Centralizer's applications to the inverse along an element

Huihui Zhu[1,2], Jianlong Chen[1]*, Pedro Patrício[2,3], Xavier Mary[4]

Abstract: In this paper, we first prove that the absorption law for one-sided inverses along an element holds, deriving the absorption law for the inverse along an element. We then apply this result to obtain the absorption law for the inverse along different elements. Also, the reverse order law and the existence criterion for the inverse along an element are given by centralizers in a ring. Finally, we characterize the Moore-Penrose inverse by one-sided invertibilities in a ring with involution.

Keywords: Absorption laws, Reverse order laws, Centralizers , Left (Right) inverses along an element, The inverse along an element

AMS Subject Classifications: 15A09, 16W10

1   Introduction

It is well-known that $a^{-1} + b^{-1} = a^{-1}(a + b)b^{-1}$ for any invertible elements a, b in a ring. The equality above is known as the absorption law. In general, the absorption laws for group inverses, Drazin inverses, Moore-Penrose inverses, {1,3}-inverses and {1,4}-inverses do not hold. So, many papers [1, 3, 5] devoted to the study of these aspects.

Recently, authors [15] introduced a new type of generalized inverse called one-sided inverses along an element, which can been seen as a generalization of group inverses, Drazin inverses, Moore-Penrose inverses and the inverse along an element. It is natural to consider whether the absorption law for such inverses holds.

---

*Corresponding author

1 Department of Mathematics, Southeast University, Nanjing 210096, China.

2 CMAT-Centro de Matemática, Universidade do Minho, Braga 4710-057, Portugal.

3 Departamento de Matemática e Aplicações, Universidade do Minho, Braga 4710-057, Portugal.

4 Université Paris-Ouest Nanterre-La Défense, Laboratoire Modal'X, 200 avenuue de la république, 92000 Nanterre, France.

Email: ahzhh08@sina.com(H. Zhu), jlchen@seu.edu.cn(J. Chen), pedro@math.uminho.pt (P. Patrício), xavier.mary@u-paris10.fr (X. Mary).

In this paper, we first prove that the absorption law for one-sided inverses along an element holds in a ring. As applications, the absorption law for the inverse along the same element, i.e., $a^{\|d} + b^{\|d} = a^{\|d}(a + b)b^{\|d}$, is obtained. We then apply this result to obtain the absorption law for the inverse along different elements. Also, the reverse order law for the inverse along an element is considered. Furthermore, we derive an existence criterion of the inverse along an element by centralizers in a ring. Finally, we characterize the Moore-Penrose inverse in terms of one-sided invertibilities, extending the results in [8, 16].

Let us now recall some notions of generalized inverses. We say that $a \in R$ is (von Neumann) regular if there exists x in R such that $a = axa$. Such x is called an inner inverse or $\{1\}$-inverse of a, and is denoted by a^-.

Following Drazin [2], an element $a \in R$ is said to be Drazin invertible if there exist $b \in R$ and positive integer k such that the following conditions hold:

$$\text{(i) } ab = ba, \text{ (ii) } b^2a = b, \text{ (iii) } a^k = a^{k+1}b.$$

The element b satisfying the above conditions (i)-(iii) is unique if it exists, and is denoted by a^D. The smallest positive integer k in condition (iii) is called the Drazin index of a and is denoted by $\text{ind}(a)$. We call a group invertible if a is Drazin invertible with $\text{ind}(a) = 1$.

Let $*$ be an involution on R, that is the involution $*$ satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. An element $a \in R$ (with involution $*$) is Moore-Penrose invertible [9] if there exists $b \in R$ satisfying the following equations

$$\text{(i) } aba = a, \text{ (ii) } bab = b, \text{ (iii) } (ab)^* = ab, \text{ (iv) } (ba)^* = ba.$$

Any element b satisfying the equations above is called a Moore-Penrose inverse of a. If such b exists, then it is unique and is denoted by $a^\dagger$. If x satisfies the equations (i) and (iii), then x is called a $\{1,3\}$-inverse of a, and is denoted by $a^{(1,3)}$. If x satisfies the equations (i) and (iv), then x is called a $\{1,4\}$-inverse of a, and is denoted by $a^{(1,4)}$.

