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THE GREENBERG FUNCTOR REVISITED

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Abstract. We extend Greenberg's original construction to arbitrary schemes over (certain types of) local artinian rings. We then establish a number of properties of the extended functor and determine, for example, its behavior under Weil restriction. We also discuss a formal analog of the functor.

Contents

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1. Introduction

As already noted by Lang in his thesis [La, p. 381], the problem of finding a zero of a polynomial $f(x_1, \ldots, x_n)$ with coefficients in a complete discrete valuation ring $\mathscr{O}$ with perfect residue field k is equivalent to finding a common zero of infinitely many polynomials with coefficients in k. Working modulo powers of the maximal ideal of $\mathscr{O}$ effectively simplifies the problem since, in this setting, one only has to deal with finitely many polinomials with coefficients in k. The general case is then treated via a limit construction. Lang's idea was developed by his student Greenberg in the papers [Gre1, Gre2], where Greenberg introduced and studied the objects that are now called the *Greenberg realization* and the *Greenberg functor*. While Greenberg was writing his thesis, Grothendieck clarified his construction as an analog of Weil restriction [CS, p. 89]. Sometime later Serre applied the Greenberg functor in an unpublished proof of the so-called Serre-Tate theorem on formal liftings of abelian varieties in the case of ordinary reduction [CS, p. 161]. Serre also used the Greenberg functor in [Se] in order to prove that, when k is algebraically closed, the abelian extensions of the fraction field of $\mathscr{O}$ correspond bijectively to isogenies of the group of units of $\mathscr{O}$, regarded as a projective limit of algebraic k-groups. Since those times the Greenberg functor has played an important role in arithmetic and algebraic geometry. See [Bég, BLR, CGP, Lip] and, more recently, [BT, NS, NS2, Sta]. The work of Greenberg came to our attention in the course of our attempts to generalize the results of Bégueri [Bég] over a non-algebraically closed residue field k. Our main difficulty in understanding Greenberg's ideas originated in his use of a pre-Grothendieck language to describe the key construction of *Greenberg algebras* [Gre1, §1]. Further, some of his original results, stated for varieties, do not easily extend to more general schemes. These problems have affected other researchers as well, since a number of errors connected with the use of the Greenberg functor have appeared in print.

In this paper we revisit Greenberg's construction using a modern scheme-theoretic language and generalize it in various ways, removing in particular certain unnecessary reducedness and finiteness conditions assumed in [Gre1, Gre2]. Further, we refine known properties of the classical Greenberg functor, establish new properties and correct certain erroneous claims about this functor that appear in the literature. We also clarify the relation that exists between the Greenberg algebra $\mathscr{R}$ associated to a local artinian ring $\mathfrak{R}$ (of a certain type) and the Greenberg module $\mathscr{I}$ associated to an ideal $\mathfrak{I}$ of $\mathfrak{R}$. We expect to use the results of this paper to investigate (elsewhere) certain interesting problems in arithmetical algebraic geometry. We should also note that the present paper is an abridged, and hopefully more readable, version of our preprint [BGA], where all tedious calculations omitted from this version have been fully worked out for the benefit of the punctilious reader.

We now describe in more detail the contents of the paper. Section 2 contains a general discussion of Greenberg modules/algebras associated to finite $W_m(k)$-modules/algebras, where $m \geq 1$ and (the field) k is assumed to be perfect and of positive characteristic if $m > 1$. Readers who are familiar with Greenberg's original construction will have noticed that this author encountered a number of technical difficulties that forced him to work only up to purely inseparable morphisms, e.g., in the proof of the fundamental theorem in [Gre2]. See Remark 2.14 and Appendix A.2. In this paper we correctly identify the ideal subscheme (2.15) of the relevant Greenberg algebra that must be chosen in order to circumvent all such technical difficulties.

In Section 3 we specialize the discussion of Section 2 to truncated discrete valuation rings and refine, for use in future applications, the presentation given in this case by Nicaise and Sebag [NS]. Incidentally, the above authors seem to have been the first to have noticed that a certain formula involving Greenberg algebras that appears in [BLR, p. 276, line -18] is incorrect. In Remark 7.4 we explain why the indicated error is (fortunately) inconsequential when working with the tower of Greenberg algebras.

Section 4 discusses the behavior of Greenberg algebras under (possibly) ramified extension of local artinian rings. The very brief Section 5 contains the definition of the Greenberg algebra associated to a discrete valuation ring and some related remarks. Section 6 introduces the Greenberg functor $\mathrm{Gr}^{\mathfrak{R}}$ in the general setting of this paper. This functor associates to an $\mathfrak{R}$-scheme X a k-scheme $\mathrm{Gr}^{\mathfrak{R}}(X)$ whose set of k-rational points is in bijection with the set of $\mathfrak{R}$-sections of X. The existence of $\mathrm{Gr}^{\mathfrak{R}}(X)$ is established via a careful discussion of the functor $h^{\mathfrak{R}}$ that is left-adjoint to $\mathrm{Gr}^{\mathfrak{R}}$. The constructions of Section 6 are then specialized to truncated discrete valuation rings in Section 7. In Section 8 we show that the change of rings morphism $\mathrm{Gr}^{\mathfrak{R}}(X) \to \mathrm{Gr}^{\mathfrak{R}'}(X')$ (8.1) is always affine, and surjective (respectively, an isomorphism) if X is smooth (respectively, étale) over $\mathfrak{R}$. In Section 9 we show that the Greenberg functor preserves a number of basic properties of morphisms. In particular, we show that it preserves quasi-projective schemes (see Proposition 9.1). Section 10 describes the behavior of the Greenberg functor under Weil restriction. See Theorem 10.2. To our knowledge, only a very specific instance of this result has appeared in print (within the context of formal geometry), namely [NS, Theorem 4.1].

In Section 11 we describe the kernel of the change of level morphism (8.3) using the Structure Theorem from Appendix A.2. In particular, we show in Remark 11.13 that [Bég, Lemma 4.1.1(2)] is false. In spite of the above, the main results of [Bég] are (fortunately) valid since [Bég] works mostly with the *perfect* Greenberg functor (discussed here in Section 12), which annihilates all possible infinitesimal error terms. See Remark 12.1 for more details.

We now observe that Sebag defined in [Seb, §3] the Greenberg realization of a separated formal scheme of topologically finite type. In Section 13 we extend his construction to the larger category of adic formal schemes and determine the behavior of the new functor under Weil restriction. In particular, we generalize [NS, Theorem 4.1]. The constructions of Section 13 are then applied in Section 14 to discuss the Greenberg realization of an R-scheme, where R is a complete discrete valuation ring.

Section 15 contains information on the Greenberg realization of a finite group scheme, which may not itself be finite over k.

Section 16 discusses the Greenberg realization of a flat, commutative and separated R-group scheme F, where R is as above, using a smooth resolution of F when one exists (this is the case if F is finite over R). In particular, we obtain results on the kernel and cokernel of the change of level morphism (see Proposition 16.3) and on certain algebraic groups related to F-torsors. See (16.7) and (16.11).

The Appendix consists of three Subsections. In subsection A.1 we discuss the Weil restriction functor and show that the hypotheses in the basic existence theorem [BLR, §7.6, Theorem 4, p. 194] can be weakened. We also record here the fundamental fact that the Weil restriction of a scheme along a finite and locally free *universal homeomorphism* always exists. In Section A.2 we extend Greenberg's structure theorem [Gre2, p. 263], showing in particular that the original version of the indicated result is unaffected by Greenberg's occasional replacement of certain Greenberg modules by inseparably-isogenous group varieties. Subsection A.3 consists of a single proposition where sufficient conditions are given for a morphism of smooth and commutative group schemes over a field to be flat.

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2. Greenberg modules and algebras

Let k be a perfect field of positive characteristic and let $m \geq 1$ be an integer. In [Lip, Appendix A], Lipman translated into scheme-theoretic language Greenberg's construction of Greenberg modules in [Gre1]. In this Section we extend Lipman's translation to other constructions/statements from [Gre1, Gre2].

For any scheme S, we will write $\mathbb{O}_S$ (or $\mathbb{O}_A$ if $S = \operatorname{Spec} A$ is affine) for the S-ring scheme $\mathbb{V}(\mathscr{O}_S) = \operatorname{Spec} \mathscr{O}_S[T]$.

2.1. **Finitely generated modules over arbitrary fields.** In this Subsection k is an arbitrary field. Let $\mathfrak{M}$ be a finitely generated k-module of rank $r \geq 1$. The *Greenberg module* associated to $\mathfrak{M}$, denoted by $\mathscr{M}$, is the affine k-scheme that represents the functor $\operatorname{Spec} A \mapsto \mathfrak{M} \otimes_k A$, where A is a k-algebra, i.e.,

$$\mathscr{M}(A) \overset{\text{def.}}{=} \operatorname{Hom}_k(\operatorname{Spec} A, \mathscr{M}) = \mathfrak{M} \otimes_k A. \tag{2.1}$$

Note that, for any choice of k-basis $\{m_1, \dots, m_r\}$ of $\mathfrak{M}$, there exists an isomorphism of $\mathbb{O}_k$-module schemes $\mathbb{A}_k^r \simeq \mathscr{M}$ given on A-sections by $A^r \xrightarrow{\sim} \mathscr{M}(A)$, $(a_i) \mapsto \sum_i m_i \otimes a_i$.

Remarks 2.2.

(a) If $A \to B$ is an injective (respectively, surjective) homomorphism of k-algebras, then the induced homomorphism of k-modules $\mathscr{M}(A) \to \mathscr{M}(B)$ is injective (respectively, surjective).

(b) If $\mathfrak{M} \to \mathfrak{M}'$ is a surjective homomorphism of finitely generated k-modules and A is a k-algebra, then the induced map $\mathscr{M}(A) \to \mathscr{M}'(A)$ is a surjective homomorphism of A-modules.

Let $\mathfrak{R}$ be a finite k-algebra. Since $\mathfrak{R}$ is a finitely generated k-module, its associated Greenberg module $\mathscr{R}$ can be defined as above. Now

$$\mathscr{R}(A) = \mathfrak{R} \otimes_k A. \tag{2.3}$$

is naturally endowed with an $\mathfrak{R}$-algebra structure. The resulting k-ring scheme $\mathscr{R}$ is called the *Greenberg algebra associated to* $\mathfrak{R}$. Note that $\mathscr{R}(k) = \mathfrak{R}$. By construction, there exists a (non-canonical) isomorphism of k-group schemes

uul (2.4)
$$\mathscr{R} \simeq \mathbb{G}_{a,k}^{\ell},$$

where $\ell = \dim_k \mathfrak{R} \geq 1$. Further, by (2.3), we have $\mathscr{R} = \mathrm{Res}_{\mathfrak{R}/k}(\mathbb{O}_{\mathfrak{R}})$ (Weil restriction. See Apprendix A.1). If $f \in A$, then $\mathscr{R}(A)_f = \mathscr{R}(A) \otimes_A A_f$, whence

very0 (2.5)
$$\mathscr{R}(A)_f = \mathscr{R}(A_f).$$

Now let $\mathfrak{R} \to \mathfrak{R}'$ be a homomorphism of finite k-algebras with kernel $\mathfrak{K}$ and let $\mathscr{R}, \mathscr{R}'$ and $\mathscr{K}$ be the Greenberg algebras/modules associated to $\mathfrak{R}, \mathfrak{R}'$ and $\mathfrak{K}$, respectively. By (2.1), the canonical exact sequence of k-modules $0 \to \mathfrak{K} \to \mathfrak{R} \to \mathfrak{R}'$ induces, for every k-algebra A, an exact sequence of $\mathfrak{R}$-A-bimodules $0 \to \mathscr{K}(A) \to \mathscr{R}(A) \to \mathscr{R}'(A)$, where

kar (2.6)
$$\mathscr{K}(A) = \mathfrak{K} \otimes_k A = \mathfrak{K}\mathscr{R}(A).$$

We conclude that

eqcase (2.7)
$$\mathscr{K} = \mathrm{Ker}\left[\mathscr{R} \to \mathscr{R}'\right].$$

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2.2. **Modules over rings of Witt vectors.** In this Subsection k is a perfect field of characteristic $p > 0$ and $\mathbb{W}_m$ denotes the k-ring scheme of Witt vectors of length $m > 1$. Let $\mathfrak{M}$ be a finitely generated $W_m(k)$-module and let $\mathbf{M}$ denote the fpqc sheaf on the category of affine k-schemes associated to the presheaf $\mathrm{Spec}\, A \mapsto \mathfrak{M} \otimes_{W(k)} W(A)$, where A is a k-algebra. By [Lip, Proposition A.1], there exists an affine $\mathbb{W}_m$-module scheme $\mathscr{M}$, called the *Greenberg module associated* to $\mathfrak{M}$, which represents $\mathbf{M}$, i.e., $\mathbf{M}(\mathrm{Spec}\, A) = \mathscr{M}(A)$, where

$$\mathscr{M}(A) \overset{\text{def.}}{=} \mathrm{Hom}_k(\mathrm{Spec}\, A, \mathscr{M}).$$

Further, by [Lip, Corollary A.2], the canonical map

lips (2.8)
$$\mathfrak{M} \otimes_{W_m(k)} W_m(A) \twoheadrightarrow \mathscr{M}(A)$$

is surjective for every k-algebra A. By construction, a choice of an isomorphism of $W_m(k)$-modules $\mathfrak{M} \simeq \prod_{i=0}^{r} W_{n_i}(k)$, where $n_i \leq m$ for every i, induces an isomorphism of $\mathbb{W}_m$-module schemes $\mathscr{M} \simeq \prod_{i=0}^{r} \mathbb{W}_{n_i}$. In particular, there exists an isomorphism of k-schemes $\mathscr{M} \simeq \mathbb{A}_k^N$, where $N = \sum_{i=1}^{r} n_i$ is the length of the $W_m(k)$-module $\mathfrak{M}$. Every homomorphism of finitely generated $W_m(k)$-modules $\mathfrak{M} \to \mathfrak{M}'$ induces a morphism of associated $\mathbb{W}_m$-module schemes $\mathscr{M} \to \mathscr{M}'$ [Lip, Proposition A.1, p. 74].

resp *Remarks* 2.9.

(a) Since $\mathscr{M} \simeq \mathbb{A}_k^N$, an injective (respectively, surjective) homomorphism of k-algebras $A \to B$ induces an injective (respectively, surjective) homomorphism of $W_m(k)$-modules $\mathscr{M}(A) \to \mathscr{M}(B)$.

(b) If $\mathfrak{M} \to \mathfrak{M}'$ is a surjective homomorphism of finitely generated $W_m(k)$-modules and A is a k-algebra, then the surjectivity of (2.8) (for both $\mathscr{M}$ and $\mathscr{M}'$) implies that the induced homomorphism $\mathscr{M}(A) \to \mathscr{M}'(A)$ is surjective.

(c) If $A = A^p$, then (2.8) is an isomorphism by [Lip, Corollary A.2]. Further, there exists a canonical isomorphism of $W_m(A)$-modules $\mathfrak{M} \otimes_{W_m(k)} W_m(A) \simeq \mathscr{M}(A)$ via the identification $W_m(k) \otimes_{W(k)} W(A) = W_m(A)$.

Let $\mathfrak{R}$ be a finite $W_m(k)$-algebra. The *Greenberg algebra associated to* $\mathfrak{R}$ is the Greenberg module associated to $\mathfrak{R}$ together with its $\mathbb{W}_m$-algebra structure induced by (2.8). Every isomorphism of $W_m(k)$-modules $\mathfrak{R} \simeq \prod_{i=0}^{r} W_{n_i}(k)$ induces an isomorphism of $\mathbb{W}_m$-module schemes $\mathscr{R} \simeq \prod_{i=0}^{r} \mathbb{W}_{n_i}$ and the k-ring scheme structure on $\mathscr{R}$ is induced by the ring structure on $\mathfrak{R}$ [Lip, Proposition A.1 and Corollary A.2]. In particular, there exists a (non-canonical) isomorphism of k-schemes

uul2 (2.10)
$$\mathscr{R} \simeq \mathbb{A}_k^{\ell} \quad (\text{where } \ell = \text{length}_{W_m(k)} \mathfrak{R})$$

and we have $\mathscr{R}(k) = \mathfrak{R}$. If $\mathfrak{R} = W_m(k)$, then $\mathscr{R} = \mathbb{W}_m$. Further, every finitely generated $\mathfrak{R}$-module $\mathfrak{B}$ defines an $\mathscr{R}$-module scheme $\mathscr{B}$ and every homomorphism $\mathfrak{B} \to \mathfrak{C}$ of finitely generated $\mathfrak{R}$-modules induces a k-morphism $\mathscr{B} \to \mathscr{C}$ of associated $\mathscr{R}$-module schemes.

uch *Remark* 2.11.

(a) If $\mathfrak{K}$ is an ideal of $\mathfrak{R}$, then the image of the canonical homomorphism $\mathscr{K}(A) \to \mathscr{R}(A)$ equals $\mathfrak{K}\mathscr{R}(A)$, as follows from the surjectivity of (2.8) (for both $\mathscr{K}$ and $\mathscr{R}$).

(b) If $A = A^p$, then the homomorphism of $W_m(A)$-algebras $\mathfrak{R} \otimes_{W_m(k)} W_m(A) \to \mathscr{R}(A)$ (2.8) is an isomorphism. Further, there exists a canonical isomorphism of $\mathfrak{R}$-$W(A)$-bialgebras $\mathfrak{R} \otimes_{W(k)} W(A) \simeq \mathscr{R}(A)$. See Remark 2.9(c).

Together with (2.5), the following proposition is the key to establishing the representability of the Greenberg functor (6.9) in a general scheme-theoretic setting.

very **Proposition 2.12.** *Let $\mathfrak{R}$ be a finite $W_m(k)$-algebra with associated Greenberg algebra $\mathscr{R}$ and let A be a k-algebra. For every $f \in A$, there exists a canonical isomorphism of $\mathscr{R}(A)$-algebras*

$$\mathscr{R}(A)_{[f]} \xrightarrow{\sim} \mathscr{R}(A_f)$$

where $[f] = (f, 0, \dots, 0) \in W_m(A)$.

Proof. By [Ill, (1.1.9), p. 505, and (1.5.3), p. 512], the homomorphism

$$W_m(A)_{[f]} \xrightarrow{\sim} W_m(A_f), (a_0, \dots, a_{m-1})/[f]^r \mapsto (a_0, \dots, a_{m-1}) \cdot [1/f^r],$$

is an isomorphism. Thus, by Remark 2.11(b), the proposition holds if $A = A^p$. The general case follows by using the existence of faithfully flat extensions $A \to B$ with $B = B^p$ [Lip, Lemma 0.1, p. 18]. See [BGA2, Proposition 3.16] for the details. □

We discuss next the k-morphism $\mathscr{B} \to \mathscr{C}$ induced by an inclusion of finitely generated $W_m(k)$-modules $\mathfrak{B} \subseteq \mathfrak{C}$. We begin with an example.

Example 2.13. Let $\mathfrak{B} = pW_m(k)$ and $\mathfrak{C} = W_m(k)$. The isomorphism of $W_m(k)$-modules

$$W_{m-1}(k) \xrightarrow{\sim} pW_m(k), (a_0, \dots, a_{m-2}) \mapsto (0, a_0^p, \dots, a_{m-2}^p),$$

induces an isomorphism of $\mathbb{W}_m$-module schemes $\mathbb{W}_{m-1} \simeq \mathscr{B}$. On the other hand, the morphism of $\mathbb{W}_m$-module schemes $\mathscr{B} \to \mathscr{C}$ induced by the inclusion $\mathfrak{B} \subset \mathfrak{C}$ corresponds to the morphism $\mathbb{W}_{m-1} \to \mathbb{W}_m$ given by

$$W_{m-1}(A) \to W_m(A), (a_0, \dots, a_{m-2}) \mapsto (0, a_0^p, \dots, a_{m-2}^p).$$

Thus, if A is not reduced, then the preceding map is not injective.

As the above example shows, we cannot expect $\mathscr{B} \to \mathscr{C}$ to be a closed immersion in general. This fact has the following undesiderable consequence. Let $\mathfrak{R} \to \mathfrak{R}'$ be a homomorphism of finite $W_m(k)$-algebras with kernel $\mathfrak{K}$. Let $\mathscr{R} \to \mathscr{R}'$ be the induced morphism of associated $\mathbb{W}_m$-module schemes and let $\mathscr{K}$ be the $\mathscr{R}$-module scheme which corresponds to $\mathfrak{K}$. Since the composite map $\mathfrak{K} \to \mathfrak{R} \to \mathfrak{R}'$ is the zero homomorphism, the composite of induced morphisms of Greenberg modules $\mathscr{K} \to \mathscr{R} \to \mathscr{R}'$ is the zero morphism. However, in contrast to (2.7), $\mathscr{K}$ may fail to be equal to the kernel of $\mathscr{R} \to \mathscr{R}'$.

Remark 2.14. The following statement appears in [Gre2, p. 257]. *Suppose that $\mathfrak{I}$ is the kernel of a surjective homomorphism* [of finite and local $W_m(k)$-algebras] $\varphi \colon \mathfrak{R} \to \mathfrak{R}'$ *and* $\mathfrak{I}\mathfrak{M} = 0$ [where $\mathfrak{M}$ is the maximal ideal of $\mathfrak{R}$]. *Then, for every pre-scheme Y over k, the homomorphism $\varphi(Y) \colon \mathscr{R}(Y) \to \mathscr{R}'(Y)$ is surjective with kernel $\mathscr{I}(Y)$ and $\mathscr{M}(Y)\mathscr{I}(Y) = 0$.* The preceding statement is false if $\mathscr{I}$ and $\mathscr{M}$ are the Greenberg module schemes associated to $\mathfrak{I}$ and $\mathfrak{M}$ or, in the terminology of [Gre1, Proposition 3, p. 628], if $\mathscr{I}$ and $\mathscr{M}$ are the Greenberg varieties *equipped with their maximal structures* associated to $\mathfrak{I}$ and $\mathfrak{M}$. We believe that Greenberg was well aware of this fact, which led him to changing the way in which a module variety is attached to a $W_m(k)$-module depending on the particular situation being considered. See [Gre1, lines above Proposition 4, p. 629] and [Gre2, p. 257, lines 5–8].

In order to obtain a correct scheme-theoretic version of Greenberg's statement just quoted, we proceed as follows.

Let $\mathfrak{R}$ be a finite $W_m(k)$-algebra and let $\mathfrak{I}$ be an ideal of $\mathfrak{R}$. The *ideal subscheme of $\mathscr{R}$ associated to* $\mathfrak{I}$ is the ideal subscheme of $\mathscr{R}$

$$\bar{\mathscr{I}} = \operatorname{Ker}\left[\mathscr{R} \to \mathscr{R}'\right], \tag{2.15}$$

where $\mathscr{R}'$ is the $\mathscr{R}$-algebra associated to $\mathfrak{R}' = \mathfrak{R}/\mathfrak{I}$. The canonical exact sequence of $W_m(k)$-modules $0 \to \mathfrak{I} \to \mathfrak{R} \to \mathfrak{R}'$ induces a complex of $\mathbb{W}_m$-module schemes $\mathscr{I} \to \mathscr{R} \to \mathscr{R}'$. Consequently, there exists a canonical morphism of $\mathscr{R}$-module schemes

$$\Theta_{\mathfrak{I}} \colon \mathscr{I} \to \bar{\mathscr{I}}. \tag{2.16}$$

By Remark 2.11(a), we have

$$\operatorname{Im}[\Theta_{\mathfrak{I}}(A) \colon \mathscr{I}(A) \to \bar{\mathscr{I}}(A)] = \mathfrak{I}\,\mathscr{R}(A) \tag{2.17}$$

for every k-algebra A.

