

Raychaudhuri equation in space-times with torsion

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Given a space-time with non-vanishing torsion, we discuss the equation for the evolution of the separation vector between infinitesimally close curves in a congruence. We show that the presence of a torsion field leads in general to tangent and orthogonal effects to the congruence; in particular, the presence of a completely generic torsion field contributes to a relative acceleration between test particles. We derive, for the first time in the literature, the Raychaudhuri equation for a congruence of time-like and null curves in a space-time with the most generic torsion field.

I. INTRODUCTION

The appearance of singularities in a physical theory ineluctably marks the pillars of Hercules of that model. General Relativity, from this point of view, is not an exception: it did not take a long time for the community of relativists to realize that even the two simplest space-time solutions of Einstein field equations, the Schwarzschild metric and the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, harbour a gravitational singularity, that is, a breakdown of the space-time structure itself! While in the early 50’s Amal Kumar Raychaudhuri was forcedly working on the properties of electronic energy bands in metals, he got interested in the debate around the nature of gravitational singularities and the generic features of Einstein’s theory of General Relativity (GR). Motivated by cosmology, Raychaudhuri started from the idea that a singularity is nothing more than an artifact of the symmetries of the matter distribution; in his seminal paper [1] he then proposed a model of time-dependent universe without assuming the cosmological principle and its implications on homogeneity and isotropy. The analysis of the flows kinematic that he carried out resulted in the renown Raychaudhuri equation¹ for the evolution of the cosmological expansion in a given background. It was Raychaudhuri himself that pointed out the relation of his work with the existence of singularities, and it should not come as a surprise that Penrose and Hawking were late inspired by Raychaudhuri’s paper to define the conditions under which their singularity theorem holds [2, 3].

Since its inception, the Raychaudhuri equation and its subsequent generalizations (see e.g. the nice review [4]) have found application not only, as already mentioned, in terms of singularity theorems but also in a vast range of different physical contexts, from the study of gravitational lensing [5], to crack formation in spherical astrophysical objects [6], and finally to some more fundamen-

tal issues as in the case of the derivation of the (modified) Einstein equations as equations of state for a (non) equilibrium spacetime thermodynamics [7–9].

In this paper we generalize, for the first time in the literature, the Raychaudhuri construction to a congruence of curves embedded in a space-time with a non-trivial, completely generic torsion tensor field. The idea of generalizing Einstein General Relativity to non-Riemannian geometries has been now around for a while. One of the prototypical example of this kind of proposals is the Einstein–Cartan–Sciama–Kibble (ECSK) theory. The ECSK theory is characterized by assuming an independent connection (using the so-called Palatini approach to find two sets of independent field equations) and further requiring the anti-symmetric part of the connection to be in general non-vanishing, defining a tensor field dubbed torsion tensor field; note, however, that the compatibility of the connection with the metric is still imposed, that is, the covariant derivative (defined with the independent connection) of the metric tensor field along any space-time curve is null.

Theories of gravity with non-vanishing torsion have been extensively considered in the literature; the attention toward these models has been initially catalyzed by the relatively simple scheme they provide to account for nontrivial quantum effects in a gravitational environment, achieved through a direct coupling between the intrinsic spin of matter and the torsion field [10–18]; however this interest has been extended to the fact that torsion appears naturally and inevitably in the low energy limit of super string theories, any theory of gravity that considers twistors and in higher dimensional Kaluza-Klein theories [19]. On top of that, gravitational theories with torsion exhibit a unique nature in comparison with other modified theories of gravity: in general, unless some particular further assumptions are imposed, the effects of a non-null torsion field cannot be recast in an effective energy-momentum tensor, in other words, the torsion field cannot be seen, in general, as an extra matter field in a Lorentzian torsion free manifold. ECSK theory is a particular exception to this property, where torsion can in fact be seen as the effect of a matter field in a Lorentzian manifold, usually called spin-torsion. Yet

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¹ Sometimes referred to as the Landau–Raychaudhuri equation.

ECSK theory by no means represents the most generic case of a theory equipped with a torsion field (see for example [19, 20]) and, although this circumstance is widely recognized, it is intriguing to note that most of the work done on gravitational theories with non-vanishing torsion only considers the ECSK setup: for instance, the very important question of whether the presence of torsion can avoid the formation of singularities resulting from gravitational collapse has only been answered for the case of a spin-torsion field [11–13]. In this paper we then start filling such a crucial gap in the literature: we will focus on the study of the effects of the most generic torsion field on the kinematics of test particles and derive the Raychaudhuri equation for a congruence of null and time-like curves in the spacetime.

The paper is organized as follows: in Sec. II we introduce the basic definitions and set the conventions that will be used throughout the article; in Sec. III we derive the evolution equation of the separation vector between test particles in space-times with torsion, define the kinematical quantities of a congruence of curves and derive the Raychaudhuri equation, both for a time-like and null congruence of curves; finally, we summarize the main results and sort out our conclusions in Sec. IV.

