NONLINEAR SYSTEM IDENTIFICATION USING CONSTELLATION BASED MULTIPLE MODEL ADAPTIVE ESTIMATORS

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ABSTRACT

This paper describes the application of the constellation based multiple model adaptive estimation (CBMMAE) algorithm to the identification and parameter estimation of nonlinear systems. The method was successfully applied to the identification of linear systems both stationary and nonstationary, being able to fine tune its parameters. The method starts by establishing a minimum set of models that are geometrically arranged in the space spanned by the unknown parameters, and adopts a strategy to adaptively update the constellation models in the parameter space in order to find the model resembling the system under identification. By downscaling the models parameters the constellation is shrunk, reducing the uncertainty of the parameters estimation. Simulations are presented to exhibit the application of the framework and the performance of the algorithm to the identification and parameters estimation of nonlinear systems.

***Index Terms*—** Dynamic systems identification: suboptimal state estimation, multiple model adaptive estimator, parameter estimation, extended Kalman filter, unscented Kalman filter.

1. INTRODUCTION

System identification and parameter tuning is a central task in science and engineering. An algorithm frequently used for system identification and parameter tuning is the multiple-model adaptive estimation (MMAE) [1]. The MMAE is a versatile and powerful algorithm used in system identification and state estimation. It uses a bank of estimators based on a set of models, termed local observers, that run in parallel and through a (posterior) probability evaluator it computes the likelihood of each model to represent the system. The set of models try to represent all possible system behavior patterns or system modes. A key step is the choice of the models to be used in the estimators. The standard approach uses an exhaustive set of models that represents every system mode so that the MMAE can pick the true one.

The constellation based multiple-model adaptive estimation (CBMMAE) uses a set of models – a constellation, whose cardinality depends on the number of unknown parameters and whose parameters can be chosen quite freely [2]. From the values for the posterior probabilities of each model, given by the posterior probability evaluator, the constellation is adaptively conformed to lower the posterior error covariance matrix. By adapting the constellation of models the parameter space is searched for the model that best mimics the system. In this paper we employ nonlinear state estimators to identify and tune the parameters of nonlinear systems. Results for the identification of linear systems, both stationary and nonstationary, with the CBMMAE can be found in [2].

The main contributions of this paper is the application of the CBMMAE to design a minimal set of models for the MMAE, requiring a minimum knowledge about the system's parameters, for the identification of nonlinear systems. The results show the effectiveness of the CBMMAE for identification and tuning the parameters of nonlinear systems.

Section 2 presents the CBMMAE, Sec. 3 describes the experimental setup and the results obtained in the identification of a nonlinear system, and Sec. 4 ends with the conclusions.

2. CONSTELLATION BASED MMAE

The dynamic model of a general nonlinear discrete time-varying system is described in the state-space representation by:

$$\mathbf{x}[t+1] = \mathbf{f}\left(\mathbf{x}[t], \mathbf{u}[t], \boldsymbol{\xi}[t], \boldsymbol{\gamma}[t], t\right), \quad t = 0, 1, 2, \ldots \quad (1)$$

where $\mathbf{x}[t] \in \mathbb{R}^n$ is the system state variables vector at the time instant t, $\mathbf{u}[t] \in \mathbb{R}^m$ is the input signal that drives/controls the system, and $\boldsymbol{\xi}[t] \in \mathbb{R}^p$ accounts for the system disturbance noise. $\boldsymbol{\gamma}[t] \in \mathbb{R}^r$ is the time-variable system's parameters vector.

The nonlinear observations of the state variables degraded by noise are given by:

$$\mathbf{z}[t] = \mathbf{h}\left(\mathbf{x}[t], \boldsymbol{\theta}[t], t\right), \quad (2)$$

$\mathbf{z}[t] \in \mathbb{R}^r$ is the observations vector, and $\boldsymbol{\theta}[t] \in \mathbb{R}^r$ is the noise associated with the measurements.

