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Spaces of matrices of constant rank and uniform vector bundles

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We consider the problem of determining $l(r, a)$, the maximal dimension of a subspace of $a \times a$ matrices of rank r. We first review, in the language of vector bundles, the known results. Then using known facts on uniform bundles we prove some new results and make a conjecture. Finally we determine $l(r; a)$ for every r, $1 \leq r \leq a$, when $a \leq 10$, showing that our conjecture holds true in this range.

0. Introduction

Let A, B be k-vector spaces of dimensions a, b (k algebraically closed, of characteristic zero). A sub-vector space $M \subset \mathcal{L}(A, B)$ is said to be of (constant) rank r if every $f \in M, f \neq 0$, has rank r. The question considered in this paper is to determine

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$l(r, a, b) := max\{\dim M \mid M \subset \mathcal{L}(A, B) \text{ has rank } r\}$. This problem has been studied some time ago by various authors [22,20,4,9] and has been recently reconsidered, especially in its (skew) symmetric version [17,18,16,5].

It is known, at least since [20], that to give a subspace M of constant rank r, dimension $n+1$, is equivalent to give an exact sequence: $0 \to F \to a.\mathcal{O}(-1) \xrightarrow{\psi} b.\mathcal{O} \to E \to 0$, on $\mathbb{P}^n$, where F, E are vector bundles of ranks $(a-r)$, $(b-r)$. Our starting point is to observe that the bundle $\mathcal{E} := Im(\psi)$, of rank r, is *uniform*, of splitting type $(-1^c, 0^{r-c})$, where $c := c_1(E)$ (Lemma 2). This had been previously observed (but not really exploited) in the cases of spaces of symmetric or skew-symmetric maps [17]. This allows us to apply the known results (and conjectures) on uniform bundles.

This paper is organized as follows. In the first section we recall some basic facts and fix the notations. Then in Section two, we set $a = b$ to fix the ideas and we survey the known results (at least those we are aware of), giving a quick, uniform (!) treatment in the language of vector bundles. In Section three, using known results on uniform bundles, we obtain a new bound on $l(r; a)$ in the range $(2a + 2)/3 > r > (a + 2)/2$ (as well as some other results, see Theorem 18). By the way we don't expect this bound to be sharp. Indeed by "translating" (see Proposition 17) a long standing conjecture on uniform bundles (Conjecture 1), we conjecture that $l(r; a) = a - r + 1$ in this range (see Conjecture 2). Finally, with some ad hoc arguments, we show in the last section, that our conjecture holds true for $a \leq 10$ (actually we determine $l(r; a)$ for every r, $1 \leq r \leq a$, when $a \leq 10$).

1. Generalities

Following [20], to give $M \subset Hom(A, B)$, a sub-space of constant rank r, with $\dim(M) = n + 1$, is equivalent to give on $\mathbb{P}^n$, an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & F_M \arrow[r] & a.\mathcal{O}(-1) \arrow[rr, "\psi_M"] \arrow[dr] & & b.\mathcal{O} \arrow[r] & E_M \arrow[r] & 0 \\ & & & \mathcal{E}_M \arrow[ur, hook] & & & \end{tikzcd} \qquad (1)$$

where $\mathcal{E}_M = Im(\psi_M), F_M, E_M$ are vector bundles of ranks $r, a - r, b - r$ (in the sequel we will drop the index M if no confusion can arise).

Indeed the inclusion $i : M \hookrightarrow Hom(A, B)$ is an element of $Hom(M, A^\vee \otimes B) \simeq M^\vee \otimes A^\vee \otimes B$ and can be seen as a morphism $\psi : A \otimes \mathcal{O} \to B \otimes \mathcal{O}(1)$ on $\mathbb{P}(M)$ (here $\mathbb{P}(M)$ is the projective space of lines of M). At every point of $\mathbb{P}(M)$, ψ has rank r, so the image, the kernel and the cokernel of ψ are vector bundles.

A different (but equivalent) description goes as follows: we can define $\psi : A \otimes \mathcal{O}(-1) \to B \otimes \mathcal{O}$ on $\mathbb{P}(M)$, by $v \otimes \lambda f \to \lambda f(v)$.

The vector bundle $\mathcal{E}_M$ is of a particular type.

Definition 1. A rank r vector bundle, E, on $\mathbb{P}^n$ is *uniform* if there exists $(a_1, ..., a_r)$ such that $E_L \simeq \bigoplus_{i=1}^r \mathcal{O}_L(a_i)$, for every line $L \subset \mathbb{P}^n$ ($(a_1, ..., a_r)$ is the splitting type of E, it is independent of L).

The vector bundle F is *homogeneous* if $g^*(F) \simeq F$, for every automorphism of $\mathbb{P}^n$.

Clearly a homogeneous bundle is uniform (but the converse is not true).

