ON ISOMETRY AND ISOMETRIC EMBEDDABILITY BETWEEN ULTRAMETRIC POLISH SPACES

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Abstract. We study the complexity with respect to Borel reducibility of the relations of isometry and isometric embeddability between ultrametric Polish spaces for which a set D of possible distances is fixed in advance. These are, respectively, an analytic equivalence relation and an analytic quasi-order and we show that their complexity depends only on the order type of D. When D contains a decreasing sequence, isometry is Borel bireducible with countable graph isomorphism and isometric embeddability has maximal complexity among analytic quasi-orders. If D is well-ordered the situation is more complex: for isometry we have an increasing sequence of Borel equivalence relations of length ω_1 which are cofinal among Borel equivalence relations classifiable by countable structures, while for isometric embeddability we have an increasing sequence of analytic quasi-orders of length at least $\omega + 3$.

We then apply our results to solve various open problems in the literature. For instance, we answer a long-standing question of Gao and Kechris by showing that the relation of isometry on locally compact ultrametric Polish spaces is Borel bireducible with countable graph isomorphism.

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2010 *Mathematics Subject Classification.* Primary: 03E15; Secondary: 54E50.

Key words and phrases. Borel reducibility; ultrametric spaces; Polish spaces; isometry; isometric embeddability.

Marcone's research was supported by PRIN 2009 Grant "Modelli e Insiemi" and PRIN 2012 Grant "Logica, Modelli e Insiemi". Until September 2014 Motto Ros was a member of the Logic Department of the Albert-Ludwigs-Universität Freiburg, which supported him at early stages of this research. After that, he was supported by the Young Researchers Program "Rita Levi Montalcini" 2012 through the project "New advances in Descriptive Set Theory".

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1. Introduction

A common problem in mathematics is to classify interesting objects up to some natural notion of equivalence. More precisely, one considers a class of objects X and an equivalence relation E on X, and tries to find a set of complete invariants I for (X, E). To be of any use, such an assignment of invariants should be as simple as possible. In most cases, both X and I carry some intrinsic Borel structures, so that it is natural to ask the assignment to be a Borel measurable map.

A classical example is the problem of classifying separable complete metric spaces, called *Polish metric spaces*, up to isometry. In [Gro99] Gromov showed for instance that one can classify compact Polish metric spaces using (essentially) elements of $\mathbb{R}$ as complete invariants; in modern terminology, we say that the corresponding classification problem is smooth. However, as pointed out by Vershik in [Ver98] the problem of classifying arbitrary Polish metric spaces is «an enormous task», in particular it is far from being smooth. Thus it is natural to ask how complicated is such a classification problem.

A natural tool for studying the complexity of classification problems is the notion of Borel reducibility introduced in [FS89] and in [HKL90]: we say that a classification problem (X, E) is *Borel reducible* to another classification problem (Y, F) (in symbols, $E \leq_B F$) if there exists a Borel measurable function $f \colon X \to Y$ such that $x \mathrel{E} x' \iff f(x) \mathrel{F} f(x')$ for all $x, x' \in X$. Intuitively, this means that the classification problem (X, E) is not more complicated than (Y, F): in fact, any assignment of complete invariants for (Y, F) may be turned into an assignment for (X, E) by composing with f. A comprehensive reference for the theory of Borel reducibility is [Gao09].

In [GK03] (see also [CGK01, Cle12]), Gao and Kechris were able to determine the exact complexity of the classification problem for isometry on arbitrary Polish metric spaces with respect to Borel reducibility: it is Borel bireducible with the most complex orbit equivalence relation (i.e. every equivalence relation induced by a Borel action of a Polish group on a Polish space Borel reduces to it). Then, extending the work of Gromov on compact Polish metric spaces, they turned their attention to some other natural subclasses of spaces. Among these, the cases of locally compact Polish metric spaces and of ultrametric[1] Polish spaces are particularly important. Notice that ultrametric Polish spaces naturally occur in various parts of mathematics and computer science. For example, the space $\mathbb{Q}_p$ of p-adic numbers with the metric induced by the evaluation map $|\cdot|_p$ (for any prime number p) and, more generally, the completion of any countable valued field are ultrametric

[1]Recall that a metric d on a space X is called an *ultrametric* if it satisfies the following strengthening of the triangular inequality: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Polish spaces. The same is true for the space ${}^\omega\Sigma$ of infinite words over a finite alphabet Σ appearing e.g. in automaton theory, which is usually equipped with the metric d measuring how much two given words $x, y \in {}^\omega\Sigma$ are close to each other, i.e. $d(x, y) = 2^{-n}$ with n least such that $x(n) \neq y(n)$. Indeed the space ${}^\omega\Sigma$ is ultrametric complete for *any* (finite or infinite) alphabet Σ, and is Polish if and only if Σ is at most countable.

While the complexity of the isometry relation on arbitrary locally compact Polish metric spaces has not yet been determined, [GK03, Theorem 4.4] shows that for the case of ultrametric Polish spaces the corresponding classification problem is Borel bireducible with isomorphism on countable graphs, the most complex isomorphism relation for classes of countable structures. Countable graph isomorphism is far from being smooth but it is strictly simpler, with respect to Borel reducibility, than isometry on arbitrary Polish metric spaces. For our purposes, it is important to notice that the proof of the result on ultrametric Polish spaces crucially uses spaces whose set of distances has 0 as a limit point and that are far from being locally compact. Motivated by these observations, Gao and Kechris devote a whole chapter of their monograph to the study of the isometry relation on Polish metric spaces that are both locally compact and ultrametric, obtaining some interesting partial results which seem to transfer this problem to the study of the isometry relation on discrete ultrametric Polish spaces. However, the latter problem remained unsolved and hence they asked:

Question 1.1. [GK03, §8C and Chapter 10] *What is the exact complexity of the isometry problem for discrete ultrametric Polish spaces and for arbitrary locally compact ultrametric Polish spaces?*

As a partial result, they isolated two lower bounds for these isometry relations, namely isomorphism of countable well-founded trees and isomorphism of countable trees with only countably many infinite branches.

Question 1.1 was raised again (and another lower bound was proposed) in [Cle07], where Clemens studies the complexity of isometry on the collection of Polish metric spaces using only distances in a set $A \subseteq \mathbb{R}^+$ fixed in advance. A related question is also raised in [GS11], where Gao and Shao consider Clemens' problem restricted to ultrametric Polish spaces:

Question 1.2. [GS11, §8] *Given a countable*[2] *set of distances $D \subseteq \mathbb{R}^+$ such that 0 is not a limit point of D, what is the complexity of the isometry relation between ultrametric Polish spaces using only distances from D?*

Since all spaces as in Question 1.2 are necessarily discrete (and hence locally compact), it is clear that the isometry relations considered there constitute lower bounds for the isometry relations of Question 1.1.

In this paper, we address Question 1.2 and show that our solution of this problem allows us to answer also Question 1.1. Moreover, we also consider the analogous problem concerning the complexity of the quasi-order of isometric embeddability between ultrametric Polish spaces using only distances from a given countable $D \subseteq \mathbb{R}^+$. (The formal setup for these problems is described in Section 4.) Concerning isometric embeddability on arbitrary ultrametric Polish spaces, it is already known

[2]The countability of D is a necessary requirement when dealing with ultrametric Polish spaces, see Lemma 4.1.

that the relation is as complicated as possible: Louveau and Rosendal ([LR05]) showed that it is a complete analytic quasi-order, and we strengthened this in [CMMR13] by showing that it is in fact invariantly universal (see definitions below). However, the proofs of the results in [LR05, CMMR13] again use in an essential way ultrametric spaces with distances converging to zero. This naturally raises the following question, which is somewhat implicit in [GS11]:

Question 1.3. *Given a countable set of distances $D \subseteq \mathbb{R}^+$ such that 0 is not a limit point of D, what is the complexity of the isometric embeddability relation between ultrametric Polish spaces using only distances from D?*

We will show that actually the answers to both Question 1.2 and 1.3 depend only on whether D contains a decreasing sequence and, when it does not, on the order type (a countable ordinal) of D. Thus isometry between ultrametric Polish spaces using only distances in D is actually the same for any ill-founded D, regardless of the limits of the decreasing sequences in D, and the same is true for isometric embeddability.

In Sections 5.1 and 6.1 we prove that if D contains a decreasing sequence of real numbers then our relations attain the maximal possible complexity: isometry is Borel bireducible with countable graph isomorphism, and isometric embeddability is invariantly universal, and therefore complete for analytic quasi-orders. This implies that the isometry relation on every class of ultrametric Polish spaces containing all spaces using only distances from some specific D with a decreasing sequence is also Borel bireducible with countable graph isomorphism. By choosing such a D so that 0 is not one of its limit points we obtain only discrete spaces. Thus our result in particular answers Question 1.1 (see Corollary 5.4): the relations of isometry on discrete ultrametric Polish spaces and on locally compact ultrametric Polish spaces are both Borel bireducible with countable graph isomorphism. This also shows that isometry on the classes of countable or σ-compact ultrametric Polish spaces is also Borel bireducible with countable graph isomorphism. We will also observe that many of the lower bounds previously isolated in the literature are in fact not sharp.

The situation is more complex when D is a well-ordered set of real numbers of order type α. In Section 5.2 we show that when α ranges over non-null countable ordinals, the corresponding isometry relations form a strictly increasing sequence of Borel equivalence relations, cofinal among Borel equivalence relations classifiable by countable structures. In Section 6.2 we prove that if $\alpha \leq \omega + 1$ then the corresponding quasi-order of isometric embeddability is not complete analytic. In fact for $1 \leq \alpha \leq \omega + 2$ we have a strictly increasing chain of quasi-orders; these are Borel exactly when $\alpha \leq \omega$. We do not know whether for some $\alpha \geq \omega + 2$ the corresponding quasi-order of isometric embeddability is already complete for analytic quasi-orders.

The main tool we use to deal with well-ordered sets of distances is a jump operator $S \mapsto S^{\mathrm{inj}}$ defined on quasi-orders, which seems to be of independent interest. This is defined and studied in Section 3, which starts with a review of Rosendal's jump operator $S \mapsto S^{\mathrm{cf}}$ in Subsection 3.1 before introducing our variant in Subsection 3.2. In particular, when S is Borel, S^{cf} is always Borel while we will show that S^{inj} can be proper analytic (Subsection 3.4) but cannot be complete for analytic quasi-orders (Subsection 3.5).

2. Terminology and notation

We now describe the basic terminology and notation used in the paper. Recall that a quasi-order is a reflexive and transitive binary relation. Any quasi-order S on a set X induces an equivalence relation on X, that we denote by E_S, defined by $x \mathrel{E_S} x'$ if and only if $x \mathrel{S} x'$ and $x' \mathrel{S} x$. A quasi-order is a *well quasi-order* (wqo for short) if it is well-founded and contains no infinite antichains. In the 1960's Nash-Williams introduced the notion of *better quasi-order* (bqo for short): the definition of bqo is quite involved, but to understand the references to this notion in this paper it suffices to know that, as the terminology suggests, every bqo is indeed a wqo, and to use as black boxes some closure properties of bqo's.

If $\mathcal{A}$ is a countably generated σ-algebra of subsets of X that separates points we refer to the members of $\mathcal{A}$ as Borel sets (indeed, as shown e.g. in [Kec95, Proposition 12.1], in this case $\mathcal{A}$ is the collection of Borel sets of some separable metrizable topology on X). A map between two sets equipped with a collection of Borel sets is Borel (measurable) if the preimages of Borel sets of the target space are Borel sets of the domain. The space $(X, \mathcal{A})$ is standard Borel if $\mathcal{A}$ is the collection of Borel sets of some Polish (i.e. separable and completely metrizable) topology on X. Except where explicitly noted, in this paper we will always deal with standard Borel spaces.

Let R and S be binary relations on spaces equipped with a collection of Borel sets X and Y, respectively. We say that R is *Borel reducible* to S, and we write $R \leq_B S$, if there is a Borel function $f \colon X \to Y$ such that $x \mathrel{R} x'$ if and only if $f(x) \mathrel{S} f(x')$ for all $x, x' \in X$. If $R \leq_B S$ and $S \leq_B R$ we say that R and S are *Borel bireducible* and we write $R \sim_B S$. If on the other hand we have $R \leq_B S$ and $S \not\leq_B R$ we write $R <_B S$.

If Γ is a class of binary relations on standard Borel spaces and $S \in \Gamma$, we say that S is *complete for* Γ if $R \leq_B S$ for all $R \in \Gamma$. Some classes Γ we consider in this paper are the collection of all analytic equivalence relations and the collection of all analytic quasi-orders. If Γ is the class of equivalence relations classifiable by countable structures, the canonical example of an equivalence relation complete for Γ is countable graph isomorphism. Thus an equivalence relation on a standard Borel space which is Borel bireducible with countable graph isomorphism is in fact complete for equivalence relations classifiable by countable structures.

Following the main result of [FMR11], in [CMMR13] we introduced the following notion.

Definition 2.1. Let the pair (S, E) consist of an analytic quasi-order S and an analytic equivalence relation $E \subseteq S$, with both relations defined on the same standard Borel space X. Then (S, E) is *invariantly universal* (for analytic quasi-orders) if for any analytic quasi-order R there is a Borel $B \subseteq X$ invariant under E such that $R \sim_B S \restriction B$.

Whenever the equivalence relation E is clear from the context (in this paper this usually means that E is isometry between metric spaces in the class under consideration) we just say that S is invariantly universal.

Notice that if (S, E) is invariantly universal, then S is, in particular, complete for analytic quasi-orders.

We now extend the notions of classwise Borel embeddability and isomorphism between equivalence relations introduced in [MR12] to pairs consisting of a quasi-order and an equivalence relation ($E \simeq_{cB} F$, respectively $E \sqsubseteq_{cB} F$, in the notation of [MR12] is the same as (E,E) is classwise Borel isomorphic to, respectively embeddable in, (F,F) in our terminology). Given a pair (S,E) consisting of a quasi-order S and an equivalence relation $E \subseteq S$ on a set X, we denote by S/E the E-quotient of S, i.e. the quasi-order on X/E induced by S. If F and E are equivalence relations on sets X and Y and $f\colon X/F \to Y/E$, then a *lifting* of f is a function $\hat{f}\colon X \to Y$ such that $[\hat{f}(x)]_E = f([x]_F)$ for every $x \in X$.

Definition 2.2. Let (R,F) and (S,E) be pairs consisting of a quasi-order and an equivalence relation on some standard Borel spaces, with $F \subseteq R$ and $E \subseteq S$.

We say that (R,F) is *classwise Borel isomorphic* to (S,E), in symbols $(R,F) \simeq_{cB} (S,E)$, if there is an isomorphism of quasi-orders f between R/F and S/E such that both f and f^{-1} admit Borel liftings.

We say that (R,F) is *classwise Borel embeddable* in (S,E), in symbols $(R,F) \sqsubseteq_{cB} (S,E)$, if there is an E-invariant Borel subset B of the domain of S such that $(R,F) \simeq_{cB} (S \restriction B, E \restriction B)$.

Again, when the equivalence relations F and E are clear from the context we just say that R is classwise Borel isomorphic to or embeddable into S, and write $R \simeq_{cB} S$ or $R \sqsubseteq_{cB} S$.

The relevance of these definitions lies in the observation that if (R,F) is invariantly universal and $(R,F) \sqsubseteq_{cB} (S,E)$ then (S,E) is invariantly universal as well. This will be used e.g. for proving Theorem 4.15 and Theorem 6.4.

We end this section by looking at Polish ultrametric preserving functions, which will be used later. Pongsriiam and Termwuttipong ([PT14]) studied *ultrametric preserving functions* (i.e. functions f such that for every ultrametric d on a space X, $f \circ d$ is still an ultrametric on X) and showed that $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is ultrametric preserving if and only if it is non-decreasing and such that $f^{-1}(0) = \{0\}$. For $f : \mathbb{R}^+ \to \mathbb{R}^+$ to send Polish ultrametrics into Polish ultrametrics it is necessary and sufficient that f is an ultrametric preserving function continuous at 0 (notice that this implies that for every sequence $(x_n)_{n\in\omega}$ such that $\lim_{n\to\infty} f(x_n) = 0$ we have $\lim_{n\to\infty} x_n = 0$).

If we consider functions $f : A \to \mathbb{R}^+$ for some $A \subseteq \mathbb{R}^+$ (in particular when 0 is not an accumulation point of A) we need to strengthen the continuity condition. Thus $f : A \to \mathbb{R}^+$ is *Polish ultrametric preserving* (i.e. for every Polish ultrametric d on a space X which uses only distances in A, $f \circ d$ is still a Polish ultrametric on X) if and only if it is non-decreasing, such that $f^{-1}(0) = \{0\}$, and satisfies $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} f(x_n) = 0$ for every sequence $(x_n)_{n\in\omega}$ of elements of A.

3. Jump operators

By a *jump operator* we mean a mapping $S \mapsto S^J$ from a collection of binary relations into itself satisfying $S \leq_B S^J$ and $R \leq_B S \implies R^J \leq_B S^J$. Typically the jump is defined on equivalence relations or quasi-orders. When we study jump operators we are interested in finding conditions on S that imply $S <_B S^J$ (notice that if $S \leq_B R <_B S^J$ then $R <_B R^J$) and in using the jump to define transfinite sequences of binary relations which are increasing with respect to Borel reducibility.

3.1. **Rosendal's jump operator $S \mapsto S^{\mathrm{cf}}$.** A jump operator on equivalence relations $E \mapsto E^+$ was introduced by H. Friedman (see [Sta85]). If E is defined on X, then E^+ is defined on ${}^\omega X$ by letting

$$(x_n)_{n\in\omega} \; E^+ \; (y_n)_{n\in\omega} \iff (\forall n \exists m \, (x_n \; E \; y_m) \wedge \forall n \exists m \, (y_n \; E \; x_m)).$$

In [Fri00] (where E^+ is denoted $\mathrm{NCS}(X,E)$) it is proved that this jump operator gives rise to a transfinite sequence of equivalence relations that is strictly increasing with respect to $\leq_B$ and cofinal among Borel equivalence relations classifiable by countable structures (see also [Gao09, §12.2], where a variant of this sequence is called the Friedman-Stanley tower). This sequence is defined as follows. Let $T(0)$ be the equivalence relation $(\omega, =)$. For $\alpha < \omega_1$, let $T(\alpha+1) = (T(\alpha))^+$. For $\lambda < \omega_1$ limit, let $T(\lambda) = \sum_{\beta<\lambda} T(\beta)$, where this sum is the disjoint union of the equivalence relations $T(\beta)$.

Another well-known sequence of equivalence relations is obtained by considering isomorphism on well-founded trees of bounded rank. Let $\mathcal{T}_\alpha$ be the set of well-founded trees on ω of rank less than α: the equivalence relation of isomorphism on $\mathcal{T}_\alpha$ is quite easily seen to be Borel. H. Friedman and Stanley proved in [FS89] that for $0 < \alpha < \beta < \omega_1$, one has that isomorphism on $\mathcal{T}_\alpha$ is $<_B$ than isomorphism on $\mathcal{T}_\beta$. The following fact is well-known and follows e.g. from [Gao09, Theorem 13.2.5].

Proposition 3.1. *For every Borel equivalence relation E classifiable by countable structures there exists $\alpha < \omega_1$ such that E is Borel reducible to isomorphism on $\mathcal{T}_\alpha$.*

In [Ros05] Rosendal introduced the following jump operator $S \mapsto S^{\mathrm{cf}}$ on quasi-orders which is an analogue in this wider context of Friedman's operator on equivalence relations $E \mapsto E^+$ (in fact, with our notation $E^+ = E_{E^{\mathrm{cf}}}$).

Definition 3.2 ([Ros05, Definition 4])**.** Let (X,S) be a quasi-order. We denote by $(X,S)^{\mathrm{cf}}$ (or even just by S^{cf}, when the space X is clear from the context) the quasi-order on ${}^\omega X$ defined by

$$(x_n)_{n\in\omega} \; S^{\mathrm{cf}} \; (y_n)_{n\in\omega} \iff \forall n \exists m (x_n \; S \; y_m).$$

It is immediate to check that $S \mapsto S^{\mathrm{cf}}$ is indeed a jump operator. Moreover S is a Borel (respectively, analytic) quasi-order if and only if so is S^{cf}.

In [Ros05], the author used the jump operator $S \mapsto S^{\mathrm{cf}}$ to define an ω_1-sequence of Borel quasi-orders which is cofinal among Borel quasi-orders.

Definition 3.3 ([Ros05])**.** Let P_0 be the quasi-order $(\omega, =)$. For $\alpha < \omega_1$ let $P_{\alpha+1} = P_\alpha^{\mathrm{cf}}$. For $\lambda < \omega_1$ limit let $P_\lambda = \prod_{\beta<\lambda} P_\beta$, where the product of relations is the relation defined componentwise on the Cartesian product of the domains.

Theorem 3.4 ([Ros05, Corollary 15])**.** *The sequence $(P_\alpha)_{\alpha<\omega_1}$ is strictly $\leq_B$-increasing and cofinal among the Borel quasi-orders, i.e. for every Borel quasi-order S there is $\alpha < \omega_1$ such that $S \leq_B P_\alpha$.*

Notice that it immediately follows that the sequence $(E_{P_\alpha})_{\alpha<\omega_1}$ of the associated equivalence relations is cofinal among all Borel equivalence relations.

Remark 3.5. By monotonicity of $S \mapsto S^{\mathrm{cf}}$ and the fact that Rosendal's sequence is strictly increasing, if S is a Borel quasi-order such that $P_\alpha \leq_B S \leq_B P_{\alpha+1}$ for some $\alpha < \omega_1$, then $S <_B S^{\mathrm{cf}}$.

To prove that the sequence $(P_\alpha)_{\alpha<\omega_1}$ is $\leq_B$-strictly increasing, one argues by induction on $\alpha < \omega_1$ using in the successor step that if S is a Borel quasi-order on X containing S-incompatible elements (i.e. x and y such that for no z we have $z \; S \; x$ and $z \; S \; y$), then $S <_B S^{\mathrm{cf}}$ (see [Ros05, Proposition 6]). For some of the results of this paper, we need to slightly improve this technical result as follows.

Lemma 3.6. *Let S be a Borel quasi-order on X and suppose that one of the following conditions holds:*

(i) there exist S-incomparable elements x, y such that the restriction of S to $\{z \in X \mid z \; S \; x \wedge z \; S \; y\}$ is well-founded;

(ii) the quotient S/E_S has a well-founded infinite downward closed subset.

Then $S <_B S^{\mathrm{cf}}$.

Proof. To simplify the notation, given $x \in X$ denote by $x^\infty \in {}^\omega X$ the constant ω-sequence with value x.

We follow the ideas of the proof of [Ros05, Proposition 6]. Suppose $f \colon {}^\omega X \to X$ witnesses $S^{\mathrm{cf}} \leq_B S$. Recall that [Ros05, Proposition 5] asserts that for every $\vec{x} = (x_n)_{n\in\omega} \in {}^\omega X$ there is $k \in \omega$ such that $f(\vec{x}) \; S \; x_k$.

Assume first that (i) holds and let x, y be incomparable elements with a well-founded set of common predecessors. Then $f(x^\infty) \; S \; x$ and $f(y^\infty) \; S \; y$; moreover, either $f(xy^\infty) \; S \; x$ or $f(xy^\infty) \; S \; y$. Since $x^\infty \; S^{\mathrm{cf}} \; xy^\infty$ and $y^\infty \; S^{\mathrm{cf}} \; xy^\infty$, one has $f(x^\infty) \; S \; f(xy^\infty)$ and $f(y^\infty) \; S \; f(xy^\infty)$. So at least one of $f(x^\infty), f(y^\infty)$ is a predecessor of both x and y. Suppose, for instance, $f(x^\infty) \; S \; x$ and $f(x^\infty) \; S \; y$ and let $z_0 = f(x^\infty)$. Now notice that z_0^∞ strictly precedes x^∞ under S^{cf} (because $x \not\!S \; y$ while $z_0 \; S \; y$), thus $z_1 = f(z_0^\infty)$ is a strict predecessor of $z_0 = f(x^\infty)$ under S. Arguing by induction in a similar way, we build a strictly decreasing sequence $z_n = f(z_{n-1}^\infty)$ of common predecessors of x and y.

If (ii) holds, let $Y \subseteq X/E_S$ be infinite, downward closed, and well-founded, as in the case assumption. If Y contains two S/E_S-incomparable elements, then we are done by case (i). Hence we can assume without loss of generality that Y is a well-order and, since it is infinite, that it contains an initial segment isomorphic to ω. Let $x_0 \; S \; x_1 \; S \; \dots$ be a strictly increasing sequence in X of representatives of such an initial segment, so that $x_0^\infty, x_1^\infty, \dots, x_\omega = (x_i)_{i\in\omega}$ is a strictly increasing sequence with respect to S^{cf}. Let $n \in \omega$ be such that $f(x_\omega) \; S \; x_n$. Then $(f(x_k^\infty))_{k\in\omega}$ would constitute a strictly increasing sequence with respect to S bounded by x_n, which is impossible. □

Remark 3.7. (1) Conditions (i) and (ii) of Lemma 3.6 are sufficient but not necessary for having $S <_B S^{\mathrm{cf}}$. To see this, consider a Borel quasi-order S whose quotient S/E_S is a copy of ω^* (the ordinal ω equipped with the reverse order) together with two incomparable elements above it.

