On the uniqueness of the limit cycle for the Liénard equation with $f(x)$ not sign-definite

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Abstract

The problem of uniqueness of limit cycles for the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ is investigated. The classical assumption of sign-definiteness of $f(x)$ is relaxed. The effectiveness of our result as a perturbation technique is illustrated by some constructive examples of small amplitude limit cycles, coming from bifurcation theory.

Keywords: Liénard equation, Limit cycles, Uniqueness.
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1. Introduction and discussion about some uniqueness results

The aim of this paper is to investigate the problem of uniqueness of the limit cycle for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous function and $g(x)$ is locally Lipschitz and satisfies the sign condition $g(x)x > 0$ for $x \neq 0$.

The first result in this direction was actually achieved by Liénard himself his pioneering paper [15] which is still a milestone in this area. Observe that in the same paper the Liénard plane

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{2}$$

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was introduced, where $F(x) = \int_0^x f(s)\,ds$. It is well known that the study of equation (1) in such a plane is equivalent to the study in the phase-plane

$$\dot{x} = y \quad \dot{y} = -f(x)y - g(x).$$

Liénard [15] proved the uniqueness of the limit cycle under the following assumption

($\mathscr{F}$) $f(0) < 0$, $F(x)$ has precisely three zeros $\alpha < 0 < \beta$, and is monotone increasing outside the interval $[\alpha, \beta]$.

It should be observed that Liénard was treating the case $f(x)$ even (and therefore $F(x)$ odd) and $g(x) = x$. This fact produces some obvious symmetries when studying (2) and is useful in the proof because it gives the property:

($\mathscr{S}$) *All the possible limit cycles cross both the lines $x = \alpha$ and $x = \beta$*

(clearly, in the case studied by Liénard it is $\alpha = -\beta$). Such uniqueness result was then improved by Levinson and Smith [14] and by Sansone [19], still keeping the symmetry property. More in detail Levinson and Smith [14] assumed $f(x)$ even and $g(x)$ odd, while Sansone [19] considered the case $g(x) = x$ and $F(\alpha) = F(\beta)$. As a consequence, in both the cases the property ($\mathscr{S}$) is satisfied. Such a property plays a crucial role as it was shown in the classical counterexample by Duff and Levinson [6]. Indeed, in [6] the authors produce an example in which $F(x)$ satisfies ($\mathscr{F}$), $g(x) = x$ but three limit cycles are present, two of them not intersecting one of the vertical lines. For this reason in the search of hypotheses ensuring the uniqueness of the limit cycle and in order to avoid the symmetry property $F(x)$ and $g(x)$ odd, one has to impose condition ($\mathscr{S}$). This fact was well known, but to our knowledge it was explicitly stated by Roberto Conti in [5] in his Italian notes for an advanced course in ODEs. Restating this result in our context, we have in fact the following:

Theorem 1.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ as above. Assume that there exist α, β with $\alpha < 0 < \beta$ such that*

($\mathscr{F}_1$) *$F(\alpha) = F(\beta) = 0$ with $F(x)x < 0$ for $x \in]\alpha, \beta[$, $x \neq 0$ and $F(x)x > 0$ for $x < \alpha$ and for $x > \beta$;*

($\mathscr{F}_2$) *$F(x)$ is monotone increasing for $x < \alpha$ and for $x > \beta$.*

Then system (2) *has at most one limit cycle provided that property* ($\mathscr{S}$) *holds.*

Notice that ($\mathscr{F}$) implies ($\mathscr{F}_1$) and ($\mathscr{F}_2$). Moreover, in Theorem 1.1 no symmetry condition on f or g is required. On the other hand, to verify assumption ($\mathscr{S}$) is not an easy task. In this light, an elegant and easy verifiable condition for ($\mathscr{S}$) is given by $G(\alpha) = G(\beta)$ (cf., for instance, [3]). Results depending on Theorem 1.1 were explicitly or implicitly used by several researchers with the main goal of producing sufficient conditions for the property ($\mathscr{S}$) and hence the uniqueness (see [1, 3, 10, 11, 12, 18, 20, 22, 23]).

