times CP$, and of the point of the plane which is in contact with P, which is $\kappa e_3 \times OP = \kappa e_3 \times (OC + CP)$. Using representatives, the condition of no slipping is thus

$$\dot{x} = g(\Omega \times \rho) + \kappa e_3^s \times (x - g\rho) \,. \tag{15}$$

The first two components of this condition define an 8-dimensional affine subbundle M of TQ which can be identified with $\mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^3 \ni (q, g, \Omega)$ (the third component of (15) is nothing but (13)).

In order to simplify the notation we identify $Q = \mathbb{R}^2 \times \mathrm{SO}(3) \ni (q, g)$ with its embedding in $\mathbb{R}^3 \times \mathrm{SO}(3) \ni (x, g)$ and $M = \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^3 \ni (q, g, \Omega)$ with its embedding in $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \ni (x, g, \Omega)$ given, in both cases, by $x(q, g) = (q_1, q_2, \gamma \cdot \rho)$. Correspondingly, we identify $TQ \equiv \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R}^3 \ni (q, g, \dot{q}, \Omega)$ with its embedding in $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (x, g, \dot{x}, \Omega)$, with x as above and $\dot{x} = (\dot{q}_1, \dot{q}_2, \dot{x}_3)$ with $\dot{x}_3$ as in (13).

The affine subbundle M corresponds to an affine distribution $\mathcal{M} = Z + \mathcal{D}$ on $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ which, once left-trivialized and embedded in $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, is given by

$$\begin{aligned} \mathcal{D}_{(q,g)} &= \big\{ \big(g(\Omega \times \rho)\,,\, \Omega\big) \;:\; \Omega \in \mathbb{R}^3 \big\} \\ Z(q, g) &= \big(\kappa e_3^s \times (x - g\rho)\,,\, 0\big) \,. \end{aligned} \tag{16}$$

5.2 The conserved moving energy. The energy is not conserved in the nonholonomic system $(L, Q, \mathcal{M})$ just constructed. However, being independent of q and depending on g only through the Poisson vector γ, the Lagrangian (14) is invariant under the lift of an action of SE(2). We may thus try to construct a moving energy using the infinitesimal generator of the action of a subgroup.

Specifically, we consider the S^1-action which (after the aforementioned embedding of Q in $\mathbb{R}^3 \times \mathrm{SO}(3)$) is given by

$$\theta.(x, g) = (R_\theta x, R_\theta g) \tag{17}$$

where R_θ is the rotation matrix in SO(3) that rotates an angle $\theta \in S^1$ about the third axis $(0,0,1)^T = e_3^s$. This action leaves the Poisson vector γ invariant. Its lift to M (as embedded in $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3$) is

$$\theta.(x, g, \Omega) = (R_\theta x, R_\theta g, \Omega) \tag{18}$$

and clearly leaves the Lagrangian (14) invariant.

The infinitesimal generator of the action (17) that corresponds to $\xi \in \mathbb{R} \cong \mathfrak{s}^1$, once left-trivialized, is the vector field with components

$$Y_\xi(q, g) = \big(\xi e_3^s \times x \,,\, \xi\gamma\big) \in \mathbb{R}^3 \times \mathbb{R}^3 \,. \tag{19}$$

Therefore

$$Y_k - Z = \big(\kappa e_3^s \times g\rho \,,\, \kappa\gamma\big) = \big(g(\kappa\gamma \times \rho) \,,\, \kappa\gamma\big) \in \mathcal{D}_{(q,g)} \,.$$

Hence, the hypotheses of Corollary 5 are satisfied and the moving energy $E_{L,Y_\kappa}|_M$ is a first integral of the system.

In order to give an expression for this moving energy we find convenient to introduce the vector function

$$K(g, \Omega) = \mathbb{I}\Omega + m\rho \times \big(\Omega \times \rho\big) \,. \tag{20}$$

Proposition 13.

$$E_{L,Y_\kappa}|_M = \frac{1}{2} K \cdot \Omega + m\mathcal{G}\rho \cdot \gamma - \kappa K \cdot \gamma + \frac{1}{2} m\kappa^2 \big(\|\rho\|^2 - \|x\|^2\big) \,. \tag{21}$$

Proof. Instead of parameterizing the embedding of M in $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3$ with (x, g, Ω), as done so far, we will parameterize it with (X, g, Ω), with $X = g^{-1}x$, the representative of OC in the body frame. Specifically, M is the the submanifold of $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3$ given by the condition $X \cdot \gamma = \rho \cdot \gamma$ (the holonomic constraint). Similarly, we parametrize $TQ \equiv \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$ with $(X, g, \dot{x}, \Omega)$ (notice that we keep the spatial representative $\dot{x}$ of the velocity of the center of mass). Thus, M is the subbundle of TQ defined by the two conditions

$$X \cdot \gamma = \rho \cdot \gamma \,, \qquad g^{-1}\dot{x} = \Omega \times \rho + \kappa\gamma \times (X - \rho) \,. \tag{22}$$

