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On the Integral Solutions of the Diophantine Equation $x^4 + y^4 = z^3$

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ABSTRACT

This paper is concerned with the existence, types and the cardinality of the integral solutions for diophantine equation $x^4 + y^4 = z^3$ where x, y and z are integers. The aim of this paper was to develop methods to be used in finding all solutions to this equation. Results of the study show the existence of infinitely many solutions to this type of diophantine equation in the ring of integers for both cases, $x = y$ and $x \neq y$. For the case when $x = y$, the form of solutions is given by $(x, y, z) = (4n^3, 4n^3, 8n^4)$, while for the case when $x \neq y$, the form of solutions is given by $(x, y, z) = (un^{3k-1}, vn^{3k-1}, n^{4k-1})$. The main result obtained is a formulation of a generalized method to find all the solutions for both types of diophantine equations.

Keywords: Integral solutions, diophantine equation, hyperbolic equation, prime power decomposition, coprime integers

INTRODUCTION

Hyperbolic equation has been studied since ancient times; one of which was started as early as 350 years ago by Fermat who showed that the equations $x^4 + y^4 = z^2$ and $x^4 + y^4 = z^4$ have no solution in integers by using the method of proof of infinite descent (for example, refer to Dudley, 1978). Nonetheless, much of work has been done to examine various kinds of hyperbolic equations until the present time and since the variation of these equations exists, this activity will be continued into the future.

Cross (1993) studied the diophantine equation $\alpha^4 + \beta^4 \neq \gamma^4$ in Gaussian integers. In his paper, the author used cyclotomic integers by considering the special case $n = 3$ of the Fermat's Last Theorem to prove that no triplet (α, β, γ) of the non-zero members of Gaussian integers exists, $\mathbb{G}$ with $\gcd(\alpha, \beta) = 1$ such that $\pm\alpha^4 \pm \beta^4 = \pm\gamma^4$. The proof given by the author is a version of Fermat's method of infinite descent.

Similarly, Dieulefait (2005) also proved that with p a prime, $p \not\equiv -1 \ (mod\ 8)$ and

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$p > 13$, the diophantine equation $x^4 + y^4 = z^p$ has no solutions x, y, z with $(x, y) = 1$ and $(x, y) \neq 1$. In the same publication, the author proved that if p is a prime, $p \geq 211$, the equation $x^4 + y^4 = z^p$ would have no primitive solution in non-zero integers. Later on, the author generalized the form of equation studied and considered the following equation, $x^4 + y^4 = qz^p$. He gave a method to solve this equation for some prime values of q and every prime p bigger than 13. The author proved that the diophantine equation, $x^4 + y^4 = qz^p$, does not have any primitive solutions for $q = 73, 89$ and 113 and $p > 13$.

Grigorov and Rizov (1998) studied the diophantine equation, $x^4 + y^4 = cz^4$. In their paper, the authors gave a precise explicit estimates for the difference of the Weil height and the Neron Tate height on the elliptic curve, $v^2 = u^3 - cu$. They later applied this to prove that if $c > 2$ is a fourth power free integer and the rank of $v^2 = u^3 - cu$ is 1, the diophantine equation $x^4 + y^4 = cz^4$ would have no non-zero solution in integers.

In his publication, Poonen (1998) proved that the equation $x^n + y^n = z^m$ has no non-trivial primitive solution for $n \geq 4$. At the same time, by assuming the Shimura-Taniyama conjecture, he also proved that the equation $x^n + y^n = z^3$ has no non-trivial primitive solution for $n \geq 3$.

Motivated by the works on the equation of Fermat's equation of degree four and its variations, this research was undertaken to study the equation $x^4 + y^4 = z^3$ and to find its solutions in the ring of integers. Our quest here was to determine whether the infinitely many solutions to the equation $x^4 + y^4 = z^3$ do exist, and to solve this diophantine equation in the integer domain.