(2) There are Borel quasi-orders S such that $S \sim_B S^{\mathrm{cf}}$: for example, let S be such that its quotient is isomorphic to a reverse well-ordering.

One may wonder whether the weaker conditions of Lemma 3.6 may be used instead of [Ros05, Proposition 6] to recursively build a $\leq_B$-increasing and cofinal ω_1-sequence of Borel quasi-orders similar to $(P_\alpha)_{\alpha<\omega_1}$. Towards this end, we first need to check that the disjunction of the conditions (i) and (ii) of Lemma 3.6 is preserved by the operator $S \mapsto S^{\mathrm{cf}}$ and by countable products. To simplify the presentation, a quasi-order satisfying such a disjunction will be called *suitable*.

Lemma 3.8. *Let S, $(S_\beta)_{\beta<\lambda}$ (for some limit $\lambda<\omega_1$) be suitable Borel quasi-orders. Then both S^{cf} and $\prod_{\beta<\lambda} S_\beta$ are suitable as well.*

Proof. Assume first that S satisfies condition (i) of Lemma 3.6, let $x, y \in X$ be S-incomparable and such that $A = \{[z] \in X/E_S \mid z \mathrel{S} x \wedge z \mathrel{S} y\}$ is well-founded. Without loss of generality, we may assume that A is actually linearly ordered: if not, simply replace x, y with $x', y' \in X$ such that

- $[x'], [y'] \in A$;
- $x', y' \in X$ are S-incomparable;
- the pair (x', y') is $S \times S$-minimal among pairs with the above two properties (such a minimal pair exists because A is well-founded).

So A is a well-order of order type some ordinal α and x^∞, y^∞ are S^{cf}-incomparable. Since S is Borel, by the boundedness theorem for analytic well founded relations ([Kec95, Theorem 31.1]), it follows that $\alpha<\omega_1$. Let $B = \{[\vec{z}] \in {}^\omega X/E_{S^{\mathrm{cf}}} \mid \vec{z} \mathrel{S^{\mathrm{cf}}} x^\infty \wedge \vec{z} \mathrel{S^{\mathrm{cf}}} y^\infty\}$, so that, in particular, for all $[\vec{z}] \in B$ the E_S-classes of all coordinates of $\vec{z}$ belong to A: we claim that B is a well-order too, so that S^{cf} satisfies condition (i) of Lemma 3.6. Indeed, if $[\vec{z}] \in B$, there are two possibilities: either there is a component in $\vec{z}$ that S-dominates all other components, in which case $\vec{z}$ is $E_{S^{\mathrm{cf}}}$-equivalent to the constant sequence with values that coordinate; or there is no S-biggest component of $\vec{z}$, so that $\vec{z}$ is S^{cf}-above all constant sequences built using its components and is S-below any sequence mentioning at least one component S-bigger than all components of $\vec{z}$. From these observations it easily follows that B is a well-order, and in fact the order type of B can be obtained from α by adding a point p_λ, for any limit ordinal $\lambda \leq \alpha$, on top of the subset $\lambda \subseteq \alpha$ (this order type is consequently either α or $\alpha+1$).

Suppose now that S satisfies condition (ii) of Lemma 3.6, and let $A \subseteq X/E_S$ be the set satisfying the condition. If A contains incomparable elements, then actually condition (i) holds for S and thus for S^{cf}. Otherwise A is a well-order. Repeating the argument of case (i), $\{[\vec{z}] \in {}^\omega X/E_{S^{\mathrm{cf}}} \mid \forall n \in \omega\ [z_n] \in A\}$ witnesses that S^{cf} fulfills condition (ii).

Let us now consider the case of countable products. Let S_β, $\beta<\lambda$, be suitable quasi-orders. If for some $\gamma<\lambda$ there are S_γ-incomparable elements a_γ and b_γ without a common S_γ-predecessor, then for every $\alpha<\lambda$ distinct from γ we fix an arbitrary element $a_\alpha = b_\alpha \in X_\alpha = \mathrm{dom}(S_\alpha)$ and we consider the sequences $\vec{a} = (a_\beta)_{\beta<\lambda}$ and $\vec{b} = (b_\beta)_{\beta<\lambda}$: clearly $\vec{a}$ and $\vec{b}$ are $\prod_{\beta<\lambda} S_\beta$-incomparable and they do not have a common $\prod_{\beta<\lambda} S_\beta$-predecessor, so that $\prod_{\beta<\lambda} S_\beta$ satisfies condition (i) of Lemma 3.6. Otherwise, for all $\beta<\lambda$ there is an S_β-minimal element a_β with some b_β strictly S_β-above a_β. Then the sequences $(a_0, b_1, a_2, a_3, \dots)$ and $(b_0, a_1, a_2, a_3, \dots)$ are $\prod_{\beta<\lambda} S_\beta$-incomparable, and the set of their common predecessors consists just of the equivalence class of $(a_0, a_1, a_2, \dots)$. Therefore $\prod_{\beta<\lambda} S_\beta$ satisfies condition (i) of Lemma 3.6 again. □

This allows us to show that in constructing Rosendal's cofinal sequence of Borel quasi-orders, we could have started with any nontrivial Borel quasi-order (instead of a Borel quasi-order satisfying the stronger assumption of [Ros05, Proposition 6]).

Proposition 3.9. *Let S be a Borel quasi-order, and recursively define the sequence $(S_\alpha)_{\alpha<\omega_1}$ by setting $S_0 = S$, $S_{\alpha+1} = S_\alpha^{\mathrm{cf}}$, and $S_\lambda = \prod_{\beta<\lambda} S_\beta$ for $\lambda<\omega_1$ limit. Then*

(1) if E_S has at least two equivalence classes, then the sequence $(S_\alpha)_{\alpha<\omega_1}$ is cofinal among the Borel quasi-orders;
(2) if moreover S is suitable, then $(S_\alpha)_{\alpha<\omega_1}$ is also strictly $\leq_B$-increasing.

Proof. To prove (1), by Theorem 3.4 it clearly suffices to show that $P_0 \leq_B S_{\omega+\omega}$, that is that there is an infinite family $(\vec{z}^{\,i})_{i\in\omega}$ of pairwise $S_{\omega+\omega}$-incomparable elements.

With an argument as the one used in the proof of the second part of Lemma 3.8, we get that S_ω contains at least two incomparable elements a_ω, b_ω. Moreover, since if x, y are incomparable with respect to a quasi-order R then x^∞, y^∞ are R^{cf}-incomparable, we also get that for each $n \in \omega$ there are $S_{\omega+n}$-incomparable elements $a_{\omega+n}$ and $b_{\omega+n}$. Finally, for $n < \omega$ we pick an arbitrary element $a_n \in X_n = \mathrm{dom}(S_n)$. For $i \in \omega$ and $\beta < \omega + \omega$ set $z^i_\beta = a_\beta$ if $\beta \neq \omega + i$ and $z^i_{\omega+i} = b_{\omega+i}$. Then the sequence obtained setting $\vec{z}^{\,i} = (z^i_\beta)_{\beta<\omega+\omega}$ is as required.

Part (2) follows from Lemmas 3.6 and 3.8. □

3.2. **The jump operator $S \mapsto S^{\mathrm{inj}}$.** Let us now consider the following variant of Rosendal's jump operator.

Definition 3.10. Let (X, S) be a quasi-order. We denote by $(X, S)^{\mathrm{inj}}$ (or just by S^{inj}, if the space X is clear from the context) the quasi-order on ${}^\omega X$ defined by

$$(x_n)_{n\in\omega} \; S^{\mathrm{inj}} \; (y_n)_{n\in\omega} \iff \exists f\colon \omega \to \omega \text{ injective such that } \forall n (x_n \; S \; y_{f(n)}).$$

Similarly to what happened for Rosendal's jump $S \mapsto S^{\mathrm{cf}}$, also the operator $S \mapsto S^{\mathrm{inj}}$ has already been implicitly considered in the literature. For example, in [FS89, §1.2.2] H. Friedman and Stanley used a jump operator for equivalence relations $E \mapsto E^\omega$ such that $E^\omega = E_{E^{\mathrm{inj}}}$, as the following lemma shows.

Lemma 3.11. *If E is an equivalence relation on X, then*

$$(x_n)_{n\in\omega} \; E_{E^{\mathrm{inj}}} \; (y_n)_{n\in\omega} \iff \exists f\colon \omega \to \omega \textit{ bijective } \forall n (x_n \; E \; y_{f(n)}).$$

Proof. The implication from right to left is immediate. Conversely, suppose there are injections $\omega \to \omega$ witnessing $(x_n)_{n\in\omega} \; E^{\mathrm{inj}} \; (y_n)_{n\in\omega}$ and $(y_n)_{n\in\omega} \; E^{\mathrm{inj}} \; (x_n)_{n\in\omega}$. This entails that for each E-equivalence class C,

$$|\{n \in \omega \mid x_n \in C\}| = |\{n \in \omega \mid y_n \in C\}|$$

implying the existence of f. □

The following result ([FS89, §1.2.2]), reformulated with our terminology using the observation above, will be used in the sequel.

Proposition 3.12. *If E is a Borel equivalence relation with at least two classes, then $E_{E^{\mathrm{inj}}}$ is Borel and $E <_B E_{E^{\mathrm{inj}}}$.*

Coming back to our operator for quasi-orders $S \mapsto S^{\mathrm{inj}}$, it is immediate to check that $S \leq_B S^{\mathrm{cf}} \leq_B S^{\mathrm{inj}}$, that S is an analytic quasi-order if and only if so is S^{inj} (but the analogous statement with analytic replaced by Borel is not true, see Propositions 3.22 and 3.23), and that $S \mapsto S^{\mathrm{inj}}$ is monotone with respect to $\leq_B$, i.e. it is really a jump operator. From these observations and Lemma 3.6 it immediately follows that in many cases the new jump operator strictly increases the complexity of the quasi-order which it is applied to.

Corollary 3.13. *If S is a suitable Borel quasi-order (i.e. S satisfies the hypothesis of Lemma 3.6), then $S <_B S^{\mathrm{inj}}$.*

We now start a detailed analysis of the new jump operator $S \mapsto S^{\mathrm{inj}}$, and compare it to the one introduced by Rosendal. In particular, we show the following.

(A) For extremely simple quasi-orders S, like equality on an arbitrary standard Borel space, we get $S^{\mathrm{cf}} \sim_B S^{\mathrm{inj}}$ (Proposition 3.16).
(B) If a Borel quasi-order S is combinatorially simple, like an equivalence relation or a wqo, then S^{inj} remains Borel (Corollaries 3.15 and 3.21).
(C) There are not too complicated Borel quasi-orders S (e.g. any linear order on ω which is isomorphic to $\mathbb{Q}$, or the inclusion relation $\subseteq$ on $\mathscr{P}(\omega)$) such that S^{inj} is analytic non-Borel. In fact, we can have both upper and lower S^{inj}-cones which are $\boldsymbol{\Sigma}^1_1$-complete, and also the associated equivalence relation $E_{S^{\mathrm{inj}}}$ can be analytic non-Borel (Propositions 3.22 and 3.23).
(D) However, if S is Borel then all $E_{S^{\mathrm{inj}}}$-equivalence classes are Borel (Theorem 3.31). In particular, S^{inj} is not complete for analytic quasi-orders.
(E) Moreover, there are also Borel quasi-orders S such that S^{inj} is analytic non-Borel but their associated equivalence relation $E_{S^{\mathrm{inj}}}$ still remains Borel (Corollary 3.33).
(F) There exist Borel quasi-orders S such that $S^{\mathrm{cf}} <_B S^{\mathrm{inj}}$ but $E_{S^{\mathrm{cf}}} \sim_B E_{S^{\mathrm{inj}}}$ (Proposition 3.35).

Most of these results will be used to analyze the isometric embeddability relation between ultrametric Polish spaces with well-ordered set of distances. However, some of them are also of independent interest. For instance, Corollary 3.33 provides simple combinatorial examples of an analytic quasi-order S such that its associated equivalence relation E_S is Borel without S being Borel itself. Proposition 3.23 and Theorem 3.31 also allow us to construct new examples of analytic quasi-orders S which are not Borel but still have the property that $S <_B S^{\mathrm{cf}}$ (Corollary 3.37): these examples are of a different type with respect to those considered in [CM07, Section 4], as they do not satisfy the hypothesis of [CM07, Corollary 4.3].

3.3. Borel quasi-orders S such that S^{inj} is Borel.

Lemma 3.14. *If E is a Borel equivalence relation then $E^{\mathrm{inj}} \leq_B (E \times (\omega, =))^{\mathrm{cf}}$.*

Proof. Suppose E is defined on the space X. Given $\vec{x} = (x_i)_{i \in \omega} \in {}^{\omega}X$, set $\varphi(\vec{x}) = (x_i, n_i)_{i \in \omega} \in {}^{\omega}(X \times \omega)$, where n_i is the number of $j < i$ with $x_j \mathrel{E} x_i$. As E is Borel the function φ is Borel.

Assume first that the function $f \colon \omega \to \omega$ witnesses $\vec{x} = (x_i)_{i \in \omega} \mathrel{E^{\mathrm{inj}}} (y_j)_{j \in \omega} = \vec{y}$. Let $\varphi(\vec{x}) = (x_i, n_i)_{i \in \omega}$ and $\varphi(\vec{y}) = (y_j, m_j)_{j \in \omega}$. Since f is injective, given $l \in \omega$ there are at least $n_l + 1$ indices j such that $y_j \mathrel{E} y_{f(l)}$, for exactly one of which $m_j = n_l$. So (x_l, n_l) and (y_j, m_j) are $(E \times =)$-related for this j, which implies that $\varphi(\vec{x}) \mathrel{(E \times =)^{\mathrm{cf}}} \varphi(\vec{y})$.

Conversely, suppose for some $\vec{x} = (x_i)_{i \in \omega}, \vec{y} = (y_j)_{j \in \omega} \in {}^{\omega}X$ we have that $\varphi(\vec{x}) = (x_i, n_i)_{i \in \omega}$ and $\varphi(\vec{y}) = (y_j, m_j)_{j \in \omega}$ are $(E \times =)^{\mathrm{cf}}$-related, so that in particular

$$\forall l \in \omega \, \exists k_l \in \omega \, (x_l \mathrel{E} y_{k_l} \wedge n_l = m_{k_l}).$$

Then the map $f \colon \omega \to \omega \colon l \mapsto k_l$ is injective and witnesses $\vec{x} \mathrel{E^{\mathrm{inj}}} \vec{y}$. $\square$

Corollary 3.15. *If E is a Borel equivalence relation, then E^{inj} is Borel as well.*

Another consequence of Lemma 3.14 is that there are some simple cases in which $S^{\mathrm{cf}} \sim_B S^{\mathrm{inj}}$.

Proposition 3.16. *(1)* $(\omega, =)^{\mathrm{inj}} \sim_B (\omega, =)^{\mathrm{cf}} \sim_B (\mathscr{P}(\omega), \subseteq)$*;*
(2) $(\mathbb{R}, =)^{\mathrm{inj}} \sim_B (\mathbb{R}, =)^{\mathrm{cf}}$*.*

Proof. All Borel reducibilities are trivial except perhaps for those of the form $E^{\mathrm{inj}} \leq_B E^{\mathrm{cf}}$, where E is one of $(\omega, =)$ or $(\mathbb{R}, =)$: these can be proved using Lemma 3.14 together with the fact that in this specific cases $E \times (\omega, =) \sim_B E$. □

More generally, by Lemma 3.14 we have that $E^{\mathrm{inj}} \sim_B E^{\mathrm{cf}}$ whenever E is a Borel equivalence relation satisfying $E \times (\omega, =) \leq_B E$. This includes e.g. the well-known equivalence relations E_0, E_1, and so on.

Corollary 3.15 provides a first example illustrating the fact that when applied to combinatorially simple quasi-orders the operator $S \mapsto S^{\mathrm{inj}}$ preserves Borelness. Another example of this kind is when S is a wqo. To see this, we first need to prove some easy facts concerning definable wqo's.

Lemma 3.17. *If S is a well-founded analytic quasi-order without uncountable antichains, then $(\mathbb{R}, =) \not\leq_B E_S$. Hence E_S has at most $\aleph_1$-many classes, and if S is Borel it has countably many classes.*

Proof. Towards a contradiction, let $f \colon \mathbb{R} \to X$ be a Borel function such that either $f(x) \not\mathrel{S} f(y)$ or $f(y) \not\mathrel{S} f(x)$ for all distinct $x, y \in \mathbb{R}$. Let $\preceq$ be defined on $\mathbb{R}$ by

$$x \preceq y \iff f(x) \mathrel{S} f(y).$$

Then $\preceq$ is an analytic well-founded partial order. Using the boundedness theorem for analytic well founded relations ([Kec95, Theorem 31.1]), $\preceq$ has countable rank. So there are uncountably many reals having the same rank with respect to $\preceq$. But this is impossible as it would yield an uncountable antichain for $\preceq$ and then for S.

The additional part follows from Burgess' trichotomy theorem for analytic equivalence relations ([Bur79, Corollary 2]) and Silver's dichotomy theorem for coanalytic equivalence relations ([Sil80]). □

It is not hard to check that if S is an analytic quasi-order on a standard Borel space X such that E_S has at most countably many classes, then S is actually Borel. This fact combined with Lemma 3.17 gives the following result.

Corollary 3.18. *Let S be a well-founded analytic quasi-order on X without uncountable antichains. Then S is Borel if and only if E_S has countably many classes.*

In particular, both Lemma 3.17 and Corollary 3.18 apply to analytic wqo's.

The proof of the next proposition is quite similar to that of [NW65, Lemma 10].

Proposition 3.19. *If S is a wqo on ω, then S^{inj} is Borel.*

Proof. For every $\vec{y} = (y_n)_{n \in \omega} \in {}^{\omega}\omega$ and $n \in \omega$, let $\vec{y}^{\,\uparrow n}$ denote the S-upper cone determined by n in $\vec{y}$, i.e. $\vec{y}^{\,\uparrow n} = \{m \in \omega \mid y_n \mathrel{S} y_m\}$. Then the set $F_{\vec{y}} = \{n \in \omega \mid \vec{y}^{\,\uparrow n} \text{ is finite}\}$ is finite by [NW65, Lemma 3]. Let $K_{\vec{y}} = \omega \setminus F_{\vec{y}}$.

Claim 3.19.1. *For every $n \in K_{\vec{y}}$ the set $\vec{y}^{\,\uparrow n}_{K_{\vec{y}}} = \{m \in K_{\vec{y}} \mid y_n \mathrel{S} y_m\}$ is infinite.*

Proof. Let $n \in \omega$ be such that $\vec{y}^{\,\uparrow n}_{K_{\vec{y}}}$ is finite. Then $\vec{y}^{\,\uparrow n} \subseteq \vec{y}^{\,\uparrow n}_{K_{\vec{y}}} \cup F_{\vec{y}}$ is finite as well, i.e. $n \in F_{\vec{y}}$. □

Given $\vec{x}, \vec{y} \in {}^{\omega}\omega$, let $K_{\vec{x},\vec{y}} = \{n \in \omega \mid \exists m \in K_{\vec{y}}\, (x_n \mathrel{S} y_m)\}$.

Claim 3.19.2. *There is an injective function $f \colon K_{\vec{x},\vec{y}} \to K_{\vec{y}}$ such that $x_n \mathrel{S} y_{f(n)}$ for every $n \in K_{\vec{x},\vec{y}}$.*

Proof. We define the function f by recursion on $n \in K_{\vec{x},\vec{y}}$. Notice that it is enough to verify that for every $n \in \omega$ the restriction $f \restriction (n \cap K_{\vec{x},\vec{y}})$ of f is injective and such that $x_k \mathrel{S} y_{f(k)}$ for every $k \in n \cap K_{\vec{x},\vec{y}}$. Suppose that $n \in K_{\vec{x},\vec{y}}$ and that $f \restriction (n \cap K_{\vec{x},\vec{y}})$ is as above. Let $m \in K_{\vec{y}}$ be such that $x_n \mathrel{S} y_m$ (which exists because $n \in K_{\vec{x},\vec{y}}$). By Claim 3.19.1 there are infinitely many $m' \in K_{\vec{y}}$ such that $y_m \mathrel{S} y_{m'}$: letting $f(n)$ be the least of these m' such that $m' \notin f(n \cap K_{\vec{x},\vec{y}})$, we get $x_n \mathrel{S} y_{f(n)}$ (by transitivity of S), and that $f \restriction (n+1 \cap K_{\vec{x},\vec{y}})$ is as required. $\square$

The next claim shows that for having $\vec{x} \mathrel{S^{\mathrm{inj}}} \vec{y}$ it is necessary and sufficient that there is a partial witness of this fact defined on $\omega \setminus K_{\vec{x},\vec{y}}$.

Claim 3.19.3. *$\vec{x} \mathrel{S^{\mathrm{inj}}} \vec{y}$ if and only if there is an injective function $g \colon \omega \setminus K_{\vec{x},\vec{y}} \to F_{\vec{y}}$ such that $x_n \mathrel{S} y_{g(n)}$ for every $n \in \omega \setminus K_{\vec{x},\vec{y}}$.*

Proof. For the forward direction, let f be a witness of $\vec{x} \mathrel{S^{\mathrm{inj}}} \vec{y}$, and let $g = f \restriction (\omega \setminus K_{\vec{x},\vec{y}})$: it suffices to show that $g(n) = f(n) \in F_{\vec{y}}$ for every $n \in \omega \setminus K_{\vec{x},\vec{y}}$. Since $x_n \mathrel{S} y_{f(n)}$, this follows from the definition of $K_{\vec{x},\vec{y}}$.

Conversely, let g be as in the statement of the claim, and let f be the map obtained in Claim 3.19.2: then $f \cup g$ witnesses $\vec{x} \mathrel{S^{\mathrm{inj}}} \vec{y}$. To see this, it suffices to show that if $n \in K_{\vec{x},\vec{y}}$ and $m \in \omega \setminus K_{\vec{x},\vec{y}}$ then $f(n) \neq g(m)$, and this follows from the fact that the range of f is contained in $K_{\vec{y}} = \omega \setminus F_{\vec{y}}$, while the range of g is contained in $F_{\vec{y}}$. $\square$

By definitions, Claim 3.19.3, and the fact that $F_{\vec{y}}$ is finite, $\vec{x} \mathrel{S^{\mathrm{inj}}} \vec{y}$ is equivalent to

$$\omega \setminus K_{\vec{x},\vec{y}} \text{ is finite} \wedge \exists g \text{ injective} \colon \omega \setminus K_{\vec{x},\vec{y}} \to F_{\vec{y}}\, \forall n \in \omega \setminus K_{\vec{x},\vec{y}}\, (x_n \mathrel{S} y_{g(n)}).$$

Since the maps $\vec{y} \mapsto F_{\vec{y}}$ and $(\vec{x}, \vec{y}) \mapsto K_{\vec{x},\vec{y}}$ are Borel, this equivalence shows that S^{inj} is Borel. $\square$

Remark 3.20. (1) If S is further assumed to be a bqo on ω, then the fact that S^{inj} is Borel follows also from Corollary 3.18 and a result of Laver ([Lav71, Theorem 4.11]). In fact in this case S^{inj} is also a bqo by Nash-Williams' theorem on transfinite sequences ([NW68]).

(2) If moreover S is linear (i.e. a well-order on ω) then:
   - (a) $S^{\mathrm{inj}} <_B (\mathscr{P}(\omega), \subseteq)$ because $f \colon {}^{\omega}\omega \to \mathscr{P}(\omega)$ defined by $f(\vec{x}) = \{\langle n, k\rangle \mid |\{i \mid n \mathrel{S} x_i\}| \geq k\}$, where $(n, k) \mapsto \langle n, k\rangle$ is a pairing function, witnesses $S^{\mathrm{inj}} \leq_B (\mathscr{P}(\omega), \subseteq)$, while $(\mathscr{P}(\omega), \subseteq) \nleq_B S^{\mathrm{inj}}$ follows from the previous point because S and thus S^{inj} are bqo's;
   - (b) if S' is another linear wqo on ω with order type strictly larger than that of S, then $S^{\mathrm{inj}} <_B (S')^{\mathrm{inj}}$ (this can be proved using the fact that every wqo contains a chain of maximal order type [Wol67]).

   In particular $(\omega, \leq)^{\mathrm{inj}} <_B (\omega+1, \leq)^{\mathrm{inj}} <_B \ldots <_B (\alpha, \leq)^{\mathrm{inj}} <_B \ldots <_B (\mathscr{P}(\omega), \subseteq)$ is a strictly increasing chain of Borel quasi-orders of length $\omega_1 + 1$.

Corollary 3.21. *Let S be a Borel quasi-order. If S is a wqo, then S^{inj} is Borel as well.*

Proof. Let S be a Borel wqo on X. By Corollary 3.18, E_S has at most countably many classes. Let $(A_i)_{i<I}$ (for some $I \leq \omega$) be an enumeration without repetitions of the E_S-equivalence classes, and consider the wqo $\tilde{S}$ on ω defined by letting $n \; \tilde{S} \; m$ if and only if one of the following conditions holds:

- $n, m < I \wedge \forall x \in A_n \forall y \in A_m \, (x \; S \; y)$; or
- $n \in \omega \setminus I \wedge m < I \wedge \forall x \in A_0 \forall y \in A_m \, (x \; S \; y)$; or
- $n < I \wedge m \in \omega \setminus I \wedge \forall x \in A_n \forall y \in A_0 \, (x \; S \; y)$; or
- $n, m \in (\omega \setminus I) \cup \{0\}$.