In the study of this problem, Lefschetz in his classical book (still in the framework of the symmetry conditions) [13, p. 272, Fig. 2] observed that if the monotonicity property of $F(x)$ outside the interval $[\alpha, \beta]$ is omitted, there is the possibility of *giving rise to a succession of "concentric" closed paths* $\Gamma_1, \Gamma_2, \ldots$ *and* Γ_i *will be orbitally stable (unstable)* alternatively. Perhaps in view of this remark, surprisingly little attention has been paid to relax the monotonicity hypothesis in this problem. Therefore, all the above quoted results must assume $f(x)$ to be positive for $|x|$ large.

In the present paper we attack the uniqueness problem for equation (1), by relaxing the monotonicity assumption ($\mathscr{F}_2$) on $F(x)$. More precisely, we will give evidence of the fact that uniqueness of the limit cycle can be guaranteed for a wide range of cases in which $F(x)$, as well as $f(x)$ oscillates or is eventually negative. Constructive examples will be presented. Throughout the article when speaking of uniqueness of limit cycle, we mean that we consider the fact that *at most* one limit cycle does exist and we do not focus our attention on the existence of limit cycles. For this latter aspect, which was widely investigated in the literature, we refer to the recent work [4] and the references therein.

For sake of completeness, we recall that another classical approach for uniqueness in the case $g(x) = x$ come from the Massera theorem, where the monotonicity assumptions are requested only on $f(x)$ (instead on $F(x)$), without any symmetry assumptions. Recently, the monotonicity assumptions in Massera theorem have been relaxed to a fixed interval in [21] (see also [11] and [2]). The case in which $f(x)$ has no fixed sign, but still keeping the hypothesis of monotonicity has been treated in [17]. However, the intriguing problem of generalizing Massera theorem to a general function $g(x)$ is still open. For an historical discussion about the theory of relaxation of the oscillation, we refer to the recent papers of Ginoux [7, 8] and the notable survey of Mawhin [16].

The plan of the paper is the following. In Section 2 we discuss in detail a proof of the uniqueness result in the monotone case, following Conti [5]. After this step, we observe that the proof still holds if the monotonicity assumption is dropped outside a certain strip including $[\alpha, \beta]$, provided that $|F(x)|$ stays bounded from below by certain values that can be explicitly produced. In Section 3 we produce some examples which give the applicability of our result as a perturbation approach which results extremely powerful in connection with some results of bifurcation.

2. The uniqueness result revisited

Let $f, g : \mathbb{R} \to \mathbb{R}$ with f continuous and g locally Lipsctitz continuous and satisfying $g(x)x > 0$ for all $x \neq 0$. We consider the Liénard system (2) with $F(x)$ satisfying only ($\mathscr{F}_1$) at the moment. Associated with equation (2) we have the conservative system $\dot{x} = y,\ \dot{y} = -g(x)$, with associated energy function $E(x, y) := \frac{1}{2}y^2 + G(x)$, for $G(x) := \int_0^x g(s)\, ds$. If we evaluate the derivative of the energy function along the trajectories of system (2) we obtain $\dot{E}(x, y) = y\dot{y} + g(x)\dot{x} = -yg(x) + g(x)(y - F(x)) = -g(x)F(x)$. As a consequence, for any limit cycle Γ we have $\oint_\Gamma -g(x(t))F(x(t))\,dt = 0$. This because if we consider a point $P = (x, y) \in \Gamma$ and follow the trajectory for its period T, we come back the the same point and therefore there is no gain or loss of energy. Being $g(x)F(x) < 0$ for all $x \neq 0$ in the interval $]\alpha, \beta[$, we find that no limit cycle lies entirely in the strip $[\alpha, \beta] \times \mathbb{R}$. In fact, for any

In the above mentioned special case in which $G(\alpha) = G(\beta)$ we have that, as the energy curve $E(x, y) = G(\alpha) = G(\beta)$ intersects the x-axis at the points $(\alpha, 0)$ and $(\beta, 0)$ we can conclude that all possible limit cycles intersect both the lines $x = -\alpha$ and $x = \beta$ and condition $(\mathscr{S})$ is fulfilled.