Throughout this paper, we assume that R is an associative ring with unity 1. Let $a, b, d \in R$. An element b is called a left (resp., right) inverse of a along d [15] if $bad = d$ (resp., $dab = b$) and $b \in Rd$ (resp., $b \in dR$). By $a_l^{\|d}$ and $a_r^{\|d}$ we denote a left and a right inverse of a along d, respectively. Furthermore, an element a is called invertible along d

[6] if there exists b such that $bad = d = dab$ and $b \in dR \cap Rd$. Such b is unique if it exists, and is denoted by $a^{\|d}$. It is known [15] that a is both left and right invertible along d if and only if it is invertible along d. More results on (one-sided) inverses along an element can be referred to [15, 16, 17].

2 Absorption laws for the inverse along an element

The main goal of this section is to illustrate that the absorption laws for (left, right) inverses along an element hold in a ring. We first begin with the following lemma.

Lemma 2.1. *Let* $a, b, d \in R$. *Then*

(i) *If* $a_l^{\|d}$ *and* $b_r^{\|d}$ *exist, then* $a_l^{\|d} a b_r^{\|d} = b_r^{\|d}$ *and* $a_l^{\|d} b b_r^{\|d} = a_l^{\|d}$.

(ii) *If* $a_r^{\|d}$ *and* $b_l^{\|d}$ *exist, then* $b_l^{\|d} b a_r^{\|d} = a_r^{\|d}$ *and* $b_l^{\|d} a a_r^{\|d} = b_l^{\|d}$.

Proof. (i) Note that $a_l^{\|d}$ can be written as the form xd for some $x \in R$. Also, there exists $y \in R$ such that $b_r^{\|d} = dy$. Hence, it follows that $a_l^{\|d} a b_r^{\|d} = a_l^{\|d} a dy = dy = b_r^{\|d}$ and $a_l^{\|d} b b_r^{\|d} = xd b b_r^{\|d} = a_l^{\|d}$.

(ii) By the symmetry of a and b. $\qquad\square$

Applying Lemma 2.1, we get the absorption laws for one-sided inverses along an element.

Theorem 2.2. *Let* $a, b, d \in R$. *Then*

(i) *If* $a_l^{\|d}$ *and* $b_r^{\|d}$ *exist, then* $a_l^{\|d} + b_r^{\|d} = a_l^{\|d}(a + b)b_r^{\|d}$.

(ii) *If* $a_r^{\|d}$ *and* $b_l^{\|d}$ *exist, then* $a_r^{\|d} + b_l^{\|d} = b_l^{\|d}(a + b)a_r^{\|d}$.

Proof. (i) It follows from Lemma 2.1(i) that

$$a_l^{\|d}(a + b)b_r^{\|d} = a_l^{\|d} a b_r^{\|d} + a_l^{\|d} b b_r^{\|d} = a_l^{\|d} + b_r^{\|d}.$$

(ii) It is a direct check by Lemma 2.1(ii). $\qquad\square$

As a corollary of Theorem 2.2, it follows that the absorption law for the inverse along an element.

Corollary 2.3. *Let* $a, b, d \in R$ *and let* $a^{\|d}$ *and* $b^{\|d}$ *exist. Then* $a^{\|d} + b^{\|d} = a^{\|d}(a + b)b^{\|d}$.

From Corollary 2.3, we know that the absorption law for the inverse along the same element holds in a ring. It is natural and interesting to consider whether the absorption law for the inverse along different elements, i.e., $a^{\|d_1} + b^{\|d_2} = a^{\|d_1}(a+b)b^{\|d_2}$ holds? In general, $a^{\|d_1} + b^{\|d_2} = a^{\|d_1}(a+b)b^{\|d_2}$ does not hold as the following example shows.

Example 2.4. Let $R = M_2(\mathbb{C})$ be the ring of all 2×2 complex matrices. Take $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $D_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We get $A^{\|D_1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ and $B^{\|D_2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ by direct calculation. However, $A^{\|D_1} + B^{\|D_2} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A^{\|D_1}(A+B)B^{\|D_2}$.

Next, we consider under what conditions, the absorption law for the inverse along different elements to hold. First, we recall the notion of centralizers.