Proposition 2.18. *Let $\mathfrak{R}$ be a finite $W_m(k)$-algebra and let $\mathfrak{I}$ be an ideal of $\mathfrak{R}$. If A is a k-algebra such that $A = A^p$, then the homomorphism of $\mathscr{R}(A)$-modules*

$$\Theta_{\mathfrak{I}}(A) \colon \mathscr{I}(A) \to \bar{\mathscr{I}}(A)$$

is surjective. Further, if A is perfect, then the preceding map is an isomorphism.

Proof. Recall $\mathfrak{R}' = \mathfrak{R}/\mathfrak{I}$. There exists a canonical commutative diagram of $W_m(A)$-modules

d.i (2.19)
$$\begin{array}{ccccccccc}
0 & \dashrightarrow & \mathfrak{I}\otimes_{W(k)} W(A) & \longrightarrow & \mathfrak{R}\otimes_{W(k)} W(A) & \longrightarrow & \mathfrak{R}'\otimes_{W(k)} W(A) & \longrightarrow & 0\\
 & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
0 & \dashrightarrow & \mathscr{I}(A) & \longrightarrow & \mathscr{R}(A) & \longrightarrow & \mathscr{R}'(A) & \longrightarrow & 0\\
 & & \downarrow\Theta_{\mathfrak{I}}(A) & & \| & & \| & & \\
0 & \longrightarrow & \overline{\mathscr{I}}(A) & \longrightarrow & \mathscr{R}(A) & \longrightarrow & \mathscr{R}'(A) & \longrightarrow & 0.
\end{array}$$

The vertical arrows in the top rectangle are isomorphisms by Remark 2.11(b). Further, the top row of the diagram (excluding the broken arrow) is exact by the right-exactness of the tensor product functor. Thus the middle row (excluding the broken arrow) is exact as well. Since the bottom row of the diagram is exact by (2.15) and Remark 2.9(b), the surjectivity of $\Theta_{\mathfrak{I}}(A)$ follows.

Now assume that A is perfect. Then the broken arrows in the above diagram can be filled in since $W(A)$ is flat over $W(k)$ [BGA2, Lemma 2.24]. The bijectivity of $\Theta_{\mathfrak{I}}(A)$ is then immediate. □

barp **Lemma 2.20.** *Let $\mathfrak{R}\to\mathfrak{R}'$ and $\mathfrak{R}\to\mathfrak{R}''$ be surjective homomorphisms of finite $W_m(k)$-algebras with kernels $\mathfrak{I}$ and $\mathfrak{J}$ which satisfy $\mathfrak{J}\mathfrak{I}=0$. Then, for every k-scheme Y, the ring homomorphism $\mathscr{R}(Y)\to\mathscr{R}'(Y)$ induced by $\mathfrak{R}\to\mathfrak{R}'$ is surjective with kernel $\overline{\mathscr{I}}(Y)$ and $\overline{\mathscr{J}}(Y)\overline{\mathscr{I}}(Y)=0$.*

Proof. Up to isomorphisms we may assume that $\mathfrak{R}'=\mathfrak{R}/\mathfrak{I}$ and $\mathfrak{R}''=\mathfrak{R}/\mathfrak{J}$. Now, since $\mathscr{R}'$ is affine, the morphism $\mathscr{R}\to\mathscr{R}'$ has a section by Remark 2.9(b) and the surjectivity of $\mathscr{R}(Y)\to\mathscr{R}'(Y)$ is clear. The kernel of the latter map is $\overline{\mathscr{I}}(Y)$ by *definition* of $\overline{\mathscr{I}}$. In order to check that $\overline{\mathscr{J}}(Y)\overline{\mathscr{I}}(Y)=0$, we may assume that $Y=\operatorname{Spec} A$, where A is a k-algebra. Since A has a faithfully flat extension B with $B^p=B$ [Lip, Lemma 0.1, p. 18], we may assume that $A=A^p$. In this case the assertion follows from diagram (2.19), and the analogous diagram for $\mathfrak{J}$ in place of $\mathfrak{I}$, using the fact that $\mathfrak{I}\mathfrak{J}=0$. □

fgt *Remark* 2.21. If $m=1$ and k is arbitrary, then the preceding considerations work equally well and the resulting Greenberg modules (respectively, algebras) coincide with those defined in the previous Subsection. In this case $\overline{\mathscr{I}}=\mathscr{I}$, $\Theta_{\mathfrak{I}}(A)$ is the identity map and Lemma 2.20 is also valid.

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2.3. **A common approach.** We discuss the two cases of the previous subsections simultaneously using the following convention: $\mathfrak{R}$ will denote a finite $W_m(k)$-algebra, where $m\geq 1$ and k is assumed to be perfect and of positive characteristic if $m>1$.

Let $\mathfrak{I}$ be an ideal of $\mathfrak{R}$, $i\geq 1$ an integer and A a k-algebra. We will write $\mathscr{I}^i$ for the $\mathbb{W}_m$-module scheme associated to the ideal $\mathfrak{I}^i$ [1].

By Lemma 2.20 and Remark 2.21, the exact sequence of $\mathfrak{R}$-modules $0\to\mathfrak{I}^i\to\mathfrak{R}\to\mathfrak{R}/\mathfrak{I}^i\to 0$ induces an exact exact sequence of $\mathscr{R}(A)$-modules

rnis (2.22)
$$0\to\overline{\mathscr{I}^i}(A)\to\mathscr{R}(A)\to\mathscr{R}^{(\mathscr{I}^i)}(A)\to 0,$$

[1] $\mathscr{I}^i$ should not be mistaken for an "i-th power of $\mathscr{I}$". The latter, in fact, cannot be defined since, in general, $\mathscr{I}$ is not an ideal subscheme of $\mathscr{R}$.

where $\mathscr{R}^{(\mathscr{I}^i)}$ is the Greenberg algebra associated to $\mathfrak{R}/\mathfrak{I}^i$. Now let $i, j \geq 1$ be integers. Applying Lemma 2.20 to the ideals $\mathfrak{I}^i/\mathfrak{I}^{i+j}$ and $\mathfrak{I}^j/\mathfrak{I}^{i+j}$ of $\mathfrak{R}/\mathfrak{I}^{i+j}$, we conclude that $\overline{\mathscr{I}^i}(A)\overline{\mathscr{I}^j}(A) \subseteq \overline{\mathscr{I}^{i+j}}(A)$. In particular, for every integer $r \geq 1$,

$$\bar{\mathscr{I}}(A)^r \subseteq \overline{\mathscr{I}^r}(A). \tag{2.23}$$

Thus, since $\mathscr{I}^n = 0$ when $\mathfrak{I}^n = 0$, we have

$$\bar{\mathscr{I}}(A)^n = 0 \qquad (\text{if } \mathfrak{I}^n = 0). \tag{2.24}$$

We will also need the following construction. By Remarks 2.2(b) and 2.9(b), if A is a k-algebra and I is a proper ideal of A, then there exists an exact sequence of $\mathfrak{R}$-modules

$$0 \to \mathscr{R}(I) \to \mathscr{R}(A) \to \mathscr{R}(A/I) \to 0, \tag{2.25}$$

where $\mathscr{R}(I) = \operatorname{Ker}[\mathscr{R}(A) \to \mathscr{R}(A/I)]$.

Lemma 2.26. *Let A be a k-algebra and let I and J be ideals of A. Then*

$$\mathscr{R}(I)\mathscr{R}(J) \subseteq \mathscr{R}(IJ).$$

Proof. This follows from the fact that the functor $\mathscr{R}(-)$ is representable. □

3. The Greenberg algebra of a truncated discrete valuation ring

In this Section we discuss the Greenberg algebras associated to truncated discrete valuation rings, which are the motivating examples of the theory. Let R be a discrete valuation ring with valuation v, maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, assumed to be perfect in the unequal characteristics case. For every $n \in \mathbb{N}$, set $R_n = R/\mathfrak{m}^n$. Since in this section we discuss constructions that depend only on the truncations R_n, *we now assume, without loss of generality, that R is complete.* Now, for every $n \in \mathbb{N}$, set $M_n = \mathfrak{m}/\mathfrak{m}^n$. If $\pi \in \mathfrak{m}$ is a uniformizer, π_n will denote the corresponding element in M_n. We will write $S = \operatorname{Spec} R$ and $S_n = \operatorname{Spec} R_n$.

Now let n, s be integers such that $n \geq s \geq 1$. Then multiplication by π^s on R induces a surjective homomorphism of R_n-modules $R_n \to M_n^s$ whose kernel is M_n^{n-s}. Thus we obtain an isomorphism of R_n-modules

$$R_{n-s} \xrightarrow{\sim} M_n^s,\ r + \mathfrak{m}^{n-s} \mapsto \pi^s r + \mathfrak{m}^n \quad (r \in R). \tag{3.1}$$

If R is an equal characteristic ring, then there exists an isomorphism $\xi \colon k[[t]] \xrightarrow{\sim} R$ and $\pi = \xi(t)$ is a uniformizer of R. Note that R_n is a finite k-algebra with basis $1, \pi_n, \dots, \pi_n^{n-1}$. In particular, the ring R_n is of the type discussed in Subsection 2.1 and the Greenberg algebra associated to R_n is the k-ring scheme

$$\mathscr{R}_n = \operatorname{Res}_{R_n/k}(\mathbb{O}_{R_n}). \tag{3.2}$$

Note that $\mathscr{R}_1 = \mathbb{O}_k$ and $\mathscr{R}_n(k) = R_n$ for every $n \geq 1$. Now, by (2.1) and (2.3), for every k-algebra A we have

$$\mathscr{R}_n(A) = R_n \otimes_k A \tag{3.3}$$

and $\mathscr{M}_n(A) = M_n \otimes_k A = \pi_n \mathscr{R}_n(A) \subseteq \mathscr{R}_n(A)$.

If R has unequal characteristics and perfect residue field k, then R is a totally ramified extension of $W(k)$ of degree $\bar{e}=v(p)\geq 1$. If $\bar{e}>1$, then there exists an isomorphism

eis
$$\xi\colon W(k)[T]/(f)\xrightarrow{\sim} R \tag{3.4}$$

where f is an Eisenstein polynomial of degree $\bar{e}$. Further, $\pi=\xi(T+(f))$ is a uniformizer of R and the artinian local ring R_n has characteristic p^m, where

m
$$m=\lceil n/\bar{e}\rceil \tag{3.5}$$

is the smallest integer that is larger than or equal to $n/\bar{e}$. As a $W_m(k)$-module, R_n can be written as an internal direct sum $W_m(k)\oplus W_m(k)\cdot\pi_n\oplus\cdots\oplus W_m(k)\cdot\pi_n^r$, with $r=\min\{\bar{e}-1,n-1\}$. Since

$$W_m(k)\cdot\pi_n^i\simeq W_{n_i}(k),\ \text{ with } n_i=\lceil (n-i)/\bar{e}\rceil,$$

there exists an isomorphism of $W_m(k)$-modules

dcp
$$R_n\simeq\prod_{i=0}^{r}W_{n_i}(k). \tag{3.6}$$

Thus the Greenberg algebra $\mathscr{R}_n$ is the $\mathbb{W}_m$-module scheme $\prod_{i=0}^{r}\mathbb{W}_{n_i}$ equipped with the ring structure induced by the rules $f(\pi_n)=\pi_n^n=0$. See also [NS, pp. 1591-94] and [NS2, §2.2]. If $R=W(k)$, then $\mathscr{R}_n=\mathbb{W}_n$.

nim0 *Remarks* 3.7.

(a) Write $n=q\bar{e}+\zeta$, where $0\leq\zeta<\bar{e}$ and $q\geq 0$. If $\zeta\neq 0$, then $n_i=m$ for $i<\zeta$ and $n_i=m-1$ for $i\geq\zeta$. If $\zeta=0$, then $n_i=m$ for all i.

(b) If $n\leq\bar{e}$, then $m=1$ and R_n is a finitely generated k-algebra. If $n>\bar{e}$, then $m>1$ and R_n is a type of ring discussed in Subsection 2.2.

Let R again be an arbitrary discrete valuation ring and let n,s be integers such that $n>s\geq 1$. Then R_n and R_s are finite $W_m(k)$-algebras, where m is given by (3.5) if R is an unequal characteristics ring and is equal to 1 otherwise. Thus we may now apply the discussion of Subsection 2.3 with $(\mathfrak{R},\mathfrak{I})=(R_n,M_n)$. Up to the identification $R_n/M_n^s=R_s$, we have an exact sequence of $\mathscr{R}_n(A)$-modules

rnms
$$0\to\overline{\mathscr{M}_n^s}(A)\to\mathscr{R}_n(A)\to\mathscr{R}_s(A)\to 0. \tag{3.8}$$

Now, by Lemma 2.20 and Remark 2.21,

ned2
$$\overline{\mathscr{M}_n^i}(A)\overline{\mathscr{M}_n^j}(A)=0\qquad\text{if } i+j\geq n. \tag{3.9}$$

Further, if $r\geq 1$ is an integer, then (2.23) yields

mincl
$$\overline{\mathscr{M}_n}(A)^r\subseteq\overline{\mathscr{M}_n^r}(A) \tag{3.10}$$

Thus, since $M_n^n=0$, we have $\overline{\mathscr{M}_n}(A)^n=0$. Now observe that, by (2.17), $\pi_n^s\mathscr{R}_n(A)\subseteq\overline{\mathscr{M}_n^s}(A)$. Next, (3.9) with $i=s$ yields

rnms1
$$\pi_n^s\overline{\mathscr{M}_n^j}(A)=0\qquad\text{if } j\geq n-s. \tag{3.11}$$

In other words, $\overline{\mathscr{M}_n^j}(A)$ is a π_n^s-torsion $\mathscr{R}_n(A)$-module for every $j\geq n-s$. We will write

ttr
$$\Theta_{n,s}\colon\mathscr{M}_n^s\to\overline{\mathscr{M}_n^s} \tag{3.12}$$

for the canonical map (2.16). Recall that, by Remark 2.21, (3.12) is the identity morphism in the equal characteristic case.

power *Remarks* 3.13.

(a) If $R = W(k)$ and V denotes the Verschiebung map, then $\overline{\mathscr{M}_n^s}(A) = V^s W_{n-s}(A) \subset W_n(A)$ for every k-algebra A.

(b) In general, the inclusion $\overline{\mathscr{M}_n}(A)^r \subseteq \overline{\mathscr{M}_n^r}(A)$ (3.10) is strict. For example, choose $R = W(k)$ and set $n = 3$ and $s = 1$ in (a). If $A \neq A^p$, then $(VW_2(A))^2$ is properly contained in the ideal $V^2W_1(A))$ of $W_3(A)$.

(c) The containment (3.10) is an equality in the unequal characteristics case if A is perfect and $n > \bar{e} = v(p)$ (so that $m > 1$ in (3.5)). Indeed, by Proposition 2.18, the map $\Theta_{n,s}(A)\colon \mathscr{M}_n^s(A) \to \overline{\mathscr{M}_n^s}(A)$ is an isomorphism for every n and $s \geq 1$. On the other hand, $\mathscr{M}_n^s(A) \simeq \pi_n^s \mathscr{R}_n(A) \simeq \mathscr{M}_n(A)^s$, as follows from Remark 2.11(b) and the flatness of $W(A)$ over $W(k)$.

(d) If R_n is a k-algebra, then (3.10) is an equality for every A. Indeed, in this case $\mathscr{M}_n^s(A) = \overline{\mathscr{M}_n^s}(A)$ by Remark 2.21 and $\mathscr{M}_n^s(A) \simeq \pi_n^s \mathscr{R}_n(A) \simeq \mathscr{M}_n(A)^s$ by (2.1).

The isomorphism of R_n-modules $R_{n-s} \xrightarrow{\sim} M_n^s$ (3.1) induces an isomorphism of $\mathscr{R}_n$-module schemes $\mathscr{R}_{n-s} \xrightarrow{\sim} \mathscr{M}_n^s$. We will write

pis
$$\varphi_{n,s}\colon \mathscr{R}_{n-s} \to \overline{\mathscr{M}_n^s} \tag{3.14}$$

for the composition $\mathscr{R}_{n-s} \xrightarrow{\sim} \mathscr{M}_n^s \to \overline{\mathscr{M}_n^s}$, where the second map is the morphism of $\mathscr{R}_n$-module schemes $\Theta_{n,s}$ (3.12).

rnm-2 **Proposition 3.15.** *If R is an equal characteristic ring, then $\varphi_{n,s}\colon \mathscr{R}_{n-s} \to \overline{\mathscr{M}_n^s}$ (3.14) is an isomorphism of $\mathscr{R}_n$-module schemes. If R is a ring of unequal characteristics and A is a k-algebra, then $\varphi_{n,s}(A)$ is a surjection if $A = A^p$ and an isomorphism if either A is perfect or $n \leq \bar{e}$.*

Proof. The fact that $\varphi_{n,s}$ is an isomorphism in the equal characteristic case follows from Remark 2.21. In the unequal characteristics case, see Proposition 2.18 and note that, by Remark 3.13(d), $\varphi_{n,s}(A)$ is an isomorphism for every A if $n \leq \bar{e}$. □

twist *Remarks* 3.16. Let k be a perfect field of characteristic $p > 0$, set $R = W(k)$ and let A be a k-algebra.

(a) The homomomorphism of $W_n(A)$-modules $\varphi_{n,s}(A)$ is the multiplication by p^s map. In particular, $\varphi_{n,n-1}(A)$ is the map $A \to V^{n-1}W_1(A) \subseteq W_n(A), a \mapsto (0,\dots,0,a^{p^{n-1}})$.

(b) By (a), for every integer $r \geq 1$, $\overline{\mathscr{M}_{r+1}^r}(A) = V^r W_1(A)$ has a canonical structure of A-module given by $a \cdot V^r(b) = (a,0,\dots,0)V^r(b) = V^r(a^{p^r}b)$. Now let ${}^{p^r}\!A$ be the ring A endowed with the A-module structure given by $a \cdot b = a^{p^r}b$ for $a, b \in A$. Then the map ${}^{p^r}\!A \to V^r W_1(A), b \mapsto V^r(b)$, is bijective and A-linear. If we identify ${}^{p^r}\!A$ and $V^r W_1(A)$ as A-modules via the preceding map, then the homomorphism of A-modules $\varphi_{r+1,r}(A)\colon W_1(A) \to V^r W_1(A)$ is identified with the A-linear map $A \to {}^{p^r}\!A, a \mapsto a^{p^r}$.

Now let ${}^{p^n}\mathbb{O}_k$ be the $\mathbb{O}_k$-module scheme given by ${}^{p^n}\mathbb{O}_k(A) = {}^{p^n}\!A$ for every k-algebra A, where ${}^{p^n}\!A$ is defined above.

gwist **Proposition 3.17.** *Let $\mathfrak{R}$ be a finite local $W(k)$-algebra of characteristic p^m, where $m \geq 1$, and let $\mathfrak{I}$ be a minimal ideal of $\mathfrak{R}$. Then there exists an isomorphism of $\mathbb{O}_k$-module schemes $\overline{\mathscr{I}} \simeq {}^{p^t}\mathbb{O}_k$, where $t \geq 0$ is a uniquely defined integer.*

Proof. There exist integers $\{n_1, \dots, n_r\}$ with $1 \leq n_1 \leq \dots \leq n_r \leq m$ and an isomorphism of $W(k)$-modules $\lambda\colon \mathfrak{R} \xrightarrow{\sim} \prod_{i=1}^r W_{n_i}(k)$. Let $\mathfrak{M}$ be the maximal ideal of $\mathfrak{R}$. The minimality hypothesis implies that $\mathfrak{I}$ is principal and $\mathfrak{M}\mathfrak{I} = 0$. Let g be a fixed generator of $\mathfrak{I}$ and write $\lambda(g) = (w_i) \in \prod_{i=1}^r W_{n_i}(k)$. Then either $w_i = 0$ or $\mathrm{ord}_p(w_i) = n_i - 1$, i.e., $w_i = p^{n_i - 1}\widetilde{w}_i$ for some $\widetilde{w}_i \in W_{n_i}(k)^{\times}$. Note that $(w_i) \neq (0, \dots, 0)$. Set $t = \min\{\mathrm{ord}_p(w_i), 1 \leq i \leq r\}$ and let q be an index where the minimum is attained, i.e.,

tnq0 (3.18) $$t = n_q - 1.$$

It is possible to construct a $W(k)$-automorphism δ of $\prod_{i=1}^r W_{n_i}(k)$ such that the composition $\delta \circ \lambda\colon \mathfrak{R} \xrightarrow{\sim} \prod_{i=1}^r W_{n_i}(k)$ induces an isomorphism $\mathfrak{I} \xrightarrow{\sim} p^{n_q - 1} W_{n_q}(k) \subset \prod_{i=1}^r W_{n_i}(k)$. See [BGA2, proof of Proposition 4.24] for the details. This isomorphism induces, for every k-algebra A, an isomorphism of A-modules $\overline{\mathscr{I}}(A) \simeq V^{n_q - 1} W_1(A)$. The proposition now follows from Remark 3.16(b). □

twist3 *Remark* 3.19. Let R be a discrete valuation ring of unequal characteristics and let $n > 1$ be an integer. Then the pair $(\mathfrak{R}, \mathfrak{I}) = (R_n, M_n^{n-1})$ satisfies the conditions of the proposition. By definition of the isomorphism (3.6), the image of π_n^{n-1} is non-trivial only on the first factor, i.e., the integer (3.18) equals $t = n_0 - 1 = m - 1$ by Remark 3.7(a). Thus there exists an isomorphism of $\mathbb{O}_k$-module schemes $\overline{\mathscr{M}_n^{n-1}} \simeq {}^{p^{m-1}}\mathbb{O}_k$ that generalizes the isomorphism $V^{n-1} W_1 \simeq {}^{p^{n-1}}\mathbb{O}_k$ described in Remark 3.16(b).

4. Greenberg algebras and ramification

ram

We keep the notation and hypotheses of the previous Section. In particular, R is a *complete* discrete valuation ring.

Let $\overline{k}$ be a fixed algebraic closure of k and let k'/k be a subextension of $\overline{k}/k$. The *extension of R of ramification index* 1 *which corresponds to* k'/k is (the complete discrete valuation ring) given by $R' = R \widehat{\otimes}_k k' \simeq k'[[t]]$ in the equal characteristic case and $R' = R \otimes_{W(k)} W(k')$ in the unequal characteristics case.

For every $n \in \mathbb{N}$, we have $R'_n = R_n \otimes_R R' = R_n \otimes_k k'$ in the equal characteristic case and

uneq (4.1) $$R'_n = R_n \otimes_R R' = R_n \otimes_{W_n(k)} W_n(k') = R_n \otimes_{W(k)} W(k'),$$

in the unequal characteristics case.

unr1 **Lemma 4.2.** *Let k'/k be a subextension of $\overline{k}/k$ and let R' be the extension of R of ramification index* 1 *that corresponds to k'/k. Then, for every $n \in \mathbb{N}$, there exists a canonical isomorphism of k'-ring schemes*

$$\mathscr{R}'_n = \mathscr{R}_n \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'.$$

In particular, $R'_n = \mathscr{R}'_n(k') = \mathscr{R}_n(k')$.

