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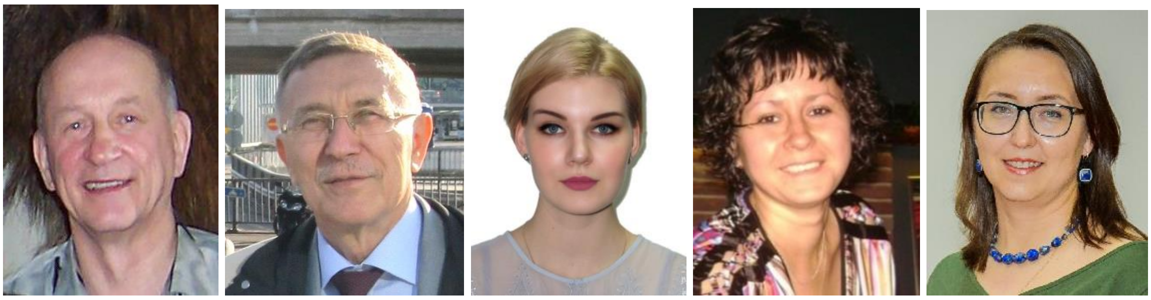
В.Г. Лабунец, В.П. Часовских, Е.А. Корх, О.А. Богословская, Малютина Л.В.

[1]Уральский государственный лесотехнический университет, г. Екатеринбург

УНИФИЦИРОВАННЫЙ ПОДХОД К КОМПЛЕМЕНТАРНЫМ ПОСЛЕДОВАТЕЛЬНОСТЯМ И ПРЕОБРАЗОВАНИЯМ. ЧАСТЬ 2. МУЛЬТИПАРАМЕТРИЧЕСКИЕ *М*-КОМПЛЕМЕНТАРНЫЕ ПРЕОБРАЗОВАНИЯ ГОЛЕЯ-РУДИНА-ШАПИРО



Ключевые слова: *обобщенные m-комплементарные последовательности, многопараметрические преобразования Фурье-Голея-Рудина-Шапиро, TDMA, FDMA, MC-CDMA, OFDM -телекоммуникационные системы.*

В данной работе мы разрабатываем новый унифицированный подход к синтезу обобщенных *m*-комплементарные последовательностей Голея-Рудина-Шапиро. Он основывается на новой итерационной генерирующей конструкции, введенной в первой части работы.

V.G. Labunets, V.P. Chasovskikh, E.A. Korkh , O.A. Bogoslovskaya, L.V. Malyutina

UNIFIED APPROACH TO COMPLEMENTARY SEQUENCES AND TRANSFORMS. PART 2. MULTIPARAMETER *M*-COMPLEMENTARY GOLAY-RUDIN-SHAPIRO TRANSFORMS

Keywords: *generalized complementary sequences, multiparameter Fourier-Golay-Rudin-Shapiro transforms. TDMA, FDMA, MC-CDMA, OFDM- telecommunication systems*.

In this paper we develop a new unified approach to the so-called generalized *m*-complementary Golay-Rudin-Shapiro (GRS) sequences. It based on a new generalized iteration generating construction, introducted in the first part of this work.

Корх Екатерина Алексеевна - магистр Уральского государственного лесотехнического университета (Екатеринбург). Тел.: 8(343)261-52-70; e-mail: katya.korkh@mail.ru

Korkh Ekaterina Alekseyevna - master of the Ural State Forestry University (Yekaterinburg). Phone: 8(343)261-52-70; e-mail: katya.korkh@mail.ru

Богословская Ольга Анатольевна - кандидат сельскохозяйственных наук, доцент кафедры менеджмента и управления качеством Института экономики и управления Уральского государственного лесотехнического университета (Екатеринбург). Тел.: (343)261-52-70; e-mail: bogoslovskayaolga@yandex.ru.

Bogoslovskaya Olga Anatolievna - PhD, Associate Professor of the Department of Quality Management at the Ural State Forest Engineering University (Yekaterinburg). Phone: 8(343)261-52-70; e-mail: bogoslovskayaolga@yandex.ru.

Малютина Людмила Владимировна - старший преподаватель кафедры менеджмента и управления качеством Института экономики и управления Уральского государственного лесотехнического университета (Екатеринбург). Тел.: 8(343)261-52-70; e-mail: ludama@yandex.ru.

Malyutina Lyudmila Vladimirovna - Senior lecturer of the Department of Quality Management at the Ural State Forest Engineering University (Yekaterinburg). Phone: 8(343)261-52-70; e-mail: ludama@yandex.ru.

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1. Introduction

Binary ± 1-valued *Golay-Rudin-Shapiro* sequences (2-GRS) associated with the cyclic group $\mathbf{Z}_2^n$ were introduced independently by M.J.E. Golay (1949, 1961, 1977), H.S. Shapiro (1951,1958) and W. Rudin (1959). M.J.E. Golay (1961) introduced the general concept of "complementary pairs" of finite sequences all of whose entries are ± 1. This was motivated by a highly non-trivial application to infrared spectrometry. Then he gave an explicit construction for binary Golay complementary pairs of length 2^m and later (Golay, 1977) noted that the construction implies the existence of at least $2^m m!/2$ binary Golay sequences of this length. They are known to exist for all lengths $N = 1^\alpha 10^\beta 26^\gamma$, where α, β, γ are integers and $\alpha, \beta, \gamma \geq 0$ (Turyn, 1974), but do not exist for any length N having a prime factor congruent to the modulo 4 (Eliahou et al., 1990). In 1951, H.S. Shapiro (1951, 1958) introduced what became known, after 1963, as the "Rudin-Shapiro" polynomial pairs. Shapiro's work was entirely in pure mathematics. S.Z. Budisin (1987, 1990, 1991) using the work of R. Sivaswamy (1978) gave a more general recursive construction for Golay complementary pairs and showed that the set of all binary Golay complementary pairs of length 2^m obtainable from it coincides with those given explicitly by Golay. For a survey of results on binary and nonbinary Golay complementary pairs, see (Byrnes, 1994) and (Fan, Darnel, 1996), respectively. In 1999, J.A Davis and J. Jedwab (1999) gave an explicit description of a large class of Golay complementary sequences in terms of certain cosets of the first order Reed-Muller codes.

Discrete classical *Fourier-Golay-Rudin-Shapiro Transforms* (FGRST) in bases of different Golay-Rudin-Shapiro sequences can be used in many signal processing applications: multiresolution by discrete orthogonal wavelet decomposition, digital audition, digital video broadcasting, communication systems (Orthogonal Frequency Division Multiplexing - OFDM, Multi-Code-Division Multiple Access - MCDA), radar, and cryptographic systems.