After these preliminary observations, we are now in position to start the proof of Theorem 1.1 following Conti's argument.

Proof. By contradiction, assume there are two limit cycles Γ_1 and Γ_2 with Γ_1 included in the open region bounded by Γ_2 crossing both the lines $x = \alpha$ and $x = \beta$. This configuration is depicted in Figure 1.

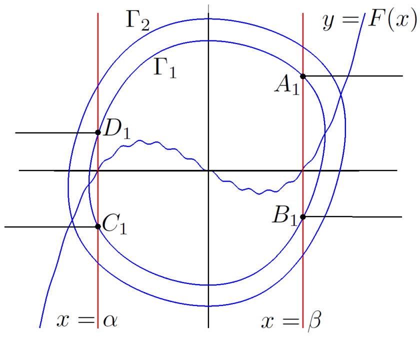

Figure 1: Example of two limit cycles Γ_1 and Γ_2, with the points A_1, B_1, C_1, D_1 in evidence.

With reference to Figure 1, for $i = 1, 2$ and counting in the clockwise sense, we denote by A_i and B_i the intersections of Γ_i with the line $x = \beta$ and by C_i and D_i the intersections of Γ_i with the line $x = \alpha$. Each limit cycle can be split into four arcs and therefore, the integral $\oint_{\Gamma_i} -g(x(t))F(x(t))\,dt$ can be expressed as sum of four integrals. The integrals for the arcs inside the strip $\alpha \leq x \leq \beta$ can be parameterized in the x-variable, while, outside the strip, we parameterize in the y-variable. We start first to compare the integrals along the arcs $\widehat{D_1A_1}$ and $\widehat{D_2A_2}$.

$$\mathcal{R}_i := \int_{\widehat{D_iA_i}} -g(x(t))F(x(t))\,dt = \int_\alpha^\beta \frac{-F(x)g(x)}{y_i - F(x)}\,dx,$$

where $y_i = y_i(x)$ is the parametrization by Dini's theorem of y as a function of x along Γ_i in the strip. From the assumption it follows that $-F(x)g(x) > 0$ for $]\alpha, \beta[$ with $x \neq 0$. Since $y_1(x) < y_2(x)$ for all $x \in [\alpha, \beta]$, we get that $\mathcal{R}_2 < \mathcal{R}_1$.

Analogously, one can see that

$$\int_{\widehat{B_2C_2}} -g(x(t))F(x(t))\,dt < \int_{\widehat{B_1C_1}} -g(x(t))F(x(t))\,dt.$$

This because, now $y_1(x) > y_2(x)$, but the strip is crossed in the opposite sense.

We compare now the integrals outside the strip which can be parameterized with respect to the y-variable.

$$\mathcal{S}_i = \int_{\widehat{A_iB_i}} -g(x(t))F(x(t))\,dt = \int_{y_{A_i}}^{y_{B_i}} F(x_i)\,dy$$

where $x_i = x_i(y)$ is the parametrization of x as a function of y along Γ_i for $x \geq \beta$. Now, $\mathcal{S}_2$ can be split as

$$\mathcal{S}_2 = \int_{y_{A_2}}^{y_{A_1}} F(x_2)\,dy + \int_{y_{A_1}}^{y_{B_1}} F(x_2)\,dy + \int_{y_{B_1}}^{y_{B_2}} F(x_2)\,dy.$$

Notice that the first and the third integrals give a negative contribution. This because $F(x) > 0$ for $x > \beta$ but $y_{A_2} > y_{A_1}$ and $y_{B_1} > y_{B_2}$. On the other hand,

$$\int_{y_{A_1}}^{y_{B_1}} F(x_1)\,dy > \int_{y_{A_1}}^{y_{B_1}} F(x_2)\,dy$$

because $y_{A_1} > y_{B_1}$ and $x_2(y) > x_1(y)$ and $F(x)$ is monotone increasing, according to $(\mathscr{F}_2)$. Therefore $\mathcal{S}_2 < \mathcal{S}_1$. Arguing in the same way, one can prove that

$$\int_{\widehat{C_2D_2}} -g(x(t))F(x(t))\,dt < \int_{\widehat{C_1D_1}} -g(x(t))F(x(t))\,dt.$$

This because $y_{C_2} < y_{C_1}$, $y_{D_1} < y_{D_2}$ but $F(x)$ is negative. Moreover, $y_{C_1} < y_{D_1}$, but now $x_2(y) < x_1(y)$ with $F(x)$ still monotone increasing.