II. CONVENTIONS AND NOTATIONS

Due to the wide variety of different conventions used in literature, let us start by gently introducing the basic definitions and setting the conventions that will be used throughout the article. Introduce a covariant derivative,

$$\nabla_\alpha U^\beta = \partial_\alpha U^\beta + C_{\alpha\sigma}^\beta U^\sigma, \quad (1)$$

constrained to be metric compatible, $\nabla_\alpha g_{\beta\gamma} = 0$, but otherwise with a completely generic connection $C_{\alpha\beta}^\gamma$. The anti-symmetric part of the connection define a tensor which is called the torsion tensor

$$S_{\alpha\beta}{}^\gamma \equiv C_{[\alpha\beta]}^\gamma = \frac{1}{2} (C_{\alpha\beta}^\gamma - C_{\beta\alpha}^\gamma). \quad (2)$$

Using such definition, it is possible to split the connection into an appropriate combination of the torsion tensor plus the usual metric Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$,

$$C_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma + S_{\alpha\beta}{}^\gamma + S^\gamma{}_{\alpha\beta} - S_\beta{}^\gamma{}_\alpha. \quad (3)$$

The sum of the three torsion pieces on the right hand side of last equation is frequently dubbed in literature as the contorsion tensor, $K_{\alpha\beta}{}^\gamma \equiv S_{\alpha\beta}{}^\gamma + S^\gamma{}_{\alpha\beta} - S_\beta{}^\gamma{}_\alpha$; using the anti-symmetry of the torsion tensor, it is an easy task to verify straightforwardly the two symmetries of contorsion,

$$K_{\alpha\beta\gamma} = -K_{\alpha\gamma\beta}, \quad (4)$$

$$K_{[\alpha\beta]}{}^\gamma = S_{\alpha\beta}{}^\gamma. \quad (5)$$

Having in mind the general affine connection, the commutator between two vectors can be expressed in terms of the torsion-full covariant derivative as

$$[u, v]^\gamma = u^\alpha \nabla_\alpha v^\gamma - v^\alpha \nabla_\alpha u^\gamma - 2S_{\alpha\beta}{}^\gamma u^\alpha v^\beta. \quad (6)$$

This last equation and the definition of the Riemann tensor associated with $C_{\alpha\beta}^\gamma$,

$$R_{\alpha\beta\gamma}{}^\rho = \partial_\beta C_{\alpha\gamma}^\rho - \partial_\alpha C_{\beta\gamma}^\rho + C_{\beta\sigma}^\rho C_{\alpha\gamma}^\sigma - C_{\alpha\sigma}^\rho C_{\beta\gamma}^\sigma. \quad (7)$$

lead to a modified version of the relation between the curvature and the commutator of two covariant derivative in the case of non vanishing torsion,

$$R_{\alpha\beta\gamma}{}^\rho w_\rho = [\nabla_\alpha, \nabla_\beta] w_\gamma + 2S_{\alpha\beta}{}^\rho \nabla_\rho w_\gamma. \quad (8)$$

The Riemann tensor for a non-symmetric connection does not have all the usual symmetries. However, from (7) we see that

$$R_{\alpha\beta\gamma}{}^\rho = -R_{\beta\alpha\gamma}{}^\rho, \quad (9)$$

and, using the symmetries of the contorsion tensor, (4) and (5),

$$R_{\alpha\beta\gamma\rho} = -R_{\alpha\beta\rho\gamma}. \quad (10)$$

So, the skew symmetry of the Riemann tensor is still verified. To define the Ricci tensor in terms of the Riemann tensor we will adopt the convention [21, 22]

$$R_{\alpha\beta} = R_{\alpha\gamma\beta}{}^\gamma, \quad (11)$$

which, using (7), can be expressed in terms of the connection coefficients as

$$R_{\alpha\beta} = \partial_\gamma C_{\alpha\beta}^\gamma - \partial_\alpha C_{\gamma\beta}^\gamma + C_{\alpha\beta}^\rho C_{\gamma\rho}^\gamma - C_{\gamma\beta}^\rho C_{\alpha\rho}^\gamma. \quad (12)$$

The Ricci scalar is defined as

$$R = g^{\alpha\beta} R_{\alpha\beta}. \quad (13)$$

Note that in the case of a general affine metric-compatible connection it is still possible to take an independent contraction of the Riemann tensor defining a 2-rank tensor, $\tilde{R}_{\alpha\beta} = g^{\gamma\delta} g_{\epsilon\beta} R_{\alpha\gamma\delta}{}^\epsilon$; however a further contraction with the metric results in $\tilde{R} = -R$, that is the Ricci scalar is unequivocally defined.

A further important remark is about the notation we will be using to describe the matter sector. For a general Riemann–Cartan theory of gravity, the matter Lagrangian can couple (non-)minimally to the torsion tensor which introduce new degrees of freedom in the problem. While the stress-energy tensor is still defined as usual as the variation of the matter action with respect to the metric, we need to introduce a new object, the intrinsic hypermomentum, defined as the variation of the action with respect to the independent connection,

$$\Delta_{\mu\nu}{}^\rho \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{Matter}}}{\delta \Gamma_{\mu\nu}{}^\rho}. \quad (14)$$

Such quantity encapsulates all the information of the microscopic structure of the particle, i.e. intrinsic spin, dilation charge and intrinsic shear.

III. RAYCHAUDHURI EQUATION

A. The separation vector and its evolution

Introduced the basic definitions and identities we are now in the position to generalize the Raychaudhuri equation for the case of an N -dimensional space-time with non-null torsion.