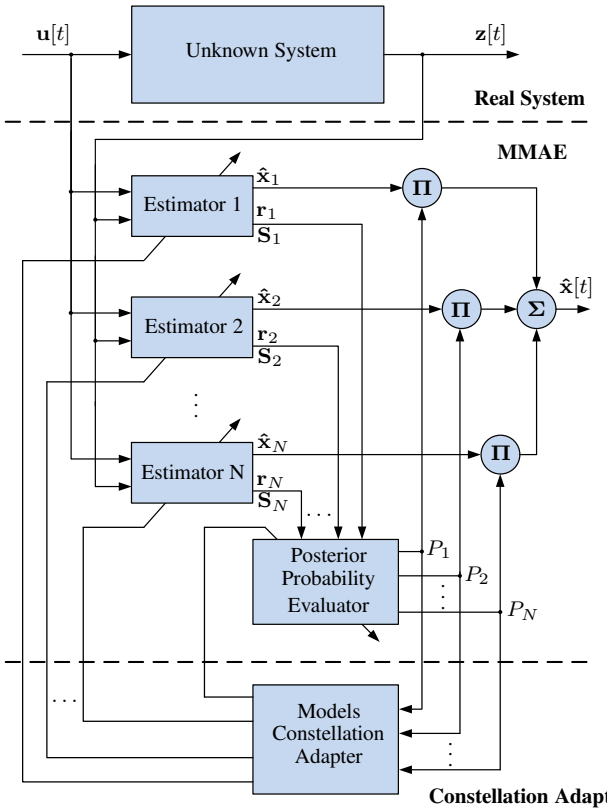

Fig. 1. Block diagram of the multiple model adaptive estimator and constellation adapter.

2.1. The MMAE Algorithm

Figure 1 displays the CBMMAE structure. The MMAE is based on a set of estimators models, that represent the possible different system's behavior patterns, that run in parallel and provide a local estimate of the system state.

The MMAE follows a Bayesian approach to calculate the posterior probabilities of each model that are combined to estimate the system state and the error covariance [1]. Using N estimators each gives the state estimate: $\hat{\mathbf{x}}_k[t]$, with error covariance $\mathbf{\Sigma}_k[t]$, for $k = 1, \ldots, N$, that are combined to compute recursively the posterior probability of the model k to represent the system as:

$$P_k[t+1] = \frac{\beta_k[t+1]\, \mathrm{e}^{-\frac{1}{2}w_k[t+1]}}{\sum_{j=1}^{N} \beta_j[t+1]\, \mathrm{e}^{-\frac{1}{2}w_j[t+1]}\, P_j[t]} P_k[t], \quad (3)$$

where w_k is the value of the matrix induced metric:

$$w_k = ||\mathbf{r}_k||_{\mathbf{S}_k^{-1}} = \mathbf{r}_k^{\mathrm{T}} \mathbf{S}_k^{-1} \mathbf{r}_k, \quad (4)$$

and β_k is defined by:

$$\beta_k = \frac{1}{(2\pi)^{m/2}\sqrt{|\mathbf{S}_k|}}. \quad (5)$$

The vector $\mathbf{r}_k[t]$ in equation (4) is the residual between the observed $\mathbf{z}$ and the predicted vector of observations $\hat{\mathbf{z}}$ from estimator k: $\mathbf{r}_k[t] = \mathbf{z}[t] - \hat{\mathbf{z}}_k[t]$ and $|\mathbf{S}_k|$ is the determinant of the residual covariance matrix. The overall state estimate is computed by combining the N estimates by:

$$\hat{\mathbf{x}}[t] = \sum_{k=1}^{N} P_k[t] \hat{\mathbf{x}}_k[t]. \quad (6)$$

The state covariance error matrix is obtained from:

$$\mathbf{\Sigma}[t] = \sum_{k=1}^{N} P_k[t] \left(\mathbf{\Sigma}_k[t] + (\hat{\mathbf{x}}_k[t] - \hat{\mathbf{x}}[t]) (\hat{\mathbf{x}}_k[t] - \hat{\mathbf{x}}[t])^{\mathrm{T}} \right). \quad (7)$$

If the i-th model matches the system the posterior probabilities of the models evolve according to:

$$\lim_{t\to\infty} P_i[t] = 1; \quad \text{while} \quad \lim_{t\to\infty} P_k[t] = 0; \quad \forall k \neq i. \quad (8)$$

