First Passgae Time

Pietro Rossi

pietro.rossi3@unibo.it

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1 Ito calculus

1.1 Basic definition

In this section we shall summarize important results from Ito's calculus without any proof. Let X_t a stochastic processes where the stochastic differential is given by:

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t \tag{1}$$

Let us make no mistakes. The meaning we give (for the time being) to eq.(1) is given by:

$$X_{t+\Delta t} - X_t = \alpha(t, X_t)\Delta t + \beta(t, X_t)(\ W_{t+\Delta t} - W_t\ )$$

and

$$(\ W_{t+\Delta t} - W_t\ ) \stackrel{d}{=} N(0, \Delta t)$$

therefore $X_{t+\Delta t} - X_t$ is conditionally normal, more precisely:

$$X_{t+\Delta t} - X_t \stackrel{d}{=} N(\alpha(t, X_t)\Delta t, \beta(t, X_t)\sqrt{\Delta t}),$$

that in differential form we can write:

$$dX_t \sim N(\alpha(t, X_t)dt, \beta(t, X_t)\sqrt{dt}).$$

1.2 The Ito's theorem in one dimension

Given any function $f(t, x)$ we can define the process

$$Z_t = f(t, X_t).$$

It can be shown that the Ito's differential dZ_t can be written as:

$$\begin{aligned} dZ_t &= \left[\frac{\partial f(t, X_t)}{\partial t} + \alpha(t, X_t)\frac{\partial f(t, X_t)}{\partial X_t} + \frac{\beta^2(t, X_t)}{2}\frac{\partial^2 f(t, X_t)}{\partial^2 X_t}\right] dt \qquad (2) \\ &+ \beta(t, X_t)\frac{\partial f(t, X_t)}{\partial X_t} dW_t, \end{aligned}$$

We will often write

$$dZ_t = \frac{\partial f(t, X_t)}{\partial t} + \mathbf{A} f(t, X_t) + \beta(t, X_t)\frac{\partial f(t, X_t)}{\partial X_t} dW_t,$$

where we have singled out the infinitesimal generator:

$$\mathbf{A} = \alpha(t, x)\frac{\partial}{\partial x} + \frac{\beta^2(t, x)}{2}\frac{\partial^2}{\partial^2 x}.$$

1.3 Stochastic differential equations

A stochastic differential equation (SDE) is given whenever we equip a stochastic differential with an initial condition:

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \geq \bar{t} \\ X_{\bar{t}} &= \bar{x}. \qquad (3) \end{aligned}$$

A problem like the one we are describing is fully determined once we know the transition probability $\mathbb{P}_{1|1}(t, x \mid \bar{t}, \bar{x})$ for the process to be at (t, x) starting from $(\bar{t}, \bar{x})$.
In term of these transition probabilities we can compute (among other things) expectation values. Given any function $g(x)$, the expectation value (at T) of the stochastic process $g(X_t)$ with respect to the measure generated by the the process (3) is given by:

$$\mathbb{E}[\, g(X_T\,)] = \int du\; g(T, u)\mathbb{P}_{1|1}(u, T \mid x, t)$$

2 Feynman-Kac Formula

2.1 Introduction

Let F a function that obeys to the PDE

$$\frac{\partial F(t,x)}{\partial t} + \mathcal{A}F(t,x) = 0, \tag{4}$$

together with the boundary condition

$$F(T,x) = \Phi(x). \tag{5}$$

where as before, the infinitesimal generator $\mathcal{A}$ is given by:

$$\mathcal{A} = \mu(t,x)\frac{\partial}{\partial x} + \frac{\sigma^2(t,x)}{2}\frac{\partial^2}{\partial x^2}.$$

2.2 Solution

Let's consider the SDE (Stochastic differential equation)

$$\begin{aligned} dX_s &= \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \quad s \geq t \\ X_t &= x. \end{aligned} \tag{6}$$