In 1952, Wendel [13] introduced the notion of left centralizers in group algebras. In 1964, Johnson [4] further gave an introduction to the theory of left and right centralizers in semigroups. Later, Vukman [11, 12] and Zalar [14] studied centralizers in operator algebras and semiprime rings, respectively. More recently, see [18], a map σ in a semigroup S is called a left (resp., right) centralizer if $\sigma(ab) = \sigma(a)b$ (resp., $\sigma(ab) = a\sigma(b)$) for all $a, b \in S$. We call σ a centralizer if it is both a left and a right centralizer, i.e., $a\sigma(b) = \sigma(ab) = \sigma(a)b$ for all $a, b \in S$.

Herein, we remind the reader some examples of left centralizers, centralizers and bijective centralizers.

Example 2.5. *Let $a, x \in R$ and let $\sigma : R \to R, \sigma(a) = xa$. Then*

    (i) *The map $\sigma : a \mapsto xa$ is a left centralizer.*

    (ii) *The map $\sigma : a \mapsto xa$ is a centralizer if x is a central element.*

    (iii) *The map $\sigma : a \mapsto xa$ is a centralizer if x is an invertible central element. Moreover, if σ is a bijective centralizer then so is σ^{-1}.*

Theorem 2.6. *Let $\sigma : R \to R$ be a bijective centralizer and let $a, b, d_1, d_2 \in R$ with $d_1 = \sigma(d_2)$. If $a^{\|d_1}$ and $b^{\|d_2}$ exist, then $a^{\|d_1} + b^{\|d_2} = a^{\|d_1}(a+b)b^{\|d_2}$.*

Proof. It is known [7] that the existence of $b^{\|d_2}$ implies that d_2 is regular. Hence $d_1 = \sigma(d_2) = \sigma(d_2 d_2^- d_2) = d_2 d_2^- \sigma(d_2) = \sigma(d_2) d_2^- d_2$. So, $d_1 R \subseteq d_2 R$ and $R d_1 \subseteq R d_2$.

As σ is bijective, then $d_2 = \sigma^{-1}(d_1)$, which also guarantees that $d_2 R \subseteq d_1 R$ and $R d_2 \subseteq R d_1$.

It follows from [7, Corollary 2.4] that if a is invertible along d_1 with $d_1 R = d_2 R$ and $R d_1 = R d_2$, then $a^{\|d_2}$ exists and $a^{\|d_1} = a^{\|d_2}$.

So the result follows from Corollary 2.3. $\qquad\square$

Next, we give two simple examples to illustrate Theorem 2.6 above.

Example 2.7. (i) Let $R = M_2(\mathbb{C})$ be the ring of all 2×2 complex matrices and let $\sigma : R \to R, \sigma(M) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} M$ for all M in R. Then σ is a bijective centralizer by Example 2.5(iii). Take $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ and $D_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. We can check $D_1 = \sigma(D_2)$, $A^{\|D_1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B^{\|D_2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ by direct calculations. Hence $A^{\|D_1} + B^{\|D_2} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ 0 & 0 \end{bmatrix} = A^{\|D_1}(A + B)B^{\|D_2}$.

(ii) Let $R = \mathbb{Z}_9$ and let $\sigma : R \to R, \sigma(x) = 2x$ for any $x \in R$. It follows from Example 2.5(iii) that σ is a bijective centralizer. Take $a = 7$, $b = 5$, $d_1 = 4$ and $d_2 = 2$. Then $d_1 = \sigma(d_2)$, $a^{\|d_1} = 7^{\|4} = 4$ and $b^{\|d_2} = 5^{\|2} = 2$. Moreover, $a^{\|d_1} + b^{\|d_2} = 6 = a^{\|d_1}(a+b)b^{\|d_2}$.

In [6], one can get $a^{\|a} = a^{\#}$, $a^{\|a^n} = a^D$ and $a^{\|a^*} = a^{\dagger}$. By $R^{\#}$, R^D and $R^{\dagger}$ we denote the sets of all group, Drazin and Moore-Penrose invertible elements in R, respectively.

