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On the Diophantine Equation $x^2 + 4{\cdot}7^b = y^{2r}$

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ABSTRACT

This paper investigates and determines the solutions for the Diophantine equation $x^2 + 4{\cdot}7^b = y^{2r}$, where x, y, b are all positive intergers and $r > 1$. By substituting the values of r and b respectively, generators of x and y^r can be determined and classified into different categories. Then, by using geometric progression method, a general formula for each category can be obtained. The necessary conditions to obtain the integral solutions of x and y are also investigated.

Keywords: Diophantine equation, generator, geometric progression

INTRODUCTION

Diophantine equation or indeterminate equation is an equation in which solutions for it are from some predetermined classes. It is one of the oldest branches of number theory, in fact, mathematics itself. It is usually difficult to tell whether a given diophantine equation is solvable. The fundamental problem when studying diophantine equation is whether a solution exists, and if it exists, how many solutions there are.

Over the years, different forms of diophantine equation have been considered and the following equation, $x^2 + C = y^n$, $x > 1$, $y > 1$, $n > 3$, is among the most popular ones. When $C = 1$, the above equation has no solution, and Lebesgue (1850) was the first to obtain non-trivial solution from that case. The only solution of the above equation is $x = 1$ and $y = 2$ with $C = -1$, a result which has been sought for many years as a special case of the Catalan's conjecture (Ko, 1965). Subsequently, Ljunggren (1943) studied the equation where $C = 2$ and showed that $x = 5$, $y = 3$ is the only solution. By setting $C = 4$, Nagell (1955) further explored the equation and discovered that $(x,y) = (2,2), (11,5)$ are the only solutions to this equation. For C in the range of $1 \le C \le 100$, Cohn developed a method by which he solved 77 values of parameter C for that equation (Cohn, 1993), followed by two additional values of $C \le 100$, namely, $C = 74$ and $C = 86$ (Mignotte &

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Weger, 1996). Nonetheless, equation $x^2 + 7 = y^n$ is still unsolved.

A different form of the above equation has been considered, i.e., the value of C is replaced by a power of a fixed prime, let $C = p^k$. Equation $x^2 + 2^m = y^n$ has been investigated (Le, 1997), while Arif and Muriefah (1998) solved for equation $x^2 + 3^m = y^n$ when m is odd. However, they also gave partial results in the case when m is even but that the general solution is even in the case m was found by Luca (2000). Subsequently, Luca completely solved the case, $C = 2^a \cdot 3^b$, where a and b denote non-negative integers (Luca, 2002). Eventually, all the integral solutions were found for constant $C = 2^a \cdot 5^b$ (Luca & Togbe, 2008).

The above results prompted us to study on the diophantine equation with constant $C = 2^a \cdot 7^b$. Therefore, in this paper, we studied and investigated the integral solutions for the equation in which $C = 4 \cdot 7^b$ and n is an even integer. The approach is by looking at the possible combinations for the product $C = 2^a \cdot 7^b$ and solving the equations simultaneously. From the results obtained, the value of $a = 2$ was substituted, followed by b to get integer the values of x and y^r under each category. General formulae for the generators of solutions are then obtained.

AN INTEGRAL SOLUTION TO THE EQUATION

In this section, the formulae for finding the integral solutions (x,y) to the equation $x^2 = 2^a \cdot 7^b$ are determined, whereby a, b, and r are positive integers. For this purpose, a generator of solutions is defined for this equation, as follows:

Definition

Let r be a positive integer. The pair of integers (x^0, y_0^r) is called a generator of solutions to the equation $x^2 + 2^a \cdot 7^b = y^{2r}$ where a and b are positive integers, if $x_0^{\,2} + 2^a \cdot 7^b = y_0^{2r}$.

In Theorem 1.1, we give the generators of solutions to the equation $x^2 + 2^a \cdot 7^b = y^{2r}$ where $a = 2$. First, we have the following assertion:

Lemma 1.1:

Let a, b and r be positive integers and, $r > 1$. Then, the generators of solutions to the equation $x^2 + 2^a \cdot 7^b = y^{2r}$ are given by:

$$x = 2^{a-p-1} \cdot 7^q - 2^{p-1} \cdot 7^{b-q}$$

$$y^r = 2^{a-p-1} \cdot 7^q + 2^{p-1} \cdot 7^{b-q}$$

or

$$x = 2^{a-p-1} \cdot 7^{b-q} - 2^{p-1} \cdot 7^q$$

$$y^r = 2^{a-p-1} \cdot 7^{b-q} + 2^{p-1} \cdot 7^q$$

where $0 < p < a,\ 0 \leq q \leq b$.