For building the classical FGRST in bases of classical Golay-Rudin-Shapiro sequences the following actors are used: 1) the Abelian group $\mathbf{Z}_2^n$, 2) 2-point Fourier transform $\mathcal{F}_2$, and 3) the complex field $\mathbf{C}$; i.e., these transforms are associated with the triple $\left(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C}\right)$. In this work, we develop a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$-, and Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $\left(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C}\right)$, but with $\left(\mathbf{Z}_m^n, \mathbf{U}_m, \mathcal{A}lg\right)$ or with $\left(\mathbf{Z}_m^n, \left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}, \mathcal{A}lg\right)$, where $\mathbf{U}_m$ or $\left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}$ are an single or a set of arbitrary unitary $(m \times m)$-transforms instead of $\mathcal{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathcal{A}lg$ is an algebras (Clifford algebras), finite rings ($\mathbf{Z}_N$) and finite Galois fields ($\mathbf{GF}(q)$) instead of the complex field $\mathbf{C}$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay-Rudin-Shapiro* m-ary sequences) is described. In Section 3, the proposed method based on two new generalized iteration constructions are explained.

2. New iteration construction for original Golay sequences

2.1. *Basic definitions*

We begin by describing the original Golay 2- and m-complementary sequences.

Definition 1. Let $\mathrm{com}^0(t):=(c_0,c_1,...,c_{N-1})$ and $\mathrm{com}^1(t)=(s_0,s_1,...,s_{N-1})$, where $c_i,s_i\in\mathbf{B}_2=\{\pm1\}$. The sequences $\mathrm{com}^0(t),\ \mathrm{com}^1(t)$ are called the *2-complementary* ((±1)-*valued*) or *Goley complementary pair* over $\{\pm1\}$, if $COR^0(\tau)+COR^1(\tau)=N\delta(\tau)$, or $\left(\left|\mathrm{COM}^0(z)\right|^2+\left|\mathrm{COM}^1(z)\right|^2\right)_{|z|=1}=N$, where $COR^0(\tau),COR^1(\tau)$ are the periodic correlation functions of $\mathrm{com}^0(t),\ \mathrm{com}^1(t)$ and $\mathrm{COM}^0(z)=\mathcal{Z}\ \mathrm{com}^0(t)$, $\mathrm{COM}^1(z)=\mathcal{Z}\ \mathrm{com}^1(t)$ are their $\mathcal{Z}$ – transforms. Any sequence, which is a member of a Golay complementary pair, is called the *Golay sequence* and its $\mathcal{Z}$ – transform $\mathrm{COM}_k(z)=\mathcal{Z}\ \mathrm{com}_k(t)$ is called the *Golay-Shapiro-Rudin polynomial (GSRP)*.

Definition 2. A generalization of Golay complementary pair, known as the *Golay m-complementary* m-element *set of complex-valued sequences* (Lei, 1991),

$$\begin{cases}\mathrm{com}_0(t) := (c_0(0),\ \ c_0(1),\ ...,\ c_0(m-1)),\\ \mathrm{com}_1(t) := (c_1(0),\ \ c_1(1),\ ...,\ c_1(m-1)),\\ \dots\dots\dots\dots\dots\dots\dots\dots,\\ \mathrm{com}_{m-1}(t):=(c_{m-1}(0),c_{m-1}(1),...,c_{m-1}(m-1))\end{cases}$$

is defined by $\sum_{k=0}^{m-1}COR_k(\tau)=m\cdot\delta(\tau)$, or $\sum_{k=0}^{m-1}\left|\mathrm{COM}_1(z)\right|^2=m$, where $\{COR_k(\tau)\}_{k=0}^{m-1}$ are the periodic autocorrelation functions of $\{\mathrm{com}_k(t)\}_{k=1}^{m}$, and $\mathrm{COM}_k(z)=\mathcal{Z}\ \mathrm{com}_k(t)$, $k=0,1,...,m-1$ are their $\mathcal{Z}$ – transforms, respectively.

2.2. *Golay matrix*

We use two symbols $\boldsymbol{\alpha}_n\in[0,m^{n-1}-1]=\mathbf{Z}_{m^n}$ and $\mathbf{t}_n\in[0,m^{n-1}-1]=\mathbf{Z}_{m^n}$ for numeration of Golay sequences and discrete time, respectively. For integer $\boldsymbol{\alpha}_n\in[0,m^{n-1}-1]$ and $\mathbf{t}_n\in[0,m^{n-1}-1]$ we shall use m-ary codes $\vec{\boldsymbol{\alpha}}_n=(\alpha_1,\alpha_2,...,\alpha_n)$, $\vec{\mathbf{t}}_n=(t_1,t_2,...,t_n)$ where $\alpha_i,t_i\in\{0,1,...,m-1\}=\mathbf{Z}_m$, $i=1,2,n$.. Let $\vec{\boldsymbol{\alpha}}_n=(\alpha_1,\alpha_2,...,\alpha_n)$ and $\vec{\mathbf{t}}_n=(t_1,t_2,...,t_n)$ be m-ary codes, then define

$$\boldsymbol{\alpha}_n=\left|\vec{\boldsymbol{\alpha}}_n\right|=\left|(\alpha_1,\alpha_2,...,\alpha_n)\right|=\sum_{i=1}^{n}\alpha_{n-i+1}m^{i-1},\ \mathbf{t}_n=\left|\vec{\mathbf{t}}_n\right|=\left|(t_1,t_2,...,t_n)\right|=\sum_{i=1}^{n}t_{n-i+1}m^{n-i}$$

be integers whose m-ary codes are $\vec{\boldsymbol{\alpha}}_n=(\alpha_1,\alpha_2,...,\alpha_n)$ and $\vec{\mathbf{t}}_n=(t_1,t_2,...,t_n)$, where α_n,t_1 are less significant bits (LSB) and α_1,t_n are most significant bits (MSB) of $\vec{\boldsymbol{\alpha}}_n=(\alpha_1,\alpha_2,...,\alpha_n)$ and $\vec{\mathbf{t}}_n=(t_1,t_2,...,t_n)$, respectively. Obviously,

$\vec{\boldsymbol{\alpha}}_1=(\alpha_1)\in\mathbf{Z}_m$,	$\boldsymbol{\alpha}_1=\alpha_1\in\mathbf{Z}_m$,	$\vec{\mathbf{t}}_1=(t_1)\in\mathbf{Z}_{m^1}$,	$\mathbf{t}_1=t_1\in\mathbf{Z}_m$,
$\vec{\boldsymbol{\alpha}}_2=(\vec{\boldsymbol{\alpha}}_1,\alpha_2)\in\mathbf{Z}_m\times\mathbf{Z}_m=\mathbf{Z}_m^2$,	$(\boldsymbol{\alpha}_1,\alpha_2)\in\mathbf{Z}_m\times\mathbf{Z}_m$,	$\vec{\mathbf{t}}_2=(\vec{\mathbf{t}}_1,t_2)\in\mathbf{Z}_m\times\mathbf{Z}_m=\mathbf{Z}_m^2$,	$(\mathbf{t}_1,t_2)\in\mathbf{Z}_m\times\mathbf{Z}_m$,
$\vec{\boldsymbol{\alpha}}_3=(\vec{\boldsymbol{\alpha}}_2,\alpha_3)\in\mathbf{Z}_m^2\times\mathbf{Z}_m=\mathbf{Z}_m^3$,	$(\boldsymbol{\alpha}_2,\alpha_3)\in\mathbf{Z}_{m^2}\times\mathbf{Z}_m$,	$\vec{\mathbf{t}}_3=(\vec{\mathbf{t}}_2,t_3)\in\mathbf{Z}_m^2\times\mathbf{Z}_m=\mathbf{Z}_m^3$,	$(\mathbf{t}_2,t_3)\in\mathbf{Z}_{m^2}\times\mathbf{Z}_m$,
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$\vec{\boldsymbol{\alpha}}_n=(\vec{\boldsymbol{\alpha}}_{n-1},\alpha_n)\in\mathbf{Z}_m^{n-1}\times\mathbf{Z}_m=\mathbf{Z}_m^n$,	$(\boldsymbol{\alpha}_{n-1},\alpha_n)\in\mathbf{Z}^1_{m^{n-1}}\times\mathbf{Z}_m$.	$\vec{\mathbf{t}}_n=(\vec{\mathbf{t}}_{n-1},t_n)\in\mathbf{Z}_m^{n-1}\times\mathbf{Z}_m=\mathbf{Z}_m^n$,	$(\mathbf{t}_{n-1},t_n)\in\mathbf{Z}^1_{m^{n-1}}\times\mathbf{Z}_m$,

