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Research Article

Nguyen Dang Minh*, Khanh To Duc, Nguyen Huy Tuan and Dang Duc Trong

A two-dimensional backward heat problem with statistical discrete data

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Abstract: We focus on the nonhomogeneous backward heat problem of finding the initial temperature $\theta = \theta(x, y) = u(x, y, 0)$ such that

$$\begin{cases} u_t - a(t)(u_{xx} + u_{yy}) = f(x, y, t), & (x, y, t) \in \Omega \times (0, T), \\ u(x, y, t) = 0, & (x, y) \in \partial\Omega \times (0, T), \\ u(x, y, T) = h(x, y), & (x, y) \in \overline{\Omega}, \end{cases}$$

where $\Omega = (0, \pi) \times (0, \pi)$. In the problem, the source $f = f(x, y, t)$ and the final data $h = h(x, y)$ are determined through random noise data $g_{ij}(t)$ and d_{ij} satisfying the regression models

$$g_{ij}(t) = f(X_i, Y_j, t) + \vartheta\xi_{ij}(t),$$
$$d_{ij} = h(X_i, Y_j) + \sigma_{ij}\varepsilon_{ij},$$

where (X_i, Y_j) are grid points of Ω. The problem is severely ill-posed. To regularize the instable solution of the problem, we use the trigonometric least squares method in nonparametric regression associated with the projection method. In addition, convergence rate is also investigated numerically.

Keywords: Backward heat problems, nonhomogeneous heat equation, ill-posed problems, nonparametric regression, statistical inverse problems

MSC 2010: 35K05, 47A52, 62G08

1 Introduction

The backward heat problem is a crucial issue in various physics and industrial applications as heat conduction theory [3], material science [21], hydrology [2, 18], groundwater contamination [23], digital remove blurred noiseless image [6]. The main task of the backward problem is of finding the initial temperature from the information of final temperature. As known, the problem is ill-posed (see [13] or Section 3) and, as classified by Cavalier [7], the ill-posedness is severe.

In the present paper, we consider the nonhomogeneous backward heat problem corresponding to the two-spatial-dimensional case. It is worth noting that the idea of this paper can be applied to the higher-dimensional problem. Let $\Omega = (0, \pi) \times (0, \pi)$, $T > 0$ and let $a : (0, T) \to \mathbb{R}$ be a Lebesgue measurable function satisfying the uniform ellipticity condition $0 < a_1 \le a(t) \le a_2 < \infty$, where a_1, a_2 are positive constants. We find a function $\theta = \theta(x, y) := u(x, y, 0)$ such that

$$u_t - a(t)(u_{xx} + u_{yy}) = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T), \tag{1}$$

***Corresponding author: Nguyen Dang Minh:** Faculty of Mathematics and Computer Science, University of Science, Vietnam National University, 227 Nguyen Van Cu, Dist. 5, Ho Chi Minh City, Vietnam, e-mail: ndminhsv1986@gmail.com
Khanh To Duc: Department Statistical Sciences, University of Padova, Via C. Battisti 241, 35121 Padova, Italy, e-mail: toduc@stat.unipd.it
Nguyen Huy Tuan, Dang Duc Trong: Faculty of Mathematics and Computer Science, University of Science, Vietnam National University, 227 Nguyen Van Cu, Dist. 5, Ho Chi Minh City, Vietnam, e-mail: thnguyen2683@gmail.com, ddtrong@hcmus.edu.vn

subject to the Dirichlet boundary condition

$$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0, \tag{2}$$

and the final condition

$$u(x, y, T) = h(x, y), \quad (x, y) \in \overline{\Omega}. \tag{3}$$

In reality, the exact values of the source f and the final data h are not available. We only have contaminated data $\tilde{f}, \tilde{h}$ that affect construction of regularization method. In most of papers, the data $\tilde{f}, \tilde{h}$ are given on the whole space domain and they are used to construct an approximation for θ. The literature for the this case of data is traditional and pretty huge. Nowadays, there are many good regularization approaches available, among them are the Tikhonov method [8, 24], quasi-boundary value method [9, 25, 26], quasi-reversibility method [19, 22], mollification [11], truncated expansion [16, 17] and the general filter regularization method [20].

In the present paper, we will consider the data from a different point of view in which the source f and the final temperature h will be measured at a discrete set of points and contain errors. These errors may be generated from controllable sources or uncontrollable sources. In the first case, the error is often deterministic and there are many papers concerned with the problem (see, e.g., [12] and references therein). If the errors are generated from uncontrollable sources as wind, rain, humidity, etc., then the model is random. On first glance, such small errors will not really make sense. Statistics handles the influence of random errors and these errors should be important enough. However, the accumulation of the small errors in the data of an ill-posed problem can make the noise of the solution to be large and, hence, cannot be ignored. This effect is considered in the theory of statistical inverse problems [1, Section 2.1.5, p. 48]. In this paper, we describe the relationship between observed data and the sources f and h by means of nonparametric regression models. Let $g_{ij}(t)$ and d_{ij} be the observed data of f and h, and let $(X_i, Y_j) = (\pi(2i-1)/2n, \pi(2j-1)/2m)$ be grid points in Ω, with $i = 1, \dots, n$, and $j = 1, \dots, m$. We consider two models

$$g_{ij}(t) = f(X_i, Y_j, t) + \vartheta \xi_{ij}(t), \tag{4}$$

$$d_{ij} = h(X_i, Y_j) + \sigma_{ij}\varepsilon_{ij}, \quad i = 1, \dots, n, \ j = 1, \dots, m, \tag{5}$$

where $\xi_{ij}(t)$ are Brownian motions, $\varepsilon_{ij} \sim \mathcal{N}(0, 1)$ and σ_{ij} are bounded by a positive constant $V_{\max}$, i.e., $0 \le \sigma_{ij} < V_{\max}$ for all i, j. The random variables $\xi_{ij}(t), \varepsilon_{ij}$ are mutually independent. Note that, in the above models, the stochastic processes $g_{ij}(t)$ and the random variables d_{ij} are observable whereas $\vartheta\xi_{ij}(t)$ and $\sigma_{ij}\varepsilon_{ij}$ are unknown. From the observations $g_{ij}(t)$ and d_{ij}, we can use the nonparametric regression method to reconstruct the final temperature h, the source f which need to estimate the initial temperature θ.

Recently, the number of articles on the statistical inverse problem and the backward problem with random data has increased significantly. In our knowledge, we can list here some related papers. Cavalier in [7] gave some theoretical examples about inverse problems with random noise. Mair and Ruymgaart [14] considered theoretical formulas for statistical inverse estimation in Hilbert scales and applied the method for some examples. Our paper is inspired from the paper by Bissantz and Holzmann [4] in which the authors considered a one-dimensional homogeneous backward problem. The very last papers are dealt with i.i.d. random noises. In the present paper, we consider the nonhomogeneous backward problem with general non-i.i.d. noises and random sources. In our opinion, it is a positive point of our paper.

To deal with the problem, we propose a “hybrid” approach in sense that it is a combination of the nonparametric least squares (NLS) method in Statistics (see, e.g., [27, p. 57]) and the projection method in the theory of inverse problem (see, e.g., [13, p. 66]). In particular, using the NLS method, the final temperature h and the source f can be approximated uniquely from the observed data d_{ij} and $g_{ij}(t)$, respectively. Then the projection method can be applied to construct estimators which stably recover the Fourier coefficients of the unknown function θ. The proposed approach seems to be a generalization of the one in [4] to the multi-dimensional and nonhomogeneous problem.

After the estimation, evaluation of the bias is an important procedure. In [4], a discretization bias of the estimators of the one-dimensional Fourier coefficients on L^2 space is stated as an assumption and no method is available for evaluating the bias in the Sobolev classes. To fill this gap in the two-dimensional case, we have to find a representation of the discretization bias by high-frequency Fourier coefficients of h, f.

The rest of the paper is divided into four parts. In Section 2, we introduce the discretization form of Fourier coefficients. Section 3 is devoted to the ill-posedness of the problem. In Section 4, we construct estimator $\hat{\theta} = \hat{\theta}(x, y)$ for the initial temperature. We also give an upper bound for the error of estimation. Finally, we present some numerical results in Section 5.

Before going to the main parts of the paper, we introduce some notations. We denote

$$L^2(\Omega) = \Big\{ g : \Omega \to \mathbb{R} : g \text{ is Lebesgue measurable and } \int_\Omega g^2(x, y)\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\},$$

with the inner product

$$\langle g_1, g_2 \rangle = \int_\Omega g_1(x, y) g_2(x, y)\,\mathrm{d}x\ \mathrm{d}y,$$

and the norm

$$\|g\| = \sqrt{\int_\Omega g^2(x, y)\ \mathrm{d}x\ \mathrm{d}y}.$$

Here, we recall that $\Omega = (0, \pi)\times(0, \pi)$. For $p, q \in \mathbb{Z}^+$, we put $\phi_p(x) = \sqrt{2/\pi} \sin px$ and $\phi_{p,q}(x, y) = \phi_p(x)\phi_q(y)$. As known, the system $\{\phi_{p,q}\}$ is completely orthonormal in $L^2(\Omega)$. For every natural numbers ℓ, k satisfying $1 \le \ell \le n$, $1 \le k \le m$, we put

$$\mathcal{V}_{\ell,k} := \operatorname{span}\{\phi_{p,q} : p = 1, \dots, \ell,\ q = 1, \dots, k\}.$$

This set is an $\ell \times k$-dimensional subspace of $L^2(\Omega)$ and $\overline{\bigcup_{\ell,k\in\mathbb{N}} \mathcal{V}_{\ell,k}} = L^2(\Omega)$. We also denote by

$$\mathcal{Q}_{\ell,k} : L^2(\Omega) \to \mathcal{V}_{\ell,k}$$

the orthogonal projection operator on $\mathcal{V}_{\ell,k}$.

