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An Evasion Game Model for Duopoly Competition

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ABSTRACT

Based on the Bertrand and Cournot economic models, we develop an evasion game model for a duopoly market with two players competing on price and quantity. We show that if the total financial strength of first mover is greater than that of the second mover, and the first mover observes the second mover perfectly, our proposed optimal strategy can be followed by the first mover to remain the market leader ahead of all competition.

Keywords: Evasion game, optimal strategy, duopoly competition, Bertrand and Cournoteconomic models

INTRODUCTION

There is a vast body of literature on first- and second-mover competition in a duopoly market (Guth *et al.*,2006; von Stengel, 2010). Dolores Alepuz and Amparo Urbano (1999) considered imperfect observation for firms in a duopoly market in real life to show that these firms were willing to learn about their environment and competitors. This competition model is used mainly for competition among firms, but it is also applicable in solving individual financial decision problems (Masud *et. al*, 2012). The assumption of perfect observation in a duopoly market is considered by Anders Udo Poulsen (2007). There is also a trend to discuss and criticise the Bertrand and Cournot oligopolistic models; for a discussion on this see Hamilton and Slutsky (1990) and van Damme and Hurkens (1999).

A basic model of commitment is to convert a two-player game in strategic form to a leadership game with the same pay-offs, in which one player (the leader) commits to

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a strategy while the other player chooses the best reply. The recent paper by Bernhard von Stengel and Shmuel Zamir (2010) studies such leadership games for games with convex strategy sets. In this paper, we set the strategy as being that the first mover remains the leader in the market based on the evasion differential games with two other players. Firms can compete on many variables. Luca Lambertini and Andrea Mantovani (2006) considered competition on quantity and sticky price based on the Cournot and Bertrand models. Here we use the same variables and propose a model along with a strategy for competition on price and quantity based on evasion differential games. To gain the result, some basic assumptions are set in our mathematical model. Each player has its own constraint and the gamr is dynamic as in the games considered by Isaacs (1999).

The paper consists of four sections including introduction, statement of the problem, main result and conclusion. In the introduction, we discuss the Bertrand and Cournot duopoly models and show that their assumptions are mostly in compliance with our model . In the second section, we state the problem by integrating the Bertrand and Cournot models, and in the third section, we propose an optimal strategy for the first mover in three different time periods. Finally, in the last section, we summarise the whole paper.

The Bertrand competition model versus the evasion model

Based on Bertrand's classical oligopoly theory of price competition, there should be at least two firms producing un-differentiated products in a market. It is assumed in this model that cooperation among firms is not allowed and market demand belongs to the firm which offers the lowest price. Marginal cost in the Bertrand duopoly model is assumed to be the same for all firms. This last assumption is not valid in our evasion model, in which the marginal cost for the first mover is less than that of the second mover due to greater financial strength of the first mover rather than that of second mover, which will be discussed later in this paper.

In Bertrand's model, the first mover enjoys a monopoly on price in an uncontested environment. As a result, the entire market share goes to the first mover, which means that there is no need for the first mover to set any strategies.

When the second mover happens to enter the market, a price war between the second and first movers ensues due to assumed homogeneity of the product. The Bertrand model illustrates that the Nash equilibrium point occurs twice during the competition. It happens once when the first and second movers try to collude to share the market equally (half-half), and again when both firms reach the same marginal cost below which competition causes them a loss.

In Bertrand's duopoly competition model, the players compete solely on prices above marginal cost rather than on monopoly price. As mentioned earlier, in our evasion model, the first mover has the advantage of a better market share, and will not let the second mover take over its place in the market. The first mover does what it can to maintain its lead in the market. Therefore, the first mover tries to keep its price lower than the second mover's. If the second mover sets its price above the monopoly price, the first mover will capture the entire market demand, but if it sets its price below the monopoly price, then, assuming perfect observation, the first mover will reduce its price, and by doing so, maintain the bigger market share.

The Cournot competition model versus the evasion model

The Cournot model,named after Antoine Augustin Cournot,rests on quantity, where firms in the market are about to set the optimal quantity to maximise their profit. Homogeneity of product is assumed in this model. The duopoly competition that is considered in this model overlapsour evasion game in the following ways.

