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An Efficiency Evaluation Problem Including Fuzzy Weights

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ABSTRACT

This paper presents a procedure to dissolve a fuzzy weights CCR model with numerical input and output data in the objective function. This technique is a combination of utilizing fuzzy operations arithmetic and traditional method in DEA in order to convert the model into two simple linear programming problems with the purpose of detecting the effect of uncertain factors on the efficiency scores of decision making units (DMUs). It is in accordance with our determination to provide a method based on data envelopment analysis (DEA), supporting efficiency evaluation problems in fold fuzziness in factor weights to assist decision making issues.

Keywords: Data Envelopment Analysis, fuzzy, weights, efficiency.

1. INTRODUCTION

Measuring the organizational performance with multiple nonhomogeneous inputs and outputs requires a mathematical method to assess the relative efficiencies of decision making units (DMUs).Data Envelopment Analysis (DEA), as germinated by Charnes, Cooper and

Rhodes (CCR) implemented this estimation using linear programming by maximizing the ratio of weighted sum of outputs to weighted sum of inputs.

Facing imprecise data or factors with uncertain values in modeling and decision making problems, particularly significance of utilizing linguistic factors directing to uncertain efficiency scores, DEA employed Fuzzy set theory where first created by Zadeh (1965). Fuzzy DEA (FDEA) models improved in theory and practice due to the necessity of analyzing these uncertainty problems.

Some of these attempts were performed by Banker *et al.* (1984), Charnes *et al.* (1978), Sengupta (1992), Triantis and Girod (1998), Kao and Liu (2000), Guo and Tanaka (2001) , Leon *et al.* (2003), Saati *et al.* (2002), Entani *et al.* (2002), Dia (2004), Saati and Memariani (2005), Liu (2008), Wang *et al.* (2009), Mansourirad *et al.* (2010). The outcomes of these models are different sets of virtual multipliers or weights accorded to each input or output factor taken into account.

The last presented model covers the efficiency evaluation problems with fuzzy output weights and scalar input weights in objective function. Scope of this paper is focused on performance evaluation of DMUs with normal data and fuzzy input and output weights. In the recommended procedure, defuzzification of fuzzy division is implemented to ease the model into two simple linear programing problems. The aim of solving this model is to find the interval efficiency scores of DMUs and the three components of triangular fuzzy weights of the output and input factors.

The paper is organized as follows: The proposed model and the recommended method of finding the efficiency score of that fuzzy weight DEA model is presented first. Then to clarify the feasibility of the suggested model, a demonstrated example is presented and lastly it closes with conclusion.

2. PROPOSED MODELED

A typical fuzzy data envelopment analysis problem refers to a relative efficiency evaluation with multiple inputs and outputs in fuzzy environment. According to the fuzzy nature of input and output factors, a set of weights has to be determined to aggregate the outputs and inputs separately to form a ratio as the efficiency. Every DMU is enabled to select their most favorable weights while requiring the resulted ratio of the weighted sum of outputs to weighted sum of inputs of all DMUs to be less than or equal to 1.

Consider a set of n DMUs, which consumes varying amounts of m different inputs to produce s different outputs, assume that x_{ij} and y_{rj} where $i=1,2,\cdots,m$ and $r=1,2,\cdots,s$ are representing the scalar inputs and outputs respectively. $\tilde{u}_r=(u_r^L,u_r^M,u_r^R)$ and $\tilde{v}_i=(v_i^L,v_i^M,v_i^R)$ which assumed as triangular fuzzy numbers, are weights corresponding to outputs and inputs. Evaluating the fuzzy efficiency score of $DMU_o(o=1,\cdots,n)$ can be done by a fuzzy fractional CCR model (model-1).

$$
\begin{aligned}
\max \quad & h_o=\frac{\sum_{r=1}^{s}\tilde{u}_r y_{ro}}{\sum_{i=1}^{m}\tilde{v}_i x_{io}} \\
\text{subject to} \quad & \frac{\sum_{r=1}^{s}\tilde{u}_r y_{rj}}{\sum_{i=1}^{m}\tilde{v}_i x_{ij}}\le\tilde{1},\quad j=1,\dots,n \\
& \tilde{u}_r,\tilde{v}_i\ge 0.
\end{aligned}
\tag{1}
$$

Apart from the restriction that no weight may be zero, weights on inputs and outputs are only restricted by the requirement that they must not make the efficiency of any DMU more than $\tilde{1}=(1^L,1,1^R)$.