Since clearly $S \sim_B \tilde{S}$, it suffices to show that $\tilde{S}^{\mathrm{inj}}$ is Borel: but this follows from Proposition 3.19, hence we are done. □

3.4. **Borel quasi-orders S such that S^{inj} is proper analytic.** We now start considering some examples of Borel quasi-orders (X, S) such that $(X, S)^{\mathrm{inj}}$ is analytic non-Borel: as customary in the subject, this is usually obtained by either directly showing that the binary relation S^{inj} is a $\mathbf{\Sigma}^1_1$-complete subset of the square ${}^\omega X \times {}^\omega X$, or else by showing that the *upper cone generated by* $\vec{p}$

$$C^{S^{\mathrm{inj}}}(\vec{p}) = \{\vec{x} \in {}^\omega X \mid \vec{p} \; S^{\mathrm{inj}} \; \vec{x}\}$$

or the *lower cone generated by* $\vec{q}$

$$C_{S^{\mathrm{inj}}}(\vec{q}) = \{\vec{x} \in {}^\omega X \mid \vec{x} \; S^{\mathrm{inj}} \; \vec{q}\}$$

are $\mathbf{\Sigma}^1_1$-complete subsets of ${}^\omega X$ (for suitable $\vec{p}, \vec{q} \in {}^\omega X$).

Our first example shows that the jump operator $S \mapsto S^{\mathrm{inj}}$ can produce analytic non-Borel quasi-orders even when applied to countable linear orders (this should be contrasted with the case of countable well-orders, see Remark 3.20(2)).

Proposition 3.22. *Consider the linear order $(\mathbb{Q}, \leq)$. Then $\leq^{\mathrm{inj}}$, considered as a subset of ${}^\omega\mathbb{Q} \times {}^\omega\mathbb{Q}$ (where the latter is endowed with the product of the discrete topology on $\mathbb{Q}$), is $\mathbf{\Sigma}^1_1$-complete and hence non-Borel. In fact the equivalence relation $E_{(\mathbb{Q},\leq)^{\mathrm{inj}}}$ is a $\mathbf{\Sigma}^1_1$-complete subset of ${}^\omega\mathbb{Q} \times {}^\omega\mathbb{Q}$.*

Therefore, for every non-scattered countable linear order L the quasi-order L^{inj} is analytic non-Borel.

Proof. Since the map $\mathbb{Q} \to \mathbb{Q} \colon q \mapsto -q$ witnesses $(\mathbb{Q}, \leq) \sim_B (\mathbb{Q}, \geq)$, it is clearly enough to prove the result for $(\mathbb{Q}, \geq)^{\mathrm{inj}}$. To this end we define a continuous reduction of NWO to $\geq^{\mathrm{inj}}$, where NWO is the set of non-well-orders (viewed as a subset of the Polish space of countable linear orders, see [Kec95, Section 27.C]), a well-known $\mathbf{\Sigma}^1_1$-complete set.

We associate to every linear order L a pair of sequences $(\vec{\alpha}_L, \vec{\beta}_L)$ of rationals in the interval $[0, 1]$. First we map continuously and in an order preserving way L into $\mathbb{Q} \cap [0, 1)$, so that we can identify L with its image. Let $\vec{\alpha}_L$ and $\vec{\beta}_L$ be injective enumerations of $L \cup \{1\}$ and L, respectively. It is straightforward to check that $L \in \mathrm{NWO}$ if and only if $\vec{\alpha}_L \geq^{\mathrm{inj}} \vec{\beta}_L$.

Since $\vec{\beta}_L \geq^{\mathrm{inj}} \vec{\alpha}_L$ for every L, this argument shows in fact that $E_{(\mathbb{Q},\geq)^{\mathrm{inj}}}$ is a $\mathbf{\Sigma}^1_1$-complete subset of ${}^\omega\mathbb{Q} \times {}^\omega\mathbb{Q}$. □

In the second example, we consider the jump of the quasi-order $(\mathscr{P}(\omega), \subseteq)$, which by Proposition 3.16(1) is Borel-bireducible with $(\omega, =)^{\mathrm{inj}}$.

Proposition 3.23. *(1) The upper cone in $(\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}}$ generated by $\vec{p} = (A_n)_{n\in\omega} \in {}^{\omega}(\mathscr{P}(\omega))$ with $A_n = \{n\}$ is $\boldsymbol{\Sigma}^1_1$-complete. In particular, $\subseteq^{\mathrm{inj}}$ is a $\boldsymbol{\Sigma}^1_1$-complete (hence non-Borel) subset of ${}^{\omega}(\mathscr{P}(\omega)) \times {}^{\omega}(\mathscr{P}(\omega))$.*

(2) The lower cone in $(\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}}$ generated by $\vec{q} = (B_n)_{n\in\omega} \in {}^{\omega}(\mathscr{P}(\omega))$ with $B_n = \omega \setminus \{n\}$ is $\boldsymbol{\Sigma}^1_1$-complete.

(3) The equivalence relation $E_{(\mathscr{P}(\omega),\subseteq)^{\mathrm{inj}}}$ is a $\boldsymbol{\Sigma}^1_1$-complete subset of ${}^{\omega}(\mathscr{P}(\omega)) \times {}^{\omega}(\mathscr{P}(\omega))$.

Proof. Fix a bijection $\omega \to {}^{<\omega}\omega \colon n \mapsto s_n$ such that $s_0 = \emptyset$, and for every $n \in \omega$ such that $n > 0$ let $n^\star$ be the unique natural number such that $s_{n^\star} = s_n \restriction (\mathrm{length}(s_n) - 1)$.

(1) We show that there is a continuous function mapping each nonempty tree $T \subseteq {}^{<\omega}\omega$ to some $\vec{q}_T \in {}^{\omega}(\mathscr{P}(\omega))$ which reduces IF to the upper cone generated by $\vec{p}$, where $\mathrm{IF} \subseteq \mathscr{P}({}^{<\omega}\omega)$ is the set of trees with at least one infinite branch (a well-known $\boldsymbol{\Sigma}^1_1$-complete set, see [Kec95, Section 27.A]). Given a nonempty tree $T \subseteq {}^{<\omega}\omega$ let $\vec{q}_T = (B_{T,n})_{n\in\omega} \in {}^{\omega}(\mathscr{P}(\omega))$ be defined by

$$B_{T,n} = \begin{cases} \emptyset & \text{if } n = 0 \\ \{n\} & \text{if } s_n \notin T \\ \{n, n^\star\} & \text{if } n \neq 0 \text{ and } s_n \in T. \end{cases}$$

Notice that if there is $m \neq n$ such that $m \in B_{T,n}$, then $\emptyset \neq s_n \in T$ and $m = n^\star$. The map $T \mapsto \vec{q}_T$ is clearly continuous, and we claim that $T \in \mathrm{IF} \iff \vec{p} \subseteq^{\mathrm{inj}} \vec{q}_T$.

Let first $T \in \mathrm{IF}$, and let $(n_k)_{k\in\omega}$ be a sequence of natural numbers such that $s_{n_k} \in T$, $\mathrm{length}(s_{n_k}) = k$, and $s_{n_k} \subseteq s_{n_{k'}}$ for $k \leq k' \in \omega$, so that, in particular, $n_0 = 0$. Define $\varphi \colon \omega \to \omega$ by setting

$$\varphi(n) = \begin{cases} n_{k+1} & \text{if } n = n_k \text{ for some } k \in \omega \\ n & \text{if } n \neq n_k \text{ for all } k \in \omega. \end{cases}$$

It is then easy to check that φ witnesses $\vec{p} \subseteq^{\mathrm{inj}} \vec{q}_T$.

Conversely, let $\varphi \colon \omega \to \omega$ be a witness of $\vec{p} \subseteq^{\mathrm{inj}} \vec{q}_T$, and recursively set $n_0 = 0$ and $n_{k+1} = \varphi(n_k)$. Using the injectivity of φ and the fact that $A_i \not\subseteq \emptyset = B_{T,0}$ for every i, one can easily check by induction that $n_k \neq n_{k+1}$ and $n_{k+1} \neq 0$ for all $k \in \omega$. These facts, together with $\{n_k\} = A_{n_k} \subseteq B_{T,\varphi(n_k)} = B_{T,n_{k+1}}$, imply that $s_{n_{k+1}} \in T$ and $n_k = n^\star_{k+1}$ (whence $s_{n_k} \subsetneq s_{n_{k+1}}$) by the observation following the definition of $\vec{q}_T$. Thus $(s_{n_k})_{k\in\omega}$ is an infinite branch through T and $T \in \mathrm{IF}$.

(2) The proof is similar to that of (1). Given a nonempty tree $T \subseteq {}^{<\omega}\omega$, let $\vec{p}_T = (A_{T,n})_{n\in\omega} \in {}^{\omega}(\mathscr{P}(\omega))$ be defined by

$$A_{T,n} = \begin{cases} \omega \setminus \{n\} & \text{if } s_n \notin T \\ \omega \setminus \{m \in \omega \mid s_m \in T \wedge m^\star = 0\} & \text{if } n = 0 \\ \omega \setminus (\{m \in \omega \mid s_m \in T \wedge m^\star = n\} \cup \{n\}) & \text{if } s_n \in T \text{ and } n \neq 0. \end{cases}$$

It is straightforward to check that exactly the same argument used in (1) shows $T \in \mathrm{IF} \iff \vec{p}_T \subseteq^{\mathrm{inj}} \vec{q}$.

(3) This follows again from a minor modification of the construction given in part (1). Given $n \in \omega$, let $\mathrm{pred}(n) = \{m \in \omega \mid s_m \subseteq s_n\}$. To a given infinite tree $T \subseteq {}^{<\omega}\omega$ associate the pair $(\vec{p}_T, \vec{q}_T)$, where $\vec{p}_T \in {}^{\omega}(\mathscr{P}(\omega))$ lists all sets in $\{\mathrm{pred}(n) \mid s_n \in T\}$, and $\vec{q}_T \in {}^{\omega}(\mathscr{P}(\omega))$ is the same as $\vec{p}_T$ except for omitting $\{0\}$, that is $\vec{q}_T$

lists the elements of $\{\mathrm{pred}(n) \mid s_n \in T \wedge n \neq 0\}$. Then $\vec{q}_T \subseteq^{\mathrm{inj}} \vec{p}_T$ for any infinite tree T, while arguing as in (1) one easily sees that $T \in \mathrm{IF} \iff \vec{p}_T \subseteq^{\mathrm{inj}} \vec{q}_T$. $\square$

Remark 3.24. The proof of Proposition 3.23(1) actually shows that the upper cone determined by $\vec{p}$ remains $\mathbf{\Sigma}^1_1$-complete even if we replace $\subseteq^{\mathrm{inj}}$ with its restriction to subsets of ω of size at most 2. Therefore we further have that for every $2 < N \leq \omega$, the analytic quasi-order $(\mathscr{P}_{<N}(\omega), \subseteq)^{\mathrm{inj}}$ still contains $\mathbf{\Sigma}^1_1$-complete upper cones.

Moreover, the proof of Proposition 3.23(3) shows that $E_{(\mathscr{P}_{<\omega}(\omega),\subseteq)^{\mathrm{inj}}}$ is already a $\mathbf{\Sigma}^1_1$-complete subset of ${}^{\omega}(\mathscr{P}_{<\omega}(\omega)) \times {}^{\omega}(\mathscr{P}_{<\omega}(\omega))$.

3.5. **The complexity of the equivalence relation associated to S^{inj}.** Throughout this subsection we fix a quasi-order S on a set X.

Definition 3.25. Let $\vec{a} = (a_n)_{n\in\omega} \in {}^{\omega}X$. By transfinite recursion we define for every $\alpha < \omega_1$ a set $I^{\vec{a}}_\alpha \subseteq \omega$ as follows:

- $I^{\vec{a}}_0 = \omega$;
- $I^{\vec{a}}_{\alpha+1} = \{n \in I^{\vec{a}}_\alpha \mid \exists^\infty m \in I^{\vec{a}}_\alpha \, (a_n \; S \; a_m)\}$;
- $I^{\vec{a}}_\lambda = \bigcap_{\alpha<\lambda} I^{\vec{a}}_\alpha$ when λ is limit.

It is obvious that $\alpha < \beta$ implies $I^{\vec{a}}_\alpha \supseteq I^{\vec{a}}_\beta$, and hence the sequence must stabilize at some countable ordinal.

Definition 3.26. For $\vec{a} \in {}^{\omega}X$ let $\rho(\vec{a})$ be the least $\alpha < \omega_1$ such that $I^{\vec{a}}_{\alpha+1} = I^{\vec{a}}_\alpha$. We write $I^{\vec{a}}$ in place of $I^{\vec{a}}_{\rho(\vec{a})}$, so that $I^{\vec{a}} = \bigcap_{\alpha<\omega_1} I^{\vec{a}}_\alpha$.

The following lemma follows immediately from the definitions.

Lemma 3.27. *Let $\vec{a} \in {}^{\omega}X$. Then*

(1) for all $\alpha < \omega_1$ and $n \in I^{\vec{a}}_\alpha$, if $a_m \; S \; a_n$ then $m \in I^{\vec{a}}_\alpha$;
(2) if $n \notin I^{\vec{a}}$ and $\alpha < \omega_1$ is such that $n \in I^{\vec{a}}_\alpha \setminus I^{\vec{a}}_{\alpha+1}$, then

$$\{m \in \omega \mid a_n \; E_S \; a_m\} = \{m \in I^{\vec{a}}_\alpha \mid a_n \; E_S \; a_m\},$$

and this set is finite;
(3) if $n \in I^{\vec{a}}$ then $\exists^\infty m \in I^{\vec{a}} \, (a_n \; S \; a_m)$.

In particular, $I^{\vec{a}}$ is either empty or infinite, and each $I^{\vec{a}}_\alpha$ is invariant under E_S, that is: if $n, m \in \omega$ are such that $a_n \; E_S \; a_m$, then $n \in I^{\vec{a}}_\alpha \iff m \in I^{\vec{a}}_\alpha$.

Lemma 3.28. *Let $\vec{a}, \vec{b} \in {}^{\omega}X$ and suppose $f, g \colon \omega \to \omega$ are injective functions witnessing respectively $\vec{a} \; S^{\mathrm{inj}} \; \vec{b}$ and $\vec{b} \; S^{\mathrm{inj}} \; \vec{a}$. Then*

(1) for every $\alpha < \omega_1$ we have $\forall n (n \in I^{\vec{a}}_\alpha \iff f(n) \in I^{\vec{b}}_\alpha)$ and $\forall n (n \in I^{\vec{b}}_\alpha \iff g(n) \in I^{\vec{a}}_\alpha)$ (so that, in particular, $\rho(\vec{a}) = \rho(\vec{b})$);
(2) $\forall n \in \omega \setminus I^{\vec{a}} (a_n \; E_S \; b_{f(n)})$ and $\forall n \in \omega \setminus I^{\vec{b}} (b_n \; E_S \; a_{g(n)})$.

Proof. To prove (1) we first notice that the right to left direction of each equivalence follows from the left to right implication of the other equivalence. Indeed if $f(n) \in I^{\vec{b}}_\alpha$ then $g(f(n)) \in I^{\vec{a}}_\alpha$. Since by definition of S^{inj} we have $a_n \; S \; b_{f(n)} \; S \; a_{g(f(n))}$, Lemma 3.27(1) implies $n \in I^{\vec{a}}_\alpha$. Similarly one shows that if $g(n) \in I^{\vec{a}}_\alpha$ then $n \in I^{\vec{b}}_\alpha$.

The proof of the forward implication of both equivalences is by induction on α. If $\alpha = 0$ the statement is obvious, and if α is a limit ordinal it suffices to apply the induction hypothesis. Thus it remains to derive the two implications for $\alpha + 1$ assuming they hold for α.

Fix $n \in I^{\vec{a}}_{\alpha+1}$. Since $n \in I^{\vec{a}}_{\alpha}$ the induction hypothesis implies that $f(n) \in I^{\vec{b}}_{\alpha}$ and to show that $f(n) \in I^{\vec{b}}_{\alpha+1}$ we need to find infinitely many $m \in I^{\vec{b}}_{\alpha}$ such that $b_{f(n)} \mathrel{S} b_m$. We have $a_n \mathrel{S} b_{f(n)}$. The proof splits into two cases.

If $b_{f(n)} \mathrel{S} a_n$ then for every $m \in I^{\vec{a}}_{\alpha}$ with $a_n \mathrel{S} a_m$ we have $b_{f(n)} \mathrel{S} a_n \mathrel{S} a_m \mathrel{S} b_{f(m)}$. Since there are infinitely many of these m (because $n \in I^{\vec{a}}_{\alpha+1}$) and for each of them $f(m) \in I^{\vec{b}}_{\alpha}$ (by induction hypothesis), by injectivity of f we have reached our goal.

Now suppose $b_{f(n)} \not\mathrel{S} a_n$. Notice that $(f \circ g)^k(f(n)) \in I^{\vec{b}}_{\alpha}$ for every k (here we are using the induction hypothesis for both f and g). Moreover, $b_{f(n)} \mathrel{S} b_{(f\circ g)^k(f(n))}$ for every k. Thus it suffices to show that the map $k \mapsto (f \circ g)^k(f(n))$ is injective. Since $f \circ g$ is injective, it is enough to show that $(f \circ g)^k(f(n)) \neq f(n)$ for every $k > 0$ and, by injectivity of f, this is equivalent to $(g \circ f)^k(n) \neq n$ for every $k > 0$. But if $k > 0$ we have $b_{f(n)} \mathrel{S} a_{(g\circ f)(n)} \mathrel{S} \ldots \mathrel{S} a_{(g\circ f)^k(n)}$. Thus $(g \circ f)^k(n) = n$ implies $b_{f(n)} \mathrel{S} a_n$ against our hypothesis.

We have thus shown the first implication. The other implication is proved by switching the roles of f and g.

To prove (2) assume that $n \in I^{\vec{a}}_{\alpha} \setminus I^{\vec{a}}_{\alpha+1}$ and $a_n \mathrel{E_S} b_{f(n)}$ fails, i.e. $b_{f(n)} \not\mathrel{S} a_n$. We showed in the second case of the preceding proof that under this hypothesis the map $k \mapsto (g \circ f)^k(n)$ is injective. Since by (1) each $(g \circ f)^k(n) \in I^{\vec{a}}_{\alpha}$ and $a_n \mathrel{S} a_{(g\circ f)^k(n)}$, this shows $n \in I^{\vec{a}}_{\alpha+1}$. Again, the second statement is proved symmetrically. □

Lemma 3.28(1) immediately implies the following corollary.

Corollary 3.29. *Let $\vec{a}, \vec{b} \in {}^{\omega}X$ be such that $\vec{a} \mathrel{E_{S^{\mathrm{inj}}}} \vec{b}$, and let $f \colon \omega \to \omega$ witness $\vec{a} \mathrel{S^{\mathrm{inj}}} \vec{b}$. Then*

(1) $f(I^{\vec{a}}_{\alpha} \setminus I^{\vec{a}}_{\alpha+1}) \subseteq I^{\vec{b}}_{\alpha} \setminus I^{\vec{b}}_{\alpha+1}$ for every $\alpha < \rho(\vec{a})$, so that, in particular, $f(\omega \setminus I^{\vec{a}}) \subseteq \omega \setminus I^{\vec{b}}$;

(2) $f(I^{\vec{a}}) \subseteq I^{\vec{b}}$.

Proof. (2) follows from $I^{\vec{a}} = \bigcap_{\alpha<\omega_1} I^{\vec{a}}_{\alpha}$ and $I^{\vec{b}} = \bigcap_{\alpha<\omega_1} I^{\vec{b}}_{\alpha}$. □

The following lemma provides a combinatorial characterization of the relation $E_{S^{\mathrm{inj}}}$.

Lemma 3.30. *For every $\vec{a}, \vec{b} \in {}^{\omega}X$ we have $\vec{a} \mathrel{E_{S^{\mathrm{inj}}}} \vec{b}$ if and only if the following conditions hold:*

(i) $\rho(\vec{a}) = \rho(\vec{b})$;

(ii) $|\{m \in \omega \mid a_n \mathrel{E_S} a_m\}| = |\{m \in \omega \mid a_n \mathrel{E_S} b_m\}|$ for all $n \in \omega \setminus I^{\vec{a}}$;

(iii) $|\{m \in \omega \mid b_n \mathrel{E_S} b_m\}| = |\{m \in \omega \mid b_n \mathrel{E_S} a_m\}|$ for all $n \in \omega \setminus I^{\vec{b}}$;

(iv) $\forall n \in I^{\vec{a}}\, \exists m \in I^{\vec{b}}\, (a_n \mathrel{S} b_m)$ and $\forall n \in I^{\vec{b}}\, \exists m \in I^{\vec{a}}\, (b_n \mathrel{S} a_m)$.

Proof. We first assume $\vec{a} \mathrel{E_{S^{\mathrm{inj}}}} \vec{b}$ and fix the injective functions $f, g \colon \omega \to \omega$ witnessing respectively $\vec{a} \mathrel{S^{\mathrm{inj}}} \vec{b}$ and $\vec{b} \mathrel{S^{\mathrm{inj}}} \vec{a}$, so that we have the same notation of Lemma 3.28 and Corollary 3.29. Then (i) follows directly from Lemma 3.28(1). Now assume $n \in \omega \setminus I^{\vec{a}}$. By Lemma 3.27(2), we have $\{m \in \omega \mid a_m \mathrel{E_S} a_n\} \subseteq \omega \setminus I^{\vec{a}}$, so that f maps injectively $\{m \in \omega \mid a_n \mathrel{E_S} a_m\}$ into $\{m \in \omega \mid a_n \mathrel{E_S} b_m\} \cap (\omega \setminus I^{\vec{b}})$ by Lemma 3.28(2) and Corollary 3.29(1). Since this last set is nonempty, by Lemma 3.27(2) again we also have $\{m \in \omega \mid a_n \mathrel{E_S} b_m\} \subseteq \omega \setminus I^{\vec{b}}$, and by Lemma 3.28(2) g maps injectively $\{m \in \omega \mid a_n \mathrel{E_S} b_m\}$ into $\{m \in \omega \mid a_n \mathrel{E_S} a_m\}$.

This shows (ii), and (iii) is obtained similarly. Finally, f maps $I^{\vec{a}}$ into $I^{\vec{b}}$ and g maps $I^{\vec{b}}$ into $I^{\vec{a}}$ by Corollary 3.29(2), which in particular implies (iv).

For the other direction assume (i)–(iv) hold. To define f witnessing $\vec{a}\ S^{\mathrm{inj}}\ \vec{b}$, we start pasting together bijections between each $\{m \in \omega \mid a_n\ E_S\ a_m\}$ and $\{m \in \omega \mid a_n\ E_S\ b_m\}$ whenever $n \in \omega \setminus I^{\vec{a}}$ is such that $\forall m < n \neg(a_n\ E_S\ a_m)$. Notice that for each such n, the set $\{m \in \omega \mid a_n\ E_S\ b_m\}$ is always contained in $\omega \setminus I^{\vec{b}}$: if not, there would be $m \in I^{\vec{b}}$ with $a_n\ E_S\ b_m$; but then by (iv) there would be $k \in I^{\vec{a}}$ with $b_m\ S\ a_k$, so that also $n \in I^{\vec{a}}$ by $a_n\ S\ b_m\ S\ a_k$ and Lemma 3.27(1). Therefore we have constructed $f \restriction (\omega \setminus I^{\vec{a}})$ with range in $\omega \setminus I^{\vec{b}}$: this leaves us with the task of defining f on $I^{\vec{a}}$, with range in $I^{\vec{b}}$. This can be done recursively as follows. Assume $n \in I^{\vec{a}}$ and we already defined $f(\ell)$ for all $\ell \in I^{\vec{a}}$ with $\ell < n$. By (iv) there exists $m \in I^{\vec{b}}$ such that $a_n\ S\ b_m$. By Lemma 3.27(3) there are infinitely many $m' \in I^{\vec{b}}$ with $b_m\ S\ b_{m'}$ and hence $a_n\ S\ b_{m'}$. If m' is the least such which is not yet in the image of f we can set $f(n) = m'$ preserving injectivity. The definition of g witnessing $\vec{b}\ S^{\mathrm{inj}}\ \vec{a}$ is symmetric. □

The following theorem shows in particular that if S is Borel then S^{inj} is far from being a complete analytic quasi-order.

Theorem 3.31. *If S is a Borel quasi-order on the standard Borel space X then for every $\vec{a} \in {}^{\omega}X$ the equivalence class $\{\vec{b} \in {}^{\omega}X \mid \vec{a}\ E_{S^{\mathrm{inj}}}\ \vec{b}\}$ is Borel.*

Proof. Fix $\vec{a} \in {}^{\omega}X$ and let $\rho = \rho(\vec{a})$. We describe a Borel procedure, based on the characterization of Lemma 3.30, for checking, given any $\vec{b} \in {}^{\omega}X$, whether $\vec{a}\ E_{S^{\mathrm{inj}}}\ \vec{b}$.

The procedure starts by computing $I^{\vec{b}}_{\alpha}$ for every $\alpha \leq \rho + 1$ (since ρ is fixed, this is Borel). If $I^{\vec{b}}_{\alpha} = I^{\vec{b}}_{\alpha+1}$ for some $\alpha < \rho$, or if $I^{\vec{b}}_{\rho} \neq I^{\vec{b}}_{\rho+1}$ then the procedure gives a negative answer (because (i) of Lemma 3.30 fails).

