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3-Point Block Backward Differentiation Formulas for Solving Fuzzy Differential Equations

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ABSTRACT

In this paper, 3-point Block Backward Differentiation Formulas (3BBDF) is used for the numerical solution of Fuzzy Differential Equations (FDEs). Implementation of 3BBDF using Newton iteration is discussed. Numerical results obtained by the 3BBDF are presented and compared with the Modified Simpson method to illustrate the ability of the 3BBDF method for solving FDEs.

Keywords: fuzzy differential equations, block backward differentiation formulas.

1. INTRODUCTION

The idea of fuzzy sets was first introduced by Zadeh (1965) where membership function was initiated and was known as the degree of an element in a particular set. Chang and Zadeh then introduced fuzzy mapping in Chang and Zadeh (1972) where it is being used as one of the most important conditions in control problems in order to achieve a control goal. Dubois and Prade in Dubois and Prade (1982) used extension principle in their work on differentiation at a fuzzy point of an ordinary function as well as differentiation at a non-fuzzy point of a fuzzy function. Seikkala in

Seikkala (1987) generalized the concept of fuzzy initial value problems (FIVPs) and it has been used widely by many researchers nowadays. Ma *et al.* (1999) was the first who introduced a numerical solution of fuzzy differential equations (FDEs) by using classical Euler method while Duraisamy used a modified Euler method in Seikkala (1987). Homotopypertubation method was used by both Allahviranloo and Ghanbari in [5, 12] where linear FDEs and FIVP involving generalized differentiability were solved. In this paper, we modify fully implicit 3-point Block Backward Differentiation Formulas (3BBDF) proposed by Ibrahim *et al.*in [14] in order to find the solutions for FIVPs. In the next section, we give some basic properties of FDEs.

2. PRELIMINARIES

A fuzzy number m can be written in parametric form as $m = \left(\underline{m}(r), \overline{m}(r)\right)$, $r \in [0,1]$ that satisfies the following conditions:

(a) $(\underline{m}(r)$ is a bounded left continuous monotonic increasing function over $[0,1]$,
(b) $\overline{m}(r)$ is a bounded right continuous monotonic decreasing function over $[0,1]$, and
(c) $\underline{m}(r) \leq \overline{m}(r)$, $0 \leq r \leq 1$.

A triangular fuzzy number, n is defined by three numbers k_1, k_2 and k_3 where $k_1 < k_2 < k_3$. The membership function of n is a triangle with base $[k_1, k_3]$ and vertex at k_2.

In this paper, we consider the following first-order fuzzy initial value differential equation given by

$$\begin{cases} y'(t) = f\left(t, y(t)\right), & t \in [\, t_0, T\,] \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where y is a fuzzy function of t, $f(t, y(t))$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. The fuzzy function $y$by $y = [\,\underline{y}, \overline{y}\,]$. This means that the r-level set of $y(t)$ for $t \,\epsilon\, [\, t_0, T\,]$is $[\, y(t)\,]_r = [\,\underline{y}(t; r), \overline{y}(t; r)\,]$.

Also

$$[y'(t)]_r = \left[\underline{y}'(t;r), \overline{y}'(t;r)\right]$$
$$\left[f\left(t, y(t)\right)\right]_r = \left[\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)\right].$$

Write $f(t,y) = [\underline{f}(t,y), \overline{f}(t,y)]$ and $\underline{f}(t,y) = F[t, \underline{y}, \overline{y}]$, $\overline{f}(t,y) = G[t, \underline{y}, \overline{y}]$. Since $y' = f(t,y)$, then

$$\underline{y}'(t;r) = \underline{f}(t, y(t); r) = F\left[t, \underline{y}(t;r), \overline{y}(t;r)\right]$$
$$\overline{y}'(t;r) = \overline{f}(t, y(t); r) = G\left[t, \underline{y}(t;r), \overline{y}(t;r)\right].$$

Also,

$$[y(t_0)]_r = \left[\underline{y}(t_0;r), \overline{y}(t_0;r)\right]$$
$$[y_0]_r = [\underline{y}_0(r), \overline{y}_0(r)]$$
$$\underline{y}(t_0;r) = \underline{y}_0(r), \quad \overline{y}(t_0;r) = \overline{y}_0(r)$$

