Eötvös Loránd University

Faculty of Science

Doctoral School of Physics

Statistical Physics, Biological Physics and Physics of Quantum Systems

Stochastic models in complex system

Application of linear algebraic methods to some concrete problems

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Statements of the Thesis

1 Random hopping particle queuing networks

In various real life queuing networks, congestion phenomena have a terrific impact on the performance of these networks. Of that reason, the field witnessed an enormous research effort to understand, control and avoid congestion. Despite of that fact, only a little is known about the connection between these phenomena and the topology of the networks. The first quarter of the Doctoral Thesis seeks connections between the topology of a network and the appearance of congestion in a minimal model.

The properties of the model are the following. The overall number of the particles being prevalent in the network varies in time - particles can arrive to the network from the environment and they can also be absorbed at the nodes of the queuing networks. The routing of the particles follows the stochastic rules of random hopping: particles can jump from one node only to an adjacent one. The direction of the jump is stochastic whose transition probabilities are fixed and do not depend on the actual state of the network. The model evolves in discrete time steps. In one time step, a queue can serve at most one particle. If a particle enters to a queue it remains there until all the particles previously arrived to that queue are served and eventually moved away or absorbed. At a fixed level of the absorption rate, increasing the number of particles arriving to the network from the environment in one time step, the network can move toward the congested phase. In the congested phase, the expectation value of the number of the particles being in the network increase linearly with time on the average and the queue lengths do not have a stationary distribution. My results are the following.

1. It is a well known fact that without the application of congestion control, the stationary distribution of the system in the uncongested phase factorizes to a product form with respect to the nodes of the network. In this case, the stationarity condition is the solvability of the balance equaiton. Fixing the values of the external load, the balance equation contains only one dynamical quantity per node, the stationary probability of the queue of that node to being occupied by at least one particle. Collecting these probabilities into a single vector $\boldsymbol{\xi}$, the balance equation is

$$(\mathbb{1} - \boldsymbol{P})\boldsymbol{\xi} = \boldsymbol{p},$$

where $\boldsymbol{P}$ describes the routing in a new synchronous model that I have introduced to decrease the complexity of the simulation of the original system and $\boldsymbol{p}$ is the external load of the same model. By using the equation, I have given a simple criterion whether a single vector $\boldsymbol{p}$ determine a congested or an uncongested phase. I have given a simple algorithm for the calculation of the order parameter for any $\boldsymbol{p}$. I have validated these results with simulations observing a fairly good agreement between the theory and the simulations.

2. In order to find connections between the long time asymptotic behaviour of the random hopping particle queuing network and the structure of its graph, I have performed the approximate, analytical inversion of the balance equation. The method is based on the approximation of various terms in the spectral decomposition of the matrix $\boldsymbol{P}$. It has been

shown previously that in the case of graph ensembles, using homogeneous external load, the critical loading is

$$p_c = \frac{\mu}{\mu + (1-\mu)d_{\max}/\overline{d}}, \tag{1}$$

where μ is the absorption, p is the arrival probability and $d_{\max}$ with $\overline{d}$ are the maximal and average value of the degree of the nodes in the network, respectively. But if someone examine individual graphs, instead of graph ensembles, one finds that the performance of (1) can be rather poor. My method shows that (1) correspond to neglecting the first genuine graph specific terms of the spectral decomposition of $\boldsymbol{P}$. I have shown how the approximation can be improved to obtain as high accuracy as $10^{-3} \ldots 10^{-5}$ relative error in p_c. I have also examined tree graphs that were previously unmentioned in the literature. With the new formulae in hand, I was able to achieve my original goals - it turned out that beyond equation (1) which suggests that p_c can be calculated based on the knowledge of the degree sequence of the graph, the higher precision needs a better understanding of the structure of the neighbourhood of the nodes.

3. I have preformed numerical calculations on Erdős–Rényi and Barabási–Albert graphs in order to clarify the conditions of the applicability of the formulae mentioned above. I have found numerical evidence that the significant graph parameter here is the mixing rate of the graph. The numerical calculations have shown that for an individual graph, equation (1) is a valid approximation of p_c if the edge density of the graph is relatively high (a rarely fulfilled condition when one considers real world graphs).

2 Examination of an SIS process with stochastic stopping time on the complete graph

Consider an SIS process taking place on the complete graph K_n. It is a well known fact that the prevalence curves obtained by various mean field approximations and which describe the parameter dependence of the long time limit of the expectation value of the prevalence cannot be calculated in the framework of the theory of Markov processes, if the population has a finite size. The reason behind this is the existence of an absorbing state in the state space of the SIS process. When the population is finite, the system will sooner or later reach the absorbing state from which it cannot jump out. To eliminate - at least partly - the effects of the presence of the absorbing state, I have introduced a stochastic stopping time of the process which can be also interpreted as a lifetime of the interconnections that are represented by K_n.

4. I have shown that whenever the stopping time has an exponential distribution, the expectation value of the prevalence on the final state can be calculated by the determination of the resolvent of the infinitesimal generator of the SIS process. I have shown that this expectation value is proportional to a similar expectation value of the modified SIS process. The non-trivial proportionality stands even in the $n \to \infty$ limit.

5. I have shown, that if the normalized infection rate τ is less than the curing rate δ, then the expectation value of the prevalence in the final state of the system vanishes if $n \to \infty$, independently of the average stopping time. I have shown that the same conditions imply that the stationary distribution of the modified SIS process has a vanishing average prevalence.