2.1 Solving the Fuzzy Division

As can be seen in model-2, after substituting related values, fuzzy operation arithmetic is applied to simplify the fuzzy divisions where the result is an interval that approximates the division.

$$
\begin{aligned}
\max \quad & h_o=\frac{\left(\sum_{r=1}^{s}u_r^L y_{ro},\sum_{r=1}^{s}u_r^M y_{ro},\sum_{r=1}^{s}u_r^R y_{ro}\right)}{\left(\sum_{i=1}^{m}v_i^L x_{io},\sum_{i=1}^{m}v_i^M x_{io},\sum_{i=1}^{m}v_i^R x_{io}\right)} \\
\text{subject to} \quad & \frac{\left(\sum_{r=1}^{s}u_r^L y_{rj},\sum_{r=1}^{s}u_r^M y_{rj},\sum_{r=1}^{s}u_r^R y_{rj}\right)}{\left(\sum_{i=1}^{m}v_i^L x_{ij},\sum_{i=1}^{m}v_i^M x_{ij},\sum_{i=1}^{m}v_i^R x_{ij}\right)}\le\tilde{1},\quad \text{j=1,…, n} \\
& u_r^R\ge u_r^M\ge u_r^L\ge 0 \qquad \text{for all } r, \\
& v_i^R\ge v_i^M\ge v_i^L\ge 0 \qquad \text{for all } i.
\end{aligned}
\tag{2}
$$

Using the division approximation formula in parametric fuzzy arithmetic, first developed by Giachetti and Young (1997), transforms the model-2 into model-3.

(3)

$$\max\ h_o = \left[\frac{\alpha\sum_{r=1}^{s}u_r^M y_{ro} + (1-\alpha)\sum_{r=1}^{s}u_r^L y_{ro}\ )}{\alpha\sum_{i=1}^{m}v_i^M x_{io} + (1-\alpha)\sum_{i=1}^{m}v_i^R x_{io})}, \frac{\alpha\sum_{r=1}^{s}u_r^M y_{ro} + (1-\alpha)\sum_{r=1}^{s}u_r^R y_{ro}}{\alpha\sum_{i=1}^{m}v_i^M x_{io} + (1-\alpha)\sum_{i=1}^{m}v_i^L x_{io}}\right]$$

subject to $\left[\frac{\alpha\sum_{r=1}^{s}u_r^M y_{rj} + (1-\alpha)\sum_{r=1}^{s}u_r^L y_{rj}}{\alpha\sum_{i=1}^{m}v_i^M x_{ij} + (1-\alpha)\sum_{i=1}^{m}v_i^R x_{ij}}, \frac{\alpha\sum_{r=1}^{s}u_r^M y_{rj} + (1-\alpha)\sum_{r=1}^{s}u_r^R y_{rj}}{\alpha\sum_{i=1}^{m}v_i^M x_{ij} + (1-\alpha)\sum_{i=1}^{m}v_i^L x_{ij}}\right] \leq 1$,

$j = 1,2,\dots,n$

$u_r^R \geq u_r^M \geq u_r^L \geq 0$ for all r,
$v_i^R \geq v_i^M \geq v_i^L \geq 0$ for all i.

As efficiency score could not be greater than real number one, and by considering the fact that in the interval if the right hand side is less than one so the left hand side must be less than one, we can assume real number one instead of $\tilde{1}$(Wang *et al.* (2009)).

2.2 Transforming to Two Fractional Programming Models

In the obtained interval programming, the relationship between the fractional left and right hand side is as follows:

$$\frac{\alpha\sum_{r=1}^{s}u_r^M y_{rj} + (1-\alpha)\sum_{r=1}^{s}u_r^L y_{rj}}{\alpha\sum_{i=1}^{m}v_i^M x_{ij} + (1-\alpha)\sum_{i=1}^{m}v_i^R x_{ij}} \leq \frac{\alpha\sum_{r=1}^{s}u_r^M y_{rj} + (1-\alpha)\sum_{r=1}^{s}u_r^R y_{rj}}{\alpha\sum_{i=1}^{m}v_i^M x_{ij} + (1-\alpha)\sum_{i=1}^{m}v_i^L x_{ij}}$$

Our model can be transformed into two fractional programming models and we can achieve the efficiency score by solving them. As we want to maximize the value of the interval, we must maximize both sides of the interval, so we will have two maximization problems. The first is maximizing the left hand side of the interval (model-4-1), and the second is maximizing the right hand side of the interval (model-4-2).