2 Discretization form of Fourier coefficients

In this section, we will construct the discretization form of the Fourier coefficients of the solution u of problem (1)–(2). Since the system $(\phi_{p,q})$ is an orthonormal basis of $L^2(\Omega)$, the solution u has the expansion

$$u(x, y, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} u_{p,q}(t)\phi_{p,q}(x, y),$$

where $u_{p,q}(t) = \langle u(\cdot, \cdot, t), \phi_{p,q}\rangle$. We also denote

$$\theta_{p,q} = \langle \theta, \phi_{p,q}\rangle, \quad f_{p,q}(t) = \langle f(\cdot, \cdot, t), \phi_{p,q}\rangle, \quad A(t) = \int_0^t a(\tau)\,\mathrm{d}\tau, \quad \lambda_{p,q}(t) = e^{-A(t)(p^2+q^2)}.$$

Substituting the expansion of the function $u(x, y, t)$ into (1) and solving the differential equation thus obtained, we have

$$u_{p,q}(t) = \Big(\theta_{p,q} + \int_0^t \lambda_{p,q}^{-1}(\tau) f_{p,q}(\tau)\,\mathrm{d}\tau\Big)\lambda_{p,q}(t).$$

Hence,

$$u(x, y, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Big(\theta_{p,q} + \int_0^t \lambda_{p,q}^{-1}(\tau) f_{p,q}(\tau)\,\mathrm{d}\tau\Big)\lambda_{p,q}(t)\phi_{p,q}(x, y). \tag{6}$$

Noting that

$$\theta(x, y) = u(x, y, 0) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \theta_{p,q}\phi_{p,q}(x, y),$$

we can obtain the expansion

$$h(x,y)=u(x,y,T)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\Big(\theta_{p,q}+\int_0^T \lambda_{p,q}^{-1}(\tau)f_{p,q}(\tau)\,d\tau\Big)\lambda_{p,q}(T)\phi_{p,q}(x,y).$$

It follows that

$$h_{p,q}=\Big(\theta_{p,q}+\int_0^T \lambda_{p,q}^{-1}(\tau)f_{p,q}(\tau)\,d\tau\Big)\lambda_{p,q}(T). \tag{7}$$

To establish a discretization formula for θ, we will use the least squares estimators of the final temperature functions h and of the source function f. From the Riemann sum, we claim that

$$h_{p,q}\approx\frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}h(X_i,Y_j)\phi_{p,q}(X_i,Y_j).$$

As mentioned in [4], the discretization bias

$$\gamma_{n,m,p,q}:=\frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}h(X_i,Y_j)\phi_{p,q}(X_i,Y_j)-h_{p,q} \tag{8}$$

is not easy to handle. In [4], for brevity of the presentation, the authors only assumed that the one-dimensional bias is of order $\mathcal{O}(n^{-1})$. In the present paper, we will give an explicitly estimate for the two-dimensional bias. In fact, the formulas for the discretization bias will be derived from [10, Lemma 3.5] that is:

Lemma 2.1. *Put*

$$\delta_{p,q,r,s}=\frac{1}{n}\sum_{i=1}^{n}\phi_p(X_i)\phi_r(X_i)\frac{1}{m}\sum_{j=1}^{m}\phi_q(Y_j)\phi_s(Y_j).$$

For $p=1,\dots,n-1$ and $q=1,\dots,m-1$, with $X_i=\pi(2i-1)/2n$, $Y_j=\pi(2j-1)/2m$, we have

$$\delta_{p,q,r,s}=\begin{cases}\pi^{-2}, & (r,s)\pm(p,q)=(2kn,2lm),\\ -\pi^{-2}, & (r,s)\pm(-p,q)=(2kn,2lm),\\ 0, & \text{otherwise.}\end{cases}$$

If $r=1,\dots,n-1$ and $s=1,\dots,m-1$, we obtain

$$\delta_{p,q,r,s}=\begin{cases}\pi^{-2}, & r=p \text{ and } s=q,\\ 0, & r\neq p \text{ or } s\neq q.\end{cases}$$

From the latter lemma, we can represent the discretization bias $\gamma_{n,m,p,q}$ by high-frequency Fourier coefficients of the function h. Precisely, we have:

Lemma 2.2. *Assume that $h\in C^1(\overline{\Omega})$. Then, for $p=1,\dots,n-1$, $q=1,\dots,m-1$,*

$$\gamma_{n,m,p,q}=P_{n,p,q}+Q_{m,p,q}+R_{n,m,p,q}, \tag{9}$$

with

$$P_{n,p,q}=\sum_{k=1}^{\infty}(-1)^k h_{2kn\pm p,q},\qquad Q_{m,p,q}=\sum_{l=1}^{\infty}(-1)^l h_{p,2lm\pm q},$$

$$R_{n,m,p,q}=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{k+l}\big(h_{2kn\pm p,2lm-q}+h_{2kn\pm p,2lm+q}\big).$$

Here, for any sequences $(a_{p,q})$, $(b_{p,q})$, we denote

$$\sum_{k=1}^{\infty}a_{2kn\pm p,q}:=\sum_{k=1}^{\infty}a_{2kn+p,q}+\sum_{k=1}^{\infty}a_{2kn-p,q},$$

$$\sum_{l=1}^{\infty}b_{p,2lm\pm q}:=\sum_{l=1}^{\infty}b_{p,2lm+q}+\sum_{l=1}^{\infty}b_{p,2lm-q}.$$

Proof. We have the transform

$$\frac{1}{m}\sum_{j=1}^{m} h(X_i, Y_j)\phi_q(Y_j) = \frac{1}{m}\sum_{j=1}^{m}\bigg(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} h_{r,s}\phi_r(X_i)\phi_s(Y_j)\bigg)\phi_q(Y_j) = \pi^{-1}\sum_{r=1}^{\infty} h_{r,q}\phi_r(X_i) + S_q,$$

where

$$S_q = \pi^{-1}\sum_{r=1}^{\infty}\phi_r(X_i)\sum_{l=1}^{\infty}(-1)^l h_{r,2lm\pm q}.$$

It follows that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\bigg(\frac{1}{m}\sum_{j=1}^{m} h(X_i, Y_j)\phi_q(Y_j)\bigg)\phi_p(X_i) &= \frac{1}{n}\sum_{i=1}^{n}\bigg(\pi^{-1}\sum_{r=1}^{\infty} h_{r,q}\phi_r(X_i)\bigg)\phi_p(X_i) + \frac{1}{n}\sum_{i=1}^{n} S_q\phi_p(X_i)\\ &= \pi^{-2}(h_{p,q} + P_{n,p,q} + Q_{m,p,q} + R_{n,m,p,q}).\end{aligned}$$

So equality (9) holds. □

Now, we consider the discretization bias of Fourier coefficient $f_{p,q}(t)$ of the function $f(x, y, t)$ from the data-set. For the readers convenience, we recall that

$$f_{p,q}(t) = \langle f(\cdot,\cdot,t), \phi_{p,q}\rangle, \quad f(x, y, t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} f_{p,q}(t)\phi_{p,q}(x, y).$$

As in Lemma 2.2, we can get similarly:

Lemma 2.3. *Assume that $f \in C([0, T]; C^1(\overline{\Omega}))$, $p = 1, \dots, n-1$ and $q = 1, \dots, m-1$. Put*

$$\eta_{n,m,p,q}(t) = \frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i, Y_j, t)\phi_{p,q}(X_i, Y_j) - f_{p,q}(t). \tag{10}$$

Then

$$\eta_{n,m,p,q}(t) = P'_{n,p,q}(t) + Q'_{m,p,q}(t) + R'_{n,m,p,q}(t), \tag{11}$$

with

$$\begin{aligned}P'_{n,p,q}(t) &= \sum_{k=1}^{\infty}(-1)^k f_{2kn\pm p,q}(t), \quad Q'_{m,p,q}(t) = \sum_{l=1}^{\infty}(-1)^l f_{p,2lm\pm q}(t),\\ R'_{n,m,p,q}(t) &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{l+k}(f_{2kn\pm p,2lm+q}(t) + f_{2kn\pm p,2lm-q}(t)).\end{aligned}$$

Combining equalities (7), (8) and (10), we can obtain a data-explicit form for $\theta(x, y)$:

Theorem 2.4. *Let $M, N \in \mathbb{N}$ such that $0 < N \le n$, $0 < M \le m$. Assume that the functions h, f fulfill Lemma 2.2 and Lemma 2.3. If u is as in (6), we have*

$$\begin{aligned}\theta(x, y) = \sum_{p=1}^{N}\sum_{q=1}^{M}\bigg[&\frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\bigg(h(X_i, Y_j)\lambda_{p,q}^{-1}(T) - \int_0^T \lambda_{p,q}^{-1}(\tau)f(X_i, Y_j, \tau)\,\mathrm{d}\tau\bigg)\phi_{p,q}(X_i, Y_j)\\ &- \bigg(\gamma_{n,m,p,q}\lambda_{p,q}^{-1}(T) - \int_0^T \lambda_{p,q}^{-1}(\tau)\eta_{n,m,p,q}(\tau)\,\mathrm{d}\tau\bigg)\bigg]\phi_{p,q}(x, y) + (\theta - \mathcal{Q}_{N,M}\theta)(x, y),\end{aligned}$$

where $\gamma_{n,m,p,q}$, $\eta_{n,m,p,q}$ are defined as in Lemma 2.2 and Lemma 2.3.