6. I have shown, that if the normalized infection rate τ is less than the curing rate δ, the unconstrained, full time evolution of the expectation value of the prevalence of the modified SIS process as well as the original SIS process has the everywhere vanishing function as their point-wise limit. I have shown that the role of the exponential distribution is not restrictive, choosing another distribution leads to the same result concerning the behaviour of the expectation value of the prevalence in the $n \to \infty$ limit.

3 The euclidean Dirac equation in higher spatial dimension

It is an old observation that the continuum limit of the one dimensional persistent random walk can be described by the free euclidean Dirac equation. Since a solution of the Dirac equation is always a solution of the telegrapher's equation, it seemed plausible that in higher spatial dimension the persistent random walk can serve as a phenomenological model of spreading processes which take place in complex, disordered materials and whose propagation are described by the telegrapher's equation. Unfortunately, all the efforts that has been made in this direction have been eventually failed: the continuum limit of the generalization of the persistent random walk is neither the Dirac, nor the telegrapher's equation. In the third quarter of the Thesis I will explain why the Dirac equation cannot be modelled by the persistent random walk in higher dimension.

7. I have proved that the euclidean Dirac equation can preserve the positivity of its initial data if and only if $d = 1$. In that case, the form of the equation is completely determined: the Clifford algebra generators are such that the equation describes the uncoupled motion of a multiple number of persistent random walkers.

4 Convection-diffusion process in an electrochemical cell

The surface of a piece of a metal, if it is kept on a constant voltage and placed into an electrolyte solution, can be the arena of various chemical processes that result in charge transfer between the metal and the solution. If the electrolyte gains electrons from the metal, then the process is called reduction. On the contrary, if the metal absorbs electrons, that is the electrolyte looses them, the process is called oxidation. The metal piece is called electrode. By varying the voltage of the electrode, it can be turned to a sensitive analytical tool to study chemical reactions. The rotating ring-disk electrode is such a common instrument in electrochemical inquiries which has a high analytical power. The electrode contains two, well insulated metal parts - the ring and the disk. Under the electrode, due to the rotation of the electrode, the fluid that contains the electrolytes flows in the direction of the surface of the electrode. This enables the reactants to approach the electrode in a controlled way. Unfortunately, the overlap of the electric field corresponding to the electrodes causes an electronic interference between the two which makes the results of the measurements sometimes hard to interpret. This is even more emphasized when the control of the electrode voltages is dynamic. The electronic interference is usually called crosstalk.

8. With my co-authors we built a numerical simulation which can simulate and quantify the so called IR-drop , the main source of the electrical interferences in those electrochemical cells which use rotating ring-disk electrodes with dynamical potential control. We have

demonstrated, both by performing numerical simulations and by conducting experiments that the position of the reference electrode greatly affects the magnitude of the electrical crosstalk. We observed that the numerical simulations and the experiments are in a fair agreement with each other.

Publications

The Doctoral Thesis is based on the following publications that appeared in peer-reviewed journals:

1. **Barankai, N.**, Fekete, A., Vattay, G. (2012). Effect of network structure on phase transitions in queuing networks. Physical Review E, 86(6), 066111.

2. Vesztergom, S., **Barankai, N.**, Kovács, N., Ujvári, M., Siegenthaler, H., Broekmann, P., Láng, G. G. (2016). Electrical cross-talk in four-electrode experiments. *Journal of Solid State Electrochemistry*, 20(11), 3165-3177.

3. Vesztergom, S., **Barankai, N.**, Kovács, N., Újvári, M., Broekmann, P., Siegenthaler, H., Láng, G. G. (2016). Electrical cross-talk in rotating ring–disk experiments. Electrochemistry Communications, 68, 54-58.

The material of the following publications are also incorporated in the Doctoral Thesis. These publications are currently under scrutiny in the peer-review process of the Journal of Statistical Mechanics and of the Journal of Mathematical Physics.

4. **Barankai, N.** and Stéger J., (2017). The SIS process in populations with exponential decay.

5. **Barankai N.**, On the positivity preservation of the free Euclidean Dirac equation, *arXiv:1612.05055*

The following publications are out of the scope of the Thesis but are mentioned in the sake of completeness:

6. Kallus, Z., **Barankai, N.**, Szüle, J., Vattay, G. (2015). Spatial fingerprints of community structure in human interaction network for an extensive set of large-scale regions. *PloS one*, 10(5), e0126713.

7. Vesztergom, S., **Barankai, N.**, Kovács, N., Ujvári, M., Wandlowski, T., Láng, G. G. (2014). Rotating ring–disk electrode with dual dynamic potential control: Theory and practice. *Acta Chim. Slovenica*, 61, 223-232.

8. Kallus, Z., **Barankai, N.**, Kondor, D., Dobos, L., Hanyecz, T., Szüle, J., Csabai, I. (2013). Regional properties of global communication as reflected in aggregated Twitter data. In *Cognitive Infocommunications (CogInfoCom), 2013 IEEE 4th International Conference on* (pp. 429-434). IEEE.

9. Kondor, D., Csabai, I., Dobos, L., Szüle, J., **Barankai, N.**, Hanyecz, T., Vattay, G. (2013). Using Robust PCA to estimate regional characteristics of language use from geotagged Twitter messages. In *Cognitive Infocommunications (CogInfoCom), 2013 IEEE 4th International Conference on* (pp. 393-398). IEEE.