The first remark is:

Lemma 2. *With notations as in (1), $c_1(E_M) \geq 0$ and $\mathcal{E}_M$ is a uniform bundle of splitting type $(-1^c, 0^b)$, where $c = c_1(E_M), b = r - c$.*

Proof. Since E is globally generated, $c_1(E) \geq 0$ (look at E_L). Let $\mathcal{E}_L = \bigoplus \mathcal{O}_L(a_i)$. We have $a_i \geq -1$, because $a.\mathcal{O}_L(-1) \twoheadrightarrow \mathcal{E}_L$. We have $a_i \leq 0$, because $\mathcal{E}_L \hookrightarrow b.\mathcal{O}_L$. So $-1 \leq a_i \leq 0, \forall i$. Since $c_1(\mathcal{E}) = -c_1(E)$ the splitting type is as asserted and does not depend on the line L. $\square$

The classification of rank $r \leq n + 1$ uniform bundles on $\mathbb{P}^n$, $n \geq 2$, is known [21,10,12,1]; if we denote by T, Ω the tangent and cotangent bundle on $\mathbb{P}^n$, then we have:

Theorem 3. *A rank $r \leq n+1$ uniform vector bundle on $\mathbb{P}^n$, $n \geq 2$, is one of the following: $\bigoplus^r \mathcal{O}(a_i)$, $T(a) \oplus k.\mathcal{O}(b)$, $\Omega(a) \oplus k.\mathcal{O}(b)$ $(0 \leq k \leq 1)$, $S^2 T_{\mathbb{P}^2}(a)$.*

We will use the following result (see [15,8]):

Theorem 4 *(Evans–Griffith). Let $\mathcal{F}$ be a rank r vector bundle on $\mathbb{P}^n$, then $\mathcal{F}$ is a direct sum of line bundles if and only if $H^i_*(\mathcal{F}) = 0$, for $1 \leq i \leq r - 1$.*

The first part of the following Proposition is well known, the second maybe less.

Proposition 5. *Assume $n \geq 1$.*
(1) If $a \geq b + n$ the generic morphism $a.\mathcal{O}_{\mathbb{P}^n} \to b.\mathcal{O}_{\mathbb{P}^n}(1)$ is surjective.
(2) If $a < b + n$ no morphism $a.\mathcal{O}_{\mathbb{P}^n} \to b.\mathcal{O}_{\mathbb{P}^n}(1)$ can be surjective.

Proof. (1) It is enough to treat the case $a = b + n$ and, by semi-continuity, to produce one example of surjective morphism. Consider

$$\Psi = \begin{pmatrix} x_0 & \cdots & x_n & 0 & \cdots & & 0 \\ 0 & x_0 & \cdots & x_n & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \\ 0 & \cdots & 0 & x_0 & \cdots & & x_n \end{pmatrix}$$

(each row contains $b - 1$ zeroes). It is clear that this matrix has rank b at any point. For a more conceptual (and complicated) proof see [14], Prop. 1.1.

(2) If $n = 1$, the statement is clear. Assume $n \geq 2$. If ψ is surjective we have $0 \to K \to a.\mathcal{O} \to b.\mathcal{O}(1) \to 0$ and K is a vector bundle of rank $r = a - b < n$. Clearly we have $H^i_*(K^\vee) = 0$ for $1 \leq i \leq r - 1 \leq n - 2$. By Evans–Griffith's theorem, K splits as a direct sum of line bundles, hence the exact sequence splits ($n \geq 2$) and this is absurd.

This can also be proved by a Chern class computation (see [20]). $\square$

From now on we will assume $A = B$ and write $l(r; a)$ instead of $l(r; a, a)$.

2. Known results

We begin with some general facts:

Lemma 6. *Assume the bundle $\mathcal{E}$ corresponding to $M \subset End(A)$ of constant rank r,* $\dim(A) = a$, *is a direct sum of line bundles. Then* $\dim(M) \leq a - r + 1$.

Proof. Let $\dim(M) = n + 1$ and assume $\mathcal{E} = k.\mathcal{O}(-1) \oplus (r - k).\mathcal{O}$. If $k = 0$, the surjection $a.\mathcal{O}(-1) \twoheadrightarrow \mathcal{E} \simeq r.\mathcal{O}$, shows that $a \geq r + n$ (see Proposition 5). If $k > 0$, we have $0 \to k.\mathcal{O}(-1) \to (a-r+k).\mathcal{O} \to E \to 0$. Dualizing we get: $(a-r+k).\mathcal{O} \twoheadrightarrow k.\mathcal{O}(1)$, hence (always by Proposition 5) $a - r + k \geq k + n$. So in any case $a - r \geq n$. $\square$

Lemma 7. *For every r, $1 \leq r \leq a$, we have $l(r; a) \geq a - r + 1$.*

Proof. Set $n = a - r$. On $\mathbb{P}^n$ we have a surjective morphism $a.\mathcal{O}(-1) \xrightarrow{\overline{\psi}} r.\mathcal{O}$ (Proposition 5). Composing with the inclusion $r.\mathcal{O} \hookrightarrow r.\mathcal{O} \oplus (a-r).\mathcal{O}$, we get $\psi : a.\mathcal{O}(-1) \to a.\mathcal{O}$, of constant rank r. $\square$

Finally we get:

Proposition 8. *(1) We have $l(r; a) \leq max\,\{r + 1, a - r + 1\}$.*

(2) If $a \geq 2r$, then $l(r; a) = a - r + 1$.

