A PRESCRIBED ANISOTROPIC MEAN CURVATURE EQUATION MODELING THE CORNEAL SHAPE: A PARADIGM OF NONLINEAR ANALYSIS

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Abstract. In this paper we survey, complete and refine some recent results concerning the Dirichlet problem for the prescribed anisotropic mean curvature equation

$$-\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = -au + b/\sqrt{1+|\nabla u|^2},$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, with $a, b > 0$ parameters. This equation appears in the description of the geometry of the human cornea, as well as in the modeling theory of capillarity phenomena for compressible fluids. Here we show how various techniques of nonlinear functional analysis can successfully be applied to derive a complete picture of the solvability patterns of the problem.

Contents

2010 *Mathematics Subject Classification.* Primary: 35J93, 35J25, 35J62, 35J67; Secondary: 35B09, 35B51, 35J20.

Key words and phrases. Prescribed anisotropic mean curvature equation, positive solution, Dirichlet boundary condition, generalized solution, classical solution, singular solution, existence, uniqueness, regularity, boundary behaviour, bounded variation function, implicit function theorem, topological degree, variational method, lower and upper solutions.

This paper was written under the auspices of INdAM-GNAMPA. The third and the fourth named authors have also been supported by the University of Trieste, in the frame of the 2015 FRA project "Differential Equations: Qualitative and Computational Theory".

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1. Introduction. The aim of this paper is to survey, complete and refine some results, recently obtained in [46, 47, 48, 52, 9, 50, 51, 10, 11], concerning existence, uniqueness, regularity and boundary behaviour of the solutions of the Dirichlet problem for the quasilinear elliptic equation

$$- \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \frac{b}{\sqrt{1+|\nabla u|^2}} \quad \text{in } \Omega, \tag{1.1}$$

where $a, b > 0$ are given constants and Ω is a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. We remark that the case $N = 1$ has been treated separately in [9]. Notice that (1.1) is a particular case of the general prescribed anisotropic mean curvature equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = NH(x, u, \mathcal{N}(u)) \quad \text{in } \Omega,$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}$ is the prescribed mean curvature and $\mathcal{N}(u) = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}$ is the unit upper normal to the graph of u in $\mathbb{R}^{N+1}$.

Equation (1.1) has been introduced either for modeling capillarity phenomena for compressible fluids, if supplemented with non-homogeneous conormal boundary conditions [16, 17, 4, 18, 3], or for describing the geometry of the human cornea, if supplemented with homogeneous Dirichlet boundary conditions [46, 47, 48, 52, 50, 51]. We refer to these papers for the derivation of the model, further discussion on the subject and an additional bibliography.

Besides the interest that this study has in view of the cited application, it will become evident from our subsequent discussion that this problem turns out to be very challenging also from the purely mathematical point of view, as it can be considered as a paradigm for the use of various methods of nonlinear analysis, such as the implicit function theorem, topological degree, calculus of variations, upper and lower solutions, combined with some techniques from the theory of linear and quasilinear elliptic partial differential equations and based on regularity theory, gradient estimates, use of barriers and comparison principles.

As anticipated above, here we discuss the solvability of the homogeneous Dirichlet problem for equation (1.1), that is,

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It should be pointed out that in [46, 47, 48, 52, 51] only a simplified version of (1.2) has been investigated, where the curvature operator

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

is replaced by its linearization around 0

$$\mathrm{div}(\nabla u) = \Delta u$$

and, furthermore, Ω is supposed to be an interval in $\mathbb{R}$, or a disk in $\mathbb{R}^2$. In the two papers [9, 10] we have instead considered the complete model (1.1) and we have proved the existence of a unique classical solution for any given choice of the positive parameters a, b, but still assuming that Ω is an interval in $\mathbb{R}$, or a ball in $\mathbb{R}^N$. Some numerical experiments for approximating the solution of the 1-dimensional problem have been performed in [9, 50]. Later on, in [11], we tackled the problem in arbitrary Lipschitz domains and we proved, for all $a, b > 0$, the existence and the uniqueness of a generalized solution, which is regular in the interior, but attains the Dirichlet boundary data classically only under an additional condition that relates the values of the parameters with the geometry of the domain. This is however not so much surprising. Indeed, it is a known fact that the solvability in the classical sense of the Dirichlet problem for the prescribed mean curvature equation

$$-\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = NH(x) \quad \text{in } \Omega, \tag{1.3}$$

as well as for the capillarity equation

$$-\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au \quad \text{in } \Omega, \tag{1.4}$$

with $a > 0$, is intimately related to the geometric properties of $\partial\Omega$. In [54] J. Serrin established a basic criterion for the solvability of the Dirichlet problem for the basic equations (1.3) and (1.4): a mean convexity assumption on $\partial\Omega$, introduced in [31, 54], was shown to be sufficient, and in a suitable sense also necessary, for the existence of a classical solution. In [54, p. 480] J. Serrin also emphasized "the delicacy of the situation when any but the simplest equations are treated".

When applying these ideas to the homogeneous Dirichlet problem for (1.1), they yield its solvability assuming a smallness condition on the coefficient b and an appropriate version of the Serrin's mean convexity condition on $\partial\Omega$: see, respectively, assumptions (2) and (3) in [40]. In [6, Remark 1] it was stated, yet without an explicit proof, that using the methods of [5] the mean convexity assumption might be suitably relaxed, allowing boundary points with negative mean curvature, at the expense however of requiring some smallness conditions both on the coefficients of the equation and on the size of the domain. We also refer to [29, 28, 30] and to the papers cited therein for further recent studies on the existence and the boundary behaviour of solutions of the Dirichlet problem for the prescribed mean curvature equation (1.2) in case the Serrin's condition is not satisfied.

In the light of this discussion the need of considering generalized solutions in this context becomes apparent, being dictated by the possible occurrence of singular solutions, namely solutions that are regular in the interior, but do not attain the Dirichlet condition at some points of the boundary, where in addition the normal derivative blows up. Following some ideas which trace back to some works of the seventies by A. Lichnewsky and R. Temam, or respectively by E. Giusti and M. Miranda, dealing with the prescribed mean curvature equation, we might define a solution as a minimizer of some related convex action functional; such solutions have been referred to as "pseudo-solutions" in [55, 33, 34, 35, 36, 13], or respectively as "generalized solutions" in [41, 24, 25, 42]. Yet, although (1.1) has a variational structure, the introduction of the associated action functional, which involves an anisotropic area term, does not appear very direct and the corresponding concepts of "pseudo-solution" and of "generalized solution" are not very transparent. Therefore we prefer to adopt in our context an equivalent notion of solution, which looks more in the spirit of classical solutions and has in our opinion a more intuitive geometric interpretation. It is worthy to point out at this stage that our definition of solution is somehow implicit in the work of A. Lichnewsky [35], concerning the minimal surface equation. Indeed, in [35, Proposition 4] the author introduces a concept of lower and upper solutions that precisely yields our notion of solution for any function that is simultaneously a lower and an upper solution of the problem.

The following notion of generalized solution for problem (1.2), partially inspired by [55, 34, 35, 36, 25, 42, 13], is therefore introduced.

Definition 1.1. A function $u \in W^{1,1}(\Omega)$ is a *generalized* solution of (1.2) if the following conditions hold:

- $\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^N(\Omega)$;
- u satisfies the equation in (1.2) a.e. in Ω;
- for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$,
  - either $u(x) = 0$,
  - or $u(x) > 0$ and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big](x) = -1$,
  - or $u(x) < 0$ and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big](x) = 1$,

  where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big] \in L^\infty(\partial\Omega)$ is the weakly defined trace on $\partial\Omega$ of the component of $\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ with respect to the unit outer normal ν to Ω.

Definition 1.2. A generalized solution u of (1.2) is *classical* if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u(x) = 0$ for all $x \in \partial\Omega$.

Definition 1.3. A generalized solution u of (1.2) is *singular* if it is not classical.

Remark 1. Assuming that $u \in W^{1,1}(\Omega)$ is such that $\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^N(\Omega)$ and satisfies the equation in (1.2) a.e. in Ω is equivalent to requiring that $u \in W^{1,1}(\Omega) \cap L^N(\Omega)$ and is a distributional solution of the equation in (1.2). Note that, according to [2], the vector field $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ belongs to the space $X(\Omega)_N$ and thus the weak trace $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right]$ on $\partial\Omega$ of the component of $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ with respect to the unit outer normal ν to Ω is defined.

The concept of solution expressed by Definition 1.1 looks rather natural in this context and can heuristically be interpreted as follows: the solution u is not required to satisfy the homogeneous Dirichlet boundary condition at all points of $\partial\Omega$, but at any point of $\partial\Omega$ where the zero boundary value is not attained the unit upper normal $\mathcal{N}(u)$ to the graph of u equals the unit outer normal $(\nu, 0)$ or the unit inner normal $(-\nu, 0)$, according to the sign of u; in this case, roughly speaking, the graph of the solution might be smoothly continued by vertical segments up to the zero level. This kind of boundary behaviour for solutions of the N-dimensional prescribed mean curvature equation has already been observed and discussed in [35, 24, 25, 42, 13]; more recently, but limited to dimension $N = 1$, it has been considered in [7, 8, 49, 45, 37, 38].

For the readers' convenience we plot in Figure 1 the graph of a generalized – singular, indeed – solution.

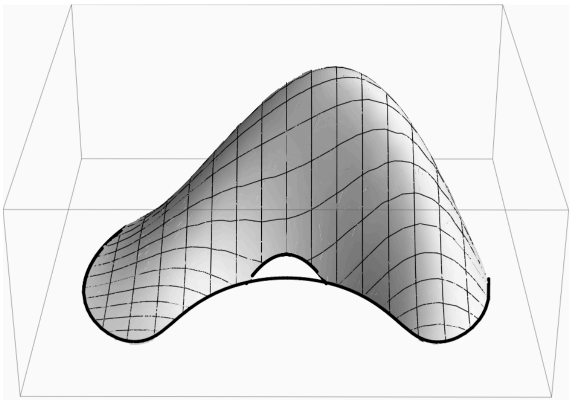

FIGURE 1. Graph of a generalized solution on an arbitrary domain.

With reference to Definition 1.1 we are able to obtain various existence, uniqueness, regularity and even stability results for problem (1.2), also showing how different analytic techniques can successfully be applied to derive a complete picture of its solvability patterns.

With respect to our previously published papers [9, 10, 11], the results in Section 2 and in Section 6 are completely new, while Section 3 and Section 5 include some new statements or proofs. Section 4 is instead basically reproduced from [11].

The remainder of this introduction is devoted to describe the contents of this paper in a schematic fashion. We point out that the presentation below, being aimed to provide a clear snapshot of our results, intentionally does not follow the structure of the rest of this paper, which instead needs a slightly different organization dictated by reasons of internal logic.

Radially symmetric solutions. This topic is discussed in Section 3 and in Section 6. Let us notice that the equation in (1.2) is invariant under orthogonal transformations, that is, if u is a solution of (1.2) and $\mathbb{U}(\Omega) = \Omega$ for some $\mathbb{U} \in \mathbb{O}(N)$, $\mathbb{O}(N)$

denoting the orthogonal group in $\mathbb{R}^N$, then $u^* = u \circ \mathbb{U}$ is still a solution. Therefore it is natural, as a first step, to look for radially symmetric solutions of (1.2) whenever the domain is either a ball, or a spherical shell. However the solvability patterns in the two cases are quite different; indeed, while in the former case we always find classical solutions, in the latter case singular solutions, not attaining the zero boundary value on the interior sphere, may appear.

In this context one looks for solutions of the form $u(x) = v(|x - x_0|)$ and the equation in (1.2) writes

$$-\left(\frac{t^{N-1}v'}{\sqrt{1+v'^2}}\right)' = t^{N-1}\left(-av + \frac{b}{\sqrt{1+v'^2}}\right), \qquad t = |x - x_0|. \tag{1.5}$$

Classical solutions on balls. This is the content of Section 3. Let $B = B(x_0, R)$ be the open ball in $\mathbb{R}^N$ of center x_0 and radius R. In this case one can exploit the validity of a one-sided Nagumo condition for the equivalent ordinary differential equation (1.5) and obtain an a priori estimate on the gradients of its possible solutions satisfying the mixed boundary condition $v'(0) = 0$, $v(R) = 0$ on the interval $[0, R]$. Then a standard application of the Leray-Schauder continuation theorem (or even of the shooting method, like in [10]) yields the existence of classical solutions.

Theorem 1.4. *For every $a > 0$, $b > 0$, there exists a unique generalized solution u of (1.2), with $\Omega = B$, which is radially symmetric and classical, with $u \in C^2(\overline{B})$. Moreover, there exists a function $v \in C^2([0, R])$, with $u(x) = v(|x - x_0|)$ for all $x \in \overline{B}$, such that*

- $0 < v(t) < b/a$ *for all* $t \in [0, R[$*;*
- $v'(t) < 0$ *for all* $t \in]0, R]$*;*
- $v''(t) < 0$ *for all* $t \in [0, R]$*.*

Singular solutions on thick spherical shells. This topic is discussed in Section 6. Let $S = S_{r,R}(x_0) = \{x \in \mathbb{R}^N \mid r < |x - x_0| < R\}$ be the spherical shell centered at x_0 and having radii r, R, with $0 < r < R$, and, as above, let $B = B(x_0, R)$ be the open ball in $\mathbb{R}^N$ of center x_0 and radius R. On thick spherical shells singular solutions may appear when $b > 0$ is large. In Figure 2 the graph of a possible singular solution is plotted.

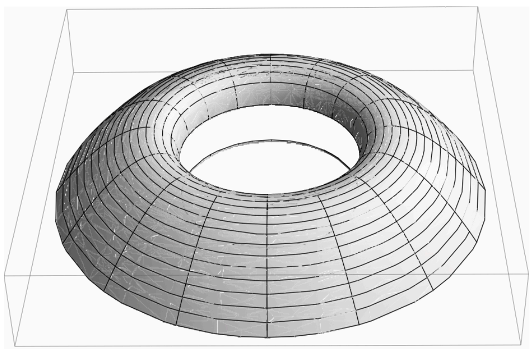

FIGURE 2. Graph of a singular solution on a thick spherical shell.

Theorem 1.5. *For any given $N \geq 2$, $a > 0$ and $r > 0$, there exist $R^* > 0$ and $b^* > 0$ such that, for all $R > R^*$ and $b > b^*$, there is a unique generalized solution u of* (1.2)*, with $\Omega = S$, which is radially symmetric, singular and satisfies*

$$\begin{aligned} &u \in C^2(S \cup \partial B),\\ &u(x) = 0 \ \text{ if } |x - x_0| = R,\\ &u(x) > 0 \ \text{ and } \Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big](x) = -1 \ \text{ if } |x - x_0| = r. \end{aligned}$$

Classical solutions on thin spherical shells. This topic is discussed in Section 6 too. The situation is instead very different if the spherical shell is thin: in this case solutions are indeed classical. Our result shows in particular that the conclusions of Theorem 1.5 fail if R is not bounded away from r, thus showing the sharpness of such statement.

Theorem 1.6. *For any given $N \geq 2$, $a > 0$, $b > 0$ and $r > 0$, there exists $R_* > 0$ such that, for all $R \in \,]r, R_*[$, there is a unique generalized solution u of* (1.2)*, with $\Omega = S$, which is radially symmetric and classical, with $u \in C^2(\overline{S})$.*

Small classical solutions on arbitrary domains. This is the content of Section 2. If Ω is an arbitrary bounded regular domain in $\mathbb{R}^N$, we are able in a rather elementary fashion to establish the existence and the uniqueness of small classical solutions, as well as to describe the structure of the solution set. Here the uniqueness is achieved by rewriting the problem as a variational inequality and exploiting the monotonicity of the zero order term, the existence of small classical solutions follows from the implicit function theorem, the existence of a maximal connected set of classical solutions emanating from the line of trivial solutions is proved via topological degree. Figure 3 graphically describes the content of Theorem 1.7.

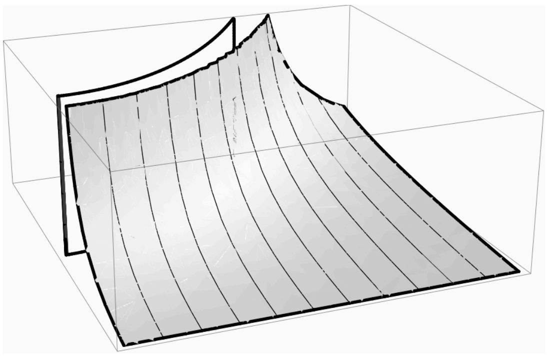

FIGURE 3. Classical solutions emanating from the trivial line: $\|\nabla u(a,b)\|_\infty$ is plotted, in applicates, versus a, in abscissas, and b, in ordinates.

Theorem 1.7. *Let Ω be a bounded domain in $\mathbb{R}^N$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in \,]0,1[$. Then, there exists a set*

$$\mathcal{E} = \bigcup_{a>0} (\{a\} \times [0, b_\infty(a)[) \subseteq \mathbb{R}^+_0 \times \mathbb{R}^+$$

such that, for any $(a, b) \in \mathcal{E} \cap (\mathbb{R}^+_0 \times \mathbb{R}^+_0)$, problem (1.2) has a unique generalized solution $u = u(a, b) \in C^{2,\alpha}(\overline{\Omega})$, which is classical, asymptotically stable, smoothly depends on the parameters (a, b) in the topology of $C^{2,\alpha}(\overline{\Omega})$, and satisfies, for every $a > 0$,

$$\lim_{b\to 0} \|u(a, b)\|_{C^{2,\alpha}} = 0$$

and, in case $b_\infty(a) < +\infty$,

$$\limsup_{b\to b_\infty(a)} \|\nabla u(a, b)\|_\infty = +\infty.$$

Generalized solutions on arbitrary domains. This is the content of Section 4. The proof of the existence of generalized solutions, which we basically reproduce from [11], is both conceptually and technically quite elaborate. It requires the study, in the space of bounded variation functions, of a suitable action functional, involving an anisotropic area term, whose minimizers give rise, via a change of variables, to the generalized solutions. The interior regularity of these bounded variation minimizers is obtained by combining a delicate approximation scheme with a "local" existence result of Serrin's type proven in [40] and with the classical gradient estimates of Ladyzhenskaya and Ural'tseva [32].