In conclusion, summing up all the steps, we get

$$0 = \oint_{\Gamma_2} -g(x(t))F(x(t))\,dt < \oint_{\Gamma_1} -g(x(t))F(x(t))\,dt = 0,$$

a contradiction. □

As already discussed in the Introduction, we observe that under the symmetry assumptions of the classical papers, condition $(\mathscr{S})$ is automatically fulfilled.

The main result of this paper, is actually based on a careful inspection of the above proof. Indeed, the monotonicity property of $F(x)$ for $x < \alpha$ and for $x > \beta$ was used only in the comparison of the integrals $\int_{y_{A_1}}^{y_{B_1}} F(x)\,dy$ and $\int_{y_{C_1}}^{y_{D_1}} F(x)\,dy$ respectively. In all the remaining integrals only assumptions on the sign of $F(x)$ were required. This means that, outside the strips $y_{B_1} \leq y \leq y_{A_1}$ and $y_{C_1} \leq y \leq y_{D_1}$, no monotonicity of $F(x)$ is required, that is no sign assumption on $f(x)$.

In order to clarify the situation, let us consider some possible behaviors for $F(x)$ for $x > \beta$, the case $x < \alpha$ being treated in the same way. There are only

two possibilities:
CASE 1. $F(x) < y_{A_1}$ for all $x > \beta$. In this situation, the monotonicity assumption cannot be relaxed to save the proof.
CASE 2. $F(x) = y_{A_1}$ for some $x = x_1 > \beta$. In this situation, it will be sufficient to require that $F(x)$ is monotone increasing on $[\beta, x_1]$ and, moreover, $F(x) > y_{A_1}$ for $x > x_1$. Notice that no monotonicity assumption will required for $x > x_1$.

To summarize all the above observations, we can state the following result.

Theorem 2.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ as above and let α, β with $\alpha < 0 < \beta$ be such that $(\mathscr{F}_1)$ and $(\mathscr{S})$ are satisfied. Suppose that there exists α_1, β_1 with $\alpha_1 < \alpha$ and $\beta_1 > \beta$ such that*

($\mathscr{F}_2'$) *$F(x)$ is monotone increasing for $\alpha_1 < x < \alpha$ and for $\beta < x < \beta_1$, with $F(x) \leq F(\alpha_1)$ for all $x \leq \alpha_1$ and $F(x) \geq F(\beta_1)$ for all $x \geq \beta_1$.*

If Γ is a limit cycle intersecting $x = \beta$ at (β, y_A) with $y_A > 0$ and $x = \alpha$ at (α, y_C) with $y_C < 0$, with $y_A \leq F(\beta_1)$ and $y_C \geq F(\alpha_1)$, then the uniqueness of the limit cycle for system (2) *is guaranteed.*

Remark 2.1. The main problem in order to apply this result is clearly to determine y_A and, symmetrically, y_C. Instead of finding y_A (resp. y_C) one can take the intersections of the limit cycle with y-axis because it is well known that these intersections are respectively the maximum and the minimum ordinates of the limit cycle. In this light, this result is more powerful if treated as a perturbation method.

More in details, Theorem 2.1 can be viewed as a perturbation result. We can start from a situation in which $F(x)$ is monotone increasing outside the interval $[\alpha, \beta]$ and therefore, in virtue of Theorem 1.1, there is exactly one limit cycle intersecting the lines $x = \alpha$ and $x = \beta$. This determines uniquely the values y_A, y_B, y_C and y_D which are the ordinates of the intersection points with the above lines. Notice that, it is possible to determine such values by rigorous numerics and nowadays this is very common in any kind of computing environment.