Otherwise the procedure looks at every $\alpha < \rho$ and $n \in I^{\vec{a}}_{\alpha} \setminus I^{\vec{a}}_{\alpha+1}$, computes the (finite) cardinality of $\{m \in \omega \mid a_n\ E_S\ a_m\} = \{m \in I^{\vec{a}}_{\alpha} \mid a_n\ E_S\ a_m\}$ (see Lemma 3.27(2)), and checks whether it coincides with the cardinality of $\{m \in \omega \mid a_n\ E_S\ b_m\}$. If this fails (so that (ii) of Lemma 3.30 is not true) the procedure gives a negative answer. Otherwise it performs a similar operation reversing the roles of $\vec{a}$ and $\vec{b}$, and checks whether (iii) holds.

If everything works out, the procedure checks whether

$$\forall n \in I^{\vec{a}}\ \exists m \in I^{\vec{b}}\,(a_n\ S\ b_m) \land \forall n \in I^{\vec{b}} \exists m \in I^{\vec{a}}\,(b_n\ S\ a_m)$$

(a Borel condition) and gives the final answer. □

In spite of Theorem 3.31, S can be a Borel quasi-order without $E_{S^{\mathrm{inj}}}$ being Borel, as Propositions 3.22 and 3.23 show. However if we restrict S^{inj} to the collection $({}^{\omega}X)_{\alpha}$ of those $\vec{a} \in {}^{\omega}X$ such that $\rho(\vec{a}) < \alpha$ for some fixed $\alpha < \omega_1$, then the proof Theorem 3.31 shows that the equivalence relation associated to $S^{\mathrm{inj}}_{\alpha} = S^{\mathrm{inj}} \restriction ({}^{\omega}X)_{\alpha}$ is Borel (notice that $({}^{\omega}X)_{\alpha}$ is Borel and $E_{S^{\mathrm{inj}}}$-invariant by condition (i) of Lemma 3.30). However even more is true, as we can show that these Borel equivalence relations have a Borel upper bound.

Proposition 3.32. *Let S be a Borel quasi-order on a standard Borel space X. There exists a Borel quasi-order R on a standard Borel space Y such that for every $\alpha < \omega_1$, $E_{S^{\mathrm{inj}}_{\alpha}} \leq_B E_{R^{\mathrm{cf}}}$. In particular $E_{S^{\mathrm{inj}}_{\alpha}}$ is Borel.*

Proof. Let (Y, R) be the direct sum

$$(\omega, =) \oplus (X, S) \oplus (X \times \omega, E_S \times =). \tag{3.1}$$

Given $\alpha < \omega_1$, let $i\colon \alpha \to \omega$ be an injection.

For any $\vec{a} = (a_n)_{n\in\omega} \in ({}^\omega X)_\alpha$ we define $f(\vec{a}) = (a'_n)_{n\in\omega} \in {}^\omega Y$ as follows. We let $a'_0 = i(\rho(\vec{a})) \in \omega$ and a'_{n+1} is defined according to whether $n \in I^{\vec{a}}$. If $n \in I^{\vec{a}}$ we let $a'_{n+1} = a_n \in X$. If $n \notin I^{\vec{a}}$ we let $a'_{n+1} = (a_n, k) \in X \times \omega$ where $k = |\{m \in \omega \mid a_n \mathrel{E_S} a_m\}|$ (this cardinality is finite by Lemma 3.27(2)).

Since α is fixed, we can recover in a Borel-in-$\vec{a}$ way $i(\rho(\vec{a}))$ and $I^{\vec{a}}$ and hence $f\colon ({}^\omega X)_\alpha \to {}^\omega Y$ is Borel.

Applying Lemma 3.30, it is straightforward to check that f witnesses $E_{S^{\mathrm{inj}}_\alpha} \leq_B E_{R^{\mathrm{cf}}}$. □

Corollary 3.33. *For every positive natural number N, the equivalence relation $E_{(\mathscr{P}_{<N}(\omega),\subseteq)^{\mathrm{inj}}}$ is Borel. Therefore, for $N > 2$, $(\mathscr{P}_{<N}(\omega), \subseteq)^{\mathrm{inj}}$ is an analytic non-Borel quasi-order whose associated equivalence relation is Borel.*

Proof. This follows from ${}^\omega(\mathscr{P}_{<N}(\omega)) \subseteq ({}^\omega(\mathscr{P}(\omega)))_{N+1}$. □

Corollary 3.33 should be contrasted with the fact that $E_{(\mathscr{P}_{<\omega}(\omega),\subseteq)^{\mathrm{inj}}}$ is already a $\mathbf{\Sigma}^1_1$-complete subset of ${}^\omega(\mathscr{P}_{<\omega}(\omega)) \times {}^\omega(\mathscr{P}_{<\omega}(\omega))$ by Remark 3.24.

It may also be interesting to notice that there exist Borel quasi-orders S such that $E_{S^{\mathrm{cf}}} \sim_B E_{S^{\mathrm{inj}}}$ even though $S^{\mathrm{cf}} <_B S^{\mathrm{inj}}$ (see the subsequent Proposition 3.35(2)). In what follows $\subseteq^{\mathrm{inj}}_\alpha$ and $\subseteq^{\mathrm{cf}}_\alpha$ stand for $(({}^\omega\mathscr{P}(\omega))_\alpha, \subseteq^{\mathrm{inj}})$ and $(({}^\omega\mathscr{P}(\omega))_\alpha, \subseteq^{\mathrm{cf}})$ respectively. If $\alpha \geq 4$, the quasi-order $\subseteq^{\mathrm{inj}}_\alpha$ is proper analytic by Remark 3.24 and the observation in the proof of Corollary 3.33, and hence $\subseteq^{\mathrm{cf}}_\alpha <_B \subseteq^{\mathrm{inj}}_\alpha$. Similarly, Remark 3.24 implies that $(\mathscr{P}_{<N}(\omega), \subseteq)^{\mathrm{cf}} <_B (\mathscr{P}_{<N}(\omega), \subseteq)^{\mathrm{inj}}$ for all $2 < N < \omega$. To show that $E_{\subseteq^{\mathrm{cf}}_\alpha} \sim_B E_{\subseteq^{\mathrm{inj}}_\alpha}$ we use the following general fact.

Fact 3.34. *There is a Borel function $f\colon {}^\omega X \to ({}^\omega X)_1$ such that $\vec{a} \mathrel{E_{S^{\mathrm{cf}}}} f(\vec{a})$ for every $\vec{a} \in {}^\omega X$.*

Proof. Define f to be a Borel function such that $f(\vec{a})$ is a sequence repeating every element of $\vec{a}$ infinitely many times. □

Proposition 3.35. *(1) For every $0 < \alpha < \omega_1$ we have*

$$E_{\subseteq^{\mathrm{cf}}_\alpha} \sim_B E_{\subseteq^{\mathrm{inj}}_\alpha} \sim_B E_{\subseteq^{\mathrm{cf}}};$$

(2) for every $2 \leq N < \omega$

$$E_{(\mathscr{P}_{<N}(\omega),\subseteq)^{\mathrm{cf}}} \sim_B E_{(\mathscr{P}_{<N}(\omega),\subseteq)^{\mathrm{inj}}}.$$

Proof. To prove part (1), it suffices to show that $E_{\subseteq^{\mathrm{inj}}_\alpha} \leq_B E_{\subseteq^{\mathrm{cf}}}$ and $E_{\subseteq^{\mathrm{cf}}} \leq_B E_{\subseteq^{\mathrm{cf}}_1}$. The latter follows from Fact 3.34. To prove the former we use Proposition 3.32 and its proof. Let (Y, R) be as in (3.1) with $(X, S) = (\mathscr{P}(\omega), \subseteq)$: by (the proof of) Proposition 3.32, it is clearly enough to show that $R \leq_B \subseteq$. We define $f\colon Y \to \mathscr{P}(\omega)$ by

$$f(a) = \begin{cases} \{\langle a, 0\rangle\}, & \text{if } a \in \omega; \\ \{\langle n, 1\rangle \mid n \in a\}, & \text{if } a \in \mathscr{P}(\omega); \\ \{\langle 2n, k+2\rangle \mid n \in b\} \cup \\ \qquad\qquad \{\langle 2n+1, k+2\rangle \mid n \notin b\}, & \text{if } a = (b, k) \in \mathscr{P}(\omega) \times \omega. \end{cases}$$

Since $E_{\subseteq}$ is equality on $\mathscr{P}(\omega)$, it is easy to check that f is the desired Borel reduction of R to $\subseteq$.

To prove the nontrivial reduction in part (2) one can use a similar argument. Let (Y, R) be as in (3.1) with $(X, S) = (\mathscr{P}_{<N}(\omega), \subseteq)$. Since ${}^{\omega}(\mathscr{P}_{<N}(\omega)) = ({}^{\omega}(\mathscr{P}_{<N}(\omega)))_{N+1}$, $E_{S^{\text{inj}}} = E_{S^{\text{inj}}_{N+1}} \leq_B E_{R^{\text{cf}}}$ by Proposition 3.32, it is enough to show that (Y, R) is Borel reducible to $(\mathscr{P}_{<N}(\omega), \subseteq)$. This is witnessed by the map

$$f(a) = \begin{cases} \{\langle a, 0\rangle\}, & \text{if } a \in \omega; \\ \{\langle n, 1\rangle \mid n \in a\}, & \text{if } a \in \mathscr{P}_{<N}(\omega); \\ \{\langle\langle n, |b|\rangle, k+2\rangle \mid n \in b\} & \text{if } a = (b, k) \in \mathscr{P}_{<N}(\omega) \times \omega. \end{cases} \qquad \square$$

Proposition 3.35(2) should be contrasted with the fact that $E_{(\mathscr{P}_{<\omega}(\omega), \subseteq)^{\text{cf}}} <_B E_{(\mathscr{P}_{<\omega}(\omega), \subseteq)^{\text{inj}}}$ because $E_{(\mathscr{P}_{<\omega}(\omega), \subseteq)^{\text{cf}}}$ is Borel while $E_{(\mathscr{P}_{<\omega}(\omega), \subseteq)^{\text{inj}}}$ is proper analytic by Remark 3.24.

3.6. **A family of proper analytic quasi-orders S with $S <_B S^{\text{inj}}$.** Combining the next simple lemma together with Proposition 3.23 and Theorem 3.31, we will construct a large class of proper analytic quasi-orders S which are not stable under the jump operators $S \mapsto S^{\text{cf}}$ and $S \mapsto S^{\text{inj}}$. Examples of this kind were already provided in [CM07, Section 4], but those considered here are different, as shown in Remark 3.38.

Lemma 3.36. *Let (X, S) be an analytic quasi-order. Then*

(1) if there are $\mathbf{\Sigma}^1_1$-complete upper cones in S, then $E_{S^{\text{cf}}}$ is $\mathbf{\Sigma}^1_1$-complete as a subset of ${}^{\omega}X \times {}^{\omega}X$;

(2) if there are $\mathbf{\Sigma}^1_1$-complete lower cones in S, then $E_{S^{\text{cf}}}$ contains a $\mathbf{\Sigma}^1_1$-complete equivalence class.

Similar results hold when S^{cf} is replaced by S^{inj}.

Proof. For part (1), let $p \in X$ be such that the upper cone $C^S(p)$ is $\mathbf{\Sigma}^1_1$-complete, and define the map $f \colon X \to {}^{\omega}X \times {}^{\omega}X$ by setting $f(x) = (p^\frown x^\infty, x^\infty)$. Since clearly $x^\infty S^{\text{cf}} p^\frown x^\infty$ for all $x \in X$ and

$$p^\frown x^\infty S^{\text{cf}} x^\infty \iff x \in C^S(p),$$

the function f continuously reduces $C^S(p)$ to $E_{S^{\text{cf}}}$.

For part (2), let $q \in X$ be such that the lower cone $C_S(q)$ is $\mathbf{\Sigma}^1_1$-complete, and define the map $g \colon X \to {}^{\omega}X$ by setting $g(x) = x^\frown q^\infty$. Arguing as above, it is easy to see that g continuously reduces $C_S(q)$ to the $E_{S^{\text{cf}}}$-equivalence class of q^∞.

The same arguments work when S^{cf} is replaced by S^{inj}. $\square$

Corollary 3.37. *Let R be a Borel quasi-order such that $(\mathscr{P}(\omega), \subseteq) \leq_B R$. Then $S = R^{\text{inj}}$ is a proper analytic quasi-order such that $S <_B S^{\text{cf}}$ (hence also $S <_B S^{\text{inj}}$).*

Proof. By Proposition 3.23(2), S contains $\mathbf{\Sigma}^1_1$-complete lower cones and in particular it is a proper analytic quasi-order. By Lemma 3.36(2) there is an $E_{S^{\text{cf}}}$-equivalence class which is proper analytic. Therefore $S^{\text{cf}} \nleq_B S$ because all E_S-equivalence classes are Borel by Theorem 3.31. $\square$

Remark 3.38. If a quasi-order of the form R^{inj} is proper analytic, it cannot satisfy the hypothesis of [CM07, Corollary 4.3]. To see this, observe that the proof of [CM07, Theorem 4.2] shows that if a directed (i.e. such that every pair of elements has un upper bound) quasi-order is not Borel then it does not satisfy that hypothesis. Since R^{inj} is always directed (and in fact even countably-directed) our examples are different as claimed.

4. Ultrametric Polish spaces with a fixed set of distances

All metric spaces we consider are always assumed to be nonempty. Let $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$. Let U be an ultrametric space with distance d_U. Notice that d_U is an ultrametric if and only if for every $x, y, z \in U$ at least two of the distances $d_U(x,y), d_U(x,z), d_U(z,y)$ equal $\max\{d_U(x,y), d_U(x,z), d_U(z,y)\}$ (i.e. all triangles are isosceles with legs not shorter than the base). Thus we have

$$d_U(x,z) < d_U(z,y) \Rightarrow d_U(x,y) = d_U(z,y). \tag{4.1}$$

Recall also that in an ultrametric space every open ball is also closed.

We say that U is an *ultrametric Polish space* if it is separable and the ultrametric d_U is complete.[3]

We denote by $D(U)$ the set of distances that are realized by points in U, i.e.

$$D(U) = \{r \in \mathbb{R}^+ \mid \exists x, y \in U (d_U(x,y) = r)\}.$$

Let $\mathcal{D}$ denote the set of all countable $D \subseteq \mathbb{R}^+$ with $0 \in D$.

Lemma 4.1. *$D(U) \in \mathcal{D}$ for every separable ultrametric space U.*

Proof. Let Q be a countable dense subset of U: it suffices to show that $D(U) = \{d_U(p,q) \mid p, q \in Q\}$. Clearly $d_U(p,p) = 0$, so let $x, y \in U$ be such that $d_U(x,y) = r \neq 0$, and let $p, q \in Q$ be such that $d_U(x,p), d_U(y,q) < r$. Then using (4.1) we get $d_U(p,y) = d_U(x,y) = r$, whence

$$d_U(p,q) = d_U(p,y) = r. \qquad \square$$

Conversely, given $D \in \mathcal{D}$ one can construct a canonical ultrametric Polish space $U(D)$ with $D(U(D)) = D$.

Definition 4.2. Let $D \in \mathcal{D}$. Then $U(D)$ is the ultrametric Polish space with domain D and distance function defined by $d_{U(D)}(r, r') = \max\{r, r'\}$ for $r \neq r'$ and $d_{U(D)}(r,r) = 0$.

Given $D \in \mathcal{D}$, we denote by $\mathcal{U}_D$ the set of all ultrametric Polish spaces U with $D(U) \subseteq D$, and by $\mathcal{U}^\star_D$ the set of all ultrametric Polish spaces U such that $D(U) = D$.

Recalling that a (nonempty) metric space is Polish if and only if it is isometric to an element of $F(\mathbb{U})$, the collection of all nonempty closed subsets of the Urysohn space $\mathbb{U}$ (this notation differs slightly from the one used in [Kec95], where $F(\mathbb{U})$ includes the empty set), we can use the following formalizations of $\mathcal{U}_D$ and $\mathcal{U}^\star_D$.

[3]Let us remark that if a Polish space X admits a compatible ultrametric, then it also admits a compatible ultrametric d which is also complete. This is because under our assumptions X must be zero-dimensional, and hence homeomorphic to a closed subset F of ${}^\omega\omega$ by [Kec95, Theorem 7.8]. Transferring back on X the complete ultrametric on F induced by the standard metric on ${}^\omega\omega$ we obtain d as required.

Definition 4.3. Let

$$\mathcal{U}_D = \{U \in F(\mathbb{U}) \mid d_U = d_{\mathbb{U}} \restriction U^2 \text{ is an ultrametric and } D(U) \subseteq D\}$$

and

$$\mathcal{U}_D^\star = \{U \in F(\mathbb{U}) \mid d_U = d_{\mathbb{U}} \restriction U^2 \text{ is an ultrametric and } D(U) = D\}.$$

Notation 4.4. Using [Kec95, Theorem 12.13], we fix once and for all a sequence of Borel functions $\psi_n \colon F(\mathbb{U}) \to \mathbb{U}$ such that for every $F \in F(\mathbb{U})$ the sequence $(\psi_n(F))_{n\in\omega}$ is an enumeration (which can be assumed without repetitions if F is infinite) of a dense subset of F. Notice that if $F \in F(\mathbb{U})$ is discrete then $(\psi_n(F))_{n\in\omega}$ is an enumeration of F.

Proposition 4.5. *The sets $\mathcal{U}_D$ and $\mathcal{U}_D^\star$ are both Borel subsets of the standard Borel space $F(\mathbb{U})$, hence they are standard Borel spaces as well.*

Proof. With the same argument as in the proof of Lemma 4.1, $U \in F(\mathbb{U})$ is in $\mathcal{U}_D$ if and only if d_U is an ultrametric[4] and $d_U(\psi_n(U), \psi_m(U)) \in D$ for all $n, m \in \omega$. To deal with $\mathcal{U}_D^\star$, add the following condition: for all $r \in D$ there exist $n, m \in \omega$ such that $d_U(\psi_n(U), \psi_m(U)) = r$. □

Another possible formalization of $\mathcal{U}_D$ and $\mathcal{U}_D^\star$ uses, instead of $\mathbb{U}$, the Polish D-ultrametric Urysohn space $\mathbb{U}_D^{\mathcal{U}}$. This is the unique (up to isometry) ultrametric Polish space with $D(\mathbb{U}_D^{\mathcal{U}}) = D$ which is ultrahomogeneous and universal for $\mathcal{U}_D$. The space $\mathbb{U}_D^{\mathcal{U}}$ can also be characterized as the unique (up to isometry) ultrametric Polish space with the D-ultrametric Urysohn property: for every finite ultrametric space B with $D(B) \subseteq D$, every $A \subseteq B$, and every isometric embedding $\varphi \colon A \to \mathbb{U}_D^{\mathcal{U}}$, there is an isometric embedding $\varphi^* \colon B \to \mathbb{U}_D^{\mathcal{U}}$ such that $\varphi^* \restriction A = \varphi$. More about $\mathbb{U}_D^{\mathcal{U}}$, including several constructions of the space, can be found in [GS11].

Now $\mathcal{U}_D$ may be identified with $F(\mathbb{U}_D^{\mathcal{U}})$, while $\mathcal{U}_D^\star$ becomes a Borel subset of $F(\mathbb{U}_D^{\mathcal{U}})$, which we denote by $F^\star(\mathbb{U}_D^{\mathcal{U}})$. Although our official definition of $\mathcal{U}_D$ and $\mathcal{U}_D^\star$ is as in Definition 4.3, it will be sometimes convenient to work instead with their counterparts $F(\mathbb{U}_D^{\mathcal{U}})$ and $F^\star(\mathbb{U}_D^{\mathcal{U}})$. The next lemma shows that the two formalizations are essentially equivalent.

Lemma 4.6. *There exist Borel maps $\Phi \colon F(\mathbb{U}_D^{\mathcal{U}}) \to \mathcal{U}_D$ and $\Psi \colon \mathcal{U}_D \to F(\mathbb{U}_D^{\mathcal{U}})$ such that $\Phi(U)$ and U are isometric for any $U \in F(\mathbb{U}_D^{\mathcal{U}})$ and $\Psi(V)$ and V are isometric for any $V \in \mathcal{U}_D$. In particular, the range of $\Phi \restriction F^\star(\mathbb{U}_D^{\mathcal{U}})$ is included in $\mathcal{U}_D^\star$ and the range of $\Psi \restriction \mathcal{U}_D^\star$ is included in $F^\star(\mathbb{U}_D^{\mathcal{U}})$.*

Proof. Any isometric embedding of $\mathbb{U}_D^{\mathcal{U}}$ in $\mathbb{U}$ induces a map Φ with the desired properties.

To define Ψ fix a countable dense subset $\{q_k \mid k \in \omega\}$ in $\mathbb{U}_D^{\mathcal{U}}$. Given $V \in \mathcal{U}_D$ consider $\{\psi_n(V) \mid n \in \omega\}$, where the functions ψ_n are as in Notation 4.4. In a Borel way we recursively define a sequence $\{k_n \mid n \in \omega\}$ such that the map $\psi_n(V) \mapsto q_{k_n}$ is an isometry. Once this is done we can define $\Psi(V)$ to be the closure of $\{q_{k_n} \mid n \in \omega\}$ in $\mathbb{U}_D^{\mathcal{U}}$.

Assuming we defined k_n for $n < m$, let k_m be the least k such that $d(\psi_m(V), \psi_n(V)) = d(q_k, q_{k_n})$ for every $n < m$. To see that such a k exists, notice that by the D-ultrametric Urysohn property there exists $y \in \mathbb{U}_D^{\mathcal{U}}$ such that $d(\psi_m(V), \psi_n(V)) =$

[4] This is a Borel condition because it is enough to check that d_U satisfies the definition of ultrametric on the dense set $\{\psi_n(U) \mid n \in \omega\} \subseteq U$.

$d(y, q_{k_n})$ for every $n < m$. If $d(q_k, y) < \min\{d(q_{k_n}, y) \mid n < m\}$, then k has the required property by (4.1). ☐

Let $\cong$ and $\sqsubseteq$ denote respectively the relations of isometry and isometric embeddability between metric spaces. When restricted to ultrametric spaces, these relations can be described as follows.

Notation 4.7. Given an ultrametric space U, $r \in \mathbb{R}^+$, and $x \in U$, let $C^x_r(U) = \{y \in U \mid d_U(x, y) = r\}$.

Lemma 4.8. *Let U, U' be two ultrametric spaces (not necessarily separable nor complete). Then the following are equivalent:*

(i) $U \cong U'$;
(ii) for every $x \in U$ there exists $x' \in U'$ such that for every $r \in D(U) \cup D(U')$, either $C^x_r(U)$ and $C^{x'}_r(U')$ are both empty, or else $C^x_r(U) \cong C^{x'}_r(U')$;
(iii) there exist $x \in U$ and $x' \in U'$ such that for every $r \in D(U) \cup D(U')$, either $C^x_r(U)$ and $C^{x'}_r(U')$ are both empty, or else $C^x_r(U) \cong C^{x'}_r(U')$.

Similarly, the following are equivalent:

(iv) $U \sqsubseteq U'$;
(v) for every $x \in U$ there exists $x' \in U'$ such that $C^x_r(U) \sqsubseteq C^{x'}_r(U')$ for every distance $r \in D(U)$ realized by x;
(vi) there exist $x \in U$ and $x' \in U'$ such that $C^x_r(U) \sqsubseteq C^{x'}_r(U')$ for every distance $r \in D(U)$ realized by x.

Proof. We first consider the part concerning isometric embeddability. To prove (iv) implies (v), let $\varphi \colon U \to U'$ be an isometric embedding and pick $x \in U$. Then setting $x' = \varphi(x)$ we get that $\varphi \restriction C^x_r(U)$ witnesses $C^x_r(U) \sqsubseteq C^{x'}_r(U')$ for every distance $r \in D(U)$ realized by x.

(v) implies (vi) is trivial, while to prove (vi) implies (iv) fix $x \in U$ and $x' \in U'$ such that for every distance $r \in D(U)$ realized by x there is an isometric embedding $\varphi_r \colon C^x_r(U) \to C^{x'}_r(U')$. We claim that $\varphi = \bigcup_r \varphi_r$ is an isometric embedding of U into U'. To see this, let $y, z \in U$. If there is $r \in D(U)$ such that $d_U(y, x) = d_U(z, x) = r$ then $y, z \in C^x_r(U)$: it follows that $\varphi(y) = \varphi_r(y)$ and $\varphi(z) = \varphi_r(z)$, whence $d_U(y, z) = d_{U'}(\varphi(y), \varphi(z))$ since φ_r is distance preserving. If instead $d_U(y, x) = r_y < r_z = d_U(z, x)$ then $d_U(y, z) = r_z$ by (4.1). Since $\varphi(y) = \varphi_{r_y}(y) \in C^{x'}_{r_y}(U')$ and $\varphi(z) = \varphi_{r_z}(z) \in C^{x'}_{r_z}(U')$, it follows that $d_{U'}(\varphi(y), x') = r_y$ and $d_{U'}(\varphi(z), x') = r_z$. Therefore $d_{U'}(\varphi(y), \varphi(z)) = r_z = d_U(y, z)$ by (4.1) again.