Let $\left\{\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})\right\}$ be m^{n+1}-element set of m-complementary sequences (of length m^{n+1}), where $\boldsymbol{\alpha}_{\mathbf{n+1}},\mathbf{t}_{\mathbf{n+1}}=\mathbf{0},\mathbf{1},\mathbf{2},\ldots,\mathbf{m}^{\mathbf{n+1}}\mathbf{-1}$. They are form rows of a $\left(m^{n+1}\times m^{n+1}\right)$-matrix

$$\mathbf{G}^{[n+1]}_{m^{n+1}}=\left[\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})\right]^{\mathbf{m}^{\mathbf{n+1}}\mathbf{-1}}_{\boldsymbol{\alpha}_{n+1},\mathbf{t}_{n+1}=\mathbf{0}}=\left[\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}\right]^{\mathbf{m}^{\mathbf{n+1}}\mathbf{-1}}_{\boldsymbol{\alpha}_{n+1}=\mathbf{0}},$$

that is called *the m-Golay matrix*. Here index $[n+1]$ shows that m-Golay matrix have been obtained on the $n+1$iteration step. We are going to group these rows (sequences) into m^n collections of m-element set of m-complementary Golay sequences

$$\mathbf{G}^{[n+1]}_{m^{n+1}}=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}-1}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n}=0}^{m^{n}-1}\left(\mathop{\boxplus}_{\alpha_{n+1}=0}^{m-1}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},\alpha_{n+1})}(\mathbf{t}_{n+1})\right)=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n}=\mathbf{0}}^{\mathbf{m}^{\mathbf{n}}\mathbf{-1}}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},1)}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},m-1)}(\mathbf{t}_{n+1})\end{bmatrix}, \qquad (1)$$

where $\left\{\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},0)}(\mathbf{t}_{n+1}),\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},1)}(\mathbf{t}_{n+1}),\ldots,\mathrm{com}^{\{n+1]}_{(\boldsymbol{\alpha}_{n},m-1)}(\mathbf{t}_{n+1})\right\}$ are m^n collections of m-element set of m-complementary Golay sequences. Let us to select the more fine structure of the m-Golay matrix:

$$\mathbf{G}^{[n+1]}_{m^{n+1}}=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}-1}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n}=\mathbf{0}}^{\mathbf{m}^{\mathbf{n}}\mathbf{-1}}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},1)}(\mathbf{t}_{n+1})\\ \ldots\ldots\ldots\ldots\ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},m-1)}(\mathbf{t}_{n+1})\end{bmatrix}=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=\mathbf{0}}^{\mathbf{m}^{\mathbf{n-1}}\mathbf{-1}}\left(\mathop{\boxplus}_{\alpha_{n}=0}^{m\text{-}1}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n 1)}(\mathbf{t}_{n+1})\\ \ldots\ldots\ldots\ldots\ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n m-1)}(\mathbf{t}_{n+1})\end{bmatrix}\right)=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=\mathbf{0}}^{\mathbf{m}^{\mathbf{n-1}}\mathbf{-1}}\left[\begin{array}{c}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,m-1)}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1})\\ \hline \vdots\\ \vdots\\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-11)}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1m-1)}(\mathbf{t}_{n+1})\end{array}\right]. \qquad (2)$$

Example 1. For $n=1$ and $n=2$ we have, respectively,

$$\mathbf{G}^{[1]}_{3^1}=\left[\mathrm{com}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)\right]^{\mathbf{2}}_{\boldsymbol{\alpha}_1,\mathbf{t}_1=\mathbf{0}}=\mathop{\boxplus}_{\boldsymbol{\alpha}_1=\mathbf{0}}^{\mathbf{2}}\mathrm{com}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)=\begin{bmatrix}\mathrm{com}_{(0)}(\mathbf{t}_1)\\ \mathrm{com}_{(1)}(\mathbf{t}_1)\\ \mathrm{com}_{(2)}(\mathbf{t}_1)\end{bmatrix},$$

$$\mathbf{G}^{[2]}_{3^2}=\left[\mathrm{com}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)\right]^{\mathbf{8}}_{\boldsymbol{\alpha}_2,\mathbf{t}_2=\mathbf{0}}=\mathop{\boxplus}_{\boldsymbol{\alpha}_2=\mathbf{0}}^{\mathbf{8}}\mathrm{com}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)=\mathop{\boxplus}_{\boldsymbol{\alpha}_1=\mathbf{0}}^{\mathbf{2}}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\\ \mathrm{com}_{(\boldsymbol{\alpha}_1,1)}(\mathbf{t}_2)\\ \mathrm{com}_{(\boldsymbol{\alpha}_1,2)}(\mathbf{t}_2)\end{bmatrix}=\left[\begin{array}{c}\mathrm{com}_{(0,0)}(\mathbf{t}_2)\\ \mathrm{com}_{(0,1)}(\mathbf{t}_2)\\ \mathrm{com}_{(0,2)}(\mathbf{t}_2)\\ \hline \mathrm{com}_{(1,0)}(\mathbf{t}_2)\\ \mathrm{com}_{(1,1)}(\mathbf{t}_2)\\ \mathrm{com}_{(1,2)}(\mathbf{t}_2)\\ \hline \mathrm{com}_{(2,0)}(\mathbf{t}_2)\\ \mathrm{com}_{(2,1)}(\mathbf{t}_2)\\ \mathrm{com}_{(2,2)}(\mathbf{t}_2)\end{array}\right].$$

The matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}$ is constructed by an iteration construction. The initial matrix $\mathbf{G}_{m^1}$ is formed by starting with an arbitrary unitary (orthogonal) $(m\times m)$-matrix

$$\mathbf{U}:=\mathbf{G}_{m^1}^{[1]}=\left[\mathrm{com}_{\boldsymbol{\alpha}_1}^{[1]}(\mathbf{t}_1)\right]_{\alpha_1,t_1=0}^{m-1}=\begin{bmatrix}\mathrm{com}_0^{[1]}(\mathbf{t}_1)\\ \mathrm{com}_1^{[1]}(\mathbf{t}_1)\\ \mathrm{com}_2^{[1]}(\mathbf{t}_1)\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{m-1}^{[1]}(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}a_0(0) & a_0(1) & a_0(2) & \ldots & a_0(m-1)\\ a_1(0) & a_1(1) & a_1(2) & \ldots & a_1(m-1)\\ a_2(0) & a_2(1) & a_2(2) & \ldots & a_2(m-1)\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ a_{m-1}(0) & a_{m-1}(1) & a_{m-1}(2) & \ldots & a_{m-1}(m-1)\end{bmatrix}.$$

where $\mathrm{com}_{\boldsymbol{\alpha}_1}^{[1]}(\mathbf{t}_1)=\left(a_{\alpha_1}(0),\ a_{\alpha_1}(1),\ a_{\alpha_1}(2),\ \ldots,\ a_{\alpha_1}(m-1)\right),\ \boldsymbol{\alpha}_1=0,1,2,\ldots,m-1.$

Example 2. The initial matrix $\mathbf{G}_{m^1}^{[1]}$ can be the Fourier transform on Abelian group $\mathbf{Z}_m$

$$\mathbf{G}_{m^1}^{[1]}=\mathcal{F}_m=\begin{bmatrix}\mathrm{com}_0^{[1]}(\mathbf{t}_1)\\ \mathrm{com}_1^{[1]}(\mathbf{t}_1)\\ \mathrm{com}_2^{[1]}(\mathbf{t}_1)\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{m-1}^{[1]}(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & \ldots & 1\\ 1 & \varepsilon^{1\cdot1} & \varepsilon^{1\cdot2} & \ldots & \varepsilon^{1\cdot(m-1)}\\ 1 & \varepsilon^{2\cdot1} & \varepsilon^{2\cdot2} & \ldots & \varepsilon^{2\cdot(m-1)}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 1 & \varepsilon^{(m-1)\cdot1} & \varepsilon^{(m-1)\cdot2} & \ldots & \varepsilon^{(m-1)\cdot(m-1)}\end{bmatrix},\tag{3}$$

where $\mathrm{com}_k^{[1]}(\mathbf{t}_1)=\left(1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\ldots,\varepsilon^{k\cdot(m-1)}\right),\ k=01,2,\ldots,m-1$ are characters of $\mathbf{Z}_m$.

□

It is easy to check that $\left(|\mathrm{COM}_0(z)|^2+|\mathrm{COM}_1(z)|^2+\ldots+|\mathrm{COM}_{m\text{-}1}(z)|^2\right)_{|z|=1}=m.$ Indeed,

$$\sum_{k=1}^{m-1}|\mathrm{COM}_k(z)|^2=\sum_{k=1}^{m-1}\mathrm{COM}_k(z)\overline{\mathrm{COM}}^k(\bar z)=\sum_{k=1}^{m-1}\left(\sum_{t=0}^{m-1}a_k(t)z^t\right)\left(\sum_{s=0}^{m-1}\bar a_k(s)\bar z^s\right)=$$

$$=\sum_{s=0}^{m-1}\sum_{t=0}^{m-1}\left(\sum_{k=0}^{m-1}a_k(t)\bar a_k(s)\right)z^t\bar z^s=\sum_{s=0}^{m-1}\sum_{t=0}^{m-1}\delta_{t-s}z^t\bar z^s=\sum_{t=0}^{m-1}|z|^{2t},$$

since $\sum_{k=0}^{m-1}a_k(t)\bar a_k(s)=\delta_{t-s}$ is true for an arbitrary unitary (orthogonal) matrix. Hence, initial sequences in the form of rows of an unitary matrix (in particular case in the form of characters $\mathrm{com}_k(\mathbf{t}_1)=\left(1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\ldots,\varepsilon^{k\cdot(m-1)}\right)$ of cyclic group $\mathbf{Z}_m$) are the Golay m-complementary sequences, since in this case $\left(\sum_{k=1}^{m-1}|\mathrm{COM}_k(z)|^2\right)\Bigg|_{|z|=1}=\left(\sum_{t=0}^{m-1}|z|^{2t}\right)_{|z|=1}=m.$

Methods

2.3. New iteration construction in time domain

The matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}$ is constructed by an iteration construction

$$\mathbf{G}_{m^1}^{[1]}\to\mathbf{G}_{m^2}^{[2]}\to\ldots.\to\mathbf{G}_{m^n}^{[n]}\to\mathbf{G}_{m^{n+1}}^{[n+1]}.\tag{4}$$

Let us to suppose that we have the m-Golay matrix $\mathbf{G}_{m^n}^{[n]}$. We need to construct the next m-Golay matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}$ using only $\mathbf{G}_{m^n}^{[n]}$ and $\mathbf{U}:=\mathbf{G}_{m^1}$. The m-Golay matrix $\mathbf{G}_{m^n}^{[n]}$ have structure similar (1)**:**

$$\mathbf{G}_{m^n}^{[n]}=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{m^n-1}\mathrm{com}_{(\boldsymbol{\alpha}_n)}^{[n]}(\mathbf{t}_n)=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{m^n\text{-}1}}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}\tag{5}$$

For constructing $\mathbf{G}^{[n+1]}_{m^{n+1}}$ from $\mathbf{G}^{[n]}_{m^n}$ we take each m-complementary set in the form of

$$\left|\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n)\right\rangle := = \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \dots\dots\dots\dots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix}$$

and construct shifted versa of their components

$$\left|{}^{(k)}\mathbf{Tcom}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n)\right\rangle := \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n+\mathbf{m}^n\cdot 0) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n+\mathbf{m}^n\cdot 1) \\ \dots\dots\dots\dots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n+\mathbf{m}^n\cdot(m-1)) \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \dots\dots\dots\dots \\ \mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} =$$

$$= \mathrm{diag}\left\{\mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n}, \mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n}, \dots, \mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n}\right\} \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \dots\dots\dots\dots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix}, \quad (6)$$

where $k = 0,1,\dots,m-1$ and $\mathbf{T}^{\mathbf{m}^n s}_{\mathbf{t}_n}$ is the shift operator on $\mathbf{m}^n s$ discrete positions in time domain $\mathbf{T}^{m^n s}_{\mathbf{t}_n} f(\mathbf{t}_n) := f(\mathbf{t}_n + m^n s)$. Now we construct the general building blocks for the Golay $\left(m^{n+1}\times m^{n+1}\right)$-matrix $\mathbf{G}_{m^{n+1}}$:

$$\mathbf{U}\cdot\left|{}^{(k)}\mathbf{Tcom}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n)\right\rangle = \mathbf{U}\cdot\mathrm{diag}\left\{\mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n}, \mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n}, \dots, \mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n}\right\}\cdot\left|\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n)\right\rangle = {}^{(k)}\mathbf{U}\cdot\left|\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n)\right\rangle, \quad (7)$$

where

$${}^{(k)}\mathbf{U} = \mathbf{U}\cdot\mathrm{diag}\left\{\mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n}, \mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n}, \dots, \mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n}\right\} =$$