Proof:

Rewriting the equation $x^2 + 2^a \cdot 7^b = y^{2r}$ as $y^{2r} - x^2 = 2^a \cdot 7^b = y^{2r}$, it follows that

$$(y^r + x)(y^r - x) = 2^a \cdot 7^b.$$

Rewriting the constant on the right hand side of the equation as $2^{a-p}.2^p.7^{b-q}.7^q$ where $0 < p < a$ and $0 \le q \le b$, the following is obtained:

$$(y^r + x)(y^r - x) = 2^{a-p}.2^p.7^{b-q}.7^q$$

Comparing the factors on the left and right sides of the expression, 8 pairs of possible expressions were successfully obtained for $(y^r + x)$ and $(y^r - x)$. Out of which, only 4 pairs of expressions will yield integral solutions and the results are as follows:

i. $y^r + x = 2^{a-p}$

$y^r - x = 2^p.7^{b-q}.7^q$

ii. $y^r + x = 2^p$

$y^r - x = 2^{a-p}.7^{b-q}.7^q$

iii. $y^r + x = 2^{a-p}.7^{b-q}$

$y^r - x = 2^p.7^q$

iv. $y^r + x = 2^{a-p}.\ 7^q$

$y^r - x = 2^p.\ 7^{b-q}$

By solving the above equations simultaneously, the generators for the above equations are given by:

$$x = 2^{a-p-1}.7^q - 2^{p-1}.7^{b-q}$$

$$y^r = 2^{a-p-1}.7^q + 2^{p-1}.7^{b-q}$$

or

$$x = 2^{a-p-1}.7^{b-q} - 2^{p-1}.7^q$$

$$y^r = 2^{a-p-1}.7^q + 2^{p-1}.7^q$$

where $0 < p < a$ and $0 \le q \le b$. In particular, when $q = 0$, the generators of solution are obtained for equations (i) and (ii).

□

In Theorem 1.1, the generators of $x_{b,i}$ and $y^r_{b,i}$ are determined, where i is the i^{th} set of non-negative integral solutions associated with each b.

Theorem 1.1:

Let b, r be positive integers, $r > 1$ and i denotes the i^{th} set of non-negative integral solutions to the equation $x^2 + 4{\cdot}7^b = y^{2r}$ associated with each b. Let $(x_{b,i}, y^r_{b,i})$ denote the generators of the i^{th} set of non-negative integral solutions to this equation. Then, $(x_{b,i}, y^r_{b,i})$ has the following form:

$$x_{b,i} = 7^{i-1}(7^{b-2i+2} - 1)$$

$$y^r_{b,i} = 7^{i-1}(7^{b-2i+2} + 1)$$

Proof:

From the equation, $x^2 + 4{\cdot}7^b = y^{2r}$, it is observed that $a = 2$. As from Lemma 1.1, the generators for each i are given by $x_{b,i} = 2^{2-p-1}.7^{b-q} - 2^{p-1}.7^q$ and $y^r_{b,i} = 2^{2-p-1}.7^{b-q} + 2^{p-1}.7^q$, where $0 <$

$p < 2$ and $0 \le q \le b$. Clearly, $p = 1$. For each value of b, the results obtained are as follows:

When $b = 1$, $0 \le q \le 1$. Hence $q = 0$ or $q = 1$.

When $q = 0$, the first ($i = 1$) set of solution is obtained, as follows:

$$x_{1,1} = 6$$

$$y^r_{1,1} = 8$$

When $q = 1$, the second ($i = 2$) set of values is derived for $x_{1,2}$ and $y^r_{1,2}$, as follows:

$$x_{1,2} = -6$$

$$y^r_{1,2} = 8$$

Nonetheless, these are ignored because $x_{1,2} < 0$.

Meanwhile, when $b = 2$, $0 \le q \le 2$, and hence, $q = 0$, 1 or 2.

When $q = 0$, the first ($i = 1$) set of solution is obtained as follows:

$$x_{2,1} = 48 = 6(7) + 6$$

$$y^r_{2,1} = 50 = 8(7) - 6$$

When $q = 1$, the second ($i = 2$) set of values is obtained for $x_{2,2}$ and $y^r_{2,2}$ as follows:

$$x_{2,2} = 0$$

$$y^r_{2,2} = 14$$

When $q = 2$, the third ($i = 3$) set of values is achieved for $x_{2,3}$ and $y^r_{2,3}$ as follows:

$$x_{2,3} = -48$$

$$y^r_{2,3} = 50$$

These are also ignored because $x_{2,3} < 0$.