More precisely, we start from the observation, already made in [16, 17, 4, 18, 5, 3], that equation (1.1) can formally be seen as the Euler equation of the functional

$$\int_\Omega e^{-bu}\sqrt{1 + |\nabla u|^2}\, dx - \frac{a}{b}\int_\Omega e^{-bu}\big(u + \tfrac{1}{b}\big)\, dx, \tag{1.6}$$

which involves the anisotropic area functional $\int_\Omega e^{-bu}\sqrt{1 + |\nabla u|^2}\, dx$. The natural change of variable $v = e^{-bu}$ transforms problem (1.2) into

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\right) = -a\log(v) - \dfrac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} & \text{in } \Omega,\\ v = 1 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

and the functional in (1.6) into

$$\int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}\, dx + \frac{a}{b^2}\int_\Omega v\,(\log(v) - 1)\, dx.$$

As the first term $\int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}\, dx$ of this functional grows linearly with respect to the gradient term, the appropriate framework where to settle its study is the space of bounded variation functions. Therefore we denote by $\int_\Omega \sqrt{v^2 + b^{-2}|Dv|^2}$ the relaxation of $\int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}\, dx$ from $W^{1,1}(\Omega)$ to $BV(\Omega)$ and we define the functional

$$\mathcal{J}(v) = \int_\Omega \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\Omega} |v - 1|\, d\mathcal{H}^{N-1},$$

where as usual (see, e.g., [26]) the term $\frac{1}{b}\int_{\partial\Omega} |v-1|\, d\mathcal{H}^{N-1}$ is introduced in order to take into account of the non-homogeneous Dirichlet boundary conditions in (1.7).

Our aim is to find a solution of (1.7) by minimizing, on the cone $BV^+(\Omega)$ of all non-negative functions in $BV(\Omega)$, the functional

$$\mathcal{I}(v) = \mathcal{J}(v) + \int_\Omega F(v)\, dx,$$

where $F(s)$ denotes the continuous extension of the function $\frac{a}{b^2}s(\log(s)-1)$ onto $[0,+\infty[$.

To carry on our argument we first need to prove various facts about $\mathcal{I}$, such as an alternative representation formula, its convexity, its Lipschitz continuity with respect to the norm of $BV(\Omega)$ and its lower semicontinuity with respect to the L^1-convergence in $BV(\Omega)$, as well as a lattice property, encoding a kind of maximum principle. We also prove a delicate approximation result, which plays a crucial role in the sequel of the proof. Once this preliminary study is completed we show the existence of a global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$. This positive minimizer v is, by the convexity of $\mathcal{I}$, unique, and it is bounded and bounded away from zero; moreover, v is the unique solution of an equivalent variational inequality.

Next we prove the interior regularity of v. This exploits an argument, which was introduced in [20] and used, e.g., in [21, 22, 3] for the study of capillarity problems. The procedure can be summarized as follows. We fix a point $x_0 \in \Omega$ and a small open ball B centered at x_0 and compactly contained in Ω. We take a sequence $(v_n)_n$ of regular functions approximating v and satisfying $\mathcal{J}(v_n) \to \mathcal{J}(v)$, whose existence is guaranteed by the above mentioned approximation property. By a result in [40] we can solve, in the classical sense, a sequence of Dirichlet problems in B for the equation in (1.7), where the boundary values are prescribed on ∂B by the restriction of each function v_n. The gradient estimates obtained in [32] and the extremality properties enjoyed by these solutions allow us to prove their convergence, possibly within a ball of smaller radius, to a regular solution of the equation in (1.7), which by uniqueness coincides with v.

By using again the extremality of v, namely the equivalent variational inequality satisfied by v, we are eventually able to conclude that $u = -\frac{1}{b}\log(v)$ is the desired solution of (1.2) according to Definition 1.1. This solution u is unique, smooth and positive in Ω.

Theorem 1.8. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then, for every $a > 0$, $b > 0$, there exists a unique generalized solution u of problem (1.2), which also satisfies:*

- *$u \in C^\infty(\Omega)$;*
- *$u \in L^\infty(\Omega)$ and $0 < u(x) < b/a$ for all $x \in \Omega$;*
- *u minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional*

$$\mathcal{H}(z) = \int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z+\tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega}|e^{-bz}-1|\,d\mathcal{H}^{N-1}.$$

The extremality property expressed by the last conclusion of Theorem 1.8 is crucial in order to infer the boundary behaviour of the solution, as required by Definition 1.1. It will actually be proved that a function $u \in W^{1,1}(\Omega)$ is a solution according to Definition 1.1 if and only if it minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional $\mathcal{H}$.

We point out that this property further witnesses that all generalized solutions of (1.2) enjoy some form of stability.

Boundary behaviour of generalized solutions. This topic is discussed in Section 5.

A geometric condition. The next theorem guarantees that the solution previously obtained attains the homogeneous Dirichlet boundary values provided that Ω satisfies an exterior sphere condition, in which the radius of the sphere is bounded from below by a constant depending on the coefficients a, b and the dimension N. This goal is achieved by first proving a comparison result valid for pairs of lower and upper solutions of problem (1.7) and then by constructing an appropriate barrier, indeed an upper solution of (1.2), vanishing at x_0. The notion of exterior sphere condition we use is as follows.

Definition 1.9. We say that an open set $\Omega \subseteq \mathbb{R}^N$ satisfies an exterior sphere condition with radius $R > 0$ at some point $x_0 \in \partial\Omega$, if there exists a point $y \in \mathbb{R}^N$ such that

$$B(y, R) \cap \Omega = \emptyset \quad \text{and} \quad x_0 \in \overline{B(y, R)} \cap \partial\Omega,$$

where $B(y, R)$ denotes the open ball of center y and radius r.

It is fairly evident that the exterior sphere condition does not imply the above mentioned Serrin's mean convexity assumption, as it permits that all principal curvatures be negative.

Theorem 1.10. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then, for every $a > 0$, $b > 0$, there exists a unique generalized solution u of* (1.2)*, which also satisfies all the conditions stated in Theorem 1.8 and*

- *at each point $x_0 \in \partial\Omega$ where an exterior sphere condition with radius $R \geq (N-1)\, b/a$ holds, u is continuous and satisfies $u(x_0) = 0$; moreover, if $R > (N-1)\, b/a$, then u also satisfies a bounded slope condition at x_0, that is $\sup_{x \in \Omega} \dfrac{u(x)}{|x - x_0|} < +\infty$.*

In particular, if an exterior sphere condition with radius $r \geq (N-1)\, b/a$ is satisfied at every point $x_0 \in \partial\Omega$, then $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ and it is a classical solution of (1.2).

Clearly, an exterior sphere condition, with arbitrary radius, holds at any point $x_0 \in \partial\Omega \cap \partial\mathrm{Conv}(\overline{\Omega})$, where $\mathrm{Conv}(\overline{\Omega})$ denotes the convex hull of $\overline{\Omega}$. Accordingly, the set of points in $\partial\Omega$, where a generalized solution attains the zero boundary condition, is always non-empty.

Classical versus singular solutions. This topic is discussed in Section 5 as well. Basically combining the previous results we are able to get a rather complete picture of the solution set of (1.2).

Theorem 1.11. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in \,]0,1[$. Then, for every $a > 0$, either for all $b > 0$ problem* (1.2) *has a unique generalized solution $u = u(a,b)$, which is classical, or there exists $b^* = b^*(a) \in \,]0, +\infty[$ such that*

- *if $b \in \,]0, b^*]$, then problem* (1.2) *has a unique generalized solution u, which is classical;*
- *if $b \in \,]b^*, +\infty[$, then problem* (1.2) *has a unique generalized solution u, which is singular.*

In addition, the following conclusions hold:

- *the map $a \mapsto b^*(a)$ is increasing, with $\inf_{a>0} b^*(a) > 0$;*
- *the map $(a,b) \mapsto u(a,b)$ is continuous from $\mathbb{R}^+_0 \times \mathbb{R}^+$ to $L^\infty(\Omega)$;*
- *for any $a > 0$, the map $b \mapsto u(a,b)$ is strictly increasing, in the sense that if $b_1 < b_2$, then $u(a,b_1) < u(a,b_2)$ in Ω;*

- *for any $b > 0$, the map $a \mapsto u(a,b)$ is strictly decreasing, in the sense that if $a_1 < a_2$, then $u(a_1,b) > u(a_2,b)$ in Ω.*

Notations. We conclude this introduction by setting some notations that are used throughout this paper. We write $\mathbb{R}^+$ and $\mathbb{R}^+_0$ to denote the intervals $[0,+\infty[$ and $]0,+\infty[$, respectively. For each $N \geq 2$, we set $1^* = \frac{N}{N-1}$. The characteristic function of any set E is denoted by χ_E. If E is a set in $\mathbb{R}^N$ having positive finite N-dimensional Lebesgue measure and $u, v : E \to \mathbb{R}$ are given functions, we write: $u \leq v$ in E (respectively, a.e. in E) whenever $u(x) \leq v(x)$ for every $x \in E$ (respectively, a.e. $x \in E$); $u < v$ in E if $u \leq v$ in E and $u \neq v$; in case E is closed, $u \ll v$ in E if there is $\varepsilon > 0$ such that $u(x) + \varepsilon \operatorname{dist}(x, \partial E) \leq v(x)$ for all $x \in E$. By $\{v < w\}$ we denote the set $\{x \in E \mid v(x) < w(x) \text{ a.e. in } E\}$. We also define $u \vee v$ and $u \wedge v$ by $(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$ for a.e. $x \in E$. The N-dimensional Lebesgue measure of E is denoted by $|E|$. If E is a set in $\mathbb{R}^N$ having positive finite $(N-1)$-dimensional Hausdorff measure and $u, v : E \to \mathbb{R}$ are given functions, we write $u \leq v$ on E (respectively, $\mathcal{H}^{N-1}$-a.e. on E) whenever $u(x) \leq v(x)$ for every $x \in E$ (respectively, $\mathcal{H}^{N-1}$-a.e. $x \in E$). The symbol δ_{ij} as usual stands for the Kronecker delta.

2. Small classical solutions on arbitrary domains.

In this section we suppose that Ω is an arbitrary bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ satisfying suitable regularity conditions. Our aim here is to establish existence and uniqueness of classical solutions emanating from the line of the trivial solutions, as well as to describe the structure of the solution set: uniqueness is achieved by rewriting the problem as a variational inequality and exploiting the monotonicity of the zero order term, the existence of small classical solutions follows from the implicit function theorem, the existence of a maximal connected set of classical solutions is proved by using topological degree.

2.1. Global uniqueness of classical solutions.

We start proving a uniqueness result.

Lemma 2.1. *Let $a > 0$ and $b > 0$ be given and let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then problem (1.2) has at most one classical solution $u \in C^2(\overline{\Omega})$.*

Proof. The proof consists of two steps.

Step 1. A variational inequality. We show that if $u \in C^2(\overline{\Omega})$ is a solution of (1.2), then $v = \exp(-bu)$ satisfies

$$\int_\Omega \sqrt{w^2 + b^{-2}|\nabla w|^2}\, dx - \int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}\, dx \geq -\int_\Omega ab^{-2} \log(v)\, (w - v)\, dx \tag{2.1}$$

for all $w \in C^1(\overline{\Omega})$ with $\min_{\overline{\Omega}} w > 0$ and $w = 1$ on $\partial\Omega$. Indeed, it is easy to verify that, if $u \in C^2(\overline{\Omega})$ is a solution of (1.2), then $v = \exp(-bu)$ satisfies

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} \right) + \dfrac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} = -a \log(v) & \text{in } \Omega, \\ v = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Pick any $w \in C^1(\overline{\Omega})$, with $\min_{\overline{\Omega}} w > 0$ and $w = 1$ on $\partial\Omega$, multiply the equation in (2.2) by $w - v$ and integrate by parts. The convexity and the differentiability in $\mathbb{R}_0^+ \times \mathbb{R}^N$ of the map $(s, \xi) \mapsto b^2\sqrt{s^2 + b^{-2}|\xi|^2}$ then yield

$$\begin{aligned} -\int_\Omega a \log(v)\,(w - v)\,dx &= \int_\Omega \frac{\nabla v\,\nabla(w - v)}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{b^2\,v\,(w - v)}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx \\ &\le \int_\Omega b^2\sqrt{w^2 + b^{-2}|\nabla w|^2}\,dx - \int_\Omega b^2\sqrt{v^2 + b^{-2}|\nabla v|^2}\,dx. \end{aligned}$$

Step 2. Uniqueness. Let us show that problem (1.2) has at most one solution $u \in C^2(\overline{\Omega})$. Suppose that u_1, $u_2 \in C^2(\overline{\Omega})$ are solutions of (1.2). Then, as $v_1 = \exp(-bu_1)$, $v_2 = \exp(-bu_2)$ satisfy (2.1), we have in particular

$$\int_\Omega \sqrt{v_2^2 + b^{-2}|\nabla v_2|^2}\,dx - \int_\Omega \sqrt{v_1^2 + b^{-2}|\nabla v_1|^2}\,dx \ge -\int_\Omega ab^{-2}\log(v_1)\,(v_2 - v_1)\,dx$$

and

$$\int_\Omega \sqrt{v_1^2 + b^{-2}|\nabla v_1|^2}\,dx - \int_\Omega \sqrt{v_2^2 + b^{-2}|\nabla v_2|^2}\,dx \ge -\int_\Omega ab^{-2}\log(v_2)\,(v_1 - v_2)\,dx.$$

Summing up and rearranging, we get

$$0 \ge \int_\Omega ab^{-2}(\log(v_2) - \log(v_1))\,(v_2 - v_1)\,dx.$$

The strict monotonicity of the logarithm function yields $v_1 = v_2$ and hence $u_1 = u_2$. □

2.2. Local existence of classical solutions. We now prove the existence of small classical solutions.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \ge 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in\,]0, 1[$. Then, for every $a_0 > 0$ there exists $\delta_0 > 0$ such that, for any $(a, b) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with $|a - a_0| < \delta_0$ and $b < \delta_0$, problem (1.2) has a unique classical solution $u = u(a, b) \in C^{2,\alpha}(\overline{\Omega})$, with*

$$0 \ll u \ll b/a \quad \text{in } \Omega, \tag{2.3}$$

which is exponentially asymptotically stable, smoothly depends on the parameters (a, b) in the topology of $C^{2,\alpha}(\overline{\Omega})$, and satisfies

$$\lim_{(a,b)\to(a_0,0)} \|u(a, b)\|_{C^{2,\alpha}} = 0. \tag{2.4}$$

Proof. Set

$$C_0^{2,\alpha}(\overline{\Omega}) = \{u \in C^{2,\alpha}(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$$

and define the operator $\mathcal{F} : C_0^{2,\alpha}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R} \to C^{0,\alpha}(\overline{\Omega})$ by

$$\mathcal{F}(u, a, b) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) - au + \frac{b}{\sqrt{1 + |\nabla u|^2}}.$$

Applying e.g. [27], it follows that $\mathcal{F}$ is of class C^∞, with partial derivative

$$\partial_u\mathcal{F}(u, a, b)[v] = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla u|^2}} - \frac{\nabla u \cdot \nabla v}{\left(\sqrt{1 + |\nabla u|^2}\right)^3}\nabla u\right) - av - \frac{b\,\nabla u \cdot \nabla v}{\left(\sqrt{1 + |\nabla u|^2}\right)^3},$$

for all $v \in C_0^{2,\alpha}(\overline{\Omega})$. We have

$$\mathcal{F}(0, a_0, 0) = 0 \quad \text{and} \quad \partial_u\mathcal{F}(0, a_0, 0)[v] = \Delta v - a_0 v.$$

As $a_0 > 0$, by [23, Theorem 6.14],

$$\partial_u \mathcal{F}(0, a_0, 0) : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$$

is a continuous isomorphism. Thus the implicit function theorem yields the existence of a constant $\delta_0 > 0$ and a map $U :]a_0 - \delta_0, a_0 + \delta_0[\times] - \delta_0, \delta_0[\to C_0^{2,\alpha}(\overline{\Omega})$ of class C^∞ such that, for all $(u, a, b) \in C_0^{2,\alpha}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$, with $\|u\|_{C^{2,\alpha}} < \delta_0$, $|a - a_0| < \delta_0$, $|b| < \delta_0$,

$$\mathcal{F}(u, a, b) = 0 \quad \text{if and only if} \quad u = U(a, b).$$

The global uniqueness result provided by Lemma 2.1 implies that, for any $(a, b) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with $|a - a_0| < \delta_0$ and $b < \delta_0$, problem (1.2) has a unique classical solution $u = u(a, b) \in C^{2,\alpha}(\overline{\Omega})$, which by [39, Section 9.1.4] is exponentially asymptotically stable. Finally, the weak maximum principle, the strong maximum principle and the Hopf boundary point lemma [23, Section 3.2] imply that $0 \ll u \ll \frac{b}{a}$ in Ω, because 0 and $\frac{b}{a}$ are respectively a lower and an upper solution of (1.2), but are not solutions. □

Remark 2. Denoting by Σ the spectrum of $-\Delta$ in $H_0^1(\Omega)$, we easily see from the proof of Theorem 2.2 that, for any $a_0 \in \mathbb{R} \setminus \Sigma$, there exists $\delta_0 > 0$ such that, for any $(a, b) \in \mathbb{R} \times \mathbb{R}$ with $|a - a_0| < \delta_0$ and $|b| < \delta_0$, problem (1.2) has a unique classical solution $u = u(a, b) \in C^{2,\alpha}(\overline{\Omega})$, with $\|u\|_{C^{2,\alpha}} < \delta_0$, which smoothly depends on the parameters (a, b) in the topology of $C^{2,\alpha}(\overline{\Omega})$ and satisfies

$$\lim_{(a,b)\to(a_0,0)} \|u(a, b)\|_{C^{2,\alpha}} = 0.$$

This holds, in particular, for $a_0 = 0$.