Proof. In the equal characteristic case the result follows from (A.2) and (3.2) since $R'_n = R_n \otimes_k k'$. In the unequal characteristics case, it suffices to check that the fpqc sheaves of sets on the category of k'-algebras which are represented by the k'-schemes $\mathscr{R}'_n$ and $\mathscr{R}_n \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ are isomorphic. Since the indicated sheaves are the sheaves associated to the functors on k'-algebras $A \mapsto R'_n \otimes_{W(k')} W(A)$ and $A \mapsto R_n \otimes_{W(k)} W(A)$ (respectively) by [Lip, Appendix A], we need only check that the canonical map $R_n \otimes_{W(k)} W(A) \to R'_n \otimes_{W(k')} W(A)$ is a bijection for every k'-algebra A. This follows from (4.1). □

In the setting of the lemma, if A is a k'-algebra, then $\mathscr{R}_n(A)$ is canonically endowed with an R'_n-algebra structure.

unr2 **Lemma 4.3.** *Let k'/k be a subextension of $\overline{k}/k$ and let R' be the extension of R of ramification index 1 which corresponds to k'/k. Then, for every $n \in \mathbb{N}$ and every k-algebra A, there exists a canonical isomorphism of R'_n-algebras*

$$\mathscr{R}'_n\big(A \otimes_k k'\big) = \mathscr{R}_n(A) \otimes_{R_n} R'_n.$$

Proof. In the equal characteristic case, (3.3) yields

$$\mathscr{R}'_n\big(A \otimes_k k'\big) = \big(A \otimes_k k'\big) \otimes_{k'} R'_n = (A \otimes_k R_n) \otimes_{R_n} R'_n = \mathscr{R}_n(A) \otimes_{R_n} R'_n.$$

Now let R be a ring of unequal characteristics and assume first that k'/k is finite. By Lemma 4.2, [Lip, Theorem C.5(i), p. 84] and (4.1), there exist natural isomorphisms of rings

$$\mathscr{R}'_n\big(A \otimes_k k'\big) \xrightarrow{\sim} \mathscr{R}_n\big(A \otimes_k k'\big) \xrightarrow{\sim} \mathscr{R}_n\big(A\big) \otimes_{W_n(k)} W_n(k') \xrightarrow{\sim} \mathscr{R}_n\big(A\big) \otimes_{R_n} R'_n.$$

By functoriality, their composition is an isomorphism of R'_n-algebras, which yields the lemma if k'/k is finite. The general case follows from the case of finite extensions via a limit argument using the fact that the functors $\mathscr{R}_n(-)$ and $W_n(-)$ commute with filtered inductive limits. □

The following lemma applies to possibly ramified *finite* extensions of R.

rne1 **Lemma 4.4.** *Let R' be a finite extension of R of ramification index e with associated residue field extension k'/k. Then, for every integer $n \geq 1$ and every k-algebra A, there exists a canonical isomorphism of R'_{ne}-algebras*

$$\mathscr{R}_n(A) \otimes_{R_n} R'_{ne} = \mathscr{R}'_{ne}\big(A \otimes_k k'\big).$$

Proof. In the equal characteristic case, the proof is similar to the proof of the corresponding case of Lemma 4.3. If R is an unequal characteristics ring and R'/R is totally ramified (respectively, of ramification index 1), then the lemma follows from [NS, Lemma 2.7, p. 1593] (respectively, Lemma 4.3). The general case now follows by combining these two cases in a well-known manner. □

gad 5. The Greenberg algebra of a discrete valuation ring

Let R be a discrete valuation ring. The *Greenberg algebra associated to* R is the affine k-scheme

$$\widetilde{\mathscr{R}} = \varprojlim_{n \in \mathbb{N}} \mathscr{R}_n\,,$$

where the transition morphisms are induced by the canonical maps $R_{n+1} \to R_n$. Note that, if $R = W(k)$, then $\widetilde{\mathscr{R}} = \mathbb{W}$. Now, if A is a k-algebra, set

ra (5.1) $$\widetilde{\mathscr{R}}(A) = \operatorname{Hom}_k\big(\operatorname{Spec} A, \widetilde{\mathscr{R}}\,\big) = \varprojlim(\mathscr{R}_n(A)).$$

If k'/k is a subextension of $\overline{k}/k$ and R' is the extension of R of ramification index 1 which corresponds to k'/k, then, by Lemma 4.2,

rk (5.2) $$\widetilde{\mathscr{R}}(k') = R'$$

Further, since the underlying scheme of $\mathscr{R}_n$ is isomorphic to $\mathbb{A}_k^n$ (see Section 3), the underlying scheme of the ring scheme $\widetilde{\mathscr{R}}$ is isomorphic to $\mathbb{A}_k^{(\mathbb{N})} = \operatorname{Spec} k[x_n; n \in \mathbb{N}]$. In particular, $\widetilde{\mathscr{R}}$ is not locally of finite type. On the other hand, $\widetilde{\mathscr{R}}$ has the properties listed in Remark 2.9(a). We also note that, since the k-algebra that represents $\widetilde{\mathscr{R}}(-)$ is not of finite presentation, the functor $\widetilde{\mathscr{R}}(-)$ does *not* commute with filtered inductive limits.

rm-ns *Remarks* 5.3.

(a) If $R \simeq k[[t]]$ and A is a k-algebra then, by (3.3),
$$\widetilde{\mathscr{R}}(A) = \varprojlim\,(R_n \otimes_k A) \simeq \varprojlim A[t]/(t^n) \simeq A[[t]] \simeq R \,\widehat{\otimes}_k\, A,$$
where the last term is the completion of $R \otimes_k A$ relative to the (t)-adic topology. Consequently, definition (5.1) coincides with that in [NS2, p. 256].

(b) Let $R \simeq W(k)[T]/(f)$ be as in (3.4) and let A be a k-algebra such that $A = A^p$. Then, by Remark 2.11(b), we have
$$\mathscr{R}_n(A) = R_n \otimes_{W(k)} W(A) \simeq W(A)[T]/(f, T^n)$$
for every $n \geq 1$. Consequently,

raw (5.4) $$\widetilde{\mathscr{R}}(A) \simeq \varprojlim W(A)[T]/(f, T^n) \simeq W(A)[T]/(f) \simeq R \otimes_{W(k)} W(A).$$

We also note that, since R is a finitely generated $W(k)$-module, definition (5.1) above generalizes the definition given in [NS2, p. 256] when $A = A^p$.

6. The Greenberg functor

gr-art

The Greenberg realization of a scheme of finite type over an artinian local ring was introduced in [Gre1]. In this Section we generalize Greenberg's construction.

We work in the setting of Subsection 2.3. Let $\mathfrak{R}$ be a local finite $W_m(k)$-algebra with maximal ideal $\mathfrak{M}$ and residue field k. Let Y be a k-scheme. We will write $\mathscr{R}(\mathscr{O}_Y)$ for the Zariski sheaf on Y defined by
$$\varGamma(U, \mathscr{R}(\mathscr{O}_Y)) = \operatorname{Hom}_k(U, \mathscr{R}) \qquad (U \subset Y \text{ open})$$

If $U = \operatorname{Spec} A$ is an affine subscheme of Y, then

use (6.1) $$\varGamma(U, \mathscr{R}(\mathscr{O}_Y)) = \mathscr{R}(A).$$

If $\mathfrak{I}$ is an ideal in $\mathfrak{R}$, we define $\bar{\mathscr{I}}(\mathscr{O}_Y)$ similarly. By (2.22), there exists a canonical exact sequence of Zariski sheaves on Y

rnm2 (6.2) $$0 \to \bar{\mathscr{I}}(\mathscr{O}_Y) \to \mathscr{R}(\mathscr{O}_Y) \to \mathscr{R}^{(\mathscr{I})}(\mathscr{O}_Y) \to 0.$$

For example, if $\mathfrak{I}=\mathfrak{M}$, then the sequence of Zariski sheaves

rr (6.3)
$$0\to\overline{\mathscr{M}}(\mathscr{O}_Y)\to\mathscr{R}(\mathscr{O}_Y)\to\mathscr{O}_Y\to 0.$$

is exact. Note that, since $\mathfrak{M}$ is a nilpotent ideal, $\overline{\mathscr{M}}(\mathscr{O}_Y)$ is a nilpotent ideal sheaf by (2.24). We now consider the locally ringed space over $\mathfrak{R}$

$$h^{\mathfrak{R}}(Y)=(|Y|,\mathscr{R}(\mathscr{O}_Y)).$$

Proposition 6.4. *Let Y be a k-scheme. Then $h^{\mathfrak{R}}(Y)$ is an $\mathfrak{R}$-scheme which is affine if Y is affine. If Y' is a closed (respectively, open) subscheme of Y, then $h^{\mathfrak{R}}(Y')$ is a closed (respectively, open) subscheme of $h^{\mathfrak{R}}(Y)$.*

Proof. Assume first that $Y=\operatorname{Spec}A$ is affine. Then $\Gamma(|h^{\mathfrak{R}}(Y)|,\mathscr{O}_{h^{\mathfrak{R}}(Y)})=\mathscr{R}(A)$ by (6.1). Let

$$\sigma^{\mathfrak{R}}\colon h^{\mathfrak{R}}(Y)\to\operatorname{Spec}\mathscr{R}(A)$$

be the morphism of locally ringed spaces which corresponds to the identity map of $\mathscr{R}(A)$ under the bijection

egabj (6.5)
$$\operatorname{Hom}_{\text{loc}}\big(h^{\mathfrak{R}}(Y),\operatorname{Spec}\mathscr{R}(A)\big)\xrightarrow{\sim}\operatorname{Hom}(\mathscr{R}(A),\mathscr{R}(A))$$

of [EGA I_{new}, Proposition 1.6.3, p. 210]. If $\mathfrak{R}=k$, then $\mathscr{R}=\mathbb{O}_k$ and $\sigma^k\colon h^k(Y)\to\operatorname{Spec}A$ is the identity morphism of Y. Now, if $\mathfrak{R}$ is arbitrary, then the identity map of $|Y|$ and the projection in (6.3) define a morphism of locally ringed spaces $\delta\colon Y\to h^{\mathfrak{R}}(Y)$. On the other hand, by (2.22), the sequence (6.3) induces a surjective homomorphism of $W_m(k)$-algebras $\mathscr{R}(A)\to A$ with nilpotent kernel $\overline{\mathscr{M}}(A)$. Thus the morphism $\varsigma\colon\operatorname{Spec}A\to\operatorname{Spec}\mathscr{R}(A)$ induced by $\mathscr{R}(A)\to A$ is a nilpotent immersion. By the functoriality of (6.5), the following diagram commutes:

dhs (6.6)
$$\begin{array}{ccc} h^k(Y) & \xrightarrow[\sim]{\sigma^k} & \operatorname{Spec}A\\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\varsigma}\\ h^{\mathfrak{R}}(Y) & \xrightarrow{\sigma^{\mathfrak{R}}} & \operatorname{Spec}\mathscr{R}(A).\end{array}$$

Since δ and ς are homeomorphisms, the above diagram shows that $\sigma^{\mathfrak{R}}$ is a homeomorphism as well. If $m>1$, then (6.6) with $Y=D(f)=\operatorname{Spec}A_f$, where $f\in A$, and Proposition 2.12 together show that $\sigma^{\mathfrak{R}}$ maps the open locally ringed subspace $h^{\mathfrak{R}}(D(f))$ of $h^{\mathfrak{R}}(Y)$ onto the open subscheme $\operatorname{Spec}\mathscr{R}(A)_{[f]}$ of $\operatorname{Spec}\mathscr{R}(A)$. Further,

$$\Gamma(|D(f)|,\mathscr{O}_{h^{\mathfrak{R}}(Y)})=\mathscr{R}(A_f)\simeq\mathscr{R}(A)_{[f]}=\Gamma\big(\sigma^{\mathfrak{R}}(|D(f)|),\mathscr{O}_{\operatorname{Spec}\mathscr{R}(A)}\big).$$

If $m=1$, the analogous result holds by (2.5). We conclude that $\sigma^{\mathfrak{R}}$ is an isomorphism of locally ringed spaces and, consequently, $h^{\mathfrak{R}}(Y)$ is a scheme.

If Y is arbitrary, let $\{Y_i\}$ be a covering of Y by open affine subschemes. By definition, the restriction of $\mathscr{R}(\mathscr{O}_Y)$ to $|Y_i|$ is $\mathscr{R}(\mathscr{O}_{Y_i})$. Thus $h^{\mathfrak{R}}(Y)$ is obtained by gluing the affine $\mathfrak{R}$-schemes $h^{\mathfrak{R}}(Y_i)$, whence $h^{\mathfrak{R}}(Y)$ is an $\mathfrak{R}$-scheme, as claimed. Consequently, if Y' is an open subscheme of Y, then $h^{\mathfrak{R}}(Y')$ is an open subscheme of $h^{\mathfrak{R}}(Y)$. Finally, the assertion on closed subschemes follows from (2.25). $\square$

It follows from the above proof that if A is a k-algebra, then

rn-aff (6.7)
$$h^{\mathfrak{R}}(\operatorname{Spec} A)=\operatorname{Spec}\mathscr{R}(A).$$

In particular, $h^{\mathfrak{R}}(\operatorname{Spec} k)=\operatorname{Spec}\mathfrak{R}$. Thus there exists a covariant functor

hra-f (6.8)
$$h^{\mathfrak{R}}\colon(\mathrm{Sch}/k)\to(\mathrm{Sch}/\mathfrak{R}),\quad Y\mapsto h^{\mathfrak{R}}(Y),$$

which respects open, closed and arbitrary immersions as well as Zariski coverings.

Now, for every $\mathfrak{R}$-scheme Z, consider the contravariant functor

fun (6.9)
$$(\mathrm{Sch}/k)\to(\mathrm{Sets}),\quad Y\mapsto \operatorname{Hom}_{\mathfrak{R}}\big(h^{\mathfrak{R}}(Y),Z\big).$$

r-def1 **Proposition-Definition 6.10.** *For every $\mathfrak{R}$-scheme Z, the functor* (6.9) *is represented by a k-scheme which is denoted by* $\mathrm{Gr}^{\mathfrak{R}}(Z)$ *and called the* Greenberg realization of Z. *The assignment*

grf (6.11)
$$\mathrm{Gr}^{\mathfrak{R}}\colon(\mathrm{Sch}/\mathfrak{R})\to(\mathrm{Sch}/k),\quad Z\mapsto \mathrm{Gr}^{\mathfrak{R}}(Z),$$

is a covariant functor called the Greenberg functor associated to $\mathfrak{R}$, *and the bijection*

adj (6.12)
$$\operatorname{Hom}_k\big(Y,\mathrm{Gr}^{\mathfrak{R}}(Z)\big)\simeq \operatorname{Hom}_{\mathfrak{R}}\big(h^{\mathfrak{R}}(Y),Z\big)$$

is functorial in the variables $Y\in(\mathrm{Sch}/k)$ and $Z\in(\mathrm{Sch}/\mathfrak{R})$. If Z is of finite type (respectively, locally of finite type), then $\mathrm{Gr}^{\mathfrak{R}}(Z)$ is of finite type (respectively, locally of finite type).

Proof. The proof of [Gre1, Theorem, p. 643] [2] shows that, if Z is (locally) of finite type over $\mathfrak{R}$, then $\mathrm{Gr}^{\mathfrak{R}}(Z)$ exists, is (locally) of finite type over k and the bijection (6.12) is bifunctorial. In [Gre1], $\mathrm{Gr}^{\mathfrak{R}}(Z)$ is constructed in a number of steps from the particular case

gr-a (6.13)
$$\mathrm{Gr}^{\mathfrak{R}}\big(\mathbb{A}^{d}_{\mathfrak{R}}\big)=\mathscr{R}^{d},$$

where $d\geq 0$ (see [Gre1, Proposition 3, p. 638] for this particular case). The *same* construction can be used to define $\mathrm{Gr}^{\mathfrak{R}}(Z)$ for any Z via possibly infinite-dimensional affine spaces, as follows. Let $\{x_i\}_{i\in I}$ be a (possibly infinite) family of independent indeterminates and set $\mathbb{A}^{(I)}_{\mathfrak{R}}=\operatorname{Spec}\mathfrak{R}[\{x_i\}_{i\in I}]$. Standard facts on projective limits show that

$$\mathrm{Gr}^{\mathfrak{R}}\big(\mathbb{A}^{(I)}_{\mathfrak{R}}\big)=\varprojlim_{J\subseteq I}\mathrm{Gr}^{\mathfrak{R}}\big(\mathbb{A}^{(J)}_{\mathfrak{R}}\big)\simeq\varprojlim_{J\subseteq I}\mathscr{R}^{|J|},$$

where the limits range over the family of finite subsets J of I and $|J|$ denotes the cardinality of J. □

The above proof shows that the k-scheme $\mathrm{Gr}^{\mathfrak{R}}(Z)$ agrees with the realization constructed in [Gre1, Proposition 7, p. 641] when Z is of finite type over $\mathfrak{R}$. Further

grnsn (6.14)
$$\mathrm{Gr}^{\mathfrak{R}}(\operatorname{Spec}\mathfrak{R})=\operatorname{Spec} k.$$

In addition, for every $\mathfrak{R}$-scheme Z and k-algebra A, (6.7) and (6.12) yield a bijection

grzz (6.15)
$$\mathrm{Gr}^{\mathfrak{R}}(Z)(A)=Z(\mathscr{R}(A)),$$

[2] Note that in [Gre1, Gre2] $h^{\mathfrak{R}}$ and $\mathrm{Gr}^{\mathfrak{R}}$ are denoted by G and F, respectively.

where $Z(\mathscr{R}(A)) = \mathrm{Hom}_{\mathfrak{R}}(\mathrm{Spec}\,\mathscr{R}(A), Z)$. More generally, let T be an $\mathfrak{R}$-scheme, Z a T-scheme and Y a $\mathrm{Gr}^{\mathfrak{R}}(T)$-scheme. Then the adjunction formula (6.12) yields a canonical bijection

g-adj (6.16)
$$\mathrm{Hom}_{\mathrm{Gr}^{\mathfrak{R}}(T)}\big(Y, \mathrm{Gr}^{\mathfrak{R}}(Z)\big) = \mathrm{Hom}_{T}\big(h^{\mathfrak{R}}(Y), Z\big).$$

rems1 *Remarks* 6.17.

(a) Both h^k and Gr^k are the identity functors on (Sch/k).

(b) The functor (6.11) transforms affine $\mathfrak{R}$-schemes into affine k-schemes and respects open, closed and arbitrary immersions. Further, if $\{Z_i\}$ is an open covering of an $\mathfrak{R}$-scheme Z, then the open subschemes $\mathrm{Gr}^{\mathfrak{R}}(Z_i)$ cover $\mathrm{Gr}^{\mathfrak{R}}(Z)$. The proofs of the preceding statements are similar to the proofs of the analogous results in [Gre1], using possibly infinite-dimensional affine spaces.

(c) Assume that $\mathfrak{R}$ is a finite k-algebra and let Z be an $\mathfrak{R}$-scheme. Since $|Y| = |Y \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\mathfrak{R}|$ for every k-scheme Y, (2.3) yields

hrny (6.18)
$$h^{\mathfrak{R}}(Y) = Y \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\mathfrak{R}.$$

Thus, in this case, (6.9) is the left adjoint of the Weil restriction functor $\mathrm{Res}_{\mathfrak{R}/k}$ (see Section A.1). Consequently, $\mathrm{Gr}^{\mathfrak{R}} = \mathrm{Res}_{\mathfrak{R}/k}$.

(d) The functor (6.11) respects fiber products (the proof of this fact is similar to that in [Gre1, Theorem, p. 643]). Consequently, $\mathrm{Gr}^{\mathfrak{R}}$ defines a covariant functor from the category of $\mathfrak{R}$-group schemes to the category of k-group schemes. Further, there exists a canonical isomorphism of k-ring schemes $\mathrm{Gr}^{\mathfrak{R}}\big(\mathbb{O}_{\mathfrak{R}}\big) = \mathscr{R}$.

(e) Let Y be a k-scheme and let $h^{\mathfrak{R}}(Y)_{\mathrm{s}}$ denote the special fiber of $h^{\mathfrak{R}}(Y)$. Note that the ideal sheaf which corresponds to $h^{\mathfrak{R}}(Y)_{\mathrm{s}}$ is $\mathfrak{M}\mathscr{R}(\mathscr{O}_Y)$. Then the composition

$$\mathscr{R}(\mathscr{O}_Y)/\mathfrak{M}\mathscr{R}(\mathscr{O}_Y) \twoheadrightarrow \mathscr{R}(\mathscr{O}_Y)/\overline{\mathscr{M}}(\mathscr{O}_Y) \xrightarrow{\sim} \mathscr{O}_Y$$

(see (2.17) and (6.3)) induces a nilpotent immersion of k-schemes $\iota_Y \colon Y \to h^{\mathfrak{R}}(Y)_{\mathrm{s}}$. If $\mathfrak{R}$ is a k-algebra, then ι_Y is an isomorphism for every Y. See(2.6) and Remark 2.21.

gberg 7. The Greenberg functor of a truncated discrete valuation ring

The definitions and constructions of the preceding Section apply, in particular, to the truncated discrete valuation rings $\mathfrak{R} = R_n$ of Section 3, where $n \in \mathbb{N}$. Let

hrn-f (7.1)
$$h_n^R = h^{R_n} \colon (\mathrm{Sch}/k) \to (\mathrm{Sch}/R_n), \quad Y \mapsto (|Y|, \mathscr{R}_n(\mathscr{O}_Y)),$$

be the functor (6.8) associated to $\mathfrak{R} = R_n = R/\mathfrak{m}^n$ and let $\mathrm{Gr}_n^R = \mathrm{Gr}^{R_n}$ be its right adjoint. Then Gr_n^R is called the *Greenberg functor of level n associated to R*. For every k-scheme Y and every R_n-scheme Z, (6.12) induces a canonical bijection

dj-bis (7.2)
$$\mathrm{Hom}_k\big(Y, \mathrm{Gr}_n^R(Z)\big) \simeq \mathrm{Hom}_{R_n}\big(h_n^R(Y), Z\big).$$

at-pts **Lemma 7.3.** *Let $n \in \mathbb{N}$ and let Z be an R_n-scheme.*

(i) *If A is a k-algebra, then $\mathrm{Gr}_n^R(Z)(A) = Z(\mathscr{R}_n(A))$.*

(ii) *If k'/k is a subextension of $\overline{k}/k$ and R' is the extension of R of ramification index 1 which corresponds to k'/k, then $\mathrm{Gr}_n^R(Z)(k') = Z(R'_n)$.*

Proof. Assertion (i) follows from (6.15) and (ii) follows from (i) using Lemma 4.2. □

s1-bis *Remark* 7.4. If $R=W(k)$, then $\mathscr{R}_n=\mathbb{W}_n$ and $h_n^R(Y)=W_n(Y)$ for every k-scheme Y, where $W_n(Y)$ is the scheme defined in [Ill, §1.5]. More generally, assume that $R/W(k)$ is totally ramified of degree $\bar{e}$ and let Y be any k-scheme such that the absolute Frobenius morphism of Y is a closed immersion. By Remark 2.11(b) and the fact that (7.1) is local for the Zariski topology, we have

for
$$h_n^R(Y)=W_m(Y)\times_{W_m(k)}S_n, \tag{7.5}$$

where $m=\lceil n/\bar{e}\,\rceil$ (3.5). We call attention to the fact that (7.5) does *not* hold for arbitrary k-schemes Y. In particular, the formula in [BLR, p. 276, line -18] is incorrect, as previously noted in [NS, p. 1592]. Note, however, that (7.5) is indeed valid for every Y provided $n=m\bar{e}$, as follows from Lemma 4.4.