When $b = 3$, $0 \le q \le 3$. Hence $q = 0$, 1, 2 or 3.

When $q = 0$, the first ($i = 1$) set of solution is retrieved, as follows:

$$x_{3,1} = 342 = 48(7) + 6 = [6(7) + 6] + 6 = 6(7^2) + 6(7) + 6$$

$$y^r_{3,1} = 344 = 50(7) - 6 = [6(7) + 6] + 6 = 6(7^2) + 6(7) + 6$$

When $q = 1$, the second ($i = 2$) set of values is achieved for $x_{3,2}$ and $y^r_{3,2}$, as follows:

$$x_{3,2} = 42 = 0(7) + 6(7)$$

$$y^r_{3,2} = 56 = 14(7) - 6(7) = 8(7)$$

When $q = 2$, the third ($i = 3$) set of values is obtained for $x_{3,3}$ and $y^r_{3,3}$, as follows:

$$x_{3,3} = -42$$

$$y^r_{3,3} = 56$$

When $q = 3$, the fourth ($i = 4$) set of values is gained for $x_{3,4}$ and $y^r_{3,4}$ as follows:

$$x_{3,4} = -342$$

$$y^r_{3,4} = 344$$

Note that the third and fourth sets are also ignored because $x_{3,3}$ and $x_{3,4}$ are negative in values.

By using the same substitution method, other values of x and y^r are obtained, as shown below:

When $b = 4$, $0 \leq q \leq 4$.

When $q = 0$, $x_{4,1} = 2400 = 6(7^3) + 6(7^2) + 6(7) + 6$

$y^r_{4,1} = 2402 = 8(7^3) - 6(7^2) - 6(7) - 6$

When $q = 1$, $x_{4,2} = 336 = 6(7^2) + 6(7)$

$y^r_{4,2} = 350 = 8(7^2) - 6(7)$

When $q = 2$, $x_{4,3} = 0$

$y^r_{4,3} = 98 = 14(7)$

When $q = 3$, $x_{4,4} = -336$

$y^r_{4,4} = 350$

When $q = 4$, $x_{4,5} = -2400$

$y^r_{4,5} = 2402$

Here, the fourth and fifth sets are neglected because $x_{4,4}$ and $x_{4,5}$ are negative in values.

When $b = 5$, $0 \leq q \leq 5$.

When $q = 0$, $x_{5,1} = 16806 = 6(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$

$y^r_{5,1} = 16808 = 8(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$

When $q = 1$, $x_{5,2} = 2394 = 6(7^3) + 6(7^2) + 6(7) + 6$

$y^r_{5,2} = 2408 = 8(7^3) - 6(7^2) - 6(7) - 6$

When $q = 2$, $x_{5,3} = 294 = 0(7) + 6(7^2)$

$y^r_{5,3} = 392 = 8(7^2)$

When $q = 3$, $x_{5,4} = -294$

$y^r_{5,4} = 392$

When $q = 4$, $x_{5,5} = -2394$

$y^r_{5,5} = 2408$

When $q = 5$, $x_{5,6} = -16806$

$y^r_{5,6} = 16808$

The fourth, fifth, and sixth sets are ignored because $x_{5,4}$, $x_{5,5}$ and $x_{5,6}$ are negative in values.

When $b = 6$, $0 \le q \le 6$.

When $q = 0$, $x_{6,1} = 117648 = 6(7^5) + 6(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$

$y^r_{6,1} = 117650 = 8(7^5) - 6(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$

When $q = 1$, $x_{6,2} = 11800 = 6(7^4) + 6(7^3) + 6(7^2) + 6(7)$

$y^r_{6,2} = 16814 = 8(7^4) - 6(7^3) - 6(7^2) - 6(7)$

When $q = 2$, $x_{6,3} = 2352 = 6(7^3) + 6(7^2)$

$y^r_{6,3} = 2450 = 8(7^3) - 6(7^2)$

When $q = 3$, $x_{6,4} = 0$

$y^r_{6,4} = 686 = 14(7^2)$

When $q = 4$, $x_{6,5} = -2352$

$y^r_{6,5} = 2450$

When $q = 5$, $x_{6,6} = -16800$

$y^r_{6,6} = 16814$

When $q = 6$, $x_{6,7} = -117648$

$y^r_{6,7} = 117650$

Similarly, the fifth, sixth, and seventh sets are ignored because $x_{6,5}$, $x_{6,6}$ and $x_{6,7}$ are negative in values.

When $b = 7$, $0 \le q \le 7$.