2.3. A maximal branch of classical solutions. We finally prove the existence of a maximal connected two-dimensional branch of classical solutions, which emanates from the line of the trivial solutions.

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in]0, 1[$. Then, there exists a set $\mathcal{E} = \bigcup_{a>0}(\{a\} \times [0, b_\infty(a)[) \subseteq \mathbb{R}_0^+ \times \mathbb{R}^+$ such that, for all $(a, b) \in \mathcal{E} \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$, problem (1.2) has a unique classical solution $u = u(a, b) \in C^{2,\alpha}(\overline{\Omega})$, which smoothly depends on the parameters (a, b) in the topology of $C^{2,\alpha}(\overline{\Omega})$, and satisfies (2.3), (2.4) for each $a_0 > 0$, and, in case $b_\infty(a_0) < +\infty$,*

$$\limsup_{(a,b)\to(a_0,b_\infty(a_0))} \|\nabla u(a, b)\|_\infty = +\infty. \tag{2.5}$$

Proof. The proof is divided into three steps.

Step 1. Reformulation of problem (1.2) as a fixed point equation. For any given $a > 0$, let $T_a : C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^+ \to C^{1,\alpha}(\overline{\Omega})$ be the operator which sends any $(u, b) \in C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^+$ onto the unique solution $v \in C^{2,\alpha}(\overline{\Omega})$ of the problem

$$\begin{cases} -(1 + |\nabla u|^2)\Delta v + (Hv\nabla u) \cdot \nabla u + av\left(\sqrt{1 + |\nabla u|^2}\right)^3 \\ \qquad\qquad\qquad\qquad = b(1 + |\nabla u|^2) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where Hv denote the Hessian matrix of v. Remark that, as

$$(1 + |\nabla u|^2)|\xi|^2 - (\xi \cdot \nabla u)^2 \geq |\xi|^2$$

for all $\xi \in \mathbb{R}^N$, the operator

$$(1 + |\nabla u|^2)\Delta v - (Hv\nabla u) \cdot \nabla u$$

is uniformly elliptic. It is clear that $u \in C^{2,\alpha}(\overline{\Omega})$ is a solution of problem (1.2) if and only if

$$u = T_a(u, b).$$

The operator $T_a : C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^+ \to C^{1,\alpha}(\overline{\Omega})$ is compact. Indeed, if $(u_n)_n$ and $(b_n)_n$ are bounded sequences in $C^{1,\alpha}(\overline{\Omega})$ and $\mathbb{R}^+$, respectively, then the Schauder theory applied to problem (2.6) implies that $(v_n)_n$, with $v_n = T_a(u_n, b_n)$, is bounded in $C^{2,\alpha}(\overline{\Omega})$ (see [23, Theorem 3.7, Theorem 6.6]). The compact imbedding of $C^{2,\alpha}(\overline{\Omega})$ into $C^2(\overline{\Omega})$ guarantees that $(v_n)_n$ has a subsequence converging in $C^2(\overline{\Omega})$ and therefore in $C^{1,\alpha}(\overline{\Omega})$. Moreover, if $(u_n)_n$ and $(b_n)_n$ are sequences converging to u and b in $C^{1,\alpha}(\overline{\Omega})$ and $\mathbb{R}^+$, respectively, then the previous conclusion implies that, denoting $v_n = T_a(u_n, b_n)$, any subsequence $(v_{n_k})_k$ of $(v_n)_n$ possesses a subsequence, still denoted by $(v_{n_k})_k$ converging in $C^2(\overline{\Omega})$ to some v. From (2.6) we get

$$\begin{aligned} -(1 + |\nabla u|^2)\Delta v &+ (Hv\nabla u) \cdot \nabla u \\ &= \lim_{k\to+\infty} \left(-(1 + |\nabla u_{n_k}|^2)\Delta v_{n_k} + (Hv_{n_k}\nabla u_{n_k}) \cdot \nabla u_{n_k}\right) \\ &= \lim_{k\to+\infty} \left(- av_{n_k}(\sqrt{1 + |\nabla u_{n_k}|^2})^3 + b_{n_k}(1 + |\nabla u_{n_k}|^2)\right) \\ &= -av(\sqrt{1 + |\nabla u|^2})^3 + b(1 + |\nabla u|^2) \quad \text{in } \Omega, \end{aligned}$$

and

$$v = \lim_{k\to+\infty} v_{n_k} = 0 \quad \text{on } \partial\Omega.$$

This means that $v = T_a(u, b)$. Hence we conclude that the whole sequence $(v_n)_n$ converges to v, that is,

$$\lim_{n\to+\infty} T_a(u_n, b_n) = T_a(u, b).$$

Step 2. Existence of maximal connected branches of solutions. Since, for any given $a > 0$, the operator $T_a : C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^+ \to C^{1,\alpha}(\overline{\Omega})$ is compact and satisfies $T_a(u, 0) = 0$ for all $u \in C^{1,\alpha}(\overline{\Omega})$, we can apply [53, Theorem 3.2]. This yields the existence of a set

$$\mathcal{C}_a \subseteq \{(u, b) \in C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^+ \mid u = T_a(u, b)\},$$

which is connected and unbounded in $C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^+$. Moreover, the weak maximum principle, the strong maximum principle and the Hopf boundary lemma imply that, if $(u, b) \in \mathcal{C}_a$, with $b > 0$, then (2.3) holds. For every $a > 0$, define

$$b_\infty = b_\infty(a) = \sup\{b \mid (u, b) \in \mathcal{C}_a\}$$

and

$$\mathcal{E} = \bigcup_{a>0} (\{a\} \times [0, b_\infty(a)[).$$

Clearly, $\mathcal{E}$ is connected and has a non-empty interior. Further, for any $(a, b) \in \mathcal{E}$, by Lemma 2.1, (1.2) has a unique solution that we denote by $u(a, b)$. Assume that, for some $a > 0$, $b_\infty(a) < +\infty$. The properties of $\mathcal{C}_a$ imply that

$$\limsup_{b\to b_\infty(a)} \|u(a, b)\|_{C^{1,\alpha}} = +\infty$$

and actually

$$\limsup_{b\to b_\infty(a)} \|u(a, b)\|_{C^1} = +\infty. \tag{2.7}$$

Indeed, if

$$\limsup_{b\to b_\infty(a)} \|u(a,b)\|_{C^1} < +\infty,$$

then we would infer from (1.2), using the elliptic L^p-regularity theory [23, Section 9.6], that $u(a,b)$ should remain bounded in $W^{2,p}(\Omega)$, for any given $p>1$, and thus in $C^{1,\alpha}(\overline{\Omega})$ by the Sobolev imbedding theorem, so getting a contradiction. Hence (2.7) is proved. Finally, from (2.7) and the bound

$$\|u\|_\infty \le b_\infty/a,$$

which follows from (2.3), we deduce that

$$\limsup_{b\to b_\infty(a)} \|\nabla u(a,b)\|_\infty = +\infty.$$

Thus, in particular, (2.5) holds.

Step 3. Existence of a smooth maximal two-dimensional branch of solutions. We prove that the map $u(\cdot,\cdot):\mathcal{E}\to C^{2,\alpha}_0(\overline{\Omega})$, which sends (a,b) onto the unique solution u of (1.2), smoothly depends on (a,b). We use the implicit function theorem, as in the proof of Theorem 2.2. Consider the operator $\mathcal{F}: C^{2,\alpha}_0(\overline{\Omega})\times\mathbb{R}\times\mathbb{R}\to C^{0,\alpha}(\overline{\Omega})$ defined in the proof of Theorem 2.2. Fix $(a^*,b^*)\in\mathcal{E}$ and let $u^*=u(a^*,b^*)$ be the corresponding solution of (1.2). We have

$$\mathcal{F}(u^*,a^*,b^*)=0$$

and

$$\partial_u\mathcal{F}(u^*,a^*,b^*)[v] = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla u^*|^2}} - \frac{\nabla u^*\cdot\nabla v}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3}\nabla u^*\right) - a^*v - \frac{b^*\,\nabla u^*\cdot\nabla v}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3},$$

for all $v\in C^{2,\alpha}_0(\overline{\Omega})$. Let $v\in C^{2,\alpha}_0(\overline{\Omega})$ satisfy

$$\partial_u\mathcal{F}(u^*,a^*,b^*)[v]=0,$$

that is,

$$\frac{\Delta v}{\sqrt{1+|\nabla u^*|^2}} - \frac{(Hv\nabla u^*)\cdot\nabla u^*}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3} - \frac{\nabla u^*\cdot\nabla v\,\Delta u^*}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3} - 2\frac{(Hu^*\nabla u^*)\cdot\nabla v}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3}$$
$$+3\frac{[(Hu^*\nabla u^*)\cdot\nabla u^*](\nabla u^*\cdot\nabla v)}{\left(\sqrt{1+|\nabla u^*|^2}\right)^5} - \frac{b^*\,\nabla u^*\cdot\nabla v}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3} - a^*v = 0 \quad \text{in } \Omega,$$

where Hu^* and Hv denote the Hessian matrices of u^* and v, respectively. As noticed in the proof of Theorem 2.3, we have

$$(1+|\nabla u^*|^2)|\xi|^2 - (\xi\cdot\nabla u^*)^2 \ge |\xi|^2$$

for all $\xi\in\mathbb{R}^N$, and thus the operator

$$\frac{(1+|\nabla u^*|^2)\Delta v - (Hv\nabla u^*)\cdot\nabla u^*}{\left(\sqrt{1+|\nabla u^*|^2}\right)^3}$$

is uniformly elliptic. Therefore the strong maximum principle implies that $v=0$. Thus

$$\partial_u\mathcal{F}(u^*,a^*,b^*): C^{2,\alpha}_0(\overline{\Omega})\to C^{0,\alpha}(\overline{\Omega})$$

is a continuous isomorphism; the implicit function theorem then applies and guarantees that the map $u(\cdot,\cdot)$, implicitly defined by the equation

$$\mathcal{F}(u,a,b)=0,$$

is smooth. □

3. Classical solutions on balls. In this section we discuss the solvability of problem (1.2) in the special case where the domain Ω is an open ball $B=B(x_0,R)$ in $\mathbb{R}^N$ of center x_0 and radius R. Since both the equation in (1.2) and the domain are rotationally invariant, it is natural to look for radially symmetric solutions. Accordingly, we establish the existence of a solution of (1.2), which is positive, classical, radially symmetric, radially decreasing and concave, by solving the problem

$$\begin{cases} -\Big(\dfrac{r^{N-1}\,v'}{\sqrt{1+v'^2}}\Big)'=r^{N-1}\Big(-av+\dfrac{b}{\sqrt{1+v'^2}}\Big) & \text{in }]0,R[,\\ v'(0)=0,\ \ v(R)=0. \end{cases} \tag{3.1}$$

This radial solution is then the unique solution of (1.2) by Lemma 2.1. We remark that the results presented in this section have been obtained in [10] by a different proof.

3.1. Properties of the solutions. We start with two preliminary results, where some properties of the possible solutions of problem (3.1) are highlighted. These properties are notable also because they allow us to get some a priori estimates which shall be used later to obtain the existence result; the validity of a one-sided Nagumo condition plays a central role here. Note that the equation in (3.1) can be written in the form

$$-v''=\Big(-av+\frac{b}{\sqrt{1+v'^2}}\Big)(1+v'^2)^{3/2}+\frac{v'}{r}(N-1)(1+v'^2). \tag{3.2}$$

Lemma 3.1. *Let $a>0$ and $b>0$ be given. Suppose that $v\in C^2(]0,R])\cap C^1([0,R])$ is a solution of (3.1). Then $v\in C^2([0,R])$.*

Proof. Let $v\in C^2(]0,R])\cap C^1([0,R])$ be a solution of (3.1). For any $r\in\,]0,R]$ we set for convenience

$$\eta(r)=\int_0^r\Big(\frac{t}{r}\Big)^{N-1}\Big(-av(t)+\frac{b}{\sqrt{1+v'(t)^2}}\Big)\,dt. \tag{3.3}$$

Observe that

$$\lim_{r\to0^+}\eta(r)=0$$

and, by de l'Hôpital's rule,

$$\lim_{r\to0^+}\frac{\eta(r)}{r}=\lim_{r\to0^+}\frac{1}{r^N}\int_0^r t^{N-1}\Big(-av(t)+\frac{b}{\sqrt{1+v'(t)^2}}\Big)\,dt=\frac{-av(0)+b}{N}.$$

Integrating over $[0,r]$ the equation in (3.1) we obtain

$$-\frac{v'(r)}{\sqrt{1+v'(r)^2}}=\eta(r).$$

Let $\psi(z)=\frac{z}{\sqrt{1-z^2}}$ be the inverse function of $\phi(\zeta)=\frac{\zeta}{\sqrt{1+\zeta^2}}$. Then we have

$$v''(0)=\lim_{r\to0^+}\frac{v'(r)}{r}=-\lim_{r\to0^+}\frac{\psi(\eta(r))}{\eta(r)}\,\frac{\eta(r)}{r}=\frac{av(0)-b}{N}.$$

Finally, we compute, using (3.2),

$$\lim_{r\to 0^+} v''(r) = \lim_{r\to 0^+} \Big(\Big(av(r) - \frac{b}{\sqrt{1+v'(r)^2}}\Big)\big(1+v'(r)^2\big)^{3/2} - \frac{v'(r)}{r}(N-1)\big(1+v'(r)^2\big)\Big)$$
$$= (av(0)-b) - (N-1)v''(0) = v''(0).$$

Thus the conclusion follows. □

Lemma 3.2. *Let $a>0$ and $b>0$ be given. Suppose that $v \in C^2([0,R])$ is a solution of* (3.1)*. Then v satisfies*

(*i*) $0 < v(r) < b/a$ *for all* $r \in [0,R[$;

(*ii*) $v'(r) < 0$ *for all* $r \in]0,R]$;

(*iii*) $v''(r) < 0$ *for all* $r \in [0,R]$;

(*iv*) $|v'(r)| < \sqrt{\exp(\frac{2b^2}{a}) - 1}$ *for all* $r \in [0,R]$.

Proof. Assume that $v \in C^2([0,R])$ is a solution of (3.1).

Let us prove (i). Suppose, by contradiction, that v attains its minimum at some point $\hat r \in [0,R[$, with $v(\hat r) \le 0$. If $\hat r = 0$ we have $v''(0) = \frac{av(0)-b}{N} < 0$; if $\hat r \in]0,R[$, by (3.2), we have $v''(\hat r) = av(\hat r) - b < 0$. Since $\hat r$ is a minimum point we have $v''(\hat r) \ge 0$; therefore in both cases we obtain a contradiction. Similarly, suppose that v attains its maximum at some point $\hat r \in [0,R[$, with $v(\hat r) \ge \frac{b}{a}$. As above we observe that $v''(\hat r) = \frac{av(0)-b}{N} \ge 0$ if $\hat r = 0$ and $v''(\hat r) = av(\hat r) - b \ge 0$ if $\hat r \in]0,R[$. Since $\hat r$ is a maximum point we have $v''(\hat r) \le 0$; in any case we obtain therefore $v''(\hat r) = 0$ and $v(\hat r) = \frac{b}{a}$. Since v is a solution of the equation (3.2), satisfying the initial conditions $v(\hat r) = \frac{b}{a}$ and $v'(\hat r) = 0$, by the uniqueness of the solution of the Cauchy problem (cf. [10, Proposition 2.2]) we infer that v is the constant function $v = \frac{b}{a}$, which is a contradiction.

Let us prove (ii). Observe that $v''(r) = av(r) - b < 0$ at any point $r \in]0,R]$ where $v'(r) = 0$. Therefore there exists at most one point $r \in [0,R]$ where $v'(r) = 0$, and necessarily $r = 0$. Hence we conclude that $v'(r) < 0$ for all $r \in]0,R]$.

Let us prove (iii). Suppose, by contradiction, that there exists $r_0 \in]0,R]$ such that $v''(r_0) \ge 0$. As observed in (i), we have $v''(0) < 0$, hence there exists $\hat r \in]0,R]$ such that $v''(\hat r) = 0$ and $v''(r) < 0$ in $[0,\hat r[$. Computing and evaluating at $\hat r$ the derivative in (3.2), we obtain

$$v'''(\hat r) = av'(\hat r)(1+v'(\hat r)^2)^{3/2} + \frac{N-1}{\hat r^2} v'(\hat r)(1+v'(\hat r)^2) < 0.$$

As $v''(\hat r) = 0$ we deduce that $v''(r) > 0$ in a left neighborhood of $\hat r$, which is a contradiction.

Let us prove (iv). Observe that, from (3.2) and taking into account (i) and (ii), for all $r \in]0,R]$ we have $v''(r) \ge -b(1+v'(r)^2)$. Therefore

$$\frac{v'(r)v''(r)}{1+v'(r)^2} \le -bv'(r).$$

Integrating over $[0,R]$, we obtain

$$\frac{1}{2}\log\big(1+v'(R)^2\big) \le b\,v(0) < \frac{b^2}{a},$$

and the conclusion follows from (iii). □

3.2. Esistence of the solution. We are ready to prove our existence result. Thanks to the gradient estimates obtained in Lemma 3.2 we can replace the degenerate problem (3.1) with a uniformly elliptic one. Next we apply a standard version of the Leray-Schauder continuation theorem to the modified problem.