Equation (8) states that if a model matches the system it is identified with probability one, while the posterior probabilities of the other models go to zero [3]. If none of the models match the system then the posterior probability of the closest model according to metric of equation (4) tends to one [1]. From equation (3) if one starts with $\sum_{k=1}^{N} P_k[0] = 1$ then: $\sum_{k=1}^{N} P_k[t] = 1, \quad \forall t$. The estimate of the parameters vector and its error covariance matrix are given by:

$$\hat{\boldsymbol{\gamma}}[t] = \sum_{k=1}^{N} P_k[t] \boldsymbol{\gamma}_k, \quad (9)$$

$$\mathbf{\Sigma}_{\hat{\boldsymbol{\gamma}}}[t] = \sum_{k=1}^{N} P_k[t] \left(\boldsymbol{\gamma}_k - \hat{\boldsymbol{\gamma}}[t]\right) \left(\boldsymbol{\gamma}_k - \hat{\boldsymbol{\gamma}}[t]\right)^{\mathrm{T}}. \quad (10)$$

The updates of the probabilities of (3) can be obtained by developing the dynamic evolution of the probabilities:

$$P_k[t+1] = \frac{p(\mathbf{z}[t+1]|\boldsymbol{\gamma}_k, \mathbf{Z}[t])}{\sum_{i=1}^{N} P_i[t] p(\mathbf{z}[t+1]|\boldsymbol{\gamma}_i, \mathbf{Z}[t])} P_k[t], \quad (11)$$

where $\mathbf{Z}[t] = \{\mathbf{u}[0], \mathbf{u}[1], \ldots, \mathbf{u}[t-1], \mathbf{z}[1], \ldots, \mathbf{z}[t-1]\}$ is the set of previous inputs and past observations, and $\boldsymbol{\gamma}_i$ is the current vector of parameters. This expression is derived by making no assumption on which type of system is considered so that it is valid both for linear and nonlinear systems [1]. However, the posterior probability density $p(\mathbf{z}[t+1]|\boldsymbol{\gamma}_k, \mathbf{Z}[t])$ cannot be obtained exactly for nonlinear systems, and it can only be approximated with a sub-optimal filter, like the extended Kalman filter (EKF) or the unscented Kalman filter (UKF), to compute the state estimates $\hat{\mathbf{x}}_k[t|t]$ and the error matrix $\mathbf{\Sigma}_k[t|t]$. The optimum state estimate and its covariance matrix are also given by equation (6) and equation (7). The selection of the type of the non-optimal filter is usually

based on the physical problem, on the performance required, and on the available computer resources [4].

The Gaussian Sum (GS) filter approximates the non-gaussian probability density state of a nonlinear system by a finite sum of Gaussian functions, each having a different mean and covariance [1,5]. Although the GS filter is based on different assumptions and follows a different philosophy concerning the MMAE algorithm, its structure is identical to the MMAE. The original GS filter explicitly employs the EKF for each estimator in Fig. 1 to obtain the individual state estimates, $\hat{\mathbf{x}}_k[t|t]$, and the covariance matrices, $\mathbf{\Sigma}_k[t|t]$ of each model. The state probability density function is computed by:

$$p(\mathbf{x}[t]|\mathbf{Z}[t]) = \sum_{k=1}^{N} \alpha_k N(\mathbf{x}[t]; \hat{\mathbf{x}}_k[t|t], \mathbf{\Sigma}[t|t]), \qquad (12)$$

where $N(\cdot)$ is the normal density probability function. The values for the weights α_k for each Gaussian density in the mixture are obtained by equation (3) (with $\alpha_k(t) = P_k(t)$), α_k is not interpreted as a probability. Similarly to the MMAE we have the properties: $\sum_{k=1}^{N} \alpha_k[t] = 1$, and $\alpha_k[t] \geq 0, \quad \forall t$. The overall state estimate and the error covariance matrix are also obtained from equation (6) and equation (7). These results justifies the application of the CBMMAE to the identification and parameters estimation of nonlinear systems .

2.2. CBMMAE Overview

The CBMMAE algorithm comprises three stages : *i)* constellation design – establish a proper set of models in the space spanned by the unknown parameters; *ii)* tracking and bracketing – search the parameter space, by moving the models constellation, to localize the region containing the system parameters; *iii)* shrinking process – reduce the interval range for the constellation models' parameters. These steps are illustrated graphically in Fig. 2 for the case of a search in a two-dimensional parameter space.