When $q = 0$, $x_{7,1} = 823542 = 6(7^6) + 6(7^5) + 6(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$

$y^r_{7,1} = 823544 = 8(7^5) - 6(7^5) - 6(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$

When $q = 1$, $x_{7,2} = 117642 = 6(7^4) + 6(7^4) + 6(7^3) + 6(7^2) + 6(7)$

$y^r_{7,2} = 117656 = 8(7^5) - 6(7^4) - 6(7^3) - 6(7^2) - 6(7)$

When $q = 2$, $x_{7,3} = 16758 = 6(7^4) + 6(7^3) + 6(7^2)$

$y^r_{7,3} = 16856 = 8(7^4) - 6(7^2) - 6(7^2)$

When $q = 3$, $x_{7,4} = 2058 = 0(7) + 6(7^3)$

$y^r_{7,4} = 2744 = 8(7^3)$

When $q = 4$, $x_{7,5} = -2058$

$y^r_{7,5} = 2744$

When $q = 5$, $x_{7,6} = -16758$

$y^r_{7,6} = 16856$

When $q = 6$, $x_{7,7} = -117642$

$y^r_{7,7} = 117656$

When $q = 6$, $x_{7,8} = -823542$

$y^r_{7,8} = 823544$

Here, the fifth, sixth, seventh and eighth sets are ignored because $x_{7,5}$, $x_{7,6}$, $x_{7,7}$ and $x_{7,8}$ are negative in values.

This sequence will continue with the value of $b > 7$ due to the infinite values of b and i. Meanwhile, collecting the generators $x_{b,i}$ and $y^r_{b,i}$ from the above for the different values of b and i for each b, we obtained the following:

$$x_{1,1} = 6$$
$$x_{2,1} = 6(7) + 6$$
$$x_{3,1} = 6(7^2) + 6(7) + 6$$
$$x_{4,1} = 6(7^3) + 6(7^2) + 6(7) + 6$$
$$x_{5,1} = 6(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$$
$$\vdots$$

$$y^r_{1,1} = 8$$
$$y^r_{2,1} = 8(7) - 6$$
$$y^r_{3,1} = 8(7^2) - 6(7) - 6$$
$$y^r_{4,1} = 8(7^3) - 6(7^2) - 6(7) - 6$$
$$y^r_{5,1} = 8(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$$
$$\vdots$$

The general form of $x_{b,1}$ and $y^r_{b,1}$, $b = 1, 2, 3, \cdots$ can be obtained by using the mathematical induction method, as follows:

Let $x_{b,1} = 6(7^{b-1}) + 6(7^{b-2}) + \cdots + 6(7) + 6$.

Clearly, the case is true when $b = 1$, since $x_{1,1} = 6$.

Based on the assumption that $x_{k,1} = 6(7^{k-1}) + 6(7^{k-2}) + \cdots\ 6(7) + 6$, it can be seen that:

$$x_{k+1,1} = 6\,(7^k) + x_{k,1}$$
$$= 6(7^k) + 6(7^{k-1}) + 6(7^{k-2}) + \cdots + 6(7) + 6$$

Hence, $x_{b,1} = 6(7^{b-1}) + 6(7^{b-2}) + \cdots + 6(7) + 6$.

Let $y^r_{b,1} = 8(7^{b-1}) - 6(7^{b-2}) - \cdots - 6(7) - 6$.

Through a similar induction process on b in $y^r_{b,1}$ it can be shown that $y^r_{b,1} = 8$.

And, based on the assumption that $y^r_{k,1} = 8(7^{k-1}) - 6(7^{k-2}) - \cdots - 6(7) - 6$ is true,

$$
\begin{aligned}
y^r_{k+1,1} &= 6(7^k) + y^r_{k,1} \\
&= 6(7^k) + 8(7^{k-1}) - 6(7^{k-2}) - \cdots - 6(7) - 6 \\
&= 6(7^k) + 2(7^k) - 2(7^k) + 8(7^{k-1}) - 6(7^{k-2}) - \cdots - 6(7) - 6 \\
&= 8(7^k) - 14(7^{k-1}) + 8(7^{k-1}) - 6(7^{k-2}) - \cdots - 6(7) - 6 \\
&= 8(7^k) - 6(7^{k-1}) - 6(7^{k-2}) - \cdots - 6(7) - 6 \\
&= 8(7^{(k+1)-1}) - 6(7^{(k+1)-2}) - 6(7^{(k+1)-3}) - \cdots - 6(7) - 6
\end{aligned}
$$

Hence, $y^r_{b,1} = 8(7^{b-1}) - 6(7^{b-2}) - \cdots - 6(7) - 6$.