Proposition 3.1. *For any given a, $b > 0$ and any $R > 0$, problem (3.1) has a solution $v \in C^2([0,R])$ satisfying the conditions (i)-(iv) listed in Lemma 3.2.*

Proof. We divide the proof into two steps.

Step 1. A modified problem. Set $c = \sqrt{\exp(2b^2/a) - 1}$ and define a C^1-homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(s) = \begin{cases} \dfrac{s}{\sqrt{1+s^2}} & \text{if } |s| \le c, \\ \dfrac{s + \operatorname{sgn}(s)\, c^3}{(1+c^2)^{3/2}} & \text{if } |s| > c. \end{cases}$$

Note that

$$\varphi'(s) = \begin{cases} \dfrac{1}{(1+s^2)^{3/2}} & \text{if } |s| \le c, \\ \dfrac{1}{(1+c^2)^{3/2}} & \text{if } |s| > c \end{cases}$$

is bounded, bounded away from 0 and satisfies, for all $s \in \mathbb{R}$,

$$\varphi'(s) \ge \frac{1}{(1+s^2)^{3/2}}. \tag{3.4}$$

Next we introduce the modified problem

$$\begin{cases} -(r^{N-1}\varphi(v'))' = r^{N-1}\Big(- av + \dfrac{b}{\sqrt{1+v'^2}}\Big) & \text{in }]0,R[, \\ v'(0) = 0, \;\; v(R) = 0, \end{cases} \tag{3.5}$$

and the operator $\mathcal{S} : C^1([0,R]) \to C^1([0,R])$ defined by

$$\mathcal{S}(w)(r) = \int_r^R \varphi^{-1}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} \Big(- aw(t) + \frac{b}{\sqrt{1+w'(t)^2}}\Big)\, dt\Big)\, ds.$$

For any given $w \in C^1([0,R])$, it is convenient to write, like in (3.3),

$$\eta(s) = \int_0^s \Big(\frac{t}{s}\Big)^{N-1} \Big(- aw(t) + \frac{b}{\sqrt{1+w'(t)^2}}\Big)\, dt. \tag{3.6}$$

We see, arguing as in Lemma 3.1, that $\eta \in C^1([0,R])$, with

$$\begin{aligned} \eta'(r) &= -aw(r) + \frac{b}{\sqrt{1+w'(r)^2}} - (N-1)\frac{\eta(r)}{r} \;\; \text{in }]0,R], \\ \eta'(0) &= \frac{1}{N}\Big(- aw(0) + \frac{b}{\sqrt{1+w'(0)^2}}\Big). \end{aligned} \tag{3.7}$$

Observe that

$$\frac{d}{dr}\mathcal{S}(w)(r) = -\varphi^{-1}\big(\eta(r)\big),$$

for all $r \in [0,R]$, and

$$\frac{d^2}{dr^2}\mathcal{S}(w)(r) = -(\varphi^{-1})'\big(\eta(r)\big)\, \eta'(r),$$

for all $r \in]0, R]$. Actually, arguing as in Lemma 3.1, we easily verify that $\mathcal{S}(w) \in C^2([0,R])$ and

$$\frac{d^2}{dr^2}\mathcal{S}(w)(0) = \frac{1}{N}\Big(aw(0) - \frac{b}{\sqrt{1+w'(0)^2}}\Big).$$

Finally, observe that $v = \mathcal{S}(w)$ is the unique solution of the problem

$$\begin{cases} -(r^{N-1}\varphi(v'))' = r^{N-1}\Big(-aw + \dfrac{b}{\sqrt{1+w'^2}}\Big) & \text{in }]0,R[, \\ v'(0) = 0, \; v(R) = 0, \end{cases}$$

and a function $v \in C^2([0,R])$ is a solution of (3.5) if and only if v is a fixed point of $\mathcal{S}$.

Step 2. Existence of a solution. We prove the existence of a fixed point v of $\mathcal{S}$ in the open bounded subset of $C^1([0,R])$

$$\mathcal{O} = \{w \in C^1([0,R]) \mid \|w\|_\infty < b/a, \|w'\|_\infty < c\}.$$

This implies in particular that v is a solution of (3.1). To apply the Leray-Schauder continuation method, we introduce the homotopy $\mathcal{T} : \overline{\mathcal{O}} \times [0,1] \to C^1([0,R])$ by setting

$$\mathcal{T}(w,\lambda)(r) = \int_r^R \varphi^{-1}\Big(\lambda \int_0^s \Big(\frac{t}{s}\Big)^{N-1}\Big(-aw(t) + \frac{b}{\sqrt{1+w'(t)^2}}\Big)\,dt\Big)\,ds \quad \text{in } [0,R].$$

Note that $\mathcal{T}(\cdot,0) = 0$, $\mathcal{T}(\cdot,1) = \mathcal{S}$ and $v \in \overline{\mathcal{O}}$ satisfies $v = \mathcal{T}(v,\lambda)$, for some $\lambda \in [0,1]$, if and only if $v \in C^2([0,R])$ and solves

$$\begin{cases} -(r^{N-1}\varphi(v'))' = \lambda r^{N-1}\Big(-av + \dfrac{b}{\sqrt{1+v'^2}}\Big) & \text{in }]0,R[, \\ v'(0) = 0, \; v(R) = 0. \end{cases} \tag{3.8}$$

We first verify that $\mathcal{T}$ is compact. The continuity of $\mathcal{T}$ is an obvious consequence of its definition. Therefore let us check that the range of $\mathcal{T}$ is relatively compact. Using (3.6) and (3.7) and arguing as above, we see that, for all $w \in \overline{\mathcal{O}}$ and $\lambda \in [0,1]$,

$$\frac{d}{dr}\mathcal{T}(w,\lambda)(r) = -\varphi^{-1}\big(\lambda\,\eta(r)\big) \quad \text{in } [0,R],$$

and

$$\frac{d^2}{dr^2}\mathcal{T}(w,\lambda)(r) = -\lambda(\varphi^{-1})'\big(\lambda\,\eta(r)\big)\,\eta'(r) \quad \text{in } [0,R].$$

Hence we infer that the set $\mathcal{T}(\overline{\mathcal{O}} \times [0,1])$ is bounded in $C^2([0,R])$ and, by the Arzelà-Ascoli theorem, it is relatively compact in $C^1([0,R])$.

Next we show that there is no fixed point of $\mathcal{T}(\cdot,\lambda)$ on $\partial\mathcal{O}$, for any $\lambda \in [0,1]$. Let $v \in \overline{\mathcal{O}}$ be a fixed point of $\mathcal{T}(\cdot,\lambda)$, for some $\lambda \in [0,1]$. The conclusion is obvious if $\lambda = 0$. Therefore suppose that $\lambda > 0$. Since v satisfies (3.8), we have in particular

$$-\varphi'(v'(r))\,v''(r) = \frac{N-1}{r}\,\varphi(v'(r)) + \lambda\Big(-av(r) + \frac{b}{\sqrt{1+v'(r)^2}}\Big)$$

for all $r \in]0,R]$. Repeating the argument we used in Lemma 3.2 to prove (i) and (ii), we see that $0 < v(r) < b/a$ for all $r \in [0,R[$ and $v'(r) < 0$ for all $r \in]0,R]$. Furthermore, using (3.4), we see, as in Lemma 3.2, that

$$v''(s) \geq -\lambda b(1+v'(s)^2) \geq -b(1+v'(s)^2),$$

for all $s \in]0, R]$. Integrating this relation on $[0, r]$, for any given $r \in [0, R]$, we obtain

$$\frac{1}{2}\log(1 + v'(r)^2) \leq b(v(0) - v(r)) < \frac{b^2}{a},$$

i.e., $v'(r) > -c$ for all $r \in [0, R]$. Hence we conclude that $v \in \mathcal{O}$. The Leray-Schauder continuation theorem therefore implies that problem (3.5) has a solution $v \in \mathcal{O}$. □

We can now state the main result of this section.

Theorem 3.3. *Let $a > 0$ and $b > 0$ be given and let $B = B(x_0, R)$ be the open ball in $\mathbb{R}^N$ of center x_0 and radius R. Then there exists a unique solution $u \in C^2(\overline{B})$ of (1.2), which in addition satisfies:*

- *there exists a function $v \in C^2([0, R])$ such that $u(x) = v(|x - x_0|)$ for all $x \in \overline{B}$;*
- *$0 < v(r) < b/a$ for all $r \in [0, R[$;*
- *$v'(r) < 0$ for all $r \in]0, R]$;*
- *$v''(r) < 0$ for all $r \in [0, R]$.*

Proof. Let $v \in C^2([0, R])$ be the solution of problem (3.1), whose existence is guaranteed by Proposition 3.1. The function v satisfies all properties listed in Lemma 3.2. Define $u \in C^1(\overline{B})$ by setting $u(x) = v(|x - x_0|)$ for all $x \in \overline{B}$. An easy calculation shows that $u \in C^2(\overline{B})$ and it is a solution of (1.2), with $\Omega = B$. The uniqueness of the solution finally follows from Lemma 2.1. □

4. Generalized solutions on arbitrary domains.

In this section we prove existence, uniqueness and interior regularity of generalized solutions of problem (1.2) in an arbitrary bounded domain Ω of $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Our analysis mainly relies on the study, in the space of bounded variation functions, of a suitable action functional, involving an anisotropic area term, whose minimizers give rise, via a variational inequality and a natural change of variables, to the generalized solutions of (1.2). The interior regularity of these bounded variation minimizers is obtained by combining a delicate approximation scheme with a "local" existence result of Serrin's type proven in [40] and with the classical gradient estimates of Ladyzhenskaya and Ural'tseva [32]. All the details that are not provided in the proofs below can be found in [11].

4.1. Variational setting and auxiliary results.

Throughout we suppose that $b > 0$ is a given constant and $\mathcal{O}$ and $\mathcal{U}$ are two open bounded sets in $\mathbb{R}^N$, such that $\overline{\mathcal{U}} \subseteq \mathcal{O}$ and $\mathcal{U}$ has a Lipschitz boundary $\partial\mathcal{U}$.

Anisotropic area functionals. We define some functionals that are relevant for our analysis.

Definition 4.1. For all $w \in BV(\mathcal{O})$, we set

$$\int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} = \sup\Big\{ \int_{\mathcal{O}} w\big(g_{N+1} + \tfrac{1}{b}\operatorname{div}\tilde{g}\big)\, dx \mid$$
$$g = (\tilde{g}, g_{N+1}) = (g_1, \ldots, g_N, g_{N+1}) \in C_0^1(\mathcal{O}; \mathbb{R}^{N+1}), |g|^2 = \sum_{i=1}^{N+1} g_i^2 \leq 1 \text{ in } \mathcal{O} \Big\}.$$

Remark 3. We can verify that, for all $w \in C^1(\overline{\mathcal{O}})$,

$$\int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} = \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|\nabla w|^2}\, dx.$$

The following lower and upper estimates can be deduced from Definition 4.1.

Proposition 4.1. *For all $w \in BV(\mathcal{O})$, we have*

$$\max\left\{\int_{\mathcal{O}} |w|\, dx, \frac{1}{b}\int_{\mathcal{O}} |Dw|\right\} \le \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} \le \int_{\mathcal{O}} |w|\, dx + \frac{1}{b}\int_{\mathcal{O}} |Dw|.$$

Proposition 4.1 immediately yields the Lipschitz continuity of the functional with respect to the BV-norm.

Proposition 4.2. *For all v, $w \in BV(\mathcal{O})$, we have*

$$\left|\int_{\mathcal{O}} \sqrt{v^2 + b^{-2}|Dv|^2} - \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2}\right| \le \int_{\mathcal{O}} |v - w|\, dx + \frac{1}{b}\int_{\mathcal{O}} |D(v - w)|.$$

In order to take into account of the Dirichlet boundary conditions, we introduce the following functional.

Definition 4.2. Let $\varphi \in L^1(\partial\mathcal{U})$ be given. For all $v \in BV(\mathcal{U})$ we define

$$\mathcal{J}_\varphi(v) = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\mathcal{U}} |v - \varphi|\, d\mathcal{H}^{N-1}.$$

In case $\varphi = 1$ we simply write $\mathcal{J}_\varphi = \mathcal{J}$, i.e.,

$$\mathcal{J}(v) = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\mathcal{U}} |v - 1|\, d\mathcal{H}^{N-1}.$$

Also by using [14, Section 5.4, Theorem 1], we can prove the following additivity property of $\mathcal{J}_\varphi$.

Proposition 4.3. *For any $v \in BV(\mathcal{U})$ and $w \in BV(\mathcal{O} \setminus \overline{\mathcal{U}})$, define $z : \mathcal{O} \to \mathbb{R}$, by setting*

$$z = \begin{cases} v & \text{a.e. in } \mathcal{U}, \\ w & \text{a.e. in } \mathcal{O} \setminus \overline{\mathcal{U}}. \end{cases}$$

Then $z \in BV(\mathcal{O})$ and satisfies

$$\int_{\mathcal{O}} \sqrt{z^2 + b^{-2}|Dz|^2} = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \int_{\mathcal{O}\setminus\overline{\mathcal{U}}} \sqrt{w^2 + b^{-2}|Dw|^2} + \frac{1}{b}\int_{\partial\mathcal{U}} |v - w|\, d\mathcal{H}^{N-1}.$$

Proposition 4.3, together with [19, Teorema 1.II], allows to prove the convexity and the lower semicontinuity of $\mathcal{J}_\varphi$.

Proposition 4.4. *Let $\varphi \in L^1(\partial\mathcal{U})$ be given. Then the following properties hold:*

(*i*) *$\mathcal{J}_\varphi$ is convex;*
(*ii*) *$\mathcal{J}_\varphi$ is lower semicontinuous with respect to the L^1-convergence in $BV(\mathcal{U})$, i.e., if $(v_n)_n$ is a sequence in $BV(\mathcal{U})$, which converges in $L^1(\mathcal{U})$ to $v \in BV(\mathcal{U})$, then*

$$\mathcal{J}_\varphi(v) \le \liminf_{n\to+\infty} \mathcal{J}_\varphi(v_n).$$

An approximation and a lattice property. The following *approximation property* plays a crucial role in the sequel; it generalizes the classical approximation property in the space of bounded variation functions with respect to the strict convergence (see, e.g., [26, Theorem 1.17]).

Proposition 4.5. *Let $\varphi \in L^1(\partial\mathcal{U})$ and $w \in BV(\mathcal{U})$ be given. Then, for each $p \in [1, 1^*[$, there exists a sequence $(w_n)_n$ in $C^\infty(\mathcal{U}) \cap W^{1,1}(\mathcal{U})$ such that*

$$\lim_{n\to+\infty} w_n = w \quad \text{in } L^p(\mathcal{U}),$$
$$\lim_{n\to+\infty} \mathcal{J}_\varphi(w_n) = \mathcal{J}_\varphi(w),$$
$$w_n = \varphi \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\mathcal{U}, \quad \text{for all } n.$$

Moreover, if there exist $c, d \in \mathbb{R}$, with $c \le w \le d$ a.e. in $\mathcal{U}$ and $c \le \varphi \le d$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, then, for each $\sigma > 0$, a sequence $(w_n)_n$, satisfying the previous conditions, can be selected such that, for all n,

$$c - \sigma \le w_n \le d + \sigma \quad \text{in } \mathcal{U}, \quad \text{for all } n.$$

In the particular case where $\varphi = 1$, i.e., $\mathcal{J}_\varphi = \mathcal{J}$, we can restate Proposition 4.5 as follows.

Corollary 4.1. *Let $w \in BV(\mathcal{U})$ be given. Then, for each $p \in [1, 1^*[$, there exists a sequence $(w_n)_n$ in $C^\infty(\mathcal{U}) \cap W^{1,1}(\mathcal{U})$ such that*

$$\lim_{n\to+\infty} w_n = w \quad \text{in } L^p(\mathcal{U}),$$
$$\lim_{n\to+\infty} \mathcal{J}(w_n) = \mathcal{J}(w),$$
$$w_n = 1 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\mathcal{U}, \quad \text{for all } n.$$

Moreover, if there exist $c, d \in \mathbb{R}$, with $c \le 1 \le d$ and $c \le w \le d$ a.e. in $\mathcal{U}$, then, for each $\sigma > 0$, a sequence $(w_n)_n$, satisfying the previous conditions, can be selected such that

$$c - \sigma \le w_n \le d + \sigma \quad \text{in } \mathcal{U}, \quad \text{for all } n.$$

The lower semicontinuity of the functional $\mathcal{J}_\varphi$ and the above stated approximation property are the essential ingredients for proving the validity of the following *lattice property*, which encodes a kind of maximum principle.

Proposition 4.6. *Let $\varphi \in L^1(\partial\mathcal{U})$ be given. For any v, $w \in BV(\mathcal{U})$, we have*

$$\mathcal{J}_\varphi(v \wedge w) + \mathcal{J}_\varphi(v \vee w) \le \mathcal{J}_\varphi(v) + \mathcal{J}_\varphi(w).$$

4.2. Global minimization. In this subsection we prove that the action functional, naturally associated with the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\right) = -a\log(v) - \dfrac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} & \text{in } \Omega, \\ v = 1 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

has a unique global minimizer in the cone of non-negative functions of $BV(\Omega)$, which is bounded, strictly positive and regular in Ω, and in addition it satisfies a suitable variational inequality.

Definition 4.3. Let us set

$$BV^+(\Omega) = \{w \in BV(\Omega) \mid w \ge 0 \text{ a.e. in } \Omega\}$$

and define the functional $\mathcal{I} : BV^+(\Omega) \to \mathbb{R}$ by setting

$$\mathcal{I}(v) = \mathcal{J}(v) + \mathcal{F}(v),$$

where $\mathcal{J}$ has been introduced in Definition 4.2, with $\mathcal{U} = \Omega$, and $\mathcal{F} : BV^+(\Omega) \to \mathbb{R}$ is the potential functional

$$\mathcal{F}(v) = \int_\Omega F(v)\, dx,$$

with $F : [0, +\infty[\to \mathbb{R}$ the continuous extension of the function $\frac{a}{b^2} s(\log(s) - 1)$.