Proof. (1) Assume $r + 1 \geq a - r + 1$. If $\dim(M) = l(r, a) = n + 1$ and if $r < n$, then [12] $\mathcal{E}$ is a direct sum of line bundles and $n \leq a - r$. But then $r < n \leq a - r$, against our assumption. So $r + 1 \geq n + 1 = l(r; a)$.

Now assume $a - r \geq r$. If $n > a - r$, then $n > r$ and this implies that $\mathcal{E}$ is a direct sum of line bundles. Hence $n \leq a - r$.

(2) We have $max\,\{r + 1, a - r + 1\} = a - r + 1$ if $a \geq 2r$. So $l(r; a) \leq a - r + 1$ by (1). We conclude with Lemma 7. $\square$

Remark 9. Proposition 8 was first proved (by a different method) by Beasley [4].

Very few indecomposable rank r vector bundles with $r < n$ are known on $\mathbb{P}^n$ ($n > 4$). One of these is the bundle of Tango (see [19], p. 84 for details). We will use it to prove:

Lemma 10. *We have $l(t+1;2t+1) = t+2$.*

Proof. By Proposition 8 we know that $l(t+1;2t+1) \leq t+2$. So it is enough to give an example. Set $n = t+1$ and assume first $n \geq 3$. If $\mathcal{T}$ denotes the Tango bundle, then we have: $0 \to T(-2) \to (2n-1).\mathcal{O} \to \mathcal{T} \to 0$. Dualizing we get $0 \to \mathcal{T}^\vee(-1) \to (2n-1).\mathcal{O}(-1) \to \Omega(1) \to 0$. Combining with the exact sequence: $0 \to \Omega(1) \to (n+1).\mathcal{O} \oplus (n-2).\mathcal{O} \to \mathcal{O}(1) \oplus (n-2).\mathcal{O} \to 0$, we get a morphism $(2n-1).\mathcal{O}(-1) \to (2n-1).\mathcal{O}$, of constant rank n.

If $n = 2$, using the fact that $T(-2) \simeq \Omega(1)$, from Euler's sequence, we get $3.\mathcal{O}(-1) \to 3.\mathcal{O}$, whose image is $T(-2)$. $\square$

Remark 11. Lemma 10 was first proved by Beasley [4], by a different method.

Finally on the opposite side, when r is big compared with a, we have:

Proposition 12. *(Sylvester [20]) We have:*

$$l(a-1;a) = \begin{cases} 2 & \textit{if } a \textit{ is even} \\ 3 & \textit{if } a \textit{ is odd} \end{cases}$$

The proof is a Chern classes computation. The next case $a = r-2$ is more involved and there are only partial results:

Proposition 13. *(Westwick [24]) We have $3 \leq l(a-2;a) \leq 5$. Moreover:*

(1) *$l(a-2;a) \leq 4$ except if $a \equiv 2, 10 \pmod{12}$ where it could be $l(a-2;a) = 5$.*
(2) *If $a \equiv 0 \pmod 3$, then $l(a-2;a) = 3$.*
(3) *If $a \equiv 1 \pmod 3$, we have $l(2;4) = 3$ and $l(8;10) = 4$ (so a doesn't determine $l(a-2;a)$).*
(4) *If $a \equiv 2 \pmod 3$, then $l(a-2;a) \geq 4$. Moreover if $a \not\equiv 2 \pmod 4$, then $l(a-2;a) = 4$.*

Proof. We denote by $\mathcal{C}(G) = 1 + c_1 h + ... + c_n h^n$ the Chern polynomial of G (computations are made in $\mathbb{Z}[h]/(h^{n+1})$). We recall that if G is a vector bundle of rank r, then $c_i = 0$ if $i > r$. We have $C(F) = C(E)(1-h)^a$. Let $C(F) = 1 + s_1 h + s_2 h^2$, $C(E) = 1 + t_1 h + t_2 h^2$. We get $s_1 = t_1 - a$ (coefficient of h); $s_2 = a(a-1)/2 - at_1 + t_2$ (coefficient of h^2). From the coefficient of h^3 it follows that: $t_2 = (a-1)[3t_1 - a + 2]/6$. The coefficient of h^4 yields after some computations: $(a+1)(a-2-2t_1) = 0$. It follows that $t_1 = \dfrac{(a-2)}{2}$ and $t_2 = \dfrac{(a-1)(a-2)}{12}$ if we are on $\mathbb{P}^n, n \geq 4$. Finally the coefficient of h^5 gives $(a+1)(a+2) = 0$, showing that $l(a-2;a) \leq 5$.