By using the extension principle defined by Zadeh (1965), the membership function is

$$f(t, y(t))(s) = sup\{\, y(t)(\tau) \mid s = f(t,\tau) \,\}, s \in \mathbb{R}$$

From this, it follows that

$$[f(t, y(t))]_r = \left[\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)\right], r \in [0, 1]$$

where

$$\underline{f}(t, y(t); r) = min\{\, f(t,u) | u \in [y(t)]r \,\}$$
$$\overline{f}(t, y(t); r) = max\{\, f(t,u) | u \in [y(t)]r \,\}.$$

3. REVIEW OF 3-POINT BBDF

In this section, we review the derivation of implicit 3BBDF by Ibrahim *et al.* in [14]. Consider an initial value problem for the first order ODE of the form:

$$y' = f(x, y), y(a) = y_0, a \leq x \leq b \tag{2}$$

The method computes three approximation values, y_{n+1}, y_{n+2} and y_{n+3} simultaneously using one earlier block as shown in Figure 1.

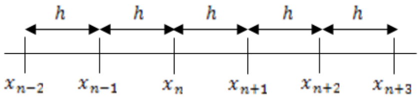

Figure 1: 3-Point Block Method of Constant Step Size

The coefficients of 3BBDF are generated by the backward difference representation of the interpolating polynomial $P_{5,n+3}(x)$ which interpolates $f(x,y)$ at points y_{n-2}, y_{n-1}, y_n, y_{n+1}, y_{n+2} and y_{n+3} has the form,

$$P_{5,n+3}(x) = \sum_{m=0}^{5} (-1)^m \binom{-s}{m} \nabla^m y_{n+3} \tag{3}$$

where

$$s = \frac{x - x_{n+3}}{h}.$$

The result of differentiating (3) once at the point $x = x_{n+3}$ gives

$$P'_{5,n+3}(x) = \frac{1}{h} \sum_{m=0}^{5} \delta_{1,m} \nabla^m y_{n+3} \tag{4}$$

Therefore, for the case $j = 1$, it follows that

$$D_1(t) = \sum_{m=0}^{\infty} \delta_{1,m} \mathrm{t}^m = -\log(1 - t) \tag{5}$$

Equation (5) can be represented in the form of infinite series as follows

$$-\log(1 - t) = t + \frac{1}{2} t^2 + \frac{1}{3} t^3 + \cdots$$

Then equating coefficients of t^m in (5), results in the following relationship:

$$\begin{aligned} \sum_{m=0}^{5} \delta_{1,m} \mathrm{t}^m &= \delta_{1,0} \mathrm{t}^0 + \delta_{1,1} \mathrm{t}^1 + \delta_{1,2} \mathrm{t}^2 + \delta_{1,3} \mathrm{t}^3 + \delta_{1,4} \mathrm{t}^4 + \delta_{1,5} \mathrm{t}^5 \\ &= t + \tfrac{1}{2} t^2 + \tfrac{1}{3} t^3 + \tfrac{1}{4} t^4 + \tfrac{1}{5} t^5 \end{aligned} \tag{6}$$

where

$$\delta_{1,0} = 0, \delta_{1,1} = 1, \delta_{1,2} = \frac{1}{2}, \delta_{1,3} = \frac{1}{3}, \delta_{1,4} = \frac{1}{4}, \delta_{1,5} = \frac{1}{5}.$$