Let us consider the process $F(t,X_t)$ where F s a determionistic function that satisfies eq.(4) together with the boundary condition. From Ito's formula we get:

$$\begin{aligned} dF_s &= \frac{\partial F(s,X_s)}{\partial s} + \mathcal{A}F(s,X_s) + \sigma(s,x)\frac{\partial F(s,X_s)}{\partial x}dW_s, \quad s \geq t \\ F(t,X_t) &= F(t,x). \end{aligned}$$

Making use of eq.(4) and the boundary condition (5) we get:

$$F(T,X_T) = F(t,x) + \int_t^T \sigma(t,x)\frac{\partial F(t,x)}{\partial x}dW_s$$

and averaging with respect to the measure induced by eq.(6) we get:

$$F(t,x) = \mathbb{E}\left[\Phi(X_T)\,|\,t,x\right] \tag{7}$$

3 The Kolmogorov equations for the transition probabilities

3.1 The backward Kolmogorov equation

Let us assume that the transition probability is described by a probability density $p(y, T|x, t)$, that is:

$$Pr(X_T < y|X_t = x) = \int_{-\infty}^{y} dz\, p_{1|1}(z, T\,|\,x, t),$$

then from (7)

$$F(t, x) = \int dz\, \Phi(z)\, p_{1|1}(z, T|x, t)$$

If

$$\Phi(z) = \delta(z - y)$$

we get:

$$F(t, x) = p_{1|1}(y, T\,|\,x, t),$$

and we conclude that the transition probability $p(y, T|x, t)$ satisfies the equation:

$$\begin{aligned} \frac{\partial}{\partial t} p_{1|1}(y, T|x, t) \quad &+ \quad \mu(t, x) \frac{\partial}{\partial x} p_{1|1}(y, T|x, t) + \frac{1}{2}\sigma^2(t, x) \frac{\partial^2}{\partial x^2} p_{1|1}(y, T|x, t) = 0, \\ \lim_{t \uparrow T} p_{1|1}(y, T|x, t) \quad &= \quad \delta(x - y). \end{aligned}$$

3.2 The Forward Kolmogorov equation

For any function $g(X_t)$ of the stochastic process X_t the expectation $E_{t,x}[g]$ with respect to the measure generated by by 4 can be written as:

$$E_{t_0,x_0}[g] = \int_{-\infty}^{+\infty} dy\, g(x)\, p(x, t|x_0, t_0)$$

Let us consider a trial function $h(t, x)$ with compact support, then the stochastic process $h(t, X_t)$ will satisfy:

$$h(T, X_T) = h(t_0, x_0) + \int_{t_0}^{T} dt\, \left[\frac{\partial h}{\partial t} + \mathbf{A}h\right] + \int_{t_0}^{T} \sigma(t, X_t) \frac{\partial h(t, X_t)}{\partial X_t} dW_t,$$

since the support is compact as long as we keep t_0 and T outside of the domain we have

$$h(T, X_T) = h(t_, x_0) = 0.$$

Taking averages

$$0 \quad = \quad \int_{t_0}^{T} dt \int_{-\infty}^{+\infty} dx\, p_{1|1}(x, t|x_0, t_0) \left[\frac{\partial}{\partial t} + \mathbf{A}\right] h(x, t)$$

integratiing by parts the r.h.s we have:

$$0 = \int_{t_0}^{T} dt \int_{-\infty}^{+\infty} dx\, h(x,t) \left[-\frac{\partial p_{1|1}(x,t|x_0,t_0)}{\partial t} + \mathbf{A}^{\dagger} p_{1|1}(x,t|x_0,t_0) \right]$$

Sing thi equation must hold for any $h(t,x)$ we can conclude that the integrand must be null and this yields the forward Kolmogorov equation:

$$\begin{aligned} \frac{\partial}{\partial t} p_{1|1}(x,t|x_0,t_0) &= \frac{\partial}{\partial x}\left[-\mu(t,x) + \frac{1}{2}\frac{\partial}{\partial x}\sigma^2(t,x) \right] p_{1|1}(x,t|x_0,t_0), \\ \lim_{t \downarrow t_0} p_{1|1}(x,t|x_0,t_0) &= \delta(x - x_0). \end{aligned} \tag{8}$$