If we are on $\mathbb{P}^4$, from $t_1 = (a-2)/2$ we see that a is even. From $t_2 = (a-1)(a-2)/12$, we get $a^2 + 2 - 3a \equiv 0 \pmod{12}$. This implies $a \equiv 1, 2, 5, 10 \pmod{12}$. Since a is even we get $a \equiv 2, 10 \pmod{12}$.

If $a = 3m$ and if we are on $\mathbb{P}^3$, then $6t_2 = (3m-1)(3t_1 - 3m + 2) \equiv 0 \pmod 6$, which is never satisfied. So $l(a-2, a) \leq 3$ in this case.

The other statements follow from the construction of suitable examples, see [24]. □

Remark 14. On $\mathbb{P}^4$ a rank two vector bundle with $c_1 = 0$ has to verify the Schwarzenberger condition $c_2(c_2 + 1) \equiv 0 \pmod{12}$. If $l(a-2; a) = 5$ for some a, then $a = 12m + 2$ or $a = 12m + 10$. In the first case the condition yields $m \equiv 0, 5, 8, 9 \pmod{12}$, in the second case $m \equiv 2, 3, 6, 11 \pmod{12}$. So, as already noticed in [24], the lowest possible value of a is $a = 34$. This would give an indecomposable rank two vector bundle with Chern classes $c_1 = 0, c_2 = 24$. Indeed if we have an exact sequence (1), E and F cannot be both a direct sum of line bundles (because, by Theorem 4, $\mathcal{E}$ would also be a direct sum of line bundles, which is impossible).

Finally we have:

Proposition 15. *(Westwick [23]) For every a, r, $l(r; a) \leq 2a - 2r + 1$.*

As noticed in [17] (Theorem 1.4) this follows directly from a result of Lazarsfeld on ample vector bundles. We will come back later on this bound.

3. Further results and a conjecture

There are examples, for every $n \geq 2$, of uniform but non-homogeneous vector bundles on $\mathbb{P}^n$ of rank $2n$ [6]. However it is a long standing conjecture that every uniform vector bundle of rank $r < 2n$ is homogeneous. Homogeneous vector bundles of rank $r < 2n$ on $\mathbb{P}^n$ are classified [2], so the conjecture can be formulated as follows:

Conjecture 1. *Every rank $r < 2n$ uniform vector bundle on $\mathbb{P}^n$ is a direct sum of bundles chosen among: $S^2T_{\mathbb{P}^2}(a)$, $\wedge^2 T_{\mathbb{P}^4}(b)$, $T_{\mathbb{P}^n}(c)$, $\Omega_{\mathbb{P}^n}(d)$, $\mathcal{O}_{\mathbb{P}^n}(e)$; where $a, b, ..., e$ are integers.*

The conjecture holds true if $n \leq 3$ [10,3].

Before to go on we point out an obvious but useful remark.

Remark 16. Clearly an exact sequence (1) exists if and only if the dual sequence twisted by $\mathcal{O}(-1)$ exists. So we may replace $\mathcal{E}$ by $\mathcal{E}^\vee(-1)$. If $\mathcal{E}$ has splitting type $(-1^c, 0^b)$, $\mathcal{E}^\vee(-1)$ has splitting type $(0^c, -1^b)$.

Proposition 17. *(1) Take r, n such that $n \leq r < 2n$. Assume $a - r < n$ and that every rank r uniform bundle on $\mathbb{P}^n$ is homogeneous. Then $l(r; a) \leq n$, except if $r = n, a = 2n-1$ in which case $l(n; 2n-1) = n + 1$.*

(2) Assume Conjecture 1 is true. Then $l(r;a) = a - r + 1$ for $r < (2a+2)/3$, except if $r = (a+1)/2$, in which case $l(r;a) = a - r + 2$.

Proof. (1) In order to prove the statement it is enough to show that there exists no subspace M of constant rank r and dimension $n+1$ under the assumption $a - r < n$, $n \leq r < 2n$ (except if $r = n, a = 2n-1$, in which case $l(n; 2n-1) = n+1$ by Lemma 10).

Such a space would give an exact sequence (1) with $\mathcal{E}$ uniform of rank $r < 2n$ on $\mathbb{P}^n$. If $\mathcal{E}$ is a direct sum of line bundles, by Lemma 6 we get $\dim(M) = n+1 \leq a-r+1 < n+1$. Hence $\mathcal{E}$ is not a direct sum of line bundles. Since the splitting type of $\mathcal{E}$ is $(-1^c, 0^{r-c})$ (Lemma 2), we see that: $\mathcal{E} \simeq \Omega(1) \oplus k.\mathcal{O} \oplus (r-k-n).\mathcal{O}(-1)$, $\mathcal{E} \simeq T(-2) \oplus t.\mathcal{O} \oplus (r-t-n).\mathcal{O}(-1)$, or, if $n = 4$, $\mathcal{E} \simeq (\wedge^2\Omega)(2)$.