Existence, uniqueness and localization of the global minimizer. The following proposition proves the existence, the uniqueness and the localization of the global minimizer in $BV^+(\Omega)$ of the action functional $\mathcal{I}$ associated with problem (4.1).

Proposition 4.7. *The functional $\mathcal{I}$ has a unique global minimizer $v \in BV^+(\Omega)$, which also satisfies*

$$\exp\big(-\tfrac{b^2}{a}\big) \le v \le 1 \quad a.e.\ in\ \Omega.$$

Proof. Let us fix $p \in]1, 1^*[$.

Step 1. $\mathcal{I}$ is lower semicontinuous with respect to the L^p-convergence. As there exists $c > 0$ such that F satisfies

$$|F(s)| \le c(|s|^p + 1),$$

for all $s \ge 0$, we deduce by, e.g., [15, Theorem 2.3], that $\mathcal{F}$ is continuous with respect to the L^p-convergence in $BV^+(\Omega)$. Hence the conclusion follows from Proposition 4.4.

Step 2. Existence of a global minimizer. Let $(v_n)_n$ be a minimizing sequence of $\mathcal{I}$ in $BV^+(\Omega)$. By Proposition 4.1, we have

$$\max\left\{\int_\Omega |v_n|\, dx, \frac{1}{b}\int_\Omega |Dv_n|\right\} \le \int_\Omega \sqrt{v_n^2 + b^{-2}|Dv_n|^2}$$
$$\le \mathcal{J}(v_n) + \int_\Omega \big(F(v_n) - \min_{[0,+\infty[} F\big)\, dx = \mathcal{I}(v_n) + \frac{a}{b^2}|\Omega|.$$

Hence $(v_n)_n$ is bounded in $BV(\Omega)$. By [1, Corollary 3.49, Proposition 3.6], there exists a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, and $v \in BV^+(\Omega)$ such that $\lim_{n\to+\infty} v_n = v$ in $L^p(\Omega)$. By the lower semicontinuity of $\mathcal{I}$ with respect to the L^p-convergence we conclude that v is a global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$.

Step 3. Uniqueness of the global minimizer. Since $\mathcal{J}$ is convex in $BV(\Omega)$ and, due to the strict convexity of F in $[0, +\infty[$, $\mathcal{F}$ is strictly convex in $BV^+(\Omega)$, the functional $\mathcal{I}$ is strictly convex in $BV^+(\Omega)$. This implies the uniqueness of the global minimizer.

Step 4. We have $v \ge \exp\big(-\frac{b^2}{a}\big)$ a.e. in Ω. Let us set, for convenience, $\varepsilon = \exp\big(-\frac{b^2}{a}\big)$. As v is a global minimizer, by Proposition 4.6, we have

$$0 \le \mathcal{I}(v \vee \varepsilon) - \mathcal{I}(v) \le \mathcal{J}(\varepsilon) - \mathcal{J}(v \wedge \varepsilon) + \mathcal{F}(v \vee \varepsilon) - \mathcal{F}(v).$$

Using Proposition 4.1 and $\varepsilon \in]0, 1]$, we prove that

$$\mathcal{J}(\varepsilon) - \mathcal{J}(v \wedge \varepsilon) \le \int_{\{v<\varepsilon\}} (\varepsilon - v)\, dx.$$

Thus we have

$$0 \le \mathcal{I}(v \vee \varepsilon) - \mathcal{I}(v) \le \int_{\{v<\varepsilon\}} (\varepsilon - v + F(\varepsilon) - F(v))\, dx.$$

Since the function $G : [0, +\infty[\to \mathbb{R}$, defined by $G(s) = s + F(s)$, is strictly decreasing in $[0, \varepsilon]$, we conclude that

$$0 \le \mathcal{I}(v \vee \varepsilon) - \mathcal{I}(v) \le \int_{\{v<\varepsilon\}} (G(\varepsilon) - G(v))\, dx \le 0,$$

where the last inequality is strict if $|\{v < \varepsilon\}| > 0$. This implies that $|\{v < \varepsilon\}| = 0$, i.e., $v \ge \varepsilon = \exp\big(-\frac{b^2}{a}\big)$ a.e. in Ω.

Step 4. We have $v \le 1$ a.e. in Ω. By Proposition 4.6, we have

$$\mathcal{I}(v \wedge 1) - \mathcal{I}(v) \le \mathcal{J}(1) - \mathcal{J}(v \vee 1) + \mathcal{F}(v \wedge 1) - \mathcal{F}(v).$$

On the one hand, by Proposition 4.1, we get

$$\mathcal{J}(1) \; - \mathcal{J}(v \vee 1) \le \int_\Omega (1 - |v \vee 1|)\, dx \le 0.$$

On the other hand, since F is increasing in $[1, +\infty[$, we infer

$$\mathcal{F}(v \wedge 1) - \mathcal{F}(v) = \int_{\{v \ge 1\}} (F(1) - F(v))\, dx \le 0.$$

We then obtain

$$\mathcal{I}(v \wedge 1) \le \mathcal{I}(v).$$

As v is the unique global minimizer of $\mathcal{I}$, this implies that $v \wedge 1 = v$, i.e., $v \le 1$ a.e. in Ω. □

Interior $C^{1,\alpha}$-regularity of the global minimizer. In order to prove the local $C^{1,\alpha}$-regularity in Ω of the global minimizer v of $\mathcal{I}$, we use an argument which requires a preliminary study of the problem

$$\begin{cases} -\mathrm{div}\Big(\dfrac{\nabla z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\Big) = -a \log(z) - \dfrac{b^2 z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}} & \text{in } B_r, \\ z = \psi & \text{on } \partial B_r, \end{cases} \tag{4.2}$$

where $B_r = B(x_0, r)$ is the ball of center $x_0 \in \Omega$ and radius $r > 0$, with $\overline{B}_r \subseteq \Omega$, and $\psi \in C^{2,\alpha}(\overline{B_r})$, for some $\alpha \in]0, 1[$, is a given function, with

$$\tfrac{1}{2} \exp\big(-\tfrac{b^2}{a}\big) \le \psi \le \tfrac{3}{2} \quad \text{in } \overline{B_r}. \tag{4.3}$$

We associate with problem (4.2) the functional $\mathcal{I}_r : BV^+(B_r) \to \mathbb{R}$, defined by

$$\mathcal{I}_r(w) = \int_{B_r} \sqrt{w^2 + b^{-2}|Dw|^2} + \frac{1}{b}\int_{\partial B_r} |w - \psi|\, d\mathcal{H}^{N-1} + \int_{B_r} F(w)\, dx,$$

where $BV^+(B_r) = \{w \in BV(B_r) \mid w \ge 0 \text{ a.e. in } B_r\}$ and F has been introduced in Definition 4.3.

Our first result is based on [40, Corollary 1] and on [32, Theorem 4].

Lemma 4.4. *Fix any $x_0 \in \Omega$. Then there exists $r_0 > 0$ such that, for any given $r \in]0, r_0[$ and every $\psi \in C^{2,\alpha}(\overline{B_r})$ satisfying (4.3), problem (4.2) has a unique solution $z \in C^{2,\alpha}(\overline{B_r})$ such that*

(i) $\frac{1}{2} \exp\big(-\frac{b^2}{a}\big) \le z \le \frac{3}{2}$ *in B_r;*

(*ii*) *there exist* $\beta = \beta(a, b, N, r) > 0$ *and* $C = C(a, b, N, r) > 0$*, independent of* ψ*, such that* $\|z\|_{C^{1,\beta}(\overline{B_{r/4}})} \leq C$*;*

(*iii*) z *is a global minimizer of* $\mathcal{I}_r$ *in* $BV^+(B_r)$*.*

With the help of this lemma, we can prove the interior regularity of the global minimizer v of $\mathcal{I}$.

Proposition 4.8. *The global minimizer* $v \in BV^+(\Omega)$ *of* $\mathcal{I}$ *belongs to* $W^{1,1}(\Omega)$*, and, for every open set* Ω_1*, with* $\overline{\Omega_1} \subseteq \Omega$*, there exists* $\alpha > 0$ *such that* $v \in C^{1,\alpha}(\overline{\Omega_1})$*.*

Proof. Let $p \in]1, 1^*[$ be fixed. By Corollary 4.1, there exists a sequence $(v_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\lim_{n\to+\infty} v_n = v \quad \text{in } L^p(\Omega),$$
$$\lim_{n\to+\infty} \mathcal{J}(v_n) = \mathcal{J}(v),$$
$$\tfrac{1}{2}\exp\big(-\tfrac{b^2}{a}\big) \leq v_n \leq \tfrac{3}{2} \quad \text{in } \Omega, \quad \text{for all } n.$$

The continuity of the potential operator $\mathcal{F}$ in $L^p(\Omega)$ also implies

$$\lim_{n\to+\infty} \mathcal{I}(v_n) = \mathcal{I}(v). \tag{4.4}$$

Let now fix $x_0 \in \Omega$ and take $r \in]0, r_0[$, with $r_0 > 0$ given by Lemma 4.4. For each n, consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\right) = -a\log(z) - \dfrac{b^2 z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}} & \text{in } B_r, \\ z = v_n & \text{on } \partial B_r \end{cases}$$

and denote by $z_n \in C^2(\overline{B_r})$ its unique solution provided by Lemma 4.4. Define the sequence $(w_n)_n$ in $BV(\Omega)$ by setting

$$w_n = \begin{cases} z_n & \text{in } \overline{B_r}, \\ v_n & \text{in } \Omega \setminus \overline{B_r}. \end{cases}$$

Step 1. The sequence $(w_n)_n$ *is bounded in* $BV(\Omega)$ *and satisfies* $\lim_{n\to+\infty} w_n = v$ *in* $L^p(\Omega)$*.* Using Proposition 4.3 and conclusion (*iii*) in Lemma 4.4 with $\psi = v_n$, we obtain

$$\begin{aligned} &\mathcal{I}(w_n) \\ &= \mathcal{I}_r(z_n) + \int_{\Omega\setminus\overline{B_r}} \sqrt{v_n^2 + b^{-2}|Dv_n|^2} + \frac{1}{b}\int_{\partial\Omega} |v_n - 1|\, d\mathcal{H}^{N-1} + \int_{\Omega\setminus B_r} F(v_n)\, dx \\ &\leq \mathcal{I}_r(v_n) + \int_{\Omega\setminus\overline{B_r}} \sqrt{v_n^2 + b^{-2}|Dv_n|^2} + \frac{1}{b}\int_{\partial\Omega} |v_n - 1|\, d\mathcal{H}^{N-1} + \int_{\Omega\setminus B_r} F(v_n)\, dx \\ &= \mathcal{I}(v_n). \end{aligned}$$

As a consequence of (4.4), we may assume that $\mathcal{I}(v_n) \leq \mathcal{I}(v) + 1$, for all n. Hence, by Proposition 4.1, we obtain, as in Proposition 4.7, the boundedness of $(w_n)_n$ in $BV(\Omega)$. By [1, Corollary 3.49, Proposition 3.6], there exists a subsequence of $(w_n)_n$, still denoted by $(w_n)_n$, which converges in $L^p(\Omega)$ and a.e. in Ω to some $w \in BV(\Omega)$. As $w_n \geq \frac{1}{2}\exp\big(-\frac{b^2}{a}\big)$ in Ω for all n, we have that $w \in BV^+(\Omega)$. The lower semicontinuity of $\mathcal{I}$ with respect to the L^p-norm then yields

$$\mathcal{I}(w) \leq \liminf_{n\to+\infty} \mathcal{I}(w_n) \leq \liminf_{n\to+\infty} \mathcal{I}(v_n) = \mathcal{I}(v).$$

We finally conclude that $v = w$ by uniqueness of the minimizer of $\mathcal{I}$ in $BV^+(\Omega)$.

Step 2. For every open set Ω_1, with $\overline{\Omega_1} \subseteq \Omega$, there exists $\alpha > 0$ such that $v \in C^{1,\alpha}(\overline{\Omega_1})$. This can be easily deduced from Lemma 4.4 (ii) and the compactness of $\overline{\Omega_1}$.

Step 3. v belongs to $W^{1,1}(\Omega)$. As $v \in C^1(\Omega) \cap BV(\Omega)$, we have $Dv = \nabla v\, dx$ and $\int_\Omega |\nabla v|\, dx = \int_\Omega |Dv|$ and then $v \in W^{1,1}(\Omega)$. □

A variational inequality. We prove now a characterization of the global minimizer v of $\mathcal{I}$ as a solution of an associated variational inequality.

Proposition 4.9. *Let $v \in BV(\Omega)$ be such that $\operatorname{ess\,inf}_\Omega v > 0$. Then v is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$ if and only if v satisfies the variational inequality*

$$\mathcal{J}(w) - \mathcal{J}(v) \geq -\frac{a}{b^2} \int_\Omega \log(v)\,(w - v)\, dx \tag{4.5}$$

for all $w \in BV(\Omega)$.

Proof. The proof consists of three steps.

Step 1. If $v \in BV^+(\Omega)$ is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$, then v satisfies (4.5) for all $w \in BV(\Omega) \cap L^\infty(\Omega)$. Let $w \in BV(\Omega) \cap L^\infty(\Omega)$ be fixed. By Proposition 4.7 we know that $v \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_\Omega v > 0$. Hence there exists $\bar{t} > 0$ such that, for all $t \in [0, \bar{t}[$,

$$\tfrac{1}{2} \operatorname{ess\,inf}_\Omega v \leq v + t(w - v) \leq 2 \operatorname{ess\,sup}_\Omega v \quad \text{a.e. in } \Omega. \tag{4.6}$$

As $\mathcal{J}$ is convex and v is a global minimizer of $\mathcal{I} = \mathcal{J} + \mathcal{F}$ in $BV^+(\Omega)$, we have, for all $t \in]0, \bar{t}[$

$$\mathcal{J}(w) - \mathcal{J}(v) \geq \frac{\mathcal{J}(v + t(w - v)) - \mathcal{J}(v)}{t} \geq -\int_\Omega \frac{F(v + t(w - v)) - F(v)}{t}\, dx. \tag{4.7}$$

On the other hand, as $F :]0, +\infty[\to \mathbb{R}$ is continuously differentiable, with $F'(s) = \frac{a}{b^2}\log(s)$, and (4.6) holds, we get, by the dominated convergence theorem,

$$\lim_{t \to 0^+} \int_\Omega \frac{F(v + t(w - v)) - F(v)}{t}\, dx = \frac{a}{b^2} \int_\Omega \log(v)\,(w - v)\, dx. \tag{4.8}$$

The conclusion then follows from (4.7) and (4.8).

Step 2. If $v \in BV^+(\Omega)$ is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$, then v satisfies (4.5) for all $w \in BV(\Omega)$. Let $w \in BV(\Omega)$ be fixed. By Corollary 4.1, there exists a sequence $(w_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\lim_{n \to +\infty} w_n = w \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \lim_{n \to +\infty} \mathcal{J}(w_n) = \mathcal{J}(w).$$

For each n, let us define $\tilde{w}_n = (w_n \wedge n) \vee -n$. We have $\tilde{w}_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $\mathcal{J}(\tilde{w}_n) \leq \mathcal{J}(w_n)$. Therefore, from Step 1, we infer

$$\mathcal{J}(w_n) - \mathcal{J}(v) \geq \mathcal{J}(\tilde{w}_n) - \mathcal{J}(v) \geq -\frac{a}{b^2} \int_\Omega \log(v)\,(\tilde{w}_n - v)\, dx. \tag{4.9}$$

Since $\lim_{n \to +\infty} \tilde{w}_n = w$ in $L^1(\Omega)$, we get

$$\lim_{n \to +\infty} \int_\Omega \log(v)\,(\tilde{w}_n - v)\, dx = \int_\Omega \log(v)\,(w - v)\, dx.$$

By passing to the limit in (4.9), we conclude that (4.5) holds.

Step 3. If v satisfies (4.5) for all $w \in BV(\Omega)$, then v is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$. Since F is convex and continuously differentiable in $]0,+\infty[$, with $F'(s) = \frac{a}{b^2}\log(s)$, and $\operatorname*{ess\,inf}_{\Omega} v > 0$, from (4.5) we get

$$\mathcal{I}(w) \geq \mathcal{J}(w) + \int_\Omega F(v)\,dx + \int_\Omega F'(v)(w-v)\,dx \geq \mathcal{J}(v) + \int_\Omega F(v)\,dx = \mathcal{I}(v),$$

for all $w \in BV^+(\Omega)$. Hence v is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$. □

As a consequence of Proposition 4.9 we can show that v satisfies the equation in (4.1) in the weak sense.

Corollary 4.2. *The global minimizer $v \in W^{1,1}(\Omega)$ of $\mathcal{I}$ in $BV^+(\Omega)$ satisfies*

$$\int_\Omega \frac{\nabla v \nabla \phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{b^2 v\,\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + a\int_\Omega \log(v)\,\phi\,dx = 0 \tag{4.10}$$

for all $\phi \in C_0^1(\Omega)$.

Proof. Pick $\phi \in C_0^1(\Omega)$. As $v \in W^{1,1}(\Omega)$ satisfies (4.5), we have, for all $t > 0$,

$$\int_\Omega \frac{1}{t}\Big(\sqrt{(v+t\phi)^2 + b^{-2}|\nabla(v+t\phi)|^2} - \sqrt{v^2 + b^{-2}|\nabla v|^2}\Big)\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx \geq 0.$$

Using $\operatorname*{ess\,inf}_{\Omega} v > 0$, we can pass to the limit as $t \to 0^+$ and get

$$\frac{1}{b^2}\int_\Omega \frac{\nabla v\,\nabla\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx \geq 0.$$

By replacing ϕ with $-\phi$, we then conclude that (4.10) holds. □

Interior smoothness of the global minimizer. We are finally in position of proving the smoothness in Ω of the global minimizer v of $\mathcal{I}$.