A crucial role will be now played by y_A and y_C. Given y_A we look wether there exists $\beta_1 > \beta$ such that $F(\beta_1) \geq y_A$. In this situation occurs, we can modify $F(x)$ in the interval $[\beta_1, +\infty)$, provided that the modified function remains above the level $F(\beta_1) \geq y_A$. Similarly, if there is $\alpha_1 < \alpha$ such that $F(\alpha_1) \leq y_C$, we can modify $F(x)$ in the interval $(-\infty, \alpha_1]$ provided that the modified function remains below the level $F(\alpha_1) \leq y_C$. With this procedure we can provide a broad family of non-monotone functions $F(x)$ for which the uniqueness of the limit cycle is guaranteed. We just remark that this theorem generalizes most of the results present in the literature.

A different way to read Theorem 2.1 is the following. Assume that, by means of some a priori bounds, we are able to determine an horizontal strip $\mathbb{R} \times [\phi, \psi]$ such that all the possible limit cycles are contained in such a strip. This is a common procedure in order to apply the Poincaré-Bendixson theorem and again, in many concrete situations, finding such a priori bounds is not an hard task. Suppose also that the equation $F(x) = \phi$ has a unique solution $x = \alpha_1$ which is simple and, similarly, the equation $F(x) = \psi$ has a unique solution $x = \beta_1$ which is simple, too. In the light of Theorem 2.1 to get the uniqueness of the limit cycle it is sufficient to suppose the monotonicity of $F(x)$ in the interval $[\alpha_1, \alpha]$ and $[\beta, \beta_1]$. As a side remark, we observe that, while, in the classical case, we ask the limit cycles to cross the vertical strip $\alpha \le x \le \beta]$, in this approach we ask the limit cicles to be confined in the horizontal strip $\phi \le y \le \psi]$.

A few examples are given in the next section.

3. Examples and applications

As mentioned above, Theorem 2.1 can be read as a perturbation result. In this light it is interesting the case in which there is a unique limit cycle of small amplitude, because this clearly enlarges the possibility of perturbing the function F outside a small region near the origin. In this light we consider the Hopf bifurcation from the origin and the classical Van der Pol equation. Namely, we consider, respectively, the equations

$$(E_1) \quad \ddot{x}+(x^2-a)\dot{x}+x = 0 \ (a > 0) \qquad \text{and} \quad (E_2) \quad \ddot{x}+\mu(x^2-1)\dot{x}+x = 0 \ (\mu > 0).$$

It is well known that both equations have a unique stable limit cycle, in virtue of Liénard theorem or Massera theorem. In the first case, we have a Hopf bifurcation from the origin, therefore, when $a > 0$ is small, the limit cycle is small. In the second case, a bifurcation from the circle of radius 2 occurs as μ goes to 0 (see [9, p. 190]. Therefore, in both case, it is easy to determine the range in which $F(x)$ may be modified. As a result, the phase-portrait may dramatically change but no new limit cycle appears and the existing limit cycle does not move. Finally, we observe that $F(x)$ may be perturbed in such a way that it becomes eventually decreasing, provided that $|F(x)|$ is bounded away from a certain constant bounding the amplitude of the limit cycle. This clearly produces examples in which $f(x)$ is eventually negative and the classical uniqueness results cannot be applied.

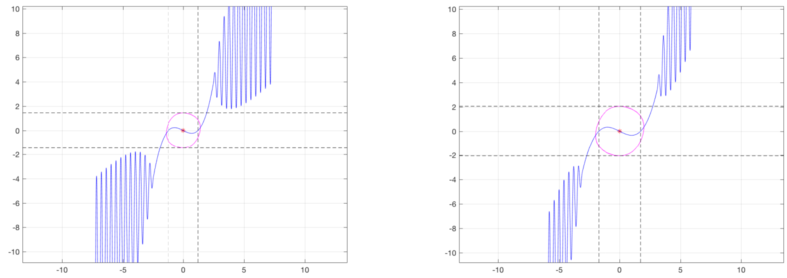

Figure 2: Example of equations (E_1) (left) and (E_2) (right) with $a = 0.5$ and $\mu = 0,5.$. The function $F(x)$ is drastically modified outside an interval whose length can be explicitly determined. No new limit cycle appears according to Theorem 2.1. The simulations are produced using MATLAB® software.

