Malaysian Journal of Mathematical Sciences 6(2): 213-223 (2012)

Spanning Trees of 2-Complexes from Diagram Groups over the Construction of Semigroup Presentation of Integers using Lifting Method

[1,2]Yousof Gheisari, [1]Abd Ghafur Bin Ahmad, [3]S. Seddigh Chaharborj, [3]Siti Hasana Sapar and [3]Kamel Ariffin Mohd Atan

[1]*School of Mathematical Science, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia*

[2]*Department of Mathematics, Islamic Azad University, Bushehr Branch, Bushehr, Iran*

[3]*Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

E-mail: yousofgheisari54@gmail.com

ABSTRACT

For any given semigroup presentation we may obtain the fundamental group. In this paper we will determine spanning trees for the 2-complexes of the fundamental groups obtained from the union of two semigroup presentations with finite different initial generators using lifting method. The spanning trees will be systematically selected by using lifting method according to the length of words. Also the general formula for all lifts of spanning trees and the number of edges in the spanning trees will be computed.

Keywords: Fundamental group, semigroup presentation, generators, spanning tree

1. INTRODUCTION

The construction for the spanning trees in graphs of semigroup presentations of integers with three and n initial generators namely, ${}^{3}S=\langle x,y,z \mid x=y, y=z, x=z\rangle$ and ${}^{n}S=\langle x_1,x_2,x_3,\ldots,x_n \mid x_i=x_j;1\le i<j\le n\rangle$ can be obtained in (see Gheisari and Ghafur (2010) and (2011)) using lifting method. In this research we want to determine spanning tree from fundamental groups over the union of two semigroup presentations of integers with s and t different initial generators by adding a relation.

For given any semigroup presentation $S=\langle X \mid R\rangle$ we may obtain fundamental group $\pi_1(K(S))$. Then we can determine the generators from

$\pi_1(K(S),U)$ with the basepoint U. Thus if $S_1 = \langle X_1 | R_1 \rangle$ and $S_2 = \langle X_2 | R_2 \rangle$, the we compute the $\pi_1(K(S_1 \cup S_2),U)$.

Guba and Sapir (1997) have shown that if we consider the semigroup presentation S, obtained from union of initial generators and relations of two semigroup presentations S_1 and S_2 by adding relation $x_1 = a_1$, then $D(S,U_1)$ isomorphic to direct product of $D(S_1,U_1)$ and $D(S_2,U_2)$. Also they proved in 1997, that if we consider $S = \langle X_1 \cup X_2 | R_1 \cup R_2 \cup \{U_1 = U_2\} \rangle$ where X_1, X_2 disjoint sets, and the congruence class of U_i modulo S_i does not contain words of the form $xU_i y$ and x, y are words and X_1, X_2 is not empty. Then $D(S,U_1)$ isomorphic to free product of $D(S_1,U_1)$ and $D(S_2,U_2)$. Now in this paper, we consider the semigroup presentation $S = \langle X_1 \cup X_2 | R_1 \cup R_2 \cup \{U_1 = U_2\} \rangle$ for our method.

Let the two semigroup presentations

$$^{s}S = \langle x_1, x_2, \dots, x_s \,|\, x_i = x_j, 1 \le i < j \rangle$$

and

$$^{t}S = \langle a_1, a_2, \dots a_t \,|\, a_i = a_j, 1 \le i < j \le t \rangle$$

with s and t initial generators. Now we consider the new semigroup presentation

$$S = \langle x_1, x_2, \dots, x_s, a_1, a_2, \dots a_t \,|\, x_i = x_j, 1 \le i < j \le s, a_i = a_j, 1 \le i < j \le t, x_1 = a_1 \rangle$$

which is obtained from the union of initial generators and relations of ^{s}S and ^{t}S by adding a relation $x_1 = a_1$. In this paper we will determine the spanning trees and their lifts of S.

In second section we have some preliminaries about diagram groups and semigroup presentation and lifting method. In third section, we will determine the graphs $\Gamma_n(S)(n \in N)$ using lifting method. In section results and discussion, we will determine spanning trees of semigroup presentation S according to the length of words, in the graphs $\Gamma_n(S)$. Also the general formula of all lift of spanning trees and the number of edges in spanning trees will be computed.