$$= \begin{bmatrix} a_0(0) & a_0(1) & \dots & a_0(m-1) \\ a_1(0) & a_1(1) & \dots & a_1(m-1) \\ \dots & \dots & \dots & \dots \\ a_{m-1}(0) & a_{m-1}(1) & \dots & a_{m-1}(m-1) \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n} & & & \\ & \mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n} & & \\ & & \dots & \\ & & & \mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n} \end{bmatrix} =$$

$$= \begin{bmatrix} a_0(0)\mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n} & a_0(1)\mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n} & \dots & a_0(m-1)\mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n} \\ a_1(0)\mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n} & a_1(1)\mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n} & \dots & a_1(m-1)\mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n} \\ \dots & \dots & \dots & \dots \\ a_{m-1}(0)\mathbf{T}^{\mathbf{m}^n(0\oplus k)}_{\mathbf{t}_n} & a_{m-1}(1)\mathbf{T}^{\mathbf{m}^n(1\oplus k)}_{\mathbf{t}_n} & \dots & a_{m-1}(m-1)\mathbf{T}^{\mathbf{m}^n((m-1)\oplus k)}_{\mathbf{t}_n} \end{bmatrix}.$$

Using building blocks of $\left(m^n\times m^n\right)$-matrix $\mathbf{G}_{m^n}$ we construct the Golay $\left(m^{n+1}\times m^{n+1}\right)$-matrix $\mathbf{G}_{m^{n+1}}$ according to the following iteration rule:

$$\left|\mathbf{com}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n)\right\rangle := \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} \begin{matrix} \nearrow \\ \\ \rightarrow \\ \\ \searrow \end{matrix} \left[\begin{array}{c} {}^{(0)}U\cdot\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline {}^{(1)}U\cdot\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline \ldots \\ \hline {}^{(m-1)}U\cdot\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} \end{array}\right] = \left[\begin{array}{c} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1}) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,m-1)}(\mathbf{t}_{n+1}) \end{bmatrix} \\ \hline \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1}) \end{bmatrix} \\ \hline \ldots \\ \hline \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-11)}(\mathbf{t}_{n+1}) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1m-1)}(\mathbf{t}_{n+1}) \end{bmatrix} \end{array}\right] = \mathbf{G}_{m^{n+1}},$$

where

$${}^{(0)}U\cdot\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} a_0(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} & a_0(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 1} & \ldots & a_0(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-1)} \\ a_1(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} & a_1(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 1} & \ldots & a_1(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-1)} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m-1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} & a_{m-1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 1} & \ldots & a_{m-1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-1)} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix},$$

$${}^{(1)}U\cdot\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} a_0(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 1} & a_0(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 2} & \ldots & a_0(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} \\ a_1(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 1} & a_1(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 2} & \ldots & a_1(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m-1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 1} & a_{m-1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 2} & \ldots & a_{m-1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix},$$

$$\ldots\ldots\ldots\ldots\ldots$$

$${}^{(m-1)}U\cdot\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} a_0(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-1)} & a_0(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} & \ldots & a_0(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-2)} \\ a_1(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-1)} & a_1(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} & \ldots & a_1(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-2)} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m-1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-1)} & a_{m-1}(1)_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot 0} & \ldots & a_{m-1}(m-1)_{\mathbf{t}_n}^{\mathbf{m}^{\mathbf{n}}\cdot(m-2)} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \ldots\ldots\ldots\ldots \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} \text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}) \\ \text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}) \\ \dots\dots\dots\dots \\ \text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1}) \end{bmatrix} = {}^{(k)}U \cdot \begin{bmatrix} \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \dots\dots\dots\dots \\ \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix} =$$

$$= \begin{bmatrix} a_0(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & a_0(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \dots & a_0(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n} \\ a_1(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & a_1(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \dots & a_1(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n} \\ \dots & \dots & \dots & \dots \\ a_{m-1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & a_{m-1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \dots & a_{m-1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n} \end{bmatrix} \cdot \begin{bmatrix} \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \dots\dots\dots\dots \\ \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n) \end{bmatrix},$$

or,

$$\text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) = \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(\beta_n \underset{m}{\oplus} \alpha_n)}_{\mathbf{t}_n} \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n).$$

Since $\mathbf{t}_{n+1} = (\mathbf{t}_n, t_{n+1})$, then believing $t_{n+1} = \alpha_n \underset{m}{\oplus} \beta_n$, we obtain as:

$$\text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) = \text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1}) = \sum_{t_{n+1}=0}^{m-1} a_{\alpha_{n+1}}(\alpha_n \underset{m}{\oplus} t_{n+1})\mathbf{T}^{m^n t_{n+1}}_{\mathbf{t}_n} \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{m}{\oplus} t_{n+1})}(\mathbf{t}_n) =$$

$$= \sum_{t_{n+1}=0}^{m-1} a_{\alpha_{n+1}}(\alpha_n \underset{m}{\oplus} t_{n+1}) \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{m}{\oplus} t_{n+1})}(\mathbf{t}_n + m^n t_{n+1}). \qquad (8)$$

It is finally recurrent relation between m-complementary sequences of $\mathbf{G}^{[n+1]}_{m^{n+1}}$ and $\mathbf{G}^{[n]}_{m^n}$. In particular, for the initial matrix in the form of the Fourier matrix $\mathbf{G}^{[1]}_{m^1} = [\varepsilon_m^{\alpha t}]$ we have

$$\text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) = \text{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1}) = \sum_{t_{n+1}=0}^{m-1} \varepsilon_m^{\alpha_{n+1}(\alpha_n \underset{m}{\oplus} t_{n+1})} \mathbf{T}^{m^n t_{n+1}}_{\mathbf{t}_n} \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{m}{\oplus} t_{n+1})}(\mathbf{t}_n) =$$

$$= \varepsilon_m^{\alpha_n \alpha_{n+1}} \sum_{t_{n+1}=0}^{m-1} \varepsilon_m^{\alpha_{n+1} t_{n+1}} \text{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{m}{\oplus} t_{n+1})}(\mathbf{t}_n + m^n t_{n+1}). \qquad (9)$$

Example 3. Let $n = 2$, $m = 3$ and

$$\mathbf{U} \equiv G^{[1]}_{3^1} = \left[\text{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)\right]^2_{\boldsymbol{\alpha}_1,\mathbf{t}_1=\mathbf{0}} = \begin{bmatrix} \text{com}^{[1]}_0(\mathbf{t}_1) \\ \text{com}^{[1]}_1(\mathbf{t}_1) \\ \text{com}^{[1]}_2(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0) & a_0(1) & a_0(2) \\ a_1(0) & a_1(1) & a_1(2) \\ a_2(0) & a_2(1) & a_2(2) \end{bmatrix}.$$

$$\mathbf{G}^{[2]}_{3^2} = \left[\text{com}^{[2]}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)\right]^{\mathbf{2}}_{\boldsymbol{\alpha}_2,\mathbf{t}_2=\mathbf{0}} = \left[\text{com}^{[2]}_{(\alpha_1,\alpha_2)}(\mathbf{t}_2)\right]^{\mathbf{2}}_{\boldsymbol{\alpha}_2,\mathbf{t}_2=\mathbf{0}} = \underset{\boldsymbol{\alpha}_1=\mathbf{0}}{\overset{2}{\boxplus}} \begin{bmatrix} \text{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(\boldsymbol{\alpha}_1,1)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(\boldsymbol{\alpha}_1,2)}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} \text{com}^{[2]}_{(0,0)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(0,1)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(0,1)}(\mathbf{t}_2) \\ \hline \text{com}^{[2]}_{(1,0)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(1,1)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(1,2)}(\mathbf{t}_2) \\ \hline \text{com}^{[2]}_{(2,0)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(2,1)}(\mathbf{t}_2) \\ \text{com}^{[2]}_{(2,2)}(\mathbf{t}_2) \end{bmatrix}.$$