4 The diffusion equation

In the following we will find ourselves confronted with a p.d.e. (Fokker-Planck or forward Kolmogorov) of the type

$$\frac{\partial}{\partial t}f(x,t)+\frac{\partial}{\partial x}\left(\mu-\frac{\sigma^2}{2}\frac{\partial}{\partial x}\right)f(x,t)=0, \tag{9}$$

and let us assume that the function $f(t,x)$, besides the equation (9) will satisfy the following initial condition:

$$f(x,t_0)=0, \tag{10}$$

with g some arbitrary function.
There is no particular meaning in writing $\sigma^2(t)$ rather that $\sigma(t)$ except for that fact that with the notation $\sigma^2(t)$ we stress the fact that the coefficient of the second order derivative term (the diffusive term) is a positive number.
We look for a solution of (9) in the domain: $\{(x,t)\in[t_0,T]\times[\lambda,\Lambda]\}$ where we allow the possibility that $\lambda\to-\infty$ and $\Lambda\to+\infty$.
Let us multiply both sides of (9) by f and integrate over the domain $[t_0,T]\times[\lambda,\Lambda]$, then we get:

$$\int_\lambda^\Lambda dx\ \int_{t_0}^T dt\ f(t,x)\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\left(\mu(t)-\frac{\sigma^2(t)}{2}\frac{\partial}{\partial x}\right)\right]f(t,x)=0. \tag{11}$$

Let us define:

$$\begin{aligned} I_1 &:= \int_\lambda^\Lambda dx\ \int_{t_0}^T dt\ f(x,t)\frac{\partial}{\partial t}f(x,t)\\ I_2 &:= \int_{t_0}^T dt\ \int_\lambda^\Lambda dx\ f(t,x)\frac{\partial}{\partial x}\left(\mu(t)-\frac{\sigma^2(t)}{2}\frac{\partial}{\partial x}\right)f(x,t). \end{aligned}$$

Clearly we have:

$$I_1=\frac{1}{2}\int_\lambda^\Lambda dx\ \int_{t_0}^T dt\ \frac{\partial}{\partial t}f^2(t,x),$$

after integration by part, and exploiting the initial condition we get:

$$I_1=\frac{1}{2}\int_\lambda^\Lambda dx\ f^2(T,x).$$

As far as I_2 is concerned we have:

$$\begin{aligned} I_2 &= \frac{1}{2}\int_{t_0}^T dt\ \int_\lambda^\Lambda dx\ \left[\mu\frac{\partial}{\partial x}f^2(t,x)-\sigma^2 f(t,x)\frac{\partial^2}{\partial x^2}f(x,t)\right]\\ &= \frac{\sigma^2}{2}\int_{t_0}^T dt\ \int_\lambda^\Lambda dx\ \left[\frac{\mu}{\sigma^2}\frac{\partial}{\partial x}f^2(t,x)-f(t,x)\frac{\partial^2}{\partial x^2}f(x,t)\right]\\ &= \frac{\sigma^2}{2}\int_{t_0}^T dt\ \int_\lambda^\Lambda dx\ \frac{\partial}{\partial x}\left[\frac{\mu}{\sigma^2}f^2(t,x)-f(t,x)\frac{\partial}{\partial x}f(x,t)\right]+\frac{\sigma^2}{2}\int_{t_0}^T dt\ \int_\lambda^\Lambda dx\ \left(\frac{\partial}{\partial x}f(x,t)\right)^2 \end{aligned}$$

that after integration by parts becomes:

$$\begin{aligned} I_2 &= \frac{1}{2}\int_{t_0}^{T} dt\, \sigma^2(t) f(t,\Lambda)\left[\frac{\mu(t)}{\sigma^2(t)} f(t,\Lambda) - f'(t,\Lambda)\right] \\ &- \frac{1}{2}\int_{t_0}^{T} dt\, \sigma^2(t) f(t,\lambda)\left[\frac{\mu(t)}{\sigma^2(t)} f(t,\lambda) - f'(t,\lambda)\right] \\ &+ \frac{1}{2}\int_{t_0}^{T} dt \int_{\lambda}^{\Lambda} dx\, \sigma^2(t)\left(\frac{\partial}{\partial x} f(t,x)\right)^2 \end{aligned}$$

where the notation $f'(t,\lambda)$ stands for:

$$f'(t,\lambda) := \left.\frac{\partial}{\partial x} f(t,x)\right|_{x=\lambda}.$$