2. PRELIMINARIES

Let $S = \langle X \mid R \rangle$ be a semigroup presentation. Then we may obtain the diagram group $D(S,W)$ where W is a word on X as defined by Guba and Sapir (1997). The 2-complex, associated with presentation S is denoted by $K(S)$. As the 2-complex we may obtain the fundamental group $\pi_1(K(S),W)$ with a basepoint W. Kilibarda (1994,1997) has shown that the fundamental group $\pi_1(K(S),W)$ is isomorphic to diagram group $D(S,W)$. Thus it is sufficient to consider $\pi_1(K(S),W)$ instead of $D(S,W)$.

We will consider the fundamental group $\pi_1(K(S),W)$ constructed from the semigroup presentation of integers, ${}^nS = \langle x_1, x_2, \ldots, x_n \mid x_i = x_j; 1 \le i < j \le n \rangle$. Guba and Sapir (1997) have shown that $\pi_1(K({}^3S),x)$ is an infinite cyclic, for ${}^3S = \langle x, y, z \mid x = y, y = z, x = z \rangle$.

As the 2-complexes $K(S)$ we may obtain spanning trees of graphs ${}^n\Gamma_m(S)$ depending on the length of words. Then we determine the mapping between ${}^n\Gamma_m(S)$ and ${}^n\Gamma_{m+1}(S)$. Once we found for ${}^n\Gamma_1(S)$, the rest of the graphs are just the lift of ${}^n\Gamma_1(S)$.

We will also show that the 2-complex $K(S)$ obtained from semigroup presentation S is actually a union of the graphs ${}^n\Gamma_m(S)$ where ${}^n\Gamma_m(S)$ contains all vertices of length *m*. Here n refers to the number of initial generators $x_1, x_2, \ldots, x_n$, in the semigroup presentation of integers ${}^nS = \langle x_1, x_2, \ldots, x_n \mid x_i = x_j;\ 1 \le i < j \le n \rangle$. Note that any 2-complex contains vertices, edges, and 2-cells. Thus a 2-complex without 2-cells is simply a graph.

For the semigroup presentation of integers, the 2-complex consists of infinitely connected component ${}^{n}\Gamma_{m}(S)$ for all $m,n\in N$, where N is a set of the Natural numbers. Note that all vertices in ${}^{n}\Gamma_{i}(S)$ are words of length i. Ahmad and Al-Odhari (2004) proved that if length $(U)=$ length (V) then $\pi_1(K(S),U)$ isomorphic to $\pi_1(K(S),V)$.

As a group, it is sufficient to determine its generators and relations. The generators of this group can be determined from the 2-complex $K(S)$ by identifying the of a spanning tree T. Fix a vertex v, where v belong to $K(S)$ and let e be any edge such that $e\notin T$. Then $\gamma_{t(e)}e\gamma^{-1}\tau(e)$ is the generator, where $\gamma_{t(e)},\gamma_{\tau(e)}$ are paths in a spanning tree T from $v\in K(S)$, to the initial and terminal of e respectively.

Let U_i be a word of length i. We will show that the generator for $\pi_1(K(S),U_{i+1})$ can be obtained from the generator of $\pi_1(K(S),U_i)$. This is a lifting method. Hence it is sufficient to determine the generator for $\pi_1(K(S),x_1)$. Lifting method can determine all generators for the whole groups $\pi_1(K(S),U_i)$ for all basepoint U_i belongs to X. Also using lifting method we can determine the spanning trees of the graphs $\Gamma_n(S)$.

3. ALGORITHM FOR THE GRAPHS $\Gamma_n(S)(n\in N)$

In this section we explain the Algorithm for determining the graphs $\Gamma_n(S)$.