Proposition 4.10. *The global minimizer $v \in BV^+(\Omega)$ of $\mathcal{I}$ belongs to $C^\infty(\Omega) \cap W^{1,1}(\Omega)$.*

Proof. As $\operatorname*{ess\,inf}_{\Omega} v > 0$, using Corollary 4.2 and Proposition 4.8, we have that, for any smooth subdomain Ω_1, with $\overline{\Omega_1} \subseteq \Omega$, v is a weak solution of the linear Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^N a^{ij}\partial_{x_i x_j} z = g & \text{in } \Omega_1, \\ z = v & \text{on } \partial\Omega_1, \end{cases}$$

with coefficients

$$a^{ij} = \frac{\delta_{ij}}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} - \frac{\partial_{x_i} v\,\partial_{x_j} v}{b^2(v^2 + b^{-2}|\nabla v|^2)^{3/2}},$$

for $i, j \in \{1, \dots, N\}$, and

$$g = \frac{v\,|\nabla v|^2}{(v^2 + b^{-2}|\nabla v|^2)^{3/2}} + a\,\log(v) + \frac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}$$

belonging to $C^{0,\alpha}(\overline{\Omega_1})$. The result can then be deduced from [23, Theorem 6.13] and iterated applications of [23, Theorem 6.17]. □

4.3. From minimizers to generalized solutions. We show here the equivalence between problem (1.2) and the variational inequality (4.5), which by Proposition 4.9 is in turn equivalent to the minimization of $\mathcal{I}$ in $BV^+(\Omega)$. We start proving a localization result for any generalized solution of (1.2).

Proposition 4.11. *Let u be a generalized solution of (1.2). Then $u \in L^\infty(\Omega)$ and $0 \le u \le b/a$ a.e. in Ω.*

Proof. From the equation in (1.2), we see that $u \in L^N(\Omega)$. Then multiplying the equation by u^-, using the integration by parts formula in [2, Proposition 1.3], which holds according to Remark 1, and the boundary conditions satisfied by u, we get

$$\begin{aligned} 0 &\ge -\int_\Omega \frac{|\nabla u^-|^2}{\sqrt{1+|\nabla u^-|^2}}\,dx - \int_{\partial\Omega} u^-\,d\mathcal{H}^{N-1} \\ &= \int_\Omega \frac{\nabla u\,\nabla u^-}{\sqrt{1+|\nabla u|^2}}\,dx - \int_{\partial\Omega}\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}},\nu\Big]u^-\,d\mathcal{H}^{N-1} \\ &= -\int_\Omega a u u^-\,dx + \int_\Omega \frac{b\,u^-}{\sqrt{1+|\nabla u|^2}}\,dx \ge 0 \end{aligned}$$

and hence $u(x) \ge 0$ for a.e. $x \in \Omega$. In a completely similar way, multiplying now by $(u - \frac{b}{a})^+$, we prove that $u(x) \le b/a$ for a.e. $x \in \Omega$. $\square$

Proposition 4.12. *Let $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, with $0 < \operatorname*{ess\,inf}_\Omega v \le \operatorname*{ess\,sup}_\Omega v \le 1$, satisfy (4.5) for all $w \in BV(\Omega)$. Then $u = -\frac{1}{b}\log(v) \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ is a generalized solution of (1.2).*

Proof. The proof is divided into two steps.
Step 1. The function u is such that $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^\infty(\Omega)$ and u satisfies the equation in (1.2) a.e. in Ω. As $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $0 < \operatorname*{ess\,inf}_\Omega v \le \operatorname*{ess\,sup}_\Omega v \le 1$, we have $u = -\frac{1}{b}\log(v) \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $u \ge 0$ a.e. in Ω. By Proposition 4.9 and Corollary 4.2 we know that, for any $\phi \in C_0^\infty(\Omega)$, v satisfies (4.10) and hence u satisfies

$$-\frac{1}{b}\int_\Omega \frac{\nabla u\,\nabla\phi}{\sqrt{1+|\nabla u|^2}}\,dx + \int_\Omega \frac{\phi}{\sqrt{1+|\nabla u|^2}}\,dx - \frac{a}{b}\int_\Omega u\,\phi\,dx = 0.$$

We then conclude that $\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^\infty(\Omega)$ and

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = -au + \frac{b}{\sqrt{1+|\nabla u|^2}}, \tag{4.11}$$

a.e. in Ω.
Step 2. For $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $u(x) = 0$, or both $u(x) > 0$ and $\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}},\nu\Big](x) = -1$ hold. Let us fix $\phi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, $\phi(x) = 0$ whenever $v(x) = 1$. Pick $t > 0$. By assumption v satisfies (4.5) and hence we have

$$\begin{aligned} &\int_\Omega \frac{1}{t}\Big(\sqrt{(v+t\phi)^2 + b^{-2}|\nabla(v+t\phi)|^2} - \int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}\Big)\,dx \\ &\quad + \frac{1}{b}\int_{\partial\Omega} \frac{1}{t}\Big(|v+t\phi-1| - |v-1|\Big)\,d\mathcal{H}^{N-1} + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx \ge 0. \end{aligned} \tag{4.12}$$

Since $v \in W^{1,1}(\Omega)$ satisfies $v \le 1$ a.e. in Ω, it also satisfies $v \le 1$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ (see [14, Theorem 5.3.2]). The dominated convergence theorem can be applied to prove that

$$\lim_{t\to 0^+} \frac{1}{b}\int_{\partial\Omega} \frac{1}{t}\Big(|v+t\phi-1| - |v-1|\Big)\, d\mathcal{H}^{N-1} = -\frac{1}{b}\int_{\partial\Omega} \phi\, d\mathcal{H}^{N-1}.$$

Accordingly, passing to the limit as $t \to 0^+$ in (4.12), we get

$$\frac{1}{b^2}\int_\Omega \frac{\nabla v\,\nabla\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx - \frac{1}{b}\int_{\partial\Omega}\phi\,d\mathcal{H}^{N-1} \ge 0.$$

Replacing ϕ with $-\phi$, we obtain

$$\frac{1}{b^2}\int_\Omega \frac{\nabla v\,\nabla\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx - \frac{1}{b}\int_{\partial\Omega}\phi\,d\mathcal{H}^{N-1} = 0.$$

The change of variable $u = -\frac{1}{b}\log(v)$ gives

$$\int_\Omega \frac{\nabla u\,\nabla\phi}{\sqrt{1+|\nabla u|^2}}\,dx = \int_\Omega\Big(-au + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\phi\,dx - \int_{\partial\Omega}\phi\,d\mathcal{H}^{N-1}.$$

By the integration by parts formula in [2, Proposition 1.3], we infer

$$\int_\Omega\Big(\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) - au + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\phi\,dx = \int_{\partial\Omega}\Big(\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}},\nu\Big]+1\Big)\phi\,d\mathcal{H}^{N-1}.$$

Hence, using (4.11), we have

$$\int_{\partial\Omega}\Big(\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}},\nu\Big]+1\Big)\phi\,d\mathcal{H}^{N-1} = 0. \tag{4.13}$$

Since (4.13) holds for all $\phi \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ such that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, $\phi(x) = 0$ whenever $u(x) = 0$, we conclude that

$$\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}},\nu\Big](x) = -1,$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ such that $u(x) > 0$. □

The converse of the previous result holds.

Proposition 4.13. *Let u be a generalized solution of problem (1.2). Then $v = e^{-bu} \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ satisfies (4.5) for all $w \in BV(\Omega)$.*

Proof. It follows by Proposition 4.11 that $v = e^{-bu} \in W^{1,1}(\Omega)$ and $\exp(-\frac{b^2}{a}) \le \operatorname*{ess\,inf}_\Omega v \le \operatorname*{ess\,sup}_\Omega v \le 1$.

Step 1. Inequality (4.5) holds for all $w \in W^{1,1}(\Omega)$ such that $w = 1$ $\mathcal{H}^{N-1}$-a.e. on Ω. Let $w \in W^{1,1}(\Omega)$ satisfy $w = 1$ $\mathcal{H}^{N-1}$-a.e. on Ω and set $\phi = w - v$. Observe that $\phi \in W^{1,1}(\Omega)$ is such that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, $\phi(x) = 0$ whenever $u(x) = 0$, or equivalently $v(x) = 1$. According to the boundary behaviour of u, we have

$$\int_{\partial\Omega} \left(\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu \right] + \operatorname{sgn}(u) \right) \phi \, d\mathcal{H}^{N-1} = 0.$$

On the other hand, multiplying by ϕ the equation in (1.2), integrating over Ω and applying the integration by parts formula in [2, Proposition 1.3], we obtain

$$\begin{aligned}\int_\Omega \frac{\nabla u \, \nabla \phi}{\sqrt{1+|\nabla u|^2}} \, dx &= \int_\Omega \left(-au + \frac{b}{\sqrt{1+|\nabla u|^2}} \right) \phi \, dx + \int_{\partial\Omega} \left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu \right] \phi \, d\mathcal{H}^{N-1} \\ &= \int_\Omega \left(-au + \frac{b}{\sqrt{1+|\nabla u|^2}} \right) \phi \, dx - \int_{\partial\Omega} \operatorname{sgn}(u) \, \phi \, d\mathcal{H}^{N-1}.\end{aligned}$$

The change of variable $v = e^{-bu}$ yields

$$\begin{aligned}\frac{1}{b^2} \int_\Omega \frac{\nabla v \, \nabla \phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} \, dx &+ \int_\Omega \frac{v \, \phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} \, dx \\ &= -\frac{a}{b^2} \int_\Omega \log(v) \, \phi \, dx + \frac{1}{b} \int_{\partial\Omega} \operatorname{sgn}(1-v) \, \phi \, d\mathcal{H}^{N-1}.\end{aligned}$$

Then the convexity in $[0, +\infty[\times \mathbb{R}^N$ and the differentiability in $]0, +\infty[\times \mathbb{R}^N$ of the map $(s, \xi) \mapsto \sqrt{s^2 + b^{-2}|\xi|^2}$, together with the condition $\operatorname{ess\,inf}_\Omega v > 0$, yield

$$\begin{aligned}\int_\Omega \sqrt{(v+\phi)^2 + b^{-2}|\nabla(v+\phi)|^2} \, dx &- \int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2} \, dx \\ &\geq -\frac{a}{b^2} \int_\Omega \log(v) \, \phi \, dx + \frac{1}{b} \int_{\partial\Omega} \operatorname{sgn}(1-v) \, \phi \, d\mathcal{H}^{N-1}.\end{aligned}$$

Since

$$\operatorname{sgn}(1-v) \, \phi + |v + \phi - 1| - |v - 1| \geq 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega,$$

we infer that

$$\mathcal{J}(v+\phi) - \mathcal{J}(v) \geq -\frac{a}{b^2} \int_\Omega \log(v) \, \phi \, dx,$$

which is (4.5) as $v + \phi = w$.

Step 2. Inequality (4.5) *holds for all* $w \in BV(\Omega)$. Pick $w \in BV(\Omega)$. According to Proposition 4.5, there exists a sequence $(w_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\lim_{n \to +\infty} w_n = w \quad \text{in } L^1(\Omega), \qquad \lim_{n \to +\infty} \mathcal{J}(w_n) = \mathcal{J}(w),$$
$$w_n = 1 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \qquad \text{for all } n.$$

By Step 1, for all n we have

$$\mathcal{J}(w_n) - \mathcal{J}(v) \geq -\frac{a}{b^2} \int_\Omega \log(v) \, (w_n - v) \, dx.$$

Then, passing to the limit as $n \to +\infty$, we obtain (4.5). □

The following is the main result of this section.

Theorem 4.5. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then, for every a, $b > 0$, problem* (1.2) *has a unique generalized solution u, which also satisfies*

(*i*) $u \in C^\infty(\Omega)$;
(*ii*) $0 \le u(x) \le b/a$ *for all* $x \in \Omega$;
(*iii*) u *minimizes in* $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *the functional*

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z+\tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega}|e^{-bz}-1|\,d\mathcal{H}^{N-1}.$$

Proof. The proof is divided into two steps.

Step 1. Problem (1.2) *has a unique generalized solution* u *satisfying* (*i*) *and* (*ii*). By Proposition 4.7 and Proposition 4.10, we know that the functional $\mathcal{I}$ admits a unique global minimizer v in $BV^+(\Omega)$, which satisfies $v \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ and $\exp\big(-\frac{b^2}{a}\big) \le v \le 1$ in Ω. Hence Proposition 4.9, Proposition 4.11, Proposition 4.12 and Proposition 4.13 yield the conclusion.

Step 2. The function u *minimizes in* $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *the functional*

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z+\tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega}|e^{-bz}-1|\,d\mathcal{H}^{N-1}.$$

This can be easily deduced from the fact that

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z+\tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega}|e^{-bz}-1|\,d\mathcal{H}^{N-1} = \mathcal{I}(e^{-bz})$$

and $v = e^{-bu}$ minimizes $\mathcal{I}$ in $BV^+(\Omega)$. □

Remark 4. As a consequence of the structure of equation (1.1), classical results, such as [43, Theorem 5.8.6], guarantee that the solution u is actually analytic in Ω.

Remark 5. The last conclusion of Theorem 4.5 shows that all generalized solutions enjoy some form of stability: we refer to [12] for a discussion of this matter.

5. Boundary behaviour: classical versus singular solutions. In this section we discuss the behaviour at the boundary of Ω of the generalized solutions of problem (1.2), whose existence and interior regularity have been proved in Section 4. This is achieved first by proving a comparison principle for upper and lower solutions and then by exhibiting upper and lower solutions which satisfy the homogeneous Dirichlet boundary conditions. Again, we refer the readers to [11] for additional details.

5.1. A comparison principle. We present here a comparison principle and we state some of its consequences.

Proposition 5.1. *Let* γ, $\delta \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *satisfy* $\operatorname*{ess\,inf}_\Omega \gamma > 0$, $\operatorname*{ess\,inf}_\Omega \delta > 0$,

$$\mathcal{J}(\gamma + z) - \mathcal{J}(\gamma) \ge -\frac{a}{b^2}\int_\Omega \log(\gamma)\,z\,dx,$$

for all $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *with* $z \le 0$ *a.e. in* Ω, *and*

$$\mathcal{J}(\delta + z) - \mathcal{J}(\delta) \ge -\frac{a}{b^2}\int_\Omega \log(\delta)\,z\,dx,$$

for all $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, *with* $z \ge 0$ *a.e. in* Ω. *Then* $\gamma \le \delta$ *a.e. in* Ω.

Proof. The proof follows using similar ideas as in Step 2 of the proof of Lemma 2.1. □

5.2. Upper and lower solutions. We introduce a notion of upper and lower solutions for problem (1.2), which has already been considered in [35, Proposition 4] for studying the minimal surface equation.

Definition 5.1. Let $\beta \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ be such that $\operatorname{div}\Big(\frac{\nabla \beta}{\sqrt{1+|\nabla\beta|^2}}\Big) \in L^N(\Omega)$. We say that β is an upper solution of problem (1.2) if

$$-\operatorname{div}\Big(\frac{\nabla \beta}{\sqrt{1+|\nabla\beta|^2}}\Big) \geq -a\beta + \frac{b}{\sqrt{1+|\nabla\beta|^2}} \quad \text{a.e. in } \Omega \tag{5.1}$$

and, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $\beta(x) \geq 0$ or both $\beta(x) < 0$ and $\Big[\frac{\nabla \beta}{\sqrt{1+|\nabla\beta|^2}}, \nu\Big](x) = 1$.

A lower solution α is defined similarly by reversing the inequality in (5.1) and assuming that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $\alpha(x) \leq 0$ or both $\alpha(x) > 0$ and $\Big[\frac{\nabla \alpha}{\sqrt{1+|\nabla\alpha|^2}}, \nu\Big](x) = -1$.

Remark 6. It is clear that a function u is a solution of problem (1.2) if and only if u is simultaneously an upper solution and a lower solution of the problem.

Lemma 5.2. *Let β be an upper solution of (1.2) and set $\gamma = e^{-b\beta}$. Then $\gamma \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $\operatorname*{ess\,inf}_{\Omega} \gamma > 0$, $\operatorname{div}\Big(\frac{\nabla \gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big) \in L^N(\Omega)$ and*

$$\mathcal{J}(\gamma + z) - \mathcal{J}(\gamma) \geq -\frac{a}{b^2}\int_\Omega \log(\gamma)\, z\, dx, \tag{5.2}$$

for all $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, with $z \leq 0$ a.e. in Ω.