By applying the same mathematical induction on $x_{b,i}$ and $y^r_{b,i}$ for the different values of b and i = 2, 3, 4 as shown above, the general forms of $x_{b,i}$ and $y^r_{b,i}$ are as follows:

When $i = 2$,

$$x_{b,2} = 6(7^{b-2}) + 6(7^{b-3}) + \cdots + 6(7)$$

$$y^r_{b,2} = 8(7^{b-2}) - 6(7^{b-3}) - \cdots - 6(7)$$

When $i = 3$,

$$x_{b,3} = 6(7^{b-3}) + 6(7^{b-4}) + \cdots + 6(7^2)$$

$$y^r_{b,3} = 8(7^{b-3}) - 6(7^{b-4}) - \cdots - 6(7^2)$$

When $i = 4$,

$$x_{b,4} = 6(7^{b-4}) + 6(7^{b-5}) + \cdots + 6(7^3)$$

$$y^r_{b,4} = 8(7^{b-4}) - 6(7^{b-5}) - \cdots - 6(7^3)$$

The general form of generators $x_{b,i}$ and $y^r_{b,i}$ are obtained by applying induction on i for each b, as follows:

Let $x_{b,i} = 6(7^{b-i}) + 6(7^{b-i-1}) + 6(7^{b-i-2}) + \cdots + 6(7^i) + 6(7^{i-1})$.

Clearly, the case is true when $i = 1$, since

$$x_{b,1} = 6(7^{b-1}) + 6(7^{b-2}) + \cdots + 6(7^2) + 6.$$

On the assumption that,

$$x_{b,k} = 6(7^{b-k}) + 6(7^{b-k-1}) + 6(7^{b-k-2}) + \cdots + 6(7^k) + 6(7^{k-1}).$$

It can be seen that,

$$
\begin{aligned}
x_{b,k+1} &= x_{b,k} - 6(7^{b-k}) - 6(7^{k-1}) \\
&= 6(7^{b-k}) + 6(7^{b-k-1}) + 6(7^{b-k-2}) + \cdots + 6(7^k) + 6(7^{k-1}) - 6(7^{b-k}) - 6(7^{k-1}) \\
&= 6(7^{b-k-1}) + 6(7^{b-k-2}) + \cdots + 6(7^k) \\
&= 6(7^{b-(k+1)}) + 6(7^{b-(k+1)-1)}) + \cdots + 6(7^{(k+1)}) + 6(7^{(k+1)-1)})
\end{aligned}
$$

Hence, $x_{b,i} = 6(7^{b-i}) + 6(7^{b-i-1}) + 6(7^{b-i-2}) + \cdots + 6(7^i) + 6(7^{i-1})$.

That is,

$$x_{b,i} = 7^{\,i-1}(7^{b-2i+2} - 1) \qquad [1.1]$$

Let $y^r_{b,i} = 8(7^{b-i}) - 6(7^{b-i-1}) - 6(7^{b-i-2}) - \cdots - 6(7^i) - 6(7^{i-1})$.

Through a similar induction process on i in $y^r_{b,i}$, it can be shown that,

$$y^r_{b,i} = 8(7^{b-i}) - 6(7^{b-2}) - 6(7^{b-3}) - \cdots - 6(7) - 6.$$

On the assumption that,

$$y^r_{b,k} = 8(7^{b-k}) - 6(7^{b-k-1}) - 6(7^{b-k-2}) - \cdots - 6(7^k) - 6(7^{k-1})$$

is true.

$$\begin{aligned} y^r_{b,k+1} &= y^r_{b,k} - 6(7^{b-k}) + 6(7^{k-1}) \\ &= 8(7^{b-k}) - 6(7^{b-k-1}) - 6(7^{b-k-2}) - \cdots - 6(7^k) - 6(7^{k-1}) - 6(7^{b-1}) + 6(7^{k-1}) \\ &= 2(7^{b-k}) - 6(7^{b-k-1}) - 6(7^{b-k-2}) - \cdots - 6(7^k) \\ &= 14(7^{b-k-1}) - 6(7^{b-k-1}) - 6(7^{b-k-2}) - \cdots - 6(7^k) \\ &= 8(7^{b-k-1}) - 6(7^{b-k-2}) - \cdots - 6(7^k) \\ &= 8(7^{b-(k+1)}) - 6(7^{b(k+1)-1}) - 6(7^{b(k+1)-2}) - \cdots - 6(7^{b(k+1)-1}) \end{aligned}$$

Hence, $y^r_{b,i} = 8(7^{b-1}) - 6(7^{b-i-1}) - 6(7^{b-i-2}) - \cdots - 6(7^i) - 6(7^{i-1})$.

