ONE-DIMENSIONAL VON KÁRMÁN MODELS FOR ELASTIC RIBBONS

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ABSTRACT. By means of a variational approach we rigorously deduce three one-dimensional models for elastic ribbons from the theory of von Kármán plates, passing to the limit as the width of the plate goes to zero. The one-dimensional model found starting from the "linearized" von Kármán energy corresponds to that of a linearly elastic beam that can twist but can deform in just one plane; while the model found from the von Kármán energy is a non-linear model that comprises stretching, bendings, and twisting. The "constrained" von Kármán energy, instead, leads to a new Sadowsky type of model.

KEYWORDS: Elastic ribbons, von Kármán plates, Sadowsky functional, Gamma-convergence

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1. INTRODUCTION

Geometrically a ribbon is a body with three length scales: it is a parallelepiped whose length ℓ is much larger than the width ε, which, in turn, is much larger than the thickness h. That is, $\ell \gg \varepsilon \gg h$. Since two characteristic dimensions are much smaller than the length, ribbons can be efficiently modelled as a one-dimensional continuum, see [14]. In the literature, two types of one-dimensional models are found: rod models and "Sadowsky type" models. We shall mainly discuss the latter, since we work within that framework; for rod type models we refer to [7] and the references therein.

So far, "Sadowsky type" models have been deduced starting from a plate model, that is, from a two-dimensional model obtained from a three-dimensional problem by letting the thickness h go to zero. Starting from a Kirchhoff plate model, a one-dimensional model for an isotropic elastic ribbon was proposed by Sadowsky in 1930, [15, 20]. The model was formally justified in 1962 by Wunderlich, [23, 26], by considering the Kirchhoff model for a plate of length ℓ and width ε, and by letting ε go to zero. The justification given was only formal, since it was based on an ansatz on the deformation. Wunderlich's technique is quite ingenious, but it leads to a singular energy density; we refer to [17] for a rigorous analysis of the so-called Wunderlich energy. A corrected Sadowsky type of energy was derived in [9] and generalized in [1, 10].

A third approach, which partly justifies the two approaches mentioned above, is to let the width ε and the thickness h go to zero simultaneously. By appropriately tuning the rates at which ε and h converge to zero, one obtains a hierarchy of one-dimensional models: in [11, 12] several rod models have been deduced, and in a forthcoming paper we will show that also "Sadowsky type" models can be obtained.

Before describing the contents of the present paper, we point out that the literature on ribbons is really blooming in several interesting directions, see, for instance, [2, 3, 4, 6, 8, 18, 21, 22].

Our starting point are the von Kármán plate models, whereas the papers quoted above have the Kirchhoff plate model as a starting point. The von Kármán model for plates has been successfully used in [5] to describe the plethora of morphological instabilities observed in a stretched and twisted ribbon.

The von Kármán plate equations, formulated more than a hundred years ago [24], have been recently justified by Friesecke, James, and Müller [13]. These authors consider a three-dimensional non-linear hyper-elastic material in a reference configuration $\Omega_h = S_\varepsilon \times (-\frac{h}{2}, \frac{h}{2})$ with a stored energy density $W : \mathbb{R}^{3\times 3} \to [0, +\infty)$ satisfying standard regularity and growth conditions. In [13] the set S_ε is quite general, but in this introduction, in order to be consistent with the previous discussion, we take $S_\varepsilon = (-\ell/2, \ell/2)\times(-\varepsilon/2, \varepsilon/2)$. Then, the energy associated with a deformation

$y : \Omega_h \to \mathbb{R}^3$ is given by

$$\mathcal{E}^h(y) = \int_{\Omega_h} W(\nabla y)\, dx.$$

By scaling the elastic energy per unit volume $\mathcal{E}^h/h \sim h^\beta$, with β a positive real parameter, in [13] a hierarchy of plate models has been derived (by letting h go to zero) by means of Γ-convergence theory. The larger β is, the smaller the energy becomes. Therefore, heuristically, for large β the limit of the rescaled energy should produce the linear plate equation. This is indeed corroborated in [13]. Still in the same paper it is shown that for $\beta = 2\alpha - 2$ and for the regimes $\alpha > 3$, $\alpha = 3$, and $2 < \alpha < 3$ three different Γ-limits are obtained that correspond to von Kármán type of energies.

Precisely, denoting by $u : S_\varepsilon \to \mathbb{R}^2$ and $v : S_\varepsilon \to \mathbb{R}$ the in-plane and the out-of-plane displacement fields, respectively, the three asymptotic energies are as follows (see [13, Theorem 2]):

$(LvK)_\varepsilon$ for $\alpha > 3$ we have the "linearized" von Kármán theory, where $u = 0$ and v minimizes the functional

$$I^{LvK}_\varepsilon(v) := \frac{1}{24}\int_{S_\varepsilon} Q_2(\nabla^2 v)\, dx,$$

with $Q_2 : \mathbb{R}^{2\times 2}_{\rm sym} \to [0, +\infty)$ the positive definite quadratic form of linearized elasticity, see Remark 2.5 for a precise definition;

$(vK)_\varepsilon$ for $\alpha = 3$ we have the von Kármán theory, where the in-plane and the out-of-plane displacements u and v minimize the functional

$$I^{vK}_\varepsilon(u, v) := \frac{1}{2}\int_{S_\varepsilon} Q_2\Big(\frac{1}{2}[\nabla u + (\nabla u)^T + \nabla v \otimes \nabla v]\Big)\, dx + \frac{1}{24}\int_{S_\varepsilon} Q_2(\nabla^2 v)\, dx;$$

$(CvK)_\varepsilon$ for $2 < \alpha < 3$ we have the "constrained" von Kármán theory, in which the functional

$$I^{CvK}_\varepsilon(v) := \frac{1}{24}\int_{S_\varepsilon} Q_2(\nabla^2 v)\, dx$$

has to be minimized under the non-linear constraint

$$\nabla u + (\nabla u)^T + \nabla v \otimes \nabla v = 0, \tag{1.1}$$

or, equivalently, the functional I^{CvK}_ε has to be minimized under the constraint

$$\det(\nabla^2 v) = 0$$

(which is, in turn, necessary and sufficient for the existence of a map u satisfying (1.1)).

The existence of minimizers and the characterization of the Euler equations for constrained von Kármán plates have been studied in [16].

By letting h go to zero, the three-dimensional domain $\Omega_h = S_\varepsilon \times (-\frac{h}{2}, \frac{h}{2})$ is "squeezed" to become S_ε. In this paper, we consider the von Kármán energies and we let ε go to zero, still by means of Γ-convergence, to find one-dimensional models for elastic ribbons in the von Kármán regimes. In this way, the two-dimensional domain $S_\varepsilon = (-\ell/2, \ell/2) \times (-\varepsilon/2, \varepsilon/2)$ is "squeezed" to the segment $I = (-\ell/2, \ell/2)$, which we parametrize with the coordinate x_1. In the limit, the in-plane displacement $u : S_\varepsilon \to \mathbb{R}^2$ generates two displacements: an axial displacement $\xi_1 : I \to \mathbb{R}$, and an orthogonal "in-plane" displacement $\xi_2 : I \to \mathbb{R}$. The out-of-plane displacement $v : S_\varepsilon \to \mathbb{R}$, in turn, generates an "out-of-plane" displacement $w : I \to \mathbb{R}$ and the derivative of v in the direction orthogonal to the axis leads to a rotation $\vartheta : I \to \mathbb{R}$. The limit energies that we find in the three regimes are the following:

(LvK) the limit of the "linearized" von Kármán energy is

$$J^{LvK}(w, \vartheta) := \frac{1}{24}\int_I Q_1(w'', \vartheta')\, dx_1;$$

(vK) the limit of the von Kármán energy is

$$J^{vK}(\xi, w, \vartheta) := \frac{1}{2}\int_I Q_0\Big(\xi_1' + \frac{|w'|^2}{2}\Big)\, dx_1 + \frac{1}{24}\int_I \big(Q_0(\xi_2'') + Q_1(w'', \vartheta')\big)\, dx_1;$$

(CvK) the limit of the "constrained" von Kármán energy is

$$J^{CvK}(w,\vartheta) := \frac{1}{24}\int_I \overline{Q}(w'',\vartheta')\,dx_1.$$

Here Q_1, Q_0, and $\overline{Q}$ are energy densities whose precise definition can be found in Section 2; see Remark 2.5 for the specialization of these energies in the isotropic case.