The same proof works for isometry as well: it is enough to notice that in this case all the isometric embeddings involved are automatically surjective. ☐

Notation 4.9. Let $\cong_D$ and $\cong^\star_D$ denote the restrictions of $\cong$ to spaces in $\mathcal{U}_D$ and $\mathcal{U}^\star_D$, respectively. Similarly, let $\sqsubseteq_D$ and $\sqsubseteq^\star_D$ be the restrictions of $\sqsubseteq$ to $\mathcal{U}_D$ and $\mathcal{U}^\star_D$.

Using Proposition 4.5, it is straightforward to see that all the relations $(\mathcal{U}_D, \cong_D)$, $(\mathcal{U}^\star_D, \cong^\star_D)$, $(\mathcal{U}_D, \sqsubseteq_D)$, and $(\mathcal{U}^\star_D, \sqsubseteq^\star_D)$ are analytic.

Remark 4.10. By Lemma 4.6 the equivalence relations $\cong_D$ and $\cong^\star_D$ are classwise Borel isomorphic to $\cong\restriction F(\mathbb{U}^{\mathcal{U}}_D)$ and $\cong\restriction F^\star(\mathbb{U}^{\mathcal{U}}_D)$ respectively. Similarly for the quasi-orders $\sqsubseteq_D$ and $\sqsubseteq^\star_D$.

Our aim is to study the complexity of the above relations for various $D \in \mathcal{D}$. Notice that since $\mathcal{U}^\star_D \subseteq \mathcal{U}_D$, the identity function on $\mathcal{U}^\star_D$ witnesses that $\cong^\star_D \leq_B \cong_D$

and $\sqsubseteq^\star_D \leq_B \sqsubseteq_D$ for every $D \in \mathcal{D}$. We will show later (Corollaries 5.10 and 6.5), using some nontrivial results, that the converse reductions hold as well.

The following lemma extends [GS11, Theorem 8.2]. Another strengthening, without the hypotheses of continuity in 0, will be given in Corollary 5.11.

Lemma 4.11. *Let $D, D' \in \mathcal{D}$.*

(1) If there exists a Polish ultrametric preserving injection $f \colon D \to D'$ then $\cong_D$ classwise Borel embeds into $\cong_{D'}$ and $\sqsubseteq_D$ classwise Borel embeds into $\sqsubseteq_{D'}$;

(2) If there exists a Polish ultrametric preserving bijection $f \colon D \to D'$ then the relations $\cong_D$, $\cong^\star_D$, $\sqsubseteq_D$, and $\sqsubseteq^\star_D$ are classwise Borel isomorphic[5] *to $\cong_{D'}$, $\cong^\star_{D'}$, $\sqsubseteq_{D'}$, and $\sqsubseteq^\star_{D'}$, respectively.*

Proof. We explicitly consider only the case of isometry, but the same proof works for isometric embeddability as well. Let $f \colon D \to D'$ be a Polish ultrametric preserving injection and $D'' \subseteq D'$ be the range of f. We show that $\cong_D$ is classwise Borel isomorphic to $\cong_{D''}$. By Remark 4.10 we can work with $\cong \restriction F(\mathbb{U}^{\mathcal{U}}_D)$ and $\cong \restriction F(\mathbb{U}^{\mathcal{U}}_{D''})$. Let $C \in F(\mathbb{U}^{\mathcal{U}}_{D''})$ be such that there exists a homeomorphism $\varphi \colon \mathbb{U}^{\mathcal{U}}_D \to C$ satisfying $d(\varphi(x), \varphi(y)) = f(d(x,y))$ for every $x, y \in \mathbb{U}^{\mathcal{U}}_D$ (the existence of C follows from the fact that f is Polish ultrametric preserving). The map φ induces the Borel bijection $\Phi \colon F(\mathbb{U}^{\mathcal{U}}_D) \to F(C) \subseteq F(\mathbb{U}^{\mathcal{U}}_{D''})$. Notice that, by injectivity of f, Φ reduces $\cong \restriction F(\mathbb{U}^{\mathcal{U}}_D)$ to $\cong \restriction F(\mathbb{U}^{\mathcal{U}}_{D''})$.

Using f^{-1} we analogously define $C' \in F(\mathbb{U}^{\mathcal{U}}_D)$ and a Borel $\Psi \colon F(\mathbb{U}^{\mathcal{U}}_{D''}) \to F(C') \subseteq F(\mathbb{U}^{\mathcal{U}}_D)$. By construction $\Psi(\Phi(U)) \cong U$ and $\Phi(\Psi(V)) \cong V$ for all $U \in F(\mathbb{U}^{\mathcal{U}}_D)$ and $V \in F(\mathbb{U}^{\mathcal{U}}_{D''})$. This shows that the closures under isometries of the ranges of Φ and Ψ are $F(\mathbb{U}^{\mathcal{U}}_{D''})$ and $F(\mathbb{U}^{\mathcal{U}}_D)$ respectively, completing the proof that $\cong \restriction F(\mathbb{U}^{\mathcal{U}}_D)$ and $\cong \restriction F(\mathbb{U}^{\mathcal{U}}_{D''})$ are classwise Borel isomorphic.

This gives (1) and the part of (2) concerning $\cong_D$ e $\cong_{D'}$, since in the latter case $D'' = D'$. For the part of (2) concerning $\cong^\star_D$ e $\cong^\star_{D'}$, notice that the restriction of Φ to $F^\star(\mathbb{U}^{\mathcal{U}}_D)$ has range in $F^\star(\mathbb{U}^{\mathcal{U}}_{D''}) = F^\star(\mathbb{U}^{\mathcal{U}}_{D'})$ and similarly for Ψ. $\square$

For some specific $D \in \mathcal{D}$, the complexity of the relations $\cong_D$ and $\sqsubseteq_D$ (and sometimes also of $\cong^\star_D$ and $\sqsubseteq^\star_D$) has already been considered in the literature, so let us end this section by discussing the known results.

Recall from [GK03, GS11] the following facts concerning the complexity of the relation of isometry on ultrametric Polish spaces.

Proposition 4.12. *(1) Isometry on arbitrary ultrametric Polish spaces is Borel reducible to countable graph isomorphism; thus so is each of the relations $\cong_D$ for $D \in \mathcal{D}$.*

(2) If 0 is a limit point of $D \in \mathcal{D}$, then $\cong_D$ is in fact Borel bireducible with countable graph isomorphism; thus so is the relation of isometry on arbitrary ultrametric Polish spaces.

(3) Isometry on discrete ultrametric Polish spaces (and thus also any $\cong_D$ with $D \in \mathcal{D}$ bounded away from 0) is Borel reducible to isometry on locally compact ultrametric Polish spaces, which in turn is Borel reducible to isometry on arbitrary ultrametric Polish spaces.

(4) If $D \in \mathcal{D}$ is finite of cardinality $n \in \omega$, then $\cong_D$ is Borel bireducible with isomorphism between countable trees of height n, and thus is Borel.

[5] In fact, they are Borel isomorphic.

The exact complexity of the equivalence relations of isometry on discrete or locally compact ultrametric Polish spaces considered in (3) remained unknown, leading to Question 1.1. Gao and Kechris provided in [GK03, Section 8] a first lower bound for such complexity by showing that isomorphism between trees on ω with countably many infinite branches (equivalently, isometry between countable closed subsets of ${}^{\omega}\omega$) is Borel reducible to isometry on discrete ultrametric Polish spaces. Moreover they observed that isomorphism between well-founded trees on ω (equivalently, isometry between discrete subsets of ${}^{\omega}\omega$) Borel reduces to the above isomorphism relation, and thus is a second lower bound for the complexity of isometry restricted to the classes of ultrametric Polish spaces mentioned above. They asked whether these lower bounds are sharp. Another somewhat artificial lower bound (namely: isomorphism between reverse trees, see the end of Section 5.2) was isolated by Clemens in [Cle07], and he asked as well whether this other lower bound is sharp. (We will partially answer these questions on lower bounds after Corollary 5.4 and Theorem 5.15.) More precisely, combining the proofs of [GK03, Theorem 8.10 and Proposition 8.11] and [Cle07, Proposition 16] with Lemma 4.11 we actually get that the above isomorphism relations isolated by Gao-Kechris and Clemens are lower bounds for the complexity of $\cong_D$ for, respectively, $D \in \mathcal{D}$ ill-founded (and possibly bounded away from 0) and $D \in \mathcal{D}$ well-ordered and infinite.

For what concerns isometric embeddability, we already know the following facts.

Proposition 4.13. *If $D \in \mathcal{D}$ is finite then both $\sqsubseteq_D$ and $\sqsubseteq^\star_D$ are bqo's, and therefore very simple Borel quasi-orders (such as $(\omega, =)$ and $(\omega, \geq)$) are not Borel reducible to either of them.*

Sketch of the proof. If D has cardinality $n > 0$, then, as noticed in [GS11, Theorem 8.4], $\sqsubseteq^\star_D$ is Borel bireducible to the relation of embeddability on subtrees of ${}^{<\omega}\omega$ of height n. Hence the result follows from a classic result of Nash-Williams ([NW65, Theorem 2]). $\square$

The proof of [NW65, Theorem 2] makes an essential use of a strong form of the Axiom of Choice AC (see the proof of [NW65, Lemma 38], in which one argues by induction on a well-ordering of all barriers). However, only very weak versions of AC are actually needed to obtain Proposition 4.13: indeed, in Section 6.2 we will provide an alternative proof of this result in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$ alone (where $\mathsf{AC}_\omega(\mathbb{R})$ is the Axiom of Countable Choice over the Reals).

Proposition 4.14 ([GS11, Theorem 8.3])**.** *If D contains a decreasing sequence converging to 0 then $\sqsubseteq_D$ is complete for analytic quasi-orders.*

Proof. This is essentially [LR05, Proposition 4.2], whose proof involves only spaces in $\mathcal{U}_D$ for $D = \{0, 2^{-n} \mid n \in \omega\}$. For the general case we choose from D a strictly decreasing sequence $(r_n)_{n \in \omega}$ converging to 0, and then in the construction of Louveau and Rosendal we systematically replace the distance 2^{-n} with r_n. $\square$

This has been (essentially) strengthened in [CMMR13] to the following result.

Theorem 4.15 ([CMMR13, Theorem 5.19])**.** *If D contains a decreasing sequence converging to 0 then $\sqsubseteq_D$ is invariantly universal.*

Proof. For $D = \{0, 2^{-n} \mid n \in \omega\}$ this follows from the proof of [CMMR13, Theorem 5.19]. The general case follows from Lemma 4.11. $\square$

The proofs of Proposition 4.14 and Theorem 4.15 do not say anything about the relation $\sqsubseteq_D^\star$: even when $D = \{0\} \cup \{2^{-n} \mid n \in \omega\}$ the spaces involved in those arguments do not realize all distances from D.

5. The complexity of isometry

In this section we fully answer Question 1.2, and as a by-product also Question 1.1.

5.1. **Ill-founded sets of distances.** We first focus on the study of the complexity of the relations $\cong_D$ and $\cong_D^\star$ for an ill-founded $D \in \mathcal{D}$. To this aim, we will use the following combinatorial objects. A *rooted combinatorial tree* G is a connected acyclic graph with a distinguished vertex called the root of G. The collection of all rooted combinatorial trees with universe ω forms a Borel subset RCT of the Polish space $\mathrm{Mod}_{\mathcal{L}}$ of $\mathcal{L}$-structures on ω (for a suitable language $\mathcal{L}$). The relation of isomorphism on RCT is easily seen to be Borel bireducible with isomorphism on trees on ω, and the latter is Borel bireducible to countable graph isomorphism by [FS89].

Each $G \in \mathrm{RCT}$ can be identified in a Borel-in-G way with a graph G' having as domain a subset of ${}^{<\omega}\omega$ closed under subsequences and such that its root is $\emptyset$ and its edge relation coincide with the successor relation, i.e. for $s, t \in G' \subseteq {}^{<\omega}\omega$ we have $s \mathrel{G'} t$ if and only if $s \restriction \mathrm{length}(s) - 1 = t$ or $t \restriction \mathrm{length}(t) - 1 = s$.

Let $D \in \mathcal{D}$ contain a strictly decreasing sequence $(r_n)_{n\in\omega}$ with $r_n \to r > 0$. We define a distance on ${}^{<\omega}\omega$ with values in $\{0\} \cup \{r_n \mid n \in \omega\} \subseteq D$ by

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ r_n & \text{if } s \neq t \text{ and } n \text{ is greatest such that } s \restriction n = t \restriction n. \end{cases}$$

It is easy to check that d is an ultrametric. The completeness of d follows from the fact that $r > 0$.[6] Fix an isometric embedding ρ of $({}^{<\omega}\omega, d)$ into the Urysohn space $\mathbb{U}$. Then ρ induces a Borel map from the subsets of ${}^{<\omega}\omega$ to $\mathcal{U}_D$.

Given $G \in \mathrm{RCT}$, we construct an ultrametric Polish space U_G with distances in D as follows. The domain of U_G is $G' \subseteq {}^{<\omega}\omega$ and the distance d_G is the restriction of d to G'. Notice that $s \in G'$ realizes a distance r_n (i.e. $d_G(s,t) = r_n$ for some $t \in G'$) if and only if either $n < \mathrm{length}(s)$ or $n = \mathrm{length}(s)$ and s is not a terminal node of G' (i.e. there exists $t \in G'$ with $s \subsetneq t$). Notice also that

$$s \subsetneq t \text{ if and only if } d_G(s,t) = r_{\mathrm{length}(s)}. \tag{5.1}$$

Definition 5.1. Let $\theta\colon \mathrm{RCT} \to \mathcal{U}_D$ be the composition of the map sending G to U_G with the map induced by ρ.

Clearly θ is a Borel map.

Theorem 5.2. *The function θ simultaneously reduces isomorphism to isometry, and embeddability to isometric embeddability.*

Proof. Fix first $G, H \in \mathrm{RCT}$ and suppose φ embeds G into H. This induces an embedding $\varphi'\colon G' \to H'$. Since $\varphi'(\emptyset) = \emptyset$ (because $\emptyset$ is the root of both G' and H'), it is easy to check by induction on $\mathrm{length}(s)$ that $\mathrm{length}(\varphi'(s)) = \mathrm{length}(s)$ for every

[6] Had we defined d using a sequence $(r_n)_{n\in\omega}$ converging to 0, the resulting metric would have been a non-complete ultrametric, as if $x \in {}^{\omega}\omega$ then $(x \restriction n)_{n\in\omega}$ would be a non-converging d-Cauchy sequence.

$s \in G'$. Arguing by induction on length(t) and using the previous observation, it follows that $s \subseteq t \iff \varphi'(s) \subseteq \varphi'(t)$ for every $s, t \in G'$. Therefore φ' is a distance preserving map witnessing $U_G \sqsubseteq U_H$, whence $\theta(G) \sqsubseteq \theta(H)$. If moreover φ is assumed to be an isomorphism, then φ' is surjective too, whence $\theta(G) \cong \theta(H)$.

Assume now that φ is an isometric embedding between U_G and U_H. Fix an enumeration without repetitions $(s_n)_{n\in\omega}$ of G' such that $s_n \subseteq s_m \Rightarrow n \leq m$. We will recursively construct a sequence of maps $\varphi_n \colon G' \to H'$ with the following properties:

(i) φ_n is distance preserving (i.e. an isometric embedding);
(ii) $\varphi_n(s_i) = \varphi_i(s_i)$ for every $i < n$;
(iii) length$(\varphi_n(s_n)) =$ length(s_n).

Given such a sequence $(\varphi_n)_{n\in\omega}$, define $\tilde{\varphi} \colon G' \to H'$ by setting $\tilde{\varphi}(s_n) = \varphi_n(s_n)$ for every $n \in \omega$. By (i) and (ii) above, $\tilde{\varphi}$ preserves distances between the spaces U_G and U_H. We now claim that $\tilde{\varphi}$ is a graph embedding of G' into H' (so that G embeds into H). Let $s, t \in G'$ be linked by an edge in G', and assume without loss of generality that $s \subsetneq t$ (so that, in particular, length$(t) =$ length$(s) + 1$). Then by (5.1) we have $d_G(s,t) = r_{\mathrm{length}(s)}$. Since $\tilde{\varphi}$ preserves distances, $d_H(\tilde{\varphi}(s), \tilde{\varphi}(t)) = r_{\mathrm{length}(s)}$. By (iii) above, length$(\tilde{\varphi}(s)) =$ length(s), and therefore $\tilde{\varphi}(t) \supsetneq \tilde{\varphi}(s)$ by (5.1). Since length$(\tilde{\varphi}(t)) =$ length$(t) =$ length$(s) + 1 =$ length$(\tilde{\varphi}(s)) + 1$ by (iii) again, it follows that $\tilde{\varphi}(s)$ and $\tilde{\varphi}(t)$ are linked by an edge in H'. A similar argument shows that if $s, t \in G'$ are such that $\tilde{\varphi}(s)$ and $\tilde{\varphi}(t)$ are linked by an edge in H' then s and t are linked by an edge in G'.

We show next how to recursively construct the sequence $(\varphi_n)_{n\in\omega}$ with the desired properties (i)–(iii). Set $\varphi_{-1} = \varphi$. Let now $n \in \omega$ and suppose that φ_i has been defined for every $0 \leq i < n$ and that it satisfies conditions (i)–(iii) above. We first show that length$(\varphi_{n-1}(s_n)) \geq$ length(s_n). This is clear if $n = 0$, as $s_0 = \emptyset$ by the choice of the enumeration $(s_n)_{n\in\omega}$. If $n > 0$, suppose toward a contradiction that $k =$ length$(\varphi_{n-1}(s_n)) <$ length(s_n), and let $i \in \omega$ be such that $s_i = s_n \restriction k$. Then $i < n$ (since $s_i \subsetneq s_n$), so that length$(\varphi_{n-1}(s_i)) =$ length$(s_i) = k$ by inductive hypothesis (conditions (ii) and (iii)). Therefore length$(\varphi_{n-1}(s_i)) =$ length$(\varphi_{n-1}(s_n)) = k$, whence $d_H(\varphi_{n-1}(s_i), \varphi_{n-1}(s_n)) > r_k$ by injectivity of φ_{n-1}. Since $d_G(s_i, s_n) = r_k$ and φ_{n-1} is distance preserving, this is a contradiction.

We now define φ_n by redefining φ_{n-1} on s_n (and possibly on another sequence), and to do this we distinguish various cases according to the behavior of φ_{n-1}. Let $k =$ length(s_n).

Case 1: length$(\varphi_{n-1}(s_n)) = k$. Then we set $\varphi_n = \varphi_{n-1}$: conditions (i)–(iii) are then trivially satisfied by φ_n by case assumption and inductive hypothesis.

Case 2: length$(\varphi_{n-1}(s_n)) > k$ but $\varphi_{n-1}(s_n) \restriction k$ is not in the range of φ_{n-1}. Then we set $\varphi_n(s_n) = \varphi_{n-1}(s_n) \restriction k$ and $\varphi_n(t) = \varphi_{n-1}(t)$ for every $t \neq s_n$. Then (ii)–(iii) are automatically satisfied. To check that φ_n is still distance preserving, for $t \neq s_n$ let $i \in \omega$ be such that $d_G(t, s_n) = r_i$. Then $i \leq k$ and $d_H(\varphi_{n-1}(t), \varphi_{n-1}(s_n)) = r_i$ (since φ_{n-1} is distance preserving), whence by definition of d_H

$$d_H(\varphi_n(t), \varphi_n(s_n)) = d_H(\varphi_{n-1}(t), \varphi_{n-1}(s_n) \restriction k) = d_H(\varphi_{n-1}(t), \varphi_{n-1}(s_n)) = r_i.$$

Case 3: length$(\varphi_{n-1}(s_n)) > k$ and there is $s_i \neq s_n$ such that $\varphi_{n-1}(s_i) = \varphi_{n-1}(s_n) \restriction k$. In this case we set $\varphi_n(s_n) = \varphi_{n-1}(s_i) = \varphi_{n-1}(s_n) \restriction k$, $\varphi_n(s_i) = \varphi_{n-1}(s_n)$, and $\varphi_n(t) = \varphi_{n-1}(t)$ for every $t \in G' \setminus \{s_n, s_i\}$. It is clear that condition

(iii) is satisfied by definition. Since $d_H(\varphi_{n-1}(s_n), \varphi_{n-1}(s_i)) = r_k$ and φ_{n-1} is distance preserving, (5.1) implies that $s_n \subsetneq s_i$. This implies $n < i$, so that also (ii) is satisfied by φ_n (φ_n coincides with φ_{n-1} on each s_j with $j \neq i, n$, hence in particular on every s_j with $j < n$). It remains only to show that φ_n is distance preserving. Assume first towards a contradiction that there is some $t \in G'$ such that $s_n \subsetneq t \subsetneq s_i$. Then $d_H(\varphi_{n-1}(t), \varphi_{n-1}(s_i)) = d_G(t, s_i) = r_{\mathrm{length}(t)} < r_k$, while $\varphi_{n-1}(s_i)$ can realize, besides 0, only distances $\geq r_k$ in U_H (by $\mathrm{length}(\varphi_{n-1}(s_i)) = k$ and the definition of d_H): this gives the desired contradiction. Similarly, if $t \in G'$ is such that $s_i \subsetneq t$, then $d_H(\varphi_{n-1}(t), \varphi_{n-1}(s_i)) = d_G(t, s_i) = r_{\mathrm{length}(s_i)} < r_k$, contradicting again the fact that in U_H the point $\varphi_{n-1}(s_i)$ can realize, besides 0, only distances $\geq r_k$. Therefore s_i is an immediate successor of s_n (i.e. $\mathrm{length}(s_i) = k + 1$) and is a terminal node in G'. By definition of d_G, this implies that $d_G(s_i, t) = d_G(s_n, t)$ for every $t \in G' \setminus \{s_n, s_i\}$. Since $d_H(\varphi_n(s_i), \varphi_n(t)) = d_H(\varphi_{n-1}(s_n), \varphi_{n-1}(t)) = d_G(s_n, t)$ and $d_H(\varphi_n(s_n), \varphi_n(t)) = d_H(\varphi_{n-1}(s_i), \varphi_{n-1}(t)) = d_G(s_i, t)$, we get $d_H(\varphi_n(s_i), \varphi_n(t)) = d_G(s_i, t)$ and $d_H(\varphi_n(s_n), \varphi_n(t)) = d_G(s_n, t)$. Moreover,

$$d_H(\varphi_n(s_n), \varphi_n(s_i)) = d_H(\varphi_{n-1}(s_i), \varphi_{n-1}(s_n)) = d_G(s_i, s_n)$$

(because φ_{n-1} is distance preserving). Since $\varphi_n \restriction (G \setminus \{s_n, s_i\}) = \varphi_{n-1} \restriction (G \setminus \{s_n, s_i\})$ and φ_{n-1} satisfies (i) by inductive hypothesis, we obtain that φ_n is distance preserving as well, hence we are done.

If moreover φ is an isometry between U_G and U_H, it will be proved that $\tilde{\varphi}$ is surjective: by the previous arguments, this implies that $\tilde{\varphi}$ is an isomorphism between G' and H', whence G and H are isomorphic. Since φ is surjective, it is easy to show by induction on $n \in \omega$ that so are all the φ_n because Case 2 cannot occur. Moreover, whenever $\varphi_n \neq \varphi_{n-1}$, which means that φ_n has been defined according to Case 3, one also has $\mathrm{length}(\varphi_{n-1}(s_n)) = \mathrm{length}(s_n) + 1$: otherwise $\varphi_{n-1}(s_n)$ would realize in H' the distance $r_{\mathrm{length}(s_n)+1}$, while this does not happen for s_n in G'. So φ_n can differ from φ_{n-1} only on two sequences (namely, by the previous argument, s_n and some immediate successor s_i of it) on which φ_{n-1} is not order preserving, but φ_n is. This implies that for every $t \in H'$ the sequence $(\varphi_n^{-1}(t))_{n \in \omega}$ is eventually constant — in fact it takes at most two values. Clearly, its eventual value is $\tilde{\varphi}^{-1}(t)$. □

Corollary 5.3. *Let $D \in \mathcal{D}$ be ill-founded. Then the relation of isometry on $\mathcal{U}_D$ is Borel bireducible with countable graph isomorphism.*

Proof. The relation $\cong_D$ Borel reduces to countable graph isomorphism by Proposition 4.12, so let us show that countable graph isomorphism (or, equivalently, isomorphism between trees on ω, or isomorphism on RCT) Borel reduces to $\cong_D$. If D contains an ill-founded subset bounded away from 0, then apply Theorem 5.2. Otherwise, D contains a decreasing infinitesimal sequence r_n. Then apply the proof of [GK03, Theorem 4.4] noticing that even if such a proof uses the sequence $r_n = 2^{-n}$, it actually works for any decreasing infinitesimal sequence (essentially, this amounts to using Lemma 4.11 above). □

We can now answer Question 1.1. Notice that both classes of ultrametric Polish spaces considered in the next corollary are Borel $\mathbf{\Pi}^1_1$-complete[7].