Let

$$s=\left\langle x_1,x_2,\ldots,x_s,a_1,a_2,\ldots,a_t \,\middle|\, x_i=x_j,1\le i<j\le s,a_i=a_j,1\le 1<j\le t,x_1=a_1\right\rangle$$

be a semigroup presentation which is obtained from the union of initial generators and relations of ${}^{s}S$ and ${}^{t}S$ by adding a relation $x_1=a_1$. Associated with semigroup presentation $Q=\left\langle X\,|\,R\right\rangle$ we have a graph Γ where the vertices are words on X and the edges are of the form $e=(T_1,R_\varepsilon\to R_{-\varepsilon},T_2)$ such that $t(e)=T_1R_\varepsilon T_2$, $\tau(e)=T_1R_{-\varepsilon}T_2$. The graph obtained from Q is collections of subgraphs Γ_n. Note that the graph $\Gamma({}^sS)$ obtained from ${}^sS$ is just a collection of subgraphs $\Gamma_n({}^sS)$ where $\Gamma_n({}^sS)$ contains all vertices of length n and respective edges. Similarly we obtain $\Gamma_n({}^tS)$ for ${}^tS$.

Now for S, the graph $\Gamma_n(S) = \Gamma_n({}^sS) \cup \Gamma_n({}^tS) \cup \{(u, x_1 \to a_1, v)\}$ such that the length $uv = n-1$. If T_n is a vertex in $\Gamma_n(S)$ then $T_n g, (g \in \{x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t\})$ is a vertex in $\Gamma_{n+1}(S)$. Similarly if $(u, R_\varepsilon \to R_{-\varepsilon}, v)$ is a edge in $\Gamma_n(S)$, then $(u, R_\varepsilon \to R_{-\varepsilon}, vg)$ is the respective edges in $\Gamma_{n+1}(S)$. Thus $\Gamma_{n+1}(S)$ is just $(s+t)$ copies of $\Gamma_n(S)$ together with $(s+t)$ vertices $(u, x_1 \to a_1, vg)$ $(g \in \{x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t\})$.

For example consider graph $\Gamma_1(S)(V_1, E_1)$, where $V_1 = X = \{x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t\}$ is a set of vertices and $E_1 = \{e_{1x} \cup e_{1a} \cup x_1 = a_1\}$ is set of edges, where $e_{1x} = \{(1, x_i \to x_j, 1), (1 \le i < j \le s)\}$, $e_{1a} = \{(1, a_1 \to a_j, 1), (1 \le i < j \le t)\}$ (see Figure 1).

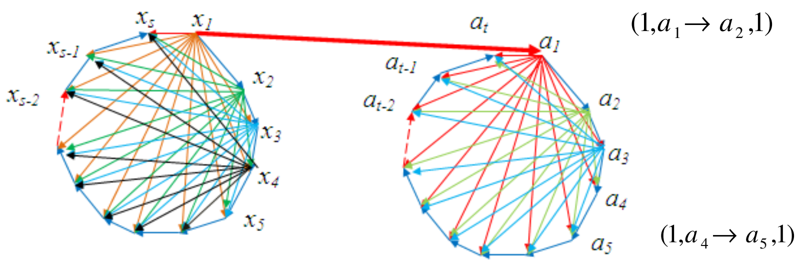

Figure 1: Graph of $\Gamma_1(S)$

Note that $\Gamma_2(S)$ is $(s+t)$ copies of $\Gamma_1(S)$ and each vertex in each copy are joined together, respectively by considering the relation $x_1 = a_1$. Similarly, with $(s+t)$ copies of $\Gamma_2(S)$, we may obtain $\Gamma_3(S)$. Repeating similar procedures for obtain $\Gamma_4(S)$ and so on.

Algorithm

Step 1: Determine the graph of $\Gamma_1(S)$.

Step 2: The graph $\Gamma_2(S)$ is $(s+t)$ copies of $\Gamma_1(S)$ similar procedures for obtaining $\Gamma_n(S)$ which are $(s+t)$ copies of $\Gamma_{n-1}(S)$.

4. RESULTS AND DISCUSSION

In this section we will determine spanning trees in $\Gamma_n(S)$. Also the general formula of all lifts of spanning tree and the number of edges in spanning trees will be provided and proved.