Let's first get rid of this last case. The assumption $a - r < n$ implies $a \leq 9$. It is enough to show that there is no exact sequence (1) on $\mathbb{P}^4$, with $\mathcal{E} = (\wedge^2\Omega)(2)$ and $a = 9$. From $0 \to \mathcal{E} \to 9.\mathcal{O} \to E \to 0$, we get $\mathcal{C}(E) = \mathcal{C}(\mathcal{E})^{-1}$. From the Koszul complex we have $0 \to \mathcal{E} \to \wedge^2 V \otimes \mathcal{O} \to \Omega(2) \to 0$. It follows that $\mathcal{C}(E) = \mathcal{C}(\Omega(2))$. Since $rk(E) = 3$ and $c_4(\Omega_{\mathbb{P}^4}(2)) = 1$, we get a contradiction.

So we may assume $\mathcal{E} \simeq \Omega(1) \oplus k.\mathcal{O} \oplus (r-k-n).\mathcal{O}(-1)$ or $\mathcal{E} \simeq T(-2) \oplus t.\mathcal{O} \oplus (r-t-n).\mathcal{O}(-1)$. By dualizing the exact sequence (1), we may assume $\mathcal{E} \simeq \Omega(1) \oplus k.\mathcal{O} \oplus (r-k-n).\mathcal{O}(-1)$. The exact sequence (1) yields:

$$0 \to \Omega(1) \oplus (r-n-k).\mathcal{O}(-1) \to (a-k).\mathcal{O} \to E \to 0 \tag{2}$$

Since $H^i_*(\Omega) = 0$ for $2 \leq i \leq n-1$, from the exact sequence (2) we get $H^i_*(E) = 0$, for $1 \leq i \leq n-2$. Since $rk(E) = a - r < n$, it follows from Evans–Griffith's theorem that $E \simeq \bigoplus \mathcal{O}(a_i)$. We have $a_i \geq 0$, $\forall i$, because E is globally generated. Moreover one a_i at least must be equal to 1 (otherwise $h^1(E^\vee \otimes \mathcal{E}) = 0$ and the sequence (2) splits, which is impossible). So $a_1 = 1, a_i \geq 0, i > 1$. It follows that $h^0(E) \geq (n+1)+(a-r-1) = n+a-r$. On the other hand $h^0(E) = a - k$ from (2).

If $k < r - n$, we see that one of the a_i's, $i > 1$, must be > 0. This implies $h^0(E) \geq 2(n+1) + (a-r-2) = 2n + a - r$. So $a - k = h^0(E) \geq 2n + a - r$. Since $a \geq a - k$, it follows that $a \geq 2n + a - r$ and so $r \geq 2n$, against our assumption.

We conclude that $k = r - n$ and $E = \mathcal{O}(1) \oplus (a-r-1).\mathcal{O}$. In particular $\mathcal{E} = \Omega(1) \oplus (r-n).\mathcal{O}$ ((2) is Euler's sequence plus some isomorphisms). We turn now to the other exact sequence:

$$0 \to F \to a.\mathcal{O}(-1) \to \Omega(1) \oplus (r-n).\mathcal{O} \to 0 \tag{3}$$

We have $\mathcal{C}(F) = (1-h)^a.\mathcal{C}(\Omega(1))^{-1}$. From the Euler sequence $\mathcal{C}(\Omega(1))^{-1} = 1 + h$. It follows that:

$$\mathcal{C}(F) = (1+h).\left(\sum_{i=0}^{a} \binom{a}{i} (-1)^i h^i\right)$$

Since $rk(F) = a - r < n$, $c_n(F) = 0$. Since $a \geq r \geq n$, it follows that $\binom{a}{n} = \binom{a}{n-1}$. This implies $a = 2n - 1$.

Observe that $r \geq n$ (because $k = r - n \geq 0$). If $r \geq n + 1$, then $rk(F) \leq n - 2$, hence $c_{n-1}(F) = 0$. This implies: $\binom{2n-1}{n-1} = \binom{2n-1}{n-2}$, which is impossible.

We conclude that $r = n$ and $a = 2n-1$, so we are looking at $l(n; 2n-1)$. By Lemma 10 we know that $l(n; 2n - 1) = n + 1$.

This proves (1).