.alpha *Example* 7.6. Let k be a field of positive characteristic p and let $R\simeq k[[t]]$. By Remark (6.17)(c) and [BLR, §7.6, proof of Theorem 4, pp. 194-195], $\mathrm{Gr}_n^R(\mathbb{A}^1_{R_n})=\mathrm{Res}_{R_n/k}(\mathbb{A}^1_{R_n})=\mathbb{A}^n_k$. On the other hand, by (6.18), we have $h_n^R(\mathbb{A}^n_k)=\mathbb{A}^n_{R_n}$. Now (7.2) or, equivalently, Appendix (A.1), yields a canonical morphism

$$h_n^R(\mathrm{Gr}_n^R(\mathbb{A}^1_{R_n}))\to\mathbb{A}^1_{R_n}$$

which is induced by the ring homomorphism $q^{(n)}\colon R_n[x]\to R_n[x_0,\dots,x_{n-1}]$ given by the formula $q^{(n)}(x)=\sum_{i=0}^{n-1}x_it^i$. Since $t^j=0$ in R_n for $j\geq n$, we have $q^{(n)}(x^p)=\sum_{i=0}^{\lfloor(n-1)/p\rfloor}x_i^pt^{ip}$. We conclude that

alp
$$\mathrm{Gr}_n^R(\mathrm{Spec}(R_n[x]/(x^p)))\simeq\mathrm{Spec}\left(k[x_0,\dots,x_{n-1}]/(x_i^p,i\leq(n-1)/p)\right). \tag{7.7}$$

Compare with [BLR, §7.6, proof of Proposition 2(ii), pp. 193-194]. In particular, (7.7) is not a finite k-scheme for any $n\geq 2$.

s-cr 8. The change of rings morphism

We return to the setting of Section 6. Thus $\mathfrak{R}$ is a local finite $W_m(k)$-algebra with residue field k.

Let $\mathfrak{I}$ be a nilpotent ideal of $\mathfrak{R}$, write $\mathfrak{R}'$ for the artinian local ring $\mathfrak{R}/\mathfrak{I}$ and let $\mathscr{R}'$ be the corresponding Greenberg algebra. Let X be an $\mathfrak{R}$-scheme and write X' for $X_{\mathfrak{R}'}$. If A is a k-algebra, the canonical homomorphism $\mathscr{R}(A)\to\mathscr{R}'(A)$ induces a map $X(\mathscr{R}(A))\to X'(\mathscr{R}'(A))$ and thus a map $\mathrm{Gr}^{\mathfrak{R}}(X)(A)\to\mathrm{Gr}^{\mathfrak{R}'}(X')(A)$ (6.15). In particular, there exists a morphism of k-schemes

tr0
$$\varrho_X^{\mathfrak{R},\mathfrak{R}'}\colon\mathrm{Gr}^{\mathfrak{R}}(X)\to\mathrm{Gr}^{\mathfrak{R}'}(X') \tag{8.1}$$

which is called the *change of rings morphism associated to* X. By (6.14), we have

$$\varrho_{\mathrm{Spec}\,\mathfrak{R}}^{\mathfrak{R},\mathfrak{R}'}=1_{\mathrm{Spec}\,k}.$$

Further, if $\mathfrak{J}$ is an ideal of $\mathfrak{R}$ which contains $\mathfrak{I}$ and $\mathfrak{R}''=\mathfrak{R}/\mathfrak{J}$, then

vrkb
$$\varrho_X^{\mathfrak{R},\mathfrak{R}''}=\varrho_{X'}^{\mathfrak{R}',\mathfrak{R}''}\circ\varrho_X^{\mathfrak{R},\mathfrak{R}'}. \tag{8.2}$$

In particular, if R is a complete discrete valuation ring and R_{n+i} and R_n are the truncations associated to a pair of integers $n\geq 1, i\geq 0$, then $\varrho_Z^{R_{n+i},R_n}$ is defined for every

R_{n+i}-scheme Z. The preceding map will be called the *change of level morphism associated to* Z and denoted by

clm0 (8.3)
$$\varrho^{i}_{n,Z}\colon \mathrm{Gr}^{R}_{n+i}(Z)\to \mathrm{Gr}^{R}_{n}(Z),$$

where we have written $\mathrm{Gr}^{R}_{n}(Z)$ for $\mathrm{Gr}^{R}_{n}(Z\times_{R_{n+i}}\mathrm{Spec}\, R_n)$.

For every k-scheme Y and $\mathfrak{R}$-scheme Z, let

vphi (8.4)
$$\varphi^{\mathfrak{R}}_{Y,Z}\colon \mathrm{Hom}_k\big(Y,\mathrm{Gr}^{\mathfrak{R}}(Z)\big)\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{R}}\big(h^{\mathfrak{R}}(Y),Z\big)$$

be the bijection (6.12). Now recall the schemes $h^{\mathfrak{R}}(Y)$ and $h^{\mathfrak{R}'}(Y)$ introduced in Section 6 and the surjective morphism (of Zariski sheaves on Y) $\mathscr{R}(\mathscr{O}_Y)\to\mathscr{R}'(\mathscr{O}_Y)$ with nilpotent kernel $\bar{\mathscr{I}}(\mathscr{O}_Y)$ (see (6.2) and (2.24)). The preceding map induces a nilpotent immersion

dlt (8.5)
$$\delta^{\mathfrak{R},\mathfrak{R}'}_{Y}\colon h^{\mathfrak{R}'}(Y)\to h^{\mathfrak{R}}(Y)$$

which is functorial in Y. By standard applications of the adjunction isomorphisms (8.4) for $\mathfrak{R}$ and $\mathfrak{R}'$ (see [BGA2, Section 9] for more details), the following holds.

rdad **Proposition 8.6.** *Let Y be a k-scheme, X an $\mathfrak{R}$-scheme and $u\colon Y\to\mathrm{Gr}^{\mathfrak{R}}(X)$ a morphism of k-schemes. Then $\varrho^{\mathfrak{R},\mathfrak{R}'}_{X}\circ u$ is the unique morphism of k-schemes $a\colon Y\to\mathrm{Gr}^{\mathfrak{R}'}(X')$ such that the diagram*

$$\begin{array}{ccc}
h^{\mathfrak{R}'}(Y) & \xrightarrow{\varphi^{\mathfrak{R}'}_{Y,X'}(a)} & X' \\
\downarrow{\scriptstyle \delta^{\mathfrak{R},\mathfrak{R}'}_{Y}} & & \downarrow{\scriptstyle \mathrm{pr}_X} \\
h^{\mathfrak{R}}(Y) & \xrightarrow{\varphi^{\mathfrak{R}}_{Y,X}(u)} & X
\end{array}$$

commutes. □

We now discuss the functoriality of the assignment $X\mapsto\varrho^{\mathfrak{R},\mathfrak{R}'}_{X}$.

Let $f\colon Z\to X$ be a morphism of $\mathfrak{R}$-schemes and write $f'\colon Z'\to X'$ for $f_{\mathfrak{R}'}$. By definition of the change of rings morphism (8.1), the following diagram commutes

funct (8.7)
$$\begin{array}{ccc}
\mathrm{Gr}^{\mathfrak{R}}(Z) & \xrightarrow{\varrho^{\mathfrak{R},\mathfrak{R}'}_{Z}} & \mathrm{Gr}^{\mathfrak{R}'}(Z') \\
\downarrow{\scriptstyle \mathrm{Gr}^{\mathfrak{R}}(f)} & & \downarrow{\scriptstyle \mathrm{Gr}^{\mathfrak{R}'}(f')} \\
\mathrm{Gr}^{\mathfrak{R}}(X) & \xrightarrow{\varrho^{\mathfrak{R},\mathfrak{R}'}_{X}} & \mathrm{Gr}^{\mathfrak{R}'}(X').
\end{array}$$

In particular, if X is an $\mathfrak{R}$-group scheme, then the change of rings morphism (8.1) is a morphism of k-group schemes.

f-et **Proposition 8.8.** *Let $f\colon Z\to X$ be a formally étale morphism of $\mathfrak{R}$-schemes. Then the diagram (8.7) is cartesian. Consequently, there exists a canonical isomorphism of k-schemes*

$$\mathrm{Gr}^{\mathfrak{R}}(Z)=\mathrm{Gr}^{\mathfrak{R}}(X)\times_{\mathrm{Gr}^{\mathfrak{R}'}(X')}\mathrm{Gr}^{\mathfrak{R}'}(Z').$$

Proof. It suffices to check that (8.7) satisfies the required universal property on A-sections for every k-algebra A. This follows from (6.15) since

$$\operatorname{Hom}_X(\operatorname{Spec}\mathscr{R}(A), Z) \to \operatorname{Hom}_X(\operatorname{Spec}\mathscr{R}'(A), Z), v \mapsto v \circ \delta^{\mathfrak{R},\mathfrak{R}'}_{\operatorname{Spec}A},$$

is a bijection by (2.22), (2.24) and the assumption on f. □

ffet **Corollary 8.9.** *Let $f\colon Z \to X$ be a formally étale morphism of $\mathfrak{R}$-schemes. Then there exists a canonical isomorphism of k-schemes*

$$\operatorname{Gr}^{\mathfrak{R}}(Z) = Z_{\mathrm{s}} \times_{X_{\mathrm{s}}} \operatorname{Gr}^{\mathfrak{R}}(X).$$

Consequently, $\operatorname{Gr}^{\mathfrak{R}}(f)\colon \operatorname{Gr}^{\mathfrak{R}}(Z) \to \operatorname{Gr}^{\mathfrak{R}}(X)$ *can be identified with* $f_{\mathrm{s}} \times_{X_{\mathrm{s}}} \operatorname{Gr}^{\mathfrak{R}}(X)$.

fetR **Corollary 8.10.** *Let Z be a formally étale $\mathfrak{R}$-scheme. Then the change of rings morphism $\varrho_Z^{\mathfrak{R},\mathfrak{R}'}\colon \operatorname{Gr}^{\mathfrak{R}}(Z) \to \operatorname{Gr}^{\mathfrak{R}'}(Z')$ is an isomorphism.*

Next we derive some properties of the change of rings morphism.

aff **Proposition 8.11.** *Let Z be an $\mathfrak{R}$-scheme. Then $\varrho_Z^{\mathfrak{R},\mathfrak{R}'}\colon \operatorname{Gr}^{\mathfrak{R}}(Z) \to \operatorname{Gr}^{\mathfrak{R}'}(Z')$ is affine.*

Proof. Recall that $Z' \to Z$ is a nilpotent immersion. Let U be an open subscheme of Z and let U' be the corresponding subscheme of Z'. By Proposition 8.8

$$\big(\varrho_Z^{\mathfrak{R},\mathfrak{R}'}\big)^{-1}(\operatorname{Gr}^{\mathfrak{R}'}(U')) = \operatorname{Gr}^{\mathfrak{R}}(Z) \times_{\operatorname{Gr}^{\mathfrak{R}'}(Z')} \operatorname{Gr}^{\mathfrak{R}'}(U') = \operatorname{Gr}^{\mathfrak{R}}(U).$$

Since the functor $\operatorname{Gr}^{\mathfrak{R}}$ maps open affine coverings to open affine coverings (see Remark 6.17(b)), the proposition follows. □

m-surj **Proposition 8.12.** *Let Z be a formally smooth $\mathfrak{R}$-scheme. Then the change of rings morphism $\varrho_Z^{\mathfrak{R},\mathfrak{R}'}\colon \operatorname{Gr}^{\mathfrak{R}}(Z) \to \operatorname{Gr}^{\mathfrak{R}'}(Z')$ is surjective.*

Proof. By definition of $\varrho_Z^{\mathfrak{R},\mathfrak{R}'}$, it suffices to check that the natural map $Z(\mathscr{R}(A)) \to Z'(\mathscr{R}'(A))$ is surjective for every k-algebra A. This follows from the universal property of formal smoothness, since $\mathscr{R}(A) \to \mathscr{R}'(A)$ is a surjective map with nilpotent kernel by (2.22) and (2.24). □

9. Basic properties of the Greenberg functor

bas

We keep the notation of the previous Section. In this Section we discuss properties of schemes/morphisms which are preserved by the functor $\operatorname{Gr}^{\mathfrak{R}}$ (properties that are *not* preserved by $\operatorname{Gr}^{\mathfrak{R}}$ include flatness, properness and finiteness, for which the reader is referred to [BGA2, Examples 11.10]).

q-proj **Proposition 9.1.** *Let Z be a quasi-projective $\mathfrak{R}$-scheme. Then $\operatorname{Gr}^{\mathfrak{R}}(Z)$ is a quasi-projective k-scheme.*

Proof. The commutativity of diagram (8.7) for $X = \operatorname{Spec}\mathfrak{R}$ shows that the structure morphism $\operatorname{Gr}^{\mathfrak{R}}(Z) \to \operatorname{Spec}k$ factors through the change of rings morphism $\varrho_Z^{\mathfrak{R},k}\colon \operatorname{Gr}^{\mathfrak{R}}(Z) \to Z_{\mathrm{s}}$, which is affine and of finite type by Proposition 8.11. Thus, since $Z_{\mathrm{s}} \to \operatorname{Spec}k$ is quasi-projective, $\operatorname{Gr}^{\mathfrak{R}}(Z) \to \operatorname{Spec}k$ is quasi-projective as well. □

The preceding result is new in the unequal characteristics case. In the equal characteristic case, the Greenberg functor of level n coincides with the Weil restriction functor $\mathrm{Res}_{R_n/k}$ and the corresponding result is a particular case of [CGP, Proposition A.5.8].

r-prop **Proposition 9.2.** *Consider, for a morphism of schemes, the property of being:*

(i) *quasi-compact;*
(ii) *quasi-separated;*
(iii) *separated;*
(iv) *locally of finite type;*
(v) *of finite type;*
(vi) *affine.*

If $\boldsymbol{P}$ *denotes one of the above properties and the* $\mathfrak{R}$*-morphism* $f\colon X\to Y$ *has property* $\boldsymbol{P}$*, then the* k*-morphism* $\mathrm{Gr}^{\mathfrak{R}}(f)\colon \mathrm{Gr}^{\mathfrak{R}}(X)\to\mathrm{Gr}^{\mathfrak{R}}(Y)$ *has property* $\boldsymbol{P}$ *as well.*

Proof. Recall diagram (8.7) with $\mathfrak{R}'=\mathfrak{R}/\mathfrak{M}=k$:

$$\begin{array}{ccc} \mathrm{Gr}^{\mathfrak{R}}(X) & \xrightarrow{\varrho_X^{\mathfrak{R},k}} & X_{\mathrm{s}} \\ {\scriptstyle \mathrm{Gr}^{\mathfrak{R}}(f)}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{s}}} \\ \mathrm{Gr}^{\mathfrak{R}}(Y) & \xrightarrow{\varrho_Y^{\mathfrak{R},k}} & Y_{\mathrm{s}}. \end{array}$$

By Proposition 8.11, the horizontal morphisms in the above diagram are affine and therefore separated and quasi-compact. Thus (i) follows from the diagram using [EGA $\mathrm{I}_{\mathrm{new}}$, Propositions 6.1.4 and 6.1.5(v), p. 291]. To establish the proposition for properties (ii) and (iii), assume that the diagonal morphism $\Delta_f\colon X\to X\times_Y X$ is quasi-compact (respectively, a closed immersion). Then, by Remarks 6.17, (b) and (d), and the first part of the proof,

$$\mathrm{Gr}^{\mathfrak{R}}(\Delta_f)=\Delta_{\mathrm{Gr}^{\mathfrak{R}}(f)}\colon \mathrm{Gr}^{\mathfrak{R}}(X)\to \mathrm{Gr}^{\mathfrak{R}}(X)\times_{\mathrm{Gr}^{\mathfrak{R}}(Y)}\mathrm{Gr}^{\mathfrak{R}}(X)$$

is quasi-compact (respectively, a closed immersion). Since $\mathrm{Gr}^{\mathfrak{R}}$ respects open and closed immersions, to prove the proposition for property (iv) we may assume that $Y=\mathrm{Spec}\, B$ and $X=\mathbb{A}^d_B$, where B is an $\mathfrak{R}$-algebra. In this case f is the map $\mathbb{A}^d_{\mathfrak{R}}\times_{\mathfrak{R}}\mathrm{Spec}\, B\to\mathrm{Spec}\, B$, whence (by Remark 6.17(d)) $\mathrm{Gr}^{\mathfrak{R}}(f)$ is the base change along $\mathrm{Gr}^{\mathfrak{R}}(\mathrm{Spec}\, B)\to\mathrm{Spec}\, k$ of the canonical morphism $\mathscr{R}^d\to\mathrm{Spec}\, k$, which is clearly a morphism of finite type. The proposition holds for property (v) since it holds for properties (i) and (iv). Finally, by Remark 6.17(b), $\mathrm{Gr}^{\mathfrak{R}}(Y)$ is covered by affine open subschemes of the form $\mathrm{Gr}^{\mathfrak{R}}(U)$, where U is an affine open subscheme of Y. Since $\mathrm{Gr}^{\mathfrak{R}}(X)\times_{\mathrm{Gr}^{\mathfrak{R}}(Y)}\mathrm{Gr}^{\mathfrak{R}}(U)=\mathrm{Gr}^{\mathfrak{R}}(X\times_Y U)$ is affine, the proof is complete. $\square$

Proposition 9.3. *Let* $f\colon Z\to Z'$ *be a formally smooth (respectively, formally unramified, formally étale) morphism of* $\mathfrak{R}$*-schemes. Then the induced* k*-morphism* $\mathrm{Gr}^{\mathfrak{R}}(f)\colon \mathrm{Gr}^{\mathfrak{R}}(Z)\to\mathrm{Gr}^{\mathfrak{R}}(Z')$ *is formally smooth (respectively, formally unramified, formally étale).*

Proof. We need to show that, if $Y=\mathrm{Spec}\, A$ is an affine scheme and $J\subset A$ is a nilpotent ideal, then the canonical map

$$\mathrm{Hom}_{\mathrm{Gr}^{\mathfrak{R}}(Z')}(\mathrm{Spec}\, A,\mathrm{Gr}^{\mathfrak{R}}(Z))\to\mathrm{Hom}_{\mathrm{Gr}^{\mathfrak{R}}(Z')}(\mathrm{Spec}\,(A/J),\mathrm{Gr}^{\mathfrak{R}}(Z))$$

is surjective (respectively, injective, bijective). By (6.7) and (6.16), the above map may be identified with the map $\mathrm{Hom}_{Z'}(\mathrm{Spec}\,\mathscr{R}(A), Z) \to \mathrm{Hom}_{Z'}(\mathrm{Spec}\,\mathscr{R}(A/J), Z)$, which is surjective (respectively, injective, bijective) since the kernel of $\mathscr{R}(A) \to \mathscr{R}(A/J)$ is a nilpotent ideal by Lemma 2.26. □

gr-sm **Corollary 9.4.** *Let $f\colon Z \to Z'$ be a smooth (respectively, unramified, étale) $\mathfrak{R}$-morphism. Then $\mathrm{Gr}^{\mathfrak{R}}(f)\colon \mathrm{Gr}^{\mathfrak{R}}(Z) \to \mathrm{Gr}^{\mathfrak{R}}(Z')$ is a smooth (respectively, unramified, étale) k-morphism.*

Proof. This follows by combining the proposition and Proposition 9.2(iv). □

It follows from Proposition 9.2(vi) that, if (Z_λ) is a projective system of $\mathfrak{R}$-schemes with affine transition morphisms, then so also is $(\mathrm{Gr}^{\mathfrak{R}}(Z_\lambda))$ and $\varprojlim \mathrm{Gr}^{\mathfrak{R}}(Z_\lambda)$ exists in the category of k-schemes. Thus, by (6.12) and the universal property of projective limits, we obtain the following statement.

rojlim **Proposition 9.5.** *The functor $\mathrm{Gr}^{\mathfrak{R}}$ commutes with the formation of projective limits of schemes with affine transition morphisms.*

10. Weil restriction and the Greenberg functor

wrbe

For $n \in \mathbb{N}$ let R_n be the n-th truncation of a complete discrete valuation ring R and recall $S_n = \mathrm{Spec}\, R_n$. Recall also the functors h_n^R and Gr_n^R introduced in Section 7. If R' is an extension of R, let k'/k denote the corresponding residue field extension and set $S' = \mathrm{Spec}\, R'$.

hne1 **Lemma 10.1.** *Let R' be a finite extension of R of ramification index e. Then, for every k-scheme Y,*

$$h_n^R(Y) \times_{S_n} S'_{ne} = h_{ne}^{R'}\big(Y \times_k \mathrm{Spec}\, k'\big)$$

Proof. Since h_n^R is local for the Zariski topology, we may assume that $Y = \mathrm{Spec}\, A$. In this case $h_n^R(Y) = \mathrm{Spec}\,\mathscr{R}_n(A)$ (6.7) and the lemma follows from Lemma 4.4. □

The following is the main result of this section. For the meaning of the term "admissible", see Definition A.6.

wr-gr **Theorem 10.2.** *Let R' be a finite extension of R of ramification index e. If Z is an S'_{ne}-scheme which is admissible relative to $S'_{ne} \to S_n$, then $\mathrm{Res}_{k'/k}\big(\mathrm{Gr}_{ne}^{R'}(Z)\big)$ and $\mathrm{Res}_{S'_{ne}/S_n}(Z)$ exist and*

rgr

$$\mathrm{Res}_{k'/k}\big(\mathrm{Gr}_{ne}^{R'}(Z)\big) = \mathrm{Gr}_n^R\big(\mathrm{Res}_{S'_{ne}/S_n}(Z)\big). \tag{10.3}$$

Proof. By Lemma A.13, $Z \times_{S'_{ne}} S'_1$ is admissible relative to k'/k. Thus, since

$$\mathrm{Gr}_{ne}^{R'}(Z) \to \mathrm{Gr}_1^{R'}(Z \times_{S'_{ne}} S'_1) = Z \times_{S'_{ne}} S'_1$$

is an affine morphism of k'-schemes by Proposition 8.11, $\mathrm{Gr}_{ne}^{R'}(Z)$ is admissible relative to k'/k. The existence assertions now follow from Theorem A.8. On the other hand, (10.3) follows from Lemma 10.1 using the adjunction formula (6.12), the definition of the Weil restriction functor (A.1) and Yoneda's lemma. □

Remark 10.4. In the equal characteristic case (10.3) is a particular case of the well-known transitivity of the Weil restriction functor (A.3). On the other hand, Remark A.7(a) shows that the admissibility condition on Z is satisfied if Z is quasi-projective over R'_{ne}.

tot-gr **Proposition 10.5.** *Let R' be a finite and totally ramified extension of R of degree e and let Z be an arbitrary S'_{ne}-scheme. Then $\mathrm{Res}_{S'_{ne}/S_n}(Z)$ exists and*

$$\mathrm{Gr}_n^R\big(\mathrm{Res}_{S'_{ne}/S_n}(Z)\big)=\mathrm{Gr}_{ne}^{R'}(Z).$$

Proof. The existence assertion is Remark A.12. The formula now follows as in the proof of Theorem 10.2. □

The behavior of the functor Gr_n^R under finite extensions of R was discussed in [NS, Theorem 3.1] for R_n-schemes of finite type. We now extend the indicated theorem to arbitrary R_n-schemes.

unr3 **Proposition 10.6.** *Let k'/k be a subextension of $\overline{k}/k$ and let R' be the extension of R of ramification index 1 which corresponds to k'/k. Then, for every S_n-scheme Z, there exists a canonical isomorphism of k'-schemes*

$$\mathrm{Gr}_n^R(Z)\times_k \mathrm{Spec}\, k'=\mathrm{Gr}_n^{R'}\big(Z\times_{S_n}S'_n\big).$$

Proof. By Lemma 4.2, we have $\mathscr{R}_n(A)=\mathscr{R}'_n(A)$ for every k'-algebra A. Thus, for every k'-scheme T, there exists a canonical isomorphism of S_n-schemes $h_n^{R'}(T)=h_n^R(T)$. The proposition now follows from (6.12). □

b-c **Proposition 10.7.** *Let R' be a finite extension of R of ramification index e. Then, for every S_n-scheme Z, there exists a canonical closed immersion of k'-schemes*

$$\mathrm{Gr}_n^R(Z)\times_k \mathrm{Spec}\, k'\hookrightarrow \mathrm{Gr}_{ne}^{R'}\big(Z\times_{S_n}S'_{ne}\big)$$

which is an isomorphism if $e=1$.