Using $\begin{bmatrix} \text{com}^{[1]}_0(\mathbf{t}_1) \\ \text{com}^{[1]}_1(\mathbf{t}_1) \\ \text{com}^{[1]}_2(\mathbf{t}_1) \end{bmatrix}$, we construct the Golay $(3^2 \times 3^2)$-matrix $\mathbf{G}^{[2]}_{3^2}$:

$$\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} \begin{matrix} \nearrow \\ \rightarrow \\ \searrow \end{matrix} \left[\begin{array}{c} {}^{(0)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} \\ \hline {}^{(1)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} \\ \hline {}^{2}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} \end{array} \right] = \left[\begin{array}{c} \begin{bmatrix} \mathrm{com}_{(00)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(01)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(02)}^{[2]}(\mathbf{t}_2) \end{bmatrix} \\ \hline \begin{bmatrix} \mathrm{com}_{(10)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(11)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(12)}^{[2]}(\mathbf{t}_2) \end{bmatrix} \\ \hline \begin{bmatrix} \mathrm{com}_{(20)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(21)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(22)}^{[2]}(\mathbf{t}_2) \end{bmatrix} \end{array} \right] = \mathbf{G}_{3^2},$$

where

$$\begin{bmatrix} \mathrm{com}_{(00)}^{[2]}(\mathbf{t}_1) \\ \mathrm{com}_{(01)}^{[2]}(\mathbf{t}_1) \\ \mathrm{com}_{(02)}^{[2]}(\mathbf{t}_1) \end{bmatrix} = {}^{(0)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} & a_0(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} & a_0(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} \\ a_1(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} & a_1(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} & a_1(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} \\ a_2(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} & a_2(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} & a_2(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix},$$

$$\begin{bmatrix} \mathrm{com}_{(10)}^{[2]}(\mathbf{t}_1) \\ \mathrm{com}_{(11)}^{[2]}(\mathbf{t}_1) \\ \mathrm{com}_{(12)}^{[2]}(\mathbf{t}_1) \end{bmatrix} = {}^{(1)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} & a_0(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} & a_0(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} \\ a_1(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} & a_1(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} & a_1(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} \\ a_2(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} & a_2(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} & a_2(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix},$$

$$\begin{bmatrix} \mathrm{com}_{(20)}^{[2]}(\mathbf{t}_1) \\ \mathrm{com}_{(21)}^{[2]}(\mathbf{t}_1) \\ \mathrm{com}_{(22)}^{[2]}(\mathbf{t}_1) \end{bmatrix} = {}^{(2)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} & a_0(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} & a_0(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} \\ a_1(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} & a_1(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} & a_1(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} \\ a_2(0)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 2} & a_2(1)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 0} & a_2(2)\mathbf{T}_{\mathbf{t}_1}^{3^1\cdot 1} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix}.$$

Finally,

$${}^{(0)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0) & a_0(1) & a_0(2) \\ a_1(0) & a_1(1) & a_1(2) \\ a_2(0) & a_2(1) & a_2(2) \end{bmatrix} \cdot \begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) & & \\ & \mathrm{com}_1^{[1]}(\mathbf{t}_1) & \\ & & \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} =$$

$$= \begin{bmatrix} a_0(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1), a_0(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1), a_0(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1) \\ a_1(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1), a_1(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1), a_1(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1) \\ a_2(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1), a_2(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1), a_2(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(00)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(01)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(02)}^{[2]}(\mathbf{t}_2) \end{bmatrix},$$

$${}^{(1)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0) & a_0(1) & a_0(2) \\ a_1(0) & a_1(1) & a_1(2) \\ a_2(0) & a_2(1) & a_2(2) \end{bmatrix} \cdot \begin{bmatrix} & \mathrm{com}_0^{[1]}(\mathbf{t}_1) & \\ & & \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) & & \end{bmatrix} =$$

$$= \begin{bmatrix} a_0(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1), a_0(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1), a_0(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ a_1(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1), a_1(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1), a_1(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ a_2(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1), a_2(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1), a_2(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(10)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(11)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(12)}^{[2]}(\mathbf{t}_2) \end{bmatrix},$$

□

$${}^{(2)}\mathbf{U}\cdot\begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} a_0(0) & a_0(1) & a_0(2) \\ a_1(0) & a_1(1) & a_1(2) \\ a_2(0) & a_2(1) & a_2(2) \end{bmatrix} \cdot \begin{bmatrix} & & \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) & & \\ & \mathrm{com}_2^{[1]}(\mathbf{t}_1) & \end{bmatrix} =$$

$$= \begin{bmatrix} a_0(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1), a_0(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1), a_0(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ a_1(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1), a_1(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1), a_1(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ a_2(1)\cdot\mathrm{com}_1^{[1]}(\mathbf{t}_1), a_2(2)\cdot\mathrm{com}_2^{[1]}(\mathbf{t}_1), a_2(0)\cdot\mathrm{com}_0^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(20)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(21)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(22)}^{[2]}(\mathbf{t}_2) \end{bmatrix}.$$

New sequences in (8) are orthogonal sequences. Indeed,

$$\left\langle \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) \middle| \mathrm{com}^{[n+1]}_{(\boldsymbol{\gamma}_{n-1},\gamma_n,\gamma_{n+1})}(\mathbf{t}_{n+1}) \right\rangle =$$

$$= \left\langle \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \sum_{\beta_n'=0}^{m-1} a_{\gamma_{n+1}}(\beta_n') \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(\beta_n' \underset{m}{\oplus} \gamma_n)} \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle =$$

$$= \sum_{\beta_n=0}^{m-1} \sum_{\beta_n'=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\gamma_{n+1}}(\beta_n') \left\langle \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(\beta_n' \underset{m}{\oplus} \gamma_n)} \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle =$$

$$= \sum_{\beta_n=0}^{m-1} \sum_{\beta_n'=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\gamma_{n+1}}(\beta_n') \left\langle \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}\left((\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n)\right)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle.$$