The notion of separation (sometimes deviation) vector between two infinitesimally close curves is quite intuitive: define a congruence of curves, not necessarily geodesics, such that each curve of the congruence is parameterized by an affine parameter λ . Consider a second congruence, this time of geodesics, parametrised by an affine parameter t , such that each geodesic intersects a curve of the first congruence at one and only one point of the space-time. Given two curves in the first congruence, c_1 and c_2 , and a geodesic of the second congruence, γ , let the two points p and q be the intersection points of γ with, respectively, c_1 and c_2 , with $c_1(\lambda_0) = \gamma(t_0) = p$. Let us now assume that the point q is in a small enough neighbourhood of the point p such that $q = \gamma(t_0 + \delta t) \approx p + \left. \frac{\partial \gamma}{\partial t} \right|_{t_0} \delta t$. If n is the tangent vector to the geodesic γ in p , then

$$n \equiv \delta t \left. \frac{\partial \gamma}{\partial t} \right|_{t_0} = q - p \quad (15)$$

gives also a meaningful notion of the separation between the curves c_1 and c_2 .

Let us consider a coordinate neighbourhood that contains the points p and q , such that $p = \{x^\alpha\}$, $q = \{x'^\alpha\} = \{x^\alpha + n^\alpha\}$, with

$$n^\alpha = \delta t \frac{\partial x^\alpha}{\partial t}, \quad (16)$$

and let U^α be the tangent vector to the curve c_1 (from here on we will drop the index 1) at p ,

$$U^\alpha = \frac{\partial x^\alpha}{\partial \lambda}. \quad (17)$$

In order to find the general expression for the evolution of the separation vector, n^α , we will start by computing the Lie derivative of n over the tangent vector U and vice versa. From the definition of the Lie derivative and using (16) and (17) we find that

$$\mathcal{L}_n U = \mathcal{L}_U n = 0. \quad (18)$$

Using (6) and (18) it is possible to derive an equation for the change of the separation vector along the fiducial curve c ,

$$U^\beta \nabla_\beta n^\alpha = B_\beta{}^\alpha n^\beta, \quad (19)$$

where

$$B_\beta{}^\alpha = \nabla_\beta U^\alpha + 2S_{\gamma\beta}{}^\alpha U^\gamma. \quad (20)$$

For infinitesimally close curves, the evolution of the separation vector along the fiducial curve is entirely described by the tensor field $B_{\alpha\beta}$. Let us emphasize that in the derivation of (19) and (20) we have not specified the type of the tangent vector to the fiducial curve, U^α , hence, this equations are equally valid for the case of U^α being time-like, space-like or light-like, with the fiducial curve being either a geodesic or not.

Let us now verify some physical implications of (19) and (20) in the case of the presence a non-vanishing torsion tensor. The derivative along c of the quantity $n_\alpha U^\alpha$ reads

$$\frac{D(n_\alpha U^\alpha)}{d\lambda} = n_\beta a^\beta + 2S_{\sigma\gamma\alpha} U^\sigma U^\alpha n^\gamma, \quad (21)$$

where we have defined the acceleration vector appearing for non geodesics fiducial curves as

$$a^\alpha \equiv U^\gamma \nabla_\gamma U^\alpha. \quad (22)$$

The expression (21) represents the failure of the separation vector n^α and the tangent vector U^α to stay orthogonal to each other, that is, if at a given point n^α and U^α are orthogonal to each other, a general non-null torsion, $S_{\alpha\beta\gamma}$, or a non-null acceleration, a^α , will spoil the preservation of such orthogonality along the curve. Note that a torsion field, with no further imposed symmetry, will lead to effects parallel to the direction of U^α (second term on the right hand side of (21)), contributing to a relative acceleration between two initially infinitesimally close particles.

The analysis of (21) leads to the conclusion that the tensor $B_{\alpha\beta}$ describing the behaviour of the separation vector will have, for the case of a generic torsion tensor, a non-null component tangential to the fiducial curve c . Without loss of generality, it is then possible to write $B_{\alpha\beta}$ in terms of two components, one orthogonal and the other parallel to c : given a projector $h_{\alpha\beta}$ onto the hypersurface orthogonal to the curve c at a given point, we can write

$$B_{\alpha\beta} = B_{\perp\alpha\beta} + B_{\parallel\alpha\beta}, \quad (23)$$

with

$$B_{\perp\alpha\beta} \equiv h_\alpha^\gamma h_\beta^\sigma B_{\gamma\sigma}, \quad (24)$$

$$B_{\parallel\alpha\beta} \equiv B_{\alpha\beta} - B_{\perp\alpha\beta}. \quad (25)$$

In analogy to the General Relativity case, we want to define the kinematical quantities identifying expansion, shear, and vorticity - θ , $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$, respectively - of neighbouring curves of the congruence. These quantities will only depend on the orthogonal part of the tensor $B_{\alpha\beta}$, $B_{\perp\alpha\beta}$, so that, defining the expansion, shear and vorticity as

$$\theta = B_{\perp\gamma}{}^\gamma, \quad (26)$$

$$\sigma_{\alpha\beta} = B_{\perp(\alpha\beta)} - \frac{h_{\alpha\beta}}{h_\gamma{}^\gamma} \theta, \quad (27)$$

$$\omega_{\alpha\beta} = B_{\perp[\alpha\beta]}, \quad (28)$$

$B_{\perp\alpha\beta}$ can be decomposed into

$$B_{\perp\alpha\beta} = \frac{h_{\alpha\beta}}{h_{\gamma}^{\gamma}}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}. \quad (29)$$

Before we continue and derive the Raychaudhuri equation let us emphasize that the definitions of the kinematical quantities given by (26)-(28) are always valid whenever the tensor $B_{\alpha\beta}$ is related with the variation of the separation vector between curves of a congruence by an equation such as (19). A formal proof for this assertion can be seen in [23].