The interesting term to consider is the term:

$$\frac{1}{2}\int_{t_0}^{T} dt\, b^2(t)\left\{ f(t,\Lambda)\left[\frac{a(t)}{b^2(t)} f(t,\Lambda) - f'(t,\Lambda)\right] - f(t,\lambda)\left[\frac{a(t)}{b^2(t)} f(t,\lambda) - f'(t,\lambda)\right]\right\}$$

that will turn out to be zero if one of the following conditions will hold:

- A:
$$f(t,\lambda) = 0, \quad f(t,\Lambda) = 0$$
- B:
$$f(t,\lambda) = 0, \quad f'(t,\Lambda) = \frac{\mu}{\sigma^2} f(t,\Lambda)$$
- C:
$$f'(t,\lambda) = \frac{\mu}{\sigma^2(t)} f(t,\lambda), \quad f(t,\Lambda) = 0$$
- D:
$$f'(t,\lambda) = \frac{\mu}{\sigma^2(t)} f(t,\lambda), \quad f'(t,\Lambda) = \frac{\mu}{\sigma^2(t)} f(t,\Lambda)$$

Boundary condition A (as we will see in the following) corresponds to the situation of absorbing barriers, condition D instead correspond to (partially) reflecting barriers, while B and C are a mix of the two. For the time being we will concentrate on condition of type A only, then if we supplement the initial condition (10) with boundary condition of type A, equation reduces to:

$$\frac{1}{2}\int_{\lambda}^{\Lambda} dx f^2(T,x) + \frac{1}{2}\int_{t_0}^{T} dt \int_{\lambda}^{\Lambda} dx\, b^2(t)\left(\frac{\partial}{\partial x} f(t,x)\right)^2 = 0.$$

Given that we are dealing with positive terms, we can satisfy the equation if and only if both tems are identically zero. The second one requires $f(t,x)$ constant for all values $\lambda \leq$

$x \leq \Lambda, \quad t_0 \leq t \leq T$, while the first one implies: $f(T,x) = 0, \lambda \leq x \leq \Lambda$, these two results togheter require that $f(t,x) = 0$.
This seemingly trivial result is instead rather important given that ensures that if a solution exists then it is unique.
Let us assume in fact that we have two solutions $f(x,t)$ and $(g(x,t)$ both satisfying the following b.c.

$$\begin{aligned} f(t_0,x) = g(t_0,x) &= h(x) \\ f(t,\lambda) = g(t,\lambda) &= J(t) \\ f(t,\Lambda) = g(t,\Lambda) &= K(t). \end{aligned}$$

Then the function $F(x,t) = f(x,t) - g(x,t)$ will satisfy both the equation and the homogeneous b.c., therefore $F(x,t) = 0 \to f(x,t) = g(x,t)$ Uniqueness is rather important in the sense that whatever we do to get to a solution (legitimate math or not) is rather immaterial. As long as at the end we come up with a function which is a solution and satisfy the b.c., we can rest assured that that solution is unique.