As the game is assumed to be sequential, the first mover enters the market and captures the entire demand. The quantity the product produced is based on the monopoly price until the second mover enters the market as well. When this happens, market demand will be shared between the first and second movers, while the first mover tries to remain the market leader with greater pay-offs. This makes evaluating the residual demand critical for the first mover,who needs perfect observation once again as assumed in our model.

In Cournot's model, it can be seen that the second mover has no output quantity before entering the market, and the best output is what the first mover assigns to produce to serve the market demand. When the second mover comes in and produces an output quantity, the optimal output for the first mover must be decided from an estimate of the residual demand of the market. The first mover must then reduces its output quantity up to the level where together, both players are producingthe same quantity of products as that used to be produced by the first mover originally. The second mover might try to increase output in the hope of edging out the first player or causing the first player's output to gradually decrease. As it is assumed in this paper that the first mover has the privilege of financial strength, the output quantity of the first mover is given as being greater than that of the second mover. Therefore, the first mover can maintain its lead by using our evasion model.

STATEMENT OF PROBLEM

Competition in a duopoly market is seen mainly in benefits (price) and market share (sale quantity). The major difficulty with existing models of the duopoly market is that they are restricted to competition on one of the above factors. In the Cournot model, competition is based only on quantity, whereas in the Bertrand model it is solely on price. The problem is the lack of a hybrid model which can handle competition based on the coordinates of qantity and price simultaneously.

Setup of the model

We consider both the Bertrand and Cournot economic models to propose a mathematical model for the duopoly market where the two players compete on both coordinates of price and quantity. As mentioned earlier, the first mover is assumed throughout the game to observe the second mover perfectly, so that it can set its price and residual demand based on data gathered through this observation of the second mover in order to find the optimal pay-off that will allow it to remain the market leader.

Our differential evasion game model assigns appropriate strategies in different periods of time to let the first mover take action based on the actions of the second mover to maintain its lead in the market with a pay-off. In our mathematical model, the first and second movers can take action or respond according to both the coordinates of price and quantity.

The first mover has to avoid being captured by the second mover. Based on the Bertrand and Cournot competition models, the two players (firms) might compete strategically on price and quantity in a duopoly market. As mentioned before, the Nash equilibrium in these economic models can be reached, but our mathematical model gives the first mover a maths-simulated strategy so that it can be ahead of the second mover all the time. The equilibrium point, in particular, never happens in this model, since the equilibrium is where the pursuer (second mover) catches the evader (first mover), which is prohibited in our model. In contrast to the usual firm competition models, here the aim is to simulate a mathematical model to show that it is possible for the first mover to remain the market leader considering better financial strength and our optimal strategy.

Restrictions of the model

Some of the standard assumptions of the Bertrand and Cournot models are assumed, as well as certain extra assumptions which naturally happen in the real market. These conditions are as follows: the firms are producing homogeneous products; the firms are not involved cooperation or joint ventures during the game; and the outputs affect the price i.e. each firm has market power. We only deal with two players (firms) and there are constraints both for the first and second movers which will be specified in the next section.

There are a few different games for duopoly competition including the sequential and coincidental games (Amir & Stepanova 2006). The sequential game fits with our evasion model. We assume that there are two players in our model to simulate a duopoly market,which consists of two firms. The most important assumptions are sequentiality as well as perfect observation of moves in the game.

Implications of using the model

We are not aware of any research that models the duopoly market by differential evasion game strategically in the vast amount of literature on game theory and the duopoly market. In the next section, we propose an optimal strategy for the first mover to remain advantageous over its competition. This model has a clear advantage to both the Bertrand and Cournot models as it allows change in both price and quantity during computation. Our model has some restrictions (mentioned above) but it is still applicable in many real competition setups. The main implication of our analysis of competition in the duopoly market is the rigorous proof that even if both competitors are free to play over price and quantity, if the first mover is financially privileaged at the beginning, it will remain the market leader.

MAIN RESULT

In this section we propose a mathematical model to simulate the integrated economic models of Bertrand and Cournot. This is a mathematical model of rivalry between the first and second movers in a dynamic market, which is a simple motion dynamic system based on the dynamism of a duopoly market.