B. Raychaudhuri equation for a congruence of time-like curves

The results in the previous section are quite general and valid for curves of any kind; however, the procedure that defines the projector $h_{\alpha\beta}$ strictly depends on the specific family of curves considered. Once the projector is assigned, (29) will give an actual expression in terms of the tangent vector and the torsion tensor.

Let us then start by considering the generalized Raychaudhuri equation for a congruence of time-like curves. In this case we will impose that the fiducial curve is parametrised by the proper time, τ , and, in order to avoid confusion with the general case, we will label its tangent vector as v^{α} , with $v_{\alpha}v^{\alpha} = -1$. The operator projecting onto the hypersurface orthogonal to v^{α} is given by

$$h_{\alpha\beta} \equiv g_{\alpha\beta} + v_{\alpha}v_{\beta}, \quad (30)$$

and fulfills the following conditions

$$\begin{aligned} h_{\alpha\beta}v^{\alpha} &= 0, \\ h_{\alpha}^{\gamma}h_{\gamma\beta} &= h_{\alpha\beta}, \\ h_{\sigma}^{\sigma} &= N - 1, \end{aligned} \quad (31)$$

where N is, again, the dimension of the space-time.

In order to calculate the two components, orthogonal and parallel, of $B_{\alpha\beta}$ another ingredient is necessary: we must find an expression for the trajectories along which free particles move. In a general non-Riemannian manifold, it is not an easy task to determine such physical curves. The statement usually claimed in the literature that particles follow geodesics, either of the Levi-Civita generic connection or of the metric connection, turns out to be rather unsatisfactory and *naïve*. In a manifold equipped with a completely generic affine connection, the particle trajectories can be correctly determined starting from the equations of motion, with the latter obtained themselves from the conservation laws for canonical energy-momentum and hypermomentum. A comprehensive treatment for the calculation of the propagation equations of single-pole and pole-dipole particles in metric-affine theories of gravity has been developed in

[24]² (see also [25, 26] for early single-pole approximation descriptions). A quite interesting result is that single-pole particles without intrinsic hypermomentum follow, as in general relativity, geodesics of the metric connection, $v^{\beta}\nabla_{\beta}^{(g)}v^{\alpha} = 0$, no matter what is the underlying theory of gravity.

Things become much more complicated for particles with non-vanishing intrinsic hypermomentum. We will focus on the case of Riemann-Cartan space-time, since in our setup nonmetricity is trivially zero. We will also consider particles endowed with only mass and intrinsic spin but we neglect dilaton charge and intrinsic shear for simplifying reason. This means that the hypermomentum tensor reduces (for a single-pole particle [24]) to $\Delta_{[\mu\nu]}^{\rho} = \tau_{\mu\nu}v^{\rho}$, where $\tau_{\mu\nu}$ is the anti-symmetric spin density tensor. In this case the equations of motion for a particle read

$$\begin{aligned} v^{\kappa}\nabla_{\kappa}v_{\alpha} &= -v_{\alpha}v^{\kappa}\nabla_{\kappa}\ln m - \frac{2}{m}\left[v^{\kappa}\nabla_{\kappa}(v^{\beta}v^{\sigma}\nabla_{\sigma}\tau_{\beta\alpha}) + \right. \\ &+ S_{\alpha\beta}{}^{\mu}v^{\beta}(mv_{\mu} + 2v^{\nu}v^{\sigma}\nabla_{\sigma}\tau_{\nu\mu}) + \\ &\left. + \frac{1}{2}R_{\alpha\beta\mu}{}^{\nu}v^{\beta}\tau^{\mu}{}_{\nu}\right] \equiv a_{\alpha}, \end{aligned} \quad (32)$$

$$v^{\alpha}\nabla_{\alpha}\tau_{\mu\nu} - 2v_{[\mu}v^{\sigma}v^{\kappa}\nabla_{\kappa}\tau_{\sigma|\nu]} = 0, \quad (33)$$

Notably, this equation is independent by the specific choice of the gravitational part in the action. It is important to stress that for a non-spinning particle, $\tau_{\mu\nu}$ vanishes, the second of equations (33) is trivially satisfied while the first one recovers the equation of the geodesics of the metric, $v^{\beta}\nabla_{\beta}^{(g)}v^{\alpha} = 0$. Note that the rest mass, that is the projection of the particle 4-momentum on the rest frame, is guaranteed to be constant along the congruence only in the case of non-spinning particle.

Using equations (30)-(33) in equations (24) and (25) we find

$$B_{\perp\alpha\beta} = \nabla_{\alpha}v_{\beta} + 2S_{\rho\alpha\beta}v^{\rho} + 2S_{\rho\alpha\sigma}v^{\rho}v^{\sigma}v_{\beta} + v_{\alpha}a_{\beta} \quad (34)$$

$$B_{\parallel\alpha\beta} = -2S_{\rho\alpha\sigma}v^{\rho}v^{\sigma}v_{\beta} - v_{\alpha}a_{\beta}. \quad (35)$$

For the case of a congruence of time-like curves in a N -dimensional space-time (29) can be written as

$$B_{\perp\alpha\beta} = \frac{1}{N-1}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \quad (36)$$

where (31) was used.