Hence, we get

Corollary 2.8. *Let $\sigma : R \to R$ be a bijective centralizer and let $a, b \in R$. Then*

(i) *If $a = \sigma(b)$, then $a^{\#} + b^{\#} = a^{\#}(a + b)b^{\#}$, for $a, b \in R^{\#}$.*

(ii) *If $a^n = \sigma(b^m)$ for some integers m and n, then $a^D + b^D = a^D(a + b)b^D$, for $a, b \in R^D$.*

(iii) *Let R be a ring with involution. If $a^* = \sigma(b^*)$, then $a^{\dagger} + b^{\dagger} = a^{\dagger}(a + b)b^{\dagger}$, for $a, b \in R^{\dagger}$.*

(iv) *Let R be a ring with involution. If $a = \sigma(b^*)$, then $a^{\#} + b^{\dagger} = a^{\#}(a + b)b^{\dagger}$, for $a \in R^{\#}$ and $b \in R^{\dagger}$.*

3   Characterizations of the inverse along an element

We first begin with a lemma on the commutativity of the inverse along an element.

Lemma 3.1. *Let $\sigma : R \to R$ be a bijective centralizer and let $a, d \in R$ with $ad = \sigma(da)$. If a is invertible along d, then $a^{\|d} a = a a^{\|d}$.*

Proof. We have

$$ad = \sigma(da) = \sigma(a^{\|d}ada) = a^{\|d}a\sigma(da) = a^{\|d}a^2d. \qquad (3.1)$$

Also, as σ is bijective, then

$$da = \sigma^{-1}(ad) = \sigma^{-1}(adaa^{\|d}) = \sigma^{-1}(\sigma(da)aa^{\|d}) = da^2a^{\|d}. \qquad (3.2)$$

As $a^{\|d} \in dR \cap Rd$, then there exist $x, y \in R$ such that $a^{\|d} = xd = dy$. Multiplying the equality (3.1) by y on the right yields $aa^{\|d} = ady = a^{\|d}a^2dy = a^{\|d}a^2a^{\|d}$. Multiplying the equality (3.2) by x on the left yields $a^{\|d}a = xda = xda^2a^{\|d} = a^{\|d}a^2a^{\|d}$.

Hence, $a^{\|d}a = aa^{\|d}$. $\qquad\square$

In [16, page 7], the authors gave a counterexample to illustrate that reverse order law for the inverse along an element does not hold in general.

The following theorem, extending [16, Theorem 2.14], considers the reverse order law for the inverse along an element, under a generalized commutativity condition.

Theorem 3.2. *Let $\sigma : R \to R$ be a bijective centralizer and let $a, b, d \in R$ with $ad = \sigma(da)$. If $a^{\|d}$ and $b^{\|d}$ exist, then*

    (i) $(ab)^{\|d}$ *exists and* $(ab)^{\|d} = b^{\|d}a^{\|d}$.

    (ii) $(ba)^{\|d}$ *exists and* $(ba)^{\|d} = a^{\|d}b^{\|d}$.

Proof. Applying Lemma 2.1(ii), it follows $b^{\|d}aa^{\|d} = b^{\|d}$. Also, $a^{\|d}a = aa^{\|d}$ by Lemma 3.1. Then $b^{\|d}a^{\|d}abd = b^{\|d}aa^{\|d}bd = b^{\|d}bd = d$ and $dabb^{\|d}a^{\|d} = \sigma^{-1}(ad)bb^{\|d}a^{\|d} = \sigma^{-1}(a)(dbb^{\|d})a^{\|d} = \sigma^{-1}(ad)a^{\|d} = daa^{\|d} = d$. Finally, $b^{\|d}a^{\|d} \in dR \cap Rd$ since $a^{\|d} \in dR \cap Rd$ and $b^{\|d} \in dR \cap Rd$.

Hence, ab is invertible along d and $(ab)^{\|d} = b^{\|d}a^{\|d}$.

    (ii) The proof is similar to (i). $\qquad\square$

Remark 3.3. According to proof of Theorem 3.2, one can see that it indeed holds in a semigroup.

Corollary 3.4. *Let $a, b, d \in R$ with $ad = da$. If $a^{\|d}$ and $b^{\|d}$ exist, then*

    (i) $(ab)^{\|d}$ *exists and* $(ab)^{\|d} = b^{\|d}a^{\|d}$.

    (ii) $(ba)^{\|d}$ *exists and* $(ba)^{\|d} = a^{\|d}b^{\|d}$.