(2) Now we apply (1) by setting $n := a - r + 1$. Clearly $n > a - r$. The condition $n \leq r < 2n$ translates in: $(a + 1)/2 \leq r < (2a + 2)/3$. So, under these assumptions, we get $l(r; a) \leq n = a - r + 1$, except if $r = n$, $a = 2n - 1$. In this latter case we know that $l(n; 2n - 1) = n + 1$ (Lemma 10). We conclude with Lemma 7. □

Since Conjecture 1 is true for $r \leq n + 1$ and $n = 3, r = 5$ [3], we may summarize our results as follows:

Theorem 18.
(1) If $r \leq a/2$, then $l(r, a) = a - r + 1$.
(2) If a is odd, $l(\frac{a+1}{2}; a) = \frac{a+1}{2} + 1$ $(= a - r + 2)$.
(3) If $\frac{(2a+2)}{3} > r \geq \frac{a}{2} + 1$, then $l(r; a) \leq r - 1$.
(4) If a is even: $l(\frac{a}{2} + 1; a) = \frac{a}{2}$ $(= a - r + 1)$.
(5) If $r \geq (2a + 2)/3$, then $l(r, a) \leq 2(a - r) + 1$.
(6) We have $l(5; 7) = 3$ $(= a - r + 1)$.

Proof. (1) This is Proposition 8.

(2) This is Lemma 10.

(3) Set $n = r - 1$. Uniform vector bundles of rank $r = n + 1$ on $\mathbb{P}^n$ are homogeneous. We have $n \leq r < 2n$ if $r \geq 3$ and $a - r < n$ if $r \geq (a/2) + 1$. If $r \leq 2$ and $r \geq (a/2) + 1$, then $a \leq 2$. Hence $r = a = 2$ and $l(2; 2) = 1$. So the assumptions of Proposition 17, (1) are fulfilled. We conclude that $l(r, a) \leq r - 1$.

(4) Follows from (3) and Lemma 7.

(5) This is Proposition 15.

(6) Since uniform vector bundles of rank 5 on $\mathbb{P}^3$ are homogeneous, this follows from Proposition 17 (1) and Lemma 7. □

Remark 19. Point 3 of the theorem improves the previous bound of Beasley but we don't expect this bound to be sharp (see Conjecture 2). Points 4 and 6 also are new. The bound of (5) is so far the best known bound in this range. It is reached for some values of a in the case $r = a - 1$ (Proposition 12), but already in the case $r = a - 2$ we don't know if it is sharp.

It is natural at this point to make the following:

Conjecture 2. *Let a, r be integers such that $(2a+2)/3 > r > (a/2)+1$, then $l(r;a) = a-r+1$.*

Remark 20. This conjecture should be easier to prove than Conjecture 1, indeed in terms of vector bundles it translates as follows: every rank $r < 2n$ uniform vector bundle, $\mathcal{E}$, fitting in an exact sequence (1) on $\mathbb{P}^n$ is homogeneous.

By the way the condition $r < 2n$ seems necessary. If $n = 2$ this can be seen as follows. Consider the following matrix (taken from [20]):

$$\Psi = \begin{pmatrix} 0 & -x_2 & 0 & -x_0 & 0 \\ x_2 & 0 & 0 & -x_1 & -x_0 \\ 0 & 0 & 0 & -x_2 & -x_1 \\ x_0 & x_1 & x_2 & 0 & 0 \\ 0 & x_0 & x_1 & 0 & 0 \end{pmatrix}$$

It is easy to see that Ψ has rank four at any point of $\mathbb{P}^2$, hence we get:

$$0 \to \mathcal{O}(b) \to 5.\mathcal{O}(-1) \xrightarrow{\Psi} 5.\mathcal{O} \to \mathcal{O}(c) \to 0$$

with $\mathcal{E} = Im(\Psi)$ a rank four uniform bundle. On the line L of equation $x_2 = 0$, $\Psi = (f, g)$, where $f : 3.\mathcal{O}_L(-1) \to 2.\mathcal{O}_L$ is given by $\begin{pmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & x_1 \end{pmatrix}$ and where $g : 2.\mathcal{O}_L(-1) \to 3.\mathcal{O}_L$ is given by $\begin{pmatrix} -x_0 & 0 \\ -x_1 & -x_0 \\ 0 & -x_1 \end{pmatrix}$. Clearly f is surjective and by computing the Chern classes the kernel is $\mathcal{O}_L(-3)$. Also g is injective of constant rank two, hence the cokernel is $\mathcal{O}_L(2)$.

It follows that $b = -3, c = 2$ and the splitting type of $\mathcal{E}$ is $(-1^2, 0^2)$. Now rank four homogeneous bundles on $\mathbb{P}^2$ are classified (Prop. 3, p. 18 of [7]) and are direct sum of bundles chosen among $\mathcal{O}(a), T(b), S^2T(c), S^3T(d)$. If $\mathcal{E}$ is homogeneous the only possibility is $\mathcal{E}(1) \simeq T(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}$, but in this case the exact sequence $0 \to \mathcal{O}(-2) \to 5.\mathcal{O} \to \mathcal{E}(1) \to 0$, would split, which is absurd. We conclude that $\mathcal{E}$ is not homogeneous. In fact $\mathcal{E}$ is one of the bundles found by Elencwajg [11].

Remark 21. The results of this section and the previous one determine $l(r;a)$ for $a \leq 8$, $1 \leq r \leq a$. To get a complete list for $a \leq 10$, we have to show, according to Conjecture 2, that $l(6;9) = l(7;10) = 4$. This will be done in the next section.