Taking the variation of the tensor $B_{\perp\alpha\beta}$ along the fiducial curve, and remembering the expression (8) for the

² We warn about a slight different notation in [24]: for example the definition of the torsion tensor is there twice our definition.

Riemann tensor, we find

$$\begin{aligned} \frac{D B_{\perp\alpha\beta}}{d\tau} &= v^\gamma \nabla_\gamma B_{\perp\alpha\beta} = R_{\gamma\alpha\beta\rho} v^\rho v^\gamma + \nabla_\alpha a_\beta - \\ &\quad - (\nabla_\alpha v^\gamma) (\nabla_\gamma v_\beta) - 2v^\gamma S_{\gamma\alpha}{}^\rho \nabla_\rho v_\beta + \\ &\quad + 2v^\gamma \nabla_\gamma (S_{\rho\alpha\beta} v^\rho) + v^\gamma \nabla_\gamma (v_\alpha a_\beta) + \\ &\quad + 2v^\gamma \nabla_\gamma (S_{\rho\alpha\sigma} v^\rho v^\sigma v_\beta), \end{aligned} \quad (37)$$

Shear and vorticity are, respectively, traceless and anti-symmetric, so that, contracting α and β on both sides of previous equations, we obtain

$$\begin{aligned} \frac{D\theta}{d\tau} &= -R_{\gamma\rho} v^\rho v^\gamma + \nabla_\alpha a^\alpha - (\nabla_\alpha v^\gamma) (\nabla_\gamma v^\alpha) \\ &\quad - 2v^\gamma S_{\gamma\alpha}{}^\rho \nabla_\rho v^\alpha + 2v^\gamma \nabla_\gamma (S_\rho v^\rho), \end{aligned} \quad (38)$$

where $S_\rho \equiv S_{\rho\alpha}{}^\alpha$. Taking into account that

$$\begin{aligned} B_{\perp\alpha\beta} B_{\perp}^{\beta\alpha} &= (\nabla_\alpha v^\beta) (\nabla_\beta v^\alpha) + 2v^\rho S_{\rho\beta}{}^\alpha \nabla_\alpha v^\beta + \\ &\quad + 2S_{\rho\alpha}{}^\beta v^\rho \nabla_\beta v^\alpha + 4S_{\rho\alpha}{}^\beta S_{\gamma\beta}{}^\alpha v^\rho v^\gamma + \\ &\quad + 4S_{\alpha\beta\gamma} v^\alpha v^\gamma a^\beta, \end{aligned} \quad (39)$$

and using (36) we can write (38) as

$$\begin{aligned} \frac{D\theta}{d\tau} &= -R_{\gamma\rho} v^\rho v^\gamma - \left[\frac{1}{N-1} \theta^2 + \sigma_{\alpha\beta} \sigma^{\alpha\beta} + \omega_{\alpha\beta} \omega^{\beta\alpha} \right] + \\ &\quad + 2S_{\rho\alpha}{}^\beta v^\rho \left[\frac{1}{N-1} h_\beta{}^\alpha \theta + \sigma_\beta{}^\alpha + \omega_\beta{}^\alpha \right] \\ &\quad + \nabla_\alpha a^\alpha + 2v^\gamma \nabla_\gamma (S_\rho v^\rho) + 2S_{\rho\alpha}{}^\beta v^\rho v_\beta a^\alpha. \end{aligned} \quad (40)$$

This equation represents the generalization of the Raychaudhuri equation for a time-like congruence of curves in the presence a generic torsion field. Here, we would like to make a couple of comments: first of all, this equation has been obtained using only geometrical arguments, plus the canonical energy-momentum conservation equation to define the equations of motion of the particles; this means that the result is independent by the specific geometrical theory that we are choosing: once that the theory has been assigned, then the Ricci tensor can be related with the energy-momentum tensor accordingly with the (modified) Einstein field equations. Secondly, it is interesting to stress that the extra-force responsible of the acceleration term reported in the first line of (40) is of purely geometric origin, related to the extra coupling of the intrinsic spin (*viz.* intrinsic hypermomentum in the most general setup) with post-Riemannian structures. Note also that our result differs from previous versions of the torsion-full Raychaudhuri equation available in literature. More concretely, refs. [4, 19, 27] did not take properly into account the relation between the tensor $B_{\alpha\beta}$ and the evolution of the separation vector between infinitesimally close curves of the congruence, assuming that the expression of the tensor $B_{\alpha\beta}$ is *a priori* the same as in the case of null torsion. Few quite specific models for torsion accidentally result anyway in the correct expression: retracing the properties of the intrinsic

spin, Refs. [14, 28] consider the simplifying assumption on the torsion tensor $S_{\alpha\beta}{}^\gamma = S_{\alpha\beta} v^\gamma$, with $S_{(\alpha\beta)} = 0$ and $S_{\alpha\beta} v^\alpha = 0$; in Ref. [12] instead torsion is constrained to be $S_{\alpha\beta}{}^\gamma = \eta_{\alpha\beta\sigma\epsilon} v^\gamma v^\sigma S^\epsilon$, where $\eta_{\alpha\beta\sigma\epsilon}$ is the completely anti-symmetric Levi-Civita tensor. However, in these special cases the extra symmetries imposed on the torsion tensor imply that the second term on the right hand side of (20) is null, reducing the problem to the torsion free case.

C. Raychaudhuri equation for a congruence of null curves

Let us now derive the Raychaudhuri equation for a congruence of null curves. As in the previous subsection, to avoid any confusion, we will re-label the tangent vector to the fiducial curve and call it k^α , so that $k^\alpha k_\alpha = 0$.