To track and bracket the system parameters' point the constellation's topology must possess a finite volume, different from zero; and interior point. The finite volume assures that the parameters point can be localized inside a delimited region and the interior point enables the algorithm to detect when the system point is inside the constellation by checking when the probability of the corresponding model is close to one. A topology having these characteristics is an hypercube with a center point. In a n-dimensional space an hypercube constellation has $N = 2^n + 1$ points (hypercube's vertices plus a center point, see Fig. 2 step 1). For a two-dimensional unknown parameter space $n = 2$: $N = 5$ models with parameters: $\{\begin{bmatrix}\gamma_{11} & \gamma_{21}\end{bmatrix}; \begin{bmatrix}\gamma_{11} & \gamma_{22}\end{bmatrix}; \begin{bmatrix}\gamma_{12} & \gamma_{21}\end{bmatrix}; \begin{bmatrix}\gamma_{12} & \gamma_{22}\end{bmatrix}; \begin{bmatrix}\gamma_{1c} & \gamma_{2c}\end{bmatrix}\}$ where $\begin{bmatrix}\gamma_{1c} & \gamma_{2c}\end{bmatrix}$ is the center point, and $\boldsymbol{\gamma} = \begin{bmatrix}\gamma_1^t & \gamma_2^t\end{bmatrix}$ is the system parameters' vector to be estimated. The constellation models' parameters can be initialized with physical plausible values or with an estimated range.

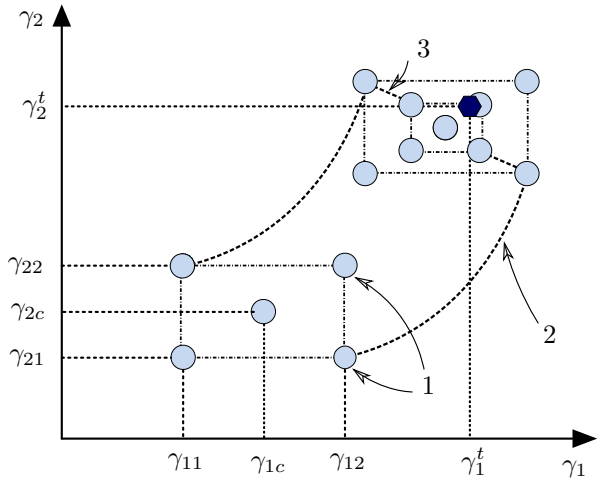

Fig. 2. Constellation topology and evolution in a two-dimensional parameter space: 1- initial constellation setup; 2- tracking and bracketing; 3- shrinking.

Each point in the topology is associated with one of the N estimators of the MMAE structure of Fig. 1 that was complemented with the constellation adapter block. When the posterior probability of a model reaches a threshold value near to one, $P_i \geq P_{th}$, it is identified as the closest to the system point (in Fig. 2, and considering the euclidian metric, the model with coordinates $\begin{bmatrix}\gamma_{12} & \gamma_{22}\end{bmatrix}$ is identified as the closest to the system point, $\begin{bmatrix}\gamma_1^t & \gamma_2^t\end{bmatrix}$), the constellation is translated by positioning its center point, $\begin{bmatrix}\gamma_{1c} & \gamma_{2c}\end{bmatrix}$, at the point previously identified as the closest to the system point. Repeating this process the center point is eventually identified as the closest to the system point that ends located inside the constellation – the system point is bracketed. The tracking and bracketing stage is illustrated by step 2 in Fig. 2.

To estimate the parameters of the system the ranges of the models' parameters of the constellation are systematically reduced and whenever the system point is bracketed the constellation volume is reduced by a shrinking process. To refine the parameters estimation the constellation volume's is shrunk by scaling the parameters' intervals by a factor $\lambda < 1$ (step 3 in Figure 2) and the system point is tracked and bracketed again.

To estimate the system state the CBMMAE uses a bank of local estimators. For linear systems the Kalman filter (KF) is optimal [6]. However, for nonlinear systems optimal state estimation is difficult since the probability density functions of the signals and noise are altered by the nonlinearities, so that the mean and standard deviation are insufficient to describe the probability density function. In this case the state estimate depends on the characteristics of the problem. We will use extended Kalman filter (EKF) [6] and the unscented Kalman filter (UKF) [4] as the state estimators for the nonlinear system.