Proof. The indicated map is an isomorphism if $e=1$ by Proposition 10.6. If Z is of finite type over S_n, the proposition was established in [NS, Theorem 3.1]. The method used in [loc.cit.] easily extends to arbitrary S_n-schemes Z provided the finite-dimensional affine space $\mathbb{A}^N_{R_n}$ considered in [NS, proof of Lemma 3.5, p. 1598] is replaced by the affine space $\mathbb{A}^{(I)}_{R_n}$ introduced in the proof of Proposition-Definition 6.10. □

gp-sch

11. The change of level morphism for smooth group schemes

Let R be a complete discrete valuation ring and let G be a smooth R-group scheme. Let $r\geq 1$ and $i\geq 0$ be integers. By Remark 6.17(d) and Corollary 9.4, the change of level morphism (8.3)

nm-hom (11.1) $$\varrho^i_{r,G}\colon \mathrm{Gr}^R_{r+i}(G)\to \mathrm{Gr}^R_r(G)$$

is a morphism of smooth k-group schemes, where $\mathrm{Gr}_n^R(G)=\mathrm{Gr}_n^R(G\times_R S_n)$. Further, by (8.2),

comp (11.2) $$\varrho^{i+1}_{r,G}=\varrho^1_{r,G}\circ\varrho^i_{r+1,G}.$$

In this Section we will describe the kernel of (11.1). To this end, let $\omega^1_{G/R}=\varepsilon^*\Omega^1_{G/R}$, where $\varepsilon\colon \operatorname{Spec} R\to G$ is the unit section of G. When $i=1$, the kernel of (11.1) is described by Corollary A.18, namely if F_k denotes the Frobenius endomorphism of $\operatorname{Spec} k$ when $\operatorname{char} k=p>0$ and $\mathbb{V}\big(\omega^1_{G_{\rm s}/k}\big)^{(p^{m-1})}=\mathbb{V}\big((F_k^{m-1})^*\omega^1_{G_{\rm s}/k}\big)$, then

uc
$$\operatorname{Ker}\varrho^1_{r,G}=\begin{cases}\mathbb{V}\big(\omega^1_{G_{\rm s}/k}\big) & \text{if } \operatorname{char} R=\operatorname{char} k,\\ \mathbb{V}\big(\omega^1_{G_{\rm s}/k}\big)^{(p^{m-1})} & \text{if } \operatorname{char} R\neq p=\operatorname{char} k, \text{ where } m=\lceil (r+1)/\bar{e}\rceil.\end{cases}\tag{11.3}$$

m-vker **Proposition 11.4.** *Let G be a smooth R-group scheme and let r,i be positive integers. Then $\varrho^i_{r,G}$ (11.1) is a smooth and surjective morphism of k-group schemes and $\operatorname{Ker}\varrho^i_{r,G}$ is smooth, connected and unipotent.*

Proof. By Propositions 8.11 and 8.12 and Corollary 9.4, $\varrho^i_{r,G}$ is an affine surjective morphism of smooth k-group schemes. Now (11.3) shows that $\varrho^1_{r,G}$ is smooth and the smoothness of $\varrho^i_{r,G}$ for arbitrary i follows by induction from (11.2). In particular, $\varrho^i_{r,G}$ is faithfully flat and the sequence

rho-sm
$$1\longrightarrow \operatorname{Ker}\varrho^i_{r,G}\longrightarrow \operatorname{Gr}^R_{r+i}(G)\overset{\varrho^i_{r,G}}{\longrightarrow}\operatorname{Gr}^R_r(G)\longrightarrow 1\tag{11.5}$$

is exact for the fppf topolgy on (Sch/k). It remains to check that $U^i_r=\operatorname{Ker}\varrho^i_{r,G}$ is connected and unipotent. By (11.3), this is the case if $i=1$. The proposition now follows by induction since there exist exact sequences for the fppf topology on (Sch/k)

$$1\longrightarrow U^i_{r+1}\longrightarrow U^{i+1}_r\overset{u}{\longrightarrow}U^1_r\longrightarrow 1,$$

where $u=\varrho^i_{r+1,G}\times_{\operatorname{Gr}^R_r(G)}\operatorname{Spec} k$. □

Note that (2.4), (2.10) and (6.13) yield a (non-canonical) isomorphism of k-schemes

obv0
$$\operatorname{Gr}^R_i(\mathbb{V}(\omega^1_{G/R}))\xrightarrow{\sim}\mathbb{A}^{id}_k,\tag{11.6}$$

where $d=\dim G_{\rm s}$. Further, if either $i\leq \bar{e}=v(p)$ or $\operatorname{char} R=\operatorname{char} k$, then (11.6) is induced by an isomorphism of k-group schemes

obv
$$\operatorname{Gr}^R_i(\mathbb{V}(\omega^1_{G/R}))\xrightarrow{\sim}\mathbb{G}^{id}_{a,k}.\tag{11.7}$$

We will now define, for an arbitrary R-group scheme G, a canonical morphism of k-group schemes

wj-1
$$\Phi^i_{r,G}\colon \operatorname{Gr}^R_i(\mathbb{V}(\omega^1_{G/R}))\to \operatorname{Ker}\varrho^i_{r,G}\qquad (1\leq i\leq r)\tag{11.8}$$

and show that it is an isomorphism under certain conditions.

For every k-algebra A, set $B=\mathscr{R}_{r+i}(A)$ and $J=\overline{\mathscr{M}^{r}_{r+i}}(A)$. By (3.8) and (3.9), we have $J^2=0$ and $\mathscr{R}_r(A)$ is isomorphic to B/J. Also recall that, by definition, $\varrho^i_{r,G}(A)$ can be identified with the canonical map $G(B)\to G(B/J)$ and therefore also with the map $G_B(B)\to G_B(B/J)$. Thus there exists a canonical isomorphism of groups

wj-0
$$\operatorname{Ker}\varrho^i_{r,G}(A)\xrightarrow{\sim}\operatorname{Hom}_{B\text{-mod}}(w^1_{G_B/B},J),\tag{11.9}$$

where $w^1_{G_B/B} = \Gamma\big(\operatorname{Spec} B, \omega^1_{G_B/B}\big)$ [DG, Theorem 3.5, p. 208]. On the other hand, we may make the identifications

$$\mathrm{Gr}_i^R(\mathbb{V}(\omega^1_{G/R}))(A) = \mathbb{V}(\omega^1_{G_C/C})(C) = \mathrm{Hom}_{C\text{-mod}}(w^1_{G_C/C}, C) = \mathrm{Hom}_{B\text{-mod}}(w^1_{G_B/B}, C),$$

where $C = \mathscr{R}_i(A)$. Now recall the homomorphism of B-modules $\varphi_{r+i,r}(A)\colon C \to J$ (3.14). Under the above identifications, $\mathrm{Hom}_{B\text{-mod}}(w^1_{G_B/B}, \varphi_{r+i,r}(A))$ can be identified with a map

wj
$$\mathrm{Gr}_i^R(\mathbb{V}(\omega^1_{G/R}))(A) \to \mathrm{Hom}_{B\text{-mod}}(w^1_{G_B/B}, J). \tag{11.10}$$

Composing the preceding map with the inverse of (11.9) and letting A vary, we obtain the canonical morphism of k-group schemes $\Phi^i_{r,G}$ (11.8).

vker1 **Proposition 11.11.** *Assume that R is an equal characteristic ring and let G be a smooth R-group scheme. Then the map $\Phi^i_{r,G}$ (11.8) is an isomorphism of k-group schemes. Consequently, $\operatorname{Ker} \varrho^i_{r,G}$ is (non-canonically) isomorphic to $\mathbb{G}^{id}_{a,k}$, where $d = \dim G_{\mathrm{s}}$.*

Proof. By Proposition 3.15, $\varphi_{r+i,r}(A)$ is an isomorphism for every k-algebra A. Consequently, the map (11.10) is an isomorphism for arbitray A and therefore $\Phi^i_{r,G}$ is an isomorphism. The proposition now follows from (11.7). □

vker **Proposition 11.12.** *Let R be a ring of unequal characteristics $(0, p)$ and let G be a smooth R-group scheme. Then the map $\Phi^i_{r,G}$ (11.8) is an isogeny of smooth, connected and unipotent k-group schemes. Its kernel is an infinitesimal k-group scheme which is trivial if $r + i \le \bar{e} = v(p)$. Further, if $i \le \bar{e}$, then $\operatorname{Ker} \varrho^i_{r,G}$ is (non-canonically) isomorphic to $\mathbb{G}^{id}_{a,k}$, where $d = \dim G_{\mathrm{s}}$.*

Proof. By (11.6) and Proposition 11.4, $\mathrm{Gr}_i^R(\mathbb{V}(\omega^1_{G/R}))$ and $\operatorname{Ker} \varrho^i_{r,G}$ are smooth, connected and unipotent k-group schemes. On the other hand, by Proposition 3.15, $\varphi_{r+i,r}(A)$ is an isomorphism of abelian groups if $r + i \le \bar{e}$ and A is any k-algebra or if $r + i > \bar{e}$ and A is perfect. Thus $\Phi^i_{r,G}$ is an isomorphism if $r + i \le \bar{e}$. When $r + i > \bar{e}$, the maps (11.10) and $\Phi^i_{r,G}(A)$ (11.8) are isomorphisms of abelian groups for every perfect k-algebra A. Consequently $(\operatorname{Ker} \Phi^i_{r,G})\big(\bar{k}\big) = \operatorname{Ker}\big(\Phi^i_{r,G}(\bar{k})\big) = \{1\}$ and $\Phi^i_{r,G}(\bar{k})$ is surjective. Thus $\operatorname{Ker} \Phi^i_{r,G}$ is an infinitesimal k-group scheme and $\Phi^i_{r,G}$ is faithfully flat. The last assertion of the proposition follows from (11.7). □

ex-beg *Remark* 11.13. The infinitesimal k-group scheme $\operatorname{Ker} \Phi^i_{r,G}$ of Proposition 11.12 can be nontrivial. In effect, let $R = W(k)$ and $G = \mathbb{G}_{a,R}$. Then $\operatorname{Ker} \Phi^1_{1,G}$ is isomorphic to the infinitesimal k-group scheme α_p since, for every k-algebra A, the map $\Phi^1_{1,G}(A)$ may be identified with the map $\varphi_{2,1}(A)\colon A \to VW_2(A), a \mapsto (0, a^p)$ (see Remark 3.16(a)). In particular, [Bég, Lemma 4.1.1(2), p. 37] is false. See also Remark 12.1 below.

The preceding considerations yield the following exactness result.

-green **Proposition 11.14.** *Let $1 \to F \to G \overset{q}{\to} H \to 1$ be a sequence of smooth R_n-group schemes. Assume that $F = \operatorname{Ker}(q)$ and q is quasi-compact and surjective. Then the induced sequence of smooth k-group schemes $1 \to \mathrm{Gr}_n^R(F) \to \mathrm{Gr}_n^R(G) \to \mathrm{Gr}_n^R(H) \to 1$ is exact.*

Proof. The indicated sequence is left-exact since Gr_n^R has a left-adjoint functor. Since $\mathrm{Gr}_n^R(q)$ is smooth, it remains only to check that $\mathrm{Gr}_n^R(q)$ is surjective. The case $n=1$ is clear. The surjectivity of q_{s} and (11.3) show that the induced morphism $\operatorname{Ker}\varrho_{r,G}^1 \to \operatorname{Ker}\varrho_{r,H}^1$ is surjective for every $r\geq 1$. The surjectivity of $\mathrm{Gr}_{n+1}^R(q)$ now follows by induction from the surjectivity of $\mathrm{Gr}_n^R(q)$ and the surjectivity of the change of level morphisms $\varrho_{n,G}^1$ and $\varrho_{n,H}^1$ established in Proposition 11.4. □

d-comp **Corollary 11.15.** *Let G be a smooth R-group scheme. Then*

(i) $\dim \mathrm{Gr}_n^R(G) = n\dim G_{\mathrm{s}}$.
(ii) $\mathrm{Gr}_n^R(G)$ *is connected if, and only if, G_{s} is connected.*
(iii) $\mathrm{Gr}_n^R(G^0) = \mathrm{Gr}_n^R(G)^0$.
(iv) $\mathrm{Gr}_n^R(\pi_0(G)) = \pi_0(\mathrm{Gr}_n^R(G))$.

Proof. Assertion (i) follows by induction from (11.5) and (11.3) and (ii) follows from Proposition 11.4. Now, since G^0 is an open subgroup scheme of G, (iii) follows from (ii) and Remark 6.17(b). Finally, (iv) follows from (iii) and Proposition 11.14. □

The results of this Section can also be proven for smooth R_n-group schemes, provided the integers r, i appearing in the first three propositions satisfy the condition $r+i\leq n$.

pfgf

12. The perfect Greenberg functor

Let R be a discrete valuation ring with perfect residue field k of positive characteristic p and write (Perf/k) for the category of perfect k-schemes. The inclusion functor $(\mathrm{Perf}/k)\to(\mathrm{Sch}/k)$ has a right-adjoint functor $(\mathrm{Sch}/k)\to(\mathrm{Perf}/k), Y\mapsto Y^{\mathrm{pf}}$, where Y^{pf} is the inverse perfection of Y, defined as the projective limit over $\mathbb{N}$ of copies of Y with all transition morphisms equal to the Frobenius endmorphism of Y. See [BGA, §5] for more details.

If $n\in\mathbb{N}$, the *perfect Greenberg functor of level n* (associated to R) is

$$\mathbf{Gr}_n^R\colon (\mathrm{Sch}/R_n)\to(\mathrm{Perf}/k),\quad Z\mapsto \mathrm{Gr}_n^R(Z)^{\mathrm{pf}}.$$

Analogs of Propositions 9.5, 10.5, 10.6 and 10.7 with $\mathbf{Gr}_n^R$ in place of Gr_n^R are easily established. Further, since the perfection functor preserves exact sequence of smooth k-group schemes by [BGA, Theorem 6.1], Proposition 11.14 holds also when Gr_n^R is replaced with $\mathbf{Gr}_n^R$.

egueri *Remark* 12.1. Since the perfection of an infinitesimal k-group scheme is the trivial k-group scheme [BGA, Lemma 5.20], Propositions 11.11 and 11.12 show that the perfection of the canonical morphism of k-group schemes $\Phi_{r,G}^i$ (11.8) is an isomorphism for every smooth R-group scheme G. It follows from the above that, despite the fact that the possibly non-trivial infinitesimal kernel of $\Phi_{r,G}^i$ is ignored in [Bég] (see Remark 11.13), the indicated oversight had no consequences for the validity of the main results of [Bég].

We will write R^{nr} for the extension of R of ramification index 1 which corresponds to $\overline{k}/k$.

Proposition 12.2. *Let $0\to F\xrightarrow{f} G\to H\to 0$ be a complex of commutative R_n-group schemes, where G and H are smooth. Assume that*

(i) *f is quasi-compact,*
(ii) *$\pi_0(G)(R_n^{\mathrm{nr}})$ is a finitely generated abelian group, and*
(iii) *the induced sequence of abelian groups $0 \to F(R_n^{\mathrm{nr}}) \to G(R_n^{\mathrm{nr}}) \to H(R_n^{\mathrm{nr}}) \to 0$ is exact.*

Then the induced complex of perfect and commutative k-group schemes

$$0 \to \mathbf{Gr}_n^R(F) \to \mathbf{Gr}_n^R(G) \to \mathbf{Gr}_n^R(H) \to 0$$

is exact for the the fpqc topology on (Perf/k).

Proof. By (iii), Lemma 7.3(ii) and Corollary 9.4, the sequence

$$0 \to \mathrm{Gr}_n^R(F) \to \mathrm{Gr}_n^R(G) \to \mathrm{Gr}_n^R(H) \to 0$$

is a complex of commutative k-group schemes such that the sequence

$$0 \to \mathrm{Gr}_n^R(F)\big(\overline{k}\big) \to \mathrm{Gr}_n^R(G)\big(\overline{k}\big) \to \mathrm{Gr}_n^R(H)\big(\overline{k}\big) \to 0$$

is exact. Thus the proposition will follow from [BGA, Proposition 6.3] once we check that the following additional conditions hold: (a) $\mathrm{Gr}_n^R(f)\colon \mathrm{Gr}_n^R(F) \to \mathrm{Gr}_n^R(G)$ is quasi-compact, and (b) $\mathrm{Gr}_n^R(G) \to \mathrm{Gr}_n^R(H)$ is flat. Condition (a) follows at once from (i) and Proposition 9.2. On the other hand, by Corollary 11.15(iv), Lemma 7.3(ii) and Lemma 4.2, we have

$$\pi_0(\mathrm{Gr}_n^R(G))\big(\overline{k}\big) = \mathrm{Gr}_n^R(\pi_0(G))\big(\overline{k}\big) = \pi_0(G)(R_n^{\mathrm{nr}}),$$

which is finitely generated by (ii). Thus, since $\mathrm{Gr}_n^R(G)(\overline{k}) \to \mathrm{Gr}_n^R(H)(\overline{k})$ is surjective, we conclude from Lemma A.19 that $\mathrm{Gr}_n^R(G) \to \mathrm{Gr}_n^R(H)$ is flat, i.e., (b) holds. □

forgr

13. The Greenberg realization of an adic formal scheme

We continue to assume that R is a *complete* discrete valuation ring with perfect residue field in the unequal characteristics case. Let $\mathfrak{S} = \widehat{S}$ be the formal completion of S along $S_1 = \operatorname{Spec} k$. We will write $(\text{Ad-For}/\mathfrak{S})$ for the category of adic formal $\mathfrak{S}$-schemes, whose objects are (also) adic in the (non-standard) terminology of [Ab, Definition 2.1.16, p. 121]. By the equivalence of [Ab, Proposition 2.2.14, p. 130], we have $\mathfrak{X} = \varinjlim \mathfrak{X}_n$ for every adic $\mathfrak{S}$-scheme $\mathfrak{X}$, where $\mathfrak{X}_n = (|\mathscr{X}|, \mathscr{O}_{\mathfrak{X}}/\mathfrak{m}^n\mathscr{O}_{\mathfrak{X}})$ for $n \in \mathbb{N}$. Further, for every $\mathfrak{S}$-adic scheme $\mathscr{Y}$, we have

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \simeq \varprojlim \mathrm{Hom}_{S_n}(\mathfrak{X}_n, \mathfrak{Y}_n).$$

Now set

grnx (13.1) $$\mathrm{Gr}_n^R(\mathfrak{X}) = \mathrm{Gr}_n^R(\mathfrak{X}_n)$$

and define

r-def2 (13.2) $$\mathrm{Gr}^R(\mathfrak{X}) = \varprojlim \mathrm{Gr}_n^R(\mathfrak{X}),$$

where the transition morphisms are the change of level morphisms, which are affine by Proposition 8.11. Then $\mathrm{Gr}^R(\mathfrak{X})$ is a k-scheme and $\mathrm{Gr}^R(\mathfrak{S}) = \operatorname{Spec} k$ by (6.14). We now generalize the adjunction formula (7.2).

Let Y be a k-scheme. Recall the R_n-schemes $h_n^R(Y) = (|Y|, \mathscr{R}_n(\mathscr{O}_Y))$ and the nilpotent immersions $\delta_Y^{i,j-i} = \delta_Y^{R_i,R_j}\colon h_i^R(Y) \to h_j^R(Y)$ (8.5), where $1 \le i \le j$. Then

$$\mathfrak{h}^R(Y) = \varinjlim h_n^R(Y)$$

is a formal $\mathfrak{S}$-scheme equal to $(|Y|, \widetilde{\mathscr{R}}(\mathscr{O}_Y))$, where $\widetilde{\mathscr{R}}(\mathscr{O}_Y)$ is the Zariski sheaf on Y defined by

$$\widetilde{\mathscr{R}}(\mathscr{O}_Y) = \varprojlim \mathscr{R}_n(\mathscr{O}_Y).$$

In particular, if $U = \operatorname{Spec} A$ is an affine subscheme of Y, then (5.1) and (6.1) yield

zsf2 (13.3) $$\Gamma(U, \widetilde{\mathscr{R}}(\mathscr{O}_Y)) = \widetilde{\mathscr{R}}(A)$$

and $h^R(U) = \operatorname{Spf} \widetilde{\mathscr{R}}(A)$. Further, if $R = W(k)$ then, using (13.3), $\mathfrak{h}^R(Y) = W(Y)$ is the formal scheme considered in [Ill, §1.5, p. 511]. Note that, as illustrated in Remark 3.13(b), the inclusion $(VW_n(\mathscr{O}_Y))^m \subseteq V^m(W_n(\mathscr{O}_Y))$ can be strict, whence $W(Y)$ is not, in general, an adic formal scheme. However, combining Remarks 2.21 and 3.13(c)-(d), (7.2) and [EGA $\mathrm{I}_{\mathrm{new}}$, Corollary 10.6.4 p. 414], the following holds.

fhr-a **Proposition 13.4.** *Let Y be a k-scheme and let $\mathfrak{X}$ be an adic formal $\mathfrak{S}$-scheme. Assume that*

(i) *R is an equal characteristic ring, or*
(ii) *R is a ring of unequal characteristics and Y is a perfect k-scheme.*

Then $\mathfrak{h}^R(Y)$ is an adic formal $\mathfrak{S}$-scheme and there exists a canonical bijection

$$\operatorname{Hom}_k\big(Y, \operatorname{Gr}^R(\mathfrak{X})\big) = \operatorname{Hom}_{(\text{Ad-For}/\mathfrak{S})}(\mathfrak{h}^R(Y), \mathfrak{X}). \quad \square$$

Consequently, if R is an *equal characteristic* ring, then the functor $\operatorname{Gr}^R\colon (\text{Ad-For}/\mathfrak{S}) \to (\text{Sch}/k)$ is right adjoint to $\mathfrak{h}^R\colon (\text{Sch}/k) \to (\text{Ad-For}/\mathfrak{S})$. The corresponding statement in the unequal characteristics case is false. However, the following generalization of [NS2, line 10, p. 256] is valid.

at-pts **Lemma 13.5.** *Let $\mathfrak{X}$ be an adic formal $\mathfrak{S}$-scheme and let A be a k-algebra which is assumed to be perfect if R is a ring of unequal characteristics. Then $\operatorname{Gr}^R(\mathfrak{X})(A) = \mathfrak{X}(\widetilde{\mathscr{R}}(A))$.*

Proof. The lemma is immediate from (13.3) and Proposition 13.4. □

pro-gr **Proposition 13.6.** *Consider, for a morphism of formal schemes, the property of being:*

(i) *quasi-compact;*
(ii) *quasi-separated;*
(iii) *separated;*
(iv) *a closed immersion;*
(v) *affine;*
(vi) *an open immersion;*
(vii) *formally étale.*

If $\boldsymbol{P}$ denotes one of the above properties and $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of adic formal $\mathfrak{S}$-schemes with property $\boldsymbol{P}$, then the morphism of k-schemes $\operatorname{Gr}^R(f)\colon \operatorname{Gr}^R(\mathfrak{X}) \to \operatorname{Gr}^R(\mathfrak{Y})$ has property $\boldsymbol{P}$ as well.