3 The ill-posedness of the problem

From the theorem, we can consider the ill-posedness of our problem. We investigate a concrete model of data and prove the instability of the solution in the case of random noise data. Suppose that $h(x, y) = f(x, y, t) \equiv 0$ and $a(t) = 1$, $u(x, y, T) = 0$. The unique solution of (1)–(2) is $u(x, y, t) \equiv 0$. Let the random noise data be

$$g_{ij}(t) = 0 + \vartheta\xi_{ij}(t), \quad d_{ij} = 0 + \varepsilon_{ij}, \quad \varepsilon_{ij} \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, n^{-1}m^{-1})$$

for $i = 1, \dots, n$, $j = 1, \dots, m$. We will construct the solution of (1)-(2) with respect to the random data. Using the idea of the nonparametric regression method (see the next section), we put

$$\overline{h}^{nm}(x,y) = \sum_{p=1}^{n-1}\sum_{q=1}^{m-1} \overline{h}^{mn}_{p,q}\phi_{p,q}(x,y), \quad \overline{f}^{nm}(x,y,t) = \sum_{p=1}^{n-1}\sum_{q=1}^{m-1} \overline{f}^{mn}_{p,q}(t)\phi_{p,q}(x,y),$$

where

$$\overline{h}^{nm}_{p,q} = \frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \varepsilon_{ij}\phi_{p,q}(X_i,Y_j), \quad \overline{f}^{nm}_{p,q}(t) = \frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \xi_{ij}(t)\phi_{p,q}(X_i,Y_j).$$

The definition implies

$$\overline{\gamma}_{n,m,p,q} := \frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \varepsilon_{ij}\phi_{p,q}(X_i,Y_j) - \overline{h}^{nm}_{p,q} = 0,$$

$$\overline{\eta}_{n,m,p,q}(t) := \frac{\pi^2\vartheta}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \xi_{ij}(t)\phi_{p,q}(X_i,Y_j) - \overline{f}^{nm}_{p,q}(t) = 0.$$

By the orthogonal property stated in Lemma 2.1, we can verify directly that

$$\overline{h}_{nm}(X_i,Y_j) = d_{ij}, \quad \overline{f}_{nm}(X_i,Y_j,t) = g_{ij}(t).$$

Let $\overline{u} = \overline{u}(x,y,t)$ be the solution of the system

$$\begin{cases} \overline{u}_t - (\overline{u}_{xx} + \overline{u}_{yy}) = \overline{f}^{nm}(x,y,t), & (x,y,t) \in \Omega \times (0,T), \\ \overline{u}(x,y,T) = \overline{h}^{nm}(x,y), & (x,y) \in \overline{\Omega}, \end{cases}$$

subject to the Dirichlet condition

$$\overline{u}(0,y,t) = \overline{u}(\pi,y,t) = \overline{u}(x,0,t) = \overline{u}(x,\pi,t) = 0.$$

We can remark that $\overline{u}(\cdot,\cdot,t)$ is a trigonometric polynomial with order $< n$ (with respect to the variable x) and order $< m$ (with respect to the variable y). Putting $\overline{\theta}^{nm}(x,y) = \overline{u}(x,y,0)$, we get in view of the remark that $\overline{\theta}^{nm}_{p,q} := \langle \overline{\theta}^{nm}, \phi_{p,q}\rangle$ for $p \geq n$ or $q \geq m$. Applying Theorem 2.4 with $N = n-1$, $M = m-1$, we obtain

$$\overline{\theta}^{nm}(x,y) = \sum_{p=1}^{n-1}\sum_{q=1}^{m-1}\bigg(\overline{h}^{nm}_{p,q} - \int_0^T \lambda^{-1}_{p,q}(\tau)\overline{f}^{nm}_{p,q}(\tau)\,d\tau\bigg)\lambda^{-1}_{p,q}(T)\phi_{p,q}(x,y),$$

thus

$$\|\overline{\theta}^{nm}\|^2 \geq \bigg(\overline{h}^{nm}_{n-1,m-1} - \int_0^T \lambda^{-1}_{n-1,m-1}(\tau)\overline{f}^{nm}_{n-1,m-1}(\tau)\,d\tau\bigg)^2 \lambda^{-2}_{n-1,m-1}(T).$$

Assuming that the random quantities ε_{ij} and $\xi_{ij}(t)$ are mutually independent, we can obtain by direct computation that $\lim_{n,m\to\infty}\mathbb{E}\|\overline{f}^{n,m}(\cdot,t)\|^2 = 0$ for all $t \in [0,T]$. Moreover, by the Parseval equality, we have

$$\|\overline{h}^{nm}\|^2 = \sum_{p=1}^{n-1}\sum_{q=1}^{m-1}(\overline{h}^{mn}_{p,q})^2 = \sum_{p=1}^{n-1}\sum_{q=1}^{m-1}\frac{\pi^4}{n^2m^2}\bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_{ij}\phi_{p,q}(X_i,Y_j)\bigg)^2.$$

Using Lemma 2.1, we obtain

$$\mathbb{E}\|\overline{h}^{nm}\|^2 = \sum_{p=1}^{n-1}\sum_{q=1}^{m-1}\frac{\pi^2}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{E}\varepsilon_{ij}^2 = \frac{(n-1)(m-1)}{n^2m^2}.$$

Thus

$$\lim_{n,m\to\infty}\mathbb{E}\|\overline{h}^{nm}\|^2 = 0.$$

On the other hand, we claim that $\mathbb{E}\|\overline{\theta}_{nm}\|^2 \to \infty$ as $n, m \to \infty$. In fact, we have

$$\mathbb{E}\|\overline{\theta}_{nm}\|^2 \geq \bigg[\mathbb{E}(\overline{h}^{nm}_{n,m})^2 + \mathbb{E}\bigg(\int_0^T e^{-\tau(n^2+m^2)}\overline{f}^{nm}_{p,q}(\tau)\,d\tau\bigg)^2\bigg]e^{2T(n^2+m^2)} \geq \frac{\pi^2 e^{2T(n^2+m^2)}}{n^2m^2}$$

and $\mathbb{E}\|\overline{\theta}_{nm}\|^2 \to +\infty$ as in $n, m \to +\infty$. From the latter inequality, we can deduce that the problem is ill-posed. Moreover, as classified in [7], the problem is severely ill-posed. Hence, a regularization method is necessary for stable reconstruction of the initial temperature.

4 Estimators and convergence results

4.1 Nonparametric least squares method

Since the final temperature h and the source f satisfy two nonparametric regression models, we will first consider a generalization of the models. Assume that g is a unknown function from Ω into $\mathbb{R}$ and that the observations Z_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$ satisfy the model

$$Z_{ij} = g(X_i, Y_j) + \varepsilon_{ij},$$

where (X_i, Y_j) are as in the introduction, ε_{ij} are mutually independent and $\mathbb{E}\varepsilon_{ij} = 0$. We will estimate the function g by the nonparametric least squares estimators. Using the idea of the statistical projection method, we will find the estimators in $\mathcal{V}_{N,M}$ with $1 \le N \le n$, $1 \le M \le m$ which are minimizers of the problem

$$\hat{g}^{\mathrm{LS}}_{n,m,N,M} = \arg\min_{\psi \in \mathcal{V}_{N,M}} \sum_{i=1}^{n} \sum_{j=1}^{m} (Z_{ij} - \psi(X_i, Y_j))^2. \tag{12}$$

From the lemma, we can obtain the explicit form of our minimizers.

Lemma 4.1. *Problem* (12) *has a unique solution*

$$\hat{g}^{\mathrm{LS}}_{n,m,N,M} := \sum_{p=1}^{N} \sum_{q=1}^{M} \left(\frac{\pi^2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij} \phi_{p,q}(X_i, Y_j) \right) \phi_{p,q}.$$

Proof. For $\psi \in \mathcal{V}_{N,M}$, we have

$$\psi(x, y) = \sum_{p=1}^{N} \sum_{q=1}^{M} c_{p,q} \phi_{p,q}(x, y).$$

Putting $c = (c_{p,q})$, $p = 1, \dots, n$, $q = 1, \dots, m$, we can rewrite problem (12) as

$$\hat{g}^{\mathrm{LS}}_{n,m,N,M} = \arg\min_{c \in \mathbb{R}^{N \times M}} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(Z_{ij} - \sum_{p=1}^{N} \sum_{q=1}^{M} c_{p,q} \phi_{p,q}(X_i, Y_j) \right)^2.$$

Denote

$$L(c) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(Z_{ij} - \sum_{p=1}^{N} \sum_{q=1}^{M} c_{p,q} \phi_{p,q}(X_i, Y_j) \right)^2.$$

At the minimize point, we have

$$\frac{\partial L}{\partial c_{\ell,k}} = -2 \sum_{i=1}^{n} \sum_{j=1}^{m} \left(Z_{ij} - \sum_{p=1}^{N} \sum_{q=1}^{M} c_{p,q} \phi_{p,q}(X_i, Y_j) \right) \phi_{\ell,k}(X_i, Y_j) = 0$$

with $\ell = 1, \dots, N$, $k = 1, \dots, M$. By Lemma 2.1, we obtain

$$\hat{c}_{\ell,k} = \frac{\pi^2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij} \phi_{\ell,k}(X_i, Y_j).$$

Hence, the nonparametric least squares estimator is

$$\hat{g}^{\mathrm{LS}}_{n,m,N,M} := \sum_{p=1}^{N} \sum_{q=1}^{M} \left(\frac{\pi^2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij} \phi_{p,q}(X_i, Y_j) \right) \phi_{p,q},$$

as desired. □

4.2 Estimators of the initial temperature

Problem (1)–(2) with the discrete conditions (4)–(5) should have many infinitely solutions. So, to regularize problem (1)–(3), one has to discretize the problem and reduce it to a finite system of linear equations. To do this end, one popular method is the projection method. Choosing two natural numbers N, M such that $1 \le N \le n$, $1 \le M \le m$, we will find a solution $w(x, y, t)$ of the problem on the subspace $\mathcal{V}_{\ell,k}$ such that

$$\begin{cases} w_t - a(t)(w_{xx} + w_{yy}) = \hat{f}^{\mathrm{LS}}_{n,m,N,M}(x, y, t), & (x, y, t) \in \Omega \times (0, T), \\ w(x, y, T) = \hat{h}^{\mathrm{LS}}_{n,m,N,M}(x, y), & (x, y) \in \overline{\Omega}, \end{cases} \tag{13}$$

where

$$\hat{h}^{\mathrm{LS}}_{n,m,N,M} = \arg \min_{h \in \mathcal{V}_{N,M}} \sum_{i=1}^{n} \sum_{j=1}^{m} (d_{ij} - h(X_i, Y_j))^2, \quad \hat{f}^{\mathrm{LS}}_{n,m,N,M} = \arg \min_{f(\cdot,t) \in \mathcal{V}_{N,M}} \sum_{i=1}^{n} \sum_{j=1}^{m} (g_{ij}(t) - f(X_i, Y_j, t))^2.$$

Using Lemma 4.1 and formula (6), we deduce that system (13) has a unique solution

$$w = \sum_{p=1}^{N} \sum_{q=1}^{M} \bigg(\hat{h}_{p,q} - \int_t^T \lambda^{-1}_{p,q}(\tau) \hat{f}_{p,q}(\tau)\, d\tau \bigg) \lambda^{-1}_{p,q}(T) \phi_{p,q}, \tag{14}$$

where

$$\hat{h}_{p,q} = \frac{\pi^2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} \phi_{p,q}(X_i, Y_j), \quad \hat{f}_{p,q} = \frac{\pi^2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij}(t) \phi_{p,q}(X_i, Y_j).$$