We note that, since Q_1 is quadratic, the functional (LvK) corresponds to the energy of a linearly elastic "three-dimensional beam" in which the section S_ε is unstretchable: the energy is simply due to the "out-of-plane" bending of the axis and to the torsion of the cross-section orthogonal to the axis. The limit functional (vK) is non-linear and penalizes stretching and both bendings of the axis, as well as the torsion of the cross-section. The functional (CvK) is sometimes called the energy of a beam with large deflections, see [25]. Despite the appearance, the energy functional (CvK) is very different from that of (LvK). Indeed, in contrast to Q_1, the energy density $\overline{Q}$ is not quadratic. It incorporates into its definition the non-linear constraint (1.1) that appears into the two-dimensional model $(CvK)_\varepsilon$. The energy density $\overline{Q}$ agrees with the corrected Sadowsky energy density found in [9] in the isotropic case, and with that found in [10] for the general anisotropic case. To the best of our knowledge, the model (CvK) is new.

We conclude this introduction by pointing out that the statements of the results and the precise definitions are given in Section 2, while Section 3 is exclusively devoted to the the proofs of these results.

2. Narrow strips

Let $\ell > 0$, let I denote the interval $(-\ell/2, \ell/2)$, and let $S_\varepsilon = I \times (-\varepsilon/2, \varepsilon/2)$ with $\varepsilon > 0$. For $u \in W^{1,2}(S_\varepsilon;\mathbb{R}^2)$ and $v \in W^{2,2}(S_\varepsilon)$ we consider the scaled von Kármán extensional and bending energies

$$\mathcal{J}^{ext}_\varepsilon(u,v) = \frac{1}{\varepsilon}\frac{1}{2}\int_{S_\varepsilon} Q_2\Big(Eu + \frac{1}{2}\nabla v\otimes\nabla v\Big)\,dx, \qquad \mathcal{J}^{ben}_\varepsilon(v) = \frac{1}{\varepsilon}\frac{1}{24}\int_{S_\varepsilon} Q_2(\nabla^2 v)\,dx,$$

where $Eu = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient of the in-plane displacement u, while $\nabla^2 v$ denotes the Hessian matrix of the out-of-plane displacement v. The energy density $Q_2 : \mathbb{R}^{2\times 2}_{\rm sym} \to [0,+\infty)$ is assumed to be a positive definite quadratic form.

To simplify our analysis we rewrite the energies over the domain $S := S_1 = I \times (-1/2, 1/2)$. More precisely, we introduce the scaled versions $y : S \to \mathbb{R}^2$ and $w : S \to \mathbb{R}$ of u and v, respectively, by setting

$$y_1(x_1,x_2) := u_1(x_1,\varepsilon x_2), \quad y_2(x_1,x_2) := \varepsilon u_2(x_1,\varepsilon x_2), \quad w(x_1,x_2) := v(x_1,\varepsilon x_2),$$

and define the scaled differential operators

$$E^\varepsilon y := \begin{pmatrix} \partial_1 y_1 & \frac{1}{2\varepsilon}(\partial_1 y_2 + \partial_2 y_1) \\ \frac{1}{2\varepsilon}(\partial_2 y_1 + \partial_1 y_2) & \frac{1}{\varepsilon^2}\partial_2 y_2 \end{pmatrix},$$

$$\nabla_\varepsilon w := \Big(\partial_1 w, \frac{1}{\varepsilon}\partial_2 w\Big), \quad \nabla^2_\varepsilon w := \begin{pmatrix} \partial^2_{11} w & \frac{1}{\varepsilon}\partial^2_{12} w \\ \frac{1}{\varepsilon}\partial^2_{21} w & \frac{1}{\varepsilon^2}\partial^2_{22} w \end{pmatrix},$$

so that

$$E^\varepsilon y(x) = Eu(x_1,\varepsilon x_2), \quad \nabla_\varepsilon w(x) = \nabla v(x_1,\varepsilon x_2), \quad \nabla^2_\varepsilon w(x) = \nabla^2 v(x_1,\varepsilon x_2).$$

By performing the change of variables in the energy integrals we have that $\mathcal{J}^{ext}_\varepsilon(u,v) = J^{ext}_\varepsilon(y,w)$ and $\mathcal{J}^{ben}_\varepsilon(v) = J^{ben}_\varepsilon(w)$, where

$$J^{ext}_\varepsilon(y,w) := \frac{1}{2}\int_S Q_2\Big(E^\varepsilon y + \frac{1}{2}\nabla_\varepsilon w\otimes\nabla_\varepsilon w\Big)\,dx, \qquad J^{ben}_\varepsilon(w) := \frac{1}{24}\int_S Q_2(\nabla^2_\varepsilon w)\,dx. \tag{2.1}$$

Since we do not impose boundary conditions, we require the displacements to have zero average and, for the out-of plane component, also zero average gradient. That is, we shall work in the following spaces: for every open set $\Omega \subset \mathbb{R}^\alpha$ with $\alpha = 1, 2$, we consider

$$W^{1,2}_{\langle 0 \rangle}(\Omega) := \Big\{ g \in W^{1,2}(\Omega) : \int_\Omega g(x)\, dx = 0 \Big\},$$
$$W^{2,2}_{\langle 0 \rangle}(\Omega) := \Big\{ g \in W^{2,2}(\Omega) : \int_\Omega g(x)\, dx = 0 \text{ and } \int_\Omega \nabla g(x)\, dx = 0 \Big\},$$

and similarly we define $W^{1,2}_{\langle 0 \rangle}(\Omega;\mathbb{R}^2)$.

Our first result is about compactness of sequences with bounded energy; the limit of the in-plane displacements will belong to the space of two-dimensional Bernoulli-Navier functions defined by

$$\begin{aligned} BN_{\langle 0 \rangle}(S;\mathbb{R}^2) :&= \{ g \in W^{1,2}_{\langle 0 \rangle}(S;\mathbb{R}^2) \ : \ (Eg)_{12} = (Eg)_{22} = 0 \} \\ &= \{ g \in W^{1,2}_{\langle 0 \rangle}(S;\mathbb{R}^2) \ : \ \exists\, \xi_1 \in W^{1,2}_{\langle 0 \rangle}(I) \text{ and } \xi_2 \in W^{1,2}_{\langle 0 \rangle}(I) \cap W^{2,2}(I) \text{ such that} \\ &\qquad g_1(x) = \xi_1(x_1) - x_2 \xi_2'(x_1),\ g_2(x) = \xi_2(x_1) \}, \end{aligned}$$

where the second characterization can be obtained by arguing as in [19, Section 4.1].