Proof. If $\boldsymbol{P}$ denotes one of properties (i)-(v) and f has property $\boldsymbol{P}$, then each $f_n\colon \mathfrak{X}_n \to \mathfrak{Y}_n$ has property $\boldsymbol{P}$ by [FK, Propositions 1.6.9, 4.6.9 and 4.4.2]. Consequently, $\operatorname{Gr}^R(f)$ has property $\boldsymbol{P}$ by [BGA, Proposition 3.2], Remark 6.17(b) and Proposition 9.2. In the case of properties (vi) and (vii), a different argument is needed since a projective limit of open immersions may not be an open immersion. If f has one of the indicated properties, then each f_n is formally étale. Thus Corollary 8.9 shows that $\mathrm{Gr}_n^R(f_n)$ and $\mathrm{Gr}^R(f)$ can be identified with $f_1 \times_{\mathfrak{Y}_1} \mathrm{Gr}_n^R(\mathfrak{Y})$ and $f_1 \times_{\mathfrak{Y}_1} \mathrm{Gr}^R(\mathfrak{Y})$, respectively. Thus, since f_1 is an open immersion (respectively, formally étale), $\mathrm{Gr}^R(f)$ is an open immersion (respectively, formally étale). □

Proposition 13.7. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be adic formal $\mathfrak{S}$-schemes. Then there exists a canonical isomorphism of k-schemes*

$$\mathrm{Gr}^R(\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}) = \mathrm{Gr}^R(\mathfrak{X}) \times_k \mathrm{Gr}^R(\mathfrak{Y}).$$

Proof. By [FK, Corollary 1.3.5, p. 267], $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} = \varinjlim (\mathfrak{X}_n \times_{S_n} \mathfrak{Y}_n)$. Thus the proposition follows from Remark 6.17(d) and the fact that $\{(n,n) \colon n \in \mathbb{N}\}$ is cofinal in $\mathbb{N} \times \mathbb{N}$. □

In particular, if $\mathfrak{X}$ is an adic formal $\mathfrak{S}$-group scheme, then $\mathrm{Gr}^R(\mathfrak{X})$ is a k-group scheme.

We will now discuss the behaviour of Gr^R under Weil restriction.

Let $R \to R'$ be an extension of complete discrete valuation rings and let $\mathfrak{S}' \to \mathfrak{S}$ be the corresponding morphism of adic formal schemes. Let $\mathfrak{X}'$ be an adic formal $\mathfrak{S}'$-scheme. We will say that *the Weil restriction of $\mathfrak{X}'$ along $\mathfrak{S}' \to \mathfrak{S}$ exists* if the contravariant functor $(\text{Ad-For}/\mathfrak{S}) \to (\text{Sets}), \mathfrak{T} \to \mathrm{Hom}_{\mathfrak{S}'}(\mathfrak{T} \times_{\mathfrak{S}} \mathfrak{S}', \mathfrak{X}')$, is represented by an adic formal $\mathfrak{S}$-scheme $\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\mathfrak{X}')$ (which will then be called *the Weil restriction of $\mathfrak{X}'$ along $\mathfrak{S}' \to \mathfrak{S}$*).

wr-gr2 **Proposition 13.8.** *Let R' be a finite extension of R of ramification index e with residue field k' and let $\mathfrak{X}' = \varinjlim \mathfrak{X}'_n$ be an adic formal $\mathfrak{S}'$-scheme such that $\mathfrak{X}'_{ne}$ is admissible relative to $S'_{ne} \to S_n$ for every $n \geq 1$ (see Definition* A.6*). Then $\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\mathfrak{X}')$ and $\mathrm{Res}_{k'/k}(\mathrm{Gr}^{R'}(\mathfrak{X}'))$ exist and*

$$\mathrm{Gr}^R(\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\mathfrak{X}')) = \mathrm{Res}_{k'/k}(\mathrm{Gr}^{R'}(\mathfrak{X}')).$$

Proof. By Theorem A.8, $\mathrm{Res}_{S'_{ne}/S_n}(\mathfrak{X}'_{ne})$ exists for every $n \in \mathbb{N}$. Further, by (A.2) and (A.11), $\mathrm{Res}_{S'_{re}/S_r}(\mathfrak{X}'_{re}) = \mathrm{Res}_{S'_{ne}/S_n}(\mathfrak{X}'_{ne}) \times_{S_n} S_r$ for $1 \leq r \leq n$. Thus

for-wr (13.9)
$$\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\mathfrak{X}') \overset{\text{def.}}{=} \varinjlim \mathrm{Res}_{S'_{ne}/S_n}(\mathfrak{X}'_{ne})$$

is the Weil restriction of $\mathfrak{X}'$ along $\mathfrak{S}' \to \mathfrak{S}$. Now, by (13.1), Theorem 10.2 and Proposition A.10, we have

$$\mathrm{Gr}^R(\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\mathfrak{X}')) = \varprojlim_{n \in \mathbb{N}} \mathrm{Res}_{k'/k}(\mathrm{Gr}_{ne}^{R'}(\mathfrak{X}'_{ne})) = \mathrm{Res}_{k'/k}(\mathrm{Gr}^{R'}(\mathfrak{X}')),$$

as claimed. □

Remark 13.10. Recall that, if R'/R is a finite and totally ramified extension of degree e and $\mathfrak{X}' = \varinjlim \mathfrak{X}'_n$ is an adic formal $\mathfrak{S}'$-scheme, then Proposition 10.5 yields a formula

$$\mathrm{Gr}_{ne}^{R'}(\mathfrak{X}') = \mathrm{Gr}_n^R(\mathrm{Res}_{S'_{ne}/S_n}(\mathfrak{X}'))$$

for every integer $n \geq 1$, where $\mathrm{Res}_{S'_{ne}/S_n}(\mathfrak{X}') \overset{\text{def.}}{=} \mathrm{Res}_{S'_{ne}/S_n}(\mathfrak{X}'_{ne})$. In particular, if $n = 1$, then $\mathrm{Gr}_e^{R'}(\mathfrak{X}') = \mathrm{Res}_{R'_e/k}(\mathfrak{X}')$, which generalizes [NS, Theorem 4.1]. Note that the hypothesis "nice" (i.e., admissible) in the statement of [NS, Theorem 4.1] is unnecessary.

r-unr3 **Proposition 13.11.** *Let k'/k be a subextension of $\overline{k}/k$ and let R' be the extension of R of ramification index 1 which corresponds to k'/k. Then, for every adic formal $\mathfrak{S}$-scheme $\mathfrak{X}$, there exists a canonical isomorphism of k'-schemes*

$$\mathrm{Gr}^{R}(\mathfrak{X})\times_{\operatorname{Spec}k}\operatorname{Spec}k'=\mathrm{Gr}^{R'}\big(\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}'\big).$$

Proof. Set $S'=\operatorname{Spec}R'$. Since $\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}'=\varinjlim\,(\mathfrak{X}_n\times_{S_n}S'_n)$ by [FK, Corollary 1.3.5, p. 267], (13.1) yields $\mathrm{Gr}^{R'}_n\big(\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}'\big)=\mathrm{Gr}^{R'}_n\big(\mathfrak{X}_n\times_{S_n}S'_n\big)$. Thus, since $\mathrm{Gr}^{R}_n(\mathfrak{X})=\mathrm{Gr}^{R}_n(\mathfrak{X}_n)$, Proposition 10.6 yields, for every $n\in\mathbb{N}$, a canonical isomorphism of k'-schemes

$$\mathrm{Gr}^{R}_n(\mathfrak{X})\times_{\operatorname{Spec}k}\operatorname{Spec}k'=\mathrm{Gr}^{R'}_n\big(\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}'\big).$$

The proposition now follows from (13.2) noting that projective limits of schemes commute with base extension. □

The following proposition generalizes [NS, Theorem 3.8].

for-bc **Proposition 13.12.** *Let $\mathfrak{X}$ be an adic formal $\mathfrak{S}$-scheme and let R' be a finite extension of R with associated residue field extension k'/k. Then there exists a canonical closed immersion of k'-schemes*

$$\mathrm{Gr}^{R}(\mathfrak{X})\times_{\operatorname{Spec}k}\operatorname{Spec}k'\hookrightarrow\mathrm{Gr}^{R'}\big(\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}'\big).$$

If R'/R has ramification index 1, then the preceding map is an isomorphism.

Proof. The second assertion is a particular case of Proposition 13.11. Let e be the ramification index of R' over R. Since $(\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}')_{ne}=\mathfrak{X}_{ne}\times_{S_{ne}}S'_{ne}=\mathfrak{X}_n\times_{S_n}S'_{ne}$, Proposition 10.7 yields, a canonical closed immersion of k'-schemes

$$\mathrm{Gr}^{R}_n(\mathfrak{X})\times_{\operatorname{Spec}k}\operatorname{Spec}k'\hookrightarrow\mathrm{Gr}^{R'}_{ne}\big(\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{S}'\big).$$

The proposition now follows by taking projective limits [BGA, Proposition 3.2(v)]. □

14. The Greenberg realization of an R-scheme

grsh

Let X be an R-scheme and let $\widehat{X}=\varinjlim\,(X\times_S S_n)$ be the formal completion of X along $X\times_S\operatorname{Spec}k$, which is an object of (Ad-For/$\mathfrak{S}$). The *Greenberg realization of X* is the k-scheme

grx2 (14.1) $$\mathrm{Gr}^{R}(X)\overset{\text{def.}}{=}\mathrm{Gr}^{R}\big(\widehat{X}\big)=\varprojlim\mathrm{Gr}^{R}_n(X),$$

where $\mathrm{Gr}^{R}_n(X)=\mathrm{Gr}^{R}_n(X\times_S S_n)$ and the transition morphisms of the limit are the change of level morphisms $\varrho^{i}_{n,X}\colon\mathrm{Gr}^{R}_{n+i}(X)\to\mathrm{Gr}^{R}_n(X)$. The resulting functor $\mathrm{Gr}^{R}\colon(\mathrm{Sch}/R)\to(\mathrm{Sch}/k), X\mapsto\mathrm{Gr}^{R}(X)$, satisfies $\mathrm{Gr}^{R}(S)=\operatorname{Spec}k$. Note that, by (14.1), $\mathrm{Gr}^{R}(\mathbb{A}^1_R)=\varprojlim\mathrm{Gr}^{R}_n(\mathbb{A}^1_{R_n})=\varprojlim\mathscr{R}_n=\mathscr{R}\simeq\mathbb{A}^{(\mathbb{N})}_k$, which is not locally of finite type.

Remark 14.2. A proof analogous to that of Proposition 13.6 shows that Gr^{R} preserves all the properties of morphisms of schemes listed there.

The following lemma is an analog of Lemma 7.3(i).

r-pts2 **Proposition 14.3.** *Let X be an R-scheme and let A be a k-algebra which is assumed to be perfect if R is a ring of unequal characteristics. Then $\mathrm{Gr}^{R}(X)(A)=X(\widetilde{\mathscr{R}}(A))$.*

Proof. By (2.5) and Proposition 2.12, we may assume that $X = \operatorname{Spec} B$ is affine. Assume first that R is a ring of unequal characteristics, so that A is perfect. Set $Y = \operatorname{Res}_{R/W(k)}(X)$, which is an affine $W(k)$-scheme. Then $\operatorname{Gr}^R(X) = \operatorname{Gr}^{W(k)}(Y)$ by Proposition 13.8. Further, by (5.4), $\widetilde{\mathscr{R}}(A)$ is canonically isomorphic to $R \otimes_{W(k)} W(A)$, which yields $X(\widetilde{\mathscr{R}}(A)) = Y(W(A))$ by (A.1). Thus $\operatorname{Gr}^R(X)(A) = X(\widetilde{\mathscr{R}}(A))$ if, and only if, $\operatorname{Gr}^{W(k)}(Y) = Y(W(A))$. In other words, we may assume that $R = W(k)$. By (14.1) and Lemma 13.5, we have $\operatorname{Gr}^{W(k)}(X)(A) = \widehat{X}(\operatorname{Spf} W(A))$, whence it remains to check that $\widehat{X}(\operatorname{Spf} W(A)) = X(W(A))$. This follows from the universal property of the p-adic completion. Finally, assume that $R \simeq k[[t]]$. Then, by (14.1), Lemma 13.5 and Remark 5.3(a), we have $\operatorname{Gr}^R(X)(A) = \widehat{X}(\operatorname{Spf} A[[t]])$. As above, the equality $\widehat{X}(\operatorname{Spf} A[[t]]) = X(A[[t]])$ follows from the universal property of the t-adic completion. □

Corollary 14.4. *Let X be an R-scheme which is separated and locally of finite type. Then $\operatorname{Gr}^R(X)(\overline{k}) = X(\widehat{R}^{\mathrm{nr}})$.*

Proof. This follows from (5.2) and the proposition. □

Lemma 14.5. *If X is a smooth R-scheme, then $\operatorname{Gr}^R(X)$ is a reduced k-scheme.*

Proof. Since $X \times_S S_n$ is smooth over S_n for every n, $\operatorname{Gr}_n^R(X)$ is smooth over k for every n by Corollary 9.4. Consequently, each $\operatorname{Gr}_n^R(X)$ is reduced and therefore $\operatorname{Gr}^R(X) = \varprojlim \operatorname{Gr}_n^R(X)$ is reduced as well by [EGA, IV$_3$, Proposition 8.7.1]. □

If k is perfect of positive characteristic and X is an R-scheme, the *perfect Greenberg realization of* X is the perfect k-scheme

pgr-pf (14.6)
$$\mathbf{Gr}^R(X) = \operatorname{Gr}^R(X)^{\mathrm{pf}}.$$

Remark 14.7. Assume that R is a ring of unequal characteristics and let X be an R-scheme such that $\operatorname{Res}_{R/W(k)}(X)$ exists. In [Bég, §4.1, p. 36] the author defined the Greenberg realization of level n of X to be

$$\underset{\sim}{\operatorname{Gr}}_n(X) = \operatorname{Gr}_n^{W(k)}(\operatorname{Res}_{R/W(k)}(X) \times_{W(k)} \operatorname{Spec} W_n(k)).$$

By (A.2) and (A.11), we have

$$\operatorname{Res}_{R/W(k)}(X) \times_{W(k)} \operatorname{Spec} W_n(k) = \operatorname{Res}_{R_{n\bar{e}}/W_n(k)}(X \times_S S_{n\bar{e}}),$$

where $\bar{e}$ denotes the ramification of $R/W(k)$, whence

new (14.8)
$$\underset{\sim}{\operatorname{Gr}}_n(X) = \operatorname{Gr}_n^{W(k)}(\operatorname{Res}_{R_{n\bar{e}}/W_n(k)}(X \times_S S_{n\bar{e}})).$$

Note that, since $R/W(k)$ is totally ramified, $\operatorname{Res}_{R_{n\bar{e}}/W_n(k)}(X_{n\bar{e}})$ exists for *every* R-scheme X by Remark A.12. Thus (14.8) may be taken to be the *definition* of $\underset{\sim}{\operatorname{Gr}}_n(X)$ when $\operatorname{Res}_{R/W(k)}(X)$ fails to exist. Now observe that, if A is *any* k-algebra, then

$$\underset{\sim}{\operatorname{Gr}}_n(X)(A) = X(R \otimes_{W(k)} W_n(A)).$$

Indeed, since $R_{n\bar{e}} = R \otimes_{W(k)} W_n(k)$, (14.8) and Lemma 7.3(i) show that

$$\begin{aligned} \underset{\sim}{\operatorname{Gr}}_n(X)(A) = \operatorname{Res}_{R_{n\bar{e}}/W_n(k)}(X \times_S S_{n\bar{e}})(W_n(A)) &= X(R_{n\bar{e}} \otimes_{W_n(k)} W_n(A)) \\ &= X(R \otimes_{W(k)} W_n(A)), \end{aligned}$$

as claimed. Next, by Proposition 10.5, (14.8) may be written as $\underset{\sim}{\mathrm{Gr}}_n(X) = \mathrm{Gr}^R_{n\bar{e}}(X)$. It follows that, if $\underset{\sim}{\mathrm{Gr}}(X) = \varprojlim \underset{\sim}{\mathrm{Gr}}_n(X)$ is the object introduced in [Bég, §4.1, p. 36], then $\underset{\sim}{\mathrm{Gr}}(X) = \mathrm{Gr}^R(X)$, where $\mathrm{Gr}^R(X)$ is the k-scheme (14.1). Further, if $\underline{\underline{G}}(X) \overset{\text{def.}}{=} \underset{\sim}{\mathrm{Gr}}(X)^{\mathrm{pf}}$ is the perfect k-scheme considered in [loc.cit.] and $\mathbf{Gr}^R(X)$ is the object (14.6), then $\underline{\underline{G}}(X) = \mathbf{Gr}^R(X)$. Regarding the latter functor, [loc.cit., p. 36, line -11] contains the (unproven) claim that, for every perfect k-algebra A,

$$\mathbf{Gr}^R(X)(A) = X(R \otimes_{W(k)} W(A)).$$

The latter is indeed valid and follows from (14.6), Proposition 14.3 and (5.4).

The next result applies to commutative R-group schemes.

Proposition 14.9. *Let $0 \to F \to G \overset{q}{\to} H \to 0$ be a sequence of smooth and commutative R-group schemes. Assume that $F = \mathrm{Ker}\, q$ and q is quasi-compact and surjective. Then the induced sequence of smooth and commutative k-group schemes*

$$0 \to \mathrm{Gr}^R(F) \longrightarrow \mathrm{Gr}^R(G) \to \mathrm{Gr}^R(H) \to 0$$

is exact for the fpqc topology.

Proof. By Proposition 11.14, the induced sequence of smooth and commutative k-group schemes

$$0 \longrightarrow \mathrm{Gr}^R_n(F) \longrightarrow \mathrm{Gr}^R_n(G) \overset{\mathrm{Gr}^R_n(q)}{\longrightarrow} \mathrm{Gr}^R_n(H) \longrightarrow 0$$

is exact. Now observe that, since $\mathrm{Gr}^R_n(q)$ is smooth, quasi-compact and surjective by Propositions 9.2 and 9.4, $\mathrm{Gr}^R_n(q)$ is faithfully flat and quasi-compact. On the other hand, since $F = G \times_H S$ is smooth over S, the transition morphisms of the system $(\mathrm{Gr}^R_n(F))$ are surjective by Proposition 8.12. We may now apply [BGA, Proposition 3.8] to complete the proof. □

If $S' = \mathrm{Spec}\, R'$, where R' is a finite extension of R of ramification index e, let $\mathfrak{S}' = \widehat{S'}$. More generally, if X' is an S'-scheme, its formal completion along its special fiber is $\widehat{X'} = \varinjlim (X' \times_{S'} S'_{ne})$.

-fcomp **Lemma 14.10.** *Let R' be a finite extension of R and let X' be an R'-scheme which is admissible relative to R'/R (see Definition A.6). Then $\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\widehat{X'})$ and $\mathrm{Res}_{R'/R}(X')$ exist and*

$$\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\widehat{X'}) = \widehat{\mathrm{Res}_{R'/R}}(X').$$

Proof. The R-scheme $\mathrm{Res}_{R'/R}(X')$ exists by Theorem A.8. Using (A.11) and Remark A.7(c), $X' \times_{S'} S'_{ne}$ is admissible relative to $S'_{ne} \to S_n$ for every $n \in \mathbb{N}$. Thus $\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\widehat{X'})$ exists by Proposition 13.8. Further, (13.9), (A.2) and (A.11) yield

$$\begin{aligned}\mathrm{Res}_{\mathfrak{S}'/\mathfrak{S}}(\widehat{X'}) &= \varinjlim \mathrm{Res}_{S'_{ne}/S_n}(X' \times_{S'} S'_{ne}) = \varinjlim \mathrm{Res}_{(S' \times_S S_n)/S_n}(X' \times_S S_n)\\ &= \varinjlim (\mathrm{Res}_{S'/S}(X') \times_S S_n) = \widehat{\mathrm{Res}_{R'/R}}(X'),\end{aligned}$$

as claimed. □

Proposition 14.11. *Let R' be a finite extension of R with associated residue field extension k'/k and let X' be an R'-scheme which is admissible relative to R'/R. Then* $\operatorname{Res}_{R'/R}(X')$ *and* $\operatorname{Res}_{k'/k}(\operatorname{Gr}^{R'}(X'))$ *exist and*

$$\operatorname{Gr}^{R}(\operatorname{Res}_{R'/R}(X')) = \operatorname{Res}_{k'/k}(\operatorname{Gr}^{R'}(X')).$$

Proof. The R-scheme $\operatorname{Res}_{R'/R}(X')$ exists by Theorem A.8. Now, as noted in the proof of Lemma 14.10, each $X' \times_{S'} S'_{ne}$ is admissible relative to $S'_{ne} \to S_n$. Thus $\operatorname{Res}_{k'/k}(\operatorname{Gr}^{R'}(X')) = \operatorname{Res}_{k'/k}(\operatorname{Gr}^{R'}(\widehat{X'}))$ exists and

$$\operatorname{Res}_{k'/k}(\operatorname{Gr}^{R'}(X')) = \operatorname{Gr}^{R}(\operatorname{Res}_{\mathfrak{S}'/\mathfrak{S}}(\widehat{X'}))$$

by Proposition 13.8. The result now follows from (14.1) and Lemma 14.10. □

fgr

15. The Greenberg realization of a finite group scheme

In this Section R is a complete discrete valuation ring with fraction field K. Recall $S = \operatorname{Spec} R$.

Let F be a finite and flat R-group scheme. By Proposition 6.10 and Remark 6.17, $\operatorname{Gr}_n^R(F) = \operatorname{Gr}_n^R(F \times_S S_n)$ is an affine and algebraic k-group scheme. Recall that, by Example 7.6, $\operatorname{Gr}_n^R(F)$ and $\operatorname{Gr}^R(F) = \varprojlim_n \operatorname{Gr}_n^R(F)$ may fail to be finite over k.