The cases of non-existence of solutions for this equation have been discussed earlier by some authors. For examples, Cohen (2002) gave an assertion as stated in Lemma 1.1. It is easy to show that the solution set of the equation $x^4 + y^4 = z^3$ is non-empty whenever $\gcd(x, y, x) > 1$. For example, $(x, y, x) = (4, 4, 8), (289, 578, 4913)$ is in the solution set. Theorem 1.1 gives a general proof of this particular assertion. Since the solution set is non-empty, we are interested to know the form of integers that satisfies this equation. However, the general forms of the solutions for such equation have not been discussed extensively. Hence, using the elementary methods in this study, we determined some explicit forms of solutions to the equation $x^4 + y^4 = z^3$ when $\gcd(x, y, x) > 1$.

RESULTS AND DISCUSSION

This paper is outlined as follows. There are three theorems discussed. Theorem 1.1 shows that the solution set of $x^4 + y^4 = z^3$ is always non-empty whenever $\gcd(x, y, x) > 1$. Theorem 1.2 deals with the case when $x = y$ and $\gcd(x, y, x) > 1$, while Theorem 1.3 deals with the case when $x \neq y$ and $\gcd(x, y, x) > 1$. We reproduced an assertion of Cohen (2002) in Lemma 1.1, which deals with the equation $x^4 + y^4 = z^3$, whereby $\gcd(x, y, x) = 1$. In addition, two corollaries are also discussed in this paper. Corollary 1.1 gives the forms of solutions for the triplet (a, b, c) when $n = u^4 + v^4$ for a pair of integers u and v is a cube in which there exist infinitely many solutions to the equation $x^4 + y^4 = z^3$. Also, Corollary 1.2 in this paper gives the form of solutions for (a, b, c), when $w = n^k$, where $w = \frac{c}{\gcd(a, b, c)}$, and n and k are positive integers.

We will show that the solution set of the equation in which $\gcd(x,y,x)>1$ is always non-empty in the following theorem. It shows that a solution set can always be constructed from a given pair of integers.

Theorem 1.1

Suppose u and v are integers and $n=u^4+v^4$. Let k be an integer such that $k\equiv 2(\bmod 3)$. Then, $a=un^k$, $b=vn^k$ and $c=n^{\frac{4k+1}{3}}$ is in the solution set of the equation $x^4+y^4=z^3$.

Proof:

By multiplying both sides of the equation $n=u^4+v^4$ with n^{4k}, the following is obtained:

$$n^{4k}\cdot n=(n^{4k}\cdot u^4)+(n^{4k}\cdot v^4)$$

or $$n^{4k+1}=(n^k u)^4+(n^k v)^4$$

Since $k\equiv 2(\bmod 3)$, we have $4k+1\equiv 0(\bmod 3)$.

Thus, there exists integer m such that $4k+1=3m$.

It follows that $$n^{3m}=(n^k u)^4+(n^k v)^4$$

or $$(n^m)^3=(n^k u)^4+(n^k v)^4$$

Hence, $a=un^k$, $b=vn^k$ and $c=n^m$ satisfy the equation $x^4+y^4=z^3$. We obtain the assertion by putting $m=\dfrac{4k+1}{3}$.

A corollary obtained from the above theorem is as shown below. The forms of solutions for the triplet (a,b,c) are obtainable when n is a cube in which case there exist infinitely many solutions to the following equation, $x^4+y^4=z^3$.

Corollary 1.1

Suppose u and v are integers and $n=u^4+v^4$. Suppose $n=r^3$ for some integer, r. Let k be a positive integer. Then $a=un^k$, $b=vn^k$ and $c=n^{\frac{4k+1}{3}}$ is in the solution set of $x^4+y^4=z^3$.

Proof:

From the proof of Theorem 1.1, it can be shown that:

$$n^{4k+1}=(n^k u)^4+(n^k v)^4$$

Let $n=r^3$, then the following will be obtained:

$$(r^{4k+1})^3=(n^k u)^4+(n^k v)^4$$

Hence, $a=un^k$, $b=vn^k$ and $c=n^{\frac{4k+1}{3}}$ is in the solution set of $x^4+y^4=z^3$, if n is a cube.