That is, $y^r_{b,i} = 8(7^{b-1}) - 6\left[\dfrac{7^{b-1-1}(1-7^{-(b-2i+1)})}{1-7^{-1}}\right]$

$$y^r_{b,i} = 8(7^{b-1}) - 7^{b-i} + 7^{i-1}$$

$$y^r_{b,i} = 7^{b-i+1} + 7^{i-1}$$

$$y^r_{b,i} = 7^{i-1}(7^{b-2i+2} + 1) \qquad [1.2]$$

□

Remarks:

It is clear that the i in the i^{th} set of generators $(x_{b,i}, y^r_{b,i})$ corresponds to the i^{th} pair of generators for solution to the equation $x^2 + 4{\cdot}7^b = y^{2r}$.

Corollary 1.1:

Let b be even and r is any positive integer. Suppose that $i = \frac{1}{2}b + 1$ is the number of generators of the non-negative integral solutions associated with each b in the equation $x^2 + 4{\cdot}7^b = y^{2r}$, then $x_{b,i} = 0$ and $y^r_{b,i} = 2{\cdot}7^{\frac{1}{2}b}$ are the generators of solutions to the equation.

Proof:

From [1.1] and [1.2] in Theorem 1.1, we obtain

$$x_{b,i} = 7^{(i-1)}(7^{b-2i+2} - 1)$$

$$x_{b,i} = 7^{(i-1)}(7^{b-2i+2} + 1)$$

Since $i = \frac{1}{2}b + 1$, we have

$$x_{b,i} = 0$$

$$y_{b,i}^{r} = 2{\cdot}7^{\frac{1}{2}b}.$$

Thus, $x_{b,i} = 0$ and $y_{b,i}^{r} = 2{\cdot}7^{\frac{1}{2}b}$ are the solutions that satisfy the following equation, $x^2 + 4{\cdot}7^b = y^{2r}$ for b even, r any positive integer and $i = \frac{1}{2}b + 1$ is the number of generators of non-negative integral solutions associated with each b. □

Meanwhile, the following lemma will determine the range of i in $x_{b,i}$ and $y_{b,i}^{r}$ for the different values of b.

Lemma 1.2:

Let b and r be positive integers. Then, the range of i, the number of generators of non-negative integral solutions associated with each b to the equation $x^2 + 4{\cdot}7^b = y^{2r}$ is given by:

$$\begin{cases} 0 < i \leq \frac{1}{2}(b+1), \text{ when } b \text{ is odd} \\ 0 < i \leq \frac{1}{2}\ b + 1, \text{ when } b \text{ is even} \end{cases}$$

Proof:

From Theorem 1.1, we have,

$$x_{b,i} = 7^{(i-1)}(7^{b-2i+2} - 1)$$

Since i denotes the number of generators of non-negative integral solutions associated with each b, we have $x_{b,i} \geq 0$, that is:

$$7^{b-2i+2} \geq 1.$$

By simplifying the inequality, we obtained:

$$i \leq \frac{1}{2}(b+2).$$

Since $i \leq \lfloor \frac{1}{2}(b+2) \rfloor$ where $\lfloor x \rfloor$ denotes the floor function of x, we obtain $i \leq \frac{1}{2}(b+1)$ when b is odd and $i \leq \frac{1}{2}(b+2)$ when b is even.

Therefore, i is given by:

$$\begin{cases} 0 < i \leq \frac{1}{2}(b+1), \text{ when } b \text{ is odd} \\ 0 < i \leq \frac{1}{2}\ b + 1, \text{ when } b \text{ is even} \end{cases}$$

□

Theorems 1.2, 1.3 and 1.4 give the forms of integral solutions for $x_{b,i}$ and $y_{b,i}$ to the equation $x^2 + 4{\cdot}7^b = y^{2r}$ when $r = 3$, $b = 6t - 5$ and $i = 3t - 2$. It is shown that there are no integral solutions when $r \neq 3$.

Theorem 1.2:
Let b, i and t be positive integers. Then, $x_{b,i} = 6{\cdot}7^{3(t-1)}$ and $y_{b,i} = 2{\cdot}7^{t-1}$ are the integral solutions to the equation, $x^2 + 4{\cdot}7^b = y^6$ if and only if $b = 6t - 5$ and $i = 3t - 2$.