But

$$\left\langle \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}\left((\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n)\right)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle =$$

$$= \begin{cases} 0, & (\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n) \neq 0, \\ \left\langle \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle, & (\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n) \neq 0, \end{cases}$$

since $\left(\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}\left((\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n)\right)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \right) \cdot \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) = 0$ if $(\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n) \neq 0$ and

$$\left\langle \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle = \delta_{\boldsymbol{\alpha}_{n-1},\boldsymbol{\gamma}_{n-1}} \delta_{\beta_n,\beta_n'}.$$

For this reason

$$\left\langle \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}\left((\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n)\right)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle = \delta_{\boldsymbol{\alpha}_{n-1},\boldsymbol{\gamma}_{n-1}} \delta_{\alpha_n,\gamma_n} \delta_{\beta_n,\beta_n'},$$

$$\left\langle \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) \middle| \mathrm{com}^{[n+1]}_{(\boldsymbol{\gamma}_{n-1},\gamma_n,\gamma_{n+1})}(\mathbf{t}_{n+1}) \right\rangle =$$

$$= \sum_{\beta_n=0}^{m-1} \sum_{\beta_n'=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\gamma_{n+1}}(\beta_n') \left\langle \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}\left((\beta_n-\beta_n') \underset{m}{\oplus} (\alpha_n-\gamma_n)\right)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \middle| \mathrm{com}^{[n]}_{(\boldsymbol{\gamma}_{n-1},\beta_n')}(\mathbf{t}_n) \right\rangle =$$

$$= \sum_{\beta_n=0}^{m-1} \sum_{\beta_n'=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\gamma_{n+1}}(\beta_n') \delta_{\boldsymbol{\alpha}_{n-1},\boldsymbol{\gamma}_{n-1}} \delta_{\alpha_n,\gamma_n} \delta_{\beta_n,\beta_n'} = \delta_{\boldsymbol{\alpha}_{n-1},\boldsymbol{\gamma}_{n-1}} \delta_{\alpha_n,\gamma_n} \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\gamma_{n+1}}(\beta_n) = \delta_{\boldsymbol{\alpha}_{n-1},\boldsymbol{\gamma}_{n-1}} \delta_{\alpha_n,\gamma_n} \delta_{\alpha_{n+1},\gamma_{n+1}}.$$

New sequences in (8) are m-complementary sequences, too. Indeed, for each m-element set $\left\{ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) \right\}_{\alpha_{n+1}=0}^{m-1}$ of sequences (8) we have the following set of Schapiro-Rudin-Golay polynomials

$$\left\{ \mathrm{COM}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(z) \right\}_{\alpha_{n+1}=0}^{m-1} = \left\{ \mathcal{Z}\left\{ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}) \right\} \right\}_{\alpha_{n+1}=0}^{m-1} =$$

$$= \left\{ \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \mathcal{Z}\left\{ \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n) \right\} \right\}_{\alpha_{n+1}=0}^{m-1} = \left\{ \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \cdot z^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \mathrm{COM}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(z) \right\}_{\alpha_{n+1}=0}^{m-1}.$$

For these polynomials

$$\sum_{\alpha_{n+1}=0}^{m-1} \mathrm{COM}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(z) \cdot \overline{\mathrm{COM}}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\bar{z}) =$$

$$= \sum_{\alpha_{n+1}=0}^{m-1} \sum_{\beta_n=0}^{m-1} \sum_{\gamma_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\alpha_{n+1}}(\gamma_n) \cdot z^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \bar{z}^{\mathbf{m^n}(\gamma_n \underset{m}{\oplus} \alpha_n)} \mathrm{COM}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(z) \overline{\mathrm{COM}}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\gamma_n)}(\bar{z}) =$$

$$= \sum_{\beta_n=0}^{m-1} \sum_{\gamma_n=0}^{m-1} \left(\sum_{\alpha_{n+1}=0}^{m-1} \mathrm{COM}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(z) \overline{\mathrm{COM}}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\gamma_n)}(\bar{z}) \right) a_{\alpha_{n+1}}(\beta_n) a_{\alpha_{n+1}}(\gamma_n) \cdot z^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \bar{z}^{\mathbf{m^n}(\gamma_n \underset{m}{\oplus} \alpha_n)} =$$

$$= m^n \sum_{\beta_n=0}^{m-1} \sum_{\gamma_n=0}^{m-1} \delta_{\beta_n,\gamma_n} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\alpha_{n+1}}(\gamma_n) \cdot z^{\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)} \bar{z}^{\mathbf{m^n}(\gamma_n \underset{m}{\oplus} \alpha_n)} = m^n \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}(\beta_n) \bar{a}_{\alpha_{n+1}}(\beta_n) \cdot |z|^{2\mathbf{m^n}(\beta_n \underset{m}{\oplus} \alpha_n)}$$

and

$$\left(\sum_{\alpha_{n+1}=0}^{m-1}\mathrm{COM}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(z)\cdot\overline{\mathrm{COM}}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\overline{z})\right)\Bigg|_{|z|=1}=$$

$$=m^n\left(\sum_{\beta_n=0}^{m-1}a_{\alpha_{n+1}}(\beta_n)\overline{a}_{\alpha_{n+1}}(\beta_n)\cdot|z|^{2\mathbf{m^n}(\beta_n\oplus\alpha_n)}\right)\Bigg|_{|z|=1}=m^n\sum_{\beta_n=0}^{m-1}a_{\alpha_{n+1}}(\beta_n)\overline{a}_{\alpha_{n+1}}(\beta_n)=m^{n+1}.$$

Hence, new Schapiro-Rudin-Golay polynomials (8) are m-complementary orthogonal sequences.

2.4. The second generalization

In this section, we introduce the second generalized Golay-Rudin-Shapiro sequences. It is based on the following iteration construction (instead of initial (4))

$$\mathbf{G}^{[1]}_{m^1}(\mathbf{U}_1)\xrightarrow{\mathbf{U}_2}\mathbf{G}^{[2]}_{m^2}(\mathbf{U}_1,\mathbf{U}_2)\xrightarrow{\mathbf{U}_3}\mathbf{G}^{[3]}_{m^3}(\mathbf{U}_1,\mathbf{U}_2,\mathbf{U}_3)\xrightarrow{\mathbf{U}_4}....\xrightarrow{\mathbf{U}_n}\mathbf{G}^{[n]}_{m^n}(\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n)\xrightarrow{\mathbf{U}_{n+1}}\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n,\mathbf{U}_{n+1}).$$

where $\mathcal{U}_n:=\{\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n\}$, $\mathcal{U}_{n+1}:=\{\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n,\mathbf{U}_{n+1}\}=\{\mathcal{U}_n,\mathbf{U}_{n+1}\}$. Here

$$\mathbf{U}_s=\begin{bmatrix}a_0^s(0)&a_0^s(1)&a_0^s(2)&...&a_0^s(m-1)\\a_1^s(0)&a_1^s(1)&a_1^s(2)&...&a_1^s(m-1)\\a_2^s(0)&a_2^s(1)&a_2^s(2)&...&a_2^s(m-1)\\...&...&...&...&...\\a_{m-1}^s(0)&a_{m-1}^s(1)&a_{m-1}^s(2)&...&a_{m-1}^s(m-1)\end{bmatrix}\in SU(m),\quad s=1,2,...,n+1$$