The following theorem gives the form of solutions to the following equation, when $x = y$ and $\gcd(a,b,c) > 1$.

Theorem 1.2:

Suppose the triplet (a,b,c) in which $a = b$ and $\gcd(a,b,c) > 1$ is a non-trivial solution to $\gcd(a,b,c) > 1$. Then a,b,c is of the form $a = b = 4n^3$ and $c = 8n^4$ for some integer, n.

Proof:

First, suppose that the triplet (a,b,c) is a solution in which $a = b$. Then, the equation will become:

$$2a^4 = c^3 \tag{1.1}$$

Clearly $2 | c^3$ which implies that c is even.

Hence, there exists q_j in the prime power decomposition of c such that $q_j = 2$.

Rearranging the prime power decomposition of c, let $q_1 = 2$, we will have the following:

$c = 2^{f_1} \prod_{j=2}^{m} q_j^{f_j}$ where q_j are odd primes and $f_j \geq 1$ for $1 \leq j \leq m$

Similarly, let $p_1 = 2$ in the prime power decomposition of a. Then, the following will be obtained:

$a = 2^{e_1} \prod_{i=2}^{l} p_i^{ei}$ in which q_j are odd primes, $e_1 \geq 0$, $e_i \geq 0$ for $2 \leq i \leq l$

Substituting the above expression for a and c into (1.1), the following will be obtained:

$$2^{4e_1+1}\left(\prod_{i=2}^{l} p_i^{4e_i}\right) = 2^{3f_1}\left(\prod_{j=2}^{m} q_j^{3f_j}\right) \tag{1.2}$$

By uniqueness of the prime power decomposition of a and c, we will have for each i, the integer j in such a way that $p_i = q_j$ and vice versa for $2 \leq i \leq l$, $2 \leq j \leq m$, and $l = m$.

Also, we will have $2^{4e_1+1} = 2^{3f_1}$, which implies that $4e_1 + 1 = 3f_1$

or $$3f_1 + (-4)e_1 = 1$$

Since $\gcd(3,-4) = 1$, many integral solutions to this equation do exist.

Meanwhile, we can see that $e_1 = 1$, and $f_1 = 3$ is a particular solution that satisfies this equation.

Thus, all the solutions for this equation are given by:

$$e_1 = 2 - 3t_1 \quad \text{and} \quad f_1 = 3 - 4t_1 \tag{1.3}$$

where t_1 is the integer.

Since e_1 and f_1 are positive integers, $t_1 \leq 0$.

Now for each $t_1 < 0$, the integer $s_1 > 0$ exists in such a way that $t_1 = -s_1$. Thus, from (1.3), we will obtain the following:

$$e_1 = 2 + 3s_1 \quad \text{and} \quad f_1 = 3 + 4s_1$$

Similarly from (1.2), by uniqueness of prime power decomposition of a and c, we will have by considering the odd prime factors of a and c that l, $p_i = q_j$ and $4e_i = 3f_j$ for some i and j, where $i, j = 2, 3, ...m$.

From here, we can see that $4 | 3f_j$ and $3 | 4e_i$.

Since $\gcd(4,3) = 1$, the integers r_j and s_i exist in such a way that $f_j = 4r_j$ and $e_i = 3s_i$.

It follows that $s_i = r_j$.

Then, the prime power decomposition of a, b and c will respectively be:

$$a = b = 2^{2+3s_1} \prod_{i=2}^{l} p_i^{3s_i} = 2^2 (2^{s_1} \prod_{i=2}^{l} p_i^{3s_i})^3$$

and

$$c = 2^{3+4s_1} \prod_{i=2}^{l} p_i^{4s_i} = 2^3 (2^{s_1} \prod_{i=2}^{l} p_i^{s_i})^4$$

Let $n = 2^{s_1} \prod_{i=2}^{l} p_i^{s_i}$ in this case.