3. EXPERIMENTAL RESULTS

This method was developed in the context of the design of bioelectronic vision systems, therefore the CBMMAE algorithm is applied to the identification and parameters' estimation of a retinal ganglion cell (RGC) neuron model. The stochastic leaky integrate-and-fire (SLIF) neuron model [7] is employed complemented with a sigmoidal nonlinear block to obtain the firing rate.

The pulse of a postsynaptic ionic current induced in a neuron in response to a presynaptic set of action potentials $\rho(t)$ is given by:

$$\frac{d^2 I_s(t)}{dt^2} + \frac{2}{\tau_\alpha}\frac{dI_s(t)}{dt} + \frac{1}{\tau_\alpha^2} I_s(t) = \alpha_0 \rho(t), \qquad (13)$$

α_0 sets the peak amplitude of the current pulse and τ_α its decay time. The input spike train is defined as a series of Dirac delta functions: $\rho(t) = \sum_i \delta(t - t_i)$ modeling the barrage of incoming action potentials from different presynaptic neurons at distinct time instants t_i. The SLIF model describes the subthreshold potential of a neuron's membrane by:

$$\tau_m \frac{dV_m(t)}{dt} = -V_m(t) + R_m I_s(t) + \sigma_m \xi_m(t), \qquad (14)$$

where $\tau_m = R_m C_m$ is the membrane's time constant and σ_m establishes the power of the membrane noise $\xi_m(t)$. Therefore, the subthreshold dynamics of the neuron according to the SLIF model is written in the state space form as:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\boldsymbol{\xi}(t), \qquad (15)$$

with the state-space variables vector:

$$\mathbf{x}(t) = \begin{bmatrix} dI_s(t)/dt + 1/\tau_\alpha I_s(t) & I_s(t) & V_m(t) \end{bmatrix}^{\mathrm{T}}, \qquad (16)$$

with the dynamic matrix:

$$\mathbf{A} = \begin{bmatrix} -1/\tau_\alpha & 0 & 0 \\ 1 & -1/\tau_\alpha & 0 \\ 0 & 1/C_m & -1/\tau_m \end{bmatrix}. \qquad (17)$$

The input vector is the neural function $\mathbf{u}(t) = \begin{bmatrix} \rho(t) \end{bmatrix}$ where the input matrix is $\mathbf{B} = \begin{bmatrix} \alpha_0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. The noise vector $\boldsymbol{\xi}(t)$ models the spontaneous activity observed in neurons in the absence of controlled input stimulus and the random variations on the ionic currents through the membrane.

The values for the parameters, given in Table 1, are taken to conform with the literature [8]. The neuron synaptic current and the membrane voltage are sampled at regular intervals of time. The discrete time measurement equation:

$$\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t) + \boldsymbol{\theta}(t), \quad t = nT_s, \;\; n \in \mathbb{Z}, \qquad (18)$$

is joined to the state-space equations. T_s is the sampling period and $\mathbf{C}$ is observation matrix. The noise vector $\boldsymbol{\theta}(t)$ in (18) models the observations' errors.

Parameter	Value
Membrane capacitance C_m	250 pF
Membrane time constant τ_m	10 ms
Synapse current peak α_0	0.38089 μA/s
Synapse current decay time τ_α	0.32564 ms

Table 1. Neuron parameters.

To evaluate the CBMMAE in the identification and parameter estimation of nonlinear systems the SLIF neuron model is extended with a nonlinear block to generate the firing rate from the subthreshold membrane potential. The nonlinear block generates the discrete firing rate $r[t]$ from the membrane potential $V_m[t]$ by applying a sigmoid, such that $r[t] = \mathcal{S}\left(V_m[t]\right)$, with the form:

$$\mathcal{S}\left(V_m[t]\right) = \frac{r_0}{1 + \mathrm{e}^{-(V_m[t] - V_0)/\tau_r}}, \qquad (19)$$

where r_0 establishes the maximum value for the firing rate, V_0 shifts the sigmoid along the horizontal axis, and τ_r establishes the slope of the sigmoid between its zero minimum value and its maximum value r_0. The discrete state-space nonlinear system model can be written as:

$$\mathbf{x}[t+1] = \begin{bmatrix} \begin{bmatrix} \mathbf{A}_d & \mathbf{0}_{3\times 1} \end{bmatrix} \mathbf{x}[t] \\ \mathcal{S}\left(\mathbf{x}_3[t]\right) \end{bmatrix} + \mathbf{B}\mathbf{u}[t] + \boldsymbol{\xi}[t], \qquad (20)$$

where $\mathbf{A}_d$ is the discrete counterpart of (17) and the state vector of equation (16) is augmented to include the firing rate:

$$\mathbf{x}[t] = \begin{bmatrix} dI_s(t)/dt + 1/\tau_\alpha I_s(t) & I_s(t) & V_m(t) & r[t] \end{bmatrix}^{\mathrm{T}}. \qquad (21)$$

The linear and nonlinear blocks of the model are apparent from (20). The parameters of the sigmoid are adjusted to the firing rate of the RGC data set referred in [9] by following the methodology described in [10]. The matrix $\boldsymbol{\xi}(t)$ includes the current, $\xi_s(t)$, voltage, $\xi_m(t)$ and firing rate, $\xi_r(t)$, noises:

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \xi_s(t) \\ \xi_m(t) \\ \xi_r(t) \end{bmatrix}, \text{ with power } \mathbf{L} = \begin{bmatrix} \sigma_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_r \end{bmatrix}. \qquad (22)$$

The noise components are considered to be Gaussian white noise with variances: $\sigma_s^2 = 10^{-18}$ A^2, $\sigma_m^2 = 10^{-6}$ V^2 and $\sigma_r^2 = 10^{-2}$ Hz^2. The input signal $\mathbf{u}[t]$ is generated from a Poisson distribution with a mean rate of 45 spikes/s. (The RGC data presented in [9] has an average of 42 spikes/s). The observation matrix is $\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, so that only the firing rate is observed. The other state variables are inferred through the nonlinearity. The observation noise has the intensity: $\boldsymbol{\Theta} = 10^{-4}$. The constellation models' probabilities were initialized with $P_k = 1/N, \;\; k = 1, \ldots, N$ $N = 2^n + 1$, with the initial mean estimate $\hat{\mathbf{x}}_k[0|0] = \mathbf{0}_{4\times 1}$ and covariance $\boldsymbol{\Sigma}_k[0|0] = 10^{-3}\mathbf{I}_{4\times 4}$. The threshold probability was

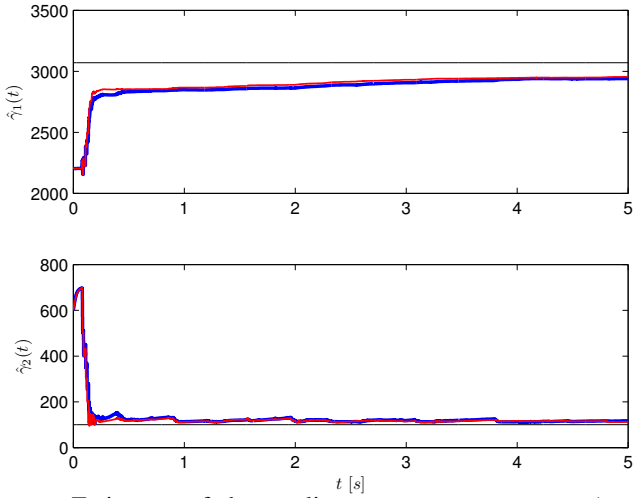

Fig. 3. Estimates of the nonlinear system parameters (EKF; UKF).

set to $P_{th} = 0.95$. The shrinking scale factor is $\lambda = 0.5$. The results were obtained by averaging $M = 25$ Monte-Carlo runs of the experiment. Figure 3 displays the evolution of the estimates the parameters of the linear block computed when observed through the nonlinear sigmoid with (9) using a constellation of $N = 2^2 + 1 = 5$ models. The estimation error in Fig. 4 is computed by $\hat{\boldsymbol{\gamma}}[t] = \frac{1}{M}\sum_{i=1}^{M}[\boldsymbol{\gamma}[t] - \hat{\boldsymbol{\gamma}}_i[t]]$. Figure 3 shows the results obtained for the estimation of the parameters vector $\boldsymbol{\gamma} = \begin{bmatrix}1/\tau_\alpha & 1/\tau_m\end{bmatrix}^{\mathrm{T}}$, that results in a constellation of $N = 5$ models for the CBMMAE, with the initial ranges: $\gamma_1 \in \begin{bmatrix}2100, & 2300\end{bmatrix}$ and $\gamma_2 \in \begin{bmatrix}500, & 700\end{bmatrix}$.