4. Some partial results

In the following lemma we relax the assumption $r < 2n$ in Proposition 17 when $c_1(\mathcal{E}(1)) = 1$.

Lemma 22. *Assume we have an exact sequence (1) on $\mathbb{P}^n$, with $rk(F) = a - r < n$ and $c_1(\mathcal{E}(1)) = 1$. Then $a = 2n - 1$ and $r = n$.*

Proof. If $c_1(\mathcal{E}(1)) = 1$, $\mathcal{E}(1)$ has splitting type $(1, 0^{r-1})$. It follows from [13], Prop. IV, 2.2, that $\mathcal{E}(1) = \mathcal{O}(1) \oplus (r-1).\mathcal{O}$ or $\mathcal{E}(1) = T(-1) \oplus (r-n).\mathcal{O}$. From the exact sequence $0 \to F(1) \to a.\mathcal{O} \to \mathcal{E}(1) \to 0$, we get $c_1(F(1)) = -1$. Since $F(1) \hookrightarrow a.\mathcal{O}$, it follows that $F(1)$ is uniform of splitting type $(-1, 0^{a-r-1})$. Since $rk(F) < n$, $F(1) = \mathcal{O}(-1) \oplus (a-r-1).\mathcal{O}$. This shows that necessarily $\mathcal{E}(1) = T(-1) \oplus (r-n).\mathcal{O}$. Now from the exact sequence: $0 \to T(-1) \oplus (r-n).\mathcal{O} \to a.\mathcal{O}(1) \to E(1) \to 0$, we get $\mathcal{C}(E(1)) = (\mathcal{C}(t(-1))^{-1}(1+h)^a$, i.e. $\mathcal{C}(E(1)) = (1-h)(1+h)^a$. Since $rk(E) < n$, we have $c_n(E(1)) = 0$ and arguing as in the proof of Proposition 17, we get $a = 2n-1, r = n$. $\square$

Remark 23. Since we know that $l(n; 2n-1) = n+1$ (Lemma 10), we may, from now on, assume $c_1(\mathcal{E}(1)) \geq 2$.

Since $\mathcal{E}(1)$ is globally generated, taking $r - 1$ general sections we get:

$$0 \to (r-1).\mathcal{O} \to \mathcal{E}(1) \to \mathcal{I}_X(b) \to 0 \tag{4}$$

Here X is a pure codimension two subscheme, which is smooth if $n \leq 5$ and which is irreducible, reduced, with singular locus of codimension ≥ 6, if $n \geq 6$.

Lemma 24. *Assume $n \geq 3$ and $rk(F) < n$. If X is arithmetically Cohen–Macaulay (aCM), i.e. if $H^i_*(\mathcal{I}_X) = 0$ for $1 \leq i \leq n-2$, then F is a direct sum of line bundles.*

Proof. From (4) we get $H^i_*(\mathcal{E}) = 0$ for $1 \leq i \leq n-2$. By Serre duality $H^i_*(\mathcal{E}^\vee) = 0$, for $2 \leq i \leq n-1$. From the exact sequence $0 \to \mathcal{E}^\vee \to a.\mathcal{O}(1) \to F^\vee \to 0$, we get $H^i_*(F^\vee) = 0$, for $1 \leq i \leq n-2$. Since $F^\vee$ has rank $< n$, by Evans–Griffith theorem we conclude that $F^\vee$ (hence also F) is a direct sum of line bundles. $\square$

Proposition 25. *Assume that we have an exact sequence (1) on $\mathbb{P}^4$ with $rk(F) < 4$. Let $(-1^c, 0^{r-c})$ be the splitting type of $\mathcal{E}$. If $r > 4$ and if F is not a direct sum of line bundles, then $c, r-c \geq 4$; in particular $rk(\mathcal{E}) \geq 8$.*

Proof. Assume c or $b := r - c < 4$. By dualizing the exact sequence (1) if necessary, we may assume $b < 4$. We have an exact sequence (4):

$$0 \to (r-1).\mathcal{O} \to \mathcal{E}(1) \to \mathcal{I}_X(b) \to 0$$

where $X \subset \mathbb{P}^4$ is a smooth surface of degree $d = c_2(\mathcal{E}(1))$. If $b < 3$, X is either a complete intersection $(1, d)$ or lies on a hyper-quadric. In any case X is a.C.M. By Lemma 24, F is a direct sum of line bundles.

Assume $b = 3$. From the classification of smooth surfaces in $\mathbb{P}^4$ we know that if $d \leq 3$, then X is a.C.M. Now X is either a complete intersection $(3, 3)$, hence a.C.M. or linked

to a smooth surface, S, of degree $9-d$ by such a complete intersection. If S is a.C.M. the same holds for X. From the classification of smooth surfaces of low degree in $\mathbb{P}^4$, if X is not a.C.M. we have two possibilities:
(i) X is a Veronese surface and S is an elliptic quintic scroll,
(ii) X is an elliptic quintic scroll and S is a Veronese surface.