(4-1)

$$\max \quad h_o^L = \frac{\alpha\sum_{r=1}^{s}u_r^M y_{ro} + (1-\alpha)\sum_{r=1}^{s}u_r^L y_{ro}}{\alpha\sum_{i=1}^{m}v_i^M x_{io} + (1-\alpha)\sum_{i=1}^{m}v_i^R x_{io}}$$

subject to $\frac{\alpha\sum_{r=1}^{s}u_r^M y_{rj} + (1-\alpha)\sum_{r=1}^{s}u_r^L y_{rj}}{\alpha\sum_{i=1}^{m}v_i^M x_{ij} + (1-\alpha)\sum_{i=1}^{m}v_i^R x_{ij}} \leq 1$, $j = 1,2,\dots,n$

$u_r^M \geq u_r^L \geq 0$ for all r,
$v_i^R \geq v_i^M \geq 0$ for all i.

(4-2)

max $$h_o^R = \frac{\alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^R y_{ro}}{\alpha \sum_{i=1}^{m} v_i^M x_{io} + (1-\alpha) \sum_{i=1}^{m} v_i^L x_{io}}$$

subject to $$\frac{\alpha \sum_{r=1}^{s} u_r^M y_{rj} + (1-\alpha) \sum_{r=1}^{s} u_r^R y_{rj}}{\alpha \sum_{i=1}^{m} v_i^M x_{ij} + (1-\alpha) \sum_{i=1}^{m} v_i^L x_{ij}} \leq 1 , \quad j = 1,2, \dots , n$$

$$u_r^R \geq u_r^M \geq 0 \qquad \text{for all } r,$$
$$v_i^M \geq v_i^L \geq 0 \qquad \text{for all } i.$$

2.3 Converting To Parametric Linear Programming Models

By using the traditional method in DEA and Setting the denominator of the objective function equal to one and adding the restriction that the right value of the interval of efficiency score must be greater than or equal with the left value we can convert the models to two parametric linear programming models as can be seen in model-5-1 and 5-2. Existence of few restrictions, yield the model to less computations, which supports it from incorrect results or infeasibility.

The Left Value of the Interval of Efficiency Score: (5-1)

max $$h_o^L = \alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^L y_{ro}$$

subject to $$\alpha \sum_{i=1}^{m} v_i^M x_{io} + (1-\alpha) \sum_{i=1}^{m} v_i^R x_{io} = 1$$

$$\alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^L y_{ro} - (\alpha \sum_{i=1}^{m} v_i^M x_{io} + (1-\alpha) \sum_{i=1}^{m} v_i^R x_{io}) \leq 0, \quad j = 1,2, \dots , n$$

$$u_r^M \geq u_r^L \geq 0 \qquad \text{for all } r,$$
$$v_i^R \geq v_i^M \geq 0 \qquad \text{for all } i.$$

The Right Value of the Interval of Efficiency Score: (5-2)

max $$h_o^R = \alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^R y_{ro}$$

subject to $$\alpha \sum_{i=1}^{m} v_i^M x_{io} + (1-\alpha) \sum_{i=1}^{m} v_i^L x_{io} = 1$$

$$\alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^R y_{ro} - (\alpha \sum_{i=1}^{m} v_i^M x_{io} + (1-\alpha) \sum_{i=1}^{m} v_i^L x_{io}) \leq 0, \quad j = 1,2, \dots , n$$

$$\alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^R y_{ro} \geq \alpha \sum_{r=1}^{s} u_r^M y_{ro} + (1-\alpha) \sum_{r=1}^{s} u_r^L y_{ro}$$

$$u_r^R \geq u_r^M \geq 0 \qquad \text{for all } r,$$
$$v_i^M \geq v_i^L \geq 0 \qquad \text{for all } i.$$

2.4 Solving Two Parametric Linear Programming Problems:

To achieve the correct results as the left and right values of the interval efficiency score, for assumed data and any parameter α ∈ (0,1], we must first solve the left hand side. Our goal is not only finding the left value of efficiency score. We also need to know the values of $u_r^L, u_r^M, v_i^M, v_i^R$. The right hand side will be done after substituting the obtained parameters from the left. In addition, various optimal solutions can be found by changing the value of α from 0 to 1.

3. NUMERICAL EXAMPLE

For more clarification of the proposed model, consider the example below. Two inputs and two outputs data related to six DMUs, are presented in table-1. Our purpose is evaluating and comparing the relative efficiency scores related to these DMUs.

TABLE 1: Inputs and Outputs of six DMUs

DMUS	Input 1	Input 2	Output 1	Output 2
1	16	31	43	15
2	14	29	64	27
3	15	33	50	17
4	16	27	63	26
5	17	30	55	18
6	19	32	61	29

As we noted before, the values of α could be in the range of (0,1] and each value of α provides a set of solutions for efficiency scores of DMUs. We used 0.1, 0.3, 0.5, 0.7 and 0.9 as the values of parameter α in this example and for each of these values, the efficiency scores of left hand side and right hand side of the efficiency interval is attained by using model-5-1 and model-5-2, which are stated in Table 2 and Table3, respectively.