Therefore,

$$\begin{aligned}\sum_{m=0}^{5} \delta_{1,m} \nabla^m y_{n+3} &\\ &= \delta_{1,0}\nabla^0 y_{n+3} + \delta_{1,1}\nabla^1 y_{n+3} + \delta_{1,2}\nabla^2 y_{n+3} + \delta_{1,3}\nabla^3 y_{n+3} \\ &+ \delta_{1,4}\nabla^4 y_{n+3} + \delta_{1,5}\nabla^5 y_{n+3} \\ &= 0 + \nabla^1 y_{n+3} + \frac{1}{2}\nabla^2 y_{n+3} + \frac{1}{3}\nabla^3 y_{n+3} + \frac{1}{4}\nabla^4 y_{n+3} \\ &+ \frac{1}{5}\nabla^5 y_{n+3} \\ &= y_{n+3} - y_{n+2} + \frac{1}{2}(y_{n+3} - 2y_{n+2} + y_{n+1}) \\ &+ \frac{1}{3}(y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n) \\ &+ \frac{1}{4}(y_{n+3} - 4y_{n+2} + 6y_{n+1} - 4y_n + y_{n-1}) \\ &+ \frac{1}{5}(y_{n+3} - 5y_{n+2} + 10y_{n+1} - 10y_n + 5y_{n-1} - y_{n-2})\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{m=0}^{5} \delta_{1,m} \nabla^m y_{n+3} = \frac{137}{60} y_{n+3} - 5y_{n+2} + 5y_{n+1} \\ -\frac{10}{3} y_n + \frac{5}{4} y_{n-1} - \frac{1}{5} y_{n-2}\end{aligned} \qquad (7)$$

Equating (7) to $f(x_{n+3}, y_{n+3})$, we obtain the discrete approximation to (2)

$$\begin{aligned}hf_{n+3} = \sum_{m=0}^{5} \delta_{1,m} \nabla^m y_{n+3} = \frac{137}{60} y_{n+3} - 5y_{n+2} \\ +5y_{n+1} - \frac{10}{3} y_n + \frac{5}{4} y_{n-1} - \frac{1}{5} y_{n-2}\end{aligned}$$

Solve for y_{n+3}, yields

$$y_{n+3} = \tfrac{12}{137} y_{n-2} - \tfrac{75}{137} y_{n-1} + \tfrac{200}{137} y_n - \tfrac{300}{137} y_{n+1} + \tfrac{300}{137} y_{n+2} + \tfrac{60}{137} hf_{n+3} \tag{8}$$

The derivation of the first point y_{n+1} and the second point y_{n+2} are derived similarly by using the method previously described. The 3BBDF methods for finding the solution (2) at x_{n+1}, x_{n+2} and x_{n+3} simultaneously have the form,

$$\begin{aligned} y_{n+1} &= \frac{1}{10} y_{n-2} - \frac{3}{4} y_{n-1} + 3y_n - \frac{3}{2} y_{n+2} + \frac{3}{20} y_{n+3} + 3hf_{n+1} \\ y_{n+2} &= -\frac{3}{65} y_{n-2} + \frac{4}{13} y_{n-1} - \frac{12}{13} y_n + \frac{24}{13} y_{n+1} - \frac{12}{65} y_{n+3} + \frac{12}{13} hf_{n+2} \\ y_{n+3} &= \frac{12}{137} y_{n-2} - \frac{75}{137} y_{n-1} + \frac{200}{137} y_n - \frac{300}{137} y_{n+1} + \frac{300}{137} y_{n+2} + \frac{60}{137} hf_{n+3} \end{aligned}$$

4. MODIFIED 3-POINT BBDF FOR SOLVING FDEs DIRECTLY

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximated solution of the FIVP given in equation (1). Let

$$\begin{aligned} [Y(t)]_r &= [\underline{Y}(t;r), \overline{Y}(t;r)], \\ [y(t)]_r &= [\underline{y}(t;r), \overline{y}(t;r)]. \end{aligned}$$

Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$$\begin{aligned} [Y(t_n)]_r &= [\underline{Y}(t_n;r), \overline{Y}(t_n;r)] \\ [y(t_n)]_r &= [\underline{y}(t_n;r), \overline{y}(t_n;r)] \end{aligned}$$

for $(0 \leq n \leq N)$.Given the initial condition of the FIVP as in equation (1)

$$[y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)]$$

It follows that

$$\begin{aligned} F(t_0;r) &= F\left[t_0, \underline{y}(t_0;r), \overline{y}(t_0;r)\right] \\ G(t_0;r) &= G\left[t_0, \underline{y}(t_0;r), \overline{y}(t_0;r)\right] \end{aligned}$$

The initial values $[y(t_1)]_r$, $[y(t_2)]_r$ and $[y(t_3)]_r$ are obtained by using Euler method,