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References

[1] T. Carletti, Uniqueness of limit cycles for a class of planar vector fields, *Qual. Theory Dyn. Syst.* **6** (2005), 31-43.

[2] T. Carletti, L. Rosati, G. Villari, Qualitative analysis of the phase portrait for a class of planar vector fields via the comparison method, *Nonlinear Anal.* **67** (2007), 39–51.

[3] T. Carletti, G. Villari, A note on existence and uniqueness of limit cycles for Liénard systems, *J. Math. Anal. Appl.* **307** (2005), 763-773.

[4] M. Cioni, G. Villari, An extension of Dragilev's Theorem for the existence of periodic solutions of the Liénard equation, *Nonlinear Analysis, TMA* **127** (2015), 55–70.

[5] R. Conti Equazioni di Van der Pol e controllo in tempo minimo, *Quaderni Istituto Matematico Ulisse Dini* **13** (1976), 46 pp.

[6] G.F. Duff, N. Levinson, On the non-uniqueness of periodic solutions for an asymmetric Liénard equation, *Quart. Appl. Math.* **10** (1952), 86-88.

[7] J.-M. Ginoux, Self-excited oscillations: from Poincaré to Andronov, *Nieuw Arch. Wiskd.* **13** (2012), 170–177.

[8] J.-M. Ginoux, Van der Pol and the history of relaxation oscillations: toward the emergence of a concept, *Chaos* **22** (2012), 023120, 15 pp.

[9] J.K. Hale, *Ordinary differential equations*, Second edition. Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[10] M. Hayashi, On the uniqueness of the closed orbit of the Liénard system, *Math. Japon.* **46** (1997), 371-376.

[11] M. Hayashi, On the improved Massera's theorem for the unique existence of the limit cycle for a Liénard equation, *Rend. Istit. Mat. Univ. Trieste* **48** (2016), 487–493.

[12] M.Hayashi, M. Tsukada, A uniqueness theorem on the Liénard system with a non-hyperbolic equilibrium point *Dynam. Systems Appl.* **9**(2000), 99-108.

[13] S. Lefschetz, *Differential equations: geometric theory*, (reprinting of the second edition) DoverPublications, Inc., New York, 1977.

[14] N. Levinson, O.K. Smith, A general equation for relaxation oscillations, *Duke Math. J.* **9** (1942), 382-403.

[15] A. Liénard, Étude des oscillations entretenues, *Revue générale de lElectricité* **23** (1928), pp. 901-912 et 946-954.

[16] J. Mawhin, Can the drinking bird explain economic cycles ? (A history of auto-oscillations and limit cycles), *Bulletin de la Classe des Sciences* **20** (2009), 49–94.

[17] L. Rosati, G. Villari, On Massera's theorem concerning the uniqueness of a periodic solution for the Liénard equation. When does such a periodic solution actually exist? *Bound. Value Probl.* **2013**, 2013:144, 12 pp.

[18] M. Sabatini, G. Villari, Limit cycle uniqueness for a class of planar dynamical systems, *Appl. Math. Lett.* **19** (2006), 1180-1184.

[19] G. Sansone, Sopra l'equazione di A. Liénard delle oscillazioni di rilassamento, *Ann. Mat. Pura Appl.* **28**, (1949). 153–181.

[20] G. Villari, Some remarks on the uniqueness of the periodic solutions for Liénard's equation, *Boll. Un. Mat. Ital. C* (6) **4** (1985), 173-182.

[21] G. Villari, An improvement of Massera's theorem for the existence and uniqueness of a periodic solution for the Liénard equation, *Rend. Istit. Mat. Univ. Trieste* **44** (2012), 187-195.

[22] G. Villari, F. Zanolin, On the uniqueness of the limit cycle for the Liénard equation, via comparison method for the energy level curves, *Dynam. Systems Appl.* **25** (2016), 321-334.

[23] D. Xiao, Z. Zhang, On the uniqueness and nonexistence of limit cycles for predator-prey systems *Nonlinearity* **16** (2003), 1185-1201.