The moves of the second mover, S, and the first mover, F, follow these equations:

$$S: \dot{x} = u, \quad x(0) = x_0, \\ F: \dot{y} = v, \quad y(0) = y_0, \tag{1}$$

where $x, y, u, v \in \mathbb{R}^2$, $x_0 \neq y_0$, u is the control parameter of the second mover, S, and v is that of the first mover, F.

Definition 1

A measurable function $v:[0,\infty) \to \mathbb{R}^2$ such that

$$\left(\int_0^\infty |v(s)|^2 ds \right)^{1/2} \leq \Omega_1,$$

is called an admissible control of the first mover, F, where Ω_1 is the financial strength of the first mover.

Similarly, a measurable function $u:[0,\infty) \to \mathbb{R}^2$ such that

$$\left(\int_0^\infty |u(s)|^2 ds \right)^{1/2} \leq \Omega_2,$$

is called an admissible control of the second mover, S, where Ω_2 is the financial strength of the second mover.

Definition 2

By the strategy of the first mover, F, is meant a function $V = V(y, x, u)$,

$$V: \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2,$$

for which the system

$$\dot{x} = u, \quad x(0) = x_0, \\ \dot{y} = V(y, x, u), \quad y(0) = y_0,$$

has a unique absolutely continuous solution $(y(t), x(t))$, for arbitrary admissible control S of the second mover, S. The strategy, V, is called admissible if each control generated by V is admissible.

Definition 3

Evasion for the first mover, S. from the second mover, S, is possible if there exists a strategy, V, of the first mover, F, such that $x(t) \neq y(t)$, $t > 0$, for any admissible control S of the second mover, S.

In the above definitions, the positions of the first and second movers at time t are denoted as $y(t)$ and $x(t)$, respectively. In our problem the position of each mover is its market share, which is less for the second mover when competition starts. In this situation, the market share privilege for the first mover causes its successful evasion from the second mover.

Proposition 4
If $\Omega_1 > \Omega_2$, then the evasion for the first mover, F, from the second mover, S, is possible in the game (1).

We prove the theorem in three steps.

1^o. Construction of the first mover's strategy

Denote the first and second coordinates of the competition strategy on quantity and price by v_q (respectively, u_q) and u_p (respectively, u_p) for the first mover (respectively, second mover). We consider the vector $v=(v_q,v_p)$ of the first mover as a function of the vector $u=(u_q,u_p)$ of the second mover. Consider the parameter $0<\delta<1/2$ as a fixed threshold of the market share defined willingly by the first mover. In addition, the cost that is imposed onthe first mover due to competing on quantity and price is denoted by ω, which will be specified later. We assume τ as the first time that the second mover takes over market share, δ.

Now, we construct the strategy of the first mover as follows:

$$v(t)=\begin{cases}(0,0) & if \quad 0\le t<\tau,\\ \left(\omega+|u_q(t)|,\omega+|u_p(t)|\right), & if \quad \tau\le t<\tau+\dfrac{\delta}{\omega},\\ (0,|u(t)|), & if \quad t\ge\tau+\dfrac{\delta}{\omega},\end{cases}$$

where ω is chosen to satisfy the condition $\int_0^\infty |v(s)|^2\,ds\le\Omega_1^2$.

In the above-mentioned strategy, 0 implies the manual strategy of the first mover regardless of the move of the second mover. In addition, $|u_q(t)|$ is the quantity that the second mover sets below that of the first mover because of its cost disadvantage. No matter how the second mover tries to increase its quantity, the first mover is able to observe the quantity of the second mover and produce the residual demand that was calculated by Cournot in his model. Moreover, $|u_p(t)|$ is price that the second mover sets below that of the first mover.

2^o. Evasion is possible for the first mover

At the first, in our mathematical model we estimate $\left|y(t)-x(t)\right|$, $\tau\le t<\tau+\dfrac{\delta}{\omega}$, which shows the difference between the position (market share) of the first and second movers in the given time interval. Since

$$\left|\int_\tau^t v(s)ds\right|\le\int_\tau^t\left|v(s)\right|ds$$

$$\le\left(\int_\tau^t\left|v(s)\right|^2ds\right)^{1/2}\left(\int_\tau^t 1^2ds\right)^{1/2}\le\Omega_1\left(t-\tau\right)^{1/2},$$

we have

$$\left|y(t)-x(t)\right|$$

$$=\left|y(\tau)+\int_\tau^t v(s)ds-x(\tau)-\int_\tau^t u(s)ds\right|$$

$$\ge\left|y(\tau)-x(\tau)\right|-\left|\int_\tau^t v(s)ds\right|$$

$$-\left|\int_\tau^t u(s)ds\right|\ge\delta-2\Omega_1\left(t-\tau\right)^{1/2}.$$

This inequality holds if $\int_\tau^t |v(s)|^2\,ds\le\Omega_1^2$, and $\int_\tau^t |u(s)|^2\,ds\le\Omega_2^2$.