From (14), the estimator of the initial temperature function has the form

$$\hat{\theta}_{N,M} = \sum_{p=1}^{N} \sum_{q=1}^{M} \hat{A}_{p,q} \phi_{p,q}, \tag{15}$$

where

$$\hat{A}_{p,q} = \bigg(\hat{h}_{p,q} - \int_0^T \lambda^{-1}_{p,q}(\tau) \hat{f}_{p,q}(\tau)\, d\tau \bigg) \lambda^{-1}_{p,q}(T).$$

4.3 Convergence rate of the estimator

Now, we study the convergence rate, which is the main result of this paper. We note that N, M are regularization parameters, namely the truncation parameters in the series estimator. If the regularization parameters are too large, the projection estimator is not convergence. Hence, we prove that a suitable choosing regularization parameters is necessary. In fact, we will verify that

$$\lim_{m,n \to \infty} \min_{1 \le N \le n,\, 1 \le M \le m} \mathbb{E} \|\hat{\theta}_{N,M} - \theta\|^2 = 0.$$

Hereafter, for any positive numbers α, β and E, we denote the Sobolev class of functions by

$$\mathcal{C}_{\alpha,\beta,E} = \bigg\{ g \in L^2(\Omega) : \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^{2\alpha} q^{2\beta} |\langle g, \phi_{p,q} \rangle|^2 \le E^2 \bigg\}.$$

The convergence rate of estimator $\hat{\theta}_{N,M}$ in (15) is presented by Theorem 4.8. In order to prove the theorem, we need the evaluation for $\mathbb{E}\|\hat{\theta}_{N,M} - \theta\|^2$. In fact, this estimate procedure has to undergo some important steps. In the first step, we have the following lemma.

Lemma 4.2. *Let the regression models* (4) *and* (5) *hold. Assume that* $\theta \in \mathcal{C}_{\alpha,\beta,E}$ *and* $0 < N < n$, $0 < M < m$. *Then*

$$\|\hat{\theta}_{N,M} - \theta\|^2 = \sum_{p=1}^{N} \sum_{q=1}^{M} \bigg[\frac{\pi^2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(\lambda^{-1}_{p,q}(T) \sigma_{ij} \varepsilon_{ij} - \vartheta \int_0^T \lambda^{-1}_{p,q}(\tau) \xi_{ij}(\tau)\, d\tau\Big) \phi_{p,q}(X_i, Y_j) \\ - \int_0^T \lambda^{-1}_{p,q}(\tau) \eta_{n,m,p,q}(\tau)\, d\tau + \gamma_{n,m,p,q} \lambda^{-1}_{p,q}(T) \bigg]^2 + \inf_{\phi \in \mathcal{V}_{N,M}} \|\theta - \phi\|^2. \tag{16}$$

Proof. By the Parseval equality, we have

$$\|\hat{\theta}_{N,M} - \theta\|^2 = \sum_{p=1}^{N}\sum_{q=1}^{M}(\hat{A}_{p,q} - \theta_{p,q})^2 + \sum_{p=N+1}^{\infty}\sum_{q=1}^{M}\theta_{p,q}^2 + \sum_{p=1}^{N}\sum_{q=M+1}^{\infty}\theta_{p,q}^2 + \sum_{p=N+1}^{\infty}\sum_{q=M+1}^{\infty}\theta_{p,q}^2.$$

From the formula of $\hat{A}_{p,q}$ and $\theta_{p,q}$, $p = 1, \dots, N$, $q = 1, \dots, M$, we get

$$\hat{A}_{p,q} - \theta_{p,q} = \frac{\pi^2}{nm}\lambda_{p,q}^{-1}(T)\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}\varepsilon_{ij}\phi_{p,q}(x_i, y_j) - \int_0^T \lambda_{p,q}^{-1}(\tau)[\hat{f}_{p,q}(\tau) - f_{p,q}(\tau)]\,\mathrm{d}\tau - \gamma_{n,m,p,q}\lambda_{p,q}^{-1}(T)$$

with

$$\hat{f}_{p,q}(t) - f_{p,q}(t) = \frac{\pi^2\vartheta}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_{ij}(t)\phi_{p,q}(x_i, y_j) + \eta_{n,m,p,q}(t).$$

Thus, we obtain (16). □

Now, we prove that $\gamma_{n,m,p,q}$ and $\eta_{n,m,p,q}$ tend to zero as $n, m \to \infty$. We first have:

Lemma 4.3. *Assume that $f(\cdot, t) \in \mathcal{C}_{\alpha,\beta,E}$ for all $t \in [0, T]$ and $\theta, h \in L^2(\Omega)$. Then*

$$|h_{p,q}| \le \|\theta\|\lambda_{p,q}(T) + \frac{E}{p^\alpha q^\beta a_1(p^2 + q^2)}.$$

Proof. From (7) and $|f_{p,q}(\cdot)| \le E/(p^\alpha q^\beta)$, we have

$$|h_{p,q}| \le \|\theta\|\lambda_{p,q}(T) + \frac{E}{p^\alpha q^\beta}\int_0^T e^{-(p^2+q^2)\int_\tau^T a(s)\,\mathrm{d}s}\,\mathrm{d}\tau.$$

Since $a(t) \ge a_1$, we deduce

$$|h_{p,q}| \le \|\theta\|\lambda_{p,q}(T) + \frac{E}{p^\alpha q^\beta a_1(p^2 + q^2)}.$$

This completes the proof. □

In the next lemma, we will give an upper bound for the discretization bias of $h_{p,q}$. Indeed, we have:

Lemma 4.4. *Suppose that $f(\cdot, \cdot, t) \in \mathcal{C}_{\alpha,\beta,E}$ and that $p = 1, \dots, n-1$, $q = 1, \dots, m-1$. With $\gamma_{n,m,p,q}$ defined by* (8)*, there is a generic constant C independent of n, m, p, q such that*

$$|\gamma_{n,m,p,q}| \le Cn^{-1-\alpha/2}m^{-1-\beta/2}. \tag{17}$$

Proof. From (9), we have

$$|\gamma_{n,m,p,q}| \le |P_{n,p,q}| + |Q_{m,p,q}| + |R_{n,m,p,q}|.$$

Using Lemma 4.3 gives

$$\begin{aligned}
|P_{n,p,q}| &\le \sum_{k=1}^{\infty}|h_{2kn\pm p,q}| \le \|\theta\|\sum_{k=1}^{\infty}\lambda_{2kn\pm p,q}(T) + \sum_{k=1}^{\infty}\frac{E}{(2kn \pm p)^\alpha q^\beta a_1((2kn \pm p)^2 + q^2)} \\
&\le \|\theta\|\sum_{k=1}^{\infty}e^{-A(T)[(2kn\pm p)^2+q^2]} + \sum_{k=1}^{\infty}\frac{E}{a_1[(2kn \pm p)^{2+\alpha} + q^{2+\beta}]} \\
&\le \|\theta\|\frac{e^{-A(T)(2n-p+q^2)} + e^{-A(T)(2n+p+q^2)}}{1 - e^{-2nA(T)}} + \sum_{k=1}^{\infty}\frac{E}{a_1(2kn)^{2+\alpha}} \\
&\le \|\theta\|\frac{2e^{-A(T)(2n-p+q^2)}}{1 - e^{-2nA(T)}} + \frac{E}{a_1 n^{2+\alpha}}\sum_{k=1}^{\infty}\frac{1}{(2k)^{2+\alpha}}.
\end{aligned}$$

Since $A(T) > a_1 T$ and $1 - e^{-2nA(T)} \ge \frac{1}{2}$ as n large, we obtain

$$|P_{n,p,q}| \le 4e^{-a_1 T(2n-p+q^2)}\|\theta\| + \frac{2E\kappa_\alpha}{a_1 n^{2+\alpha}} := K_{1,n,m}, \tag{18}$$

where we use $\kappa_\alpha := \sum_{k=1}^{\infty} \frac{1}{(2k)^{2+\alpha}} < 2$ for all $\alpha > 0$. Similarly, we get

$$|Q_{m,p,q}| \le 4e^{-a_1 T(2m-q+p^2)}\|\theta\| + \frac{2E\kappa_\beta}{a_1 m^{2+\beta}} := K_{2,n,m}. \tag{19}$$

Next, we find an upper bound for $|R_{n,m,p,q}|$. In fact, we have

$$|R_{n,m,p,q}| \le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |h_{2kn\pm p,2lm-q}| + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |h_{2kn\pm p,2lm+q}| = R^{I}_{n,m,p,q} + R^{II}_{n,m,p,q}.$$

Now we estimate the first term as follows

$$\begin{aligned} R^{I}_{n,m,p,q} &\le \|\theta\| \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} e^{-A(T)[(2kn\pm p)^2+(2lm-q)^2]} + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{E}{a_1[(2kn\pm p)^{2+\alpha} + (2lm-q)^{2+\beta}]} \\ &\le \frac{\|\theta\|(e^{-A(T)(2n+2m-p-q)} + e^{-A(T)(2n+2m+p-q)})}{[1-e^{-2nA(T)}][1-e^{-2mA(T)}]} + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{E}{a_1[(2kn)^{2+\alpha} + (2lm)^{2+\beta}]}. \end{aligned}$$

Similarly, using the inequality $x + y \ge 2\sqrt{xy}$ $(x, y \ge 0)$, we obtain

$$R^{I}_{n,m,p,q} \le 8e^{-a_1 T(2n+2m-p-q)}\|\theta\| + \frac{E\kappa_{\alpha,\beta}}{2a_1 n^{1+\alpha/2} m^{1+\beta/2}},$$

where $\kappa_{\alpha,\beta} := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(2k)^{1+\alpha/2}(2l)^{1+\beta/2}} < +\infty$ for all $\alpha, \beta > 0$. Similarly, we get

$$R^{II}_{n,m,p,q} \le 8e^{-a_1 T(2n+2m-p-q)}\|\theta\| + \frac{E\kappa_{\alpha,\beta}}{2a_1 n^{1+\alpha/2} m^{1+\beta/2}}.$$

Therefore

$$|R_{n,m,p,q}| \le 16e^{-a_1 T(2n+2m-p-q)}\|\theta\| + \frac{E\kappa_{\alpha,\beta}}{a_1 n^{1+\alpha/2} m^{1+\beta/2}} := K_{3,n,m}.$$

Noting that $2(K_{1,n,m} + K_{2,n,m}) \le Cn^{-1-\alpha/2}m^{-1-\beta/2}$ and that $K_{3,n,m} \le \mathcal{O}(n^{-1-\alpha/2}m^{-1-\beta/2})$, we get (17). □

Remark. Writing almost verbatim (in fact, easier) the above proof, we can obtain an estimation of order $\mathcal{O}(n^{-1-\alpha/2})$ for the discretization bias of one-dimensional Fourier coefficients. The order is better than the order $\mathcal{O}(n^{-1})$ assumed in [4] and it can be applied for the Sobolev class of functions. Moreover, the idea can be generalized to the n-dimensional case.