Lemma 2.1. *Let $(w_\varepsilon) \subset W^{2,2}_{\langle 0 \rangle}(S)$ be a sequence such that*

$$\sup_\varepsilon J^{ben}_\varepsilon(w_\varepsilon) < \infty. \tag{2.2}$$

Then, up to a subsequence, there exist a vertical displacement *$w \in W^{2,2}_{\langle 0 \rangle}(I)$ and a* twist function *$\vartheta \in W^{1,2}_{\langle 0 \rangle}(I)$ such that*

$$w_\varepsilon \rightharpoonup w \text{ in } W^{2,2}(S), \qquad \nabla_\varepsilon w_\varepsilon \rightharpoonup (w', \vartheta) \text{ in } W^{1,2}(S;\mathbb{R}^2), \tag{2.3}$$

and

$$\nabla^2_\varepsilon w_\varepsilon \rightharpoonup \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix} \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym}) \tag{2.4}$$

for a suitable $\gamma \in L^2(S)$.

Moreover, if $(y_\varepsilon) \subset W^{1,2}_{\langle 0 \rangle}(S;\mathbb{R}^2)$ is a further sequence such that

$$\sup_\varepsilon J^{ext}_\varepsilon(y_\varepsilon, w_\varepsilon) < \infty, \tag{2.5}$$

then, up to a subsequence, there exists $y \in BN_{\langle 0 \rangle}(S;\mathbb{R}^2)$ such that

$$y_\varepsilon \rightharpoonup y \text{ in } W^{1,2}(S;\mathbb{R}^2).$$

Also,

$$E^\varepsilon y_\varepsilon \rightharpoonup E \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$$

for a suitable $E \in L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$ such that $E_{11} = \partial_1 y_1$.

The rest of this section is devoted to state the Γ-convergence results starting from the simpler case of the linearized theory $(LvK)_\varepsilon$, and proceeding in the order of increasing difficulty to consider the standard and the constrained models $(vK)_\varepsilon$ and $(CvK)_\varepsilon$, respectively.

2.1. **The linearized von Kármán model.** In order to state our first convergence result we need to introduce some definitions. Let $Q_1 : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be defined by

$$Q_1(\kappa, \tau) := \min_{\gamma \in \mathbb{R}} \Big\{ Q_2(M) \ : \ M = \begin{pmatrix} \kappa & \tau \\ \tau & \gamma \end{pmatrix} \Big\}.$$

Let $J^{LvK} : W^{2,2}_{\langle 0 \rangle}(I) \times W^{1,2}_{\langle 0 \rangle}(I) \to \mathbb{R}$ be defined by

$$J^{LvK}(w, \vartheta) := \frac{1}{24} \int_I Q_1(w'', \vartheta')\, dx_1.$$

Theorem 2.2. *As $\varepsilon \to 0$, the functionals J^{ben}_ε Γ-converge to the functional J^{LvK} in the following sense:*

(i) (liminf inequality) *for every sequence* $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$*,* $w \in W^{2,2}_{\langle 0\rangle}(I)$*, and* $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$ *such that* $w_\varepsilon \rightharpoonup w$ *in* $W^{2,2}(S)$*, and* $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w', \vartheta)$ *in* $W^{1,2}(S;\mathbb{R}^2)$*, we have that*

$$\liminf_{\varepsilon\to 0} J^{ben}_\varepsilon(w_\varepsilon) \ge J^{LvK}(w,\vartheta);$$

(ii) (recovery sequence) *for every* $w \in W^{2,2}_{\langle 0\rangle}(I)$ *and* $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$ *there exists a sequence* $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$ *such that* $w_\varepsilon \rightharpoonup w$ *in* $W^{2,2}(S)$*,* $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w', \vartheta)$ *in* $W^{1,2}(S;\mathbb{R}^2)$*, and*

$$\limsup_{\varepsilon\to 0} J^{ben}_\varepsilon(w_\varepsilon) \le J^{LvK}(w,\vartheta).$$

2.2. **The von Kármán model.** The statement of our second convergence result needs some further definitions. Let $Q_0 : \mathbb{R} \to [0,+\infty)$ be defined by

$$Q_0(\mu) := \min_{z\in\mathbb{R}} Q_1(\mu, z) = \min_{(z_1,z_2)\in\mathbb{R}^2} \Big\{ Q_2(M) \ : \ M = \begin{pmatrix} \mu & z_1 \\ z_1 & z_2 \end{pmatrix} \Big\}.$$

Let $J^{vK} : BN_{\langle 0\rangle}(S;\mathbb{R}^2) \times W^{2,2}_{\langle 0\rangle}(I) \times W^{1,2}_{\langle 0\rangle}(I) \to \mathbb{R}$ be defined by

$$J^{vK}(y,w,\vartheta) := \frac{1}{2}\int_S Q_0\Big(\partial_1 y_1 + \frac{|w'|^2}{2}\Big)\,dx + \frac{1}{24}\int_I Q_1(w'',\vartheta')\,dx_1.$$

Theorem 2.3. *As* $\varepsilon \to 0$*, the functionals* $J^{vK}_\varepsilon := J^{ext}_\varepsilon + J^{ben}_\varepsilon$ *Γ-converge to the functional* J^{vK} *in the following sense:*

(i) (liminf inequality) *for every pair of sequences* $(y_\varepsilon) \subset W^{1,2}_{\langle 0\rangle}(S;\mathbb{R}^2)$*,* $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$*,* $y \in BN_{\langle 0\rangle}(S;\mathbb{R}^2)$*,* $w \in W^{2,2}_{\langle 0\rangle}(I)$*, and* $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$ *such that* $y_\varepsilon \rightharpoonup y$ *in* $W^{1,2}(S;\mathbb{R}^2)$*,* $w_\varepsilon \rightharpoonup w$ *in* $W^{2,2}(S)$*, and* $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w',\vartheta)$ *in* $W^{1,2}(S;\mathbb{R}^2)$*, we have that*

$$\liminf_{\varepsilon\to 0} J^{vK}_\varepsilon(y_\varepsilon, w_\varepsilon) \ge J^{vK}(y,w,\vartheta);$$

(ii) (recovery sequence) *for every* $y \in BN_{\langle 0\rangle}(S;\mathbb{R}^2)$*,* $w \in W^{2,2}_{\langle 0\rangle}(I)$ *and* $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$ *there exists a pair of sequences* $(y_\varepsilon) \subset W^{1,2}_{\langle 0\rangle}(S;\mathbb{R}^2)$*,* $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$ *such that* $y_\varepsilon \rightharpoonup y$ *in* $W^{1,2}(S;\mathbb{R}^2)$*,* $w_\varepsilon \rightharpoonup w$ *in* $W^{2,2}(S)$*,* $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w',\vartheta)$ *in* $W^{1,2}(S;\mathbb{R}^2)$*, and*

$$\limsup_{\varepsilon\to 0} J^{vK}_\varepsilon(y_\varepsilon, w_\varepsilon) \le J^{vK}(y,w,\vartheta).$$

2.3. **The constrained von Kármán model.** The constrained von Kármán energy of a displacement $v \in W^{2,2}_{\langle 0\rangle}(S_\varepsilon)$ such that $\det \nabla^2 v = 0$ a.e. in S_ε is $\mathcal{J}^{ben}_\varepsilon(v)$. We observe that the map w, defined over the rescaled domain, belongs to the space

$$W^{2,2}_{\det,\varepsilon}(S) := \big\{ w \in W^{2,2}_{\langle 0\rangle}(S) : \ \det \nabla^2_\varepsilon w = 0 \text{ a.e. in } S \big\}.$$

We set $J^{CvK}_\varepsilon : W^{2,2}_{\det,\varepsilon}(S) \to \mathbb{R}$ the functional $J^{CvK}_\varepsilon(w) = J^{ben}_\varepsilon(w)$.