Let H_r^i be the schematic image of the change of level morphism $\varrho_r^i = \varrho_{r,F}^i$ (8.3), which then factors as

$$\operatorname{Gr}_{r+i}^R(F) \twoheadrightarrow H_r^i \hookrightarrow \operatorname{Gr}_r^R(F).$$

Using Greenberg approximation, we will derive conditions on r and i so that H_r^i is a finite k-group scheme. Note that the finiteness of H_r^i implies that of H_r^l for every integer $l \geq i$.

r-appr **Lemma 15.1.** *There exist integers $c \geq 1$, $d \geq 0$ and $M \geq 0$ such that, if $r \geq M$, then*

$$\operatorname{Im}[F(R_{cr+d}) \to F(R_r)] = \operatorname{Im}[F(R) \to F(R_r)].$$

Proof. This follows at once from [Gre3, Corollary 1, p. 59], taking there $d = sc$ and $M = \max\{\lceil (N-d)/c \rceil, 0\}$, where s, c and N are the integers in [loc.cit.]. □

fdim **Proposition 15.2.** *Let $c \geq 1$, $d \geq 0$ and $M \geq 0$ be as in Lemma 15.1. If $r \geq M$ and $i \geq (c-1)r + d$, then H_r^i is finite over k.*

Proof. By Proposition 10.6 and faithfully flat and quasi-compact descent, we may assume that k is algebraically closed. By Lemma 7.3(ii), we have $H_r^i(k) = \operatorname{Im}[F(R_{r+i}) \to F(R_r)]$. Thus, by Lemma 15.1, $H_r^{(c-1)r+d}(k) = \operatorname{Im}[F(R) \to F(R_r)]$. Since $F(R)$ is finite, we conclude that $H_r^{(c-1)r+d}(k)$ is finite as well, which yields the proposition. □

The previous result can be strengthened when F is *generically étale*, i.e., $F \times_S \operatorname{Spec} K$ is étale. In this case $\omega^1_{F/R} = \varepsilon^* \Omega^1_{F/R}$ is a torsion R-module and the *defect of smoothness of F* is defined by

df (15.3) $$\delta(F) = \operatorname{length}_R(\omega^1_{F/R}).$$

We note that (15.3) behaves well with respect to extensions of R of ramification index 1 and coincides with the defect of smoothness of F (at any $R^{\rm nr}$-rational point of F) defined in [BLR, p. 65].

gr-3c **Lemma 15.4.** *Assume that F is generically étale. Then H_r^r is finite over k for every integer $r \geq \delta(F)+2$, where $\delta(F)$ is the defect of smoothness of F (15.3).*

Proof. If F is étale over R (which is the case if $\operatorname{char} k = 0$), then ϱ_r^i in an isomorphism for all r, i (cf. Corollary 8.10) and therefore $H_r^r = F_{\rm s}$ is indeed finite over k. Assume now that $\operatorname{char} k = p > 0$. Using the étale-connected sequence of F and the left exactness of the Greenberg functor, we may assume that $F = F^{\circ}$ has a connected special fiber. Choose an isomorphism $F \simeq \operatorname{Spec}(R[X_1, \dots X_n]/(\Phi_1, \dots, \Phi_n))$ as in [MR, Lemma 6.1, p. 220]. By adapting the proof of [Gre3, Lemma 2, p. 567], it is possible to show that $c = 1$, $d = \delta(F)$ and $M = \delta(F) + 2$ are valid choices in Lemma 15.1. See [BGA2, proof of Lemma 16.11] for the details. Since $M = \delta(F) + 2 \geq d = \delta(F)$, it is then possible to choose $i = r \geq M = \delta(F) + 2$ in Proposition 15.2, which yields the lemma. □

We will now discuss the affine k-scheme $\operatorname{Gr}^R(F) = \varprojlim \operatorname{Gr}_n^R(F)$.

grfin **Lemma 15.5.** *The affine k-group scheme $\operatorname{Gr}^R(F)$ has finitely many points and each of its residue fields is an algebraic extension of k. In particular, $\dim \operatorname{Gr}^R(F) = 0$.*

Proof. Let $c \geq 1, d \geq 0$ and $M \geq 0$ be as in Lemma 15.1 and let $r \geq \max\{M, d\}$ and $t \geq 0$ be integers. Since $r \geq \max\{M/c^t, d/c^t\}$, Proposition 15.2 shows that $H_{rc^t}^{rc^{t+1}}$ is a finite k-subgroup scheme of $\operatorname{Gr}_{rc^t}^R(F)$. Set $H = \varprojlim_t H_{rc^t}^{rc^{t+1}}$. By construction, H is isomorphic to $\operatorname{Gr}^R(F)$. The lemma now follows by applying [BGA, Proposition 3.6] to H. □

grfin2 **Proposition 15.6.** *Assume that k is perfect. Then $\operatorname{Gr}^R(F)_{\rm red}$ is a finite and étale k-group scheme.*

Proof. By (14.1) and Proposition 13.11, we may assume that $k = \overline{k}$. By the proof of Lemma 15.5, $\operatorname{Gr}^R(F)_{\rm red} = H_{\rm red}$ is profinite since it is an inverse limit of finite and constant k-group schemes. Since $|\operatorname{Gr}^R(F)_{\rm red}| = |\operatorname{Gr}^R(F)|$, Lemma 15.5 now shows that $\operatorname{Gr}^R(F)_{\rm red}$ is indeed finite and étale. □

Remarks 15.7. The functor Gr^R does not respect the étale-connected sequence for F, i.e., the k-scheme $\operatorname{Gr}^R(F^{\circ})$ may be disconnected. For example, let $R = W(\mathbb{F}_2)$ and consider the connected finite R-group scheme $F = F^{\circ} = \mu_{2,R}$ of square roots of unity. We have $F^{\circ}(R) = F^{\circ}(K) = \{\pm 1\}$ and $\operatorname{Gr}^R(F)_{\rm red}$ is finite and étale by Proposition 15.6. Further, by Proposition 14.3, $F^{\circ}(R) = \operatorname{Gr}^R(F^{\circ})(k) = \operatorname{Gr}^R(F^{\circ})_{\rm red}(k)$, whence $\operatorname{Gr}^R(F^{\circ})_{\rm red}$, and therefore also $\operatorname{Gr}^R(F^{\circ})$, is disconnected.

comgr

16. Commutative group schemes

Let R be a complete discrete valuation ring. Recall that, if G is an R-group scheme and $n \in \mathbb{N}$, $\operatorname{Gr}_n^R(G)$ denotes $\operatorname{Gr}_n^R(G \times_S S_n)$, where $S_n = \operatorname{Spec} R_n$ is the spectrum of the n-th truncation of R.

Let F be a flat, commutative and separated R-group scheme of finite type and assume that F has a *smooth resolution*, i.e., there exists a sequence of commutative and separated R-group schemes of finite type

sm-res (16.1)
$$0 \to F \xrightarrow{j} G \xrightarrow{q} H \to 0,$$

where G and H are smooth, q is faithfully flat and j is a closed immersion which identifies F with the scheme-theoretic kernel of q. If F is finite over S, then F has a smooth resolution by [MR, Proposition 5.1(i) and its proof, pp. 217-218]. See also [Bég, §2.2, pp. 25-27].

Now recall the change of level morphism $\varrho^{i}_{n,F}\colon \mathrm{Gr}^{R}_{n+i}(F) \to \mathrm{Gr}^{R}_{n}(F)$ (8.3) and the canonical morphism of k-group schemes (11.8)

wj-2 (16.2)
$$\Phi^{i}_{n,F}\colon \mathrm{Gr}^{R}_{i}(\mathbb{V}(\omega^{1}_{F/R})) \to \operatorname{Ker} \varrho^{i}_{n,F},$$

where $1 \le i \le n$.

vker2 **Proposition 16.3.** *Let $n \ge 1$ and $i \ge 1$ be integers. Then*

(i) $\operatorname{Ker} \varrho^{i}_{n,F}$ *is a unipotent k-group scheme of finite type.*
(ii) $\Phi^{i}_{n,F}$ *is a morphism of unipotent k-group schemes of finite type whose kernel and cokernel are unipotent and infinitesimal.*
(iii) $\Phi^{i}_{n,F}$ *is an isomorphism if R is an equal characteristic ring or if R is a ring of unequal characteristics and $n+i \le \bar{e} = v(p)$.*

Proof. Since Gr^{R}_{n} is a left-exact functor, (16.1) induces an exact and commutative diagram of k-group schemes of finite type

grfgh (16.4)
$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Gr}^{R}_{n+i}(F) & \xrightarrow{\mathrm{Gr}^{R}_{n+i}(j)} & \mathrm{Gr}^{R}_{n+i}(G) & \xrightarrow{\mathrm{Gr}^{R}_{n+i}(q)} & \mathrm{Gr}^{R}_{n+i}(H) \\
& \downarrow{\scriptstyle \varrho^{i}_{n,F}} & & \downarrow{\scriptstyle \varrho^{i}_{n,G}} & & \downarrow{\scriptstyle \varrho^{i}_{n,H}} \\
0 \longrightarrow & \mathrm{Gr}^{R}_{n}(F) & \xrightarrow{\mathrm{Gr}^{R}_{n}(j)} & \mathrm{Gr}^{R}_{n}(G) & \xrightarrow{\mathrm{Gr}^{R}_{n}(q)} & \mathrm{Gr}^{R}_{n}(H).
\end{array}$$

The above diagram induces an exact sequence of k-group schemes of finite type

vs (16.5)
$$0 \to \operatorname{Ker} \varrho^{i}_{n,F} \to \operatorname{Ker} \varrho^{i}_{n,G} \to \operatorname{Ker} \varrho^{i}_{n,H}.$$

Since $\operatorname{Ker} \varrho^{i}_{n,G}$ and $\operatorname{Ker} \varrho^{i}_{n,H}$ are unipotent and of finite type by Proposition 11.4, assertion (i) is clear. Now, by [LLR, Proposition 1.1(a), p. 459] and the left exactness of the functor Gr^{R}_{i}, the sequence (16.1) induces an exact sequence of k-group schemes of finite type

vs2 (16.6)
$$0 \to \mathrm{Gr}^{R}_{i}(\mathbb{V}(\omega^{1}_{F/R})) \to \mathrm{Gr}^{R}_{i}(\mathbb{V}(\omega^{1}_{G/R})) \to \mathrm{Gr}^{R}_{i}(\mathbb{V}(\omega^{1}_{H/R})).$$

We now assume that $1 \le i \le n$. Since G and H are smooth, Propositions 11.11 and 11.12 show that $\mathrm{Gr}^{R}_{i}(\mathbb{V}(\omega^{1}_{F/R}))$ is a unipotent k-group scheme. Now the exact and commutative diagram whose top row is (16.6), bottom row is (16.5) and vertical arrows are the morphisms $\Phi^{i}_{n,F}, \Phi^{i}_{n,G}$ and $\Phi^{i}_{n,H}$ induces an exact sequence

$$0 \to \operatorname{Ker} \Phi^{i}_{n,F} \to \operatorname{Ker} \Phi^{i}_{n,G} \to W \to \operatorname{Coker} \Phi^{i}_{n,F} \to 0$$

for some k-subgroup scheme W of $\operatorname{Ker} \Phi^{i}_{n,H}$. Using the above sequence, (ii) and (iii) follow from Propositions 11.11 and 11.12. □

Since the category of commutative k-group schemes of finite type is abelian, we may now define

h1n (16.7) $$\mathcal{H}^1(R_n, F) = \operatorname{Coker} \operatorname{Gr}_n^R(q).$$

Since $\operatorname{Gr}_n^R(H)$ is smooth, $\mathcal{H}^1(R_n, F)$ is smooth as well. Further, (16.7) is independent, up to isomorphism, of the choice of the smooth resolution (16.1) (cf. [Bég, proof of Lemma 4.2.1(b)] and [BGA, pp. 106-108]). Note also that, since $q = \operatorname{Gr}_1^R(q)$ is surjective, we have

h1=0 (16.8) $$\mathcal{H}^1(R_1, F) = \mathcal{H}^1(k, F) = 0.$$

Further, since the canonical morphism $\operatorname{Gr}_n^R(H)(\overline{k}) \to \mathcal{H}^1(R_n, F)(\overline{k})$ is surjective, Lemma 7.3(ii) yields an exact sequence of abelian groups

$$0 \to F(R_n^{\mathrm{nr}}) \to G(R_n^{\mathrm{nr}}) \to H(R_n^{\mathrm{nr}}) \to \mathcal{H}^1(R_n, F)(\overline{k}) \to 0.$$

Consequently, $\mathcal{H}^1(R_n, F)(\overline{k}) = \mathrm{H}^1_{\mathrm{fppf}}(R_n^{\mathrm{nr}}, F)$, which explains our choice of notation in (16.7).

Now, by diagram (16.4), the following diagram is exact and commutative

ggf (16.9) $$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Gr}_{m+j}^R(G)/\operatorname{Gr}_{m+j}^R(F) & \longrightarrow & \operatorname{Gr}_{m+j}^R(H) & \longrightarrow & \mathcal{H}^1(R_{m+j}, F) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\varrho}_{m,G}^{\,j}} & & \downarrow{\scriptstyle \varrho_{m,H}^{j}} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Gr}_{m}^R(G)/\operatorname{Gr}_{m}^R(F) & \longrightarrow & \operatorname{Gr}_{m}^R(H) & \longrightarrow & \mathcal{H}^1(R_{m}, F) & \longrightarrow & 0 \end{array}$$

for all integers $m \geq 1$ and $j \geq 0$, where $\overline{\varrho}_{m,G}^{\,j}$ is induced by $\varrho_{m,G}^{j}$.

Lemma 16.10. *For every $n \in \mathbb{N}$, $\mathcal{H}^1(R_n, F)$ is a smooth, commutative, connected and unipotent k-group scheme.*

Proof. Commutativity is clear and smoothness was observed above. Now set $m = 1$ and $j = n - 1$ in diagram (16.9) and use (16.8) to obtain the following exact sequence of k-group schemes of finite type:

$$0 \to \operatorname{Ker} \overline{\varrho}_{1,G}^{\,n-1} \to \operatorname{Ker} \varrho_{1,H}^{n-1} \to \mathcal{H}^1(R_n, F) \to 0.$$

The lemma now follows from Proposition 11.4. □

The lemma and diagram (16.9) yield a projective system of smooth, commutative, connected and unipotent k-group schemes $(\mathcal{H}^1(R_n, F))$. The projective limit of this system is the commutative, affine, reduced and connected k-group scheme

ucok-i (16.11) $$\mathcal{H}^1(R, F) = \varprojlim \mathcal{H}^1(R_n, F).$$

We now note that, if k' is an algebraic extension of k and R'/R is the corresponding extension of ramification index 1, then Proposition 10.6 yields a canonical isomorphism of k-group schemes of finite type

ucalg (16.12) $$\mathcal{H}^1(R_n, F) \times_{\operatorname{Spec} k} \operatorname{Spec} k' = \mathcal{H}^1(R_n', F \times_S S').$$

Since projective limits commute with base extension, the projective limit of (16.12) is an isomorphism

$$\mathcal{H}^1(R, F) \times_{\operatorname{Spec} k} \operatorname{Spec} k' = \mathcal{H}^1(R', F \times_S S').$$

Theorem 16.13. *Assume that F is generically smooth. Then there exists an integer $i_0 \in \mathbb{N}$ such that, for every integer $n \geq i_0$, the transition morphism $\mathcal{H}^1(R_{n+1}, F) \to \mathcal{H}^1(R_n, F)$ is an isomorphism of k-group schemes.*

Proof. By (16.12) and faithfully flat and quasi-compact descent, we may assume that $k = \overline{k}$. It is shown in [LLR, p. 465] (with $G' = F$, $G'' = H$, $u = q$ and $g'' = h$ in the notation of that paper) that there exists a commutative diagram of flat and commutative R-group schemes of finite type

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{F} & \longrightarrow & \widetilde{G} & \xrightarrow{\widetilde{q}} & \widetilde{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & F & \xrightarrow{j} & G & \xrightarrow{q} & H & \longrightarrow & 0, \end{array}$$

where $\widetilde{q}$ is smooth, faithfully flat and of finite presentation, and the bottom row is the sequence (16.1). For every integer $n \geq 1$, the preceding diagram induces an exact and commutative diagram of k-group schemes of finite type

gggf (16.14)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Gr}_n^R(\widetilde{F}) & \longrightarrow & \mathrm{Gr}_n^R(\widetilde{G}) & \longrightarrow & \mathrm{Gr}_n^R(\widetilde{H}) & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow \mathrm{Gr}_n^R(g) & & \downarrow \mathrm{Gr}_n^R(h) & & & & \\ 0 & \longrightarrow & \mathrm{Gr}_n^R(F) & \longrightarrow & \mathrm{Gr}_n^R(G) & \xrightarrow{\mathrm{Gr}_n^R(q)} & \mathrm{Gr}_n^R(H) & \longrightarrow & \mathcal{H}^1(R_n, F) & \longrightarrow & 0, \end{array}$$

where the top row is exact by Proposition (11.14). By the functoriality of the change of level morphism (8.7), we conclude that there exists an exact and commutative diagram of k-group schemes of finite type

$$\begin{array}{ccccccc} \mathrm{Coker}\,\mathrm{Gr}_{n+1}^R(g) & \longrightarrow & \mathrm{Coker}\,\mathrm{Gr}_{n+1}^R(h) & \longrightarrow & \mathcal{H}^1(R_{n+1}, F) & \longrightarrow & 0 \\ \downarrow \alpha_n & & \downarrow \beta_n & & \downarrow & & \\ \mathrm{Coker}\,\mathrm{Gr}_n^R(g) & \longrightarrow & \mathrm{Coker}\,\mathrm{Gr}_n^R(h) & \longrightarrow & \mathcal{H}^1(R_n, F) & \longrightarrow & 0. \end{array}$$

Now it is shown in [LLR, p. 471] (set $g_i = \alpha_{n+1}$ and $g_i'' = \beta_{n+1}$ in [loc.cit.]) that there exists an integer $i_0 \in \mathbb{N}$ such that the maps α_n and β_n appearing above are isomorphisms of smooth k-group schemes for every integer $n \geq i_0$. The theorem is now clear. □

cor1 **Corollary 16.15.** *Assume that F is generically smooth and let $i_0 \in \mathbb{N}$ be as in the theorem. Then, for every integer $n \geq i_0$, the following holds:*

(i) *The canonical projection $\mathcal{H}^1(R, F) \to \mathcal{H}^1(R_n, F)$ is an isomorphism.*
(ii) *There exists an isomorphism of k-group schemes of finite type $\mathrm{Coker}\,\varrho_{n,F}^1 \simeq \mathbb{G}_{a,k}^r$, where $\varrho_{n,F}^1$ is the change of level morphism and $r = \dim_k \mathrm{Lie}(F_{\mathrm{s}}) - \dim F_{\mathrm{s}}$.*
(iii) $\dim \mathrm{Gr}_n^R(F) = (n - i_0) \dim F_{\mathrm{s}} + \dim \mathrm{Gr}_{i_0}^R(F)$.
(iv) $\dim \mathcal{H}^1(R_n, F) = \dim \mathrm{Gr}_{i_0}^R(F) - i_0 \dim F_{\mathrm{s}}$.

Proof. Assertion (i) is immediate from the theorem. Now, by (16.4), (16.9) and the theorem, the smooth resolution (16.1) induces an exact sequence

$$0 \to \mathrm{Ker}\,\varrho_{n,F}^i \to \mathrm{Ker}\,\varrho_{n,G}^i \to \mathrm{Ker}\,\varrho_{n,H}^i \to \mathrm{Coker}\,\varrho_{n,F}^i \to 0,$$

where $n \geq i_0$ and $i \geq 1$. Consequently

$$\dim \operatorname{Ker} \varrho^1_{n,F} = \dim_k \mathbb{V}\big(\omega^1_{F_{\rm s}/k}\big) = \dim_k \operatorname{Lie}(F_{\rm s})$$

by Proposition 16.3(ii). Further, by Propositions 11.11 and 11.12, $\operatorname{Ker} \varrho^1_{n,G}$ is isomorphic to $\mathbb{G}^d_{a,k}$, where $d = \dim G_{\rm s}$, and similarly with H in place of G. In particular, since $\dim G_{\rm s} = \dim F_{\rm s} + \dim H_{\rm s}$, we conclude that $\operatorname{Coker} \varrho^1_{n,F} \simeq \mathbb{G}^r_{a,k}$ with $r = \dim_k \operatorname{Lie}(F_{\rm s}) - \dim F_{\rm s}$. This completes the proof of (ii). Now, by (ii), there exists an exact sequence

$$0 \longrightarrow \operatorname{Ker} \varrho^1_{n,F} \longrightarrow \operatorname{Gr}^R_{n+1}(F) \overset{\varrho^1_{n,F}}{\longrightarrow} \operatorname{Gr}^R_n(F) \longrightarrow \mathbb{G}^r_{a,k} \longrightarrow 0.$$

Thus, by the definition of r, $\dim \operatorname{Gr}^R_{n+1}(F) = \dim \operatorname{Gr}^R_n(F) + \dim F_{\rm s}$. Assertion (iii) now follows by induction. Assertion (iv) follows from the bottom sequence in (16.14) by combining (iii) and Corollary 11.15(i). □

In connection with the above corollary, Example 7.6 shows that $\dim \operatorname{Gr}^R_n(F)$ can be unbounded as n grows if F is not generically smooth.

Lemma 16.16. *Let n and r be integers such that $1 \leq r < n$. Then*

$$\dim \operatorname{Gr}^R_n(\mathbb{V}(R_r)) = r.$$

Proof. By (3.1) and the description of Greenberg modules in Section 2, $\dim \mathscr{M}^{n-r}_n = r$. Thus, since the perfection functor on k-schemes preserves dimensions [BGA, Remark 5.18(b)], to prove the lemma it suffices to construct a morphism of k-group schemes $\gamma\colon \mathscr{M}^{n-r}_n \to \operatorname{Gr}^R_n(\mathbb{V}(R_r))$ such that $\gamma^{\rm pf}$ is an isomorphism.

Let A be any k-algebra. By Lemma 7.3(i),

$$\operatorname{Gr}^R_n(\mathbb{V}(R_r))(A) = \mathbb{V}(R_r)(\mathscr{R}_n(A)) = \operatorname{Hom}_{R_n\text{-mod}}(R_r, \mathscr{R}_n(A)) = \mathscr{R}_n(A)_{\pi^r_n\text{-tors}}.$$

Further, by (3.11), the inclusion $\overline{\mathscr{M}^{n-r}_n}(A) \subseteq \mathscr{R}_n(A)$ factors through $\mathscr{R}_n(A)_{\pi^r_n\text{-tors}}$. Let $\gamma(A)$ be the composition of the canonical map $\mathscr{M}^{n-r}_n(A) \to \overline{\mathscr{M}^{n-r}_n}(A)$ (3.12) and the inclusion $\overline{\mathscr{M}^{n-r}_n}(A) \subseteq \mathscr{R}_n(A)_{\pi^r_n\text{-tors}}$. The preceding construction is functorial in A and defines the required morphism $\gamma\colon \mathscr{M}^{n-r}_n \to \operatorname{Gr}^R_n(\mathbb{V}(R_r))$. If R is an equal characteristic ring, then γ is, in fact, an isomorphism, which completes the proof in this case. In effect

vtor (16.17) $$\mathscr{R}_n(A)_{\pi^r\text{-tors}} = \mathscr{M}^{n-r}_n(A) = \overline{\mathscr{M}^{n-r}_n}(A)$$

by Remark 3.13(d), (2.1) and the flatness of A over k.

Now let R be a ring of unequal characteristics. Then, by Remark 3.13(c), the equality (16.17) holds if A is perfect. Consequently $\gamma^{\rm pf}$ is an isomorphism by [BGA, Remark 5.18(a)]. □

Proposition 16.18. *Assume that F is finite and generically étale. Then*

$$\dim \mathcal{H}^1(R, F) = \delta(F),$$

where $\delta(F)$ is the defect of smoothness of F (15.3).

Proof. By Corollary 16.15, (i), (iii) and (iv), we have $\dim \mathcal{H}^1(R,F)=\dim \mathrm{Gr}_r^R(F)$ for every integer $r\geq i_0$. On the other hand, by Lemma 15.4, $\varrho_{n,F}^n$ factors through a finite k-subgroup scheme of $\mathrm{Gr}_n^R(F)$ if $n\geq \delta(F)+2$. Thus, by Proposition 16.3(ii), we have $\dim \mathrm{Gr}_{2n}^R(F)=\dim \mathrm{Gr}_n^R(\mathbb{V}(\omega_{F/R}^1))$ if $n\geq r=\max\{i_0,\delta(F)+2\}$. Therefore $\dim \mathcal{H}^1(R,F)=\dim \mathrm{Gr}_n^R(\mathbb{V}(\omega_{F/R}^1))$ if $n\geq r$. Now, by the structure theorem for torsion R-modules, there exists an isomorphism of R-modules $\omega_{F/R}^1\simeq \oplus_{i=1}^t R/(\pi^{n_i})$, where $\sum n_i=\mathrm{length}_R(\omega_{F/R}^1)=\delta(F)$. Thus we are reduced to checking that $\dim \mathrm{Gr}_n^R(\mathbb{V}(R/(\pi^{n_i})))=n_i$. This follows from the previous lemma. □

Appendix

wres

A.1. **Weil restriction.** Let $f\colon S'\to S$ be a morphism of schemes and let X' be an S'-scheme. We will say that *the Weil restriction of X' along f exists* if the contravariant functor $(\mathrm{Sch}/S)\to(\mathrm{Sets}), T\mapsto \mathrm{Hom}_{S'}(T\times_S S',X')$, is representable, i.e., if there exists a pair $(\mathrm{Res}_{S'/S}(X'),q)$, where $\mathrm{Res}_{S'/S}(X')$ is an S-scheme and $q\colon \mathrm{Res}_{S'/S}(X')_{S'}\to X'$ is an S'-morphism of schemes, such that the map

wr (A.1) $$\mathrm{Hom}_S(T,\mathrm{Res}_{S'/S}(X'))\xrightarrow{\sim}\mathrm{Hom}_{S'}(T\times_S S',X'),\quad g\mapsto q\circ g_{S'}$$

is a bijection. The scheme $\mathrm{Res}_{S'/S}(X'))$ is called the *Weil restriction of X' along f*. If $S'=\mathrm{Spec}\, B$ and $S=\mathrm{Spec}\, A$ are affine, we will write $\mathrm{Res}_{B/A}(X')$ for $\mathrm{Res}_{S'/S}(X')$.