Lemma 4.5. *Assume that $f(\cdot,\cdot,t) \in \mathcal{C}_{\alpha,\beta,E}$ and $\alpha, \beta > 1$. With $\eta_{n,m,p,q}(t)$ defined by* (10), *we obtain*

$$|\eta_{n,m,p,q}(t)| \le C'(n^{-\alpha} + m^{-\beta}), \tag{20}$$

where $2 \le C' < \infty$.

Proof. From (11), the triangle inequality implies

$$|\eta_{n,m,p,q}(t)| \le |P'_{n,p,q}(t)| + |Q'_{m,p,q}(t)| + |R'_{n,m,p,q}(t)|.$$

Estimating directly the first term gives

$$|P'_{n,p,q}(t)| \le \sum_{k=1}^{\infty}(|f_{-p+2kn,q}(t)| + |f_{p+2kn,q}(t)|) \le \sum_{k=1}^{\infty} \frac{2E}{(2kn-p)^{\alpha}} \le \sum_{k=1}^{\infty} \frac{2E}{(2kn-n)^{\alpha}} \le C_\alpha n^{-\alpha}.$$

Similarly, we also have

$$|Q'_{m,p,q}(t)| \le C_\beta m^{-\beta} \quad \text{and} \quad |R'_{n,m,p,q}(t)| \le C_{\alpha,\beta} n^{-\alpha} m^{-\beta}$$

with $4 \le C_{\alpha,\beta} < \infty$. Moreover, we easily see that the upper bound of $|R'_{n,m,p,q}(t)|$ is very smaller than the upper bounds of $|P'_{n,p,q}(t)|$ and $|Q'_{m,p,q}(t)|$ as n, m tend to infinity. Hence, we get (20). □

To prepare for the proof of the main result, we need:

Lemma 4.6. *Let $L > 1$ and $k > 0$. Then*

$$\int_1^L e^{ku^2}\,\mathrm{d}u \le \frac{e^{L^2k}}{Lk}. \tag{21}$$

Proof. Putting $s = u/L$, we have

$$\int_1^L e^{ku^2}\,\mathrm{d}u = L\int_{1/L}^1 e^{L^2ks^2}\,\mathrm{d}s \le L\int_0^1 e^{L^2ks^2}\,\mathrm{d}s.$$

Then transforming variable $v = L^2k(1-s)$ gives

$$L\int_0^1 e^{L^2ks^2}\,\mathrm{d}s = \frac{e^{L^2k}}{Lk}\int_0^{L^2k} e^{L^2k((1-\frac{v}{L^2k})^2-1)}\,\mathrm{d}v.$$

Since

$$L^2k\Big(\Big(1-\frac{v}{L^2k}\Big)^2-1\Big) = v\frac{L^2k((1-\frac{v}{L^2k})^2-1)}{v} \le -v,$$

we have

$$\int_1^L e^{ku^2}\,\mathrm{d}u \le \frac{e^{L^2k}}{Lk}\int_0^{L^2k} e^{-v}\,\mathrm{d}v \le \frac{e^{L^2k}}{Lk}(1-e^{-L^2k}) \le \frac{e^{L^2k}}{Lk}.$$

Therefore, (21) holds. □

Finally, we are ready to state and prove two main theorems of our paper.

Theorem 4.7. *Let $E > 0$, $\alpha, \beta > 1$, $1 \le N \le n$, $1 \le M \le m$ and $h \in C^1(\overline{\Omega}) \cap \mathcal{C}_{\alpha,\beta,E}$, $f \in C([0,T]; C^1(\overline{\Omega}) \cap \mathcal{C}_{\alpha,\beta,E})$. Assume that system* (1)–(2) *has a (unique) solution $u \in C^1([0,1]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega))$. For $\hat{\theta}_{N,M}$ defined in* (15)*, we have*

$$\mathbb{E}\|\hat{\theta}_{N,M} - \theta\|^2 \le \frac{C_0 e^{2a_2[(N+1)^2+(M+1)^2]}}{2a_1 nm(N+1)(M+1)} + \inf_{\phi\in\mathcal{V}_{N,M}} \|\theta-\phi\|^2,$$

where the positive constant C_0 is independent of n, m, N, M. It follows that

$$\min_{1\le N\le n,\,1\le M\le m} \mathbb{E}\|\hat{\theta}_{N,M} - \theta\|^2 \le \min_{1\le N\le n,\,1\le M\le m}\Big(\frac{C_0 e^{2a_2[(N+1)^2+(M+1)^2]}}{nm(N+1)(M+1)} + \inf_{\phi\in\mathcal{V}_{N,M}} \|\theta-\phi\|^2\Big).$$

Proof. According to Lemma 4.2, we denote

$$\begin{aligned} I_1 &= \frac{3\pi^4}{n^2m^2}\sum_{p=1}^N\sum_{q=1}^M\Bigg[\Bigg(\sum_{i=1}^n\sum_{j=1}^m\Big(\lambda_{p,q}^{-1}(T)\sigma_{ij}\varepsilon_{ij} - \int_0^T \lambda_{p,q}^{-1}(\tau)\vartheta\xi_{ij}(\tau)\,\mathrm{d}\tau\Big)\phi_{p,q}(X_i,Y_j)\Bigg)^2 \\ &\qquad + \Big(\int_0^T \lambda_{p,q}^{-1}(\tau)\eta_{n,m,p,q}(\tau)\,\mathrm{d}\tau\Big)^2 + \gamma_{n,m,p,q}^2\lambda_{p,q}^{-2}(T)\Bigg] \\ &= \frac{3\pi^4}{n^2m^2}(I_{1,1}+I_{1,2}+I_{1,3}). \end{aligned}$$

We will find upper bounds for $I_{1,1}, I_{1,2}, I_{1,3}$. We first have

$$\begin{aligned} I_{1,1} &= \sum_{p=1}^N\sum_{q=1}^M\Bigg[\sum_{i=1}^n\sum_{j=1}^m\Big(\lambda_{p,q}^{-1}(T)\sigma_{ij}\varepsilon_{ij} - \int_0^T \lambda_{p,q}^{-1}(\tau)\vartheta\xi_{ij}(\tau)\,\mathrm{d}\tau\Big)\phi_{p,q}(X_i,Y_j)\Bigg]^2 \\ &\le 2\sum_{p=1}^N\sum_{q=1}^M\Bigg[\lambda_{p,q}^{-2}(T)\Big(\sum_{i=1}^n\sum_{j=1}^m \phi_{p,q}(X_i,Y_j)\sigma_{ij}\varepsilon_{ij}\Big)^2 + \Big(\sum_{i=1}^n\sum_{j=1}^m \phi_{p,q}(X_i,Y_j)\int_0^T \lambda_{p,q}^{-1}(\tau)\vartheta\xi_{ij}(\tau)\,\mathrm{d}\tau\Big)^2\Bigg]. \end{aligned}$$

From the Brownian motion properties, we known that $\mathbb{E}[\xi_{ij}(t)\xi_{kl}(t)] = 0$ for $k \neq i$, $l \neq j$ and $\mathbb{E}\xi_{ij}^2(t) = t$. By the Hölder inequality, we obtain

$$\begin{aligned}\mathbb{E}(I_{1,1}) &\leq 2\sum_{p=1}^{N}\sum_{q=1}^{M}\Big(\frac{nm}{\pi^2}V_{\max}\lambda_{p,q}^{-2}(T) + \sum_{i=1}^{n}\sum_{j=1}^{m}\phi_{p,q}^2(X_i, Y_j)\int_0^T \lambda_{p,q}^{-2}(\tau)\,d\tau\int_0^T \vartheta^2\mathbb{E}\xi_{ij}^2(\tau)\,d\tau\Big)\\ &\leq \frac{2nm}{\pi^2}\Big(V_{\max} + \frac{\vartheta^2T^3}{2}\Big)\sum_{p=1}^{N}\sum_{q=1}^{M}e^{2A(T)(p^2+q^2)}.\end{aligned}$$

Lemma 4.6 gives

$$\mathbb{E}(I_{1,1}) \leq \frac{2nm}{\pi^2}\Big(V_{\max} + \frac{\vartheta^2T^3}{2}\Big)\int_1^{N+1}\int_1^{M+1}e^{2A(T)(s^2+r^2)}\,dr\,ds.$$

Thus, we obtain

$$\mathbb{E}(I_{1,1}) \leq \frac{nme^{2A(T)[(N+1)^2+(M+1)^2]}}{2\pi^2A^2(T)(N+1)(M+1)}\Big(V_{\max} + \frac{\vartheta^2T^3}{2}\Big). \tag{22}$$

Next we evaluate $I_{1,2}$. Putting $\eta_{n,m} = \max\{|\eta_{n,m,p,q}| : p = 1,\dots,N,\ q = 1,\dots,M\}$, we obtain directly

$$I_{1,2} \leq \sum_{p=1}^{N}\sum_{q=1}^{M}\eta_{n,m}^2\Big(\int_0^T \lambda_{p,q}^{-1}(\tau)\,d\tau\Big)^2 \leq \eta_{n,m}^2\sum_{p=1}^{N}\sum_{q=1}^{M}\Big(\int_0^T e^{a_2\tau(p^2+q^2)}\,d\tau\Big)^2.$$

Hence, it follows from Lemma 4.5 that

$$I_{1,2} \leq \eta_{n,m}^2\sum_{p=1}^{N}\sum_{q=1}^{M}\frac{e^{2a_2T[p^2+q^2]}}{a_2^2(p^2+q^2)^2} \leq \frac{e^{2a_2T[(N+1)^2+(M+1)^2]}}{2a_2^3T(N+1)(M+1)}C'^2(n^{-\alpha}+m^{-\beta})^2. \tag{23}$$