Let $\overline{Q} : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$ be defined by

$$\overline{Q}(\kappa,\tau) := \min_{\gamma\in\mathbb{R}} \Big\{ Q_2(M) + \alpha^+(\det M)^+ + \alpha^-(\det M)^- : M = \begin{pmatrix} \kappa & \tau \\ \tau & \gamma \end{pmatrix} \Big\},$$

where

$$\alpha^+ := \sup\{\alpha > 0 : \ Q_2(M) + \alpha \det M \ge 0 \text{ for every } M \in \mathbb{R}^{2\times 2}_{\rm sym}\}$$

and

$$\alpha^- := \sup\{\alpha > 0 : \ Q_2(M) - \alpha \det M \ge 0 \text{ for every } M \in \mathbb{R}^{2\times 2}_{\rm sym}\}.$$

Let $J^{CvK} : W^{2,2}_{\langle 0\rangle}(I) \times W^{1,2}_{\langle 0\rangle}(I) \to \mathbb{R}$ be defined by

$$J^{CvK}(w,\vartheta) := \frac{1}{24}\int_I \overline{Q}(w'',\vartheta')\,dx_1.$$

Theorem 2.4. *As* $\varepsilon \to 0$*, the functionals* J^{CvK}_ε *Γ-converge to the functional* J^{CvK} *in the following sense:*

(i) (liminf inequality) *for every sequence* (w_ε) *with* $w_\varepsilon \in W^{2,2}_{\det,\varepsilon}(S)$, $w \in W^{2,2}_{\langle 0\rangle}(I)$, *and* $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$ *such that* $w_\varepsilon \rightharpoonup w$ *in* $W^{2,2}(S)$, *and* $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w', \vartheta)$ *in* $W^{1,2}(S;\mathbb{R}^2)$, *we have that*

$$\liminf_{\varepsilon\to 0} J^{CvK}_\varepsilon(w_\varepsilon) \geq J^{CvK}(w,\vartheta);$$

(ii) (recovery sequence) *for every* $w \in W^{2,2}_{\langle 0\rangle}(I)$ *and* $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$ *there exists a sequence* (w_ε) *with* $w_\varepsilon \in W^{2,2}_{\det,\varepsilon}(S)$ *such that* $w_\varepsilon \rightharpoonup w$ *in* $W^{2,2}(S)$, $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w',\vartheta)$ *in* $W^{1,2}(S;\mathbb{R}^2)$, *and*

$$\limsup_{\varepsilon\to 0} J^{CvK}_\varepsilon(w_\varepsilon) \leq J^{CvK}(w,\vartheta).$$

Remark 2.5. The quadratic energy density Q_2 can be computed from the non-linear energy density W of the material, also mentioned in the introduction, by first computing the quadratic energy density Q_3, see [13],

$$Q_3(F) := \frac{\partial^2 W}{\partial F^2}(I)(F,F) = \sum_{i,j,k,l=1}^{3} \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(I) F_{ij}F_{kl}, \qquad F \in \mathbb{R}^{3\times 3},$$

and then by minimizing over the third column and row:

$$Q_2(A) := \min\{Q_3(F) : F_{\alpha\beta} = A_{\alpha\beta} \quad \alpha,\beta = 1,2\}, \qquad A \in \mathbb{R}^{2\times 2}_{\rm sym}.$$

If the energy density W is isotropic, the quadratic energy density Q_3 has the following representation:

$$Q_3(F) = 2\mu|F_{\rm sym}|^2 + \lambda(F_{\rm sym}\cdot I)^2, \qquad F_{\rm sym} := \frac{F+F^T}{2} \in \mathbb{R}^{3\times 3},$$

where μ and λ are the so-called Lamé coefficients. A simple computation then leads to

$$Q_2(A) = 2\mu|A|^2 + \frac{2\mu\lambda}{2\mu+\lambda}(A\cdot I)^2, \qquad A \in \mathbb{R}^{2\times 2}_{\rm sym}.$$

The energy densities Q_1, Q_0, and $\overline{Q}$, may be found to have the following representation

$$Q_1(\kappa,\tau) = E_{\rm Y}\kappa^2 + 4\mu\tau^2,$$

where $E_{\rm Y} := \mu\frac{2\mu+3\lambda}{\mu+\lambda}$ is the Young modulus of the material,

$$Q_0(\kappa) = E_{\rm Y}\kappa^2,$$

and

$$\frac{1}{12}\overline{Q}(\kappa,\tau) = \begin{cases} \mathcal{D}\dfrac{(\kappa^2+\tau^2)^2}{\kappa^2} & \text{if } |\kappa| > |\tau|, \\ 4\mathcal{D}\tau^2 & \text{if } |\kappa| \leq |\tau|, \end{cases}$$

where $\mathcal{D} := \frac{\mu(\lambda+\mu)}{3(2\mu+\lambda)}$ is the bending stiffness.

3. Proofs

This section is devoted to prove the theorems stated in the previous section. For a given function $u \in L^1(S)$, we shall denote by $\langle u\rangle$ the integral mean value of u on S, that is,

$$\langle u\rangle := \frac{1}{\ell}\int_S u(x)\,dx.$$

We use the same notation to denote the average over I of functions defined on I.

Proof of Lemma 2.1. Let $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$ be a sequence of vertical displacements of S satisfying (2.2). This bound and the fact that Q_2 is positive definite imply that

$$\|\partial^2_{11}w_\varepsilon\|_{L^2(S)} + \|\varepsilon^{-1}\partial^2_{12}w_\varepsilon\|_{L^2(S)} + \|\varepsilon^{-2}\partial^2_{22}w_\varepsilon\|_{L^2(S)} \leq C \tag{3.1}$$

for any ε. Since

$$\int_S w_\varepsilon(x)\,dx = 0, \qquad \int_S \nabla w_\varepsilon(x)\,dx = 0$$

for every $\varepsilon > 0$, by Poincaré-Wirtinger inequality the sequence (w_ε) is uniformly bounded in $W^{2,2}(S)$. Therefore, there exists $w \in W^{2,2}_{\langle 0\rangle}(S)$ such that $w^\varepsilon \rightharpoonup w$ weakly in $W^{2,2}(S)$, up to a subsequence.

By the previous bound, $\nabla(\varepsilon^{-1}\partial_2 w_\varepsilon)$ is a bounded sequence in $L^2(S;\mathbb{R}^2)$ and, by Poincaré-Wirtinger inequality, also $(\varepsilon^{-1}\partial_2 w_\varepsilon)$ is bounded in $L^2(S)$. It follows that w is independent of x_2 and there exixts $\vartheta \in W^{1,2}_{\langle 0\rangle}(S)$ such that $\varepsilon^{-1}\partial_2 w_\varepsilon \rightharpoonup \vartheta$ weakly in $W^{1,2}(S)$, up to a subsequence. Moreover, also ϑ is independent of x_2.

By (3.1), up to subsequences, we have that $\nabla^2_\varepsilon w_\varepsilon$ converges to a matrix field A weakly in $L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$. By using the convergences established above, it follows that $A_{11} = w''$ and $A_{12} = \vartheta'$. The entry A_{22}, that cannot be identified in terms of w and ϑ, is denoted by γ in the statement. This proves (2.4).