Example 1

Let T_1 be a spanning tree in $\Gamma_1(S)$ where $T_1=(1,x_{s-1}\to x_s,1)^{-1}\cdots$ $(1,x_1\to x_2,1)^{-1}$ $(1,x_1\to a_1,1)$ $(1,a_1\to a_2,1)$ $(1,a_2\to a_3,1)$ $\cdots(1,a_{t-2}\to a_{t-1},1)$ $(1,a_{t-1}\to a_t,1)$ (see Figure 2).

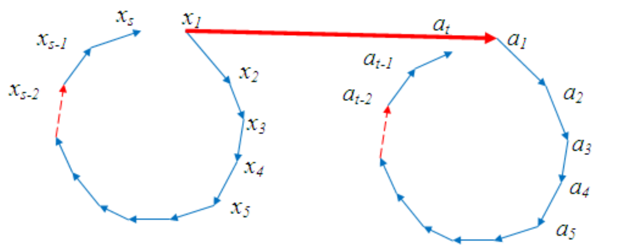

Figure 2: Spanning tree in $\Gamma_1(S)$

Then the collections all lifts of Γ_1 in $\Gamma_1(S)$ at $v_2=x_s a=\{x_s x_1, x_s x_2,\ldots,x_s{}^2,$ $x_s a_1,\ldots,x_s a_t\}$, for every $a\in X$ are as follows:

(1) Lift of T_1 at $x_s x_1$ is:

$$(1,x_{s-1}\to x_s,x_1)^{-1}\cdots(1,x_1\to x_2,x_1)^{-1}(1,x_1\to a_1,x_1)$$
$$(1,a_1\to a_2,x_1)(1,a_2\to a_3,x_1)\ldots(1,a_{t-2}\to a_{t-1},x_1)(1,a_{t-1}\to a_t,x_1)$$

(2) Lift of T_1 at $x_s x_2$ is:

$(1, x_{s-1} \to x_s, x_2)^{-1} \ldots (1, x_1 \to x_2, x_2)^{-1} (1, x_1 \to a_1, x_2)$
$(1, a_1 \to a_2, x_2)(1, a_2 \to a_3, x_2) \ldots (1, a_{t-2} \to a_{t-1}, x_2)(1, a_{t-1} \to a_t, x_2)$
$\vdots$

(3) Lift of T_1 at $x_s{}^2$ are:

$(1, x_{s-1} \to x_s, x_s)^{-1} \ldots (1, x_1 \to x_2, x_s)^{-1} (1, x_1 \to a_1, x_s)$
$(1, a_1 \to a_2, x_s)(1, a_2 \to a_3, x_s) \ldots (1, a_{t-2} \to a_{t-1}, x_s)(1, a_{t-1} \to a_t, x_s)$
and
$(x_s, x_{s-1} \to x_s, 1)^{-1} \ldots (x_s, x_1 \to x_2, 1)^{-1} (x_s, x_1 \to a_1, 1)$
$(x_s, a_1 \to a_2, 1)(x_s, a_2 \to a_3, 1) \ldots (x_s, a_{t-2} \to a_{t-1}, 1)(x_s, a_{t-1} \to a_t, 1)$

(4) Lift of T_1 at $x_s a_1$ is:

$(1, x_{s-1} \to x_s, a_1)^{-1} \ldots (1, x_1 \to x_2, a_1)^{-1} (1, x_1 \to a_1, a_1)$
$(1, a_1 \to a_2, a_1)(1, a_2 \to a_3, a_1) \ldots (1, a_{t-2} \to a_{t-1}, a_1)(1, a_{t-1} \to a_t, a_1)$

(5) Lift of T_1 at $x_s a_2$ is:

$(1, x_{s-1} \to x_s, a_2)^{-1} \ldots (1, x_1 \to x_2, a_2)^{-1} (1, x_1 \to a_1, a_2)$
$(1, a_1 \to a_2, a_2)(1, a_2 \to a_3, a_2) \ldots (1, a_{t-2} \to a_{t-1}, a_2)(1, a_{t-1} \to a_t, a_2)$
$\vdots$

(6) Lift of T_1 at $x_s a_t$ is:

$(1, x_{s-1} \to x_s, a_t)^{-1} \ldots (1, x_1 \to x_2, a_t)^{-1} (1, x_1 \to a_1, a_t)$
$(1, a_1 \to a_2, a_t)(1, a_2 \to a_3, a_t) \ldots (1, a_{t-2} \to a_{t-1}, a_t)(1, a_{t-1} \to a_t, a_t)$

Example 1 presents all lifts of T_1 at $v_1 = x_s a,\ a \in X$, which are exactly a spanning tree in $\Gamma_2(S)$.