Finally, we find an upper bound of $I_{1,3}$. Putting $\gamma_{n,m} = \max\{|\gamma_{n,m,p,q}| : p \in 1,\dots,N,\ q \in 1,\dots,M\}$ and using Lemma 4.4 we have

$$I_{1,3} \leq \gamma_{n,m}^2\lambda_{p,q}^{-2}(T) \leq 8\sum_{p=1}^{N}\sum_{q=1}^{M}(K_{1,n,m}^2 + K_{2,n,m}^2)\lambda_{p,q}^{-2}(T) = 8(I'_{1,3} + I''_{1,3}),$$

where $K_{1,n,m}$ and $K_{2,n,m}$ are defined in (18) and (19). We get

$$\begin{aligned}I'_{1,3} &\leq \sum_{p=1}^{N}\sum_{q=1}^{M}\Big[4e^{-a_1T(2n-p+q^2)}\|\theta\| + \frac{2E}{a_1n^{2+\alpha}}\Big]^2\lambda_{p,q}^{-2}(T)\\ &\leq \frac{8E^2}{a_1^2n^{4+2\alpha}}\sum_{p=1}^{N}\sum_{q=1}^{M}\frac{\lambda_{p,q}^{-2}(T)}{p^{2+2\alpha}} + 32e^{-4na_1T}\|\theta\|^2\sum_{p=1}^{N}\sum_{q=1}^{M}e^{-2a_1T(q^2-p)}\lambda_{p,q}^{-2}(T)\\ &\leq \mathcal{O}\Big(\frac{e^{2A(T)[(N+1)^2+(M+1)^2]}}{n^{4+2\alpha}(N+1)(M+1)}\Big).\end{aligned}$$

Similarly, we obtain

$$I''_{1,3} \leq \mathcal{O}\Big(\frac{e^{2A(T)[(N+1)^2+(M+1)^2]}}{m^{4+2\beta}(N+1)(M+1)}\Big).$$

Hence, we get

$$I_{1,3} \leq \mathcal{O}\Big(\frac{e^{2A(T)[(N+1)^2+(M+1)^2]}}{(N+1)(M+1)}\Big(\frac{1}{n^{4+2\alpha}} + \frac{1}{m^{4+2\beta}}\Big)\Big). \tag{24}$$

Therefore, combining (22), (23) and (24), we get

$$\mathbb{E}I_1 \leq \mathcal{O}\Big(\frac{e^{2a_2[(N+1)^2+(M+1)^2]}}{nm(N+1)(M+1)}\Big),$$

as desired. □

Theorem 4.8. *Let $E > 0$, $\alpha, \beta > 1$ and $h \in C^1(\overline{\Omega}) \cup \mathcal{C}_{\alpha,\beta,E}$, $f \in C([0,T]; C^1(\overline{\Omega}) \cap \mathcal{C}_{\alpha,\beta,E})$. Assume that system (1)–(2) has a (unique) solution $u \in C^1([0,1]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega))$. Choose*

$$N_{n,m} = M_{n,m} \sim \mathcal{O}(\lfloor \log^{1/2} nm \rfloor),$$

where $\lfloor x \rfloor$ is the greatest integer $\le x$. For $\hat{\theta}_{N_{n,m},M_{n,m}}$ defined in (15) and $\theta \in \mathcal{C}_{\alpha,\beta,E}$, we have

$$\min_{1 \le N \le n,\, 1 \le M \le m} \mathbb{E}\|\hat{\theta}_{N,M} - \theta\|^2 \le \mathbb{E}\|\hat{\theta}_{N_{n,m},M_{n,m}} - \theta\|^2 \le C_1 \log^{-\alpha_0} nm,$$

where the positive constant C_1 is independent of n, m and $\alpha_0 = \min\{\alpha, \beta\}$.

Proof. According to Theorem 4.7, we have

$$\mathbb{E}\|\hat{\theta}_{N,M} - \theta\|^2 \le C_1 \Delta_{n,m,N,M} + \inf_{\phi \in \mathcal{V}_{N,M}} \|\theta - \phi\|^2,$$

where

$$\Delta_{n,m,N,M} := \frac{e^{2a_2[(N+1)^2 + (M+1)^2]}}{nm(N+1)(M+1)}.$$

Now, we find an upper bound for the second term. In fact, we have

$$\begin{aligned}\inf_{\phi \in \mathcal{V}_{N,M}} \|\theta - \phi\|^2 &= \sum_{p=N+1}^{\infty} \sum_{q=1}^{M} p^{-2\alpha} q^{-2\beta} |\langle p^\alpha q^\beta \theta, \phi_{p,q}\rangle|^2 + \sum_{p=1}^{N} \sum_{q=M+1}^{\infty} p^{-2\alpha} q^{-2\beta} |\langle p^\alpha q^\beta \theta, \phi_{p,q}\rangle|^2 \\ &\quad + \sum_{p=N+1}^{\infty} \sum_{q=M+1}^{\infty} p^{-2\alpha} q^{-2\beta} |\langle p^\alpha q^\beta \theta, \phi_{p,q}\rangle|^2 \\ &\le E^2\big((N+1)^{-2\alpha} + (M+1)^{-2\beta} + (N+1)^{-2\alpha}(M+1)^{-2\beta}\big) \\ &\le 2E^2\big((N+1)^{-2\alpha_0} + (M+1)^{-2\alpha_0}\big) =: 2E^2 \Lambda_{n,m,N,M},\end{aligned}$$

where $\alpha_0 = \min\{\alpha, \beta\}$. Therefore

$$\mathbb{E}\|\hat{\theta}_{N,M} - \theta\|^2 \le C_0'(\Delta_{n,m,N,M} + \Lambda_{n,m,N,M}),$$

where $C_0' = \min\{C_0, 2E^2\}$. We choose the numbers N, M for minimizing the left-hand side of the latter inequality. Put

$$L(z, \omega) = \Delta_{n,m,z,\omega} + \Lambda_{n,m,z,\omega}.$$

The function $L(z, \omega)$ attains its minimum at (z_{nm}, ω_{nm}) satisfying $z_{nm}, \omega_{nm} \ge 1$ and

$$\frac{\partial L(z_{nm}, \omega_{nm})}{\partial z} = \frac{(4a_2(z_{nm}+1)^2 - 1)e^{2a_2[(z_{nm}+1)^2 + (\omega_{nm}+1)^2]}}{nm(z_{nm}+1)(\omega_{nm}+1)} - \frac{2\alpha_0}{(z_{nm}+1)^{2\alpha_0 - 1}} = 0$$

and

$$\frac{\partial L(z_{nm}, \omega_{nm})}{\partial \omega} = \frac{(4a_2(\omega_{nm}+1)^2 - 1)e^{2a_2[(z_{nm}+1)^2 + (\omega_{nm}+1)^2]}}{nm(z_{nm}+1)(\omega_{nm}+1)} - \frac{2\alpha_0}{(\omega_{nm}+1)^{2\alpha_0 - 1}} = 0.$$

We can verify that

$$\lim_{n,m\to\infty} z_{nm} = \lim_{n,m\to\infty} \omega_{nm} = \infty$$

and

$$F(z_{nm}+1) = F(\omega_{nm}+1)$$

with $F(\rho) = 4a_2\rho^{2\alpha_0+1} - \rho^{2\alpha_0-1}$. Since $F'(\rho) > 0$ for $\rho > a_2^{-1/2}$, we obtain $z_{nm} = \omega_{nm}$ for n, m large enough. Hence, we have the equation

$$\frac{(4a_2(z_{nm}+1)^2 - 1)e^{4a_2(z_{nm}+1)^2}}{nm(z_{nm}+1)^2} - \frac{2\alpha_0}{(z_{nm}+1)^{2\alpha_0-1}} = 0$$

which gives $\lim_{n,m\to\infty} \frac{\log^{1/2} nm}{z_{nm}+1} = 2\sqrt{a_2}$. Thus, we can choose $N_{n,m} = M_{n,m} = \lfloor z_{nm} \rfloor \sim \mathcal{O}(\lfloor \log^{1/2} nm \rfloor)$ and obtain

$$\mathbb{E}\|\hat{\theta}_{N_{n,m},M_{n,m}} - \theta\|^2 \le \mathcal{O}(\log^{-\alpha_0} nm),$$

as desired. □

5 Numerical results

We illustrate the theoretical results by concrete examples. We first describe a plan for computation. Let $\Omega = (0, \pi) \times (0, \pi)$, $T = 1$ and

$$\begin{cases} u_t - a(t)\Delta u = f(x, y, t), & (x, y, t) \in \Omega \times (0, 1), \\ u(x, y, t)\big|_{\partial\Omega} = 0, & 0 \le t \le 1, \\ u(x, y, 1) = h(x, y), & (x, y) \in \overline{\Omega}, \end{cases}$$

where the functions f, h are measured and the function $a : (0, 1) \to \mathbb{R}$ is known.

We will simulate the data for heat source term and final condition, respectively. In fact, at each point $(X_i, Y_j) = (\pi(2i - 1)/2n, \pi(2j - 1)/2m)$, $i = 1, \dots, n$, $j = 1, \dots, m$, using two subroutines in **FORTRAN** programs of John Burkardt (see [5]) and of Marsaglia and Tsang (see [15]), we make noises the heat source by $\vartheta\xi_{ij}(t)$ and the final data by $\sigma_{ij}\varepsilon_{ij}$, where $\xi_{ij}(t)$ are the normal Brownian motions and ε_{ij} are the standard normal random variables. Choosing $\sigma_{ij}^2 = \sigma^2 = \vartheta = 10^{-1}$ and 10^{-2}, we have two following regression models:

$$d_{ij} = h(X_i, Y_j) + \sigma\varepsilon_{ij}, \quad g_{ij}(t) = f(X_i, Y_j) + \vartheta\xi_{ij}, \quad \varepsilon_{ij} \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, 1).$$

Now, we choose some numerical methods to compare errors. The first method is the trigonometric nonparametric regression (truncated method for short) which is considered in the present paper. The second method is the quasi-boundary value (QBV) regularization. The third method is based on the classical solution (CS for short) of the backward problem.