In this case, unfortunately, if $\tilde{h}_{\alpha\beta}$ is the projector onto the hypersurface that is orthogonal to the fiducial null curve, it cannot be naively defined by (30), since it would not be orthogonal to k^α (it would be $\tilde{h}_{\alpha\beta} k^\alpha = k_\beta \neq 0$). The way out of this problem is through the introduction of an auxiliary null vector field, ξ^α , such that [29]

$$k^\alpha \xi_\alpha = -1, \quad (41)$$

$$\xi_\alpha \xi^\alpha = 0. \quad (42)$$

Using (41) and (42) we can now properly introduce a projector onto the hypersurface orthogonal to both k^α and ξ^α as

$$\tilde{h}_{\alpha\beta} = g_{\alpha\beta} + k_\alpha \xi_\beta + \xi_\alpha k_\beta, \quad (43)$$

satisfying the following properties

$$\begin{aligned} \tilde{h}_{\alpha\beta} k^\alpha &= \tilde{h}_{\alpha\beta} \xi^\alpha = 0, \\ \tilde{h}_\alpha{}^\sigma \tilde{h}_{\sigma\beta} &= \tilde{h}_{\alpha\beta}, \\ \tilde{h}_\sigma{}^\sigma &= N - 2. \end{aligned} \quad (44)$$

As in the time-like case, the extra ingredient to be taken into account is the effective particles trajectories. An important *caveat* is here due: particles that are moving along null curves are massless particle; the most natural candidates in the Standard Model are then photons. However, the minimal coupling procedure to generalize the electromagnetic field to non-Riemannian environments preserve photons by having a non-vanishing intrinsic hypermomentum; hence, photons follow the geodesics determined by the Christoffel symbols,

$$k^\alpha \nabla_\alpha^{(g)} k^\beta = 0. \quad (45)$$

As a side note, let us stress that some theories, such as the Standard Model Extension [30], allow for a non-minimal coupling of the electromagnetic field with geometry (and eventually other fields), which means a non trivial intrinsic hypermomentum: anyway, in this case the extra operators will also introduce some effective mass for the photon (the simplest case being the explicitly massive Proca field) and *a fortiori* they could not follow null curves.

In the context of the Standard Model, neutrinos deserve a separate mention; the recent confirmation of the phenomenon of neutrino oscillations [31, 32] directly imply that neutrinos might be massive [33], avoiding them to follow null paths³. In spite of that, let us take into account the behaviour of the Standard Model massless neutrino, *viz.* a single-pole, massless Dirac particle. Dirac particles have a completely anti-symmetric hypermomentum, which implies, in the single-pole approximation, a vanishing intrinsic spin density tensor, $\tau_{\mu\nu} = 0$ [34, 35]. Back to (33), this would mean that the particles follow metric geodesics, as in the non-spinning case, and as for photons.

It is finally possible to use (41)-(45) to find the orthogonal and tangential components of the tensor $B_{\alpha\beta}$ that defines the dynamics of the separation vector of the null congruence of geodesics; a straightforward calculation gives

$$\begin{aligned} B_{\perp\alpha\beta} = & \nabla_{\alpha}k_{\beta} - 2S_{\alpha\gamma\beta}k^{\gamma} - 2S_{\alpha\gamma\sigma}k^{\gamma}k^{\sigma}\xi_{\beta} + 2S_{\gamma\sigma\beta}k_{\alpha}k^{\gamma}\xi^{\sigma} \\ & + 2S_{\gamma\rho\sigma}k_{\alpha}k^{\gamma}k^{\rho}\xi_{\beta}\xi^{\sigma} + k_{\alpha}\xi^{\gamma}\nabla_{\gamma}k_{\beta} + k_{\beta}\xi^{\gamma}\nabla_{\gamma}k_{\alpha} \\ & - 2S_{\alpha\gamma\sigma}k_{\beta}k^{\gamma}\xi^{\sigma} + k_{\alpha}k_{\beta}\xi^{\gamma}\xi^{\sigma}\nabla_{\sigma}k_{\gamma} \\ & + 2S_{\beta\gamma\sigma}k^{\gamma}k^{\sigma}\xi_{\alpha} + 2S_{\gamma\sigma\rho}k_{\alpha}k_{\beta}k^{\gamma}\xi^{\rho}\xi^{\sigma} \\ & + 2k_{\beta}\xi_{\alpha}\xi^{\sigma}S_{\sigma\gamma\rho}k^{\gamma}k^{\rho} \end{aligned} \quad (46)$$

$$\begin{aligned} B_{\parallel\alpha\beta} = & -2S_{\rho\sigma\beta}k^{\rho}k_{\alpha}\xi^{\sigma} - k_{\beta}\xi^{\rho}\nabla_{\rho}k_{\alpha} + 2S_{\alpha\rho\sigma}k^{\rho}k_{\beta}\xi^{\sigma} \\ & - 2S_{\rho\sigma\gamma}k^{\rho}k_{\alpha}k_{\beta}\xi^{\sigma}\xi^{\gamma} + 2S_{\alpha\rho\sigma}k^{\rho}k^{\sigma}\xi_{\beta} \\ & - 2S_{\rho\gamma\sigma}k^{\rho}k^{\sigma}k_{\alpha}\xi^{\gamma}\xi_{\beta} - 2\xi^{\sigma}\xi_{\alpha}k_{\beta}S_{\sigma\gamma\rho}k^{\gamma}k^{\rho} \\ & - 2\xi_{\alpha}S_{\beta\gamma\sigma}k^{\gamma}k^{\sigma} - k_{\alpha}\xi^{\rho}\nabla_{\rho}k_{\beta} \\ & - k_{\alpha}k_{\beta}\xi^{\rho}\xi^{\sigma}\nabla_{\sigma}k_{\rho} \end{aligned} \quad (47)$$