It follows from the above definition that $\mathrm{Res}_{S'/S}$ is compatible with fiber products. In particular, if X' is an S'-group scheme such that $\mathrm{Res}_{S'/S}(X')$ exists, then $\mathrm{Res}_{S'/S}(X')$ is an S-group scheme. On the other hand, if $\mathrm{Res}_{S'/S}(X')$ exists and $T\to S$ is a morphism of schemes, then there exists a canonical isomorphism of T-schemes

wrbc (A.2) $$\mathrm{Res}_{S'/S}(X')\times_S T\xrightarrow{\sim}\mathrm{Res}_{S'_T/T}(X'\times_{S'}S'_T).$$

Moreover, if $S''\to S'\to S$ are morphisms of schemes, then there exists a canonical isomorphism of S-schemes

wrcomp (A.3) $$\mathrm{Res}_{S'/S}(\mathrm{Res}_{S''/S'}(X''))\xrightarrow{\sim}\mathrm{Res}_{S''/S}(X'')$$

(when the indicated Weil restrictions exist).

We now discuss existence results. Let $f\colon S'\to S$ be a finite and locally free morphism of schemes. For every $s\in S$, let

gfs (A.4) $$\gamma(f;s)=\#\big(S'\times_S \mathrm{Spec}\,\overline{k(s)}\,\big)$$

be the cardinality of the geometric fiber of s. If S has a unique point s, we will write $\gamma(f)$ for $\gamma(f;s)$.

Remarks A.5. Let $f\colon S'\to S$ be a finite and locally free morphism of schemes.

(a) If k is a field, A is a finite étale k-algebra and $f\colon \mathrm{Spec}\, A\to \mathrm{Spec}\, k$ is the corresponding morphism of schemes, then $\gamma(f)=\dim_k A$.

(b) Let $g\colon T\to S$ be a morphism of schemes and consider the finite and locally free morphism $f\times_S T\colon S'\times_S T\to T$. Let $t\in T$ and set $s=g(t)$. Then $\gamma(f\times_S T;t)=\gamma(f;s)$.

(c) Let $s \in S$ and let $g\colon T' \to S'$ be a universal homeomorphism such that $h = f \circ g\colon T' \to S$ is finite and locally free. Then $\gamma(h;s) = \gamma(f;s)$.

adm **Definition A.6.** Let $f\colon S' \to S$ be a finite and locally free morphism of schemes. An S'-scheme X' is called *admissible relative to* f if, for every point $s \in S$, every collection of $\gamma(f;s)$ points in $X' \times_S \operatorname{Spec} k(s)$ is contained in an affine open subscheme of X', where $\gamma(f;s)$ is the integer (A.4).

If $S' = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$ are affine, we will also say that X' is *admissible relative to* B/A.

ms-adm *Remarks* A.7.

(a) By [EGA, II, Definition 5.3.1 and Corollary 4.5.4], a quasi-projective S'-scheme is admissible relative to an arbitrary finite and locally free morphism $S' \to S$.

(b) If the geometric fibers of $f\colon S' \to S$ are one-point schemes, then $\gamma(f;s) = 1$ for every $s \in S$. Consequently, *every* S'-scheme is admissible relative to f. This is the case, for example, if f is a universal homeomorphism.

(c) If X' is an S'-scheme which is admissible relative to f and $g\colon T \to S$ is an *affine* morphism of schemes, then the $(S' \times_S T)$-scheme $X' \times_{S'} (S' \times_S T) = X' \times_S T$ is admissible relative to $f \times_S T\colon S' \times_S T \to T$.

(d) If X' is an S'-scheme which is admissible relative to $f\colon S' \to S$ and $g\colon T' \to S'$ is a universal homeomorphism such that $h = f \circ g\colon T' \to S$ is finite and locally free, then the T'-scheme $X' \times_{S'} T'$ is admissible relative to h.

We can now strengthen [BLR, §7.6, Theorem 4, p. 194]:

wr-rep **Theorem A.8.** *Let $f\colon S' \to S$ be a finite and locally free morphism of schemes and let X' be an S'-scheme which is admissible relative to f. Then $\operatorname{Res}_{S'/S}(X')$ exists.*

Proof. See [BLR, §7.6, Theorem 4, p. 194] and note that in the last paragraph of that proof the set of points $\{z_j\}$ in $S' \times_S T$ lying over a given point $z \in T$, where $g\colon T \to S$ is an arbitrary S-scheme, has cardinality at most $\gamma(f;s)$, where $s = g(z)$. Thus the corresponding set of points $\{x_j\} \subseteq X'$ considered in [BLR, p. 195, line -14] has cardinality at most $\gamma(f;s)$, whence it is contained in an open affine subscheme of X' by Definition A.6. This is the condition needed in [loc.cit.] to complete that proof. □

wr-uh **Corollary A.9.** *Let $f\colon S' \to S$ be a finite and locally free morphism of schemes which is a universal homeomorphism and let X' be any S'-scheme. Then $\operatorname{Res}_{S'/S}(X')$ exists.*

Proof. This is immediate from the theorem and Remark A.7(b). □

w-lim **Proposition A.10.** *Let k'/k be a finite field extension and let $(X_\lambda)_{\lambda\in\Lambda}$ be a projective system of k'-schemes, where Λ is a directed set containing an element λ_0 such that the transition morphisms $X_\mu \to X_\lambda$ are affine if $\mu \geq \lambda \geq \lambda_0$. Assume that X_{λ_0} is admissible relative to k'/k. Then $\operatorname{Res}_{k'/k}\big(\varprojlim X_\lambda\big)$ and $\varprojlim \operatorname{Res}_{k'/k}(X_\lambda)$ exist and*

$$\operatorname{Res}_{k'/k}\big(\varprojlim X_\lambda\big) = \varprojlim \operatorname{Res}_{k'/k}(X_\lambda).$$

Proof. We may assume that λ_0 is an initial element of Λ. The stated formula will follow from (A.1) once the existence assertion is established. Set $X = \varprojlim X_\lambda$. Since the canonical morphism $X \to X_{\lambda_0}$ is affine, X is also admissible relative to k'/k. Thus, by Theorem A.8, $\mathrm{Res}_{k'/k}(X)$ exists. Similarly X_λ is admissible relative to k'/k for every λ, and $\mathrm{Res}_{k'/k}(X_\lambda)$ exists. It remains only to check that the transition morphisms $\mathrm{Res}_{k'/k}(X_\mu) \to \mathrm{Res}_{k'/k}(X_\lambda)$, $\mu \geq \lambda$, are affine. Let U be an affine open subscheme of X_λ. Then $X_\mu \times_{X_\lambda} U$ is affine and therefore so also is

$$\mathrm{Res}_{k'/k}(X_\mu \times_{X_\lambda} U) = \mathrm{Res}_{k'/k}(X_\mu) \times_{\mathrm{Res}_{k'/k}(X_\lambda)} \mathrm{Res}_{k'/k}(U).$$

Since $\mathrm{Res}_{k'/k}(X_\lambda)$ is covered by affine open subschemes of the form $\mathrm{Res}_{k'/k}(U)$ [BLR, p. 195], the proposition follows. □

Let R be a complete discrete valuation ring and let R'/R be a finite extension of R with maximal ideal $\mathfrak{m}'$, residue field k' and ramification index e. Recall $S = \mathrm{Spec}\, R$ and let $S' = \mathrm{Spec}\, R'$. For every integer $n \geq 1$, set $S'_n = \mathrm{Spec}\, R'_n = \mathrm{Spec}\,(R'/(\mathfrak{m}')^n R')$. Since $\mathfrak{m} = (\mathfrak{m}')^e$, there exists a canonical isomorphism

am-ten (A.11)
$$S'_{ne} = S' \times_S S_n.$$

Now observe that $S' \to S$ is finite and locally free and therefore so also is the induced morphism $f_n \colon S'_{ne} \to S_n$. Further, $\gamma(f_n)$ (A.4) equals $[k' \colon k]_{\mathrm{sep}}$. Thus Z is admissible relative to f_n if, and only if, every set of $[k' \colon k]_{\mathrm{sep}}$ points in $Z \times_{S_n} S_1$ is contained in an open affine subscheme of Z.

tram *Remark* A.12. If R'/R is *totally ramified*, then $k' = k$ and therefore $S'_{ne} \to S_n$ is, in fact, a universal homeomorphism. Consequently, by Corollary A.9, the Weil restriction $\mathrm{Res}_{S'_{ne}/S_n}(Z)$ exists for every S'_{ne}-scheme Z.

adm1 **Lemma A.13.** *Let $n \geq 1$ be an integer and let Z be an S'_{ne}-scheme which is admissible relative to $f_n \colon S'_{ne} \to S_n$. Then the k'-scheme $Z \times_{S'_{ne}} S'_1$ is admissible relative to k'/k.*

Proof. Since $S_1 \to S_n$ is affine and $S'_{ne} \times_{S_n} S_1$ equals S'_e by (A.11), the S'_e-scheme $Z \times_{S'_{ne}} S'_e$ is admissible relative to $f_n \times_{S_n} S_1 \colon S'_e \to S_1$ by Remark A.7(c). Now, since $S'_1 \to S'_e$ is a universal homeomorphism, Remark A.7(d) shows that $(Z \times_{S'_{ne}} S'_e) \times_{S'_e} S'_1 = Z \times_{S'_{ne}} S'_1$ is, indeed, admissible relative to $S'_1 \to S_1$. □

gstr

A.2. **Greenberg's structure theorem.** Let $\mathfrak{R}$ be a finite $W_m(k)$-algebra, where $m \geq 1$ and k is assumed to be perfect and of positive characteristic if $m > 1$.

Consider the following cases and notations:

(i) $\mathfrak{R}$ is a k-algebra, $\mathfrak{I}$ an ideal of $\mathfrak{R}$ such that $\mathfrak{I}\mathfrak{M} = 0$ and $t = \dim_k \mathfrak{I}$, or

(ii) $\mathfrak{R}$ is a finite $W(k)$-algebra of characteristic p^m, where $m > 1$, $\mathfrak{I}$ is a minimal ideal of $\mathfrak{R}$ and t is the unique non-negative integer such that $\overline{\mathscr{I}} \simeq {}^{p^t}\mathbb{O}_k$ (see Proposition 3.17).

Note that $\mathfrak{I}\mathfrak{M} = 0$ in either case. In particular, since $\mathfrak{I} \subset \mathfrak{M}$, we have $\mathfrak{I}^2 = 0$. As in the main text, we will write $\mathfrak{R}' = \mathfrak{R}/\mathfrak{I}$. For every $\mathfrak{R}$-scheme X, consider the quasi-coherent $\mathscr{O}_{X_{\mathrm{s}}}$-module

$$\mathscr{E}_{X_{\mathrm{s}}/k} = \begin{cases} \displaystyle\bigoplus_{i=1}^{t} \Omega^1_{X_{\mathrm{s}}/k} & \text{in case (i)} \\ (F^t_{X_{\mathrm{s}}})^* \Omega^1_{X_{\mathrm{s}}/k} & \text{in case (ii),} \end{cases}$$

where, in case (ii), $F_{X_{\mathrm{s}}}$ denotes the absolute Frobenius endomorphism of X_{s}.

val *Remark* A.14. Let R be a discrete valuation ring with residue field k and let $n > 0$ be an integer. Then (R_n, M_n^{n-1}) is a valid choice for $(\mathfrak{R}, \mathfrak{I})$. If R is an equal characteristic ring or R has unequal characteristics $(0, p)$ and $n \leq \bar{e} = v(p)$, then (R_n, M_n^{n-1}) is of type (i) with $t = 1$. If R has unequal characteristics $(0, p)$ and $n > \bar{e} = v(p)$, then (R_n, M_n^{n-1}) is of type (ii) with $t = m - 1$, where $m = \lceil n/\bar{e} \rceil$ (as noted in Remark 3.19).

Now let the following data be given: a k-scheme Y, an $\mathfrak{R}$-scheme X and a k-morphism $u' \colon Y \to \mathrm{Gr}^{\mathfrak{R}'}(X')$, where $X' = X \times_{\mathfrak{R}} \operatorname{Spec} \mathfrak{R}'$. Note that Y is an X_{s}-scheme via the k-morphism $a \colon Y \to X_{\mathrm{s}}$ which is defined by the commutativity of the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;u'\;} & \mathrm{Gr}^{\mathfrak{R}'}(X') \\ & {\scriptstyle a}\searrow & \big\downarrow {\scriptstyle \varrho_{X'}^{\mathfrak{R}',k}} \\ & & X_{\mathrm{s}}, \end{array}$$

where $\varrho_{X'}^{\mathfrak{R}',k}$ is the change of rings morphism (8.1). Next, consider the Zariski sheaf of abelian groups on Y

$$\mathscr{H}_a = \mathscr{H}\!om_{\mathscr{O}_Y}\big(a^*\Omega^1_{X_{\mathrm{s}}/k}, \bar{\mathscr{I}}(\mathscr{O}_Y)\big).$$

Proposition A.15. *Let $\mathfrak{R}$ be as in* (i) *or* (ii) *above, let X be an $\mathfrak{R}$-scheme and let Y be a $\mathrm{Gr}^{\mathfrak{R}'}(X')$-scheme. Then there exists an isomorphism of Zariski sheaves on Y*

$$\mathscr{H}_a \simeq \mathbb{V}\big(\mathscr{E}_{X_{\mathrm{s}}/k}\big) \times_{X_{\mathrm{s}}} \mathrm{Gr}^{\mathfrak{R}'}(X'),$$

where $\mathrm{Gr}^{\mathfrak{R}'}(X')$ is regarded as an X_{s}-scheme via $\varrho_{X'}^{\mathfrak{R}',k}$.

Proof. In case (ii), the isomorphism of $\mathbb{O}_k$-modules $\bar{\mathscr{I}} \simeq {}^{p^t}\mathbb{O}_k$ of Proposition 3.17 yields an isomorphism of Zariski sheaves $\bar{\mathscr{I}}(\mathscr{O}_U) \simeq {}^{p^t}\mathscr{O}_U$ for every open subset U of Y. Thus, by [BGA, (4.12) and Caveat 4.14], we have

$$\begin{aligned} \mathscr{H}_a(U) &\simeq \operatorname{Hom}_{\mathscr{O}_U}\big((a|_U)^*\Omega^1_{X_{\mathrm{s}}/k}, {}^{p^t}\mathscr{O}_U\big) \simeq \operatorname{Hom}_{\mathscr{O}_U}\Big(\big(F_U^t\big)^*(a|_U)^*\Omega^1_{X_{\mathrm{s}}/k}, \mathscr{O}_U\Big) \\ &\simeq \operatorname{Hom}_{\mathscr{O}_U}\Big((a|_U)^*\big(F_{X_{\mathrm{s}}}^t\big)^*\,\Omega^1_{X_{\mathrm{s}}/k}, \mathscr{O}_U\Big) = \operatorname{Hom}_{\mathscr{O}_{X_{\mathrm{s}}}}\big(\mathscr{E}_{X_{\mathrm{s}}/k}, (a|_U)_*\mathscr{O}_U\big) \\ &\simeq \operatorname{Hom}_{X_{\mathrm{s}}}\big(U, \mathbb{V}(\mathscr{E}_{X_{\mathrm{s}}/k})\big) \simeq \operatorname{Hom}_{\mathrm{Gr}^{\mathfrak{R}'}(X')}\big(U, \mathbb{V}(\mathscr{E}_{X_{\mathrm{s}}/k}) \times_{X_{\mathrm{s}}} \mathrm{Gr}^{\mathfrak{R}'}(X')\big). \end{aligned}$$

Similar calculations establish the proposition in case (i). □

Now let $\mathscr{P}(u')$ be the following Zariski sheaf of sets on Y: for every open subset $U \subseteq Y$, let $\mathscr{P}(u')(U)$ be the set of k-morphisms $v \colon U \to \mathrm{Gr}^{\mathfrak{R}}(X)$ (if any exist) such that $v \circ \varrho_X^{\mathfrak{R},\mathfrak{R}'} \colon U \to \mathrm{Gr}^{\mathfrak{R}'}(X')$ equals $u'|_U$. Then, by (6.12) and Proposition 8.6, $\mathscr{P}(u')(U)$ is in bijection with the set of $\mathfrak{R}$-morphisms $f_U \colon h^{\mathfrak{R}}(U) \to X$ (if any exist) such that the following diagram commutes

$$\begin{array}{ccc} h^{\mathfrak{R}}(U) & \overset{f_U}{\dashrightarrow} & X \\ {\scriptstyle \delta_U^{\mathfrak{R},\mathfrak{R}'}}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}_X} \\ h^{\mathfrak{R}'}(U) & \xrightarrow{\varphi^{\mathfrak{R}'}_{Y,X'}(u')|_{h^{\mathfrak{R}}(U)}} & X'. \end{array}$$

Clearly, the existence of v (or f_U) is equivalent to the non-emptyness of $\mathscr{P}(u')(Y)$.

Lemma A.16. *For every (respectively, every smooth) $\mathfrak{R}$-scheme X, $\mathscr{P}(u')$ is a pseudo-torsor (respectively, torsor) under $\mathscr{H}_a = \mathscr{H}om_{\mathscr{O}_Y}\big(a^*\Omega^1_{X_s/k}, \bar{\mathscr{I}}(\mathscr{O}_Y)\big)$ on the Zariski site of Y.*

Proof. By [SGA1, III, Proposition 5.1] with $S = \operatorname{Spec}\mathfrak{R}$ and $g_0 = \mathrm{pr}_X \circ \varphi^{\mathfrak{R}'}_{Y,X'}(u')$, $\mathscr{P}(u')$ is pseudo-torsor under the sheaf $\mathscr{H}om_{\mathscr{R}'(\mathscr{O}_Y)}\big(g_0^*\Omega^1_{X/\mathfrak{R}}, \bar{\mathscr{I}}(\mathscr{O}_Y)\big)$, which is in fact isomorphic to $\mathscr{H}_a$. Note that, since $\delta_U^{\mathfrak{R},\mathfrak{R}'}$ is a nilpotent immersion, $\mathscr{P}(u')$ has non-empty fibers if X is smooth over $\mathfrak{R}$ by the lifting property in the definition of smoothness. □

The preceding result yields the existence of a bijection of fiber products of sets

$$\mathbb{V}(\mathscr{E}_{X_s/k})(Y) \times_{\{a\}} \mathrm{Gr}^{\mathfrak{R}}(X)(Y) \xrightarrow{\sim} \mathrm{Gr}^{\mathfrak{R}}(X)(Y) \times_{\{u'\}} \mathrm{Gr}^{\mathfrak{R}}(X)(Y).$$

When Y and u' vary, the latter bijections induce an isomorphism of k-schemes

$$\mathbb{V}(\mathscr{E}_{X_s/k}) \times_{X_s} \mathrm{Gr}^{\mathfrak{R}}(X) \xrightarrow{\sim} \mathrm{Gr}^{\mathfrak{R}}(X) \times_{\mathrm{Gr}^{\mathfrak{R}'}(X')} \mathrm{Gr}^{\mathfrak{R}}(X).$$

Note that, by Propositions 8.11 and 8.12 and Corollary 9.4, $\varrho_X^{\mathfrak{R},\mathfrak{R}'}$ is a quasi-compact and surjective morphism of smooth k-schemes and therefore faithfully flat and locally of finite presentation. Consequently, the following holds

Theorem A.17. *Let X be an arbitrary (respectively, smooth) $\mathfrak{R}$-scheme. Then the $\mathrm{Gr}^{\mathfrak{R}'}(X')$-scheme $\mathrm{Gr}^{\mathfrak{R}}(X)$ with structural morphism $\varrho_X^{\mathfrak{R},\mathfrak{R}'}$ is a pseudo-torsor (respectively, torsor) under $\mathbb{V}(\mathscr{E}_{X_s/k}) \times_{X_s} \mathrm{Gr}^{\mathfrak{R}'}(X')$ in the category of fppf sheaves of sets on $(\mathrm{Sch}/\mathrm{Gr}^{\mathfrak{R}'}(X'))$.*

The following corollary is now immediate from Remark A.14.

k2 **Corollary A.18.** *Let R be a discrete valuation ring and let X be a smooth R_n-scheme. Then the $\mathrm{Gr}^R_{n-1}(X)$-scheme $\mathrm{Gr}^R_n(X)$ is an fppf torsor under*

(i) $\mathbb{V}\Big(\Omega^1_{X_s/k}\Big) \times_{X_s} \mathrm{Gr}^R_{n-1}(X)$ *if R is an equal characteristic ring, or*

(ii) $\mathbb{V}\Big(\big(F^{m-1}_{X_s}\big)^*\Omega^1_{X_s/k}\Big) \times_{X_s} \mathrm{Gr}^R_{n-1}(X)$ *if R is a ring of unequal characteristics, where $m = \lceil n/\bar{e}\rceil$*

last

A.3. **A flatness result.**

flat3 **Proposition A.19.** *Let k be a field and let $q\colon G \to H$ be a morphism of smooth and commutative k-group schemes. Assume that*

(i) $q\big(\bar{k}\big)\colon G\big(\bar{k}\big) \to H\big(\bar{k}\big)$ *is surjective, and*

(ii) $\pi_0(G)\big(\bar{k}\big)$ *is a finitely generated abelian group.*

Then q is flat.

Proof. Since G^0 and H^0 are both of finite type, q^0 is a morphism of smooth and connected k-group schemes of finite type. Thus, by [SGA3$_{\text{new}}$, VI$_{\text{B}}$, Proposition 3.11 and its proof], it suffices to check that $q^0(\overline{k})\colon G^0(\overline{k}) \to H^0(\overline{k})$ is surjective. Hypothesis (ii) and the snake lemma applied to the exact and commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^0(\overline{k}) & \longrightarrow & G(\overline{k}) & \longrightarrow & \pi_0(G)(\overline{k}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q^0(\overline{k})} & & \downarrow{\scriptstyle q(\overline{k})} & & \downarrow{\scriptstyle \pi_0(q)(\overline{k})} & & \\ 0 & \longrightarrow & H^0(\overline{k}) & \longrightarrow & H(\overline{k}) & \longrightarrow & \pi_0(H)(\overline{k}) & \longrightarrow & 0 \end{array}$$

show that $C = \operatorname{Coker} q^0(\overline{k})$ is finitely generated. Since $H^0(\overline{k})$ is n-divisible for all n prime to the characteristic of k, C is also n-divisible for all such n. Thus C is trivial. □

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