Let $R = \mathbb{Z}_7$. Then 5 is invertible along 3 and $5^{\|3} = 3$. We notice the following fact, i.e., 5 is also invertible along $2 \cdot 3$ and $5^{\|2 \cdot 3} = 3 = 5^{\|3}$, and $2 \cdot 5$ is also invertible along 3 and $(2 \cdot 5)^{\|3} = 5 = 4 \cdot 3 = 2^{-1} \cdot 5^{\|3}$. Motivated by this, we prove the next proposition.

Proposition 3.5. *Let $\sigma : R \to R$ be a bijective centralizer and let $a, d \in R$. Then*

(i) *$a^{\|d}$ exists if and only if $a^{\|\sigma(d)}$ exists. Moreover, $a^{\|\sigma(d)} = a^{\|d}$.*

(ii) *$a^{\|d}$ exists if and only if $(\sigma(a))^{\|d}$ exists. Moreover, $(\sigma(a))^{\|d} = \sigma^{-1}(a^{\|d})$.*

Proof. (i) "$\Rightarrow$" As $a^{\|d}$ exists, then d is regular and hence $\sigma(d) = dd^-\sigma(d) = \sigma(d)d^-d$. So, $\sigma(d)R \subset dR$ and $R\sigma(d) \subset Rd$. Also, σ is bijective, then $d = \sigma(d)\sigma^{-1}(1) = \sigma^{-1}(1)\sigma(d)$ implies $dR \subset \sigma(d)R$ and $Rd \subset R\sigma(d)$. Hence, $dR = \sigma(d)R$ and $Rd = R\sigma(d)$. It follows from [7, Corollary 2.4] that $a^{\|\sigma(d)}$ exists and $a^{\|\sigma(d)} = a^{\|d}$.

"$\Leftarrow$" By noting that σ^{-1} is a bijective centralizer.

(ii) One can check it directly. $\qquad\square$

We next give an existence criterion of the inverse along an element by centralizers in a ring. Herein, a lemma is presented.

Lemma 3.6. *Let $a, b, c \in R$. Then*

(i) *If $(1 + ab)c = 1$, then $(1 + ba)(1 - bca) = 1$.*

(ii) *If $c(1 + ab) = 1$, then $(1 - bca)(1 + ba) = 1$.*

It follows from Lemma 3.6 that $1 + ab$ is left (right) invertible if and only if $1 + ba$ is left (right) invertible. In particular, $1 + ab$ is invertible if and only if $1 + ba$ is invertible. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$. This formula is known as Jacobson's Lemma.

Given an element $a \in R$, it is known that if a is regular then so is $\sigma(a)$ (σ is a bijective centralizer). Moreover, $(\sigma(a))^- = \sigma^{-1}(a^-)$.

Theorem 3.7. *Let $a, d \in R$ with d regular. If $\sigma : R \to R$ is a bijective centralizer, then the following conditions are equivalent:*

(i) *$a^{\|d}$ exists.*

(ii) *$u = \sigma(da) + 1 - dd^-$ is invertible.*

(iii) *$v = \sigma(ad) + 1 - d^-d$ is invertible.*

In this case, $a^{\|d} = \sigma(u^{-1})d = d\sigma(v^{-1})$.

7

Proof. (ii) $\Leftrightarrow$ (iii) As σ is a centralizer, then $\sigma(da) = d\sigma(a)$ and $\sigma(ad) = \sigma(a)d$. Take $x = -d$ and $y = \sigma(a) - d^-$. Then, $u = d\sigma(a) + 1 - dd^- = 1 - xy$ is invertible if and only if $v = \sigma(a)d + 1 - d^-d = 1 - yx$ is invertible by Jacobson's Lemma.

(i) $\Rightarrow$ (ii) Suppose that $a^{\|d}$ exists. Then $a^{\|\sigma(d)}$ exists by Proposition 3.5. Since $a^{\|\sigma(d)} \in \sigma(d)R$, there exists $x \in R$ such that $a^{\|\sigma(d)} = \sigma(d)x$ and hence $\sigma(d) = \sigma(d)aa^{\|\sigma(d)}} = \sigma(da)\sigma(d)x$. As $(\sigma(da)dd^- + 1 - dd^-)(\sigma(d)x\sigma^{-1}(d^-) + 1 - dd^-) = 1$, then $\sigma(da)dd^- + 1 - dd^-$ is right invertible. Lemma 3.6 ensures that $dd^-\sigma(da) + 1 - dd^-$, i.e., $\sigma(da) + 1 - dd^-$ is right invertible.