[7] A reduction of the set of well-founded trees to the class of locally compact subsets of Baire space is obtained by mapping the tree $T \subseteq {}^{<\omega}\omega$ to the body of the pruned tree

$$\{(2s)^\frown n^k \in {}^{<\omega}\omega \mid s \in T \wedge n \text{ is odd} \wedge k \in \omega\},$$

Corollary 5.4. *The relations of isometry on discrete ultrametric Polish spaces and on locally compact ultrametric Polish spaces are both Borel bireducible with countable graph isomorphism.*

Proof. By Proposition 4.12 it is enough to show that $\cong_D$ is bireducible with countable graph isomorphism for some $D \in \mathcal{D}$ bounded away from 0. Taking e.g. $D = \{0\} \cup \{1 + 2^{-n} \mid n \in \omega\}$ and applying Corollary 5.3, we get the desired result. $\square$

Obviously, Corollary 5.4 implies that for every class $\mathcal{A}$ of ultrametric Polish spaces containing all discrete ones, the isometry relation on $\mathcal{A}$ is Borel bireducible with countable graph isomorphism: this includes the class of σ-compact ultrametric Polish spaces and the class of countable ultrametric Polish spaces (which are both Borel $\mathbf{\Pi}^1_1$-complete, see [MR17, Proposition 2.5]).

Observe moreover that a proper analytic equivalence relation on a standard Borel space (and, consequently, any equivalence relation which is Borel bireducible with countable graph isomorphism) cannot be Borel reducible to isomorphism on the collection WF of well-founded trees on ω. Indeed, WF is a $\mathbf{\Pi}^1_1$-complete class, and the map associating to any tree in WF its rank is actually a $\mathbf{\Pi}^1_1$-rank. Since the range of any hypothetical Borel reduction as above would be an analytic subset of WF, by [Kec95, Theorem 35.23] it would be a subset of $\mathcal{T}_\alpha$ (see beginning of Section 3) for some $\alpha < \omega_1$. This is impossible since the isomorphism relation on $\mathcal{T}_\alpha$ is Borel. So the second lower bound found by Kechris and Gao (namely, isomorphism on WF) is not sharp, even though they noticed that it is absolutely $\mathbf{\Delta}^1_2$ bireducible with countable graph isomorphism.

On the other hand we still do not know whether the first lower bound is sharp, which by our results means whether isomorphism between trees on ω with countably many infinite branches is Borel bireducible with countable graph isomorphism (see Question 7.2). This class too is $\mathbf{\Pi}^1_1$-complete, and the isomorphism relation on it is again absolutely $\mathbf{\Delta}^1_2$ bireducible with countable graph isomorphism. Recall also that, as observed in [GK03, Chapter 8], this relation is the same as isometry between countable closed subsets of ${}^\omega\omega$.

Now we deal with $\mathcal{U}^\star_D$ for an ill-founded $D \in \mathcal{D}$. Fix a decreasing sequence $(r_n)_{n\in\omega}$ in D converging to some $r \geq 0$, and an element $\bar{r} \in D$ bigger than every r_n. Set $D' = \{r_n \mid n \in \omega\} \cup \{0\}$. We define a Borel map $f \colon \mathcal{U}_{D'} \to \mathcal{U}_D$ as follows. Given $U \in \mathcal{U}_{D'}$, let $U^* = (U^*, d_{U^*})$ be obtained by gluing U with the canonical ultrametric Polish space $U(D \setminus \{r_0\})$ (see Definition 4.2): this is done by setting for each $x \in U$ and $r' \in D \setminus \{r_0\}$

$$d_{U^*}(x, r') = \max\{\bar{r}, r'\}.$$

Notice that U^* realizes all distances in D, except possibly r_0.

We claim that U^* can be identified in a Borel-in-U way with an element $f(U)$ of $\mathcal{U}_D \subseteq F(\mathbb{U})$. Fix an isometric embedding ρ of $(\mathbb{U}^{\mathcal{U}}_D)^*$ in $\mathbb{U}$. By Lemma 4.6 we can

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where $2s$ is the sequence obtained by doubling every entry of s and n^k is the sequence of length k with all entries equal to n. The same reduction shows also that the class of discrete ultrametric Polish spaces is Borel $\mathbf{\Pi}^1_1$-complete. In the computable (lightface) setting Nies and Solecki recently proved that the class of locally compact ultrametric Polish spaces is Π^1_1-complete using a different construction ([NS15]).

assume $U \in F(\mathbb{U}_D^{\mathcal{U}})$, and hence also $U^* \subseteq (\mathbb{U}_D^{\mathcal{U}})^*$. Thus we can let $f(U)$ to be the image of U^* under ρ.

Theorem 5.5. *The map f reduces $\cong_{D'}$ to $\cong_D$.*

Proof. Let $U_0, U_1 \in \mathcal{U}_{D'}$. If $\varphi \colon U_0 \to U_1$ is an isometry, then $\varphi \cup \mathrm{id} \colon U_0^* \to U_1^*$ is an isometry, where id is the identity on $U(D \setminus \{r_0\})$. Conversely, let $\psi \colon U_0^* \to U_1^*$ be an isometry. Then either $\psi(U_0) \subseteq U_1$ or $\psi(U_0) \subseteq U(D \setminus \{r_0\})$: indeed, any two points in U_0 are at most r_0 apart, while any point in U_1 has distance at least $\bar{r}$ from any point in $U(D \setminus \{r_0\})$. For a similar reason, in the former case we have $\psi(U(D \setminus \{r_0\})) \subseteq U(D \setminus \{r_0\})$, so that actually $\psi(U_0) = U_1$. In the latter, there are some subcases to consider.

If U_0 is not a singleton, then $\psi(U_0) \subseteq U(D \setminus \{r_0\}) \cap [0, \bar{r}) = U((D \setminus \{r_0\}) \cap [0, \bar{r}))$, because $D(U_0) \subseteq D' \subseteq D \cap [0, \bar{r})$ while any $y \in U(D \setminus \{r_0\}) \setminus [0, \bar{r})$ realizes distances $\geq \bar{r}$. Since any $y \in U((D \setminus \{r_0\}) \cap [0, \bar{r})) \setminus \psi(U_0)$ would be less than $\bar{r}$ apart from any point of $\psi(U_0)$, while its preimage has distance $\bar{r}$ from any point of U_0, we conclude that actually $\psi(U_0) = U((D \setminus \{r_0\}) \cap [0, \bar{r})) = U(D' \setminus \{r_0\})$. Since ψ is a bijection it must then be $\psi(U(D' \setminus \{r_0\})) = U_1$, so $U_0 \cong U(D' \setminus \{r_0\}) \cong U_1$ as desired. If U_1 is not a singleton, we can apply the previous argument exchanging the roles of U_0 and U_1 and using ψ^{-1} in place of ψ. The remaining case is when U_0 and U_1 are both singletons, and hence clearly isometric. □

Let $\mathcal{A} = \{U \in \mathcal{U}_{D'} \mid \forall x \in U \, \exists y \in U \, d_{\mathbb{U}}(x, y) = r_0\}$ and notice that $\mathcal{A}$ is closed under isometries. Moreover, arguing as in Proposition 4.5, it is easy to see that $\mathcal{A}$ is Borel.

Lemma 5.6. *The restriction of $\cong$ to $\mathcal{A}$ is Borel bireducible with countable graph isomorphism.*

Proof. If $r > 0$ observe that in the proof of Theorem 5.2 for D' as the set of distances, the range of the reduction is contained in $\mathcal{A}$, since r_0 is the distance between the root of the tree and any other point.

If $r = 0$ we are going to apply the proof of [GK03, Theorem 4.4] to D' as set of distances. This proof shows that the map $T \mapsto [T]$ is a Borel reduction between isomorphism on the set $\mathcal{PT}$ of nonempty pruned subtrees of ${}^{<\omega}\omega$ and isometry on $\mathcal{U}_{D'}$. If $\mathcal{PT}'$ is the set of pruned trees containing all finite sequences all of whose entries equal 0 and no other sequence with a null entry, then isomorphism on $\mathcal{PT}'$ is still Borel bireducible with countable graph isomorphism: this is witnessed by the reduction $T \in \mathcal{PT} \to T' = \{s + 1 \mid s \in T\} \cup \{0^{(n)} \mid n \in \omega\} \in \mathcal{PT}'$, where $s + 1 = (s(i) + 1)_{i < \mathrm{length}(s)}$ and $0^{(n)}$ is the sequence of length n constantly equal to 0. Now notice that if $T \in \mathcal{PT}'$ then $[T] \in \mathcal{A}$. □

Corollary 5.7. *Let $D \in \mathcal{D}$ be ill-founded. Then the relation of isometry on $\mathcal{U}_D^\star$ is Borel bireducible with countable graph isomorphism.*

Proof. The relation $\cong_D^\star$ is Borel reducible to countable graph isomorphism by Proposition 4.12 and $\mathcal{U}_D^\star \subseteq \mathcal{U}_D$. For the other direction, notice that if $U \in \mathcal{A}$, then U^* realizes all distances in D, including r_0, so $U^* \in \mathcal{U}_D^\star$. Now apply Lemma 5.6 and Theorem 5.5. □

5.2. **Well-founded sets of distances.** Let us now consider the case of a well-ordered $D \in \mathcal{D}$. By Lemma 4.11, if D is well-ordered, then up to classwise Borel isomorphism the relations $\cong_D$, $\cong_D^\star$, $\sqsubseteq_D$, and $\sqsubseteq_D^\star$ do not depend really on D but only on its order type. Thus for each $1 \leq \alpha < \omega_1$ we can fix some $D_\alpha \in \mathcal{D}$ with order type α and let $(r_\beta)_{\beta<\alpha}$ be an increasing enumeration of D_α, so that $r_0 = 0$.

To simplify the notation we use $\mathcal{U}_\alpha$ and $\mathcal{U}^\star_\alpha$ in place of $\mathcal{U}_{D_\alpha}$ and $\mathcal{U}^\star_{D_\alpha}$. Similarly, the symbols $\cong_\alpha$, $\cong^*_\alpha$, $\sqsubseteq_\alpha$, and $\sqsubseteq^\star_\alpha$ will abbreviate the corresponding symbols with subscript D_α.

Notice that $D_1 = \{r_0\} = \{0\}$ and every $X \in \mathcal{U}^\star_1 = \mathcal{U}_1$ is a singleton, while if $\alpha > 1$ then every space in $\mathcal{U}^\star_\alpha$ has at least two points because the distance $r_1 > 0$ must be realized. Moreover, by Lemma 4.11 we have

$$\cong_\alpha \leq_B \cong_\beta \text{ and } \sqsubseteq_\alpha \leq_B \sqsubseteq_\beta \text{ for every } 1 \leq \alpha \leq \beta < \omega_1. \tag{5.2}$$

The next lemma shows in particular that (5.2) remains true if we replace all relations with their counterparts with superscript $\star$.

Lemma 5.8. *Let $1 \leq \alpha < \omega_1$. Then $\cong_\alpha \sim_B \cong^*_\alpha$ and $\sqsubseteq_\alpha \sim_B \sqsubseteq^\star_\alpha$.*

Proof. Since $\mathcal{U}_1 = \mathcal{U}^\star_1$, it can be assumed $\alpha \geq 2$. Since $\mathcal{U}^\star_\alpha \subseteq \mathcal{U}_\alpha$, we need only to prove $\cong_\alpha \leq_B \cong^*_\alpha$ and $\sqsubseteq_\alpha \leq_B \sqsubseteq^\star_\alpha$.

We first prove the assertion for $\alpha = \beta + 1$ a successor ordinal. It can be assumed $D_{\beta+1} = D_\beta \cup \{r_\beta\}$. Given $X \in \mathcal{U}_{\beta+1}$, let $X' \in \mathcal{U}^\star_{\beta+1}$ be the space obtained by taking the disjoint union of X and the space $U(D_\beta)$ from Definition 4.2, and setting $d_{X'}(x, r_\xi) = r_\beta$ for every $x \in X$ and $\xi < \beta$. Arguing as in the discussion before Theorem 5.5 one can show that the map $X \mapsto X'$ is Borel: we claim that it is the desired reduction.

Let $X, Y \in \mathcal{U}_{\beta+1}$ and $\varphi \colon X \to Y$ be a witness of $X \sqsubseteq_{\beta+1} Y$: then $\varphi \cup \mathrm{id}$ witnesses $X' \sqsubseteq^\star_{\beta+1} Y'$. If moreover φ is an isometry, then $\varphi \cup \mathrm{id}$ is an isometry as well.

Conversely, let ψ be an isometric embedding of X' into Y', and let $X_0 = \psi^{-1}(U(D_\beta))$. Notice that $d_{X'}(x_0, x_1) < r_\beta$ for every $x_0, x_1 \in X_0$ since $\psi(x_0), \psi(x_1) \in U(D_\beta)$. Hence, by construction of X', either $X_0 \cap X = \emptyset$ (i.e. $X_0 \subseteq U(D_\beta)$), or else $X_0 \subseteq X$. In the former case $\psi \restriction X$ is an isometric embedding of X into Y. If moreover ψ is surjective (i.e. an isometry), then $\psi \restriction X$ is an isometry onto Y: notice indeed that $\psi(U(D_\beta))$ cannot intersect both Y and $U(D_\beta)$, since any two points of $U(D_\beta)$ are less than r_β apart.

If instead $X_0 \subseteq X$, we claim that $\varphi = \psi \restriction (X \setminus X_0) \cup (\psi \circ \psi) \restriction X_0$ witnesses $X \sqsubseteq_{\beta+1} Y$. Notice that φ is well-defined by the definition of X_0 and the fact that $U(D_\beta) \subseteq X'$. First we check that the range of φ is contained in Y. If $x \in X \setminus X_0$ then trivially $\varphi(x) = \psi(x) \in Y$ by definition of X_0. If $x \in X_0 \subseteq X$ then $\psi(x) \in U(D_\beta) \subseteq X'$. Since $x \in X$, we have that $d_{X'}(x, \psi(x)) = r_\beta$, whence also $d_{Y'}(\psi(x), \psi(\psi(x))) = r_\beta$. Since any two points in $U(D_\beta)$ are $< r_\beta$ apart in Y' and $\psi(x) \in U(D_\beta)$, it follows that $\varphi(x) = \psi(\psi(x)) \in Y$. Finally, we check that φ preserves distances. It is clearly enough to show that for $x \in X \setminus X_0$ and $x' \in X_0$ we have $d_X(x, x') = d_Y(\varphi(x), \varphi(x'))$. Since $\psi(x) \in Y$ and $\psi(x') \in U(D_\beta)$, we have $d_{Y'}(\psi(x), \psi(x')) = r_\beta$, whence $d_X(x, x') = d_{X'}(x, x') = d_{Y'}(\psi(x), \psi(x')) = r_\beta$. Since $\psi(x') \in U(D_\beta) \subseteq X'$ and $x \in X$, $d_{X'}(x, \psi(x')) = r_\beta$, therefore $d_{Y'}(\psi(x), \psi(\psi(x'))) = r_\beta$. Since $\psi(x) = \varphi(x)$ and $\psi(\psi(x')) = \varphi(x')$ by definition of φ, we have $d_Y(\varphi(x), \varphi(x')) = d_{Y'}(\varphi(x), \varphi(x')) = r_\beta = d_X(x, x')$, as required. If moreover ψ is surjective (i.e. an isometry), by case hypothesis $\psi(U(D_\beta)) \subseteq Y$, and we also have that $\psi(X_0) = U(D_\beta)$. So if $y \in Y \setminus \psi(X \setminus X_0)$, which means $y \in \psi(U(D_\beta))$, it follows that $y = \varphi(x)$ for some $x \in X_0$. This implies that φ is surjective as well, hence it is an isometry between X and Y.

Now we prove the result for $\alpha \geq \omega$. Given $X \in \mathcal{U}_\alpha$, let $X' \in \mathcal{U}^\star_\alpha$ be the ultrametric space with domain $X \times \alpha$ and distance function $d_{X'}$ defined by setting, for distinct $(x,\xi), (y,\xi') \in X'$ such that $d_X(x,y) = r_\eta$,

$$d_{X'}((x,\xi),(y,\xi')) = \max\{r_\xi, r_{\xi'}, r_{1+\eta}\}. \tag{5.3}$$

($1+\eta < \alpha$ because α is infinite). Recalling that $r_0 = 0$ we then have in particular that

- for all distinct $x, y \in X$, if $d_X(x,y) = r_\eta$ then $d_{X'}((x,0),(y,0)) = r_{1+\eta}$;
- for all $x \in X$ and $\xi < \alpha$, $d_{X'}((x,0),(x,\xi)) = r_\xi$.

We claim that the Borel[8] map $X \mapsto X'$ reduces $\cong_\alpha$ to $\cong^\star_\alpha$ and $\sqsubseteq_\alpha$ to $\sqsubseteq^\star_\alpha$.

Let $X, Y \in \mathcal{U}_\alpha$. First assume that $\varphi\colon X \to Y$ is an isometric embedding of X into Y. Define $\psi\colon X' \to Y'$ by setting $\psi(x,\xi) = (\varphi(x),\xi)$ for every $x \in X$ and $\xi < \alpha$. Then ψ is distance preserving. Indeed, fix $x, y \in X$. If $d_X(x,y) = r_\eta$ with $0 < \eta < \alpha$, then $d_Y(\varphi(x),\varphi(y)) = r_\eta$ and hence $d_{X'}((x,0),(y,0)) = r_{1+\eta} = d_{Y'}((\varphi(x),0),(\varphi(y),0))$. Therefore, by definition of ψ and using (5.3) we get for arbitrary $\xi, \xi' < \alpha$ such that $(x,\xi) \neq (y,\xi')$

$$\begin{aligned} d_{X'}((x,\xi),(y,\xi')) &= \max\{r_\xi, r_{\xi'}, d_{X'}((x,0),(y,0))\} \\ &= \max\{r_\xi, r_{\xi'}, d_{Y'}((\varphi(x),0),(\varphi(y),0))\} \\ &= d_{Y'}((\varphi(x),\xi),(\varphi(y),\xi')) \\ &= d_{Y'}(\psi(x,\xi),\psi(y,\xi')). \end{aligned}$$

If, in addition, φ is surjective (i.e. an isometry), then ψ is surjective as well, and hence it witnesses $X' \cong Y'$.

Assume now that ψ is an isometric embedding of X' into Y'. Notice that for every $x \in X$ there is $y \in Y$ such that either $\psi(x,0) = (y,0)$ or $\psi(x,0) = (y,1)$. This is because $(x,0)$ realizes all distances in X' (i.e. for every $\xi < \alpha$ there is $x' \in X'$ with $d_{X'}((x,0),x') = r_\xi$), hence $\psi(x,0)$ must have the same property relative to Y', which implies that $\psi(x,0)$ is of the prescribed form because by (5.3) points of the form (y,ξ) realize only the distances $r_{\xi'}$ with $\xi' = 0$ or $\xi' \geq \xi$. Let $\varphi(x)$ be such y. Notice that the function $\varphi\colon X \to Y$ is injective, as if $x \neq x' \in X$ and $\varphi(x) = \varphi(x')$ then either $\psi(x,0) = (y,0)$ and $\psi(x',0) = (y,1)$, or $\psi(x,0) = (y,1)$ and $\psi(x',0) = (y,0)$ by injectivity of ψ: in both cases $d_{Y'}(\psi(x,0),\psi(x',0)) = r_1 < d_{X'}((x,0),(x',0))$, contradicting the fact that ψ is distance preserving. We claim that φ is also distance preserving (hence an isometric embedding). Let $x, x' \in X$ be distinct points. Then $d_{Y'}((\varphi(x),i),(\varphi(x'),j)) > r_1$ for every $i, j \in \{0,1\}$ by injectivity of φ and (5.3), whence $d_{Y'}(\psi(x,0),\psi(x',0)) = d_{Y'}((\varphi(x),0),(\varphi(x'),0))$ by (4.1). Since ψ preserves distances, $d_{X'}((x,0),(x',0)) = d_{Y'}((\varphi(x),0),(\varphi(x'),0))$, and hence $d_X(x,x') = d_Y(\varphi(x),\varphi(x'))$ by definition of $d_{X'}$ and $d_{Y'}$ on X' and Y', respectively. If ψ is moreover assumed to be surjective (i.e. an isometry), then we can argue as above and show that for every $y \in Y$ there exist $x \in X$ and $i \in \{0,1\}$ such that $\psi(x,i) = (y,0)$. If $i = 0$, this means $\varphi(x) = y$; if $i = 1$, then $\psi(x,0) = (y,1)$ because $(x,0)$ and $(y,1)$ are the unique points at distance r_1 from, respectively, $(x,1)$ and $(y,0)$, so again $\varphi(x) = y$. It follows that φ is surjective too, and hence it witnesses $X \cong Y$. ☐

[8] The map is Borel again by the argument before Theorem 5.5. This applies to all reductions in the subsequent proofs, so from this point on we will not recall it explicitly.

Remark 5.9. Although $\sqsubseteq_\alpha \sim_B \sqsubseteq^\star_\alpha$ the two quasi-orders are combinatorially different: $\sqsubseteq_\alpha$ has always a minimum, while $\sqsubseteq^\star_\alpha$ has at least two $\sqsubseteq$-incomparable minimal elements when $\alpha \geq 4$.

Corollary 5.10. *For every $D \in \mathcal{D}$, $\cong_D \sim_B \cong^\star_D$.*

Proof. If D is ill-founded then both $\cong_D$ and $\cong^\star_D$ are Borel bireducible with countable graph isomorphism by Corollaries 5.3 and 5.7, whence $\cong_D \sim_B \cong^\star_D$. If instead D is well-founded, then the result follows from Lemma 5.8. □

The following points out that if we restrict ourselves to Borel reducibility we can weaken the hypotheses of Lemma 4.11.

Corollary 5.11. *Let $D, D' \in \mathcal{D}$. If there is an order-preserving embedding from D into D', then $\cong_D \leq_B \cong_{D'}$ and $\cong^\star_D \leq_B \cong^\star_{D'}$.*

Proof. If D' is well-ordered, then D is well-ordered as well, and the hypothesis of Lemma 4.11 is satisfied. Otherwise, $\cong_{D'}$ and $\cong^\star_{D'}$ are Borel bireducible with countable graph isomorphism by Corollaries 5.3 and 5.7, and hence the result follows from Proposition 4.12 and $\mathcal{U}^\star_D \subseteq \mathcal{U}_D$. □

Lemma 5.12. *Let $\lambda < \omega_1$ be limit. If $\cong_\beta$ is Borel for every $\beta < \lambda$, then $\cong_\lambda$ is Borel. If $\sqsubseteq_\beta$ is Borel for every $\beta < \lambda$, then $\sqsubseteq_\lambda$ is Borel as well.*

Proof. Recall the definition of C^x_r from Notation 4.7, and notice in particular that if $U \in \mathcal{U}_\lambda$ then for every $\beta < \lambda$ and $x \in U$ realizing the distance r_β, we have $C^x_{r_\beta}(U) \in \mathcal{U}_{\beta+1}$. Given $U \in \mathcal{U}_\lambda$, one has $U = \{\psi_n(U)\}_{n\in\omega}$ (where the Borel functions ψ_n are as in Notation 4.4).

By Lemma 4.8, it follows that for every $U, U' \in \mathcal{U}_\lambda$ we have $U \cong_\lambda U'$ if and only if

$$\begin{aligned}\exists n \in \omega\, \forall \beta < \lambda\, ((C^{\psi_0(U)}_{r_\beta}(U) \neq \emptyset \iff C^{\psi_n(U')}_{r_\beta}(U') \neq \emptyset) \land \\ (C^{\psi_0(U)}_{r_\beta}(U) \neq \emptyset \implies C^{\psi_0(U)}_{r_\beta}(U) \cong_{\beta+1} C^{\psi_n(U')}_{r_\beta}(U'))).\end{aligned}$$

Similarly, we have $U \sqsubseteq_\lambda U'$ if and only if

$$\exists n \in \omega\, \forall \beta < \lambda\, (C^{\psi_0(U)}_{r_\beta}(U) \neq \emptyset \implies C^{\psi_0(U)}_{r_\beta}(U) \sqsubseteq_{\beta+1} C^{\psi_n(U')}_{r_\beta}(U')).$$

Since each $\cong_{\beta+1}$ (respectively, $\sqsubseteq_{\beta+1}$) is assumed to be Borel it follows that $\cong_\lambda$ (respectively, $\sqsubseteq_\lambda$) is Borel as well. □

Theorem 5.13. *For all $1 < \alpha < \omega_1$, $\cong_{\alpha+1} \sim_B E_{{\cong_\alpha}^{\mathrm{inj}}}$ and $\sqsubseteq_{\alpha+1} \sim_B {\sqsubseteq_\alpha}^{\mathrm{inj}}$.*

Proof. First we show ${\sqsubseteq_\alpha}^{\mathrm{inj}} \leq_B \sqsubseteq_{\alpha+1}$ and $E_{{\cong_\alpha}^{\mathrm{inj}}} \leq_B \cong_{\alpha+1}$. To each sequence of spaces $\vec{X} = (X_n)_{n\in\omega} \in {}^\omega(\mathcal{U}_\alpha)$, associate the disjoint union $\Phi(\vec{X}) = \bigcup_{n\in\omega} X_n$, where $d_{\Phi(\vec{X})}(x, x') = r_\alpha$ whenever $x \in X_i$ and $x' \in X_j$ with $i \neq j$.