We now prove the second part of the statement. The bound (2.5) implies that

$$\Big\| E^\varepsilon y_\varepsilon + \frac{1}{2}\nabla_\varepsilon w_\varepsilon \otimes \nabla_\varepsilon w_\varepsilon \Big\|_{L^2} \le C \tag{3.2}$$

for any ε. Since $(\nabla^2_\varepsilon w_\varepsilon)$ is bounded in L^2, we have

$$\begin{aligned}\|\nabla_\varepsilon w_\varepsilon \otimes \nabla_\varepsilon w_\varepsilon\|_{L^2} &\le C\||\nabla_\varepsilon w_\varepsilon|^2\|_{L^2} = C\|\nabla_\varepsilon w_\varepsilon\|^2_{L^4} \le C(\|\partial_1 w_\varepsilon\|^2_{L^4} + \|\varepsilon^{-1}\partial_2 w_\varepsilon\|^2_{L^4}) \\ &\le C(\|\nabla\partial_1 w_\varepsilon\|^2_{L^2} + \|\nabla\varepsilon^{-1}\partial_2 w_\varepsilon\|^2_{L^2}) \le C\|\nabla^2_\varepsilon w_\varepsilon\|^2_{L^2} \le C\end{aligned}$$

for any ε, and the third to last inequality follows by the imbedding $W^{1,2}(S) \subset L^q\ \forall\, q \in [2, +\infty)$ and Poincaré-Wirtinger inequality. Together with (3.2), this implies that the sequence $(E^\varepsilon y_\varepsilon)$ is bounded in L^2.

By the definition of E^ε and Korn-Poincaré inequality we have that

$$\|y_\varepsilon\|_{W^{1,2}} \le C\|Ey_\varepsilon\|_{L^2} \le C\|E^\varepsilon y_\varepsilon\|_{L^2} \le C. \tag{3.3}$$

Hence, up to subsequences, there exist $E \in L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$ and $y \in W^{1,2}_{\langle 0\rangle}(S;\mathbb{R}^2)$ such that

$$\begin{aligned} E^\varepsilon y_\varepsilon &\rightharpoonup E \quad \text{in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym}), \\ y_\varepsilon &\rightharpoonup y \quad \text{in } W^{1,2}(S;\mathbb{R}^2).\end{aligned}$$

By the definition of E^ε and (3.3) we have that

$$(Ey_\varepsilon)_{12} \rightharpoonup 0 = (Ey)_{12}, \quad (Ey_\varepsilon)_{22} \rightharpoonup 0 = (Ey)_{22};$$

hence, $y \in BN_{\langle 0\rangle}(S;\mathbb{R}^2)$. Finally, the observation that $(E^\varepsilon y_\varepsilon)_{11} = \partial_1(y_\varepsilon)_1 \rightharpoonup \partial_1 y_1$ in $L^2(S)$ concludes the proof. □

Proof of Theorem 2.2–(i). Let $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$ be such that $w_\varepsilon \rightharpoonup w$ in $W^{2,2}(S)$, and $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w', \vartheta)$ in $W^{1,2}(S;\mathbb{R}^2)$, for some $w \in W^{2,2}_{\langle 0\rangle}(I)$ and $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$. Without loss of generality, we can assume that $\liminf_{\varepsilon\to 0} J^{ben}_\varepsilon(w_\varepsilon) < +\infty$. By Lemma 2.1 we infer that, up to subsequences,

$$\nabla^2_\varepsilon w_\varepsilon \rightharpoonup \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix} =: M_\gamma \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$$

for some $\gamma \in L^2(S)$. By weak lower semicontinuity and the definition of Q_1 we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} J^{ben}_\varepsilon(w_\varepsilon) &= \liminf_{\varepsilon\to 0} \frac{1}{24}\int_S Q_2(\nabla^2_\varepsilon w_\varepsilon)\,dx \\ &\ge \frac{1}{24}\int_S Q_2(M_\gamma)\,dx \ge \frac{1}{24}\int_I Q_1(w'', \vartheta')\,dx_1 = J^{LvK}(w, \vartheta).\end{aligned}$$

□

Proof of Theorem 2.2–(ii). Let $w \in W^{2,2}_{\langle 0\rangle}(I)$ and $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$. We set

$$M_\gamma := \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix},$$

where $\gamma \in L^2(I)$ is such that

$$Q_1(w'', \vartheta') = Q_2(M_\gamma).$$

The fact that γ belongs to $L^2(I)$ follows immediately by choosing $M_0 = w'' e_1 \otimes e_1 + \vartheta'(e_1 \otimes e_2 + e_2 \otimes e_1)$ as a competitor in the definition of Q_1 and by using the positive definiteness of Q_2.

Let $\vartheta_\varepsilon \in C^\infty(\overline{I})$ be such that $\int_I \vartheta_\varepsilon(x_1)\,dx_1 = 0$, $\vartheta_\varepsilon \to \vartheta$ in $W^{1,2}(I)$, and $\varepsilon\vartheta''_\varepsilon \to 0$ in $L^2(I)$. Let $\gamma_\varepsilon \in C^\infty(\overline{I})$ be such that $\gamma_\varepsilon \to \gamma$, $\varepsilon\gamma'_\varepsilon \to 0$ and $\varepsilon^2\gamma''_\varepsilon \to 0$ in $L^2(I)$. Let

$$w_\varepsilon(x) = w(x_1) + \varepsilon x_2 \vartheta_\varepsilon(x_1) + \frac{\varepsilon^2}{2}\big(x_2^2\gamma_\varepsilon(x_1) - \langle x_2^2\gamma_\varepsilon\rangle - x_1\langle x_2^2\gamma'_\varepsilon\rangle\big). \tag{3.4}$$

It turns out that $w_\varepsilon \in W^{2,2}_{\langle 0\rangle}(S)$ and, by the convergences above, we have $w_\varepsilon \to w$ in $W^{2,2}(S)$, $\nabla_\varepsilon w_\varepsilon \to (w', \vartheta)$ in $W^{1,2}(S;\mathbb{R}^2)$, and $\nabla^2_\varepsilon w_\varepsilon \to M_\gamma$ in $L^2(S)$. Moreover, by strong continuity we have

$$\lim_{\varepsilon\to 0} J^{ben}_\varepsilon(w_\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{24}\int_S Q_2(\nabla^2_\varepsilon w_\varepsilon)\,dx = \frac{1}{24}\int_I Q_1(w'', \vartheta')\,dx_1 = J^{LvK}(w,\vartheta).$$

□

Proof of Theorem 2.3–(i). Let $(y_\varepsilon) \subset W^{1,2}_{\langle 0\rangle}(S;\mathbb{R}^2)$, $(w_\varepsilon) \subset W^{2,2}_{\langle 0\rangle}(S)$ be such that $y_\varepsilon \rightharpoonup y$ in $W^{1,2}(S;\mathbb{R}^2)$, $w_\varepsilon \rightharpoonup w$ in $W^{2,2}(S)$, and $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w',\vartheta)$ in $W^{1,2}(S;\mathbb{R}^2)$, for some $y \in BN_{\langle 0\rangle}(S;\mathbb{R}^2)$, $w \in W^{2,2}_{\langle 0\rangle}(I)$ and $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$. As usual, we can assume that $\liminf_{\varepsilon\to 0} J^{vK}_\varepsilon(y_\varepsilon, w_\varepsilon) < +\infty$ and by Lemma 2.1 we deduce that, up to subsequences,

$$E^\varepsilon y_\varepsilon \rightharpoonup E \quad\text{and}\quad \nabla^2_\varepsilon w_\varepsilon \rightharpoonup \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix} =: M_\gamma \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$$

for some $E \in L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$ with $E_{11} = \partial_1 y_1$, and $\gamma \in L^2(S)$. Moreover, by the convergences above,

$$E^\varepsilon y_\varepsilon + \frac{1}{2}\nabla_\varepsilon w_\varepsilon \otimes \nabla_\varepsilon w_\varepsilon \rightharpoonup E + \frac{1}{2}(w',\vartheta)\otimes(w',\vartheta) \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym}).$$