(i) If $X = V$ is a Veronese surface then we have an exact sequence:

$$0 \to 3.\mathcal{O} \to \Omega(2) \to \mathcal{I}_V(3) \to 0$$

It follows that $\mathcal{C}(\mathcal{E}(1)) = \mathcal{C}(\Omega(2)) = 1 + 3h + 4h^2 + 2h^3 + h^4$. So $\mathcal{C}(F(1)) = (\mathcal{C}(\mathcal{E}(1))^{-1} = 1 - 3h + 5h^2 - 5h^3$. It follows that F (and hence E also) has rank three. From $\mathcal{C}(E(1)) = (1+h)^a.\mathcal{C}(F(1))$ and $c_4(E(1)) = 0$, we get $0 = a(a-5)(a-6)(a-7)$. So $a \leq 7$. Since $a = rk(E) + r$, we get a contradiction.

(ii) If $X = E$ is an elliptic quintic scroll, then we have:

$$0 \to T(-2) \to 5.\mathcal{O} \to \mathcal{I}_E(3) \to 0$$

It follows that $\mathcal{C}(\mathcal{E}(1)) = \mathcal{C}(T(-2))^{-1}$ and $\mathcal{C}(F(1)) = \mathcal{C}(T(-2)) = 1 - 3h + 4h^2 - 2h^3 + h^4$, in contradiction with $rk(F) < 4$. $\square$

Lemma 26. *Assume we have an exact sequence (1) on $\mathbb{P}^4$ with $a - r < 4$. If $r > 4$ and if F is a direct sum of line bundles, then $rk(\mathcal{E}) \geq 8$.*

Proof. If $r = 5$ we conclude with Theorem 18, (3), (6). If $r = 6$, then $a \leq 9$ and it is enough to show that $l(6;9) \leq 4$ i.e. that there is no exact sequence (1) on $\mathbb{P}^4$. In the same way, if $r = 7$ it is enough to show that $l(7;10) \leq 4$.

If $r = 6$, we may assume that the splitting type of $\mathcal{E}$ is $(-1^1, 0^5), (-1^2, 0^4), (-1^3, 0^3)$. By dualizing and by Lemma 22 we may disregard the first case. It follows that $c_1(F) = -7$ or -6. If $r = 7$, in a similar way, we may assume that the splitting type of $\mathcal{E}$ is $(-1^2, 0^5)$ or $(-1^3, 0^4)$. So $c_1(F) = -7$ or -8.

Let $\mathcal{C}(F(1)) = (1 - f_1h)(1 - f_2h)(1 - f_3h)$. We have $\mathcal{C}(E(1)) = (1+h)^a\mathcal{C}(F(1))$. From $c_4(E(1)) = 0$ we get:

$$\psi(a) := a^3 - a^2(4s + 6) + a(12d + 12s + 11) - 12d - 8s - 24t - 6 = 0$$

where $s = -c_1(F(1)) = \sum f_i$, $d = c_2(F(1)) = \sum_{i<j} f_i f_j$, $t = -c_3(F(1)) = \prod f_i$. We have $f_i \geq 0, \forall i$ and $3 \leq s \leq 5$.

We have to check that this equality can't be satisfied for $a = 9, 10$. We have $\psi(9) = 8(42 - 28s + 12d - 37)$. If $\psi(9) = 0$ we get $3 \mid s$. It follows that $s = 3$. So the condition is: $4d - t = 14$. If one of the f_i's is zero, then $t = 0$ and we get a contradiction. So $f_i > 0, \forall i$ and the only possibility is $(f_i) = (1, 1, 1)$, but then $4d - t = 11 \neq 14$.

For $a = 10$, we get $\psi(10) = 504 - 288s + 108d - 24t$. If $f_1 = f_2 = 0$, then $d = t = 0$ and we get $s = 504/288$ which is not an integer. If $f_1 = 0$, then $t = 0$, $d = f_2 f_3$, $s = f_2 + f_3$. If $s \geq 4$, $504 + 108d = 288s \geq 1152$. It follows that $d \geq 6$. If $d = 6$ we have necessarily $s = 5$ and $\psi(10) \neq 0$. So $s = 3$ and $d = 2$, but also in this case $\psi(10) \neq 0$. We conclude that $f_i > 0, \forall i$. So we are left with $(f_i) = (1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,2,2)$. In any of these cases one easily checks that $\psi(10) \neq 0$. $\square$

Corollary 27. *We have $l(6;9) = l(7;10) = 4$. In particular $l(r,a)$ is known for $a \leq 10$ and $1 \leq r \leq a$ and Conjecture 2 holds true for $a \leq 10$.*

Proof. We have seen that $l(6;9), l(7;10) \leq 4$, by Lemma 7 we have equality. Then all the other values of $l(r;a)$ are given by Theorem 18, Proposition 12 and Proposition 13, if $a \leq 10$. $\square$

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