Theorem 2. Let T_n be a collection of all lifts of T_1 at $x_1 V_{n-1}$ in $T_n(S)$, where v_{n-1} is a word of length $(n-1)$. Then T_n is a spanning tree in $T_n(S)$.

Proof. By induction on n. Consider T_2 in $\Gamma_2(S)$. By definition T_2 is a collection of lifts and the number of vertices of T_2 equal to number of vertices in $\Gamma_2(S)$, then T_2 is a spanning tree.

Now suppose T_k is a collection of all lifts of T_1 at x_1V_{k-1} in $T_k(S)$, thus the number of vertices of T_k equal to number of vertices in $\Gamma_k(S)$, then T_k is a spanning tree. The vertex $x_1^{\ k}$ in the first copy is connected to $x_2V_{k-1}, x_3V_{k-1}, \ldots, x_nV_{k-1}, a_1V_{k-1}, a_2V_{k-1}, \ldots, a_tV_{k-1}$. This is an extra lift of T_1 at x_1V_{k-1} in $\Gamma_k(S)$. By definition T_{k+1} is $(s+t)$ copies of T_k. Similarly $\Gamma_{k+1}(S)$ is $(s+t)$ copies of $\Gamma_k(S)$. Hence it is a collection of all lifts of x_1V_k in $\Gamma_{k+1}(S)$ and the number of vertices of T_{k+1} equal to number of vertices in $\Gamma_{k+1}(S)$. Then T_{k+1} is a spanning tree (see Figure 3). □

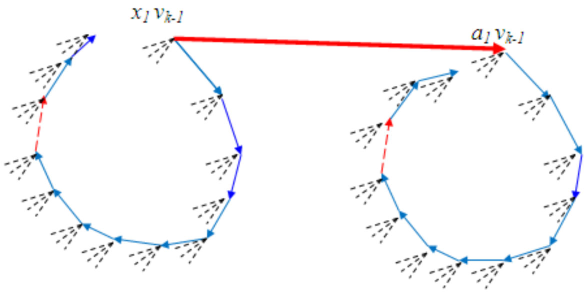

Figure 3. Spanning tree in $\Gamma_k(S)$

Next results show how to compute the total number of lifts in $\Gamma_n(S)$ and the number of edges in spanning tree in $\Gamma_n(S)$.

Corollary 3. The recurrence formula of all lifts of T_{n-1} in $\Gamma_n(S)$ is $l_n = (s+t)l_{n-1} + 1$ where l_i is the total number of lifts of $T_i, (i = 2, 3, \ldots)$ in $\Gamma_{i+1}(S)$ and $l_0 = 0$.

Proof. By induction on n. For $n = 1$ there is only one lift of $T_1 = (1, x_{s-1} \to x_s, 1)^{-1} \ldots (1, x_1 \to x_2, 1)^{-1} (1, x_1 \to a_1, 1)(1, a_1 \to a_2, 1)(1, a_2 \to a_3, 1) \ldots (1, a_{t-2} \to a_{t-1}, 1)(1, a_{t-1} \to a_t, 1)$ at $v_1 = x_s$ and we denote this number by l_1. The total number of lifts of T_2 is $(s+t)+1$, and we denote by l_2 (refer to Example 1). Now let l_k is the total number of lifts of T_{k-1} in $\Gamma_k(S)$ such that $l_k = (s+t)l_{k-1} + 1$. We will prove that l_{k+1} is the total number of lifts of T_k in $\Gamma_{k+1}(S)$ is $l_k = (s+t)l_{k-1} + 1$. By using the Algorithm T_{k+1} is $(s+t)$ copies of T_k plus one (as in proof Theorem 2). Thus, $l_{k+1} = (s+t)l_k + 1$.