On the other hand, for the points $x=(x_q,x_p)$ and $y=(y_q,y_p)$ as the positions (including quantity and price) of the second and first movers in the market competition, we have

$$\left|y(t)-x(t)\right| \geq \left|y_q(t)-x_q(t)\right|$$
$$=\left|y_q(\tau)-x_q(\tau)+\int_\tau^t(\omega+|u_q(s)|)ds-\int_\tau^t u_q(s)ds\right|$$
$$\geq \left|y_q(\tau)-x_q(\tau)\right|+\int_\tau^t(\omega+|u_q(s)|)ds$$
$$-\int_\tau^t |u_q(s)|\,ds \geq \omega(t-\tau).$$

Therefore,

$$\left|y(t)-x(t)\right| \geq max\left\{\delta-2\Omega_1\left(t-\tau\right)^{1/2},\ \omega(t-\tau)\right\}$$
$$>\frac{\omega\,\delta^2}{9\Omega_1^2},\quad (\tau \leq t < \tau+\frac{\delta}{\omega}).$$

Moreover, if $t \geq \tau+\dfrac{\delta}{\omega}$, then $y(t) \neq x(t)$, and

$$x_p(t)-y_p(t)$$
$$\leq \delta+\int_\tau^{\tau+\frac{\delta}{\omega}}u_p(s)ds+\int_{\tau+\frac{\delta}{\omega}}^t u_p(s)ds$$
$$-\int_\tau^{\tau+\frac{\delta}{\omega}}v_p(s)ds-\int_{\tau+\frac{\delta}{\omega}}^t v_p(s)ds$$
$$=\int_{\tau+\frac{\delta}{\omega}}^t u_p(s)ds-\int_{\tau+\frac{\delta}{\omega}}^t\left|u(s)\right|ds$$
$$\leq \int_{\tau+\frac{\delta}{\omega}}^t\left|u_p(s)\right|ds$$
$$-\left[\left(\int_{\tau+\frac{\delta}{\omega}}^t\left|u_q(s)\right|ds\right)^2+\left(\int_{\tau+\frac{\delta}{\omega}}^t\left|u_p(s)\right|ds\right)^2\right]^{1/2}.$$

In the above estimations, we used the following calculations.

$$\int_{\tau+\frac{\delta}{\omega}}^t |u(s)|\,ds$$
$$=\int_{\tau+\frac{\delta}{\omega}}^t |\phi(s)|\,ds \geq \left|\int_{\tau+\frac{\delta}{\omega}}^t \phi(s)ds\right|$$
$$=\left|\left(\int_{\tau+\frac{\delta}{\omega}}^t |u_q(s)|\,ds, \int_{\tau+\frac{\delta}{\omega}}^t |u_p(s)|\,ds\right)\right|$$
$$=\left[\left(\int_{\tau+\frac{\delta}{\omega}}^t |u_q(s)|\,ds\right)^2+\left(\int_{\tau+\frac{\delta}{\omega}}^t |u_p(s)|\,ds\right)^2\right]^{1/2},$$

where $\varphi(s)=\left(|u_q(s)|, |u_p(s)|\right)$.