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + hF(t_n;r)$$
$$F(t_{n+1};r) = F\left[t_{n+1}, \underline{y}(t_{n+1};r), \overline{y}(t_{n+1};r)\right]$$
$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + hG(t_n;r)$$
$$G(t_{n+1};r) = G\left[t_{n+1}, \underline{y}(t_{n+1};r), \overline{y}(t_{n+1};r)\right]$$

for $0 \le n \le 2$. The predictor formulas at t_{n+1}, t_{n+2} and t_{n+3} are

$$\left[y^{(i)}(t_{n+1})\right]_r = \left[\underline{y}^{(i)}(t_{n+1};r), \overline{y}^{(i)}(t_{n+1};r)\right],$$
$$\left[y^{(i)}(t_{n+2})\right]_r = \left[\underline{y}^{(i)}(t_{n+2};r), \overline{y}^{(i)}(t_{n+2};r)\right]$$

and

$$\left[y^{(i)}(t_{n+3})\right]_r = \left[\underline{y}^{(i)}(t_{n+3};r), \overline{y}^{(i)}(t_{n+3};r)\right]$$

respectively or can be written as

- The lower and upper parts of $\left[y^{(i)}(t_{n+1})\right]_r = \left[\underline{y}^{(i)}(t_{n+1};r), \overline{y}^{(i)}(t_{n+1};r)\right]$

$$\underline{y}^{(i)}(t_{n+1};r) = \underline{y}(t_{n-2};r) - 3\underline{y}(t_{n-1};r) + 3\underline{y}(t_n;r)$$
$$F^{(i)}(t_{n+1};r) = F\left[t_{n+1}, \underline{y}^{(i)}(t_{n+1};r), \overline{y}^{(i)}(t_{n+1};r)\right]$$
$$\overline{y}^{(i)}(t_{n+1};r) = \overline{y}(t_{n-2};r) - 3\overline{y}(t_{n-1};r) + 3\overline{y}(t_n;r)$$
$$G^{(i)}(t_{n+1};r) = G\left[t_{n+1}, \underline{y}^{(i)}(t_{n+1};r), \overline{y}^{(i)}(t_{n+1};r)\right]$$

- The lower and upper parts of $\left[y^{(i)}(t_{n+2})\right]_r = \left[\underline{y}^{(i)}(t_{n+2};r), \overline{y}^{(i)}(t_{n+2};r)\right]$

$$\underline{y}^{(i)}(t_{n+2};r) = 3\underline{y}(t_{n-2};r) - 8\underline{y}(t_{n-1};r) + 6\underline{y}(t_n;r)$$
$$F^{(i)}(t_{n+2};r) = F\left[t_{n+2}, \underline{y}^{(i)}(t_{n+2};r), \overline{y}^{(i)}(t_{n+2};r)\right]$$
$$\overline{y}^{(i)}(t_{n+2};r) = 3\overline{y}(t_{n-2};r) - 8\overline{y}(t_{n-1};r) + 6\overline{y}(t_n;r)$$
$$G^{(i)}(t_{n+2};r) = G\left[t_{n+2}, \underline{y}^{(i)}(t_{n+2};r), \overline{y}^{(i)}(t_{n+2};r)\right]$$

- The lower and upper parts of $\left[y^{(i)}(t_{n+3})\right]_r = \left[\underline{y}^{(i)}(t_{n+3};r), \overline{y}^{(i)}(t_{n+3};r)\right]$

$$\begin{aligned}
\underline{y}^{(i)}(t_{n+3};r) &= 3\underline{y}(t_{n-2};r) - 8\underline{y}(t_{n-1};r) + 6\underline{y}(t_n;r)\\
F^{(i)}(t_{n+3};r) &= F\left[t_{n+3}, \underline{y}^{(i)}(t_{n+3};r), \overline{y}^{(i)}(t_{n+3};r)\right]\\
\overline{y}^{(i)}(t_{n+3};r) &= 3\overline{y}(t_{n-2};r) - 8\overline{y}(t_{n-1};r) + 6\overline{y}(t_n;r)\\
G^{(i)}(t_{n+3};r) &= G\left[t_{n+3}, \underline{y}^{(i)}(t_{n+2};r), \overline{y}^{(i)}(t_{n+3};r)\right]
\end{aligned}$$