As can be seen in Table2, the left hand side efficiency score related to each DMU is computed separately, for each value of α. For example, D_1, for $\alpha = 0.1$ has efficiency score of 0.62852822481 and for $\alpha = 0.3$ its efficiency score becames 0.62852822583. According to the Table-2, D_2 and D_4 are efficient, because for all values of α, their efficiency score is equal to1. Table3 includes right values of efficiency scores, respecting to different values of α. By checking two tables, Table 2 and Table 3, we can have the efficiency intervals. For example D_5 with α = 0.5 has the interval efficiency score of [0.8216619967, 0.8216619973].

Due to the aim of this paper which is finding the triangular fuzzy weights while evaluating the efficiency scores for normal inputs and outputs data, Table 4 and Table 5 present the left, right and middle weights of inputs and outputs of the model-5-1 and model-5-2 respectively, when parameter $\alpha = 0.3$. For example, the left, middle and right output weights of D_2 with parameter $\alpha = 0.3$ are as follows:

u_1^L=2.99e-05	u_2^L=0.001492
u_1^M=0.027321	u_2^M=0.055047
u_1^R=0.247123	u_2^R=0.097641

And the left, middle and right input weights of D_2 with parameter $\alpha = 0.3$ are:

v_1^L=0.009283	v_2^L=0.001637
v_1^M=0.017130	v_2^M=0.006544
v_1^R=0.066970	v_2^R=0.010581

As can be seen, for each value of α, two input and two output triangular fuzzy weights are calculated for every DMUs. So, for six DMUs, we have 24 and for five different values of α, it will be 120 triangular fuzzy weights or in other words, 360 left, right and middle components and finally for 6 DMUs we have30 efficiency scores.

TABLE 2: Left Hand Side Efficiency Scores of Six DMUs

DMU/α	0.1	0.3	0.5	0.7	0.9
1	0.6285282248	0.6285282258	0.6285282250	0.6285282257	0.6285282258
2	1	1	1	1	1
3	0.7291666666	0.7291666666	0.7291666662	0.7291666665	0.7291666666
4	1	1	1	1	1
5	0.8216619980	0.8216619972	0.8216619967	0.8216619976	0.8216619979
6	0.9411057690	0.9411057680	0.9411057691	0.9411057692	0.9411057691

TABLE 3: Right Hand Side Efficiency Scores of Six DMUs

DMU/α	0.1	0.3	0.5	0.7	0.9
1	0.6285282254	0.6285282258	0.6285282256	0.6285282257	0.6285282258
2	1	1	1	1	1
3	0.7291666666	0.7291666666	0.7291666664	0.7291666665	0.7291666666
4	1	1	1	1	1
5	0.8216619980	0.8216619977	0.8216619973	0.8216619976	0.8216619980
6	0.9411057690	0.9411057680	0.9411057757	0.9411057746	0.9411057691

TABLE 4: Left, right and middle values of fuzzy input weights with α=0.3

DMU/V	v_1^L	v_2^L	v_1^M	v_2^M	v_1^R	v_2^R
1	4.79e-18	0.003752	2.98e-17	0.019967	3.61e-17	0.037525
2	0.009283	0.001637	0.017130	0.006544	0.066970	0.010581
3	0.009523	1.34e-14	0.023998	9.00e-13	0.084953	7.26e-12
4	0.007942	0.001751	0.010198	0.009544	0.048335	0.017586
5	0.008403	8.95e-12	0.016947	1.14e-11	0.076770	2.24e-10
6	1.25e-09	0.004146	1.25e-08	0.007424	1.29e-08	0.041461

TABLE 5: Left, right and middle values of fuzzy output weights with α=0.3

DMU/U	u_1^L	u_2^L	u_1^M	u_2^M	u_1^R	u_2^R
1	0.010037	1.29e-17	0.025301	5.70e-17	0.100371	1.39e-12
2	2.99e-05	0.001492	0.027321	0.055047	0.247123	0.097641
3	0.009419	1.55e-12	0.026633	7.96e-12	0.094199	6.50e-11
4	3.09e-05	0.001944	0.027578	0.056668	0.037264	0.111672
5	0.009652	5.22e-11	0.027276	2.53e-10	0.096523	4.22e-10
6	2.22e-11	0.016436	1.40e-10	0.069821	7.99e-10	0.164361

4. CONCLUSION

A fuzzy weights CCR model is presented in this paper to solve some efficiency measurement issues, which have normal data and fuzzy essence in weights of both inputs and outputs data in objective function. Weights of the model are more important than the efficiency scores while evaluating the efficiency score for normal inputs and outputs data. The model will help the manager to be aware of uncertain influences of factors on efficiency score.

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