Figure 3 and Fig. 4, where the EKF and the UKF are used as the sub-optimal state-estimators, show that the system parameters are identified even when initialized with a range of values different from the true ones.

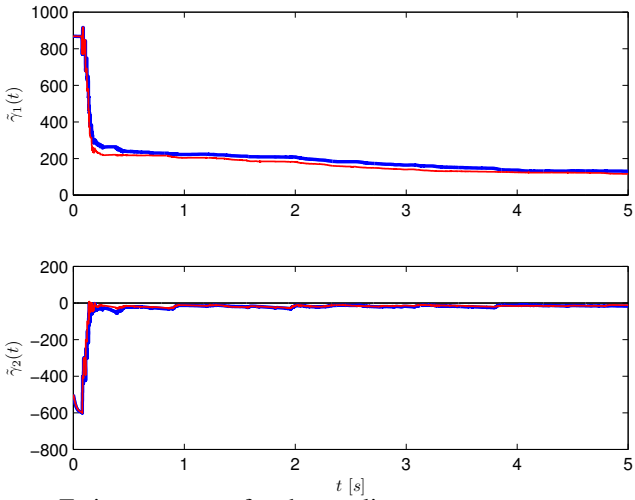

Fig. 4. Estimates errors for the nonlinear system parameters (EKF; UKF).

4. CONCLUSIONS

By using a properly designed constellation of models the CBMMAE is able to identify and tune the parameters of nonlinear systems, even when the system state variables of the state estimators are inferred through the nonlinearity. These results show the effectiveness of this use of this method in nonlinear systems. The number of estimators depend on the number of parameters than can change, and not on the number of possible different configurations of the system. The number of models in the estimators bank depends on the number of unknown and not on the number of possible system configurations, and it robust in terms the initial range of the system parameters. Further research must be done to determine the conditions on which the algorithm converges for nonlinear systems since the convergence of the MMAE is guaranteed only for linear systems.

REFERENCES

[1] B.D.O. Anderson and J.B. Moore, *Optimal Filtering*, Electrical Engineering Series. Dover, 2005.

[2] J.C. Martins, J.J. Caeiro, and L.A. Sousa, "A new approach to system identification and parameter tuning with multiple model adaptive estimators," in *7th International Symposium on Image and Signal Processing and Analysis (ISPA 2011)*, September 2011, pp. 72 –77.

[3] Y. Baram and N.R. Sandell, "An information theoretic approach to dynamical systems modeling and identification," *IEEE Transactions on Automatic Control*, vol. 23, no. 1, pp. 61 – 66, February 1978.

[4] D. Simon, *Optimal state estimation: Kalman, H_∞ and nonlinear approaches*, Wiley-Interscience, 2006.

[5] D.L. Alspach and H.W. Sorenson, "Nonlinear bayesian estimation using gaussian sum approximations," *IEEE Transactions on Automatic Control*, vol. 17, no. 4, pp. 439–448, January 1972.

[6] A. Gelb, *Applied Optimal Estimation*, MIT Press, 1974.

[7] J.C. Martins and L.A. Sousa, *Bioelectronic Vision: Retina Models, Evaluation Metrics, and System Design*, vol. 3 of *Series on Bioengineering & Biomedical Engineering*, World Scientific, Singapore, 2009.

[8] M.O. Gewaltig, M. Diesmann, and A. Aertsen, "Propagation of cortical synfire activity: survival probability in single trials and stability in the mean," *Neural Networks*, vol. 14, no. 6-7, pp. 657–673, July 2001.

[9] W. Gerstner and R. Naud, "How good are neuron models?," *Science*, vol. 326, no. 5951, pp. 379–380, October 2009.

[10] E. J. Chichilnisky, "A simple white-noise analysis of neuronal light responses," *Network: Computation in Neural Systems*, vol. 12, no. 2, pp. 199–213, 2001.