For the mentioned function a, we use the method Legendre–Gauss quadrature with the roots x_i of the Legendre polynomials $P_{512}(x)$, $x \in [-1, 1]$ to calculate

$$A_{\mathrm{GL}} = \int_0^1 a(s)\,\mathrm{d}s = \frac{1}{2}\sum_{n=1}^{512} w_i a\Big(\frac{x_i}{2} + \frac{1}{2}\Big),$$

where

$$w_i = \frac{2}{(1 + x_i^2)[P'_{512}(x_i)]^2}.$$

In the first method, we have to set up the values of $N_{n,m}$, $M_{n,m}$. With the quantity A_{GL}, we can obtain the values of $N_{n,m}$, $M_{n,m}$ from n, m by the following formula:

$$N_{n,m} = \Big\lfloor \frac{\log^{1/2} nm}{A_{\mathrm{GL}}} \Big\rfloor \quad \text{and} \quad M_{n,m} = \Big\lfloor \frac{\log^{1/2} nm}{A_{\mathrm{GL}}} \Big\rfloor.$$

In each case of variance $\sigma_{ij}^2 = \sigma^2$, we compute 30 times. To calculate the error between the exact solution and the estimator, we use the root mean squared error (RMSE) as follows:

$$\mathrm{RMSE}(\hat{\theta}; \theta) = \sqrt{\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}(\hat{\theta}(X_i, Y_j) - \theta(X_i, Y_j))^2}.$$

Then we find the average of $\mathrm{RMSE}(\hat{\theta}; \theta)$ in 30 runs order.

The second method is the quasi-boundary value (QBV) regularization with the approximation of the initial data

$$\theta_{\mathrm{QBV}}(x, y) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\bigg(\frac{\hat{h}_{p,q}}{\varepsilon(p^2 + q^2) + \lambda_{p,q}(T)} - \int_0^T \frac{\lambda_{p,q}^{-1}(\tau)\lambda_{p,q}(T)}{\varepsilon(p^2 + q^2) + \lambda_{p,q}(T)}\hat{f}_{p,q}(\tau)\,\mathrm{d}\tau\bigg)\phi_{p,q}(x, y).$$

The method is chosen since it is quite common and the stability magnitude of the regularization operator is of order $\mathcal{O}(\varepsilon^{-1})$ (see [22]). As mentioned, in the QBV method, we do not have explicit stopping indices. So, we only calculate with $p, q = 1, \dots, 20$; $\varepsilon = \sigma^2$ and use the formula

$$\theta_{\mathrm{QBV}}(x,y) \approx \sum_{p=1}^{20}\sum_{q=1}^{20}\left(\frac{\hat{h}_{p,q}}{\varepsilon(p^2+q^2)+\lambda_{p,q}(T)} - \int_0^T \frac{\lambda_{p,q}^{-1}(\tau)\lambda_{p,q}(T)}{\varepsilon(p^2+q^2)+\lambda_{p,q}(T)}\hat{f}_{p,q}(\tau)\,d\tau\right)\phi_{p,q}(x,y).$$

Finally, we consider a numerical result for the classical solution (CS for short). As the second method, we use the approximation formula

$$\theta_{CS}(x,y) \approx \sum_{p=1}^{20}\sum_{q=1}^{20}\left(\hat{h}_{p,q}\lambda_{p,q}^{-1}(T) - \int_0^T \lambda^{-1}(\tau)\hat{f}_{p,q}(\tau)\,d\tau\right)\phi_{p,q}(x,y).$$

We will illustrate the discussed plan by two examples. In Example 1, we consider the problem with an exact initial datum θ having a finite Fourier expansion. In Example 2, we compute with the function θ having an infinite Fourier expansion.

In the examples, to calculate integrals depended on the time variable t in approximation formulas, we use the generalized Simpson approximation with 101 equidistant points $0 = t_0 < t_1 < \dots < t_{101} = 1$

$$\int_0^1 v(\tau)\,d\tau = \frac{1}{100}\left[\frac{3}{8}v(t_0) + \frac{7}{6}v(t_1) + \frac{23}{24}v(t_2) + \sum_{k=3}^{n-3} v(t_k) + \frac{23}{24}v(t_{99}) + \frac{7}{6}v(t_{100}) + \frac{3}{8}v(t_{101})\right],$$

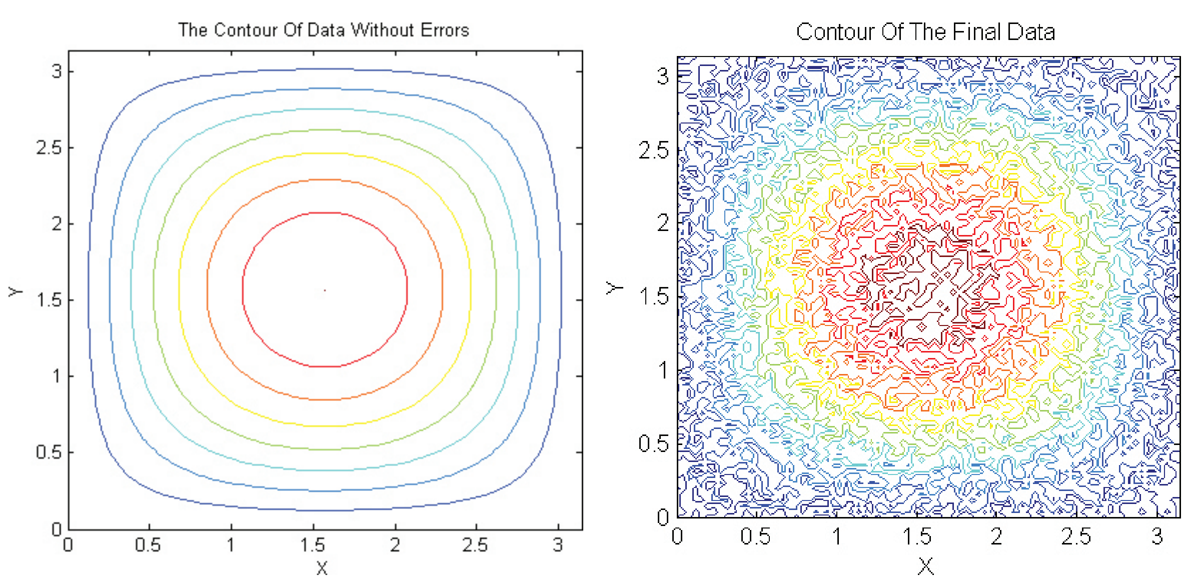
where $v(\tau) = \lambda_{p,q}^{-1}(\tau)\hat{f}_{p,q}(\tau)$.

Example 1. With $a(t) = 2 - t$, we can see that $1 = a_1 \le a(t) \le a_2 = 2$. We have $A_{\mathrm{GL}} = 1.5$. Assume that $f(x,y,t) = 2(t^3 - 2t^2 - 6t + 10)\sin(x)\sin(y)$ and $h(x,y) = 4\sin(x)\sin(y)$. The exact value of $u(x,y,0)$ is

$$\theta(x,y) = 5\sin(x)\sin(y),$$

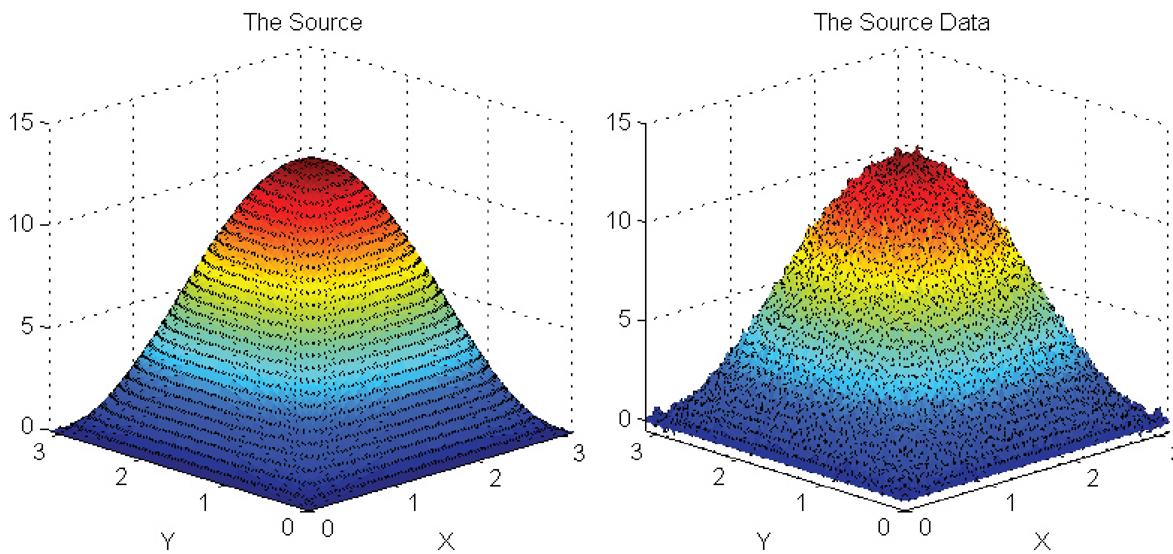
which has a finite Fourier expansion.

Figure 1 and Figure 2 present surfaces of the data and their contours (without and with noises respectively) for the final condition and the source term. They are drawn in case $\sigma^2 = 10^{-1}$, $n = m = 81$ and at the time $t = 0.5$, w.r.t. According to the figures, we can see the non-smoothness of two surfaces data in case of random noise. In fact, from the contour plot within noise of the final data, we also see that the measured data is very chaotic.



(a) True data **(b)** Observation data

Figure 1. The contour of two data set for final temperature.



(a) True data **(b)** Observation data

Figure 2. The surface of data set for heat source at $t = 0.5$.