Thus, $a = b = 4n^3$ and $c = 8n^4$ could be obtained as asserted, where n is an integer.

Now if $t_1 = 0$, we will obtain $e_1 = 2$ and $f_1 = 3$ from (1.3).

Then, the prime power decomposition of a, b and c will respectively be as follows:

$$a = b = 2^2 \prod_{i=2}^{l} p_i^{3s_i} = 2^2 (\prod_{i=2}^{l} p_i^{3s_i})^3$$

and

$$c = 2^3 \prod_{i=2}^{l} p_i^{4s_i} = 2^3 (\prod_{i=2}^{l} p_i^{s_i})^4$$

Let $n = \prod_{i=2}^{l} p_i^{s_i}$ in this case.

Thus, $a = b = 4n^3$ and $c = 8n^4$ could be obtained as asserted, where n is an integer.

Hence, the triplet $(4n^3, 4n^3, 8n^4)$ is a solution to the equation $x^4 + y^4 = z^3$, when $x = y$.

In Theorem 1.1 (i.e. where the solutions are of the form $(a,b,c) = (un^k, vn^k, n^{\frac{4k+1}{3}})$ and in Theorem 1.2 [where the solutions of this equation in which $x = y$ is $(a,b,c) = (4n^3, 4n^3, 8n^4)$], it is clear that $\gcd(a,b,c) > 1$. The subsequent Theorem 1.3 gives the forms of integral solution to the equation $x^4 + y^4 = z^3$, in which $x \neq y$ and $\gcd(a,b,c) > 1$ also. The cases in which $\gcd(x,y,z) = 1$ have been discussed by a number of earlier authors. For example, Cohen (2002) showed that non-trivial integral solutions did not exist in such cases. Thus, this assertion was reproduced in the following Lemma 1.1, the proof of which could be found in Cohen (2002).

Lemma 1.1

The equation $x^4 \pm y^4 = z^3$ has no solutions in the non-zero coprime integers, x, y, x.

The following theorem gives the form of solutions to the equation

$x^4 + y^4 = z^3$ in cases when $\gcd(a,b,c) > 1$ and $x \neq y$ in terms of n, where

$n = \left(\frac{x}{\gcd(x,y,z)}\right)^4 + \left(\frac{y}{\gcd(x,y,z)}\right)^4$ and $\gcd(x,y,z) = n^{3k-1}$, where k is a positive

integer with $k \geq 1$.

Theorem 1.3:

Suppose the triplet (a,b,c) with $a \neq b$ and $\gcd = (a,b,c) = d$ is a solution to the

equation, $x^4 + y^4 = z^3$. Let $u = \frac{a}{d}$, $v = \frac{b}{d}$ and $n = u^4 + v^4$. Let k be a positive

integer. If $d = n^{3k-1}$, then $a = un^{3k-1}$, $b = vn^{3k-1}$ and $c = n^{4k-1}$.

Proof:

Since $a = du$, $b = dv$ and $d = n^{3k-1}$, we have clearly $a = un^{3k-1}$ and $b = vn^{3k-1}$.

From the following equation $a^4 + b^4 = c^3$, it follows that:

$(un^{3k-1})^4 + (un^{3k-1})^4 = c^3$ or $(n^{3k-1})(u^4 + v^4) = c^3$

Since, $n = u^4 + v^4$, we have $(n^{4k-1})^3 = c^3$

which implies that $c = n^{4k-1}$

Hence, $a = un^{3k-1}$, $b = vn^{3k-1}$ and $c = n^{4k-1}$ if $d = n^{3k-1}$, where $n = u^4 + v^4$

A corollary was obtained from the above theorem, as shown below. The forms of solution for the triplet (a,b,c) are obtainable when $w = \frac{c}{d} = n^k$.