Similarly to what was done in (29), we can relate the component $B_{\perp\alpha\beta}$ with the expansion, shear and vorticity of the congruence of the null curves,

$$B_{\perp\alpha\beta} = \frac{\tilde{h}_{\alpha\beta}}{N-2}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \quad (48)$$

that in combination with (46) provides an expression for the expansion of the congruence of null geodesics in terms of k^{α} , ξ^{α} and $S_{\alpha\beta\gamma}$, given by

$$\theta = \nabla_{\gamma}k^{\gamma} + 2S_{\gamma}k^{\gamma} = \nabla_{\gamma}^{(g)}k^{\gamma} \equiv \tilde{\theta}. \quad (49)$$

Note that also in this case, as in general relativity, the scalar expansion θ does not depend on the auxiliary vector ξ^{μ} chosen to define a proper projection operator to treat the null curves congruence case.

Tracing Eq. (46) and computing its covariant derivative along the fiducial null curve we find

$$\begin{aligned} \frac{D\theta}{d\lambda} = & -R_{\alpha\beta}k^{\alpha}k^{\beta} - \left(\frac{1}{N-2}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\beta\alpha} \right) \\ & + 2S_{\rho\alpha}{}^{\beta}k^{\rho} \left(\frac{\tilde{h}_{\beta}{}^{\alpha}}{N-2}\theta + \sigma_{\beta}{}^{\alpha} + \omega_{\beta}{}^{\alpha} \right) + 2k^{\gamma}\nabla_{\gamma}(S_{\rho}k^{\rho}) \\ & + 2\nabla_{\alpha}(S^{\alpha}{}_{\gamma\rho}k^{\gamma}k^{\rho}) + 4(h^{\beta\gamma}S_{\alpha\beta\gamma}k^{\alpha} - S_{\alpha}k^{\alpha})^2 \\ & + 4S_{\mu}{}^{\alpha\beta}k^{\mu} \left[S_{\delta\beta\alpha}k^{\delta} - h_{\beta}^{\rho}S_{\delta\rho\alpha}k^{\delta} \right] \\ & + S^{\alpha}{}_{\mu\nu}k^{\mu} \left(\tilde{h}^{\nu\beta} - g^{\nu\beta} \right) (4B_{\parallel\alpha\beta} + 6B_{\parallel\beta\alpha}) \\ & - 2B_{\parallel\delta}{}^{\alpha}B_{\parallel\beta\alpha} \left(\tilde{h}^{\delta\beta} - g^{\delta\beta} \right), \end{aligned} \quad (50)$$

which represents the Raychaudhuri equation for a congruence of null geodesics in the presence of a torsion tensor field. Here the literature is more sparse: ref. [36] derives the Raychaudhuri equation for a null congruence of curves in the particular case of a completely anti-symmetric torsion, missing in any case the correct definition of the $B_{\alpha\beta}$ tensor (although without affecting the final result).

Looking at Eq. (50) it is not clear that the Raychaudhuri equation for a null congruence is independent of the choice of the auxiliary null vector ξ^{α} that was introduced to define the projector onto the orthogonal hypersurface to the congruence, Eq. (43): we have terms that depend on the projector $\tilde{h}^{\alpha\beta}$ and $B_{\parallel\alpha\beta}$, which themselves depend on ξ^{α} . However, since the expansion θ is a scalar quantity, its rate of variation along the congruence will be related to the the rate of variation of $\tilde{\theta}$ by

$$\frac{D\theta}{d\lambda} = k^{\mu}\nabla_{\mu}\theta = k^{\mu}\partial_{\mu}\theta = k^{\mu}\partial_{\mu}\tilde{\theta} = k^{\mu}\nabla_{\mu}^{(g)}\tilde{\theta} = \frac{\tilde{D}\tilde{\theta}}{d\lambda}. \quad (51)$$

where $\frac{\tilde{D}}{d\lambda} = k^{\mu}\nabla_{\mu}^{(g)}$ represents the covariant derivative along the fiducial curve where only the metric connection is considered. Computing the derivative, we recover the usual general relativity expression for the Raychaudhuri equation governing the evolution of the expansion:

$$\frac{\tilde{D}\tilde{\theta}}{d\lambda} = -\tilde{R}_{\alpha\beta}k^{\alpha}k^{\beta} - \left[\frac{1}{N-2}\tilde{\theta}^2 + \tilde{\sigma}_{\alpha\beta}\tilde{\sigma}^{\alpha\beta} + \tilde{\omega}_{\alpha\beta}\tilde{\omega}^{\beta\alpha} \right], \quad (52)$$

where the tilded quantities are calculated with the Christoffel connection, and

$$\begin{aligned} \tilde{\theta} = & \nabla_{\mu}^{(g)}k^{\mu}, \\ \tilde{\sigma}_{\mu\nu} = & \nabla_{(\mu}^{(g)}k_{\nu)} + \xi^{\gamma}k_{(\mu}\nabla_{\gamma}^{(g)}k_{\nu)} + \xi^{\gamma}k_{(\nu}\nabla_{\mu}^{(g)}k_{\gamma)} \\ & + k_{\mu}k_{\nu}\xi^{\gamma}\xi^{\sigma}\nabla_{\sigma}^{(g)}k_{\gamma} - \nabla_{\mu}^{(g)}k^{\mu}, \\ \tilde{\omega}_{\mu\nu} = & \nabla_{[\mu}^{(g)}k_{\nu]} + k_{[\mu}\xi^{\gamma}\nabla_{\gamma}^{(g)}k_{\nu]} + \xi^{\gamma}k_{[\nu}\nabla_{\mu}^{(g)}k_{\gamma]}. \end{aligned} \quad (53)$$