The matrix form of Newton iterations are as follows:

$$\begin{bmatrix}
1 - 3h\dfrac{\partial F(t_{n+1};r)}{\partial \underline{y}(t_{n+1};r)} & \dfrac{3}{2} - 3h\dfrac{\partial F(t_{n+1};r)}{\partial \underline{y}(t_{n+2};r)} & -\dfrac{3}{20} - 3h\dfrac{\partial F(t_{n+1};r)}{\partial \underline{y}(t_{n+3};r)}\\
-\dfrac{24}{13} - \dfrac{12}{13}h\dfrac{\partial F(t_{n+2};r)}{\partial \underline{y}(t_{n+1};r)} & 1 - \dfrac{12}{13}h\dfrac{\partial F(t_{n+2};r)}{\partial \underline{y}(t_{n+2};r)} & \dfrac{12}{65} - \dfrac{12}{13}h\dfrac{\partial F(t_{n+2};r)}{\partial \underline{y}(t_{n+3};r)}\\
\dfrac{300}{137} - \dfrac{60}{137}h\dfrac{\partial F(t_{n+3};r)}{\partial \underline{y}(t_{n+1};r)} & -\dfrac{300}{137} - \dfrac{60}{137}h\dfrac{\partial F(t_{n+3};r)}{\partial \underline{y}(t_{n+2};r)} & 1 - \dfrac{60}{137}h\dfrac{\partial F(t_{n+3};r)}{\partial \underline{y}(t_{n+3};r)}
\end{bmatrix}$$

$$\begin{bmatrix}
\underline{e}^{(i+1)}(t_{n+1};r)\\
\underline{e}^{(i+1)}(t_{n+2};r)\\
\underline{e}^{(i+1)}(t_{n+3};r)
\end{bmatrix}$$

$$= \begin{bmatrix}
-1 & -\dfrac{3}{2} & \dfrac{3}{20}\\
\dfrac{24}{13} & -1 & -\dfrac{12}{65}\\
-\dfrac{300}{137} & \dfrac{300}{137} & -1
\end{bmatrix}
\begin{bmatrix}
\underline{y}^{(i)}(t_{n+1};r)\\
\underline{y}^{(i)}(t_{n+2};r)\\
\underline{y}^{(i)}(t_{n+3};r)
\end{bmatrix}
+ h\begin{bmatrix}
3 & 0 & 0\\
0 & \dfrac{12}{13} & 0\\
0 & 0 & \dfrac{60}{137}
\end{bmatrix}
\begin{bmatrix}
F^{(i)}(t_{n+1};r)\\
F^{(i)}(t_{n+2};r)\\
F^{(i)}(t_{n+3};r)
\end{bmatrix}$$

$$+ \begin{bmatrix}
\underline{\omega}(t_{n+1};r)\\
\underline{\omega}(t_{n+2};r)\\
\underline{\omega}(t_{n+3};r)
\end{bmatrix}$$

and

$$\begin{bmatrix} 1-3h\frac{\partial G(t_{n+1};r)}{\partial \overline{y}(t_{n+1};r)} & \frac{3}{2}-3h\frac{\partial G(t_{n+1};r)}{\partial \overline{y}(t_{n+2};r)} & -\frac{3}{20}-3h\frac{\partial G(t_{n+1};r)}{\partial \overline{y}(t_{n+3};r)} \\ -\frac{24}{13}-\frac{12}{13}h\frac{\partial G(t_{n+2};r)}{\partial \overline{y}(t_{n+1};r)} & 1-\frac{12}{13}h\frac{\partial G(t_{n+2};r)}{\partial \overline{y}(t_{n+2};r)} & \frac{12}{65}-\frac{12}{13}h\frac{\partial G(t_{n+2};r)}{\partial \overline{y}(t_{n+3};r)} \\ \frac{300}{137}-\frac{60}{137}h\frac{\partial G(t_{n+3};r)}{\partial \overline{y}(t_{n+1};r)} & -\frac{300}{137}-\frac{60}{137}h\frac{\partial G(t_{n+3};r)}{\partial \overline{y}(t_{n+2};r)} & 1-\frac{60}{137}h\frac{\partial G(t_{n+3};r)}{\partial \overline{y}(t_{n+3};r)} \end{bmatrix}$$