Similarly, $\sigma(da) + 1 - dd^-$ is left invertible since $a^{\|\sigma(d)} \in R\sigma(d)$.

Therefore, $\sigma(da) + 1 - dd^- = u$ is invertible.

(ii) $\Rightarrow$ (i) As u is invertible, then v is also invertible. From $ud = \sigma(da)d = d\sigma(ad) = dv$, we get $d = u^{-1}\sigma(dad) = \sigma(u^{-1})dad = dad\sigma(v^{-1})$. Hence, $a^{\|d}$ exists in terms of [7, Theorem 2.2].

Now, we show that $m = \sigma(u^{-1})d = d\sigma(v^{-1})$ is the inverse of a along d. As $u^{-1}d = dv^{-1}$, then $\sigma(u^{-1})d = d\sigma(v^{-1})$. It follows that $mad = \sigma(u^{-1})dad = u^{-1}\sigma(dad) = d = dad\sigma(v^{-1}) = dam$ and $m \in dR \cap Rd$.

Thus, $a^{\|d} = \sigma(u^{-1})d = d\sigma(v^{-1})$. $\qquad\square$

Remark 3.8. The assumption " a bijective centralizer" in Theorem 3.7 above cannot be replaced by "a centralizer". Such as let $R = \mathbb{Z}_6$ and let $\sigma : R \rightarrow R, \sigma(x) = 3x$ for all $x \in R$. Then σ is a centralizer but not bijective. Take $a = 4$ and $d = 2$ in R. Then a is invertible along d and $a^{\|d} = 4$. However, $\sigma(da) + 1 - dd^- = 3 \cdot 2 + 1 - 2 \cdot 2 = 3$ is not invertible.

Corollary 3.9. [7, Theorem 3.2] *Let $a, d \in R$ with d regular. Then the following conditions are equivalent*:

(i) $a^{\|d}$ *exists.*

(ii) $u = da + 1 - dd^-$ *is invertible.*

(iii) $v = ad + 1 - d^-d$ *is invertible.*

In this case, $a^{\|d} = u^{-1}d = dv^{-1}$.

It is known [10, Theorem 1] that $a \in R^{\#}$ exists if and only if $a^2 + 1 - aa^-$ is invertible

if and only if $a^2 + 1 - a^- a$ is invertible, for a regular element a. Herein, taking $d = a$ in Theorem 3.7. We get a new existence criterion of the group inverse of a in a ring.

Corollary 3.10. *Let $a \in R$ be regular. If $\sigma : R \to R$ is a bijective centralizer, then the following conditions are equivalent:*

   (i) $a \in R^{\#}$.

   (ii) $u = \sigma(a^2) + 1 - aa^-$ is invertible.

   (iii) $v = \sigma(a^2) + 1 - a^- a$ is invertible.

   In this case, $a^{\#} = \sigma(u^{-1})a = a\sigma(v^{-1})$.

Setting $d = a^n$, it follows the characterizations of the Drazin inverse of a by units and centralizers.

Corollary 3.11. *Let $a \in R$ be regular. If $\sigma : R \to R$ is a bijective centralizer, then the following conditions are equivalent:*

   (i) $a \in R^D$.

   (ii) $u = \sigma(a^{n+1}) + 1 - aa^-$ is invertible, for some integer n.

   (iii) $v = \sigma(a^{n+1}) + 1 - a^- a$ is invertible, for some integer n.

   In this case, $a^D = \sigma(u^{-1})a^n = a^n\sigma(v^{-1})$.

Next, we consider the characterizations of the Moore-Penrose inverse of a regular element by centralizers in a ring.

It follows from from [15, Theorem 2.16] that $a \in R^{\dagger} \Leftrightarrow a \in aa^* aR \Leftrightarrow a \in Raa^* a$, which allow us to characterize Moore-Penrose inverses in terms of one-sided invertibilities in a ring. Also, we know [15, Theorems 2.19 and 2.20] that $a = aa^* ax$ or $a = yaa^* a$ implies that a is Moore-Penrose invertible and $a^{\dagger} = a^* ax^2 a^* = a^* y^2 aa^*$.