4.1 Solution of the equation

Let us posit a solution of the form:

$$\psi_{h,k}(x,t) = \exp\{iht + ikx\}$$

If we require:

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\left(\mu - \frac{\sigma^2}{2}\frac{\partial}{\partial x}\right)\right]\psi_{hk}(t,x) = 0,$$

we get the consistency condition:

$$ih + i\mu k + \frac{\sigma^2}{2}k^2 = 0,$$

therefore the relation:

$$ih = -i\mu k - \frac{\sigma^2 k^2}{2}$$

ensures that ψ_{hk} solves the p.d.e. The linearity of the equation guarantees that any linear combination of ψ's is also a solution.
For any set of functions $g_n(k)$ it is easy to check that:

$$f(t,x) := \sum_n \int_{-\infty}^{\infty} dk\, g_n(k) \exp\left\{ik(x - at) - \frac{\sigma^2 k^2}{2}t\right\}$$

is also a solution of the same p.d.e.
The role of eq.(9) will be (and often is) tied with the evolution of probability density functions (p.d.f.) so we require an initial condition that states that all of the probability is concentrated in a finite (maybe countable) number of points.

To embrace some sort of generality we require our equation to hold true for $t > t_0$, and we require the initial condition:

$$\lim_{t \downarrow t_0} f(t,x) = \sum_n w_n \delta(x - x_n)$$

Clearly we can fulfill our requirements by asking that

$$\lim_{t \downarrow t_0} \int_{-\infty}^{\infty} dk\ g_n(k)\ \exp\left\{ ik(\, x - \mu t\,) - \frac{\sigma^2 k^2}{2} t \right\} = w_n \delta(x - x_n).$$

This determines uniquely the function g_n

$$g_n(k) = \frac{w_n}{\sqrt{2\pi}} \exp\left\{ -ik(\, x_n - \mu t_0\,) + \frac{\sigma^2 k^2}{2} t_0 \right\}$$

and from here we can compute f

$$f(x,t) = \sum_n w_n \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left\{ -\frac{\sigma^2 k^2}{2}(t - t_0) + ik[x - x_n - \mu(t - t_0)] \right\}$$

This is a simple Gaussian integral and the result is:

$$f(x,t) = \sum_n \frac{w_n}{\sqrt{2\pi\sigma^2(t - t_0)}} \exp\left\{ -\frac{[x - x_n - \mu(t - t_0)]^2}{2\sigma^2(t - t_0)} \right\} \tag{12}$$

4.2 Boundary conditions

Now, equipped with a general solution we can attempt more realistic problems. If we couple eq.(9) with the initial condition

$$\lim_{t \downarrow 0} f(x,t) = \delta(x - x_0)$$

and the boundary condition:

$$\lim_{|x| \to \infty} f(x,t) = 0$$

we have the solution:

$$f(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(t - t_0)}} \exp\left\{ -\frac{[x - x_0 - \mu(t - t_0)]^2}{2\sigma^2(t - t_0)} \right\} \tag{13}$$

4.3 Finite distance barriers

We get a problem just a bit harder if we replace the asymptotic boundary conditions with the following down absorbing barrier condition.
This problem is defined by the p.d.e

$$\frac{\partial}{\partial t} f_\lambda(x,t) + \frac{\partial}{\partial x}\left(\mu - \frac{\sigma^2}{2}\frac{\partial}{\partial x}\right) f_\lambda(x,t) = 0, \tag{14}$$

together with the initial condition:

$$f_\lambda(0,x) = \delta(x - x_0), \quad x, x_0 > \lambda$$

and the boundary conditions:

$$\lim_{x\to+\infty} f_\lambda(x,t) = 0, \quad f_\lambda(\lambda,t) = 0.$$

Let us consider for f_λ the form:

$$\begin{aligned} f_\lambda(x,t) &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[x - x_0 - \mu t]^2}{2\sigma^2 t}\right\} + \frac{w}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[x - x_0 - z - \mu t]^2}{2\sigma^2 t}\right\} \\ &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[x - x_0 - \mu t]^2}{2\sigma^2 t}\right\} \\ &\quad \times \left[1 + w \exp\left\{-\frac{z^2 - 2z[x - x_0 - \mu t]}{2\sigma^2 t}\right\}\right] \end{aligned}$$