Run	Estimator		QBV method		Classical solution	
	$\sigma^2 = 10^{-1}$	$\sigma^2 = 10^{-2}$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$
1	0.3488	0.0855	1.8493	0.6836	9.0696E+0466	7.2832E+0467
2	0.2810	0.0098	1.7936	0.6492	5.6003E+0468	1.3249E+0467
3	0.1665	0.1151	1.6715	0.6741	4.9925E+0468	7.6606E+0467
4	0.0642	0.0555	1.8313	0.6199	1.9484E+0468	8.8691E+0466
5	0.3478	0.0795	1.7854	0.5895	3.0650E+0468	9.9884E+0467
6	0.1541	0.1344	1.7437	0.6661	1.6817E+0468	1.0375E+0466
7	1.1359	0.1045	1.9001	0.6162	1.0333E+0468	5.0317E+0467
8	0.1819	0.1116	1.8155	0.6789	4.4777E+0468	2.6705E+0467
9	0.5098	0.0794	1.9957	0.6704	8.7766E+0467	1.9412E+0467
10	0.0767	0.0819	1.7344	0.6770	1.9678E+0468	7.3191E+0466
11	0.6926	0.0509	1.8346	0.6305	2.8522E+0468	3.6677E+0467
12	0.1562	0.0650	1.8199	0.6876	9.8178E+0468	6.0419E+0467
13	0.3010	0.0133	1.6247	0.6591	1.3412E+0468	4.7005E+0467
14	0.2691	0.0549	1.9827	0.6664	4.9153E+0468	2.6146E+0466
15	0.8242	0.0784	1.8294	0.6782	2.8401E+0468	2.2989E+0467
16	0.0800	0.0897	2.0291	0.6365	3.9761E+0468	3.2519E+0467
17	0.5340	0.0694	1.8317	0.6593	5.5066E+0466	4.5486E+0467
18	0.3112	0.0560	1.7623	0.6140	5.6634E+0468	4.9512E+0467
19	0.0823	0.1052	1.8327	0.6706	4.7594E+0467	1.5004E+0467
20	0.8982	0.0593	1.7463	0.6531	4.2411E+0468	3.3806E+0467
21	1.1967	0.0919	1.8337	0.6322	6.7184E+0468	3.4589E+0467
22	0.6456	0.1117	1.6554	0.6898	3.1764E+0468	8.5158E+0467
23	0.7978	0.0921	1.8755	0.6289	1.9857E+0468	1.3291E+0467
24	0.7382	0.0732	1.8330	0.6568	1.4733E+0468	1.6599E+0467
25	0.2039	0.1161	1.7400	0.6372	2.2766E+0468	2.3429E+0467
26	0.1441	0.1000	1.8158	0.6410	9.6333E+0467	3.4518E+0467
27	1.3111	0.1097	1.7632	0.6621	2.3796E+0468	3.9224E+0467
28	0.3626	0.1020	1.8254	0.6583	4.7331E+0468	7.5024E+0466
29	0.2833	0.0173	1.7640	0.6552	8.1452E+0467	1.4300E+0467
30	0.8313	0.0414	1.9595	0.6774	4.0568E+0468	4.4984E+0467
Average	0.4643	0.0785	1.8160	0.6540	divergence	divergence

Table 1. Example 1: Comparing errors between methods: $\sigma^2 = 10^{-1}, 10^{-2}$ and $n = m = 21$.

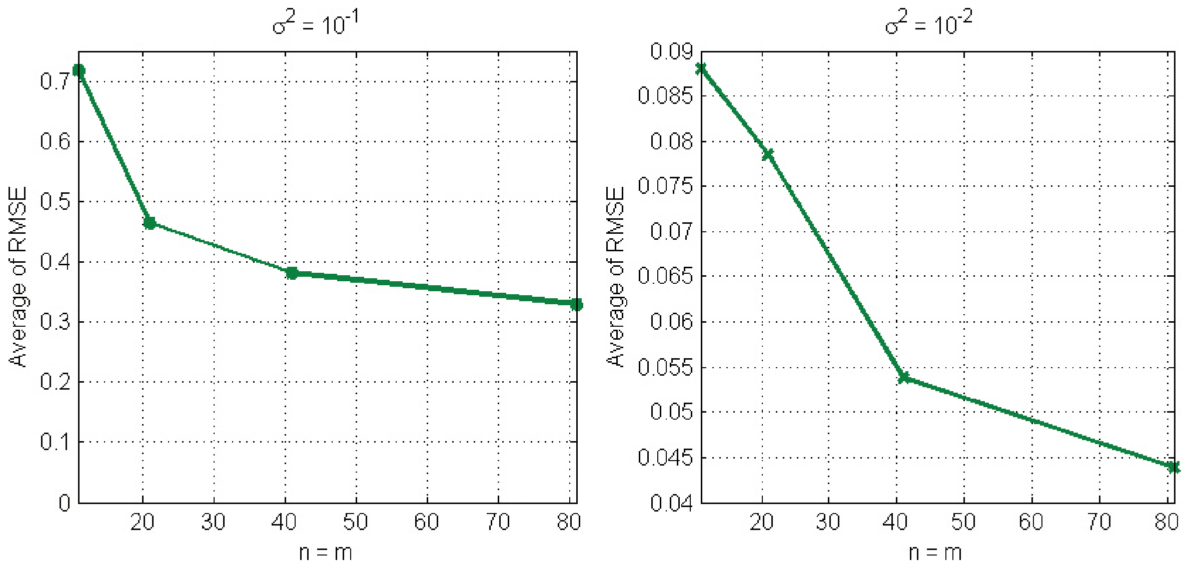


Figure 3. Example 1: The graphics of the average of RMSE in two cases $\sigma^2 = 10^{-1}$ and $\sigma^2 = 10^{-2}$.

In case of $\sigma^2 = 10^{-1}$, the error of the estimation is quite large, while, the error in case of $\sigma^2 = 10^{-2}$ is smaller. In addition, we see that the errors (in two cases of the variance $\sigma_{ij}^2 = \sigma^2$) are decreased when n, m are increased (see Figures 3). Table 1 shows the error of the method. We see that the error between the exact solution with the classical solution grows very fast. In fact, the error data is quite small $\varepsilon = 10^{-1}, 10^{-2}$ but the error solution is large $\approx 10^{466}$. This illustrates numerically the ill-posedness of our problem. On the other hand, the error in Table 1 of the truncated method is better than the one of the QBV method.

Example 2. Let $a(t) = 0.5e^{-t}$ and $e^{-1} = a_1 \le a(t) \le a_2 = 1$. Then we calculate $A_{\mathrm{GL}} = 0.3161$. Suppose that

$$f(x, y, t) = \frac{e^{-t}}{\pi}\left[(2e^{-t} + (4e^{-t} - 1)\sin 2y) + (1 - 10e^{-t})\sin 3x \sin y\right]$$

and

$$h(x, y) = \frac{e^{-1}}{\pi}\left[x(\pi - x)\sin y - \sin 3x \sin y\right].$$

We easily see that the exact value of $u(x, y, 0)$ is

$$\theta(x, y) = \frac{1}{\pi}\left[x(\pi - x)\sin y - \sin 3x \sin y\right],$$

which has an infinite Fourier expansion.

The results of Example 2 have error as in Table 2. From the results, we can obtain the same conclusions as in Example 1.

6 Conclusion

In paper, we consider a nonhomogeneous backward problem with final data and source having random noises. We first approximate the final data and the source by using nonparametric least squares regression methods in statistics. The estimate of bias of the discretization is given explicitly. On the other hand, our problem is ill-posed. Hence, a regularization is in order. We have used the projection method to approximate stably the unknown initial temperature θ. Finally, we illustrate the theoretical part by comparing computation results of the method presented in the paper, QBV and classical solution methods.

Run	Estimator		QBV method		Classical solution	
	$\sigma^2 = 10^{-1}$	$\sigma^2 = 10^{-2}$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$
1	0.2702	0.1533	0.2518	0.1431	4.79E+097	4.55E+095
2	0.2111	0.1536	0.3038	0.1512	4.97E+096	1.22E+096
3	0.1865	0.1540	0.2881	0.1402	9.17E+094	2.98E+096
4	0.3827	0.1539	0.3039	0.1350	3.80E+097	2.34E+096
5	0.2872	0.1525	0.3161	0.1521	1.18E+096	2.27E+096
6	0.2492	0.1564	0.3135	0.1525	7.43E+096	1.23E+096
7	0.2468	0.1539	0.2858	0.1289	1.28E+097	1.30E+096
8	0.6985	0.1531	0.2876	0.1436	2.11E+097	3.79E+095
9	0.2923	0.1534	0.3187	0.1421	2.04E+097	2.85E+095
10	0.3177	0.1549	0.3104	0.1484	4.02E+097	1.14E+096
11	0.1931	0.1534	0.2909	0.1386	1.21E+097	6.71E+095
12	0.1957	0.1563	0.3355	0.1821	1.60E+097	6.91E+095
13	0.1964	0.1532	0.3139	0.1313	7.14E+096	1.68E+096
14	0.2875	0.1553	0.3018	0.1570	3.96E+096	1.24E+096
15	0.2700	0.1540	0.3195	0.1403	4.51E+097	2.42E+096
16	0.2558	0.1545	0.2985	0.1362	1.68E+097	5.06E+096
17	0.1976	0.1535	0.3589	0.1466	1.60E+096	2.91E+096
18	0.4981	0.1548	0.3853	0.1594	3.85E+096	2.65E+096
19	0.2723	0.1539	0.3200	0.1540	9.70E+096	2.71E+096
20	0.3152	0.1534	0.3312	0.1466	2.04E+097	1.77E+096
21	0.3284	0.1544	0.3303	0.1442	1.60E+097	1.76E+096
22	0.4009	0.1526	0.3173	0.1274	1.01E+097	3.65E+096
23	0.3175	0.1532	0.3005	0.1475	9.57E+096	1.65E+096
24	0.4426	0.1526	0.3132	0.1537	3.06E+096	8.29E+095
25	0.3158	0.1528	0.3234	0.1353	4.21E+097	2.98E+096
26	0.2715	0.1545	0.3443	0.1428	1.22E+097	2.06E+096
27	0.1848	0.1527	0.3517	0.1312	2.46E+097	2.31E+096
28	0.2695	0.1555	0.3104	0.1470	1.05E+097	1.81E+096
29	0.5497	0.1637	0.3126	0.1300	7.88E+096	4.31E+096
30	0.3161	0.1530	0.3047	0.1414	4.69E+096	8.49E+095
Average	0.3074	0.1542	0.3148	0.1443	divergence	divergence

Table 2. Example 2: Comparing errors between methods: $\sigma^2 = 10^{-1}, 10^{-2}$ and $n = m = 21$.

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