Corollary 4. The total number of lifts of T_{n-1} in $\Gamma_n(S)$ is $l_n = \dfrac{(s+t)^n - 1}{(s+t)-1}$.

Proof. We will prove that by induction. For $n=1$ we have $l_1 = \dfrac{(s+t)-1}{(s+t)-1}$.

Then $l_1 = 1$ so its true for $n=1$. Assume true for $n=k$, so $l_k = \dfrac{(s+t)^k - 1}{(s+t)-1}$. For $n = k+1$ applying Corollary 3, we have

$$l_{k+1} = (s+t)l_k + 1 = (s+t)\cdot\frac{(s+t)^k - 1}{(s+t)-1} + 1 = \frac{(s+t)^{k+1} - 1}{(s+t)-1}. \ \square$$

Corollary 5. The recurrence formula of all edges in spanning tree of graph $\Gamma_n(S)$ is $e_n = (s+t)e_{n-1} + (s+t-1)$, where e_n is the total number of edges in spanning tree of $\Gamma_n(S)$ and $e_0 = 0$.

Proof. We argue by induction on *n*. For $n=1$, as in Figure 2, the total number of edges in spanning tree of $\Gamma_1(S)$ is $(s-1)+(t-1)+1 = (s+t-1)$. Now let e_k is the total number of edges in spanning tree of $\Gamma_n(S)$, that is $e_k = (s+t)e_{k-1} + (s+t-1)$ so the formula works when $n=1$. By using the Algorithm and assumption of induction e_{k+1} are $(s+t)$ copies of e_k plus $(s+t-1)$ (as in proof Theorem 2). Thus, $e_{k+1} = (s+t)e_k + (s+t-1)$. □

Corollary 6. The total number of edges in the spanning tree T_n in $T_n(S)$ is $e_n = ((s+t)^n - 1)$.

Proof. By induction on *n*. For $n=1$ we have $e_1 = ((s+t)^1 - 1) = (s+t-1)$ (refer to Figure 2). Now let $e_k = ((s+t)^k - 1)$. To prove that $e_{k+1} = ((s+t)^{k+1} - 1)$.

By Corollary 5, we conclude

$$e_{k+1} = (s+t)e_k + (s+t-1) = (s+t)\cdot(s+t)^k - 1) + (s+t-1)$$

$$= ((s+t)^{k+1} - 1).$$

5. CONCLUSIONS

In this study we determined the new method namely lifting method for finding spanning trees in for 2-complexes of fundamental groups obtained from the union of two semigroup presentations of integers. We also obtained the general formula of all lifts of spanning trees and the number of edges in spanning trees.

ACKNOWLEDGEMENTS

The financial support received from the UKM is gratefully acknowledged.

REFERENCES

Ahmad, A. G. B. and Al-Odhari, A. M. 2004. The graph of diagram groups constructed from natural numbers semigroup with a repeating generator. *Jour. of Inst. of Math & Com. Sci.(Math.Ser.)*. **17**: 67-69.

Guba, V. and Sapir, M. 1997. Diagram Groups. *Memoirs of the American Mathematical Society.* **130**:1-117.

Kilibarda, V. 1994. On the algebra of semigroup diagram. *PhD. Thesis. University of Nebraska.*

Kilibarda, V. 1997. On the algebra of semigroup diagrams, *Int.J of Alg. and Comput.* **7**: 313-338.

Gheisari, Y. and Ahmad, A. G. B. 2011. Lift of spanning trees of 2-complexes from fundamental groups over the semigroup presentations of integers. *Proceeding of Biennial International Group Theory Conference , Universiti Teknologi Malaysia, Johor Bahru, Malaysia,* 101-102.

Gheisari, Y. and Ahmad, A. G. B. 2010. Lift of Spanning trees for diagram groups over the semigroup presentation $P = \langle x, y, z \mid x = y, x = z, y = z \rangle$. *International Journal of Algebra*. **13**: 647-654.