Proof. From the assumptions on β it is easy to deduce that $\gamma = e^{-b\beta} \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfies $\operatorname*{ess\,inf}_{\Omega} \gamma > 0$, $\operatorname{div}\Big(\frac{\nabla \gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big) \in L^N(\Omega)$,

$$-\operatorname{div}\Big(\frac{\nabla \gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big) \leq -a\log(\gamma) - \frac{b^2\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}} \quad \text{a.e. in } \Omega \tag{5.3}$$

and, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $\gamma(x) \leq 1$, or both $\gamma(x) > 1$ and

$$\Big[\frac{\nabla \gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}, \nu\Big](x) = -b. \tag{5.4}$$

Relation (5.4), in case $\gamma(x) > 1$, can be easily deduced from the equality

$$\Big[\frac{\nabla \gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}, \nu\Big] = -b\Big[\frac{\nabla \beta}{\sqrt{1+|\nabla\beta|^2}}, \nu\Big].$$

In order to prove (5.2), let us fix $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ with $z \leq 0$ a.e. in Ω. Multiplying (5.3) by z, integrating over Ω and using again formula (1.9) in [2, Proposition 1.3], as well as the convexity in $[0, +\infty[\, \times \mathbb{R}^N$ and the differentiability in $]0, +\infty[\, \times \mathbb{R}^N$ of the map $(s, \xi) \mapsto \sqrt{s^2 + b^{-2}|\xi|^2}$, together with the condition $\operatorname*{ess\,inf}_{\Omega} \gamma > 0$, we get

$$\mathcal{J}(\gamma + z) - \mathcal{J}(\gamma) \geq -\frac{a}{b^2}\int_\Omega \log(\gamma) z\, dx + \frac{1}{b^2}\int_{\partial\Omega}\Big[\frac{\nabla \gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}, \nu\Big] z\, d\mathcal{H}^{N-1}$$
$$+ \frac{1}{b}\Big(\int_{\partial\Omega} |\gamma + z - 1|\, d\mathcal{H}^{N-1} - \int_{\partial\Omega} |\gamma - 1|\, d\mathcal{H}^{N-1}\Big).$$

Observe that on the set $\partial\Omega\cap\{\gamma\le 1\}$, as $z\le 0$ a.e. in Ω, we have $|\gamma+z-1|-|\gamma-1| = |z|$ and, by [2, Theorem 1.1],

$$\begin{aligned}\frac{1}{b^2}\Big|\int_{\partial\Omega\cap\{\gamma\le 1\}}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big]z\,d\mathcal{H}^{N-1}\Big|&\\ \le \frac{1}{b^2}\Big\|\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big\|_{L^\infty(\Omega)}\int_{\partial\Omega\cap\{\gamma\le 1\}}|z|\,d\mathcal{H}^{N-1}&\\ \le \frac{1}{b}\int_{\partial\Omega\cap\{\gamma\le 1\}}|z|\,d\mathcal{H}^{N-1}.&\end{aligned}$$

On the other hand, on $\partial\Omega\cap\{\gamma>1\}$, using the condition $z\le 0$ a.e. in Ω, we see that

$$\begin{aligned}\frac{1}{b^2}\int_{\partial\Omega\cap\{\gamma>1\}}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big]z\,d\mathcal{H}^{N-1}&\\ +\frac{1}{b}\int_{\partial\Omega\cap\{\gamma>1\}}\Big(|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}&\\ =\frac{1}{b}\int_{\partial\Omega\cap\{\gamma>1\}}\Big(-z+|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}\ge 0.&\end{aligned}$$

This implies that, for all $z\in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ with $z\le 0$ a.e. in Ω,

$$\mathcal{J}(\gamma+z)-\mathcal{J}(\gamma)\ge -\frac{a}{b^2}\int_\Omega \log(\gamma)\,z\,dx,$$

which is the conclusion. □

Proposition 5.2. *Let β be an upper solution of (1.2) and u be a solution of (1.2). Then $u\le\beta$ a.e. in Ω.*

Proof. The conclusion follows from Lemma 5.2, Proposition 4.13 and Proposition 5.1. □

A similar statement holds for lower solutions as well.

Proposition 5.3. *Let α be a lower solution of (1.2) and u be a solution of (1.2). Then $u\ge\alpha$ a.e. in Ω.*

5.3. Boundary behaviour of generalized solutions. We start by constructing an upper solution which vanishes at those boundary points of Ω where a suitable exterior sphere condition holds.

Lemma 5.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N\ge 2$, having a Lipschitz boundary $\partial\Omega$, which satisfies an exterior sphere condition with radius $r\ge (N-1)b/a$ at some $x_0\in\partial\Omega$. Then there exists an upper solution β of problem (1.2) such that $\beta(x_0)=0$. In case $r>(N-1)b/a$, the upper solution can be chosen in such a way to satisfy a bounded slope condition at x_0, that is, $\sup_{x\in\Omega}\dfrac{\beta(x)}{|x-x_0|}<+\infty$.*

Proof. According to Definition 1.9, there exist $r\ge (N-1)b/a$ and $y\in\mathbb{R}^N$ with $B(y,r)\cap\Omega=\emptyset$ and $x_0\in\overline{B(y,r)}\cap\partial\Omega$. Pick a constant $R\ge r+\frac{b}{a}$ such that

$$\Omega\subseteq S_{r,R}=\{x\in\mathbb{R}^N\mid r<|x-y|<R\}.$$

Next define a function $\eta : [r, R] \to \mathbb{R}$, by

$$\eta(t) = \begin{cases} \sqrt{\left(\frac{b}{a}\right)^2 - \left(t - \left(r + \frac{b}{a}\right)\right)^2} & \text{if } r \leq t < r + \frac{b}{a}, \\ \frac{b}{a} & \text{if } r + \frac{b}{a} \leq t \leq R, \end{cases}$$

and a function $\beta : \overline{S_{r,R}} \to \mathbb{R}$ by

$$\beta(x) = \eta(|x - y|). \tag{5.5}$$

Then a simple direct calculation shows that, for a.e. $t \in]r, R[$,

$$-\left(\frac{t^{N-1}\eta'(t)}{\sqrt{1+\eta'(t)^2}}\right)' \geq t^{N-1}\left(-a\eta(t) + \frac{b}{\sqrt{1+\eta'(t)^2}}\right).$$

Hence we conclude that β is an upper solution of (1.2) which satisfies $\beta(x_0) = 0$ and $\beta \geq 0$ on $\partial\Omega$. In Figure 4 the profile of such an upper solution β is plotted.

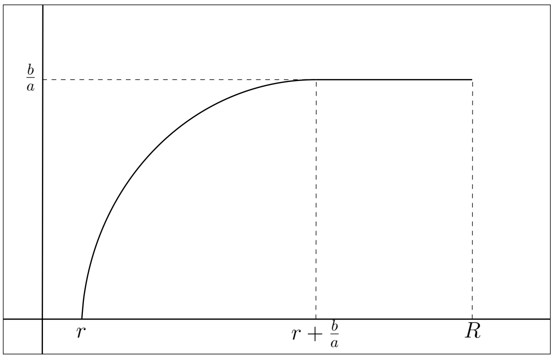

FIGURE 4. Profile of the upper solution.

In case $r > (N-1)b/a$, we modify the definition of η as follows

$$\eta(t) = \begin{cases} c\sqrt{\left(\frac{b}{a}\right)^2 + \varepsilon^2 - \left(t - \left(r + \frac{b}{a}\right)\right)^2} - \varepsilon c & \text{if } r \leq t < r + \frac{b}{a}, \\ \frac{b}{a} & \text{if } r + \frac{b}{a} \leq t \leq R, \end{cases}$$

where $c = \frac{a}{b}\left(\varepsilon + \sqrt{\left(\frac{b}{a}\right)^2 + \varepsilon^2}\right)$, for some $\varepsilon > 0$ suitably chosen. It is then easy to see that the function β defined by (5.5) is an upper solution of (1.2), which satisfies $\beta(x_0) = 0$ as well as a bounded slope condition at x_0. □

Hence the following result holds.

Theorem 5.4. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then, for every $a > 0$, $b > 0$, problem (1.2) has a unique generalized solution u, which also satisfies*

(i) $u \in C^\infty(\Omega)$;

(*ii*) *at each point* $x_0 \in \partial\Omega$ *where an exterior sphere condition with radius* $R \geq (N-1)\, b/a$ *holds,* u *is continuous and satisfies* $u(x_0) = 0$*; moreover, if* $R > (N-1)\, b/a$*, then* u *also satisfies a bounded slope condition at* x_0*, that is* $\sup_{x\in\Omega} \dfrac{u(x)}{|x - x_0|} < +\infty$*;*

(*iii*) $u \in L^\infty(\Omega)$*, with* $0 < u(x) < \frac{b}{a}$ *for all* $x \in \Omega$*;*

(*iv*) u *minimizes in* $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *the functional*

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\, dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z + \tfrac{1}{b}\big)\, dx + \frac{1}{b}\int_{\partial\Omega} |e^{-bz} - 1|\, d\mathcal{H}^{N-1}.$$

Proof. The proof is divided into four steps.

Step 1. Existence and uniqueness of a generalized solution u*.* Existence and uniqueness of a generalized solution, also satisfying (i), (iv) and $0 \leq u \leq \frac{b}{a}$ in Ω, follow from Theorem 4.5.

Step 2. The solution u *is such that* $u(x) > 0$ *for all* $x \in \Omega$*.* We already know that $u(x) \geq 0$ for all $x \in \Omega$. Assume by contradiction that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Note that the equation in (1.2) can be written as

$$\frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \sum_{i,j=1}^N \frac{\partial_{x_i} u\, \partial_{x_j} u\, \partial^2_{x_j x_i} u}{(1+|\nabla u|^2)^{3/2}} = au - \frac{b}{\sqrt{1+|\nabla u|^2}} \quad \text{in } \Omega. \tag{5.6}$$

By evaluating (5.6) at x_0, we obtain $\Delta u(x_0) = -b < 0$, thus contradicting the fact that x_0 is a minimum point of u in Ω.

Step 3. The solution u *is such that* $u(x) < b/a$ *for all* $x \in \Omega$*.* Let B be an open ball in $\mathbb{R}^N$ such that $\overline{\Omega} \subseteq B$. According to Theorem 3.3 there exists a unique solution $\beta \in C^2(\overline{B})$ of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

which in addition satisfies $\beta(x) < b/a$ for all $x \in \overline{B}$. In particular, β is an upper solution of (1.2). The conclusion then follows from Proposition 5.2.

Step 4. The solution u *satisfies condition* (ii)*.* This can be easily deduced from Lemma 5.3 and Proposition 5.2. □

From Theorem 5.4 we deduce the following two simple corollaries.

Corollary 5.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$*, with* $N \geq 2$*, having a Lipschitz boundary* $\partial\Omega$*. Then, for every* $a > 0$*,* $b > 0$*, problem* (1.2) *has a unique generalized solution* u*, which also satisfies* (i)*,* (iii)*,* (iv) *of Theorem 5.4 and*

(ii') *the set of points* $x_0 \in \partial\Omega$*, where* u *satisfies a bounded slope condition and* $u(x_0) = 0$*, is non-empty.*

Proof. In order to verify (ii') it is enough to observe that, at each point $x_0 \in \partial\Omega \cap \partial\mathrm{Conv}(\overline{\Omega})$, where $\mathrm{Conv}(\overline{\Omega})$ denotes the convex hull of $\overline{\Omega}$, an exterior sphere condition holds for any given radius $R > 0$. □

Corollary 5.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$*, with* $N \geq 2$*, having a Lipschitz boundary* $\partial\Omega$*, which satisfies, at each point* $x_0 \in \partial\Omega$*, an exterior sphere condition with radius* $R \geq (N-1)\, b/a$*. Then, for every* $a > 0$*,* $b > 0$*, problem* (1.2) *has a unique generalized solution* u*, which also satisfies* (i)*,* (iii)*,* (iv) *of Theorem 5.4 and is classical.*

5.4. Classical versus singular solutions. We conclude this section by combining the results from Section 2 and Section 4 so as to provide a rather complete picture of the structure of the solution set of (1.2), in case of an arbitrary regular domain Ω. Namely, we show that, for all $a > 0$, there is a threshold $b^* > 0$, possibly depending on a, such that, for $0 < b \leq b^*$, problem (1.2) has a unique generalized solution, which is classical, while, for $b > b^*$, problem (1.2) has a unique generalized solution, which is singular.

Theorem 5.5. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in]0,1[$. Then, for every $a > 0$, either for all $b > 0$ problem (1.2) has a unique generalized solution $u = u(a,b)$, which is classical, or there exists $b^* = b^*(a) \in]0,+\infty[$ such that*

- *if $b \in]0, b^*]$, then problem (1.2) has a unique generalized solution u, which is classical;*
- *if $b \in]b^*, +\infty[$, then problem (1.2) has a unique generalized solution u, which is singular.*

In addition, the following conclusions hold:

- *the map $a \mapsto b^*(a)$ is increasing, with $\inf_{a>0} b^*(a) > 0$;*
- *the map $(a,b) \mapsto u(a,b)$ is continuous from $\mathbb{R}^+_0 \times \mathbb{R}^+$ to $L^\infty(\Omega)$;*
- *for any $a > 0$, the map $b \mapsto u(a,b)$ is strictly increasing, in the sense that if $b_1 < b_2$, then $u(a,b_1) < u(a,b_2)$ in Ω;*
- *for any $b > 0$, the map $a \mapsto u(a,b)$ is strictly decreasing, in the sense that if $a_1 < a_2$, then $u(a_1,b) > u(a_2,b)$ in Ω.*

Proof. From Theorem 5.4 we know that, for every a, $b > 0$, problem (1.2) has a unique generalized solution $u = u(a,b)$. On the other hand, Theorem 2.3 guarantees, for each $a > 0$, the existence of $b_\infty = b_\infty(a) \in]0,+\infty]$ such that, if $b \in [0, b_\infty[$, such a solution $u(a,b)$ is classical and continuously depends on the parameters in the topology of $L^\infty(\Omega)$.

Let us fix $a > 0$ and take b_1, $b_2 > 0$, with $b_1 < b_2$. It is clear that the solution $u(a,b_1)$ of (1.2), with $b = b_1$, is a lower solution of (1.2), with $b = b_2$. Hence Proposition 5.3 implies that $u(a,b_1) < u(a,b_2)$ in Ω, that is, the map $b \mapsto u(a,b)$ is strictly increasing.

Similarly, if we fix $b > 0$ and we take a_1, $a_2 > 0$, with $a_1 < a_2$, then the solution $u(a_1,b)$ of (1.2), with $a = a_1$, is an upper solution of (1.2), with $a = a_2$. Thus we get by Proposition 5.2 that $u(a_1,b) > u(a_2,b)$ in Ω, that is, the map $a \mapsto u(a,b)$ is strictly decreasing.

Hence we also deduce that, if for some (a_0,b_0), with a_0, $b_0 > 0$, the solution $u(a_0,b_0)$ is classical, then, for all (a,b) such that $a \geq a_0$ and $0 < b \leq b_0$, $u(a,b)$ is still classical. Therefore, defining for each $a > 0$

$$b^* = b^*(a) = \sup\{b \mid \text{problem (1.2) has a classical solution}\} \, (\geq b_\infty(a)),$$

we conclude that, if $b \in]0, b^*[$, then $u(a,b)$ is classical, whereas, if $b^* < +\infty$ and $b \in]b^*, +\infty[$, then $u(a,b)$ is singular. In addition, we infer that the map $a \mapsto b^*(a)$ is increasing. Further, as from Remark 2 we know that, for $a = 0$, problem (1.2) has classical solutions for all small $b > 0$, we easily conclude that

$$\inf_{a>0} b^*(a) > 0.$$

Next we show that the map $(a,b) \mapsto u(a,b)$ is continuous from $\mathbb{R}^+_0 \times \mathbb{R}^+_0$ to $L^\infty(\Omega)$. Let us fix (a_0,b_0), with a_0, $b_0 > 0$, and let $\varepsilon \in]0,1[$ be given. Set $\alpha = u(a_0,b_0) - \varepsilon$

and $\beta = u(a_0, b_0) + \varepsilon$. It is easy to verify that α and β are, respectively, a lower solution and an upper solution of problem (1.2), for all (a, b) satisfying $a, b > 0$ and

$$(1 + \tfrac{b_0}{a_0})|a - a_0| + |b - b_0| \le a_0\, \varepsilon.$$

Proposition 5.2 and 5.3 then imply that, for all such (a, b),

$$u(a_0, b_0) - \varepsilon = \alpha \le u(a, b) \le \beta = u(a_0, b_0) + \varepsilon \quad \text{in } \Omega$$

and thus the conclusion. We finally notice that the continuity of the map $(a, b) \mapsto u(a, b)$ implies in particular that, for each $a > 0$, the solution $u(a, b^*)$ is classical. $\square$

6. Singular solutions on spherical shells. The aim of this section is to show that, in dimension $N \ge 2$, problem (1.2) may have singular solutions which do not attain the Dirichlet boundary condition. Of course, this will happen at those points of $\partial\Omega$ for which the exterior sphere condition considered in Theorem 5.4 is not satisfied. To this end, let us introduce, for any given r, R, with $0 < r < R$, the spherical shell

$$S_{r,R} = \{x \in \mathbb{R}^N \mid r < |x| < R\}$$

and consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1 + |\nabla u|^2}} & \text{in } S_{r,R}, \\ u = 0 & \text{on } \partial S_{r,R}, \end{cases} \tag{6.1}$$

where a, $b > 0$ are given constants.

We begin by observing that the solutions of (6.1) are regular in $S_{r,R} \cup \partial B$, where B denotes the open ball of center 0 and radius R.