Theorem 3.12. *Let R be a ring with involution and let $a \in R$ be regular. If $\sigma : R \to R$ is a bijective centralizer, then the following conditions are equivalent:*

   (i) $a \in R^{\dagger}$.

   (ii) $u = \sigma(aa^*) + 1 - aa^-$ is right (left) invertible.

   (iii) $v = \sigma(a^* a) + 1 - a^- a$ is right (left) invertible.

   In this case, $a^{\dagger} = a^*(\sigma(u_l^{-1}))^2 aa^* = a^* a(\sigma(v_r^{-1}))^2 a^*$, where u_l^{-1}, v_r^{-1} denote a left inverse of u and a right inverse of v, respectively.

9

Proof. For simplicity, we only prove the case of right invertibility.

(ii) $\Leftrightarrow$ (iii) As σ is a centralizer, then $\sigma(aa^*) = a\sigma(a^*)$ and $\sigma(a^*a) = \sigma(a^*)a$. Pose $x = -a$ and $y = \sigma(a^*) - a^-$. Then $1 - xy = u$ is right (left) invertible if and only if $1 - yx = v$ is right (left) invertible by Lemma 3.6.

(i) $\Rightarrow$ (ii) Since $a \in R^\dagger$, there exists $x \in R$ such that $a = aa^*ax$ from [15, Theorem 2.16]. As

$$(\sigma(aa^*)aa^- + 1 - aa^-)(a\sigma^{-1}(x)a^- + 1 - aa^-)$$
$$= \sigma(aa^*)a\sigma^{-1}(x)a^- + 1 - aa^-$$
$$= \sigma(aa^*a)\sigma^{-1}(x)a^- + 1 - aa^-$$
$$= aa^*axa^- + 1 - aa^-$$
$$= aa^- + 1 - aa^-$$
$$= 1,$$

then $\sigma(aa^*)aa^- + 1 - aa^-$ is right invertible. Again, Lemma 3.6 guarantees that $aa^-\sigma(aa^*) + 1 - aa^- = u$ is right invertible.

(ii) $\Rightarrow$ (i) Note that u and hence v are both right invertible. We have $av = \sigma(aa^*a)$. Hence, $a = \sigma(aa^*a)v_r^{-1} = aa^*a\sigma(v_r^{-1}) \in aa^*aR$, which means that $a \in R^\dagger$ by [15, Theorem 2.16].

As $ua = \sigma(aa^*a)$, then $a = u_l^{-1}\sigma(aa^*a) = \sigma(u_l^{-1})aa^*a$. Hence, $a^\dagger = a^*(\sigma(u_l^{-1}))^2aa^*$ in terms of [15, Theorem 2.20].

Similarly, $a = aa^*a\sigma(v_r^{-1})$ ensures that $a^\dagger = a^*a(\sigma(v_r^{-1}))^2a^*$ in terms of [15, Theorem 2.19]. $\square$

Remark 3.13. The expressions for the Moore-Penrose inverse $a^\dagger$ in Theorem 3.12 can also be given by u_r^{-1} or v_l^{-1}. Indeed, by Lemma 3.6, we have $v_r^{-1} = 1 - (\sigma(a^*) - a^-)u_r^{-1}a$. If we replace v_r^{-1} by $1 - (\sigma(a^*) - a^-)u_r^{-1}a$ in the equality $a^\dagger = a^*a(v_r^{-1})^2a^*$. then the formula of $a^\dagger$ can be presented by u_r^{-1}. The similar process for u_l^{-1} follows the formula of $a^\dagger$ by v_l^{-1}.

Corollary 3.14. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent*:

10

(i) $a \in R^{\dagger}$.

(ii) $u = aa^* + 1 - aa^-$ is right (left) invertible.

(iii) $v = a^*a + 1 - a^-a$ is right (left) invertible.

In this case, $a^{\dagger} = a^*(u_l^{-1})^2 aa^* = a^*a(v_r^{-1})^2 a^*$, where u_l^{-1}, v_r^{-1} denote a left inverse of u and a right inverse of v, respectively.