If $\vec{X} = (X_n)_{n\in\omega}$ and $\vec{Y} = (Y_n)_{n\in\omega}$ are such that $\vec{X} \mathrel{{\sqsubseteq_\alpha}^{\mathrm{inj}}} \vec{Y}$, let $f \colon \omega \to \omega$ be an injection with $\varphi_n \colon X_n \to Y_{f(n)}$ an isometric embedding for all $n \in \omega$. Then $\bigcup_{n\in\omega} \varphi_n \colon \Phi(\vec{X}) \to \Phi(\vec{Y})$ is an isometric embedding. If moreover $\vec{X} \mathrel{E_{{\cong_\alpha}^{\mathrm{inj}}}} \vec{Y}$, then by Lemma 3.11 this is witnessed by a bijection $f \colon \omega \to \omega$ such that for every $n \in \omega$ there is an isometry $\varphi_n \colon X_n \to Y_{f(n)}$. Thus $\bigcup_{n\in\omega} \varphi_n \colon \Phi(\vec{X}) \to \Phi(\vec{Y})$ is an isometry.

Conversely, let $\varphi \colon \Phi(\vec{X}) \to \Phi(\vec{Y})$ be an isometric embedding. Then for each $n \in \omega$ there is $f(n) \in \omega$ such that $\varphi(X_n) \subseteq Y_{f(n)}$: indeed two points in X_n

are closer to each other than r_α, which is the distance between any two points belonging to distinct Y_m. For a similar reason, the function $f\colon \omega \to \omega$ is injective, as points $x \in X_n$, $x' \in X_{n'}$ for $n \neq n'$ have distance r_α, which is bigger than any distance between points from a single Y_m. The restriction of φ to each X_n is then an isometric embedding into $Y_{f(n)}$, so $\vec{X} \sqsubseteq_\alpha{}^{\mathrm{inj}} \vec{Y}$. If moreover φ is an isometry onto $\Phi(\vec{Y})$, then f is surjective too and $\varphi \restriction X_n \colon X_n \to Y_{f(n)}$ is an isometry. Hence f witnesses $\vec{X}\ E_{\cong_\alpha{}^{\mathrm{inj}}}\ \vec{Y}$.

We now show that $\sqsubseteq_{\alpha+1} \leq_B \sqsubseteq_\alpha{}^{\mathrm{inj}}$ and $\cong_{\alpha+1} \leq_B E_{\cong_\alpha{}^{\mathrm{inj}}}$. For each $X \in \mathcal{U}_{\alpha+1}$ and $x, x' \in X$, define $x\ E_X\ x' \Leftrightarrow d_X(x,x') < r_\alpha$. This is an equivalence relation on X which, by countability of X, has countably many equivalence classes. Let $(X_n)_{n\in\omega}$ be an enumeration of the E_X-equivalence classes where, if there are finitely many of them, say m, then $X_n = \emptyset$ for $n \geq m$.

Suppose first that $\alpha = \beta+1$ is a successor ordinal. Let $X'_n = X_n \cup \{*_n\}$, where $*_n$ are new elements and extend the distance on X_n by setting $d_{X'_n}(x, *_n) = r_\beta$ for all $x \in X_n$ (here we use $\alpha > 1$). We consider the map $X \mapsto (X'_n)_{n\in\omega}$.

If $\varphi\colon X \to Y$ is an isometric embedding, then it injectively maps E_X-classes into E_Y-classes, thus inducing a well-defined *partial* injective map f satisfying $\varphi(X_n) \subseteq Y_{f(n)}$ for every n such that $X_n \neq \emptyset$. By its definition, if the domain of f is not already all of ω, such domain is finite and thus f can be arbitrarily extended to an injection of ω into itself which will still be denoted by f. Finally define the isometric embedding $\varphi_n\colon X'_n \to Y'_{f(n)}$ by extending $\varphi \restriction X_n$ with the addition of the condition $\varphi_n(*_n) = *_{f(n)}$: this shows that f witnesses $(X'_n)_{n\in\omega} \sqsubseteq_\alpha{}^{\mathrm{inj}} (Y'_n)_{n\in\omega}$. If φ is in addition an isometry onto Y, then the number of E_X-classes equals the number of E_Y-classes, say $m \leq \omega$, the function f is a permutation of m, and every $\varphi \restriction X_n$ is an isometry onto $Y_{f(n)}$. Extending f to a bijection $f\colon \omega \to \omega$, the functions φ_n defined above turn out to be isometries as well, so that f witnesses $(X'_n)_{n\in\omega} E_{\cong_\alpha{}^{\mathrm{inj}}} (Y'_n)_{n\in\omega}$.

Conversely, assume $(X'_n)_{n\in\omega} \sqsubseteq_\alpha{}^{\mathrm{inj}} (Y'_n)_{n\in\omega}$ so that there are isometric embeddings $\varphi_n\colon X'_n \to Y'_{f(n)}$ for some injective $f\colon \omega \to \omega$. If $\varphi_n(X_n) \subseteq Y_{f(n)}$, then let $\varphi'_n = \varphi_n \restriction X_n$. Otherwise, let $a \in X_n$ with $\varphi_n(a) = *_{f(n)}$. In this case, for $x \in X_n$ define

$$\varphi'_n(x) = \begin{cases} \varphi_n(x) & \text{if } x \neq a \\ \varphi_n(*_n) & \text{if } x = a. \end{cases}$$

It is not hard to check that, whenever $X_n \neq \emptyset$, $\varphi'_n\colon X_n \to Y_{f(n)}$ is still an isometric embedding. So $\varphi = \bigcup_n \varphi'_n$ is an isometric embedding $X \to Y$. If the stronger condition $(X'_n)_{n\in\omega}\ E_{\cong_\alpha{}^{\mathrm{inj}}}\ (Y'_n)_{n\in\omega}$ holds then, by Lemma 3.11, f can be assumed to be a bijection with each φ_n an isometry onto $Y'_{f(n)}$. So the functions φ'_n, as well as φ, turn out to be isometries.

Let now α be infinite. Define a new distance d'_{X_n} on X_n by letting $d'_{X_n}(x,x') = r_{1+\beta}$ for distinct x, x', where $r_\beta = d(x,x')$. Let $X'_n = X_n \cup \{x^* \mid x \in X_n\}$ be a disjoint union of two copies of X_n, where the ultrametric d'_{X_n} is extended by declaring $d_{X'_n}(x,x^*) = r_1$ for all $x \in X_n$. (Thus $d_{X'_n}(x^*,y) = d_{X'_n}(x,y^*) = d'_{X_n}(x,y)$ when $x \neq y$.) Finally, let X''_n be X'_n if this is nonempty, and consist of exactly one point if $X_n = X'_n = \emptyset$. Notice that X''_n is a singleton if and only if $X_n = \emptyset$, and that if $X_n \neq \emptyset$ then for every $x, y \in X''_n$ at distance r_1 we have that either $x = y^*$ or $y = x^*$. We claim that the map $X \mapsto (X''_n)_{n\in\omega}$ is the desired reduction.

Assume $\varphi\colon X \to Y$ is an isometric embedding. Again, this induces a partial injection f such that $\varphi(X_n) \subseteq Y_{f(n)}$ for all $n \in \omega$ for which $X_n \neq \emptyset$, which can then be extended arbitrarily to an injection $f\colon \omega \to \omega$. If $X_n \neq \emptyset$, then $\varphi_n = \varphi \restriction X_n$ can be extended to an isometric embedding $\varphi'_n\colon X''_n \to Y''_{f(n)}$ by letting $\varphi'_n(x^*) = (\varphi_n(x))^*$ for all $x \in X_n$. If instead $X_n = \emptyset$, then any function $\varphi'_n\colon X''_n \to Y''_{f(n)}$ is an isometric embedding. Thus the injection f and the isometric embeddings φ'_n witness $(X''_n)_{n\in\omega} \sqsubseteq_\alpha{}^{\mathrm{inj}} (Y''_n)_{n\in\omega}$. In case f is an isometry, then exactly as before one gets a bijection f and isometries φ'_n witnessing $(X''_n)_{n\in\omega}\ E_{\cong_\alpha{}^{\mathrm{inj}}}\ (Y''_n)_{n\in\omega}$.

Conversely, let $f\colon \omega \to \omega$ be an injection and $\varphi_n\colon X''_n \to Y''_{f(n)}$ be isometric embeddings witnessing $(X''_n)_{n\in\omega} \sqsubseteq_\alpha{}^{\mathrm{inj}} (Y''_n)_{n\in\omega}$. This implies $Y_{f(n)} \neq \emptyset$ whenever $X_n \neq \emptyset$. Moreover, for such an n and any $x \in X_n$, we have $\{\varphi_n(x), \varphi_n(x^*)\} = \{y, y^*\}$ for some $y \in Y_{f(n)}$. This defines an isometric embedding $\psi_n\colon X_n \to Y_{f(n)}\colon x \mapsto y$. Then $\bigcup_n \psi_n\colon X \to Y$ is an isometric embedding. As before, the stronger hypothesis $(X''_n)_{n\in\omega}\ E_{\cong_\alpha{}^{\mathrm{inj}}}\ (Y''_n)_{n\in\omega}$ allows us to assume that f is bijective (use Lemma 3.11) with each φ_n an isometry. Consequently, every ψ_n (for n such that $X_n \neq \emptyset$) is an isometry, implying that $\bigcup_n \psi_n$ is an isometry as well. $\square$

Remark 5.14. Although $\sqsubseteq_{\alpha+1}$ and $\sqsubseteq_\alpha{}^{\mathrm{inj}}$ are Borel bireducible, their quotient orders are not isomorphic. They both have a bottom element $\bot$: however, in the former every immediate successor x of $\bot$ has a unique immediate successor whose predecessors are exactly $\bot$ and x, while in the latter this property fails.

Theorem 5.15. *The relations $\cong_\alpha$, for $1 \leq \alpha < \omega_1$, form a strictly increasing chain of Borel equivalence relations which is cofinal among Borel equivalence relations classifiable by countable structures.*

Proof. Inductively, all relations $\cong_\alpha$ are Borel by applying Theorem 5.13 and Proposition 3.12 in the successor step, and Lemma 5.12 in the limit step. That $\cong_\alpha$ is a strictly increasing sequence follows from (5.2) and Theorem 5.13 together with Proposition 3.12.

To verify cofinality, by Proposition 3.1 it is enough to show that the isomorphism relation on $\mathcal{T}_\alpha$ Borel reduces to $\cong_{1+\alpha}$. Given any well-founded tree T, let rk_T be the function assigning to each node of T its rank in T, which is an ordinal smaller than the rank of T. Given $T \in \mathcal{T}_\alpha$, let

$$T' = T \cup \{s^\frown 0 \mid s \text{ is a terminal node in } T\}.$$

Then for every $t \in T$ we have $\mathrm{rk}_{T'}(t) = 1 + \mathrm{rk}_T(t) < 1 + \alpha$. Now define an ultrametric $d_{T'}$ on T' by letting, for distinct s and t, $d_{T'}(s,t) = r_{\mathrm{rk}_{T'}(u)}$, where u is the longest common initial segment of s and t. Notice that

$$d_{T'}(t,u) = r_1 \iff \exists s \text{ terminal node of } T \text{ such that } \{t,u\} = \{s, s^\frown 0\}. \tag{5.4}$$

On the other hand, if s is not a terminal node of T, then the least non-null distance realized by s is $r_{\mathrm{rk}_{T'}(s)} = r_{1+\mathrm{rk}_T(s)} > r_1$.

By construction, if T_0 and T_1 are isomorphic trees then $(T'_0, d_{T'_0})$ and $(T'_1, d_{T'_1})$ are isometric.

Conversely, let φ be an isometry between the spaces $(T'_0, d_{T'_0})$ and $(T'_1, d_{T'_1})$. Let $\psi\colon T_0 \to T_1$ be defined by letting $\psi(t) = \varphi(t)$ if t is not terminal in T_0, and

$$\psi(s) = \text{the unique terminal node } s' \text{ of } T_1 \text{ such that } \varphi(s) \in \{s', s'^\frown 0\}$$

if s is terminal in T_0 (the existence of such an s' is guaranteed by (5.4)). Notice that the map ψ still preserves distances, and that it is a bijection between T_0 and T_1. We claim that ψ is an isomorphism, i.e. that it preserves the tree-ordering relation (namely, inclusion) on T_0 and T_1. First notice that $\mathrm{rk}_{T'_0}(s) = \mathrm{rk}_{T'_1}(\psi(s))$ for all $s \in T_0$: for terminal nodes this is obvious from the definition of ψ, while if s is not terminal it follows from the above observation about the smallest non-null distance realized by s. Let now $s \subsetneq t \in T_0$. Then $d_{T'_0}(s,t) = \mathrm{rk}_{T'_0}(s) = \mathrm{rk}_{T'_1}(\psi(s))$. Let u be the longest common subsequence of $\psi(s)$ and $\psi(t)$, so that by definition $d_{T'_1}(\psi(s), \psi(t)) = \mathrm{rk}_{T'_1}(u)$. Then by the above computations we get $\mathrm{rk}_{T'_1}(u) = \mathrm{rk}_{T'_1}(\psi(s))$ because ψ preserves distances. But since $u \subseteq \psi(s)$, this implies $u = \psi(s)$, whence $\psi(s) = u \subsetneq \psi(t)$. A similar argument shows that if $\psi(s) \subsetneq \psi(t)$ then $s \subsetneq t$, hence we are done. □

In [GS11, Section 6], Gao and Shao provide a faithful translation of ultrametric Polish spaces as a certain kind of combinatorial objects. More precisely, given a countable $D \in \mathcal{D}$ they define the class $\mathcal{T}_D$ of *D-trees* and show that there are two maps $\Phi \colon \mathcal{U}_D \to \mathcal{T}_D$ and $\Psi \colon \mathcal{T}_D \to \mathcal{U}_D$ such that the following holds:

- for all $U \in \mathcal{U}_D$ and $T \in \mathcal{T}_D$, $\Psi(\Phi(U))$ is isometric to U and $\Phi(\Psi(T))$ is isomorphic to T (as structures in the appropriate language);
- Φ reduces isometry (respectively, isometric embeddability) on $\mathcal{U}_D$ to isomorphism (respectively, embeddability) on $\mathcal{T}_D$;
- Ψ reduces isomorphism (respectively, embeddability) on $\mathcal{T}_D$ to isometry (respectively, isometric embeddability) on $\mathcal{U}_D$.

Therefore our results on $\cong_D$ (in particular Corollary 5.3 and Theorem 5.15) can be translated as results about isomorphism between D-trees, and an analogous observation will apply to our results on $\sqsubseteq_D$ (in particular Theorems 6.3 and 6.11). Notice also that when D has order type ω the class $\mathcal{T}_D$ essentially coincides with the class of *reverse trees* considered by Clemens in [Cle07], where he observes that isomorphism between reverse trees is another lower bound for the complexity of isometry on discrete ultrametric Polish spaces. Since Theorem 5.15 shows in particular that such a relation is Borel, also this lower bound is not sharp.

6. The complexity of isometric embeddability

This section is devoted to Question 1.3, answering it in most cases.

6.1. **Ill-founded sets of distances.**

In this subsection we complete the study, started in Section 5.1, of the complexity with respect to Borel reducibility of $\sqsubseteq_D$ and $\sqsubseteq^\star_D$ when $D \in \mathcal{D}$ is not well-founded. In fact we will show that they are both invariantly universal (hence also complete for analytic quasi-orders), improving in this way Proposition 4.14 and Theorem 4.15.

Building on previous work in [FMR11], in [CMMR13, Section 3] the authors constructed a class $\mathbb{G}$ of countable rooted combinatorial trees with the following properties (see [CMMR13, Corollaries 3.2 and 3.4]):

Fact 6.1. *(1) $\mathbb{G}$ is a Borel subset of* $\mathrm{Mod}_{\mathcal{L}} = {}^{\omega\times\omega}2$ *(the Polish space of countable $\mathcal{L}$-structures, where $\mathcal{L}$ is the language consisting of one binary relation symbol), so it is a standard Borel space;*

(2) on $\mathbb{G}$ equality and isomorphism coincide;

(3) each $G \in \mathbb{G}$ is a graph which is infinite and rigid, i.e. its unique automorphism is the identity function.

(4) every $G \in \mathbb{G}$ has a distinguished vertex $r(G)$ (the root *of G) such that every embedding of G into $H \in \mathbb{G}$ must send $r(G)$ to $r(H)$.*

Thus $\mathbb{G}$ can be actually considered as a Borel subset of RCT (see the beginning of section 5.1), so that each $G \in \mathbb{G}$ can be identified in a Borel-in-G way with a graph G' on ${}^{<\omega}\omega$ with $r(G)$ identified with the empty sequence $\emptyset \in G'$. The class of graphs $\mathbb{G}$ was the key tool for the method developed in [CMMR13] for proving invariant universality of a given pair (S, E); this is summarized in the following theorem.

Theorem 6.2. *Let S be an analytic quasi-order on a standard Borel space Z, and $E \subseteq S$ be an analytic equivalence relation on the same space. Denote by $\sqsubseteq_{\mathbb{G}}$ and $\cong_{\mathbb{G}}$ the restrictions to $\mathbb{G}$ of the embeddability and isomorphism relations, respectively. Suppose there exists a Borel function $f\colon \mathbb{G} \to Z$ which simultaneously witnesses $\sqsubseteq_{\mathbb{G}} \leq_B S$ and $=_{\mathbb{G}} \leq_B E$ (which is the same as $\cong_{\mathbb{G}} \leq_B E$). Furthermore, let Y be a Polish group, $a\colon Y \times W \to W$ a Borel action of Y on a standard Borel space W, and $g\colon Z \to W$ witness $E \leq_B E_a$. Consider the map $\Sigma\colon \mathbb{G} \to F(Y)$ which assigns to $G \in \mathbb{G}$ the stabilizer of $(g \circ f)(G)$ with respect to a, i.e.*

$$\Sigma(G) = \{y \in Y \mid a(y, (g \circ f)(G)) = (g \circ f)(G)\}.$$

If Σ is Borel, then the pair (S, E) is invariantly universal.

Theorem 6.3. *If D is ill-founded, then $\sqsubseteq_D$ is invariantly universal (hence also complete for analytic quasi-orders).*

Proof. By Theorem 4.15, we can assume that 0 is not a limit point of D and that D contains a strictly decreasing sequence of distances $(r_n)_{n\in\omega}$ converging to some $r \neq 0$. In particular, the spaces in $\mathcal{U}_D$ are discrete. To prove the theorem, we will apply Theorem 6.2 with $S = \sqsubseteq_D \restriction Z$ and $E = \cong_D \restriction Z$ for Z a suitable Borel subset of $\mathcal{U}_D$ closed under isometry; then the invariant universality of (S, E) implies that $(\sqsubseteq_D, \cong_D)$ is invariantly universal as well, as desired.

Set

$$Z = \{U \in \mathcal{U}_D \mid U \text{ is infinite}\},$$

so that Z satisfies the required conditions. Recall the definition of the Borel function $\theta\colon \mathrm{RCT} \to \mathcal{U}_D$ from Definition 5.1, and notice that $\theta(\mathbb{G}) \subseteq Z$ because each $G \in \mathbb{G}$ is infinite. By Theorem 5.2, the map θ simultaneously reduces $=_{\mathbb{G}}$ to $\cong_D$ and $\sqsubseteq_{\mathbb{G}}$ to $\sqsubseteq_D$. In particular, this already shows that $\sqsubseteq_D$ is complete.

The next step to apply Theorem 6.2 is to reduce the relation $E = \cong_D \restriction Z$ to a Borel group action. Consider the countable language $\Lambda = \{R_q \mid q \in D\}$, where each R_q is a binary predicate. To each $U \in Z$ associate the Λ-structure $S(U)$ on ω by letting

$$R_q^{S(U)}(i, j) \iff d_U(\psi_i(U), \psi_j(U)) = q,$$

where the functions ψ_n are as in Notation 4.4. Recall also that since every $U \in Z$ is discrete and infinite, $(\psi_n(U))_{n\in\omega}$ is an enumeration without repetitions of all points in U. It is clear that if $U_0, U_1 \in Z$ and φ is an isometry between them, then the map $f_\varphi\colon \omega \to \omega$ sending i to the unique j such that $\varphi(\psi_i(U_0)) = \psi_j(U_1)$ is an isomorphism between $S(U_0)$ and $S(U_1)$. Conversely, if f is an isomorphism between $S(U_0)$ and $S(U_1)$ then the map $\varphi_f \colon U_0 \to U_1$ sending $\psi_i(U_0)$ to $\psi_{f(i)}(U_1)$ is an isometry. Therefore

$$g \colon Z \to \mathrm{Mod}_\Lambda \colon U \mapsto S(U)$$

is a Borel map reducing the relation of isometry on Z to the isomorphism relation on Mod_Λ, which is the orbit equivalence induced by the continuous logic action j_Λ of the Polish group S_∞ of all permutations of ω on the Polish space Mod_Λ of Λ-models with domain ω.

Therefore, to apply Theorem 6.2 it remains only to show that the map Σ assigning to each $G \in \mathbb{G}$ the stabilizer of $S(\theta(G))$ with respect to j_Λ is a Borel map. To see this, notice that by the above discussion every automorphism h of $S(\theta(G))$ can be identified with an isometry of $\theta(G)$ and hence with an isometry φ_h of U_G into itself. Conversely, every isometry φ of U_G into itself can be identified with an automorphism h_φ of $S(\theta(G))$. Since each $G \in \mathbb{G}$ is rigid, then there are no distinct terminal nodes in G' sharing the same immediate predecessor. From this fact and the analysis performed in the proof of Theorem 5.2, it follows that if φ is an isometry of U_G into itself and $s \in G'$ is such that $\varphi(s) \neq s$, then one of $s, \varphi(s)$ is a terminal node of G' and the other is its immediate predecessor, and moreover $\varphi(\varphi(s)) = s$. Therefore φ can differ from the identity function only in that it may switch some terminal nodes of G' with their immediate predecessor. Recall the definition of ρ from the beginning of Section 5.1 and notice that the maps sending $G \in \mathbb{G}$ to, respectively,

$$T_G = \{n \in \omega \mid (\rho^{-1} \circ \psi_n \circ \theta)(G) \text{ is terminal in } G'\}$$

and

$$P_G = \{(n,m) \in \omega \times \omega \mid (\rho^{-1} \circ \psi_n \circ \theta)(G) \text{ is an immediate predecessor of } (\rho^{-1} \circ \psi_m \circ \theta)(G) \text{ in } G'\}$$

are Borel. We then get for $h \in S_\infty$ that $h \in \Sigma(G)$ if and only if for all $n \in \omega$ such that $h(n) \neq n$

$$n \in T_G \wedge (h(n), n) \in P_G \quad \text{or} \quad h(n) \in T_G \wedge (n, h(n)) \in P_G.$$

Thus Σ is Borel and therefore (S, E) (hence also $(\sqsubseteq_D, \cong_D)$) is invariantly universal by Theorem 6.2. □

Theorem 6.3 can be further improved to the following.

Theorem 6.4. *If D is ill-founded, then $\sqsubseteq^\star_D$ is invariantly universal (hence also complete for analytic quasi-orders).*

Proof. As done before Theorem 5.5, fix a decreasing sequence $(r_n)_{n\in\omega}$ in D converging to some $r \geq 0$, and an element $\bar{r} \in D$ bigger than every r_n. Set $D' = \{r_n \mid n \in \omega\} \cup \{0\}$. Set again, as before Lemma 5.6, $\mathcal{A} = \{U \in \mathcal{U}_{D'} \mid \forall x \in U\, \exists y \in U\, d_{\mathbb{U}}(x,y) = r_0\}$ and recall that it is Borel and closed under isometries. Notice that the proofs of Theorem 4.15 (if $r = 0$) and Theorem 6.3 (if $r > 0$) actually show that $\sqsubseteq \restriction \mathcal{A}$ is invariantly universal (in the latter case because the range of θ is contained in $\mathcal{A}$). Thus, by the observation after Definition 2.2, it is enough to show that $\sqsubseteq \restriction \mathcal{A}$ classwise Borel embeds into $\sqsubseteq^\star_D$, where both quasi-orders are paired with isometry. This amounts to show the existence of a class $\mathcal{B} \subseteq \mathcal{U}^\star_D$ and two maps $f \colon \mathcal{A} \to \mathcal{B}$ and $g \colon \mathcal{B} \to \mathcal{A}$ such that:

(i) $\mathcal{B}$ is a Borel subset of $\mathcal{U}^\star_D$ (hence also of $F(\mathbb{U})$) closed under isometries;

(ii) f and g are both Borel;
(iii) for each $U \in \mathcal{A}$, $g(f(U))$ is isometric to U;
(iv) for each $F \in \mathcal{B}$, $f(g(F))$ is isometric to F;
(v) f simultaneously reduces $\cong \restriction \mathcal{A}$ to $\cong \restriction \mathcal{B}$ and $\sqsubseteq \restriction \mathcal{A}$ to $\sqsubseteq \restriction \mathcal{B}$.

Notice that from (iv) and (v) it also follows that g is a reduction of $\cong \restriction \mathcal{B}$ to $\cong \restriction \mathcal{A}$ and of $\sqsubseteq \restriction \mathcal{B}$ to $\sqsubseteq \restriction \mathcal{A}$.