Note that, in spite of the explicit appearance of the auxiliary vector ξ_{μ} in the two expressions for $\tilde{\sigma}_{\mu\nu}$ and $\tilde{\omega}_{\mu\nu}$,

³ For the sake of completeness, we want to mention that also in this case there is still room for the (unlikely) scenario with one out of three neutrino species exactly massless.

the total expression (52) does not depend on ξ_μ because it is possible to find that

$$\begin{aligned}\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} &= -\frac{1}{N-2}\tilde{\theta}^2 + \frac{2}{N-2}\tilde{\theta} + \frac{1}{2}\nabla_\mu^{(g)}k^\nu \cdot \nabla_\nu^{(g)}k^\mu \\ &\quad + \frac{1}{2}\nabla_\mu^{(g)}k^\nu \cdot \nabla^{(g)\mu}k_\nu, \\ \tilde{\omega}_{\mu\nu}\tilde{\omega}^{\nu\mu} &= \frac{1}{2}\nabla_\mu^{(g)}k^\nu \cdot \nabla_\nu^{(g)}k^\mu - \frac{1}{2}\nabla_\mu^{(g)}k^\nu \cdot \nabla^{(g)\mu}k_\nu,\end{aligned}\tag{54}$$

showing that Eq. (52) is independent of the auxiliary null vector ξ^μ . Since Eq. (52) is equivalent to Eq. (50), we find that the Raychaudhuri equation for a null congruence is independent of the vector field ξ^μ .

As a final comment, let us address the following important remark: one could be tempted to naively think that, since (52) does not depend explicitly of the torsion tensor, then the evolution of the expansion of the curves followed by massless particles does not depend on the presence of a torsion field. This conclusion is not correct since although the torsion tensor does not explicitly appear in (52), it does affect the geometry of the space-time in a way described by the (modified) Einstein field equations; the metric solution of the Einstein equations will be itself affected by the such modification⁴, and so will be the corresponding (metric) Ricci tensor $\tilde{R}_{\mu\nu}$ appearing in (52).

IV. CONCLUSIONS

In this paper we derived the equation for the evolution of the separation vector between infinitesimally close curves of a congruence in space-times with non-null generic torsion field, clarifying some of the ambiguities lingering in the literature about the role of the torsion tensor. We concluded that the presence of a torsion field leads in general to tangent and orthogonal effects to the congruence, in particular, the presence of a generic torsion field contributes to a relative acceleration between test particles. This effects happen either for free-falling or accelerated particles following time-like, null or space-like curves.

The evolution equation of the separation vector can be further separated and be used to study the kinematical quantities that characterize a congruence of curves, namely the expansion, shear and vorticity. We derived, for the first time in the literature, how such kinematical quantities depend on a completely generic torsion field.

Knowing how the kinematical decomposition of the geodesics congruence is influenced by the torsion tensor allows the possibility to test models equipped with a nontrivial Riemannian connection through the study of the motion of test particles. Let us expand a bit on this point: the matter source appearing on the right hand side of the equation of motion for the torsion tensor (what is usually dubbed hypermomentum, obtained from the variation of the classical matter action with respect to the independent connection) depends on the specific coupling between matter and torsion itself: different models will lead to different field equations for the torsion and henceforth to different solutions. This means that the eventual contributions of torsion to the various kinematical quantities and their evolution will be dependent on the specific chosen model; the analysis then of the evolution of a matter fluid in a region of space-time will give a chance to distinguish between different allowed couplings.

One of the most important achievements of this paper is the generalization of Raychaudhuri equation - the equation for the evolution of the expansion of a congruence of curves - for the case of time-like and null curves in an N -dimensional space-time with the most generic torsion field. The study of the evolution of the expansion of time-like and null curves in spacetime is obviously important due to its role in defining the evolution of gravitational collapse and possible formation of singularities. Moreover, as was shown in Ref. [36], the expansion of null light-rays also plays a preponderant role on the definition of the throats of (dynamical) wormholes, namely exotic solutions requiring the violation of energy conditions; a still open question is whether or not the degrees of freedom of a completely generic torsion field can somehow avoid the violation of the *null energy condition* of whatever matter present at the dynamical wormhole's throat. We firmly believe that the study of such possibilities will be of great interest in the near future.

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⁴ Just as a workable example, consider the simple Einstein–Cartan case: one of the equations of motion describes algebraically the torsion tensor as a function of the spin tensor; the other equa-

tion, rewritten in terms of the Christoffel connection eliminating the torsion tensor, relates the (Christoffel) Einstein tensor to a combined version of the stress-energy tensor, that now takes into account also spin density terms [10].

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