Proof:

First, suppose that $x_{b,i} = 6{\cdot}7^{3(t-1)}$ and $y_{b,i} = 2{\cdot}7^{t-1}$ are the integral solutions to the equation $x^2 + 4{\cdot}7^b = y^6$, we will have:

$$(6{\cdot}7^{3(t-1)})^2 + 4{\cdot}7^b = (2{\cdot}7^{t-1})^6 \qquad [1.3]$$

By simplifying the equation [1.3], we obtained:

$$b = 6t - 5.$$

Since $x_{b,i} = 6{\cdot}7^{3(t-1)}$, we have from Theorem 1.1with $r = 3$,

$$7^{(i-1)}(7^{b-2i+2} - 1) = 6{\cdot}7^{3(t-1)}.$$

Since $b = 6t - 5$,

$$7^{6t-i-4)} - 7^{i-1} = 6{\cdot}7^{3(t-1)}. \qquad [1.4]$$

Multiplying [1.4] by 7^{-i+1}, we obtained:

$$7^{2(3t-i)-3} - 1 = 6{\cdot}7^{(3t-1)-2}. \qquad [1.5]$$

Now, let $x = 7^{3t-i}$; rearranging [1.5], we will have:

$$7^{-3}{\cdot}\, x^2 - 7^{-2}{\cdot}6x - 1 = 0$$

$$x^2 - 42x - 343 = 0$$

That is,

$$(x + 7)(x - 49) = 0,$$

from which, we can see that;

$$x = 7^{3t-i} = -7$$

There is an inconsistency since 7^{3t-i} is always positive. Hence, this case need not be considered. Secondly, we have,

$$x = 7^{3t-i} = 49,$$

from which, we see that,

$$3t - i = 2$$

or

$$i = 3t - 2.$$

Thus, $b = 6t - 5$ and $i = 3t - 2$ when $x_{b,i} = 6\cdot 7^{3(t-1)}$ and $y_{b,i} = 2\cdot 7^{t-1}$ are the integral solutions to the following equation, $x^2 + 4\cdot 7^b = y^6$.

Conversely, let $b = 6t - 5$ and $i = 3t - 2$. From Theorem 1.1, when $a = 2$ and $r = 3$, the generators of the solutions $x_{b,i}, y_{b,i}^3$ to the equation $x^2 + 4\cdot 7^b = y^6$ are given by:

$$x_{b,i} = 7^{i-1}(7^{b-2i+2} - 1)$$

$$y_{b,i}^3 = 7^{i-1}(7^{b-2i+2} + 1).$$

Hence, the integral solutions for $x_{b,i}$ and $y_{b,i}^3$, in which $b = 6t - 5$ and $i = 3t - 2$, are:

$$x_{6t-5,3t-2} = 7^{(3t-2)-1}(7^{(6t-5)-2(3t-2)+2} - 1) \quad [1.6]$$

$$y_{6t-5,3t-2}^3 = 7^{(3t-2)-1}(7^{(6t-5)-2(3t-2)+2} + 1). \quad [1.7]$$

By simplifying [1.6] and [1.7], we have

$$x_{b,i} = 6\cdot 7^{3(t-1)}$$

$$y_{b,i} = 2\cdot 7^{t-1}.$$

Clearly, $x_{b,i}$ and $y_{b,i}$ are integers for any positive number, t.

Thus, $x_{b,i} = 6\cdot 7^{3(t-1)}$ and $y_{b,i} = 2\cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 4\cdot 7^b = y^6$, when $b = 6t - 5$ and $i = 3t - 2$. □

Theorem 1.3:
Let t be a positive integer, $b = 6t - 5$ and $i = 3t - 2$. Then, $x_{b,i} = 6\cdot 7^{3(t-1)}$ and $y_{b,i} = 2\cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 4\cdot 7^{2r} = y^6$ if and only if $r = 3$.

Proof:

Firstly, let $x_{b,i} = 6\cdot 7^{3(t-1)}$ and $y_{b,i} = 2\cdot 7^{t-1}$ be the integral solutions to the equation, $x^2 + 4\cdot 7^b = y^{2r}$. Then,

$$(6\cdot 7^{3(t-1)})^2 + 4\cdot 7^{6t-5} = (2\cdot 7^{t-1})^{2r} \quad [1.8]$$

By simplifying equation [1.8], we have:

$$(2\cdot 7^{t-1})^6 = (2\cdot 7^{t-1})^{2r}$$

Therefore,

$$2r = 6$$

Thus, $r = 3$ when $x_{b,i} = 6\cdot 7^{3(t-1)}$ and $y_{b,i} = 2\cdot 7^{t-1}$ are the integral solutions to the equation, $x^2 + 4\cdot 7^b = y^{2r}$, where $b = 6t - 5$ and $i = 3t - 2$, and t are positive integers.