$$\begin{bmatrix} \overline{e}^{(i+1)}(t_{n+1};r) \\ \overline{e}^{(i+1)}(t_{n+2};r) \\ \overline{e}^{(i+1)}(t_{n+3};r) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{3}{2} & \frac{3}{20} \\ \frac{24}{13} & -1 & -\frac{12}{65} \\ -\frac{300}{137} & \frac{300}{137} & -1 \end{bmatrix} \begin{bmatrix} \overline{y}^{(i)}(t_{n+1};r) \\ \overline{y}^{(i)}(t_{n+2};r) \\ \overline{y}^{(i)}(t_{n+3};r) \end{bmatrix} + h \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{12}{13} & 0 \\ 0 & 0 & \frac{60}{137} \end{bmatrix} \begin{bmatrix} G^{(i)}(t_{n+1};r) \\ G^{(i)}(t_{n+2};r) \\ G^{(i)}(t_{n+3};r) \end{bmatrix}$$

$$+ \begin{bmatrix} \overline{\omega}(t_{n+1};r) \\ \overline{\omega}(t_{n+2};r) \\ \overline{\omega}(t_{n+2};r) \end{bmatrix}$$

where

- $[\, e^{(i+1)}(t_{n+1})\,]_r$, $[\, e^{(i+1)}(t_{n+2})\,]_r$and $[\, e^{(i+1)}(t_{n+3})\,]_r$ represent the increment of$[\, y(t_{n+1})\,]_r$,$[\, y(t_{n+2})\,]_r$and $[\, y(t_{n+3})\,]_r$respectively.

- $[\omega(t_{n+1})]_r = \left[\underline{\omega}(t_{n+1};r), \overline{\omega}(t_{n+1};r)\right]$, $[\omega(t_{n+2})]_r = \left[\underline{\omega}(t_{n+2};r), \overline{\omega}(t_{n+2};r)\right]$ and $[\omega(t_{n+3})]_r = [\underline{\omega}(t_{n+3};r),$ $\overline{\omega}(t_{n+3};r)]$ represent the back values at t_{n+1}, t_{n+2} and t_{n+3} respectively.

The corrector formulae, $\left[y^{(i+1)}(t_{n+1})\right]_r$,$\left[y^{(i+1)}(t_{n+2})\right]_r$ and $\left[y^{(i+1)}(t_{n+3})\right]_r$ are written as follows

- The lower and upper parts of $\left[y^{(i+1)}(t_{n+1})\right]_r$ are

$$\underline{y}^{(i+1)}(t_{n+1};r) = \underline{y}^{(i)}(t_{n+1};r) + \underline{e}^{(i+1)}(t_{n+1};r)$$

$$F^{(i+1)}(t_{n+1};r) = F\left[t_{n+1}, \underline{y}^{(i+1)}(t_{n+1};r), \overline{y}^{(i+1)}(t_{n+1};r)\right]$$
$$\overline{y}^{(i+1)}(t_{n+1};r) = \overline{y}^{(i)}(t_{n+1};r) + \overline{e}^{(i+1)}(t_{n+1};r)$$
$$G^{(i+1)}(t_{n+1};r) = G\left[t_{n+1}, \underline{y}^{(i+1)}(t_{n+1};r), \overline{y}^{(i+1)}(t_{n+1};r)\right]$$

- The lower and upper parts of $\left[y^{(i+1)}(t_{n+2})\right]_r$ are

$$\underline{y}^{(i+1)}(t_{n+2};r) = \underline{y}^{(i)}(t_{n+2};r) + \underline{e}^{(i+1)}(t_{n+2};r)$$
$$F^{(i+1)}(t_{n+2};r) = F\left[t_{n+2}, \underline{y}^{(i+1)}(t_{n+2};r), \overline{y}^{(i+1)}(t_{n+2};r)\right]$$
$$\overline{y}^{(i+1)}(t_{n+2};r) = \overline{y}^{(i)}(t_{n+2};r) + \overline{e}^{(i+1)}(t_{n+2};r)$$
$$G^{(i+1)}(t_{n+2};r) = G\left[t_{n+2}, \underline{y}^{(i+1)}(t_{n+2};r), \overline{y}^{(i+1)}(t_{n+2};r)\right]$$