A function f reducing $\cong_{D'}$ to $\cong_D$ has been defined before Theorem 5.5. As our f here we take the restriction of that one to $\mathcal{A}$, recalling that $f(\mathcal{A}) \subseteq \mathcal{U}^\star_D$. We next define $\mathcal{B}$ as the closure (in $F(\mathbb{U})$) under isometry of $f(\mathcal{A})$: since $\mathcal{U}^\star_D$ itself is closed under isometry, $\mathcal{B} \subseteq \mathcal{U}^\star_D$. To see that $\mathcal{B}$ is Borel, first notice that, by definition of $\mathcal{A}$, the elements in $U \subseteq U^*$ can be characterized by the fact that they realize the distance r_0. Moreover, the clopen ball $B(x, \bar{r})$ of U^* centered in $x \in U$ always coincide with U. These observations lead to the following Borel description of $\mathcal{B}$. A space $F \in \mathcal{U}^\star_D$ is in $\mathcal{B}$ if and only if

- for some/every n in the set
$$C_F = \{n \in \omega \mid \exists k \in \omega \, d_{\mathbb{U}}(\psi_k(F), \psi_n(F)) = r_0\},$$
the subspace $B(\psi_n(F), \bar{r}) \cap F$ of F is in $\mathcal{A}$;
- for some/any $n \in C_F$, the subspace $F \setminus B(\psi_n(F), \bar{r})$ of F is isometric to $U(D \setminus \{r_0\})$;[9]
- for all $n \in C_F$ and all $m \in \omega \setminus C_F$, $d_{\mathbb{U}}(\psi_n(F), \psi_m(F)) \geq \bar{r}$;
- for all $r \in D$ with $r > \bar{r}$, all $n \in C_F$ and all $m \in \omega \setminus C_F$,
$$\begin{aligned} d_{\mathbb{U}}(\psi_n(F), \psi_m(F)) = r \iff & \forall k \in \omega \setminus (C_F \cup \{m\}) \, (d_{\mathbb{U}}(\psi_k(F), \psi_m(F)) \geq r) \\ & \wedge \exists k \in \omega \setminus C_F \, (d_{\mathbb{U}}(\psi_k(F), \psi_m(F)) = r). \end{aligned}$$

Finally, we define $g \colon \mathcal{B} \to \mathcal{A}$ by setting $g(F) = B(\psi_n(F), \bar{r}) \cap F$ for some/any $n \in C_F$, so that g is clearly a Borel function.

We already showed that (i) and (ii) are satisfied. Conditions (iii) and (iv) easily follow from the definition of $\mathcal{B}$ and the observations preceding it (together with the definitions of f and g, of course). As for (v), it has already been proved in Theorem 5.5 that $U_0 \cong U_1$ if and only if $U_0^* \cong U_1^*$. So we only need to prove that for all $U_0, U_1 \in \mathcal{A}$
$$U_0 \sqsubseteq U_1 \iff U_0^* \sqsubseteq U_1^*.$$
Since, as already observed, $U_i \subseteq U_i^*$ (for $i = 0, 1$) can be characterized as the collection of all points in U_i^* which realize r_0, any isometric embedding φ^* of U_0^* into U_1^* must map points in U_0 to points in U_1. Hence if $\varphi^* \colon U_0^* \to U_1^*$ is an isometric embedding, then so is $\varphi = \varphi^* \restriction U_0 \colon U_0 \to U_1$. Conversely, notice that the subspace $U_i^* \setminus U_i = U(D \setminus \{r_0\})$ of U_i^* (for $i = 0, 1$) does not depend on U_i, and that the same is true for each distance between a point in $U_i^* \setminus U_i$ and an arbitrary point in U_i^*. Therefore, if $\varphi \colon U_0 \to U_1$ is an isometric embedding, then so is the map $\varphi^* = \varphi \cup (\mathrm{id} \restriction (D \setminus \{r_0\})) \colon U_0^* \to U_1^*$. This concludes our proof. □

Corollary 6.5. *For every $D \in \mathcal{D}$, $\sqsubseteq_D \sim_B \sqsubseteq^\star_D$.*

Proof. If D is ill-founded then both $\sqsubseteq_D$ and $\sqsubseteq^\star_D$ are complete by Theorems 6.3 and 6.4, whence $\sqsubseteq_D \sim_B \sqsubseteq^\star_D$. If instead D is well-founded, then the result is contained in Lemma 5.8. □

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[9] This is a Borel condition because the isometry relation on $F(\mathbb{U})$, being reducible to an orbit equivalence relation, has only Borel classes (see e.g. [Kec95, Theorem 15.14]).

6.2. **Well-founded sets of distances.** In this section we investigate more closely the relations of isometric embeddability between Polish ultrametric spaces using a well-ordered set of distances. By (5.2) we have $\sqsubseteq_\alpha \leq_B \sqsubseteq_\beta$ for every $1 \leq \alpha \leq \beta < \omega_1$, whence also $\sqsubseteq^\star_\alpha \leq_B \sqsubseteq^\star_\beta$ by Lemma 5.8. Moreover, by Proposition 4.13 both $\sqsubseteq_n$ and $\sqsubseteq^\star_n$ are bqo's. We are now ready to provide an alternative proof of this fact which uses only $\mathsf{AC}_\omega(\mathbb{R})$. The unique result we will need from [NW65] is Corollary 28A, whose proof does not use any strong form of AC. If S is a quasi-order on X, we let $S^\#$ be the quasi-order on $\mathscr{P}(X)$ obtained by setting for every $A, B \subseteq X$

$$A \; S^\# \; B \iff \exists f \colon A \to B \text{ injective such that } \forall x \in A\, (x \; S \; f(x)).$$

Proof of Proposition 4.13. Notice first that if a quasi-order (Borel) embeds in a bqo then it is a bqo. Then, Lemmas 4.11 and 5.8 imply that it is enough to show by induction on $n \geq 1$ that the relation $\sqsubseteq_n$ is a bqo. The case $n = 1$ is clear. Assume now that $\sqsubseteq_n$ is a bqo: since by [NW65, Corollary 28A] the quasi-order ${\sqsubseteq_n}^\#$ on $\mathscr{P}(\mathcal{U}_n)$ is a bqo as well, it is enough to prove that $\sqsubseteq_{n+1}$ is reducible to ${\sqsubseteq_n}^\#$, i.e. that there is a function $f \colon \mathcal{U}_{n+1} \to \mathscr{P}(\mathcal{U}_n)$ such that $X \sqsubseteq_{n+1} Y \iff f(X) \, {\sqsubseteq_n}^\# \, f(Y)$ for every $X, Y \in \mathcal{U}_{n+1}$. Since by Theorem 5.13 we already know that $\sqsubseteq_{n+1}$ is (Borel) reducible to ${\sqsubseteq_n}^{\mathrm{inj}}$, we just need to show that ${\sqsubseteq_n}^{\mathrm{inj}}$ reduces to ${\sqsubseteq_n}^\#$. To see this, consider the map g replacing each sequence $(X_i)_{i \in \omega}$ of spaces in $\mathcal{U}_n$ with a set $\{X'_i \mid i \in \omega\} \subseteq \mathcal{U}_n$ such that each X'_i is isometric to X_i and $X'_i \neq X'_j$ for distinct $i, j \in \omega$: then g clearly reduces ${\sqsubseteq_n}^{\mathrm{inj}}$ to ${\sqsubseteq_n}^\#$, as required. □

We now show that if instead α is infinite, the situation is quite different.

Proposition 6.6. *Let $D \in \mathcal{D}$. Then the partial order $(\mathscr{P}(D \setminus \{0\}), \subseteq)$ embeds into $\sqsubseteq_D$ (hence also into $\sqsubseteq^\star_D$ by Corollary 6.5).*

Proof. The map i sending $X \subseteq D \setminus \{0\}$ into $U(X \cup \{0\})$ is the desired embedding. Indeed, if $X \subseteq Y$ then the identity function witnesses $i(X) \sqsubseteq i(Y)$. Conversely, assume that there is an isometric embedding f of $i(X)$ into $i(Y)$. Let $r \in X$, so that in particular $r > 0$. Then the distance r is realized in $i(X)$, and therefore in $i(Y)$ too; so $r \in Y$ by definition of $i(Y)$. It follows $X \subseteq Y$, as desired. □

Corollary 6.7. *Let $\omega \leq \alpha < \omega_1$. Then $(\mathscr{P}(\omega), \subseteq) \leq_B \sqsubseteq_\alpha$, and hence also $(\mathscr{P}(\omega), \subseteq) \leq_B \sqsubseteq^\star_\alpha$. In particular, both $\sqsubseteq_\alpha$ and $\sqsubseteq^\star_\alpha$ contain infinite antichains and infinite decreasing chains.*

Proof. Identify $D_\alpha \setminus \{0\}$ with ω, so that $\mathscr{P}(D_\alpha \setminus \{0\})$ is naturally identified with $\mathscr{P}(\omega)$. Then the map i defined in the proof of Proposition 6.6 is Borel, and hence witnesses $(\mathscr{P}(\omega), \subseteq) \leq_B \sqsubseteq_\alpha$. □

Theorem 6.8. *For every $1 \leq \alpha < \omega_1$, if $\sqsubseteq_\alpha$ is Borel then $\sqsubseteq_\alpha <_B \sqsubseteq_{\alpha+1}$.*

Proof. We already observed that $\sqsubseteq_\alpha \leq_B \sqsubseteq_\beta$ for every $1 \leq \alpha \leq \beta < \omega_1$. If $\alpha = 1$, then the quotient order of $\sqsubseteq_1$ consists of one point, while if $\alpha = 2$, then the quotient order of $\sqsubseteq_2$ is a linear order of order type $\omega + 1$ (the different $E_{\sqsubseteq_2}$ classes correspond to the possible cardinalities of the spaces in $\mathcal{U}_2$). Therefore $\sqsubseteq_2 \not\leq_B \sqsubseteq_1$, and, by Corollary 3.13 and Theorem 5.13, $\sqsubseteq_2 <_B {\sqsubseteq_2}^{\mathrm{inj}} \sim_B \sqsubseteq_3$.

For the case $3 \leq \alpha < \omega_1$, by Theorem 5.13 it is enough to show that $\sqsubseteq_\alpha <_B {\sqsubseteq_\alpha}^{\mathrm{inj}}$ (assuming that $\sqsubseteq_\alpha$ is Borel). To see this, consider the spaces $U_0, U_1 \in \mathcal{U}_\alpha$ with domain $\{x, y\}$ and distances defined by $d_{U_0}(x, y) = r_1$ and $d_{U_1}(x, y) = r_2$. Then clearly $U_0 \not\sqsubseteq U_1$, $U_1 \not\sqsubseteq U_0$, and $\{U \in \mathcal{U}_\alpha \mid U \sqsubseteq U_0, U_1\}$ is the $E_{\sqsubseteq_\alpha}$-equivalence class of the space consisting of just one point. The desired result then follows from Corollary 3.13. ☐

Lemma 6.9. *Let $(S_n)_{n\in\omega}$ be a sequence of analytic quasi-orders and λ an infinite countable ordinal. If for every $n \in \omega$ there exists $\beta_n < \lambda$ such that $S_n \leq_B \sqsubseteq_{\beta_n}$, then $\prod_{n\in\omega} S_n \leq_B \sqsubseteq_\lambda$.*

Proof. Using (5.2) and Lemma 5.8, it is enough to show $\prod_{n\in\omega} \sqsubseteq^\star_{\beta_n} \leq_B \sqsubseteq_\lambda$ with $\beta_n > 1$ for all $n \in \omega$. We distinguish two cases, depending on whether λ is a successor or not.

Assume first that $\lambda = \lambda' + 1$, so that, in particular, $\lambda' \geq \omega$. Fix a bijection $\langle\cdot,\cdot\rangle \colon \omega \times (\omega \setminus \{0\}) \to \omega \setminus \{0\}$, increasing with respect to its second argument. Given a sequence of spaces $(X_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}^\star_{\beta_n}$, let X' be the space in $\mathcal{U}_\lambda$ whose domain is the disjoint union of the X_n and whose distance $d_{X'}$ is defined by letting, for distinct $x, y \in X'$,

$$d_{X'}(x,y) = \begin{cases} r_{\lambda'} & \text{if } x \in X_n \text{ and } y \in X_m \text{ with } n \neq m; \\ r_\xi & \text{if } x, y \in X_n \text{ and } d_{X_n}(x,y) = r_\xi \text{ with } \xi \geq \omega; \\ r_{\langle n,i\rangle} & \text{if } x, y \in X_n \text{ and } d_{X_n}(x,y) = r_i \text{ with } i \in \omega. \end{cases}$$

Pick two sequences $(X_n)_{n\in\omega}, (Y_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}^\star_{\beta_n}$. If $X_n \sqsubseteq Y_n$ for every n then, gluing together the embeddings, we get an isometric embedding of X' into Y'. Conversely, assume $\varphi \colon X' \to Y'$ is an isometric embedding. Since $\beta_n < \lambda = \lambda' + 1$, the distance $r_{\lambda'}$ cannot be realized in X_n (as a subspace of X'), and hence for every $n \in \omega$ there is $k(n) \in \omega$ such that $\varphi(X_n) \subseteq Y_{k(n)}$. Moreover, since the distances with finite positive index used in X_n (as a subspace of X') are not used in any Y_m (as a subspace of Y') with $m \neq n$, we easily get $k(n) = n$, so that $\varphi(X_n) \subseteq Y_n$ for every $n \in \omega$ (here we are using $X_n \in \mathcal{U}^\star_{\beta_n}$ and $\beta_n > 1$). It then easily follows that for all $n \in \omega$ the restriction of φ to X_n witnesses $X_n \sqsubseteq^\star_{\beta_n} Y_n$.

For the case λ limit the argument is similar. For each $n \in \omega$, let $\lambda_n = \max\{\beta_i \mid i \leq n\} < \lambda$. Given $(X_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}^\star_{\beta_n}$, let X' be the space in $\mathcal{U}_\lambda$ whose domain is the disjoint union of the X_n and whose distance $d_{X'}$ is defined by letting, for distinct $x, y \in X'$,

$$d_{X'}(x,y) = \begin{cases} r_{2\cdot\lambda_{\max\{n,m\}}+1} & \text{if } x \in X_n \text{ and } y \in X_m \text{ with } n \neq m; \\ r_{2\cdot\xi} & \text{if } x, y \in X_n \text{ and } d_{X_n}(x,y) = r_\xi \text{ with } \xi \geq \omega; \\ r_{2\cdot\langle n,i\rangle} & \text{if } x, y \in X_n \text{ and } d_{X_n}(x,y) = r_i \text{ with } i \in \omega. \end{cases}$$

Let $(X_n)_{n\in\omega}, (Y_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}^\star_{\beta_n}$. If each X_n embeds isometrically into Y_n, then gluing together the embeddings we get an isometric embedding of X' into Y'. Conversely, assume $\varphi \colon X' \to Y'$ is an isometric embedding. Since no distance with odd index can be realized within a single X_n (as a subspace of X'), for every $n \in \omega$ there is $k(n) \in \omega$ such that $\varphi(X_n) \subseteq Y_{k(n)}$, and arguing as above we easily get $k(n) = n$. Thus $\varphi(X_n) \subseteq Y_n$ for every $n \in \omega$, whence $X_n \sqsubseteq^\star_{\beta_n} Y_n$ for every $n \in \omega$. ☐

Recall the definition of the Rosendal's sequence P_α, $\alpha < \omega_1$, given in Definition 3.3.

Corollary 6.10. *For all $\alpha < \omega_1$, $P_{1+\alpha} \leq_B \sqsubseteq_{\omega+\alpha}$.*

Proof. By induction on $\alpha < \omega_1$, using Corollary 6.7 (together with the fact that $P_1 \sim_B (\mathscr{P}(\omega), \subseteq)$ by Proposition 3.16(1)), Theorem 5.13 (together with the fact that $P_{1+\alpha+1} = P^{\mathrm{cf}}_{1+\alpha} \leq_B P^{\mathrm{inj}}_{1+\alpha}$), and Lemma 6.9. □

Theorem 6.11. *Let $1 \leq \alpha < \omega_1$. Then*

(1) if $\alpha \leq \omega$, then $\sqsubseteq_\alpha$ is Borel;
(2) if $\alpha > \omega$, $\sqsubseteq_\alpha$ contains both upper and lower cones that are $\boldsymbol{\Sigma}^1_1$-complete, and hence $\sqsubseteq_\alpha$ is analytic non-Borel;
(3) all $E_{\sqsubseteq_{\omega+1}}$-equivalence classes are Borel, hence $\sqsubseteq_{\omega+1}$ is not complete for analytic quasi-orders;
(4) for all $\alpha < \beta \leq \omega + 2$, $\sqsubseteq_\alpha <_B \sqsubseteq_\beta$.

Proof. (1) By Lemma 5.12 it is enough to prove the result for $\alpha < \omega$, and this will be shown by induction on $\alpha \geq 1$. The case $\alpha = 1$ is clear and the induction step is immediate using Proposition 4.13, Corollary 3.21 and Theorem 5.13.

(2) By Corollary 6.7 and Theorem 5.13, we get $(\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}} \leq_B \sqsubseteq_{\omega+1}$, so that the claim is true for $\alpha = \omega + 1$ by Proposition 3.23. For an arbitrary $\alpha > \omega + 1$, use the fact that $\sqsubseteq_{\omega+1} \leq_B \sqsubseteq_\alpha$ by (5.2).

(3) Since $\sqsubseteq_{\omega+1} \sim_B {\sqsubseteq_\omega}^{\mathrm{inj}}$ by Theorem 5.13 and $\sqsubseteq_\omega$ is Borel by part (1), the result follows from Theorem 3.31.

(4) Clearly $\sqsubseteq_\alpha \leq_B \sqsubseteq_\beta$ by (5.2), so only the inequality $\sqsubseteq_\beta \not\leq_B \sqsubseteq_\alpha$ needs to be proved. For $\alpha \leq \omega$ this follows from Theorem 6.8 and (1). For $\alpha = \omega + 1$ and $\beta = \omega + 2$, first observe that if $\sqsubseteq_{\omega+2} \leq_B \sqsubseteq_{\omega+1}$, then any witness to this would also witness $E_{\sqsubseteq_{\omega+2}} \leq_B E_{\sqsubseteq_{\omega+1}}$. But this is impossible because $E_{\sqsubseteq_{\omega+1}}$ has only Borel equivalence classes by part (3), while $E_{\sqsubseteq_{\omega+2}}$ has a $\boldsymbol{\Sigma}^1_1$-complete class by part (2) applied with $\alpha = \omega + 1$, Lemma 3.36(2), and the fact that $\sqsubseteq_{\omega+2} \sim_B {\sqsubseteq_{\omega+1}}^{\mathrm{inj}}$ by Theorem 5.13. □

7. Open problems

7.1. **Isometry.** In this paper we have given a fairly complete treatement of the relation of isometry on ultrametric Polish spaces; these are, in particular, zero-dimensional spaces. However, one of the main questions asked by [GK03] and still unanswered is the following:

Question 7.1. *What is the complexity of isometry between zero-dimensional Polish metric spaces?*

Clemens [Cle12] showed that this relation is strictly above countable graph isomorphism. It is conjectured in [GK03, Chapter 10] that it is Borel bireducible with any complete orbit equivalence relation.

The same question might be asked for any other topological dimension. For infinite-dimensional spaces the isometry relation is Borel bireducible with any complete orbit equivalence relation by the proofs of [GK03, Theorem 1] and [Cle12, Theorem 7]. For other dimensions the problem is open, but it is easy to observe that if $\alpha \leq \alpha' < \omega_1$ then the relation of isometry on spaces of dimension α is Borel reducible to the same relation on spaces of dimension α'.

Corollary 5.4 answers what appears to be the main question from [GK03, Chapter 8]: the relations of isomorphism on discrete Polish ultrametric spaces and on locally compact Polish ultrametric spaces are both Borel bireducible to countable graph isomorphism. As discussed after Corollary 5.4, among the lower bounds proposed in the literature for the complexity of these relations the only one that could still be sharp is isomorphism between trees on ω with countably many infinite branches or, equivalently, isometry on countable closed subspaces of ${}^{\omega}\omega$.

Question 7.2. *Is the relation of isomorphism between trees on ω with countably many infinite branches Borel bireducible with countable graph isomorphism?*

7.2. **Isometric embeddability.** The main problem left open by Theorem 6.11 is the following:

Question 7.3. *Does there exist $\alpha \geq \omega + 2$ such that $\sqsubseteq_\alpha$ is complete/invariantly universal for analytic quasi-orders? In particular, what about $\sqsubseteq_{\omega+2}$?*

In the previous question one can equivalently replace $\sqsubseteq_\alpha$ with $\sqsubseteq^\star_\alpha$. For completeness this is immediate from Lemma 5.8. For invariant universality this follows from the observation after Definition 2.2 and the fact that one can sharpen the argument of the proof of Lemma 5.8 to show that each of $\sqsubseteq_\alpha$ and $\sqsubseteq^\star_\alpha$ classwise Borel embeds into the other one.

To the best of our knowledge, the techniques to prove that an analytic quasi-order S is not complete are the following: show that S is combinatorially simple (e.g. a wqo); show that S is topologically simple (i.e. Borel); show that the equivalence relation E_S is simple (e.g. almost all its equivalence classes are Borel). None of these apply to $\sqsubseteq_{\omega+2}$. In fact, combinatorially already $\sqsubseteq_{\omega+1}$ (which is not complete for analytic quasi-orders) is very complicated because it embeds all partial orders P of size ω_1 (and thus, assuming the Continuum Hypothesis, the quotient order of any quasi-order on a standard Borel space). To see this, use the fact that by [Par63] any such P can be embedded in the relation $(\mathscr{P}(\omega), \subseteq^*)$ of inclusion modulo finite sets, together with the chain of reducibilities $(\mathscr{P}(\omega), \subseteq^*) \leq_B (\mathscr{P}(\omega), \subseteq)^{\mathrm{cf}} \leq_B \sqsubseteq_{\omega+1}$. The same argument also shows that it is not possible to answer negatively the subsequent Question 7.4 using only combinatorial arguments. Moreover, $\sqsubseteq_{\omega+2}$ is not Borel by Theorem 6.11(2), and its associated equivalence relation contains (many) non-Borel classes, as shown in the proof of Theorem 6.11(4).

A possible way to obtain a positive answer to Question 7.3 is to answer positively the following question:

Question 7.4. *Is $((\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}})^{\mathrm{inj}}$ complete/invariantly universal for analytic quasi-orders?*

Recall that $(\mathscr{P}(\omega), \subseteq) \leq_B \sqsubseteq_\omega$ by Corollary 6.7, and hence we have also $(\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}} \leq_B \sqsubseteq_{\omega+1}$ and $((\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}})^{\mathrm{inj}} \leq_B \sqsubseteq_{\omega+2}$ by Theorem 5.13.

Question 7.5. *Is it true that $\sqsubseteq_{\omega+1} \sim_B (\mathscr{P}(\omega), \subseteq)^{\mathrm{inj}}$? What about $\sqsubseteq_\omega \sim_B (\mathscr{P}(\omega), \subseteq)$?*

Notice that a positive answer to the second part of Question 7.5 implies a positive answer to the first part by Theorem 5.13. By the same theorem, a positive answer to the latter would imply that Question 7.4 is equivalent to the completeness part of Question 7.3 for $\alpha = \omega + 2$.

Notice also that an upper bound for the complexity of $\sqsubseteq_\omega$ is $(\mathscr{P}(\omega), \subseteq)^{\mathrm{cf}}$ (whence also $\sqsubseteq_{\omega+1} \leq_B ((\mathscr{P}(\omega), \subseteq)^{\mathrm{cf}})^{\mathrm{inj}}$). To see this, first observe that $\sqsubseteq_n <_B (\mathscr{P}(\omega), \subseteq)$ (because $\sqsubseteq_n$ is a bqo with only countably many $E_{\sqsubseteq_n}$-classes by Corollary 3.18 and Proposition 4.13), and then use Lemma 4.8 to get $\sqsubseteq_\omega \leq_B (\mathscr{P}(\omega), \subseteq)^{\mathrm{cf}}$. However, since $(\mathscr{P}(\omega), \subseteq) <_B (\mathscr{P}(\omega), \subseteq)^{\mathrm{cf}}$ by Lemma 3.6, the exact complexity of $\sqsubseteq_\omega$ remains undetermined.

7.3. **Classwise Borel embeddability.** The only known tool for showing that a pair (S, E) is invariantly universal is Theorem 6.2. Its proof in [CMMR13] actually shows that the conclusion can be strengthened to: $(R, =) \sqsubseteq_{cB} (S, E)$ for every analytic quasi-order R. This naturally leads to consider the following notion: a pair (S, E) as in Definition 2.2 is $\sqsubseteq_{cB}$*-complete* if $(R, F) \sqsubseteq_{cB} (S, E)$ for every pair (R, F). Notice that if a pair (S, E) is $\sqsubseteq_{cB}$-complete then, in particular, both S and E are $\leq_B$-complete for analytic quasi-orders and analytic equivalence relations, respectively. Thus none of the invariantly universal pairs (S, E) considered in this paper and in [CMMR13] is $\sqsubseteq_{cB}$-complete, because to apply our Theorem 6.2 we need that E be Borel reducible to an orbit equivalence relation (whence E cannot be $\leq_B$-complete). However, using standard arguments it is not hard to construct a $\sqsubseteq_{cB}$-complete pair. It is thus natural to ask the following:

Question 7.6. *Do there exist "natural" examples of* $\sqsubseteq_{cB}$*-complete pairs? In particular, for which of the invariantly universal quasi-orders* S *considered in this paper and in* [CMMR13] *we have that* (S, E_S) *is also* $\sqsubseteq_{cB}$*-complete?*

Of course this question is strongly related to [CMMR13, Question 6.4], which still remains wide open.

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