The boundary condition

$$f_\lambda(t,\lambda) = 0$$

is now an equation for w and z that is:

$$1 + w \exp\left\{-\frac{z^2 - 2z(\lambda - x_0)}{2\sigma^2 t} - \frac{z\mu}{\sigma^2}\right\} = 0 \tag{15}$$

that solving independently for the time dependent and time independent part produces:

$$\begin{aligned} 0 &= z - 2(\lambda - x_0) \\ w &= -\exp\left\{\frac{z\mu}{\sigma^2}\right\} \end{aligned}$$

that is:

$$z = 2(\lambda - x_0), \quad w = -\exp\left\{\frac{2(\lambda - x_0)\mu}{\sigma^2}\right\}, \quad x_0 > \lambda$$

In conclusion:

$$\begin{aligned} f_\lambda(x,t) &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[x - x_0 - \mu t]^2}{2\sigma^2 t}\right\} \\ &\quad - \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{2(x_0 - \lambda)\mu}{\sigma^2}\right\} \exp\left\{-\frac{[x - x_0 + 2(x_0 - \lambda) - \mu t]^2}{2\sigma^2 t}\right\}, \quad x, x_0 > \lambda \end{aligned} \tag{16}$$

5 Passage Time

Let's focus on the function:

$$\begin{aligned} f_\lambda(x,t) &= \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{[x-x_0-\mu t]^2}{2\sigma^2 t}\right) \\ &\quad -\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{2(x_0-\lambda)\mu}{\sigma^2}\right)\exp\left(-\frac{[x-x_0+2(x_0-\lambda)-\mu t]^2}{2\sigma^2 t}\right), \quad x, x_0 > \lambda \end{aligned}$$

that describes the probability to be at (x,t) starting from (x_0,t_0) without ever having crossed the boundary $x=\lambda$.
Let's introduce:

$$\underline{X}_T := \min(x(t), \quad t \le T)$$

then:

$$f_\lambda(t,x) = \mathbb{P}(x,t;\, \underline{X}_t > \lambda \,|\, x_0, t_0).$$

We can at once compute:

$$\mathbb{P}(\underline{X}_t > \lambda \,|\, t_0, x_0) = \int_\lambda^\infty dx\; f_\lambda(t,x)$$

A quick computation produces:

$$\begin{aligned} &\mathbb{P}(\underline{X}_t > \lambda \,|\, t_0, x_0) \\ =\; & N\left(\frac{x_0-\lambda+\mu(t-t_0)}{\sqrt{\sigma^2(t-t_0)}}\right) - \exp\left(-\frac{2(x_0-\lambda)\mu}{\sigma^2}\right) N\left(-\frac{x_0-\lambda-\mu(t-t_0)}{\sqrt{\sigma^2(t-t_0)}}\right) \end{aligned}$$

Let τ the random time corresponfing to the firstpassage through the barrier, then

$$\mathbb{P}(\underline{X}_t > \lambda \,|\, t_0, x_0) = \mathbb{P}(\tau > t) = 1 - \mathbb{P}(\tau < t)$$

Exercise 5.1. *Let*

$$\frac{dS(t)}{S(t)} = r dt + \sigma dW, \quad S(0) = S_o.$$

Let τ the first time for S to hit the barrier B_l, compute $\mathbb{P}(\tau < t)$.

Solution We define $S = S_o e^X$, then

$$dX(t) = \mu dt + \sigma dW, \quad \mu = r - \frac{\sigma^2}{2}, \quad X_0 = 0.$$

and

$$\mathbb{P}(\underline{B} > B_l) = \mathbb{P}\left(\underline{X} > \log\left(\frac{B_l}{S_o}\right)\right)$$

$$\mathbb{P}(\tau > t) = N\left(\frac{\mu t - \lambda}{\sqrt{\sigma^2 t}}\right) - \exp\left(\frac{2\lambda\mu}{\sigma^2}\right) N\left(\frac{\lambda + \mu t}{\sqrt{\sigma^2 t}}\right)$$

where:

$$\lambda = \log\left(\frac{B_l}{S_o}\right), \quad , \mu = r - \frac{\sigma^2}{s}$$