Conversely, let suppose that $r = 3$. Then, the following equation $x^2 + 4\cdot 7^b = y^6$ is obtained. Since $b = 6t - 5$, $i = 3t - 2$, and t are positive integers, we have from Theorem 1.2, $x_{b,i} = 6\cdot 7^{3(t-1)}$ and $y_{b,i} = 2\cdot 7^{t-1}$, which are the integral solutions to the equation, $x^2 + 4\cdot 7^b = y^6$.

Thus, $x_{b,i} = 6{\cdot}7^{3(t-1)}$ and $y_{b,i} = 2{\cdot}7^{t-1}$ are the integral solutions to the equation $x^2 + 4{\cdot}7^b = y^{2r}$ when $r = 3$, where $b = 6t - 5$, $i = 3t - 2$, and t are positive integers.

□

Theorem 1.4:

Let $b = 6t - 5$, $i = 3t - 2$, $r > 1$, and t be positive integers. If $r \neq 3$, the equation $x^2 + 4{\cdot}7^b = y^{2r}$ has no integral solution.

Proof:

From Theorem 1.1, we obtained:

$$x_{b,i} = 7^{i-1}(7^{b-2i+2} - 1)$$
$$y^r_{b,i} = 7^{i-1}(7^{b-2i+2} + 1)$$

Since $r > 1$ and $r \neq 3$, the possible values of r are as in the following table.

TABLE 1
The possible values of r to the equation $x^2 + 4{\cdot}7^b = y^{2r}$

Cases		Values of r
Case 2.1.1	$r \equiv 0 \pmod 3$,	$r = 3s$, $s > 1$
Case 2.1.2	$r \equiv 1 \pmod 3$,	$r = 1 + 3s$, $s \geq 1$
Case 2.1.3	$r \equiv 1 \pmod 3$,	$r = 2 + 3s$, $s \leq 1$

By substituting the values of b and i into equation [1.2], we have

$$x_{b,i} = 2{\cdot}(7^{(t-1)})^{\frac{3}{r}}. \qquad [1.9]$$

By substituting the three different forms of r values into [1.9], we have from Table 1,

Case 1.1.1: when $r = 3s$.

$$y_{b,i} = (2{\cdot}7^{(t-1)})^{\frac{1}{s}}.$$

Since $s > 1$, $2^{\frac{1}{s}}$ is an irrational number, there exists no integral values for $y_{b,i}$. It follows that there is no integral solution to this equation when $r = 3s$.

Case 1.1.2: when $r = 1 + 3s$.

$$y_{b,i} = (2{\cdot}7^{(t-1)})^{\frac{3}{1+3s}}.$$

Since $s \geq 1$, $2^{\frac{3}{1+3s}}$ is an irrational number. It follows that there exists no integral solution for the equation when $r = 1 + 3s$.

Case 1.1.3: when $r = 2 + 3s$.

$$y_{b,i} = (2{\cdot}7^{(t-1)})^{\frac{3}{2+3s}}.$$

Since $s \geq 0$, $2^{\frac{3}{2+3s}}$ is an irrational number. Clearly, the integral values do not exist for $y_{b,i}$. Consequently, there is no integral solution to the equation when $r = 2 + 3s$.

By combining all the three cases discussed above, we can therefore conclude that the following equation, $x^2 + 4 \cdot 7^b = y^{2r}$, has no integral solution if $r \neq 3$ when $b = 6t - 5$, $i = 3t - 2$, and t are positive integers. □

CONCLUSIONS

From the results above, only a pair of generators for x and y^r are obtained for the following equation, $x^2 + 4 \cdot 7^b = y^{2r}$, that is:

$$x_{b,i} = 7^{i-1}(7^{b-2i+2} - 1)$$

$$y^r_{b,i} = 7^{i-1}(7^{b-2i+2} + 1).$$

However, from Lemma 1.2, it can be noticed that the number of generator pairs increases in relation to the values of b, and it will continue to do so due to the infinite value of b. On the other hand, when $r = 3$, the equation $x^2 + 4 \cdot 7^b = y^6$ is obtained. With values of $b = 6t - 5$ and $i = 3t - 2$ and t are positive integers as necessary conditions, the integral solutions to the equation for each value of t are $x_{b,i} = 6 \cdot 7^{3(t-1)}$, $y_{b,i} = 2 \cdot 7^{t-1}$. It can be obtained by finding the r-th root of the generator y^r. It has also been proven that there are no integral solutions if the above conditions are altered.

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