Then, by lower semicontinuity, we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0} J^{vK}_\varepsilon(y_\varepsilon, w_\varepsilon) &\geq \liminf_{\varepsilon\to 0}\frac{1}{2}J_\varepsilon(y_\varepsilon, w_\varepsilon) + \liminf_{\varepsilon\to 0}\frac{1}{24}J^{\rm lin}_\varepsilon(w_\varepsilon)\\
&= \liminf_{\varepsilon\to 0}\frac{1}{2}\int_S Q_2\Big(E^\varepsilon y_\varepsilon + \frac{1}{2}\nabla_\varepsilon w_\varepsilon\otimes\nabla_\varepsilon w_\varepsilon\Big)\,dx + \liminf_{\varepsilon\to 0}\frac{1}{24}\int_S Q_2(\nabla^2_\varepsilon w_\varepsilon)\,dx\\
&\geq \frac{1}{2}\int_S Q_2\Big(E + \frac{1}{2}(w',\vartheta)\otimes(w',\vartheta)\Big)\,dx + \frac{1}{24}\int_S Q_2(M_\gamma)\,dx\\
&\geq \frac{1}{2}\int_S Q_0\Big(\partial_1 y_1 + \frac{1}{2}|w'|^2\Big)\,dx + \frac{1}{24}\int_I Q_1(w'',\vartheta')\,dx_1\\
&= J^{vK}(y, w, \vartheta).
\end{aligned}$$

□

Proof of Theorem 2.3–(ii). Let $y \in BN_{\langle 0\rangle}(S;\mathbb{R}^2)$, $w \in W^{2,2}_{\langle 0\rangle}(I)$, and $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$. As before, there exists $\gamma \in L^2(I)$ such that the matrix

$$M_\gamma := \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix}$$

satisfies

$$Q_1(w'',\vartheta') = Q_2(M_\gamma).$$

There exist $\xi_1 \in W^{1,2}_{\langle 0\rangle}(I)$ and $\xi_2 \in W^{1,2}_{\langle 0\rangle}(I) \cap W^{2,2}(I)$ such that $y_1(x) = \xi_1(x_1) - x_2\xi'_2(x_1)$ and $y_2(x) = \xi_2(x_1)$. Moreover, there exists $z \in L^2(S;\mathbb{R}^2)$ such that the matrix

$$M_z := \begin{pmatrix} \partial_1 y_1 + \frac{1}{2}|w'|^2 & z_1 \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} \xi'_1(x_1) - x_2\xi''_2(x_1) + \frac{1}{2}|w'(x_1)|^2 & z_1 \\ z_1 & z_2 \end{pmatrix}$$

satisfies

$$Q_0\Big(\partial_1 y_1 + \frac{1}{2}|w'|^2\Big) = Q_2(M_z).$$

It is easily seen that z_1 and z_2 depend linearly on $\partial_1 y_1 + \frac{1}{2}|w'|^2$. Since $\partial_1 y_1(x_1,x_2) + \frac{1}{2}|w'(x_1)|^2 = \xi_1'(x_1) - x_2\xi_2''(x_1) + \frac{1}{2}|w'(x_1)|^2$, there exist $\zeta_\alpha \in L^2(I)$ and $\eta_\alpha \in L^2(I)$ such that

$$z_\alpha(x_1,x_2) = \zeta_\alpha(x_1) + x_2\eta_\alpha(x_1), \qquad \alpha = 1,2.$$

Let w_ε be as in the proof of Theorem 2.2–(ii) (see (3.4)), and let ζ_α^ε, $\eta_\alpha^\varepsilon \in C^\infty(\overline{I})$ be such that $\zeta_\alpha^\varepsilon \to \zeta_\alpha$ and $\eta_\alpha^\varepsilon \to \eta_\alpha$ in $L^2(I)$ and $\varepsilon{\zeta_\alpha^\varepsilon}' \to 0$ and $\varepsilon{\eta_\alpha^\varepsilon}' \to 0$ in $L^2(I)$. Let us define

$$\begin{aligned}
(y_\varepsilon)_1(x_1,x_2) &:= \xi_1(x_1) - x_2\xi_2'(x_1) + \varepsilon\big(x_2^2\eta_1^\varepsilon(x_1) - \langle x_2^2\eta_1^\varepsilon\rangle\big),\\
(y_\varepsilon)_2(x_1,x_2) &:= \xi_2(x_1) + \varepsilon\Big(\int_0^{x_1}\big(2\zeta_1^\varepsilon(s) - w'(s)\vartheta(s)\big)\,ds - \langle\int_0^{x_1}\big(2\zeta_1^\varepsilon(s) - w'(s)\vartheta(s)\big)\,ds\rangle\Big)\\
&\quad + \frac{\varepsilon^2}{2}x_2\big(2\zeta_2^\varepsilon(x_1) - \vartheta^2(x_1)\big) + \frac{\varepsilon^2}{2}\big(x_2^2\eta_2^\varepsilon(x_1) - \langle x_2^2\eta_2^\varepsilon\rangle\big).
\end{aligned}$$

Then it is easy to check that

$$E^\varepsilon y_\varepsilon + \frac{1}{2}\nabla_\varepsilon w_\varepsilon \otimes \nabla_\varepsilon w_\varepsilon \to M_z \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym}).$$

Thus, by strong continuity,

$$\begin{aligned}
\lim_{\varepsilon\to 0} J_\varepsilon^{vK}(y_\varepsilon,w_\varepsilon) &= \lim_{\varepsilon\to 0}\Big(\frac{1}{2}J_\varepsilon^{ext}(y_\varepsilon,w_\varepsilon) + \frac{1}{24}J_\varepsilon^{ben}(w_\varepsilon)\Big)\\
&= \lim_{\varepsilon\to 0}\Big(\frac{1}{2}\int_S Q_2\Big(E^\varepsilon y_\varepsilon + \frac{1}{2}\nabla_\varepsilon w_\varepsilon\otimes\nabla_\varepsilon w_\varepsilon\Big)\,dx + \frac{1}{24}\int_S Q_2(\nabla_\varepsilon^2 w_\varepsilon)\,dx\Big)\\
&= \frac{1}{2}\int_S Q_2(M_z)\,dx + \frac{1}{24}\int_S Q_2(M_\gamma)\,dx\\
&= \frac{1}{2}\int_S Q_0\Big(\partial_1 y_1 + \frac{1}{2}|w'|^2\Big)\,dx + \frac{1}{24}\int_I Q_1(w'',\vartheta')\,dx_1\\
&= J^{vK}(y,w,\vartheta).
\end{aligned}$$

□

The proof of the Γ-convergence theorem 2.4 is based on a relaxation result for a quadratic integral functional with a constraint on the determinant, that has been proved in [10, Proposition 9] and recalled here for reader's convenience.