- The lower and upper parts of $\left[y^{(i+1)}(t_{n+3})\right]_r$ are

$$\underline{y}^{(i+1)}(t_{n+3};r) = \underline{y}^{(i)}(t_{n+3};r) + \underline{e}^{(i+1)}(t_{n+3};r)$$
$$F^{(i+1)}(t_{n+3};r) = F\left[t_{n+3}, \underline{y}^{(i+1)}(t_{n+3};r), \overline{y}^{(i+1)}(t_{n+3};r)\right]$$
$$\overline{y}^{(i+1)}(t_{n+3};r) = \overline{y}^{(i)}(t_{n+3};r) + \overline{e}^{(i+1)}(t_{n+3};r)$$
$$G^{(i+1)}(t_{n+3};r) = G\left[t_{n+3}, \underline{y}^{(i+1)}(t_{n+3};r), \overline{y}^{(i+1)}(t_{n+3};r)\right]$$

5. NUMERICAL EXPERIMENT

We consider the following FIVP

$$y'(t) = y(t)t \in [0,1]$$

with initial condition

$$y(0) = (0.75 + 0.25r, 1.125 - 0.125r).$$

The exact solution at $t = 1$ is given by

$$Y(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e]$$

Source: Ma *et al.* (1999).

In this example, our method 3BBDF is compared with the method used by Moghadam and Dahaghin (2004). The result is obtained as shown in Table 1.

The following notations are used in the tables:
$\boldsymbol{h}$ = Step size
$\boldsymbol{r} = r$ −level set of $y(t)$ for $t \in [0,1]$
3BBDF = 3-point BBDF
MS = Modified Simpson

The errors in the computed values of $\underline{y}$ and $\overline{y}$ are calculated as follows:

$$Error = \left|\underline{Y}(1;r) - \underline{y}(1;r)\right| + \left|\overline{Y}(1;r) - \overline{y}(1;r)\right|$$

TABLE 1.Error Comparison between 3BBDF and Modified Simpson

h	***r***	**3BBDF**	**MS**
10^{-1}	0	7.74E-02	4.46E-03
	0.2	7.84E-02	4.52E-03
	0.4	7.94E-02	4.58E-03
	0.6	8.05E-02	4.64E-03
	0.8	8.15E-02	4.70E-03
	1	8.25E-02	4.76E-03
10^{-4}	0	8.41E-08	0.00E+00
	0.2	8.52E-08	1.00E-07
	0.4	8.63E-08	1.00E-07
	0.6	8.75E-08	0.00E+00
	0.8	8.86E-08	1.00E-07
	1	8.97E-08	1.00E-07

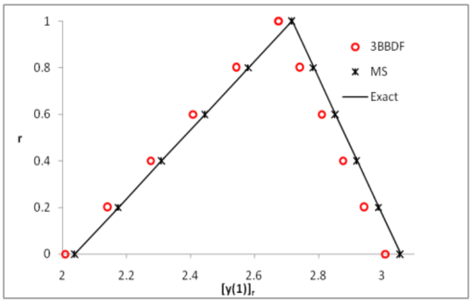

Figure 2: Comparison between Approximate Solutions and Exact Solutions at h=10^{-1}.

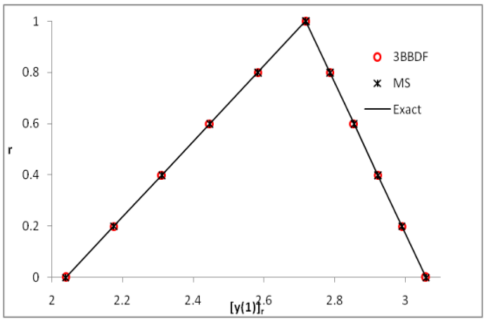

Figure 3: Comparison between Approximate Solutions and Exact Solutions at h=10^{-4}.

6. CONCLUSION

In this work, we modified and implemented3BBDF from solving ODEs to FDEs. We have shown that for certain problems, 3BBDF outperform Modified Simpson in terms of accuracy. This indicates that further research on 3BBDF using variation in step size is valuable for FDEs.

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