The energy E_L of the system is the sum of the kinetic and potential energies:

$$E_L = \frac{1}{2}\mathbb{I}\Omega \cdot \Omega + \frac{m}{2}\|\dot{x}\|^2 + m\mathcal{G}\gamma \cdot \rho$$

Its restriction to M is given by

$$\begin{aligned} E_L|_M \;&=\; \frac{1}{2}\mathbb{I}\Omega \cdot \Omega + m\mathcal{G}\gamma \cdot \rho + \frac{m}{2}\|\rho \times \Omega\|^2 + m\kappa\,(\Omega \times \rho) \cdot (\gamma \times (X - \rho)) + \frac{m\kappa^2}{2}\|\gamma \times (X - \rho)\|^2 \\ &=\; \frac{1}{2}K \cdot \Omega + m\mathcal{G}\gamma \cdot \rho - m\kappa\gamma \cdot (\rho \times (\Omega \times \rho)) + m\kappa\,(\Omega \times \rho) \cdot (\gamma \times X) + \frac{m\kappa^2}{2}\|\gamma \times X\|^2 \\ &\qquad + \frac{m\kappa^2}{2}\|\gamma \times \rho\|^2 - m\kappa^2(\gamma \times X) \cdot (\gamma \times \rho). \end{aligned}$$

On the other hand, given (19) and the form of the kinetic energy metric defined by (14), we find

$$J_{Y_\kappa} \;=\; m\kappa(q_1\dot{q}_2 - q_2\dot{q}_1) + \kappa\mathbb{I}\Omega \cdot \gamma \;=\; m\kappa\gamma \cdot (X \times (g^{-1}\dot{x})) + \kappa\mathbb{I}\Omega \cdot \gamma.$$

Its restriction to M is computed to be

$$J_{Y_\kappa}|_M \;=\; m\kappa(\Omega \times \rho) \cdot (\gamma \times X) - m\kappa^2(\gamma \times X) \cdot (\gamma \times \rho) + m\kappa^2\|\gamma \times X\|^2 + \kappa\mathbb{I}\Omega \cdot \gamma \,.$$

Hence the moving energy $E_{L,Y_\kappa}|_M = E_L|_M - J_{Y_\kappa}|_M$ is given by

$$E_{L,Y_\kappa}|_M = \frac{1}{2}K \cdot \Omega + m\mathcal{G}\rho \cdot \gamma - \kappa(K \cdot \gamma) + \frac{m\kappa^2}{2}\left(\|\gamma \times \rho\|^2 - \|\gamma \times X\|^2\right) .$$

This is equivalent to (21) because, in M, $X \cdot \gamma = \rho \cdot \gamma$ and hence $\|\gamma \times \rho\|^2 - \|\gamma \times X\|^2 = \|\rho\|^2 - \|X\|^2$, and because $\|X\| = \|x\|$. □

We note that the existence of the moving energy $E_{L,Y_\kappa}|_M$ has the following dynamical consequence:

Corollary 14. *If the motion of the rolling body is unbounded, then its angular velocity* Ω *satisfies* $\limsup_{t\to\infty} \|\Omega\| = \infty$.

Proof. Since ρ and γ are bounded, the only way in which the conserved moving energy (21) remains bounded as $\|x\|$ becomes large is that $\|\Omega\|$ becomes large. □

Remark. The distribution $\mathcal{D}$ and the vector field Z are invariant under the lifted S^1-action (18). Therefore, the system is invariant under this action. In the appendix, we give for completeness the reduced equations of motion on M/S^1.

6 The n-dimensional Chaplygin sphere that rolls on a steadily rotating hyperplane

It is natural to expect that the discussion of the previous section admits a multi-dimensional generalization. Here we outline the particular case in which the body is a sphere and the center of mass coincides with its geometric center. If the hyperplane where the rolling takes plane is not rotating we recover the n-dimensional Chaplygin sphere problem introduced in [17].

Let $x \in \mathbb{R}^n$ denote the position of the center of mass of the sphere written with respect to an inertial frame $\Sigma_s = \{O; e_1, \ldots, e_n\}$. We assume that the hyperplane where the rolling takes place passes through O and has e_n as its normal vector. Moreover, we assume that the sphere is 'above' this hyperplane so at all times the holonomic constraint $x_n = r$ is satisfied, where r is the radius of the sphere.

The configuration space is $Q = \mathbb{R}^{n-1} \times \mathrm{SO}(n) \ni (q, g)$, where $q = (x_1, \ldots, x_{n-1})$. For convenience, in the rest of the section we embed $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ by putting $x_n = r$. We will also work with the induced embedding of tangent bundles $TQ \hookrightarrow T(\mathbb{R}^n \times \mathrm{SO}(n))$ defined by the simultaneous relations $x_n = r$ and $\dot{x}_n = 0$.