are a sequence of unitary (orthogonal) $(m\times m)$-transforms, belonging to the special unitary group $SU(m)$. Let us assume that we have m-Golay matrix $\mathbf{G}^{[n]}_{m^n}(\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n)=\mathbf{G}^{[n]}_{m^n}(\mathcal{U}_n)$ (depending on n previous transforms $\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n$. We need to construct the next m-Golay matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n,\mathbf{U}_{n+1})==\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathcal{U}_{n+1})$ using only $\mathbf{G}^{[n]}_{m^n}(\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n)$ and $\mathbf{U}_{n+1}$. We are going to use for m-Golay matrix $\mathbf{G}^{[n]}_{m^n}(\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_n)$ the same structure as in (5)

$$\mathbf{G}^{[n]}_{m^n}(\mathcal{U}_n)=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{m^n-1}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n\,|\,\mathcal{U}_n)=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{m^n-1}}\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\,|\,\mathcal{U}_n)\\\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\,|\,\mathcal{U}_n)\\......................\\\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\,|\,\mathcal{U}_n)\end{bmatrix}$$

General “building blocks” for the Golay$\left(m^{n+1}\times m^{n+1}\right)$-matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathcal{U}_{n+1})$ are:

$$\mathbf{U}_{n+1}\cdot\left|{}^{(k)}\mathbf{Tcom}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n\,|\,\mathcal{U}_{n+1})\right\rangle=\mathbf{U}_{n+1}\cdot\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)},\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)},...,\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\right\}\cdot\left|\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n\,|\,\mathcal{U}_n)\right\rangle=$$

$$={}^{(k)}\mathbf{U}_{n+1}\cdot\left|\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n\,|\,\mathcal{U}_n)\right\rangle,\qquad(10)$$

where

$${}^{(k)}\mathbf{U}_{n+1}=\mathbf{U}_{n+1}\cdot\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)},\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)},...,\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\right\}$$

$$=\begin{bmatrix}a_0^{n+1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)}&a_0^{n+1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)}&...&a_0^{n+1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\\a_1^{n+1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)}&a_1^{n+1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)}&...&a_1^{n+1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\\...&...&...&...\\a_{m-1}^{n+1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)}&a_{m-1}^{n+1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)}&...&a_{m-1}^{n+1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\end{bmatrix}.$$

Using "blocks" (10) of $\left(m^n \times m^n\right)$-matrix $\mathbf{G}_{m^n}(\mathcal{U}_n)$ we construct the following Golay $\left(m^{n+1} \times m^{n+1}\right)$-matrix $\mathbf{G}_{m^{n+1}}(\mathcal{U}_{n+1})$ according to the following iteration rule:

$$
\begin{bmatrix}
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)
\end{bmatrix}
\begin{array}{c}\nearrow\\ \rightarrow\\ \searrow\end{array}
\left[\begin{array}{c}
{}^{(0)}\mathbf{U}_{n+1}\cdot
\begin{bmatrix}
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)
\end{bmatrix}\\ \hline
{}^{(1)}\mathbf{U}_{n+1}\cdot
\begin{bmatrix}
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)
\end{bmatrix}\\ \hline
\dots\\ \hline
{}^{(m-1)}\mathbf{U}_{n+1}\cdot
\begin{bmatrix}
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)
\end{bmatrix}
\end{array}\right]
=
\left[\begin{array}{c}
\begin{bmatrix}
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,m-1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})
\end{bmatrix}\\ \hline
\begin{bmatrix}
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})
\end{bmatrix}\\ \hline
\dots\\ \hline
\begin{bmatrix}
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1,0)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-11)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1m-1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})
\end{bmatrix}
\end{array}\right]
= \mathbf{G}_{m^{n+1}}(\mathcal{U}_{n+1}),
$$

where

$$
\begin{bmatrix}
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1})
\end{bmatrix}
= {}^{(k)}\mathbf{U}_{n+1}\cdot
\begin{bmatrix}
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)
\end{bmatrix}
=
$$

$$
=\begin{bmatrix}
a_0^{n+1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)} & a_0^{n+1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)} & \dots & a_0^{n+1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\\
a_1^{n+1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)} & a_1^{n+1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)} & \dots & a_1^{n+1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\\
\dots & \dots & \dots & \dots\\
a_{m-1}^{n+1}(0)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)} & a_{m-1}^{n+1}(1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)} & \dots & a_{m-1}^{n+1}(m-1)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}
\end{bmatrix}
\cdot
\begin{bmatrix}
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)\\
\dots\dots\dots\dots\\
\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n \,|\, \mathcal{U}_n)
\end{bmatrix},
$$

Hence,

$$
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1}) = \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}^{n+1}(\beta_n)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(\beta_n\oplus\alpha_n)}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n \,|\, \mathcal{U}_n).
$$

Since $\mathbf{t}_{n+1} = \left(\mathbf{t}_n, t_{n+1}\right)$, then believing $t_{n+1} = \alpha_n \underset{m}{\oplus} \beta_n$, we obtain as:

$$
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1} \,|\, \mathcal{U}_{n+1}) = \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1} \,|\, \mathcal{U}_{n+1}) =
$$

$$
= \sum_{t_{n+1}=0}^{m-1} a_{\alpha_{n+1}}^{n+1}(\alpha_n \underset{m}{\oplus} t_{n+1})\mathbf{T}_{\mathbf{t}_n}^{m^n t_{n+1}}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{m}{\oplus} t_{n+1})}(\mathbf{t}_n \,|\, \mathcal{U}_n) =
$$

$$
= \sum_{t_{n+1}=0}^{m-1} a_{\alpha_{n+1}}^{n+1}(\alpha_n \underset{m}{\oplus} t_{n+1})\mathbf{T}_{\mathbf{t}_n}^{m^n t_{n+1}}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{m}{\oplus} t_{n+1})}(\mathbf{t}_n + m^n t_{n+1} \,|\, \mathcal{U}_n). \tag{11}
$$

It is finally recurrent relation between m-complementary sequences of $\mathbf{G}^{[n+1]}_{m^{n+1}}[\mathcal{U}_{n+1}]$ and $\mathbf{G}^{[n]}_{m^n}[\mathcal{U}_n]$.

Conclusion

In this paper, we have shown a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$-,or Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure.

This construction has a rich algebraic structure. It is associated not with the triple $\left(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C}\right)$, but with $\left(\mathbf{Z}_m^n, \mathbf{U}_m, \mathcal{A}lg\right)$ or with $\left(\mathbf{Z}_m^n, \left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}, \mathcal{A}lg\right)$, where $\mathbf{U}_m$ or $\left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}$ are an single or a set of arbitrary unitary $(m \times m)$-transforms instead of $\mathcal{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathcal{A}lg$ is an algebras (Clifford algebras), finite rings ($\mathbf{Z}_N$) and finite Galois fields ($\mathbf{GF}(q)$) instead of the complex field $\mathbf{C}$.

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Рецензент статьи: доктор технических наук, профессор Института радиоэлектроники и информационных технологий Уральского федерального университета Л.Г. Доросинский.