Let $\mathcal{B}$ be a bounded open subset of $\mathbb{R}^n$. Let $Q : \mathcal{B}\times\mathbb{R}^{2\times 2}_{\rm sym} \to [0,+\infty)$ be measurable in the first variable and quadratic in the second. Define the functional

$$\mathcal{F} : L^2\left(\mathcal{B};\mathbb{R}^{2\times 2}_{\rm sym}\right) \to [0,+\infty]$$

by

$$\mathcal{F}(M) := \begin{cases} \displaystyle\int_{\mathcal{B}} Q(x,M(x))\,dx & \text{if } \det M = 0 \text{ a.e. in } \mathcal{B},\\ +\infty & \text{otherwise.}\end{cases}$$

Theorem 3.1 ([10])**.** *The weak-L^2 lower semicontinuous envelope of $\mathcal{F}$ is the functional*

$$\overline{\mathcal{F}} : L^2\left(\mathcal{B};\mathbb{R}^{2\times 2}_{\rm sym}\right) \to [0,+\infty)$$

given by

$$\overline{\mathcal{F}}(M) = \int_{\mathcal{B}}\left(Q(x,M(x)) + \alpha^+(x)(\det M(x))^+ + \alpha^-(x)(\det M(x))^-\right)\,dx,$$

where for every $x\in\mathcal{B}$

$$\alpha^+(x) := \sup\{\alpha > 0 :\ Q(x,M) + \alpha\det M \ge 0 \textit{ for every } M \in \mathbb{R}^{2\times 2}_{\rm sym}\}$$

and

$$\alpha^-(x) := \sup\{\alpha > 0 :\ Q(x,M) - \alpha\det M \ge 0 \textit{ for every } M \in \mathbb{R}^{2\times 2}_{\rm sym}\}.$$

Proof of Theorem 2.4–(i). Let (w_ε) be such that $w_\varepsilon \in W^{2,2}_{\det,\varepsilon}(S)$, $w_\varepsilon \rightharpoonup w$ in $W^{2,2}(S)$, and $\nabla_\varepsilon w_\varepsilon \rightharpoonup (w', \vartheta)$ in $W^{1,2}(S;\mathbb{R}^2)$, for some $w \in W^{2,2}_{\langle 0\rangle}(I)$ and $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$. Under the assumption that $\liminf_{\varepsilon\to 0} J^{vK}_\varepsilon(w_\varepsilon) < +\infty$, by Lemma 2.1 we deduce that, up to subsequences,

$$\nabla^2_\varepsilon w_\varepsilon \rightharpoonup \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix} \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\rm sym})$$

for some $\gamma \in L^2(S)$. Since $\det \nabla^2_\varepsilon w_\varepsilon = 0$, an application of Theorem 3.1 with $Q(x,M) := Q_2(M)$ and $\mathcal{B} = S$ shows that

$$\liminf_{\varepsilon\to 0} J^{CvK}_\varepsilon(w_\varepsilon) = \liminf_{\varepsilon\to 0} \frac{1}{24}\mathcal{F}(\nabla^2_\varepsilon w_\varepsilon) \geq \frac{1}{24}\overline{\mathcal{F}}\begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix} \geq \frac{1}{24}\int_I \overline{Q}(w'',\vartheta')\,dx_1.$$

Therefore, we conclude that

$$\liminf_{\varepsilon\to 0} J^{CvK}_\varepsilon(w_\varepsilon) \geq J(w,\vartheta).$$

□

Proof of Theorem 2.4–(ii). Let $w \in W^{2,2}_{\langle 0\rangle}(I)$ and $\vartheta \in W^{1,2}_{\langle 0\rangle}(I)$. We set

$$M := \begin{pmatrix} w'' & \vartheta' \\ \vartheta' & \gamma \end{pmatrix},$$

where $\gamma \in L^2(I)$ is such that

$$\overline{Q}(w'',\vartheta') = Q_2(M) + \alpha^+(\det M)^+ + \alpha^-(\det M)^-.$$

As before, the fact that γ belongs to $L^2(I)$ follows immediately by choosing $M_0 = w'' e_1 \otimes e_1 + \vartheta'(e_1 \otimes e_2 + e_2 \otimes e_1)$ as a competitor in the definition of $\overline{Q}$ and by using the positive definiteness of Q_2.

By Theorem 3.1 with $Q(x,M) := Q_2(M)$ and $\mathcal{B} = I$, there exist $M^j \in L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$ with $\det M^j = 0$ and such that $M^j \rightharpoonup M$ weakly in $L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$ and $\mathcal{F}(M^j) \to \overline{\mathcal{F}}(M)$, as $j \to \infty$. We can also assume that $M^j \in C^\infty(\bar I;\mathbb{R}^{2\times 2}_{\rm sym})$. The proof of this fact relies on a construction described in [9, Theorem 2.2–(ii)]. We give here full details for convenience of the reader. Suppose that (M_n) be a sequence of matrices with the same properties of (M^j) apart from the regularity, and denote by $\lambda_n \in L^2(I)$ the trace of M_n. Since M_n is symmetric with $\det M_n = 0$, there exists $\beta_n = \beta_n(x_1) \in (-\pi/2, \pi/2]$ such that

$$M_n = \begin{pmatrix} \cos\beta_n & -\sin\beta_n \\ \sin\beta_n & \cos\beta_n \end{pmatrix}\begin{pmatrix} \lambda_n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\beta_n & \sin\beta_n \\ -\sin\beta_n & \cos\beta_n \end{pmatrix},$$

and β_n is uniquely determined if $\lambda_n \neq 0$. When $\lambda_n(x_1) = 0$, we set $\beta_n(x_1) = 0$. We may assume without loss of generality that $\lambda_n \in L^\infty(I)$, possibly after truncating λ_n in modulus by n, while M_n still enjoys the same properties as before. We can find $\lambda_{n,k} \in C^\infty(\bar I)$ and $\beta_{n,k} \in C^\infty(\bar I; (-\pi/2, \pi/2))$ such that, as $k \to \infty$, $\lambda_{n,k} \to \lambda_n$ and $\beta_{n,k} \to \beta_n$ in $L^p(I)$ for every $p < +\infty$. Set

$$M_{n,k} := \begin{pmatrix} \cos\beta_{n,k} & -\sin\beta_{n,k} \\ \sin\beta_{n,k} & \cos\beta_{n,k} \end{pmatrix}\begin{pmatrix} \lambda_{n,k} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\beta_{n,k} & \sin\beta_{n,k} \\ -\sin\beta_{n,k} & \cos\beta_{n,k} \end{pmatrix}.$$

Then, $\det M_{n,k} = 0$ for every n, k and $M_{n,k} \to M_n$ in $L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$, as $k \to \infty$.

Thus, by a diagonal argument, we may assume that there exist $\lambda^j \in C^\infty(\bar I)$ and $\beta^j \in C^\infty(\bar I)$ such that $|\beta^j| < \pi/2$ on $\bar I$, and with

$$\begin{aligned} M^j \;&:=\; \begin{pmatrix} \cos\beta^j & -\sin\beta^j \\ \sin\beta^j & \cos\beta^j \end{pmatrix}\begin{pmatrix} \lambda^j & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\beta^j & \sin\beta^j \\ -\sin\beta^j & \cos\beta^j \end{pmatrix} \\ &=\; \lambda^j \begin{pmatrix} \cos^2\beta^j & \sin\beta^j\cos\beta^j \\ \sin\beta^j\cos\beta^j & \sin^2\beta^j \end{pmatrix} \end{aligned}$$

we have that $M^j \in C^\infty(\bar I;\mathbb{R}^{2\times 2}_{\rm sym})$, $\det M^j = 0$ for every j, $M^j \rightharpoonup M$ in $L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$, and $\mathcal{F}(M_j) \to \overline{\mathcal{F}}(M)$, as $j \to \infty$.