Corollary 1.2

Suppose the triplet (a,b,c) with $a \neq b$ and $\gcd = (a,b,c) = d$ is the solution to the

equation $x^4 + y^4 = z^3$. Let $u = \frac{a}{d}$, $v = \frac{b}{d}$, $w = \frac{c}{d}$ and $n = u^4 + v^4$. If $w = n^k$,

then $a = un^{3k-1}$, $b = vn^{3k-1}$ and $c = n^{4k-1}$.

Proof:

From the equation, we have $a^4 + b^4 = c^3$.

It follows that $d^4(u^4 + v^4) = d^3 n^{3k}$.

Since $u^4 + v^4 = n$, we will have the following:

$$dn = n^{3k} \qquad \text{or} \qquad d = n^{3k-1}$$

From Theorem 1.3, we have then $a = un^{3k-1}$, $b = vn^{3k-1}$ and $c = n^{4k-1}$ since $d = n^{3k-1}$.

Table 1 below shows some example solutions $(a,b,c) = (un^{3k-1}, vn^{3k-1}, n^{4k-1})$ that were obtained for Theorem 1.3 for various values of k.

TABLE 1: Some examples of the integer solutions for the diophantine equation $x^4 + y^4 = z^3$ when $x \neq y$ and $\gcd(x,y,z) > 1$ for various values of k.

k	$a = un^{3k-1}$	$b = vn^{3k-1}$	$c = n^{4k-1}$	$d = n^{3k-1}$
1	$a = un^2$	$b = vn^2$	$c = n^3$	$d = n^2$
2	$a = un^5$	$b = vn^5$	$c = n^7$	$d = n^5$
3	$a = un^8$	$b = vn^8$	$c = n^{11}$	$d = n^8$
4	$a = un^{11}$	$b = vn^{11}$	$c = n^{15}$	$d = n^{11}$
5	$a = un^{14}$	$b = vn^{14}$	$c = n^{19}$	$d = n^{14}$
⋮	⋮	⋮	⋮	⋮

Note: The solution set obtained from Theorem 1.3, where $d = n^{3k-1}$ with $k \geq 1$ as represented in Table 1 is also the solution set obtainable from the forms of solutions in Theorem 1.1, in which $k \equiv 2 \pmod 3$. In this study, the assertions of Theorem 1.1, Theorem 1.2 and Theorem 1.3 have been illustrated by showing the actual solutions for the integers (x, y, z), as in Table 2 that satisfy this equation using the C language in integer domain.

TABLE 2: Some examples of the integer solutions generated by the C language for the diophantine equation $x^4 + y^4 = z^3$ when $x = y$ and $x \neq y$.

x	y	z
4	4	8
32	32	128
108	108	648
256	256	2048
289	578	4913
500	500	5000
4000	4000	80000
6724	20172	551368
18818	28227	912673
391876	1959380	245314376
821762	2054405	263374721
340707	454276	38272753
66049	264196	16974593
⋮	⋮	⋮

CONCLUSION

From the discussion above, the non-trivial integral solutions for the diophantine equation $x^4 + y^4 = z^3$ was shown to exist when $\gcd(x, y, z) > 1$. Suppose u and v are the integers and $n = u^4 + v^4$ and k are the integer in such that $k \equiv 2(\text{mod}\,3)$. Then, $a = un^k$, $b = vn^k$ and $c = n^{\frac{4k+1}{3}}$ are the solutions to this equation. Also, suppose u and v are the integers and $n = u^4 + v^4$. If n is a cube, the many solutions of the forms $a = un^k$, $b = vn^k$ and $c = n^{\frac{4k+1}{3}}$ would infinitely exist. As for the case when $x = y$, the triplet $(x, y, z) = (4n^3, 4n^3, 8n^4)$ where n is any integer is a solution to this equation. For the case when $x \neq y$, we have $x = un^{3k-1}$, $y = vn^{3k-1}$ and $z = n^{4k-1}$, where $n = u^4 + v^4$ and $d = n^{3k-1}$. This work points towards a future direction in the determination of solutions to a more generalized equation $x^4 + y^4 = p^k z^3$, where p is a prime and k is any positive integer.

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