It follows from [16, Theorem 3.3] that another Moore-Penrose inverse of a in Corollary 3.14 can also be given as $a^{\dagger} = (u_l^{-1}a)^*a(u_l^{-1}a)^* = (av_r^{-1})^*a(av_r^{-1})^*$.

Given a regular element $a \in R$, it follows from Theorem 3.12 that $a^{\dagger}$ exists if and only if u (resp., v) is right invertible if and only if u (resp., v) is left invertible. Hence, we have the following corollary.

Corollary 3.15. *Let R be a ring with involution and let $a \in R$ be regular. If $\sigma : R \to R$ is a bijective centralizer, then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$.

(ii) $u = \sigma(aa^*) + 1 - aa^-$ is invertible.

(iii) $v = \sigma(a^*a) + 1 - a^-a$ is invertible.

In this case, $a^{\dagger} = a^*(\sigma(u^{-1}))^2 aa^* = a^*a(\sigma(v^{-1}))^2 a$.

Remark 3.16. In Corollary 3.15 above, the expressions of $a^{\dagger}$ can also be given by $\sigma(u^{-1})a^* = a^*\sigma(v^{-1})$ by Theorem 3.7.

Setting the centralizer σ to be 1, it follows that

Corollary 3.17. [8, Theorem 2.1] *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$.

(ii) $u = aa^* + 1 - aa^-$ is invertible.

(iii) $v = a^*a + 1 - a^-a$ is invertible.

In this case, $a^{\dagger} = a^*(u^{-1})^2 aa^* = a^*a(v^{-1})^2 a$.

ACKNOWLEDGMENTS

11

References

[1] X.J. Chen, J. Zhang, W.B. Guo, The absorption law of {1,3}, {1,4} inverse of matrix, J. Inn. Mong. Norm. Univ. 2008;37: 337-339.

[2] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly. 1958;65: 506-514.

[3] H.W. Jin, J. Benitez, The absorption laws for the generalized inverses in rings, Electron. J. Linear Algebra. 2015;30: 827-842.

[4] B. E. Johnson, An introduction to the theory of centralizers, Proc. London Math. Soc. 1964;14: 299-320.

[5] X.J. Liu, H.W. Jin, D.S. Cvetković-Ilić, The absorption laws for the generalized inverses, Appl. Math. Comput. 2012;219: 2053-2059.

[6] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 2011;434: 1836-1844.

[7] X. Mary, P. Patrício, Generalized inverses modulo $\mathcal{H}$ in semigroups and rings, Linear Multilinear Algebra 2013;61: 886-891.

[8] P. Patrício, C. Mendes Araújo, Moore-Penrose inverse in involutory rings: the case $aa^\dagger = bb^\dagger$, Linear Multilinear Algebra 2010;58: 445-452.

[9] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 1955;51: 406-413.

[10] R. Puystjens, R.E. Hartwig, The group inverse of a companion matrix, Linear Multilinear Algebra 1997;43: 137-150.

[11] J. Vukman, Centralizers on semiprime rings, Comment. Math. Univ. Carolin. 2001;42: 237-245.

[12] J. Vukman, An equation on operator algebras and semisimple H^*-algebras, Glasnik Mat. 2005;40: 201-206.

[13] J.G. Wendel, Left centralizers and isomorphisms of group algebras, Pacific J. Math. 1952;2: 251-261.

[14] B. Zalar, On centralizers of semiprime rings, Comment. Math. Univ. Carolin. 1991;32: 609-614.

[15] H.H. Zhu, J.L. Chen, P. Patrício, Further results on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 2016;64: 393-403.

[16] H.H. Zhu, J.L. Chen, P. Patrício, Reverse order law for the inverse along an element, Linear Multilinear Algebra DOI: 10.1080/03081087.2016.1178209.

[17] H.H. Zhu, P. Patrício, J.L. Chen, Y.L. Zhang, The inverse along a product and its applications, Linear Multilinear Algebra 2016;64: 834-841.

[18] H.H. Zhu, X.X. Zhang, J.L. Chen, Centralizers and Their Applications to Generalized Inverses, Linear Algebra Appl. 2014;458: 291-300.