For all $j = 1, 2, \dots$ and all $k,\, l \in \{1, 2\}$ we define $\overline{M}^j_{kl}(x_1) := \int_0^{x_1} M^j_{kl}(s)\, ds$,

$$w^j(x_1) := \int_0^{x_1} (x_1 - s) M^j_{11}(s)\, ds - \frac{1}{\ell} \int_I \Big(\int_0^t (t-s) M^j_{11}(s)\, ds \Big)\, dt - x_1 \langle \overline{M}^j_{11} \rangle,$$

and

$$\vartheta^j(x_1) := \overline{M}^j_{12}(x_1) - \langle \overline{M}^j_{12} \rangle.$$

It is clear that $w^j \rightharpoonup w$ weakly in $W^{2,2}(I)$ and $\vartheta^j \rightharpoonup \vartheta$ weakly in $W^{1,2}(I)$, as $j \to \infty$. Moreover, $w^j \in W^{2,2}_{\langle 0 \rangle}(I)$ and $\vartheta^j \in W^{1,2}_{\langle 0 \rangle}(I)$.

After extending β^j smoothly to all of $\mathbb{R}$, still satisfying $|\beta^j| < \pi/2$, we define $\alpha^j := \frac{\pi}{2} + \beta^j$,

$$\widetilde{b}^j(\xi_1) := \cos \alpha^j(\xi_1) e_1 + \sin \alpha^j(\xi_1) e_2 \quad \text{and} \quad \Phi^j(\xi_1, \xi_2) := \xi_1 e_1 + \xi_2 \widetilde{b}^j(\xi_1).$$

Observe that, by the definition of $\widetilde{b}^j$ and since $(w^j)'' = M^j_{11}$ and $(\vartheta^j)' = M^j_{12}$,

$$\begin{pmatrix} (w^j)'' \\ (\vartheta^j)' \end{pmatrix} \cdot \widetilde{b}^j = 0. \tag{3.5}$$

Arguing as in [10, Lemma 12], we see that for every $\varepsilon \le \varepsilon^j$ the matrix $\nabla \Phi^j(\xi_1, \xi_2)$ is invertible for $|\xi_2| \le \varepsilon$, and the map $(\Phi^j)^{-1} : S_\varepsilon \to \mathbb{R}^2$ is well defined. For such ε define $z^j : S_\varepsilon \to \mathbb{R}$ by setting

$$z^j\left(\Phi^j(\xi_1, \xi_2)\right) = w^j(\xi_1) + \xi_2 \widetilde{b}^j(\xi_1) \cdot \begin{pmatrix} (w^j)'(\xi_1) \\ \vartheta^j(\xi_1) \end{pmatrix}. \tag{3.6}$$

We clearly have

$$z^j(\cdot, 0) = w^j. \tag{3.7}$$

Moreover, taking derivatives in (3.6) and using (3.5), we obtain

$$\begin{aligned} \nabla z^j (\Phi^j)^T \nabla \Phi^j &= \left((w^j)'(\xi_1) + \xi_2 (\widetilde{b}^j)'(\xi_1) \cdot \begin{pmatrix} (w^j)'(\xi_1) \\ \vartheta^j(\xi_1) \end{pmatrix},\ \widetilde{b}^j(\xi_1) \cdot \begin{pmatrix} (w^j)'(\xi_1) \\ \vartheta^j(\xi_1) \end{pmatrix} \right) \\ &= \begin{pmatrix} (w^j)'(\xi_1) \\ \vartheta^j(\xi_1) \end{pmatrix}^T \nabla \Phi^j. \end{aligned}$$

Since $\nabla \Phi^j$ is invertible for small $|\xi_2|$, we conclude that for small $|\xi_2|$ and all ξ_1

$$\nabla z^j\left(\Phi^j(\xi_1, \xi_2)\right) = \begin{pmatrix} (w^j)'(\xi_1) \\ \vartheta^j(\xi_1) \end{pmatrix}. \tag{3.8}$$

Taking the derivative with respect to ξ_2, we conclude that

$$\nabla^2 z^j(\Phi^j(\xi_1, \xi_2))\, \widetilde{b}^j(\xi_1) = 0.$$

In particular, the kernel is nontrivial, so $\det \nabla^2 z^j = 0$ on S_ε. Since $M^j \widetilde{b}^j = 0$, in particular we have that $(\nabla^2 z^j(\cdot, 0) - M^j)\, \widetilde{b}^j = 0$. But from (3.7) we see that

$$e_1 \cdot \left(\nabla^2 z^j(\cdot, 0) - M^j \right) e_1 = \partial^2_{11} z^j(\cdot, 0) - (w^j)'' = 0.$$

Since $e_2 \cdot \widetilde{b}^j \neq 0$ and since $(\nabla^2 z^j(\cdot, 0) - M^j)$ is symmetric, we conclude that

$$\nabla^2 z^j(\cdot, 0) = M^j. \tag{3.9}$$

Finally, for ε small enough we define $\widetilde{w}^j_\varepsilon : S \to \mathbb{R}$ by $\widetilde{w}^j_\varepsilon(x_1, x_2) = z^j(x_1, \varepsilon x_2)$. From (3.7) it follows immediately that $\widetilde{w}^j_\varepsilon \to w^j$ strongly in $L^2(S)$, as $\varepsilon \to 0$. Moreover, since

$$\nabla_\varepsilon \widetilde{w}^j_\varepsilon(x) = \nabla z^j(x_1, \varepsilon x_2),$$

equation (3.8) implies that $\nabla_\varepsilon \widetilde{w}^j_\varepsilon \to ((w^j)', \vartheta^j)$ strongly in $W^{1,2}(S; \mathbb{R}^2)$, as $\varepsilon \to 0$. In particular, denoting by $F_\varepsilon \in \mathbb{R}^2$ the average of $\nabla_\varepsilon \widetilde{w}^j_\varepsilon$ over S, we have

$$\lim_{\varepsilon \to 0} F_\varepsilon = \frac{1}{\ell} \int_I \begin{pmatrix} (w^j)' \\ \vartheta^j \end{pmatrix} (x_1)\, dx_1 = 0,$$

by definition of w^j and ϑ^j. Similarly, denoting by c_ε the average of $\widetilde{w}_\varepsilon^j$ over S, we have $c_\varepsilon \to 0$. Hence the functions $w_\varepsilon^j : S \to \mathbb{R}$ defined by

$$w_\varepsilon^j(x) := \widetilde{w}_\varepsilon^j(x) - F_\varepsilon \cdot \begin{pmatrix} x_1 \\ \varepsilon x_2 \end{pmatrix} - c_\varepsilon,$$

still satisfy $\widetilde{w}_\varepsilon^j \to w^j$ strongly in $L^2(S)$ and $\nabla_\varepsilon \widetilde{w}_\varepsilon^j \to ((w^j)', \vartheta^j)$ strongly in $W^{1,2}(S;\mathbb{R}^2)$. Moreover, $w_\varepsilon^j \in W^{2,2}_{\langle 0\rangle}(S)$ by definition of F_ε and c_ε.

Finally, since

$$\nabla_\varepsilon^2 w_\varepsilon^j(x) = \nabla_\varepsilon^2 \widetilde{w}_\varepsilon^j(x) = \nabla^2 z^j(x_1, \varepsilon x_2),$$

we have that $w_\varepsilon^j \in W^{2,2}_{\det,\varepsilon}(S)$. By (3.9) we deduce that $\nabla_\varepsilon^2 w_\varepsilon^j \to M^j$ strongly in $L^2(S;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, as $\varepsilon \to 0$. Hence,

$$\lim_{\varepsilon\to 0} J_\varepsilon^{CvK}(w_\varepsilon^j) = \lim_{\varepsilon\to 0} \frac{1}{24}\int_S Q_2(\nabla_\varepsilon^2 w_\varepsilon^j(x))\,dx = \frac{1}{24}\int_S Q_2(M^j(x))\,dx = \frac{1}{24}\mathcal{F}(M^j).$$

Therefore, by taking diagonal sequences we obtain the desired maps. □

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