The Lagrangian $L : T(\mathbb{R}^n \times \mathrm{SO}(n)) \to \mathbb{R}$ is written in the left-trivialization as

$$L(x, g, \dot{x}, \Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega \rangle + m\|\dot{x}\|^2. \tag{23}$$

As usual $\Omega = g^{-1}\dot{g} \in \mathfrak{so}(n)$ is the angular velocity written in body coordinates (the left-trivialization of the velocity). The pairing $\langle \cdot, \cdot \rangle$ in (23) denotes the Killing metric

$$\langle \zeta_1, \zeta_2 \rangle = -\frac{1}{2}\mathrm{Trace}(\zeta_1\zeta_2), \qquad \zeta_1, \zeta_2 \in \mathfrak{so}(n),$$

and the inertia tensor $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is a positive definite symmetric linear operator.

The steady rotation of the hyperplane where the rolling takes place is specified by a fixed element $\eta \in \mathfrak{so}(n)$ that satisfies

$$\eta e_n = 0. \tag{24}$$

The nonholonomic constraints of rolling without slipping are

$$\dot{x} = r\omega e_n + \eta x, \tag{25}$$

where $\omega = \dot{g}g^{-1}$ is the angular velocity written in space coordinates (the right-trivialization of the velocity) that satisfies $\omega = \mathrm{Ad}_g\Omega$. Note that the last component of (25) reads $\dot{x}_n = 0$ so the equation defines an affine constraint subbundle of $TQ \subset T(\mathbb{R}^n \times \mathrm{SO}(n))$.

The constraint may be rewritten as

$$\dot{x} = r(\mathrm{Ad}_g\Omega)e_n + \eta x,$$

that defines an affine distribution $\mathcal{M} = Z + \mathcal{D}$ on $\mathbb{R}^n \times \mathrm{SO}(n)$ that is given in the left-trivialization by

$$\begin{aligned} \mathcal{D}_{(x,g)} &= \{(\dot{x}, \Omega) \, : \, \dot{x} = r(\mathrm{Ad}_g\Omega)e_n\} \, , \\ Z(x,g) &= \big(\eta x, 0\big) \, . \end{aligned}$$

We now consider the action of $\mathrm{SO}(n-1)$ on $\mathbb{R}^n \times \mathrm{SO}(n)$ defined by $h \cdot (x,g) = (\tilde{h}x, g)$, where for $h \in \mathrm{SO}(n-1)$ we denote

$$\tilde{h} = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(n). \tag{26}$$

This action leaves Q invariant and its tangent lift clearly preserves the Lagrangian (23).

Using (24) and the embedding $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$ given by (26), we can naturally think of η as an element in $\mathfrak{so}(n-1)$. Its infinitesimal generator Y_η is readily computed to be the vector field $Y_\eta(x,g) = (\eta x, 0)$. Therefore, $Y_\eta - Z = 0$ and by Corollary 5 the moving energy $E_{L,Y_\eta}|_M$ is preserved.

The procedure to compute the equations of motion and the moving energy for this problem is similar to the case of the solid of revolution rolling on a rotating plane. We state the final result in the following.

Proposition 15. *The equations of motion for an n dimensional Chaplygin ball that rolls without slipping on a hyperplane that steadily rotates with angular velocity $\eta \in \mathfrak{so}(n)$ are given by* [6]

$$\begin{aligned} \dot{K} &= [K, \Omega] - mr(g^{-1}\eta\dot{x}) \wedge E_n, \\ \dot{x} &= r(\mathrm{Ad}_g\Omega)e_n + \eta x, \\ \dot{g} &= g\Omega, \end{aligned} \tag{27}$$

where now

$$K = \mathbb{I}\Omega + mr^2(E_nE_n^T\Omega + \Omega E_nE_n^T)$$

and $E_n = g^{-1}e_n$. Equations (27) preserve the moving energy

$$E_{L,Y_\eta}|_M(x, g, \dot{x}, \Omega) = \frac{1}{2}\langle K, \Omega\rangle - \frac{m}{2}\|\eta x\|^2.$$

7 Appendix: The S^1-reduced equations of motion of the convex body that rolls on a rotating plane

In the system studied in section 5, not only the Lagrangian (14) but also the distribution $\mathcal{D}$ and the vector field Z as in (16) are invariant under the lift of the S^1-action (18) to TQ. Therefore, this

[6] In equations (27) the symbol $[\cdot,\cdot]$ denotes the matrix commutator in $\mathfrak{so}(n)$ and the wedge product of vectors $a, b \in \mathbb{R}^n$ is defined by $a \wedge b = ab^T - ba^T$.

lifted action can be restricted to the 8-dimensional phase space M, and the dynamics is equivariant. For completeness, we give here the reduced equations of motion on the quotient space M/S^1.