If $\int_{\tau+\frac{\delta}{\omega}}^t\left|u_q(s)\right|ds>0$, then it is clear from the above result that $x_p(t)-y_p(t)<0$. If $\int_{\tau+\frac{\delta}{\omega}}^t\left|u_q(s)\right|ds=0$, and for some $t \geq \tau+\dfrac{\delta}{\omega}$, then $u_q(s)=0$ almost everywhere on $[\tau+\dfrac{\delta}{\omega},t]$. In this case,

$y_p(t) \geq x_p(t)$, $t \geq \tau+\dfrac{\delta}{\omega}$, and

$$\left|y(t)-x(t)\right|$$
$$=\left|y\left(\tau+\frac{\delta}{\omega}\right)+\int_{\tau+\frac{\delta}{\omega}}^t\left(0,v_q(s)\right)ds\right.$$
$$\left.-x\left(\tau+\frac{\delta}{\omega}\right)-\int_{\tau+\frac{\delta}{\omega}}^t\left(u_q(s),u_p(s)\right)ds\right|$$
$$=[\left(y_q\left(\tau+\frac{\delta}{\omega}\right)-x_q\left(\tau+\frac{\delta}{\omega}\right)\right)^2$$
$$+(y_p\left(\tau+\frac{\delta}{\omega}\right)-x_p\left(\tau+\frac{\delta}{\omega}\right)$$

$$+\int_{\tau+\frac{\delta}{\omega}}^{t}\left(v_p(s)-u_p(s)\right)ds)^2]^{1/2}$$

$$\geq [\left(y_q\left(\tau+\frac{\delta}{\omega}\right)-x_q\left(\tau+\frac{\delta}{\omega}\right)\right)^2$$

$$+\left(y_p\left(\tau+\frac{\delta}{\omega}\right)-x_p\left(\tau+\frac{\delta}{\omega}\right)\right)^2]^{1/2}$$

$$=\left|y\left(\tau+\frac{\delta}{\omega}\right)-x\left(\tau+\frac{\delta}{\omega}\right)\right|>\frac{\delta\ \omega^2}{9\Omega_1^2}.$$

Consequently, the absolute value of difference between market share of the first and second movers remains always positive in the given time periods. In other words, there is no time, t, in the intervals in which the difference becomes zero. As a result, the market share of the second mover will never be greater than or even equal to that of the first mover.

3^o. Admissibility

The admissibility of the proposed strategy means that it should satisfy the integral constraint, which is estimated using the Cauchy-Schwartz inequality as follows:

$$\int_0^\infty |v(s)|^2\ ds$$

$$=\int_\tau^{\tau+\frac{\delta}{\omega}}\left|\left(\omega+\left|u_q(s)\right|,\omega+\left|u_p(s)\right|\right)\right|^2$$

$$ds+\int_{\tau+\frac{\delta}{\omega}}^{\infty}\left|\left(0,\left|u(s)\right|\right)\right|^2 ds$$

$$\leq 2\omega\ \delta+2\omega\{[\left(\int_\tau^{\tau+\frac{\delta}{\omega}}|u_q(s)|^2\ ds\right)^{1/2}$$

$$+\left(\int_\tau^{\tau+\frac{\delta}{\omega}}|u_p(s)|^2\ ds\right)^{1/2}]$$

$$\cdot[\int_\tau^{\tau+\frac{\delta}{\omega}}1^2 ds]^{1/2}\}+\int_\tau^\infty\left(u_q^2(s)+u_p^2(s)\right)ds$$

$$\leq 2\omega+4\Omega_1\sqrt{\omega}+\Omega_2^2\leq\Omega_1^2,$$

where $2\omega+4\Omega_1\sqrt{\omega}\leq\Omega_1^2-\Omega_2^2$ which shows a relationship between ω, as the cost that is imposed on the first mover due to competing on quantity and price, and the financial strength of both players.

Therefore, the first mover remains the market leader in a duopoly competition throughout the game, and the proof of the proposition is complete.

CONCLUSION

In the duopoly market, there are many uncontrollable conditions which make the rivalry game somewhat unpredictable. However, investigating a two-player game in real-life business needs some assumptions, such as perfect observation of both players and sequentiality of moves. Still, in reality, regardless of which strategy a firm might embark on, nearly all of the strategies are based on competing on quantity and price of products or services offered by the firms. Bertrand and Cournot were two researchers with different models about competition on price and quantity along with some assumptions as mentioned in this paper.

In this paper on duopoly competition, the privilege of the first mover in financial strength is assumed and an optimal strategy is set, such that the first mover remains ahead of the game throughout. However, it needs to bear the coordination of cost to do so and enjoy uncontested competition. We proposed here a mathematical model, in which duopoly competition was simulated, optimal strategy of the first mover was constructed and admissibility of the strategy was proved.

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