Lemma 6.1. *Let $a > 0$, $b > 0$ and r, R, with $0 < r < R$ be given. Let u be the solution of (6.1). Then there exists $v \in C^2(]r, R])$ such that $u(x) = v(|x|)$ for all $x \in S_{r,R} \cup \partial B$, which satisfies $v(R) = 0$, either $v(r) = 0$, or $v(r) > 0$ and $v'(r) = +\infty$, and $v''(t) < 0$ for all $t \in]r, R[$.*

Proof. The invariance under rotations both of the equation in (6.1) and of the domain $S_{r,R}$ implies that the solution u of (6.1), provided by Theorem 5.4, is radially symmetric. Hence there exists $v \in C^2(]r, R[)$ such that

$$u(x) = v(|x|)$$

for all $x \in S_{r,R}$. It is plain that v satisfies

$$-\left(\frac{t^{N-1} v'}{\sqrt{1 + v'^2}}\right)' = t^{N-1}\left(-av + \frac{b}{\sqrt{1 + v'^2}}\right) \quad \text{in }]r, R[,$$

or, equivalently,

$$v'' = \left(av - \frac{b}{\sqrt{1 + v'^2}}\right)(1 + v'^2)^{3/2} - \frac{N-1}{t} v'(1 + v'^2) \quad \text{in }]r, R[, \tag{6.2}$$

and, as the exterior sphere condition holds at the points of ∂B,

$$v(R) = 0, \qquad \text{either} \quad v(r) = 0, \quad \text{or} \quad v(r) > 0 \text{ and } v'(r) = +\infty.$$

Since $v(t) < b/a$ for all $t \in]r, R[$, we see from (6.2) that v satisfies $v''(t_0) < 0$, at any critical point $t_0 \in]r, R[$. The positivity and the boundary behaviour then imply that v has a unique maximum point, say, at $t_0 \in]r, R[$, with $v'(t) > 0$ for all $t \in]r, t_0[$ and $v'(t) < 0$ for all $t \in]t_0, R[$. Let us prove that $v''(t) < 0$ for all $t \in]r, R[$. Assume, by contradiction, that there exists $\bar{t} \in]r, R[$ such that $v''(\bar{t}) \ge 0$. Suppose first that $\bar t \in\,]r, t_0[$. We can assume, without loss of generality, that $v''(\bar t) = 0$ and $v''(t) < 0$ for all $t \in\,]\bar t, t_0]$. Computing and evaluating v''' at $\bar t$ by means of (6.2), we obtain

$$v'''(\bar t) = av'(\bar t)(1 + v'(\bar t)^2)^{3/2} + \frac{N-1}{\bar t^2} v'(\bar t)(1 + v'(\bar t)^2) > 0.$$

As $v''(\bar t) = 0$ we deduce that $v''(t) > 0$ for all t belonging to a right neighborhood of $\bar t$, which is a contradiction. Similarly, we find a contradiction if we suppose that $\bar t \in\,]t_0, R[$. Since u satisfies a bounded slope condition at all points x with $|x| = R$, we conclude, by the concavity of v, that $v'(R) = \lim_{t\to R^-} v'(t)$ is finite. From (6.2) we infer that $v \in C^2(]r, R])$. □

6.1. Nonexistence of classical solutions on thick shells.

Proposition 6.1. *For any given $N \geq 2$, $a > 0$ and $r > 0$, there exist $R^* > 0$ and $b^* > 0$ such that, for all $R > R^*$ and $b > b^*$, the solution u of problem (6.1) is a radially symmetric function satisfying $u \in C^2(S \cup \partial B)$,*

$$\begin{aligned} &u(x) = 0 \quad \text{if } |x| = R,\\ &u(x) > 0 \quad \text{and} \quad \Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big](x) = -1 \quad \text{if } |x| = r. \end{aligned} \tag{6.3}$$

Proof. Let $N \geq 2$, $a > 0$ and $r > 0$ be fixed. By Lemma 6.1 we know that any solution u of problem (6.1) is radially symmetric and belongs to $C^2(S \cup \partial B)$. We prove the existence of $R^* > 0$ and $b^* > 0$ such that, for all $R > R^*$ and $b > b^*$, the solution u of (6.1) satisfies (6.3). This is achieved by exhibiting a lower solution α of (6.1) such that

$$\alpha(x) > 0 \quad \text{and} \quad \Big[\frac{\nabla \alpha}{\sqrt{1+|\nabla \alpha|^2}}, \nu\Big](x) = -1 \quad \text{if } |x| = r.$$

The comparison principle stated in Proposition 5.3 implies that $u \geq \alpha$ in $S_{r,R}$ and thus the conclusion follows.

The remainder of this proof is devoted to the construction of such a lower solution: in Figure 5 we plot its profile with reference to steps 1, 2, 3 below.

Step 1. Beginning the construction with an arc of circle. For every δ, $\eta > 0$ let us define, for all $t \in [r, r + 2\delta]$,

$$w(t) = \eta + \sqrt{\delta^2 - (r + \delta - t)^2}.$$

It is clear that $w \in W^{1,1}(r, r + 2\delta) \cap C^\infty(]r, r + 2\delta[)$ and, for all $t \in [r, r + 2\delta]$,

$$\frac{1}{\sqrt{1 + w'(t)^2}} = \frac{1}{\delta}(w(t) - \eta), \qquad \frac{w'(t)}{\sqrt{1 + w'(t)^2}} = \frac{1}{\delta}(r + \delta - t),$$

$$-\Big(\frac{t^{N-1} w'(t)}{\sqrt{1 + w'(t)^2}}\Big)' = \frac{1}{\delta} t^{N-2}\big(Nt - (N-1)(r+\delta)\big),$$

$$t^{N-1}\Big(-aw(t) + \frac{b}{\sqrt{1 + w'(t)^2}}\Big) = t^{N-1}\Big(\Big(\frac{b}{\delta} - a\Big)w(t) - \frac{b}{\delta}\eta\Big).$$

If we pick δ, η such that

$$\delta > \frac{r}{N-1} \quad \text{and} \quad 0 < \eta < \frac{(N-1)\delta - r}{ar\delta},$$

we get

$$r(1 + a\delta\eta) < (N-1)\delta$$

or, equivalently,

$$\frac{1}{\delta} r^{N-2}\Big(Nr - (N-1)(r+\delta)\Big) < r^{N-1}\bigg(\Big(\frac{b}{\delta} - a\Big)\eta - \frac{b}{\delta}\eta\bigg).$$

As $w(r) = \eta$, by continuity there exists $\varepsilon \in\,]0, \delta[$ such that, for all $t \in [r, r+\varepsilon]$,

$$\begin{aligned} -\bigg(\frac{t^{N-1} w'(t)}{\sqrt{1+w'(t)^2}}\bigg)' &= \frac{1}{\delta} t^{N-2}\big(Nt - (N-1)(r+\delta)\big) \\ &< t^{N-1}\bigg(\Big(\frac{b}{\delta} - a\Big) w(t) - \frac{b}{\delta}\eta\bigg) \\ &= t^{N-1}\bigg(-a w(t) + \frac{b}{\sqrt{1+w'(t)^2}}\bigg). \end{aligned}$$

In addition, there exists $\theta > 0$ such that, for all $t \in [r+\varepsilon, r+2\delta-\varepsilon]$,

$$w(t) - \eta \ge \theta.$$

If we take

$$b > \frac{(r+\delta)\big(1 + a\delta(\eta+\delta)\big)}{r\theta},$$

we get, for all $t \in [r+\varepsilon, r+\delta]$,

$$\begin{aligned} Nt - (N-1)(r+\delta) \le r + \delta &< br\theta - a\delta(r+\delta)(\eta+\delta) \\ &\le bt\big(w(t) - \eta\big) - a\delta t w(t) \end{aligned}$$

and hence

$$\begin{aligned} -\bigg(\frac{t^{N-1} w'(t)}{\sqrt{1+w'(t)^2}}\bigg)' &= \frac{1}{\delta} t^{N-2}\big(Nt - (N-1)(r+\delta)\big) \\ &< t^{N-1}\Big(\frac{b}{\delta}\big(w(t) - \eta\big) - a w(t)\Big) \\ &= t^{N-1}\bigg(-a w(t) + \frac{b}{\sqrt{1+w'(t)^2}}\bigg). \end{aligned}$$

Further, by continuity, we can find $r^* \in\,]r+\delta, r+\delta+\frac{\delta}{\sqrt{2}}[$ such that the above inequality holds in $[r, r^*]$ too. We can also suppose that, for all $s \in [r+\delta, r^*]$,

$$|w'(s)| < 1.$$

Note finally that r^* can be chosen to be an increasing function of b.

Step 2. Continuing the construction with a segment. Let us set

$$R^* = r^* - \frac{w(r^*)}{w'(r^*)}$$

and, for all $t \in\,]r+\delta, r^*]$,

$$\varphi(t) = t - \frac{w(t)}{w'(t)} = t + \frac{w(t)\big(w(t) - \eta\big)}{t - (r+\delta)}.$$

As the range of the function φ is the interval $[R^*, +\infty[$, for every $R \ge R^*$, there is $s \in\,]r+\delta, r^*]$ such that

$$s - \frac{w(s)}{w'(s)} = R.$$

Let us define, for all $t \in [s,R]$,

$$z(t) = w'(s)(t-s) + w(s).$$

If we take

$$b > b^* = \max\left\{ \frac{(r+\delta)\big(1 + a\delta(\eta+\delta)\big)}{r\theta}, \Big(\frac{N-1}{r+\delta} + a(\eta+\delta)\Big)\frac{\delta}{s-(r+\delta)} \right\},$$

we have, in particular,

$$N - 1 \le (r+\delta)\Big(- a(\eta+\delta) + \frac{b}{\delta}\big(s - (r+\delta)\big)\Big)$$

and hence, using also $|w'(s)| < 1$, for all $t \in [s,R]$,

$$\begin{aligned}
-\bigg(\frac{t^{N-1}z'(t)}{\sqrt{1+z'(t)^2}}\bigg)' &= -\bigg(\frac{t^{N-1}w'(s)}{\sqrt{1+w'(s)^2}}\bigg)' = \frac{(N-1)t^{N-2}|w'(s)|}{\sqrt{1+w'(s)^2}} \le (N-1)t^{N-2} \\
&\le t^{N-2}(r+\delta)\Big(- a(\eta+\delta) + \frac{b}{\delta}\big(s-(r+\delta)\big)\Big) \\
&\le t^{N-1}\bigg(-aw(s) + \frac{b\,|w'(s)|}{\sqrt{1+w'(s)^2}}\bigg) \\
&\le t^{N-1}\bigg(-az(t) + \frac{b}{\sqrt{1+z'(t)^2}}\bigg).
\end{aligned}$$

Step 3. Concluding the construction. Let us set, for all $t \in [r,R]$,

$$v(t) = \begin{cases} w(t) & \text{if } r \le t \le s, \\ z(t) & \text{if } s \le t \le R. \end{cases}$$

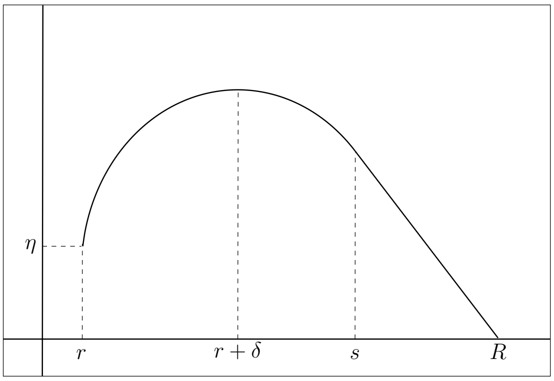

FIGURE 5. Profile of the lower solution.

Clearly, $v \in W^{1,1}(r,R) \cap W^{2,1}_{\mathrm{loc}}(]r,R[)$ and satisfies

$$\begin{cases} -\bigg(\dfrac{t^{N-1}v'(t)}{\sqrt{1+v'(t)^2}}\bigg)' \le t^{N-1}\bigg(-av(t) + \dfrac{b}{\sqrt{1+v'(t)^2}}\bigg) & \text{a.e. in }]r,R[\\ v(r) > 0, \quad v'(r) = +\infty, \quad v(R) = 0. \end{cases}$$

Then the function $\alpha : S_{r,R} \to \mathbb{R}$ defined by $\alpha(x) = v(|x|)$ is such that $\alpha \in W^{1,1}(S_{r,R})$ with a distributional divergence $\operatorname{div}\left(\frac{\nabla\alpha}{\sqrt{1+|\nabla\alpha|^2}}\right) \in L^\infty(S_{r,R})$. Indeed, we have for a.e. $x \in S_{r,R}$, setting $t = |x|$,

$$\operatorname{div}\left(\frac{\nabla\alpha(x)}{\sqrt{1+|\nabla\alpha(x)|^2}}\right) = t^{1-N}\left(\frac{t^{N-1}v'(t)}{\sqrt{1+v'(t)^2}}\right)' = \begin{cases} \dfrac{(N-1)(r+\delta)-Nt}{\delta t} & \text{if } r < t < s, \\ \dfrac{(N-1)}{t}\dfrac{|w'(s)|}{\sqrt{1+w'(s)^2}} & \text{if } s < t < R. \end{cases}$$

Moreover, the trace on $\partial S_{r,R}$ of the component of $\frac{\nabla\alpha}{\sqrt{1+|\nabla\alpha|^2}}$ with respect to the unit outer normal $\nu(x)$ to $S_{r,R}$ at any point x, with $|x| = r$, is

$$\Big[\frac{\nabla\alpha}{\sqrt{1+|\nabla\alpha|^2}}, \nu\Big](x) = -\frac{v'(r)}{\sqrt{1+v'(r)^2}} = -1.$$

Finally, as

$$\begin{aligned} &-\operatorname{div}\left(\frac{\nabla\alpha(x)}{\sqrt{1+|\nabla\alpha(x)|^2}}\right) \le -a\alpha(x) + \frac{b}{\sqrt{1+|\nabla\alpha(x)|^2}} \quad \text{a.e. in } S_{r,R}, \\ &\alpha(x) > 0 \quad \text{and} \quad \Big[\frac{\nabla\alpha}{\sqrt{1+|\nabla\alpha|^2}}, \nu\Big](x) = -1 \quad \text{if } |x| = r, \\ &\alpha(R) = 0, \end{aligned}$$

we can conclude that α is a lower solution of (6.1). □

6.2. Existence of classical solutions on thin shells. It is worth observing that the conclusions of Proposition 6.1 fail if R is not taken bounded away from r. This is a consequence of the following statement that proves the existence of classical smooth solutions on thin spherical shells. As a consequence, our results about the existence and the nonexistence of classical solutions are in some sense sharp, at least on spherical shells.

Proposition 6.2. *For any given $N \ge 2$, $a > 0$, $b > 0$ and $r > 0$, there exists $R_* > 0$ such that, for all $R \in\,]r, R_*[$, the solution u of problem (6.1) is classical, with $u \in C^2(\overline{S}_{r,R})$.*

Proof. Let $N \ge 2$, $a > 0$, $b > 0$ and $r > 0$ be fixed. We prove the existence of a constant $R_* > 0$ such that, for all $R \in\,]r, R_*[$, the solution u of (6.1), provided by Theorem 5.4, is classical and belongs to $C^2(\overline{S}_{r,R})$. From Lemma 6.1 we know that there exists $v \in C^2(]r, R])$ such that $u(x) = v(|x|)$ in $S_{r,R} \cup \partial B$. In addition, v is concave in $]r, R[$ and hence $v'(r)$ exists, possibly infinite. Then, taking R sufficiently small, we construct an upper solution $\beta \in C^2(\overline{S}_{r,R})$ of (6.1), with $\beta = 0$ on $\partial S_{r,R}$. Then the comparison principle stated in Proposition 5.3 implies that $0 \le u \le \beta$ in $S_{r,R}$. Thus $v'(r)$ is finite and the conclusion follows arguing as in the final part of the proof of Lemma 6.1.

The remainder of this proof is devoted to the construction of such an upper solution. For each $\delta > 0$ let us define, for all $t \in [r, r+2\delta]$,

$$w(t) = -\sqrt{3}\delta + \sqrt{4\delta^2 - (r+\delta-t)^2}.$$

It is clear that $w \in C^\infty([r, r+2\delta])$ and $w(r) = w(r+2\delta) = 0$. As we have, for all $t \in [r, r+2\delta]$,

$$-\left(\frac{t^{N-1}w'(t)}{\sqrt{1+w'(t)^2}}\right)' = \frac{1}{2\delta}t^{N-2}\big(Nt-(N-1)(r+\delta)\big)$$

and

$$t^{N-1}\left(-aw(t)+\frac{b}{\sqrt{1+w'(t)^2}}\right) = t^{N-1}\left(\Big(\frac{b}{2\delta}-a\Big)w(t)+b\frac{\sqrt{3}}{2}\right),$$

the inequality

$$-\left(\frac{t^{N-1}w'(t)}{\sqrt{1+w'(t)^2}}\right)' \geq t^{N-1}\left(-aw(t)+\frac{b}{\sqrt{1+w'(t)^2}}\right)$$

holds if and only if

$$\frac{1}{2\delta}\big(Nt-(N-1)(r+\delta)\big) \geq t\left(\Big(\frac{b}{2\delta}-a\Big)w(t)+\frac{\sqrt{3}}{2}b\right). \tag{6.4}$$

Take $\delta^* > 0$ satisfying both

$$b - 2\delta^* a > 0$$

and

$$\delta^*\big(4b\delta^* + 2br + N - 1\big) \leq r.$$

Then we have, for all $\delta \in \,]0, \delta^*[$,

$$(r+2\delta)2\delta b \leq r-(N-1)\delta = Nr-(N-1)(r+\delta).$$

Hence, we get, for all $t \in [r, r+2\delta]$, using $w(t) \leq (2-\sqrt{3})\delta$,

$$\begin{aligned} t\big((b-2\delta a)w(t)+b\sqrt{3}\delta\big) &\leq (r+2\delta)\big((b-2\delta a)(2-\sqrt{3})\delta+b\sqrt{3}\delta\big) \\ &= (r+2\delta)\big(2\delta b-2a\delta^2(2-\sqrt{3})\big) \\ &\leq (r+2\delta)2\delta b \leq Nr-(N-1)(r+\delta) \\ &\leq Nt-(N-1)(r+\delta), \end{aligned}$$

i.e. (6.4) holds. We set, for all $x \in S_{r,R}$,

$$\beta(x) = w(|x|)$$

and, arguing as in the last step of the proof of Proposition 6.1, we see that β is an upper solution of (6.1). Finally, setting $R_* = r + 2\delta_* > 0$, it follows that, for all $R \in \,]r, R_*[$, the solution u of (6.1) is classical and belongs to $C^2(\overline{S}_{r,R})$. □

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