The S^1-action (18) on M is free. The 7-dimensional quotient manifold M/S^1 can be identified with $\mathbb{R}^2 \times S^2 \times \mathbb{R}^3 \ni (q, \gamma, \Omega)$, with projection

$$(q, g, \Omega) \mapsto \big(q, \gamma(g), \Omega\big) .$$

We embed M/S^1 in $\mathbb{R}^9 \ni (X, \gamma, \Omega)$, as the submanifold given by

$$\|\gamma\| = 1 , \qquad (X - \rho) \cdot \gamma = 0 \tag{28}$$

where ρ stands for $\rho = F(\gamma)$ (see the proof of Proposition 13).

The definition (20) of K can be inverted to give

$$\Omega(\gamma, K) = AK + \frac{mA\rho \cdot K}{1 - mA\rho \cdot \rho} A\rho, \tag{29}$$

where $A = (\mathbb{I} + m\|\rho\|^2 \mathbb{1})^{-1}$ and $\rho = F(\gamma)$. Therefore, as (global) coordinates on $\mathbb{R}^9$ we may use (X, γ, K) instead of (X, γ, Ω).

Proposition 16. *The equations of motion of the S^1-reduced system are the restriction to the submanifold (28) of the equations*

$$\begin{aligned} \dot{X} &= (X - \rho) \times (\Omega - \kappa\gamma) \\ \dot{\gamma} &= \gamma \times \Omega \\ \dot{K} &= K \times \Omega + m\dot{\rho} \times (\Omega \times \rho) + m\mathcal{G}\gamma \times \rho + m\kappa\rho \times (\kappa X - \dot{\rho} \times \gamma) \end{aligned} \tag{30}$$

on $\mathbb{R}^9 \ni (X, \gamma, K)$, where $\Omega = \Omega(\gamma, K)$ is as in (29) and $\dot{\rho}$ is shorthand for $DF(\gamma)(\gamma \times \Omega)$.[7]

The equation for γ is the well-known evolution equation of the Poisson vector γ that can be deduced by direct differentiation of the defining relation $\gamma = g^{-1} e_3^s$. The evolution equation for X follows by differentiating $X = g^{-1}x$ and using the nonholonomic constraint (22). Both of these equations are kinematical. The equation for K is a balance of momentum. The full dynamics of the system on M is obtained by adjoining the reconstruction equation $\dot{g} = g\hat{\Omega}$.

Proof. We begin by writing the equations of motion as

$$m\ddot{x} = -m\mathcal{G}e_3 + R_1, \qquad \frac{d}{dt}(\mathbb{I}\Omega) = \mathbb{I}\Omega \times \Omega + R_2, \tag{31}$$

where $R = (R_1, R_2)$ is the nonholonomic constraint force/torque. D'Alembert's principle states that (R_1, R_2) should annihilate any vector in the distribution $\mathcal{D}$. In view of (16) one finds that $R_2 = (g^{-1}R_1) \times \rho$. On the other hand, differentiating the constraint (22) and combining it with the first of the above equations yields

$$g^{-1}R_1 = m\mathcal{G}\gamma + m\Omega \times (\Omega \times \rho) + m\dot{\Omega} \times \rho + m\Omega \times \dot{\rho} + m\kappa\gamma \times (g^{-1}\dot{x} - \Omega \times \rho - \dot{\rho})$$

Using again (22) and (28) this simplifies to

$$g^{-1}R_1 = m\mathcal{G}\gamma + m\Omega \times (\Omega \times \rho) + m\dot{\Omega} \times \rho + m\Omega \times \dot{\rho} + m\kappa\dot{\rho} \times \gamma - m\kappa^2(X - \rho).$$

[7] It is immediate to check that both $\|\gamma\|^2$ and $\gamma \cdot (X - \rho)$ are first integrals of (30). For the latter one should use the kinematic relation $\dot{\rho} \cdot \gamma = 0$.

Using this expression and substituting $R_2 = (g^{-1}R_1) \times \rho$ in the second equation of (31) gives

$$\frac{d}{dt}(\mathbb{I}\Omega) = \mathbb{I}\Omega \times \Omega + m\mathcal{G}\gamma \times \rho + m(\Omega \times (\Omega \times \rho)) \times \rho + m(\dot{\Omega} \times \rho) \times \rho + m(\Omega \times \dot{\rho}) \times \rho$$
$$+ m\kappa\rho \times (\kappa X - \dot{\rho} \times \gamma)\,.$$

A simple calculation that uses the definition of K shows that the above relation is equivalent to the last equation in (30). □

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